Cahit Arf and his invariant.

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Preface

Originally this manuscript was prepared for my talk at the workshop on Sequences, Curves and Codes in Antalya, 25-29 September 2009. Later I had given a talk with the same title on October 5, 2009 at the conference on Positivity, Valuations, and Quadratic Forms in Konstanz.

In the discussion after the talk I learned that there may be an error in Arf's paper and perhaps his main theorem has to be modified. I am indebted to Karim Johannes Becher for this comment. Indeed, after another check I found the error in the proof of one of Arf's lemmas. Accordingly the manuscript had to be corrected, taking care of the situation and clarifying the scope of Arf's theorems after the correction. We have done this in two papers [LR10, LR11].[1]

The present article is the result of unifying and re-editing our two papers mentioned above. For the convenience of the reader we have included an appendix containing very simple proofs of Arf's main results in the corrected form. It seems to be of interest that all the facts and methods which we use in these proofs can be found in Arf's paper already.

[1] We are indebted to Detlev Hoffmann for providing us with relevant information. Also we would like to thank K. Conrad and S. Garibaldi for their help in this matter. We are indebted to the referee for his informative comments, who called our attention to the book [EKM08], in particular §39.

1 Introduction

In January 2009 I received a letter from the organizers of the workshop in Antalya on Sequences, Curves and Codes, with a friendly invitation to participate. The letter was accompanied by a bank note of 10 Turkish Lira. Reading the letter I found out that this was not meant as an advance honorarium for my talk, but it was to tell me that the note carried the portrait of the Turkish mathematician Cahit Arf (1910-1997). Besides the portrait there appears some mathematical text pointing to Arf's discovery of what today is called the *Arf invariant*. Accordingly the organizers in their letter suggested that perhaps I would want to talk about the Arf invariant of quadratic forms.

Although I do not consider myself as a specialist on quadratic forms, it was my pleasure to follow this suggestion. Cahit Arf had been a Ph.D. student of Helmut Hasse in 1937/38. Arf's thesis [Arf39] has become widely known, where he had obtained a generalization of a former theorem of Hasse about the ramification behavior of abelian number fields; today this is known as the "Hasse-Arf theorem".[2] His next paper, after his thesis, contains the "Arf invariant" which is our concern here. This work too was inspired by a suggestion of Hasse. So this report about the Arf invariant fits into my general project to investigate the mathematical contacts of Hasse with various other mathematicians, including Emil Artin, Emmy Noether, Richard Brauer and others, and now with Cahit Arf.

Much of what I know in this respect is based upon the letters between Hasse and his correspondence partners. Those letters are kept in the *Handschriftenabteilung* of the Göttingen University Library, they contain a rich

[2]This generalization had been asked for by Artin in a letter to Hasse. For details from the historic perspective see, e.g., section 6 of [Roq00].

source for those who are interested in the development of algebraic number theory in the 20th century. Among those documents there are preserved about 65 letters between Arf and Hasse from 1939 until 1975.[3] We can see from them that in the course of time there developed a heartfelt friendship between the two.

2 Arf's first letter

The first ten letters are concerned with Arf's work on quadratic forms in characteristic 2. But where are the earlier letters, those about Arf's thesis? The answer is easy: There were no earlier letters, for during his graduate studies while composing his thesis, Arf worked at Göttingen University where Hasse was teaching. And people at the same university usually don't write letters but talk to each other.

Fortunately for us, when Arf worked on quadratic forms in characteristic 2 he was back in Istanbul, and therefore the communication with his former academic teacher travelled by means of letters. On October 12, 1939 Arf wrote to Hasse:[4]

> Sehr geehrter Herr Professor,
>
> Ich habe Ihren Brief vom 29. 9. 39 mit grosser Freude erhalten ...
> Ich habe jetzt eine unschöne Arbeit über quadratische Formen fast fertig geschrieben. Diese Arbeit wollte ich Ihnen vorlegen. Ich glaube aber, dass Sie jetzt wenig Zeit haben. Es handelt sich kurz um folgendes:
>
> Sie hatten einmal den Wunsch geäußert, die Geschlechtsinvarianten einer quadratischen Form mit Hilfe der Algebrentheorie begründet zu sehen. Ich habe versucht dies zu tun. Da die Aufstellung dieser Invarianten für $p \neq 2$ fast trivial ist, habe ich gedacht, dass es nützlich sein würde wenn man zunächst die Theorie in Körpern von der Charakteristik 2 zu übertragen versucht. In der genannten Arbeit über-

[3] In addition there are about 90 letters between Hasse and Arf's wife Halide, mostly in Turkish language, in which Hasse tried to practice and improve his mastery of the Turkish language.

[4] Observe the date of this letter. On September 1, 1939 World War II had started. Perhaps this was the reason why Arf believed that Hasse would not be able to devote much time to deal with Arf's paper since in war time Hasse may have been assigned other duties.

trage ich die Ergebnisse von Witt durch passende Änderungen in den Körpern von der Charakteristik 2 und ich gebe dann die vollständigen Invariantensysteme für arithmetische Äquivalenz der ternären und quaternären Formen in einem Potenzreihenkörper $k((t))$, wobei der Koeffizientenkörper k die Charakteristik 2 hat und vollkommen ist ...

Dear Professor,

I am very glad to have received your letter of September 29, 1939... I have almost completed the draft of a paper on quadratic forms. I had intended to submit it to you. But I believe that now you will not have much time for it. In short, the situation is as follows:

You had once expressed your wish to see the genus invariants of a quadratic form be established with the help of the theory of algebras. This I have tried to do. Since the compilation of those invariants is almost trivial in characterisctic $\neq 2$ *I thought it would be useful at first to try to transfer the theory to fields of characteristic* 2*. In the above mentioned paper I transfer the results of Witt by suitable modifications to fields of characteristic* 2*. And then I give a complete system of invariants for the arithmetic equivalence of ternary and quaternary forms in a power series field* $k((t))$[5] *where the field of coefficients* k *is perfect of characteristic* 2 ...

From this we learn that it had been Hasse who had suggested the topic of Arf's investigation. Hasse's interest in quadratic forms stems from the time of his own thesis, 1923/24, when he had proved the *Local-Global Principle* for quadratic forms over number fields [Has23, Has24]. Later he had established the Local-Global Principle for *central simple algebras* over number fields, in cooperation with Emmy Noether and Richard Brauer [BHN32]. There arose the question as to the mutual interrelation between the theory of quadratic forms and the theory of algebras. Perhaps it would be possible to deduce the Local-Global Principle for quadratic forms from that for algebras?

This question (and more) had been answered beautifully in Witt's seminal paper [Wit37a]. At that time Witt held the position as assistant professor in Göttingen, and he was the leading member of the *Arbeitsgemeinschaft* in

[5] I will use throughout my own notation and do not always follow the various notations in the original letters and papers.

cooperation with Hasse. In his paper Witt associates to every quadratic form f a central simple algebra $S(f)$ of 2-power index, called the *Hasse algebra* which, together with the dimension and the discriminant of the form, makes a complete set of invariants at least over global and local function fields. (Over number fields there is an additional invariant, namely the signature of a quadratic form over the real localizations of the field.)

Witt's paper represents a watershed in the theory of quadratic forms. It provided the basis of the subsequent enormous expansion of the theory of quadratic forms. His biographer Ina Kersten says that this paper "*ranks as one of his most famous works*" [Ker00]. However, Witt's theory covered only forms over a field of characteristic $\neq 2$. This is the point where Arf's paper comes in. He extended Witt's theory to fields of characteristic 2. In particular this applies to the case of local and global function fields of characteristic 2.

The desire to extend Witt's result to characteristic 2 had also been expressed by A. A. Albert in a paper which had just appeared in 1938 in the *Annals of Mathematics* [Alb38]. There had been some letters exchanged between Albert and Hasse during the years 1931-1935 and we know that Albert's interest in the theory of quadratic forms over global fields had been encouraged by Hasse.[6] The above mentioned paper by Albert shows that this interest continued. His paper is a follow-up on Witt's [Wit37a] on quadratic forms: Albert first reproves, in his own way, some of Witt's results for global function fields and then shows that these hold also in the characteristic 2 case: namely that every quadratic form in 5 or more variables is isotropic. And then, in a footnote, he says about Witt's general theory:

> *The results of Witt on quadratic forms on a field of characteristic not two may probably be obtained for the characteristic two case only for forms with cross product terms.*[7] *It would be very interesting to study the analogues of Witt's results for our characteristic two case but the author has not yet done so.*

We see that Arf did just what was proposed here. I do not know whether Arf knew about Albert's paper and its footnote. In his own paper he cites Witt only and says in the introduction:

> Die Anregung zu dieser Arbeit verdanke ich H. Hasse.

[6] See chapter 8 in [Roq05].

[7] It appears that Albert means quadratic forms which are "regular" in the sense as defined below (and "completely regular" in Arf's terminology). See section 4.

I owe to H. Hasse the suggestion for this work.

As we learn from Arf's letter, there was a second part in his manuscript where he investigates quadratic forms over rings of power series and their *arithmetic invariants* , at least for quadratic forms of low dimension. Again, the motivation for this comes from number theory. Due to his results in this second part Arf can be regarded as a forerunner of the general theory of quadratic forms over *rings*, not necessarily fields.[8]

But in the end it turned out that Parts 1 and 2 were published separately. Part 1 appeared in Crelle's Journal where Hasse was editor [Arf41]. Hasse would have liked to get also Part 2 for Crelle's Journal but it seems that there arose difficulties with the printing due to paper shortage in war times and so the second part appeared in the journal of the University of Istanbul.[9]

Here I will discuss only the first paper of Arf [Arf41] where he introduces his Arf invariant. I has turned out that there is an error in Arf's paper which on the one hand reduces the scope of his main result but on the other hand has led to an interesting development in the theory of quaternion algebras over fields of characteristic 2. We shall discuss this in due course.

But before going into details let us familiarize a little with the people involved and with the time of the game.

3 Some personal data

Cahit Arf was born in 1910 in the town of Selanik which today is Thessaloniki. At that time it belonged to the Ottoman empire. But in the course of the Balkan war 1912 his home town was affected and the family escaped to Istanbul, later in 1919 to Ankara and finally moved to Izmir.[10]

Cahit Arf's childhood encompassed the Balkan wars, the World

[8]This is evident when we look at the book by Knus who gives a survey on quadratic forms over rings [Knu80] where the notion of Arf invariant over rings is systematically treated. (However we do not find Arf's Part 2 [Arf43] cited in the bibliography of Knus' book.) It seems that the most recent survey on the topic is contained in the book [EKM08].

[9]See [Arf43]. The paper is usually cited for the year 1943 but at the end of the paper we read that it has been received on May 15, 1944.

[10]The following citations are taken from the Arf biography written by Ali Sinan Sertöz [Ser08]. There one can find more interesting information about the life, the work and the personality of Cahit Arf.

War I, the grand war at Gallipoli, the Greek invasion of western Anatolia and the invasion of Istanbul by the Allied Powers. When finally Turkey emerged as a new independent parliamentary republic in 1923 Cahit Arf was 13 years old. It was the beginning of a new era. The new Republic was hopeful, determined and full of invincible self confidence. These traits were also deeply trenched in young Arf. This would shape his attitude towards mathematics in the future.

In public school Arf's ease in mathematics was soon to be noticed by his teachers. In 1926 his father sent him to France to finish his secondary education at the prestigious St. Louis Lycée. Because of his extraordinary grades in mathematics he graduated in two years instead of the expected three years. Then he obtained a state scholarship to continue his studies at the École Normale Supérieur, again in France.

After his return to Turkey in 1932 Arf taught at high school and since 1933 he worked as instructor at Istanbul University.

It did not take him long to realize that he needed graduate study in mathematics. In 1937 he arrived at Göttingen University to study with Helmut Hasse.

At that time Arf was 27 years of age. I do not know why he had chosen Göttingen as his place of graduate study. Although Göttingen used to be an excellent mathematical center which was attractive to students throughout the world, that period had ended in 1933 when in consequence of the new Nazi policy many of the professors had left Göttingen; this had disastrous effects to the mathematical scene there. It seems improbable that Arf had not heard about the political situation in Germany and its consequences for the academic life in Göttingen. Perhaps his mathematical interests at that time leaned towards algebra and arithmetics and he had found out (maybe someone had advised him) that in Göttingen there was Hasse who was known as an outstanding mathematician in those fields. In fact, measured by the high standard of Arf's thesis which he completed within one year after his appearance in Göttingen, it seems that already in Istanbul he had acquired a profound knowledge in the basics of modern algebra and algebraic number theory, and accordingly he may have chosen Göttingen because of Hasse's presence there.

Helmut Hasse was 38 years of age when in 1937 Arf arrived in Göttingen. Hasse was known as a leading figure in the development of algebraic number theory, in particular of class field theory. Just one year earlier in 1936 he had been chosen as an invited speaker at the International Congress of Mathematicians in Oslo. There he reported on his proof of the Riemann hypothesis for elliptic function fields over finite fields of constants; this proof had appeared in three parts in the 1936 issue of Crelle's Journal [Has36a, Has36b, Has36c]. I would say that in those years Hasse was at the height of his mathematical power (notwithstanding a certain peak of his mathematical activities in the years after World War II). In 1934 Hasse had decided to leave the University of Marburg and to accept an offer to Göttingen. Hasse was not a Nazi but he described his political position as being patriotic. He strongly disagreed with the policy of expelling so many scientists from Germany; he considered this as a tragic loss of intellectual power in Germany. And he tried to do what he could to counteract this. When he decided to move from Marburg to Göttingen he did this with the expressed intention to restore, at least to a certain extent, the glory of Göttingen as an international place for mathematics. Although he spent a lot of time and energy on this, he could not be successful in the political situation. [11]

However, on a relatively small scale Hasse's activity in Göttingen had remarkable success. He managed to attract a number of highly motivated students to his seminar and the *Arbeitsgemeinschaft*. The latter was organized by Witt but Hasse participated at the meetings and led the mathematical direction of the work.

The high scientific level of the work in the *Arbeitsgemeinschaft* is documented in a number of publications in Crelle's Journal and other mathematical journals. Here we only mention volume 176 of Crelle's Journal which appeared just in 1937 when Arf came to Göttingen. A whole part of this volume[12] contains papers which arose in the *Arbeitsgemeinschaft* and in the Seminar, of which Witt's famous paper on the so-called Witt vectors is to be regarded as a highlight. In the same volume (but in another part) appeared Witt's paper on quadratic forms [Wit37a] which, as said earlier already, has decisively influenced Arf's paper on his invariant [Arf41].

Ernst Witt was 26 when he met Arf, hence one year younger. He had studied in Göttingen since 1930. He had received the topic of his Ph.D. thesis from Emmy Noether but since she had been dismissed she could not

[11]For more facts from Hasse's biography see, e.g., Frei's biography [Fre85], as well as Frei's recollections about Hasse in [FR08]. Hasse's involvement with the Nazi regime is discussed, e.g., in [Seg03].

[12]Each volume of Crelle's Journal appeared in 4 parts (*4 Hefte*).

act as his thesis referee.[13] The thesis was concerned with central simple algebras over function fields in the course of which he proved the Riemann-Roch theorem for algebras, a ground breaking paper [Wit34] which nowadays attracts new interest in the setting of non-commutative algebraic geometry.[14] When Hasse came to Göttingen in 1934 he accepted Witt as his assistant on the recommendation of Emmy Noether. Witt has not many publications when compared to the work of other mathematicians, but each one is of high level and witnesses a profound insight into matematical structure. We have already mentioned his 1937 paper where he introduces "Witt vectors" [Wit37b].[15] Earlier the same year there had appeared his paper on quadratic forms [Wit37a] which actually constituted his *Habilitation thesis*.[16]

It seems fortunate that Arf in Göttingen had the chance to join the inspiring and motivated group of young mathematicians around Hasse, and among them Witt. There arose a friendship between the two which lasted for many years. It is without doubt that Arf's paper on quadratic forms in characteristic 2 has been influenced by Witt's in characteristic $\neq 2$.

4 Quadratic spaces

Let K be a field. Classically, a quadratic form over K is given by an expression

$$q(x) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j \qquad \text{with} \qquad a_{ij} \in K\,. \tag{1}$$

Two quadratic forms are said to be *equivalent* if one is obtained from the other by a non-degenerate K-linear transformation of the variables $x = (x_1, \ldots, x_n)$. An *invariant* is a mathematical entity attached to quadratic forms which does not change if a form is replaced by an equivalent form.

Witt had replaced the above notion of quadratic form by the notion of

[13]This was Herglotz with whom Witt maintained close relationship.

[14]It is said that Witt completed the manuscript of his thesis within one week.

[15]It should not be forgotten that Witt vectors had been discovered somewhat earlier already by H. L Schmid [Sch36] who also was one of Hasse's assistants at that time. H. L. Schmid however worked with the main vector components only (*Hauptkomponenten*) where the formulas for addition and multiplication are quite cumbersome. It was Witt who observed that the structural operations for Witt vectors can be described quite easily in terms of the ghost components (*Nebenkomponenten*). In this way he made the calculus of Witt vectors widely applicable.

[16]More biographic information about Witt can be obtained from Ina Kersten's biography [Ker00] and the articles cited there.

quadratic space which was adapted to the "Modern Algebra" of the time [Wit37a]. A quadratic space over K is a vector space V equipped with a function $q : V \to K$ and a bilinear function $\beta : V \times V \to K$ subject to the following conditions:

$$\left.\begin{aligned} q(\lambda x) &= \lambda^2 q(x) \\ q(x+y) &= q(x) + q(y) + \beta(x,y) \end{aligned}\right\} \qquad \text{for } \lambda \in K\,, \quad x, y \in V. \tag{2}$$

We assume V to be of finite dimension n. If $u_1, \ldots, u_n$ is a basis of V then any $x \in V$ may be written as $x = x_1 u_1 + \cdots x_n u_n$ with $x_i \in K$ and then $q(x)$ appears in the form (1) with $a_{ij} = \beta(u_i, u_j)$ for $i < j$ and $a_{ii} = q(u_i)$. In Witt's setup the notion of "invariant" now refers to isomorphisms of quadratic spaces instead of equivalences of quadratic forms.[17]

It is common to interpret $q(x)$ as the "length" of the vector $x \in V$, more precisely as the square of its length. In fact, Witt and also Arf write $|x|^2$ instead of $q(x)$. Similarly $\beta(x, y)$ is interpreted as the "inner product" of the vectors x and y and accordingly Arf writes $x \cdot y$ for it. Witt however writes $x \cdot y$ for $\frac{1}{2}\beta(x, y)$ which is possible in characteristic $\neq 2$ and corresponds more to our geometric intuition, for then one has $x \cdot x = |x|^2$. In characteristic 2 however this is not possible and so we have $x \cdot x = 2|x|^2 = 0$. In other words, in characteristic 2 we have to live with the fact that every vector is orthogonal to itself.

This has consequences. The first observation is that the process of *diagonalization* is not generally possible in characteristic 2. Recall that in characteristic $\neq 2$ every quadratic form admits an equivalent "diagonal" form:

$$q(x) = \sum_{1 \le i \le n} a_i x_i^2 \qquad \text{with} \qquad a_i \in K\,. \tag{3}$$

In Witt's terminology this means that every quadratic space V admits an orthogonal basis $u_1, \ldots, u_n$, where $q(u_i) = a_i$. Thus V splits as an orthogonal direct sum of one-dimensional quadratic subspaces:

$$V = \perp\!\!\!\!\!\!\sum_{1 \le i \le n} K u_i \qquad\qquad \text{in char.} \neq 2\,. \tag{4}$$

But in characteristic 2 this is not always possible. Arf observed that one has to admit also two-dimensional subspaces:

$$V = \perp\!\!\!\!\!\!\sum_{1 \le i \le r} (K u_i + K v_i) + \perp\!\!\!\!\!\!\sum_{1 \le j \le s} K w_j \qquad\qquad \text{in char. } 2\,, \tag{5}$$

[17] Sometimes equivalent quadratic forms are called "isomorphic", meaning that their quadratic spaces are isomorphic.

where u_i and v_i are not orthogonal to each other, i.e., $\beta(u_i, v_i) \neq 0$. After suitable normalization we may assume that

$$\beta(u_i, v_i) = 1.^{18}$$

The dimension n of V is $n = 2r + s$. Thus in characteristic 2 any quadratic form admits an equivalent form as follows:

$$\begin{aligned} q(x) &= \sum_{1\leq i\leq r} \left(a_i x_i^2 + x_i y_i + b_i y_i^2\right) + \sum_{1\leq j\leq s} c_j z_j^2 \\ \text{for} \quad x &= \sum_{1\leq i\leq r} (x_i u_i + y_i v_i) \ + \sum_{1\leq j\leq s} z_j w_j \end{aligned} \tag{6}$$

where we have put

$$a_i = q(u_i)\,, \quad b_i = q(v_i)\,, \quad c_j = q(w_j)\,.$$

Arf speaks of "quasi-diagonalization" since only the second sum in (6) is in pure "diagonal form" whereas the matrix of the first sum splits into 2×2 submatrices along the diagonal. Note that in characteristic 2 the square operator is additive; accordingly the second sum in (6) is called the *quasi-linear part* of q. The second sum in (5) consists of all $z \in V$ which are orthogonal to V; therefore it is denoted by $V^{\perp}$.

In characteristic $\neq 2$ a quadratic form (1) is called *regular*, or equivalently *non-singular*, if it has no equivalent form which can be written in fewer than n variables; this means that in the diagonal form (3) all the coefficients $a_i \neq 0$. In characteristic 2 things are different. Arf retained the above definition of "regular"; in characteristic 2 this means that the coefficients c_j in the quasi-linear part of (6) are linearly independent over the subfield K^2. If there does not appear a quasi-linear part, i.e., if $s = 0$, then Arf called the quadratic form "completely regular" (*vollregulär*).

Today the terminology has changed. Instead of Arf's "completely regular" one says "regular" (or "non-singular") whereas Arf's "regular" is now referred to as "semi-regular". This reflects the experience that in the new terminology, the "regular" forms in characteristic 2 behave in many respects similar to the regular forms in characteristic $\neq 2$.

We shall use here the terminology of today, deviating from Arf's.

[18] Arf however does not assume this and he admits for $\beta(u_i, v_i)$ any non-zero element in K. Therefore his formulas for the Arf invariant look a little more complicated than ours.

The notions of "regular" und "semi-regular" are invariant, and can also be applied to a quadratic space: V is "regular" if $V^\perp = 0$, and V is "semi-regular" if $V^\perp$ does not contain a vector $x \neq 0$ with $q(x) = 0$, i.e., if $V^\perp$ is anisotropic.

Every quadratic space (V, q) can be scaled. If $0 \neq c \in K$ then the scaled quadratic space with scaling factor c is defined to be (V, cq). Notation: $V^{(c)}$. Thus in $V^{(c)}$ the "length" of every vector x is $c \cdot q(x)$ whereas in V it is $q(x)$.

5 Clifford algebras

The *Clifford algebra* $C(V)$ of a quadratic space V is defined as an associative K-algebra (not commutative in general) generated by the K-module V and with the *defining relations*:

$$x^2 = q(x) \qquad \text{for} \qquad x \in V\,. \tag{7}$$

In view of (2) this implies

$$xy + yx = \beta(x, y) \qquad \text{for} \qquad x, y \in V\,. \tag{8}$$

If $u_1, \ldots, u_n$ is a K-basis of V then a K-basis of $C(V)$ is given by the products $u_{i_1} u_{i_2} \cdots u_{i_k}$ with $i_1 < i_2 < \cdots < i_k$ and $0 \leq k \leq n$. The K-dimension of $C(V)$ is 2^n.

In view of its definition $C(V)$ is an *invariant* of V. So is the subalgebra $C_0(V) \subset C(V)$ which is generated by the products $u_{i_1} u_{i_2} \cdots u_{i_k}$ with an *even* number k of factors. The invariance of $C_0(V)$ is a consequence of the fact that the defining relations (7) are of degree 2. The dimension of $C_0(V)$ is 2^{n-1}.

If K is of characteristic 2 then we have the following rule:

$$\textit{If} \quad V = V_1 \perp V_2 \quad \textit{then} \quad C(V) = C(V_1) \otimes C(V_2)\,. \tag{9}$$

Here I have written $V_1 \perp V_2$ to indicate the orthogonal direct sum of V_1 and V_2. The tensor product is taken over K as the base field. The validity of (9) is immediate if we observe that any $x_1 \in V_1$ and $x_2 \in V_2$ are orthogonal to each other, i.e., $\beta(x_1, x_2) = 0$. Hence from (8) we conclude that $x_1 x_2 = -x_2 x_1 = x_2 x_1$ showing that $C(V_1)$ and $C(V_2)$ commute elementwise. (At the same time we see that in characteristic $\neq 2$ this is not the case since the appearing minus sign cannot be disregarded.)

In view of the decomposition (4) we obtain for characteristic 2 that the Clifford algebra $C(V)$ decomposes into the tensor product of r factors $C(Ku_i + Kv_i)$ of dimension 4, and the factor $C(V^{\perp})$ of dimension 2^s. The latter is the center of $C(V)$ and does not appear if V is regular.

6 Binary quadratic spaces

First Arf investigates the Clifford algebra of a regular quadratic space which is *binary*, i.e., of dimension 2. This discussion is quite elementary but it is fundamental for all of Arf's results.

A binary regular quadratic space is generated by 2 elements u, v with the relations

$$q(u) = a\,, \quad q(v) = b\,, \quad \beta(u,v) = 1\,. \tag{10}$$

Its Clifford algebra $C(V)$ is given by the defining relations

$$u^2 = a\,, \qquad v^2 = b\,, \qquad uv + vu = 1\,. \tag{11}$$

This is a central simple algebra of dimension 4, i.e., a quaternion algebra, with the K-basis $1, u, v, uv$.

The even subalgebra $C_0(V)$ is of dimension 2 and has the basis elements 1 and uv with the relation:

$$(uv)^2 + uv = uv(vu+1) + uv = u^2v^2 = ab\,.$$

Putting $w = uv$ and introducing the Artin-Schreier operator $\wp(X) = X^2 - X$ we may write this as:

$$C_0(V) = K(w) \qquad \text{with} \qquad \wp(w) = ab\,. \tag{12}$$

Let us first assume that $ab \not\equiv 0 \bmod \wp(K)$. Then $K(w)$ is a separable quadratic field extension of Artin-Schreier type. Hence it is uniquely determined by the class of ab modulo the additive subgroup $\wp(K)$ of K. This class consists of all elements $c \in K$ of the form $c = \wp(z)$ with $z \in K(w)$. The nontrivial automorphism of $K(w)$ is induced by transformation with u, since $u^{-1}wu = w + 1$ according to (11.) We conclude that $C(V)$ is the cyclic crossed product of $K(w)$ with the factor $a \in K^{\times}$. The usual notation for such crossed product is $C(V) = (K(w), a)$. The factor a is uniquely determined up to norms from $K(w)$.[19]

If $ab \equiv 0 \bmod \wp(K)$ then the the situation is quite similar except that in this case $K(w)$ is not a field but a commutative separable K-algebra which decomposes into the direct product of two copies of K. Writing $ab = \wp(c)$ with $c \in K$ and putting $e_1 = w + c$ and $e_2 = e_1 + 1$, we obtain orthogonal idempotents e_1 and e_2, hence $K(w) = Ke_1 \oplus Ke_2$. By suitable choice of the basis u, v of V one can achieve that $u^2 = a \neq 0$; then u admits an inverse in $C(V)$ and conjugation with u induces the automorphism of $K(w)$ which permutes e_1 and e_2. Hence again, $C(V)$ is a crossed product of $K(w)$ and we write $C(V) = (K(w), a)$.[20] Again, a is determined modulo norms from $K(w)$ but in this case this is trivial since *every* element in K is a norm from $K(w)$.

The above discussion shows in particular that the class of ab modulo the additive subgroup $\wp(K)$ is an invariant of V. This is defined to be the **Arf invariant**:

$$Arf(V) \equiv ab \bmod \wp(K)\,. \qquad ^{21}$$

Arf showed that the original quadratic form $q(x_1, x_2)$ of V can be rediscovered (up to equivalence) from these data by the norm function $N : K(w) \to K$ as follows:

$$\begin{aligned} a \cdot N(x_1 + x_2 w) &= a \cdot (x_1 + x_2 w)(x_1 + x_2(w+1)) \\ &= ax_1^2 + x_1(ax_2) + b(ax_2)^2 \\ &= q(x_1, ax_2)\,. \end{aligned} \tag{13}$$

Although this is not the original quadratic form $q(x_1, x_2)$, it is equivalent to it.

Formula (13) can be interpreted as follows: We may regard $K(w)$ as a quadratic space with respect to the norm function $N : K(w) \to K$. Consider

[19]Let $L|K$ is a separable quadratic field extension and (L, a), (L, a') are two crossed product algebras. If (L, a) and (L, a') are isomorphic then $a' = ac$ with $c \in N(L)$, and conversely. – Quite generally, for cyclic algebras we refer the reader to [LR03] and to [Lor08].

[20]$K(w)$ is a commutative separable algebra over K with an automorphism group G of order 2 (the group interchanging the two copies Ke_1 and Ke_2 of K). Thus $K(w)$ is a quadratic "Galois algebra". The theory of crossed products of Galois algebras can be developed in complete analogy to the theory of crossed products for Galois field extensions. The first who had done this explicitly seems to be Teichmüller in his paper [Tei36b]. His terminology was "Normalring" for what today is called "Galois algebra".

[21]Arf in his paper writes $\Delta(V)$.

the scaled space $K(w)^{(a)}$ which has the quadratic form $aN : K(w) \to K$. Then formula (13) says that $K(w)^{(a)}$ as a quadratic space is isomorphic to V. This isomorphsm is given by $1 \mapsto u$, $w \mapsto a^{-1}v$.

We observe that the extension $K(w)$ of K is uniquely determined by the Arf invariant $Arf(V)$, according to the theory of Artin-Schreier extensions. Moreover, since $C(V)$ is the crossed product $C(V) = (K(w), a)$, it follows from the theory of crossed products that the factor a is uniquely determined (up to norms from $K(w)$) by the algebra $C(V)$ and its splitting field $K(w)$.[22] Combining this information we conclude from (13):

> **Theorem:** *Every regular binary space V is uniquely determined (up to isomorphism) by its Arf invariant and its Clifford algebra.*

A quadratic space (V, q) is called *isotropic* if there exists a vector $x \neq 0$ in V such that $q(x) = 0$. Otherwise V is *anisotropic*. As a consequence of formula (13) we note:

> *A regular binary quadratic space V is isotropic if and only if $Arf(V) \equiv 0 \bmod \wp(K)$.*

For, if $Arf(V) \not\equiv 0 \bmod \wp(K)$ then $K(w)$ is a field extension of K and hence the norm $N(z) \neq 0$ for every $0 \neq z \in K(w)$. If $Arf(V) \equiv 0 \bmod \wp(K)$ then the quadratic form is equivalent to $q(x_1, x_2) = x_1 x_2$. The corresponding quadratic space is called the *hyperbolic plane* and denoted by H.

Remark: Central simple algebras in arbitrary characteristic $p > 0$ with defining relations

$$u^p = a\,, \qquad \wp(w) = c\,, \qquad uw = (w+1)u\,. \tag{14}$$

had been systematically studied earlier, in particular by Teichmüller in his paper [Tei36a]. Such an algebra is called *p-algebra*. Teichmüller denotes it by $(a, c]$.[23]

[22] If $Arf(w)$=0 then $K(w)$ is not a field but a quadratic Galois algebra, as explained above. In this case $V \cong H$; see below.

[23] Apparently this notation had been chosen to signalize the fact that the symbol $(a, c]$ is not symmetric in a and c. Compare it with the notation (a, b) for a quaternion algebra in characteristic $\neq 2$, given by the defining relations (16) below. That symbol is symmetric in the sense that $(a, b) = (b, a)$.

In the case $p = 2$ we obtain a quaternion algebra A. In view of the above considerations we see that $A = (a, c]$ is the Clifford algebra $C(V)$ of the binary quadratic space $V = Ku + Kv$ with (10) for $b = a^{-1}c$. Whereas the relations (11) represent the description of A as the Clifford algebra of the given quadratic space V, the relations (14) put into evidence the description of A as a crossed product of some separable quadratic extension $K(w)$ of K with $\wp(w) = c$. The connection between the two is given by the fact that c is the Arf invariant of the space V, while the quadratic form of V is given by the norm form of $K(w)$ scaled by a; see (13).

The paper of Teichmüller mentioned above appeared in 1936, one year before Arf came to Göttingen. In this paper Teichmüller studies, among other things, the conditions for two such p-algebras $(a, c]$ and $(a, c']$ to be isomorphic. If $p = 2$ then this result has some bearing on Arf's investigations. It would have been desirable that Arf cites Teichmüller's paper and points out the connection between his and Teichmüller's investigation. However Arf did not do this. Why not? Did he not know Teichmüller's paper?

Teichmüller had been a very active member of the Göttingen *Arbeitsgemeinschaft* but he had left for Berlin in early 1937. Hence Arf had probably not met Teichmüller. But certainly Teichmüller's results were known and valued in Göttingen and Arf must have heard about it. I find an explanation for Arf's silence about Teichmüller's work in a certain character trait of Arf which is mentioned in the biography of Sertöz [Ser08]:[24]

> *Arf was in the habit of encouraging young mathematicians to discover mathematics by themselves rather than to learn it from others. To support his cause he would tell how in his university years, i.e., his École Normale years in Paris, he would never attend classes ... but proceed to develop that theory himself.*

It seems that during his stay in Göttingen Arf had proceeded similarly, for Sertöz reports in his biography:

> *Years later in Silivri, Turkey, Hasse would recall that after taking his problem*[25] *Arf had disappeared from the scene for a few months only to come back with the solution.*

[24] The story has been confirmed to me by several Turkish colleagues who had known Cahit Arf personally.

[25] Namely the problem for his Ph.D. thesis.

This suggests to me that when Arf in 1939 was back in Istanbul and worked on quadratic forms then again he had proceeded similarly, i.e., discovering the solution of his problem by himself and not consulting other people or papers. In fact, in his paper [Arf41] Arf cites only one paper explicitly, namely Witt's on quadratic forms [Wit37a]. –

The above Theorem holds in characteristic 2. Let us briefly compare it with the similar situation in Witt's paper for characteristic $\neq 2$: In this case a binary quadratic space is of the form $V = \langle u, v\rangle$ with mutually orthogonal vectors u and v and instead of (10), (11) we have

$$q(u) = a\,, \quad q(v) = b\,, \quad \beta(u,v) = 0 \tag{15}$$

The Clifford algebra $C(V)$ is now given by the defining relations

$$u^2 = a\,, \qquad v^2 = b\,, \qquad uv = -vu\,. \tag{16}$$

Again, this is a quaternion algebra. In the theory of algebras it is often denoted by (a, b). The even subalgebra $C_0(V)$ is of dimension 2 and has the basis $1, uv$ but this time with the relation:

$$(uv)^2 = -uv(vu) = -u^2v^2 = -ab = d\,.$$

where d is the discriminant of V. Thus

$$C_0(V) = K(\sqrt{d})\,.$$

If $d \notin K^{\times 2}$ then[26] this is a quadratic field extension whose non-trivial automorphism is given by transformation with u. And again, we conclude that $C(V)$ is a crossed product of $K(\sqrt{d})$ which splits if and only if a is a norm from L.

If $d \in K^{\times 2}$ then $K(uv)$ is not a field but the direct product of two copies of K. In this case and only in this case the quadratic space is isotropic, and it turns out that in this case the corrsponding quadratic form is equivalent to $q(x,y) = xy$, the hyperbolic plane.

So we see that for binary quadratic spaces Arf's situation in characteristic 2 is quite similar to Witt's situation in characteristic $\neq 2$, the only difference being quite natural, namely that the quadratic splitting field of $C(V)$ is generated by $\sqrt{-ab}$ in the case of characteristic $\neq 2$, whereas in characteristic 2 it is generated by a root of the Artin-Schreier equation $\wp(x) = ab$. And we see already here in the binary case:

[26] $K^\times$ denotes the multiplicative group of the field K and $K^{\times 2}$ is the group of squares.

In characteristic 2 *the Arf invariant* $\mathrm{Arf}(V) \in K/\wp(K)$ *is the analogue of the discriminant* $d(V) \in K^\times/K^{\times 2}$ *in characteristic* $\neq 2$.

This was the guiding idea of Arf when he wrote his paper.

7 Higher dimensional quadratic spaces

Now let V be an arbitrary regular quadratic space over a field K of characteristic 2. From (5) we know that V decomposes into an orthogonal direct sum of two-dimensional spaces:

$$V = \perp\!\!\!\!\sum_{1\leq i\leq r} V_i \qquad \text{where} \qquad V_i = \langle u_i, v_i\rangle \tag{17}$$

and

$$q(u_i) = a_i\,, \quad q(v_i) = b_i\,, \qquad \beta(u_i, v_i) = 1 \qquad\qquad (1 \leq i \leq r)\,. \tag{18}$$

Definition of Arf invariant:

$$\mathrm{Arf}(V) \equiv \sum_{1\leq i\leq r} \mathrm{Arf}(V_i) \quad \mathrm{mod} \;\; \wp(K)\,. \tag{19}$$

Recall that by definition $\mathrm{Arf}(V_i) \equiv a_i b_i \bmod \wp(K)$ so that this definition can also be written as:

$$\mathrm{Arf}(V) \equiv \sum_{1\leq i\leq r} q(u_i)q(v_i) \quad \mathrm{mod} \;\; \wp(K)\,. \tag{20}$$

This formula is printed on the Turkish 10-Lira note where, however, the underlying field is restricted to be $K = \mathbb{F}_2$, the prime field in characteristic 2. In that case $\wp(\mathbb{F}_2) = 0$ and hence the congruence sign $\equiv$ in (20) can be replaced by equality.

If $r > 1$ then it is not clear *a priori* that $\mathrm{Arf}(V)$ is an invariant of V. For, the definition (20) depends on how V is decomposed into orthogonal subspaces V_i in the form (17). One has to show that for every two such decompositions the corresponding sums in (19) are in the same class modulo $\wp(K)$. Arf does it in his paper but the proof requires some cumbersome computations. In later years Witt [Wit54] and Klingenberg [KW54] have given simplified descriptions of $\mathrm{Arf}(V)$ from which one can see more directly its invariance. In the comments to Witt's paper in [Wit98] the editor Ina Kersten reports:

It was Witt's concern in the fifties to eliminate the assumption that the characteristic of the ground field is different from 2.

We interpret this such that Witt had carefully read Arf's paper and tried not only to simplify Arf's proof but also to build a unified theory of quadratic forms, independent of the characteristic. In particular Kersten mentions Witt's cancellation theorem (see section 8 below) and his attempts to investigate in detail the geometric situation which guarantees its validity.

Today we would verify the invariance of $\mathrm{Arf}(V)$ by investigating in more detail the structure of the Clifford algebra $C(V)$. We have already said in section 5 that $C(V)$ contains a subalgebra $C_0(V)$ which is canonically defined by V, namely: $C_0(V)$ is generated by the products with an *even* number k of factors in V. And in section 6 we have seen that in the binary case, $C_0(V_i) = K(w_i)$ is a quadratic extension defined by the relation $w_i^2 + w_i = a_i b_i$ which shows, using Artin-Schreier theory, that the class $\mathrm{Arf}(V_i)$ of $a_i b_i$ is an invariant of $C_0(V_i)$, hence of V_i. But Arf did not consider the subalgebra $C_0(V)$, probably he was not aware at that time that $C_0(V)$ was canonically defined by the quadratic space V. Therefore he had to use somewhat cumbersome explicit computations.

But using the invariance of $C_0(V)$, the following statement immediately shows that $\mathrm{Arf}(V)$ is an invariant:[27]

Proposition:*Let V be a regular quadratic space, represented as an orthogonal sum of two-dimensional spaces as in* (17), (18)*. For each i let $C_0(V_i) = K(w_i)$ with $\wp(w_i) = a_i b_i$. Put $w = \sum_i w_i$, so that $\wp(w) = \sum_i a_i b_i$. Then the quadratic extension $K(w)$ equals the center of $C_0(V)$, and hence by Artin-Schreier theory the class of $\sum_i a_i b_i$ is an invariant of V.*[28]

The essential part of the proof consists in verifying w to commute with every element in $C_0(V)$. I recommend to verify this for $r = 2$ (and then use induction). One has to use that

$$\begin{aligned} C_0(V) &= C_0(V_1) \otimes C_0(V_2) + C_1(V_1) \otimes C_1(V_2) \\ &= K(w_1, w_2) + V_1 \otimes V_2 \end{aligned}$$

[27]See Kneser's paper [Kne54].

[28]If $\wp(w) \equiv 0 \bmod \wp(K)$ then $K(w)$ is not a field but the direct sum of two fields isomorphic to K. We have discussed this situation already in the case of two-dimensional quadratic spaces.

where $C_1(V_i)$ denotes the K-space generated by all products of an odd number of elements in V_i, hence $C_1(V_i) = V_i$. Show that $w = w_1 + w_2$ commutes with w_1, with w_2 and with every product x_1x_2 with $x_i \in V_i$. (Use the fact that $w_ix_i = x_iw_i + x_i$).

Let us mention that in Witt's situation of characteristic $\neq 2$ there arises a problem with the Clifford algebra $C(V)$. For, in general this is not a central simple algebra and it is not a product of quaternion algebras. For this reason in characteristic $\neq 2$ Witt replaced the Clifford algebra $C(V)$ by another algebra $S(V)$ which Witt has called "*Hasse algebra*"; this is defined as follows: First recall the notation (a, b) for the quaternion algebra defined by the relations (16) in characteristic $\neq 2$. Now consider the coefficients a_i appearing in the diagonal form (3) and put $d_i = a_1a_2\cdots a_i$. Then the Hasse algebra is defined as the n-fold tensor product

$$S(V) = \otimes\prod_{1\leq i\leq n}(d_i, a_i) \sim \otimes\prod_{1\leq i\leq j\leq n}(a_i, a_j) \quad (21)$$ [29]

This is a central simple K-algebra and plays a role in Witt's theory of quadratic forms in characteristic $\neq 2$, analoguous to the Clifford algebra in characteristic 2. But its definition (21) depends on the coefficients a_i in the diagonal form (3). In order to show that it is an invariant, it is necessary to study the transformation from one diagonal form to an equivalent one. Witt's computations for this are similar to Arf's computations for the invariance of $\mathrm{Arf}(V)$ in characteristic 2. It seems to me that Arf had modelled his invariance proof for $\mathrm{Arf}(V)$ after Witt's invariance proof for $S(V)$.[30]

8 Witt equivalence

For any type of mathematical structures, the quest for invariants is motivated by the hope to be able to characterize the structures by their invariants (up to isomorphisms), and thus to obtain a classification of the structures under investigation. Arf was concerned with quadratic spaces V in characteristic 2 and in particular with regular spaces. We now know three invariants:

[29] Quite generally we write $A \sim B$ if A, B are central simple K-algebras which determine the same element in the Brauer group $\mathrm{Br}(K)$.

[30] We follow a suggestion of the referee and remark that the Hasse algebra $S(V)$ in characteristic $\neq 2$ is not an invariant of the class of V in the Witt ring $\mathrm{WQ}(K)$ – contrary to the situation in characteristic 2 with the Clifford algebra $C(V)$ (see the next section).

1. the dimension $\dim(V)$,
2. The Clifford algebra $C(V)$ in the Brauer group $\mathrm{Br}(K)$,
3. the Arf invariant $\mathrm{Arf}(V)$ in the additive group $K/\wp(K)$.

For arbitrary fields we cannot expect that these three invariants characterize V up to isomorphisms. But Arf wished to show that for special fields K this is indeed possible. Although, as we shall explain, his main result cannot be upheld in its full generality, it turns out that the theorem is valid, e.g., over global and local fields K in characteristic 2. In order to approach this problem, Arf follows Witt who had discovered the "Witt ring" by introducing a certain equivalence relation.

Recall that a quadratic space V is called "*isotropic*" if there exists a non-zero vector $x \in V$ with $q(x) = 0$. The prototype of an isotropic regular space is the hyperbolic plane H already introduced in section 6. The Arf invariant of H is $\mathrm{Arf}(H) = 0$, and the Clifford algebra is $C(H) \sim 1$, which means that $C(H)$ splits. Arf proves the following

> **Kernel Theorem**: **(i)** *If the regular quadratic space V is isotropic then V is isomorphic to the orthogonal sum $H \perp W$ where W is uniquely determined (up to isomorphisms) by V.*[31]
>
> **(ii)** *Consequently, every regular quadratic space V can be decomposed into an orthogonal sum of a number of spaces isomorphic to H and a space V^* which is anisotropic, and V^* is uniquely determined by V (up to isomorphisms).*

The space V^* is called the *anisotropic kernel* of V. Its quadratic form is called the *kernel form*[32] of V.

As a consequence of this result Arf proves for regular quadratic spaces V, W', W'' the general

> **Cancellation Theorem**: *If $V \perp W \cong V \perp W'$ then $W \cong W'$.*

In characteristic $\neq 2$ this famous cancellation theorem was contained in Witt's paper. Arf has observed that it holds also in characteristic 2 for regular spaces.

[31] Arf proves this under the weaker assumption that V is semi-regular in the sense as defined in section 4. Accordingly, in the cancellation theorem below the spaces W', W'' may be semi-regular – but V is to be regular there.

[32] In German: *Grundform.*

The kernel theorem is essentially a special case of the cancellation theorem. For, the *existence* of a decomposition $V \cong H \perp W$ is easily seen; the main task is to show the *uniqueness* of W. In characteristic $\neq 2$ Witt had proved the cancellation theorem first and then deduced from it the kernel theorem. It appears that Arf did not succeed to do the same in characteristic 2, but he could do the cancellation theorem in the special case of the hyperbolic plane. And that afterwards he had an idea how to deduce from it the general case. This idea is quite nice and since it is short let us present it here in order that it will not be forgotten.

Arf's proof of the cancellation theorem: We may assume that V is of dimension 2. (Otherwise decompose V into an orthogonal sum of binary regular spaces and use induction.) From $V \perp W \cong V \perp W'$ it follows

$$V \perp V \perp W \cong V \perp V \perp W'.$$

Since $V \perp V$ is isotropic, the kernel theorem shows $V \perp V \cong H \perp V'$ with V' of dimension 2. Now we compute

$$\begin{aligned} Arf(V \perp V) &= Arf(V) + Arf(V) = 0 \qquad \text{(in characteristic 2)} \\ =& Arf(H + V') = Arf(H) + Arf(V') = Arf(V') \end{aligned}$$

hence $Arf(V') = 0$. This implies $V' \cong H$ (see page 16.) It follows

$$V \perp V \perp W \cong H \perp H \perp W,$$

and similarly for W' in place of W. Applying twice the kernel theorem we conclude $W \cong W'$.[33]

The anisotropic kernel V^* is a new invariant of the regular quadratic space. The original space V is obtained from V^* by adding an orthogonal sum of a number of hyperbolic planes, as many as the dimension of V requires. We note that

$$C(V) \sim C(V^*) \qquad \text{and} \qquad \mathrm{Arf}(V) \equiv \mathrm{Arf}(V^*) \bmod \wp(K)\,. \qquad (22)$$

since $C(H) \sim 1$ and $\mathrm{Arf}(H) \equiv 0 \bmod \wp(K)$. We conclude:

> *In order to classify the regular quadratic spaces it is sufficient to classify the anisotropic spaces.*

[33] We have already mentioned that in later years Witt has provided new and more general proofs of the cancellation theorem. In view of this it seems that Arf's proof is now considered to be superseded by more powerful methods. We refer to the first chapter of Witt's "Collected Papers" [Wit98], in particular to the remarks by Sigrid Böge on pages 32-34.

It is useful to work with the following

> **Definition of Witt equivalence**: *Two regular quadratic spaces V, W (or quadratic forms) are Witt equivalent if they have isomorphic anisotropic kernels.* Notation: $V \sim W$.

This is indeed an equivalence relation. It blends with the orthogonal sum, i.e., if $V_1 \sim W_1$ and $V_2 \sim W_2$ then $V_1 \perp V_2 \sim W_1 \perp W_2$.[34] The Witt classes of regular quadratic spaces with the operation $\perp$ form a group which we denote by $\mathrm{WQ}(K)$. We have $V \perp V \sim 0$, i.e., the elements of this group are of order 2.

9 Arf's Theorems

Now we are able to state the main result of Arf's paper. For an algebraic function field or power series field K over a perfect base field of characteristic 2 he wished to prove that the above three invariants completely characterize the regular quadratic spaces. The relevant property of these fields was, in his opinion, the following which concerns the Brauer group of K:

> (Q): *The quaternion algebras over K form a group within the Brauer group* $\mathrm{Br}(K)$. In other words: *If A and B are any quaternion algebras over K then $A \otimes B \sim C$ where C is a quaternion algebra again.*

It is not difficult to show that in characteristic 2 function fields and power series fields over a perfect base field have the property (Q). But it seems that Hasse had not seen it immediately and so he asked Arf about it, who replied in a letter of March 29, 1940:

> Wenn A und B normale einfache Algebren vom Grade 2 sind, so ist $A \otimes B$ höchstens vom Index 4. Da aber $[K^{\frac{1}{2}} : K] = 2$ so enthalten A und B Teilkörper die zu $K^{\frac{1}{2}}$ isomorph sind. A und B enthalten

[34] If V and W are semi-regular but not regular then $V \perp W$ need not be semi-regular. Arf considers also this situation but then the relation $V \perp V \sim 0$ does not hold generally.

also Elemente u, v mit $u^2 = v^2 \in K$ die nicht zu K gehören. Es gilt daher

$$(u-v)^2 = 0 \qquad \text{ohne, dass} \quad u - v = 0 \quad \text{gilt.}$$

$A \otimes B$ enthält also ein nilpotentes Element. Der Index von $A \otimes B$ ist daher höchstens 2.

If A and B are central[35] *simple algebras of degree* 2 *then the index of $A \otimes B$ is at most* 4. *But since* $[K^{\frac{1}{2}} : K] = 2$, *both A and B contain subfields which are isomorphic to* $K^{\frac{1}{2}}$. *Hence A and B contain elements u and v respectively with $u^2 = v^2 \in K$, and u, v do not belong to K. Hence we have*

$$(u-v)^2 = 0 \qquad \textit{but not} \quad u - v = 0\,.$$

Thus $A \otimes B$ contains a nilpotent element. Therefore the index of $A \otimes B$ is at most 2.

This settled Hasse's question but at the same time it showed that property (Q) holds for all fields with $[K^{\frac{1}{2}} : K] = 2$.

Arf stated his main results in the form of two theorems. As indicated earlier, there is an error in the proof of his first theorem and in fact there do exist counterexamples. Hence his first theorem has to be corrected. Nevertheless let us first state it as it appears in Arf's paper.

Arf's Theorem 1:[36] **(to be corrected)** *Assume that the field K of characteristic* 2 *satisfies property* (Q). *Then any regular quadratic space V of dimension > 4 is isotropic. Consequently, its anisotropic kernel V^* is of dimension* ≤ 4.[37]

Arf's Theorem 2: *Assume that the field K of characteristic* 2 *has the property that every regular quadratic space of dimension > 4 is isotropic. Then every regular quadratic space over*

[35] Arf used the terminology "normal" but nowadays it is usually said "central" to indicate that the center of the algebra equals the base field. – The K-dimension of a central simple K-algebra A is a square n^2. The number n is called the "degree" of A. The "index" of A is defined to be the degree of the division algebra $D \sim A$.

[36] In Arf's paper [Arf41] this is Theorem 11, and our next Theorem 2 is numbered as Theorem 12 there.

[37] Arf also considered quadratic spaces V which are semi-regular but not regular. For those he claimed that the regular part of V^* is of dimension ≤ 2.

K is uniquely determined, up to isomorphism, by its dimension, its Clifford algebra and its Arf invariant.

Certainly, Arf regarded his second theorem as the highlight of his paper. He had been able to accomplish his aim, namely to characterize quadratic forms by their invariants. His first theorem was to give a sufficient criterion, in terms of the Brauer group, for the field K which implies the characterization.

It can be easily verified that the condition (Q) is *necessary* for the validity of the assertion in Arf's first theorem. But Arf was wrong to believe that it is also sufficient. In order to find the correct condition, necessary *and* sufficient, we first remark that condition (Q) is well known to be equivalent to the following condition:

(S) *Any two non-split quaternion algebras A, B over K admit a common quadratic splitting field.*

If two non-split quaternion algebras have a common quadratic splitting field then they are called "linked". If every two non-split quaternion algebras over K are linked, i.e., if K satisfies condition (S), then K is called "linked".

Every non-split quaternion algebra in characteristic 2 has two kinds of quadratic splitting fields: *separable* and *inseparable* ones. If the quaternion algebras A, B have a common inseparable quadratic splitting field then there is also a common separable quadratic splitting field. This seems to have first been observed by Draxl [Dra75]. A short and easy proof can be found in Lam's paper [Lam02].[38] See also section 13.4 below.

But now comes the surprise: the converse does not hold. If A, B have a common separable quadratic splitting field then they do not necessarily have a common inseparable quadratic splitting field. This has been observed by [Lam02].[39] In view of this the following condition appears stronger than (S):

(S_{ins}) *Any two non-split quaternion algebras A, B over K admit a common inseparable quadratic splitting field.*

[38] Perhaps it is not without interest to note that the formulas for quaternion algebras which have been used in [Lam02] are special cases for $p = 2$ of formulas which have been stated 1936 by Teichmüller for p-algebras in characteristic p [Tei36a].

[39] Lam cites [Tit93] and [Knu93] but his example is simpler and easier to verify.

Fields K with this property may be called "inseparably linked". R. Aravire and B. Jacob [AJ95, AJ96] have shown that the iterated power series field $K = \mathbb{F}_2((X))((Y))$ is linked; but it can be shown that there exist anisotropic regular quadratic spaces dimension > 4 which implies by Baeza's theorem below that K is not inseparably linked. We conclude that condition ($\mathrm{S}_{\mathrm{ins}}$) is properly stronger than (S), hence also properly stronger than Arf's condition (Q) which is equivalent to (S).

It turns out that the proper correction of Arf's first theorem consists of replacing his condition (Q) by the stronger condition ($\mathrm{S}_{\mathrm{ins}}$). This has been shown by Baeza [Bae82]:

> **Baeza's Theorem: (i)** *If K satisfies condition* ($\mathrm{S}_{\mathrm{ins}}$) *then every regular quadratic form over K of dimension > 4 is isotropic.*
> **(ii)** *Conversely, if every regular quadratic form over K of dimension > 4 is isotropic then* ($\mathrm{S}_{\mathrm{ins}}$) *holds.*

Quite generally, the so-called u-invariant $u(K)$ of a field K is defined to be the smallest number u such that every regular quadratic form of dimension $> u$ is isotropic. Thus Baeza's theorem can be formulated as follows:

> *If K is inseparably linked then $u(K) \leq 4$, and conversely.*

If we observe that Arf's first (incorrect) theorem can be formulated as: "*If K is linked then $u(K) \leq 4$,*" then we see that Arf's essential difference to Baeza's theorem consists in the absence of the inseparability condition. Apparently Arf was not aware of the fact that there is a difference of the linkage behavior of quaternions according to separability or inseparability. In fact, this question was first raised in 1974 only, by Draxl [Dra75].

If K is a function field of one variable over a perfect field of constants then there is only one inseparable quadratic field over K, namely $K^{\frac{1}{2}}$. Hence K is inseparably linked and Baeza's theorem is applicable. In fact, in this case this is almost trivial; see our appendix.

The classical fact that $[K^{\frac{1}{2}} : K] = 2$ for a function field K over a perfact base field, was also observed by Albert in his paper [Alb38] which we have cited above already. On this basis Albert had already proved that quadratic forms of 5 variables over such function fields are isotropic, i.e., that $u(K) \leq 4$.

Apparently Arf did not know Albert's paper. When O.F.G. Schilling reviewed Arf's paper in the "Mathematical Reviews" he wrote: "*The author* *is unaware of the work of A.A.Albert*". We observe that Schilling did say this as a statement, not as a guess. Schilling had been a student of Emmy Noether in Göttingen and after Noether's emigration got his Ph.D. with Helmut Hasse in Marburg. Later he went to the USA.[40] At the time when he wrote this review he held a position at the University of Chicago with Albert. He had kept contact to Hasse by mail, and on these occasions he had asked for information about the results in Hasse's Göttingen mathematical circle. It seems likely that he had been informed by Hasse or by someone else from Göttingen about Arf and his results; this enabled him to state that Arf "*was not aware*" of Albert's work, and that he did not add "*apparently*" or something like this. Certainly Schilling himself knew Albert's papers.

Arf's (erroneous) proof of theorem 1 is not easy or straightforward but it is well arranged. It seems to me that Arf's style in his paper was much influenced by the suggestions and the advice of his academic teacher Hasse. For, I have found in Arf's paper a footnote which Hasse, being the editor of Crelle's Journal, had placed at the end of the introduction:

> Anmerkung des Herausgebers: Im Einverständnis mit dem Verfasser habe ich dessen ursprüngliches Ms. überarbeitet.
>
> *Note by the editor: With the consent of the author I have revised his original manuscript.*

We see that Hasse did with Arf's manuscript what he always did as an editor of Crelle's Journal, namely checking manuscripts carefully. As Rohrbach reports in [Roh98]:

> *With his* [Hasse's] *characteristic conscientiousness, he meticulously read and checked the manuscripts word by word and formula by formula. Thus he very often was able to give all kinds of suggestions to the authors, concerning contents as well as form ...*

So he did with Arf's paper. In the Hasse-Arf correspondence we read several times that Arf responds to changes suggested by Hasse, both approvingly and critically. Finally on February 8, 1941 Arf returned the final version to Hasse and wrote:

[40] He first stayed at the Institute for Advanced Study in Princeton where he had been accepted on the recommendation of Hasse who had written a letter to Hermann Weyl.

> Mit gleicher Post schicke ich Ihnen die Korrekturbogen und das Manuskript der Arbeit über quadratische Formen zurück. Die Änderungen an drei Stellen die Sie vorgenommen haben scheinen mir unrichtig. Meine Gründe habe ich am Rand des Manuskripts geschrieben.
>
> *At the same time I am returning the galley proofs and the manuscript on quadratic forms. At three instances your proposed changes seem not to be correct. I have explained my reasons at the margin of the manuscript.*[41]

The paper appeared in the same year 1941.

It seems curious that Hasse had not detected Arf's error although he was quite interested in the subject and had closely examined Arf's paper. This is even more curious since the error is of the same kind which many years ago, in 1927, Emmy Noether had committed in a similar situation and there resulted a close correspondence between Hasse and Emmy Noether about it. This correspondence finally led to their renowned theorem about cyclicity of algebras over number fields. It appears that in 1940 Hasse had forgotten that incident.

The situation back in 1927 had been as follows:[42] Emmy Noether, in a letter to Richard Brauer of March 28, 1927, wrote to him that every minimal splitting field of a division algebra can be embedded into the algebra. Brauer knew that this was not the case and provided her with a counterexample. But this example seemed unnecessarily complicated to Emmy Noether; so she wrote to Hasse, in a postcard of October 4, 1927 asking whether he could construct easy counterexamples for quaternions. Hasse did so: He constructed fields of arbitrary high degree (over $\mathbb{Q}$) which were splitting fields of the classical quaternions but no proper subfield had this splitting property. Thus Noether as well as Hasse learned that one has to distinguish between minimal splitting fields and splitting fields of minimal degree; the latter indeed can be embedded into the algebra.

Now, in Arf's proof in the year 1940, a situation ocurred which was quite similar to the earlier one in 1927. On page 164, in the second paragraph of the proof of his "*Satz 11*", Arf considered two quaternion algebras A_1, A_2 (i.e., Clifford algebras of binary quadratic spaces V_1, V_2). He took separable quadratic extensions K_1, K_2 of K which were splitting fields of A_1 and A_2 respectively, and he assumed $K_1 \neq K_2$. Then $A_1 \otimes A_2$ is split by the field

[41] The manuscript is not preserved. I do not know which changes Hasse had proposed.

[42] I have told this story in detail in [Roq05].

compositum K_1K_2 which has degree 4 over K. In view of his hypothesis (Q) Arf knew that $A_1 \otimes A_2$ is similar to some quaternion algebra, hence it admits a splitting field of degree 2. And now he argued that "necessarily" such splitting field can be found within K_1K_2. But this is not necessarily the case. In other words: K_1K_2 may be a minimal splitting field of $A_1 \otimes A_2$ although it is not a splitting field of minimal degree.

Although Hasse's example of 1927 had referred to quaternions over $\mathbb{Q}$ we have here in characteristic 2 a similar situation. Why had Hasse not seen this error? We will never know. We know that in 1940 it was wartime and Hasse had been drafted to the Navy. He worked at a Navy research institute in Berlin and could attend to his activities as an editor of Crelle's Journal in the evenings and on weekends only. So it seems that he did not check Arf's paper as thoroughly as he was used to in earlier times with other papers for Crelle's Journal.

10 An assessment of Arf's paper

Arf's paper [Arf41] is the first in which quadratic forms over arbitrary fields of characteristic 2 are systematically studied. It is true that there was already some work before Arf which was concerned with quadratic forms in characteristic 2, e.g., Dickson in [Dic01] and Albert [Alb38] – but these discussed only special base fields: finite fields and function fields of one variable, respectively. Due to Arf's general *Ansatz* he has opened the door to an extensive expansion of the theory of quadratic forms, not only over fields but also over arbitrary (commutative) rings.

Arf used the structural language, "modern" at his time, which had been introduced by Witt into the theory of quadratic forms. Thus he spoke of quadratic "spaces" instead of "quadratic forms". Arf was able to extend a good part of Witt's seminal results in [Wit37a] to the case of characteristic 2. He showed the possibility of quasi-diagonalization, he extended Witt's important cancelling theorem (*Kürzungssatz*) to characteristic 2, he investigated the role of Clifford algebras for quadratic forms, and he defined his "Arf invariant" in characteristic 2 as a substitute for the discriminant in characteristic 0.

Arf's theorems were meant to find conditions for fields K of characteristic 2 which guarantee that all regular quadratic spaces over K are characterized (up to isomorphism) by their three invariants:

Dimension,
Clifford algebra,
Arf invariant.

His condition of linkage concerns the structure of the group of quaternion algebras inside the Brauer group over K. Although the proof of his first theorem contained an error and the theorem had to be modified, his paper still keeps its importance.

The error occurred due to the pathological and unforeseen behavior of quaternion algebras in characteristic 2, which nobody of the time was aware of. In fact, it took many years until this was discovered and cleared up, and thus Arf's first theorem could be corrected [Bae82].

Due to Arf's number theoretical background his main interest was directed to global and local fields K. In a letter to Hasse dated March 29, 1940 (from which we have already cited on page 24) he had explained that any two quaternion algebras over K have a common inseparable quadratic splitting field, namely $K^{\frac{1}{2}}$. From this he concluded that K is linked; this is true but it does not lead to the conclusion of Arf's first theorem, namely $u(K) \leq 4$. But in fact, what he showed in his letter is that K is inseparably linked and this, due to Baeza'a theorem, is sufficient for $u(K) \leq 4$.

It seems not impossible that Arf's first version of proof worked for local and global fields of characteristic 2, and that he used correctly this inseparable linkage which he had shown to Hasse. In any case, as will be put into evidence in our appendix, Arf had all the ingredients of such proof at his disposal.

But since Arf did not realize the difference between linked and separably linked quaternions, in his attempt to generalize his argument, he started with the linkage condition (S) (or rather its equivalent condition (Q)), and since he could not prove that this implies inseparable linkage (which we know today is not true), he tried to use separable quadratic splitting fields. And so, since he was convinced of his theorem due to his experience with global fields, he stumbled into his error. And even Hasse, Witt, Albert and O.F.G. Schilling (among others) did not detect his error.

There are many examples in the history of mathematics which show that if people, even respected and competent mathematicians, are convinced of the validity of a theorem then they are apt to accept any decent looking proof, even at the cost of overlooking some little detail which then may necessitate a correction of the theorem.[43]

11 Perfect base fields

If K is perfect of characteristic 2 then there exists only one quaternion algebra over K (up to isomorphisms), namely the split one. The linkage conditions (S) and also (S_{ins}) are trivially satisfied. Therefore, in the characterization of regular quadratic spaces the Clifford algebra can be omitted (since there is only one). Hence for a perfect base field, every regular quadratic space V is characterized by its dimension and its Arf invariant.

But this result does not need Arf's two theorems for its proof. If the field K is perfect then every element in K is a square and hence the set of values $q(V) = K$ for every quadratic space V over K. It follows that the orthogonal sum of any two spaces is isotropic, hence every non-vanishing anisotropic regular quadratic space is of dimension 2 and is characterized by its Arf invariant. An arbitrary regular quadratic space V is the orthogonal sum of its kernel V^* and a number of copies of H, as many as the dimension of V requires.

In the special case when K is finite then $\wp(K)$ is an additive subgroup of index 2 in K and hence there is essentially one anisotropic quadratic space. Its quadratic form is $q(x, y) = x^2 + xy + by^2$ where $b \notin \wp(K)$. If $K = \mathbb{F}_2$ then $b = 1$.

In the literature Arf's Theorem is often reduced to this case. For instance, in the "Wikipedia"[44] we read under the heading "Arf invariant" the following:

> *In mathematics, the Arf invariant of a nonsingular quadratic form over the 2-element field $\mathbb{F}_2$ is the element of $\mathbb{F}_2$ which occurs most often among the values of the form. Two nonsingular quadratic forms over $\mathbb{F}_2$ are isomorphic if and only if they have the same Arf invariant. The invariant was essentially known to Dickson* (1901) *and rediscovered by Cahit Arf* (1941).[45]

[43] One of those examples is Grunwald's theorem in class field theory (1933), which was accepted by Artin, R. Brauer, Hasse and Albert (among others) until Wang presented a counterexample (1948). See, e.g., section 5 of [Roq05].

[44] English version, August 20, 2009. – In the version of June 14, 2011 the text of the Wikipedia article is amended, now mentioning some of Arf's main results.

[45] It seems that the author tacitly assumes that both these quadratic forms have the same number of variables.

Certainly, this is all true. But does it give an idea about the main discoveries of Arf in connection with his invariant? In Arf's paper the field $\mathbb{F}_2$ is not mentioned at all, nor are finite fields. In a small remark, covering four lines only, Arf mentions how his theory applies easily and almost trivially in the case of perfect fields of characteristic 2. The main motivation and the main results of the theory of Arf invariants are concerned with fields which are not finite, not even perfect. Arf studied the role of central simple algebras in the theory of quadratic forms in characteristic 2. This aspect is not even scratched in this article of Wikipedia. Moreover, the definition of Arf invariant as given in this article is valid only for the base $\mathbb{F}_2$ and does not apply to other fields of characteristic 2.[46]

The cited Wikipedia text seems to be written exclusively in view of the application of Arf's theory in topology. For, several of those applications are mentioned in the article. And indeed in topology one has to compute cohomology and other functors with coefficients modulo 2 which means that the base field is $\mathbb{F}_2$. An overview of the application of Arf invariants in topology is given by Turgut Önder in the appendix of the Collected Papers of Cahit Arf [Arf90]. But, as said above, this is not representative of Arf's work which is meant to exhibit the role of central simple algebras in the theory of quadratic forms in characteristic 2.

By the way, in the 1901 book of Dickson on linear groups [Dic01] which is mentioned in the Wikipedia article, also the case of an arbitrary finite base field of characteristic 2 is treated, not only $\mathbb{F}_2$. The fact that Arf did not cite this book may have one of two reasons: either he knew Dickson's book and found it is of no relevance for his investigation (which would be understandable), or he did not know it (which is more probable in view of his particular character trait which we have mentioned in section 6).

In any case, the statement that Arf has "'rediscovered"' what Dickson had known is misleading. Arf discussed a quite different theorem, and in a very special case this implies the statement of Dickson.

12 Epilog

After my conference talk I was asked about Arf's biography for the years after his paper on quadratic forms. I will not repeat here what is said in

[46] It is a good exercise to identify this defintion with Arf's definition when the base field is $\mathbb{F}_2$.

his biographies contained in [Arf90, Ser08]. Let me only mention that he became a prominent member of the Turkish scientific community (which is documented by the fact that his portrait decorates an official banknote) – but he also was a dedicated teacher. Many younger mathematicians in Turkey had been introduced by him into mathematics, he had encouraged them and showed them understandingly the way into our science. He is widely remembered in the mathematical community of Turkey. Robert Langlands, in his article about his impressions in Turkey, remembers warmly his discussions with Arf [Lan04]. In particular Arf had directed Langlands' attention to a paper by Hasse on the local decomposition of the ε-factors; these factors appear in the functional equation of Artin's L-series. As Langlands says (English translation):

> "*I had rapid advance in my research having read Hasse's paper...*" and: "*...thanks to Cahit bey, I solved this problem during my stay in Ankara and proved the existence of the local ε-factor.*"

I was also asked to report more extensively on the correspondence between Arf and Hasse, in particular the letters after 1941. I plan to do this some time in the future. Let me only mention that these letters, although they do not discuss any more mathematics proper, show a growing friendship between the two. Hasse visited Turkey several times between 1957 and 1975. The last two preserved letters, dated March 1975, concern the proposal to have an international colloquium on the structure of absolute Galois groups. This colloquium was planned by Arf jointly with M. Ikeda (who had earlier got a position in Turkey on the recommendation of Hasse). This conference took place in September 1975 in Silivri, a small village on the beach of the Marmara sea. I had the chance to participate in this conference and was able to observe the close friendly relationship between the two mathematicians, Arf and Hasse.

13 Appendix: Proofs

The aim of this section[47] is to put into evidence that the proof of Baeza's theorem, i.e., the correction of Arf's first theorem, can be done solely with the arguments which can be found in Arf's paper. Thus Arf could well have proved Baeza's theorem, avoiding his error, if only he would have recognized the difference between linkage of separable and inseparable type.

[47] This section has been written jointly with Falko Lorenz.

We also add a simple proof of Arf's second theorem, as well as of Draxl's lemma.

We believe that our proofs, based on Arf's paper, are simpler than any of those which can be found in the literature.

13.1 Baeza's Theorem (i)

K denotes a field of characteristic 2. We use the following notation:

Let V be a regular quadratic space of dimension 2 over K. There is a K-basis u, v of V with

$$q(u) = a\,, \quad q(v) = b\,, \quad \beta(u,v) = 1 \qquad (a, b \in K)\,. \tag{23}$$

Here, $q : V \to K$ denotes the quadratic form of V and β is the corresponding bilinear form. Let $A = C(V)$ be the Clifford algebra of V. This is a quaternion algebra over K with basis $1, u, v, w$ and the relations

$$u^2 = a\,, \quad v^2 = b\,, \quad uv + vu = 1\,, \quad w = uv\,. \tag{24}$$

We identify the quadratic space V with the subspace of A generated by u and v, and then $q(x) = x^2$ for all $x \in V$. If $x^2 \notin K^2$ then $K(x)$ is an inseparable quadratic subfield of A.

Lemma 1. *Let $y \in A$. Then $y^2 \in K$ if and only if $y \in V + K$.*

If A does not split then this lemma gives a complete description of the inseparable quadratic subfields $L = K(y) \subset A$: then $y = x + c$ with suitable $x \in V$, $c \in K$.

The statement of Lemma 1 can be found in Arf's paper [Arf41] on page 161. Arf does not formulate the statement in the form of a lemma, he just performs the computations which we give in the proof below and uses them in his text.[48]

Proof of Lemma 1:

We represent $y \in A$ in the form

$$y = x + z \qquad \text{with} \quad x \in V, \quad z = c_0 + c_1 w \in K(w), \quad c_0, c_1 \in K \tag{25}$$

[48] Arf's notation is different from our's.

and compute

$$\begin{aligned} y^2 &= x^2 + z^2 + xz + zx \\ &= x^2 + z^2 + c_1(xw + wx) \\ &= x^2 + z^2 + c_1 x \end{aligned} \tag{26}$$

where we have used that $xw + wx = x$ for $x \in V$, which is a consequence of the relations (24). Now, $x^2 = q(x) \in K$, $z^2 \in K(w)$ and $c_1 x \in V$. Since $V \cap K(w) = 0$ we conclude that $y^2 \in K$ if and only if $c_1 x = 0$ and $z^2 \in K$, hence $c_1 = 0$ since $K(w)|K$ is separable.
□

In the following we regard K as a 1-dimensional quadratic space with the quadratic form $q(c) = c^2$ for $c \in K$. The sum $V + K \subset A$ is direct and can be regarded as the orthogonal sum $V \perp K$ of quadratic spaces.

Lemma 2. *Let V, V' be regular quadratic spaces of dimension 2 and A, A' their Clifford algebras. Assume that A and A' do not split. Then A, A' have a common inseparable quadratic splitting field if and only if $V \perp V' \perp K$ is isotropic.*

Proof:

Let $K(y) \subset A$ and $K(y') \subset A'$ be isomorphic inseparable quadratic subfields of A and A' respectively. We choose the generators y, y' in such a way that they correspond to each other in the isomorphism $K(y) \cong K(y')$, so that $y^2 = y'^2$. By Lemma 1 we have $y = x + c$ with $0 \neq x \in V$ and $c \in K$. Similarly $y' = x' + c'$ with $0 \neq x' \in V'$ and $c' \in K$. We conclude

$$x^2 + c^2 = x'^2 + c'^2 \,.$$

Putting $d = c + c'$ we obtain

$$x^2 + x'^2 + d^2 = 0 \tag{27}$$

which shows that the quadratic space $V \perp V' \perp K$ is isotropic.

Conversely, assume $V \perp V' \perp K$ is isotropic. There is a nontrivial relation of the form (27) with $x \in V$, $x' \in V'$, $d \in K$. It follows

$$x^2 = (x' + d)^2 \,. \tag{28}$$

Let, say, $x \neq 0$. Since A does not split we have $x^2 \notin K^2$. Hence $K(x)$ is a quadratic inseparable subfield of A. From (28) follows that $K(x' + d) \subset A'$

is isomorphic to $K(x)$.
□

Proof of Baeza's theorem, part (i) (see page 27) :

We assume that every two non-split quaternion algebras over K have a common inseparable quadratic splitting field. We claim that every regular quadratic space of dimension > 4 is isotropic. Write this space in the form

$$V \perp V' \perp W$$

where V and V' are of dimension 2 and W of dimension > 0. We may assume that V and V' do not split.

Let $y \in W$ and assume first that $q(y) = 1$. Then the subspace $Ky \subset W$ is isomorphic to K as a quadratic space. Since the Clifford algebras $C(V)$ and $C(V')$ have a common inseparable quadratic splitting field (by assumption) we infer from Lemma 2 that $V \perp V' \perp Ky$ is isotropic. Hence $V \perp V' \perp W$ is isotropic too.

But W may not contain a vector y with $q(y) = 1$. In this case we use the method of scaling (see page 13). If $0 \neq c \in K$ then the scaled quadratic space $V^{(c)} \perp V'^{(c)} \perp W^{(c)}$ is isotropic if and only if $V \perp V' \perp W$ is isotropic. Now choose $y \in W$ with $q(y) \neq 0$ and take the scaling factor $c = q(y)^{-1}$. Then $c \cdot q(y) = 1$. From what has been shown above it follows that $V^{(c)} \perp V'^{(c)} \perp W^{(c)}$ is isotropic, hence so is $V \perp V' \perp W$.
□

13.2 Arf's second theorem

We assume that every regular quadratic space of dimension > 4 over K is isotropic. We claim that every regular quadratic space over K is uniquely determined (up to isomorphism) by its dimension, the Brauer class of its Clifford algebra and its Arf invariant.

Proof:

Recall that $\mathrm{WQ}(K)$ denotes the (additive) group of Witt classes of regular quadratic spaces over K; see section 8. The Witt class of the space V is represented by its anisotropic kernel V^*. The whole space V arises from V^* by adding a number of hyperbolic planes H, as many as the dimension of V requires.

Hence our claim reduces to the claim that *the Witt class of V, i.e., the anisotropic space V^*, is uniquely determined by its Brauer class of $C(V)$ and the Arf invariant* $\mathrm{Arf}(V)$.

We recall that

$$C(V^*) \sim C(V) \qquad \text{and} \qquad \mathrm{Arf}(V^*) = \mathrm{Arf}(V)\,.$$

The Clifford algebra yields a homomorphism $V \mapsto C(V)$ of $\mathrm{WQ}(K)$ into the Brauer group $\mathrm{Br}(K)$. And the Arf invariant yields a homomorphism $V \mapsto \mathrm{Arf}(V)$ of $\mathrm{WQ}(K)$ into the additive group $K/\wp(K)$. We have to show:

If $C(V) \sim 1$ *in* $\mathrm{Br}(K)$ *and* $\mathrm{Arf}(V) = 0$ *then* $V \sim 0$ *in* $\mathrm{WQ}(K)$.

We have $\dim V^* \leq 4$ by hypothesis. Since V^* is regular we have $\dim V^* = 0, 2$ or 4. The case $\dim V^* = 2$ is not possible since $\mathrm{Arf}(V^*) = 0$ implies $V^* = H$, the hyperbolic plane, hence V^* would not be anisotropic.

Suppose $\dim V^* = 4$ and write $V^* = V_1 \perp V_2$ as the orthogonal sum of two binary spaces. Since $C(V^*) \sim 1$ we have $C(V_1) \sim C(V_2)$. Since both algebras have the same dimension they are isomorphic: $C(V_1) = C(V_2)$. Also, since $\mathrm{Arf}(V^*) = 0$ we have $\mathrm{Arf}(V_1) = \mathrm{Arf}(V_2)$. We have seen in section 6 that a binary regular space is uniquely determined by its Clifford algebra and its Arf invariant. It follows $V_1 = V_2$ hence $V^* = V_1 \perp V_1 \sim 0$, so again V^* would not be anisotropic.

□

13.3 Baeza's theorem (ii)

The following lemma is the separable analogue to Lemma 1. The situation is the same as in Lemma 1.

Lemma 3. *Let $y \in A$. Then $\wp(y) \in K$ if and only if $y \in V + K + w$.*

If A does not split then this lemma gives a complete description of the separable quadratic subfields $L = K(y) \subset A$: then $y = x + c + w$ with suitable $x \in V$, $c \in K$.

The statement of Lemma 3 can also be found in Arf's paper [Arf41] on page 165. Arf does not formulate this statement in the form of a lemma, he just performs the computations which we give in the proof below and uses them in his text.[49]

[49] Our notation differs from Arf's notation.

Proof of Lemma 3:

We represent $y \in A$ in the form as before in (25) and compute:

$$\begin{aligned} \wp(y) = y^2 - y &= x^2 - x + z^2 - z + c_1 x \\ &= x^2 + (c_1 - 1)x + \wp(z) \end{aligned} \tag{29}$$

where we have used that $xz + zx = c_1 x$. We conclude that $\wp(y) \in K$ if and only if $c_1 = 1$ and so $z = c_0 + w$, hence $y \in V + K + w$.

Changing notation, we write $c = c_0$ and hence $y = x + c + w$. Observing that $\wp(c + w) = \wp(c) + \wp(w)$ we have shown that

$$\wp(y) = \wp(x + c + w) \equiv x^2 + \wp(w) \mod \wp(K)\,. \tag{30}$$

We will have occasion to use this formula later.
$\square$

In the following Lemma we consider $K(w + w') \subset A \otimes A'$ as a quadratic space with respect to its norm function $N : K(w + w') \to K$. Explicitly we have

$$N(c_0 + c_1(w + w')) = c_0^2 + c_0 c_1 + c_1^2 \wp(w + w')\,. \tag{31}$$

The sum $V + V' + K(w + w') \subset A \otimes A'$ is direct and can be regarded as the orthogonal sum $V \perp V' \perp K(w + w')$ of quadratic spaces.

Lemma 4. *Let V, V' be regular quadratic spaces of dimension 2 and A, A' their Clifford algebras. Assume that A and A' do not split. Then A, A' have a common separable quadratic splitting field if and only if $V \perp V' \perp K(w+w')$ is isotropic.*

Proof:

Let $L \subset A$ and $L' \subset A'$ be isomorphic separable quadratic subfields of A and A' respectively. We may write $L = K(y)$ and $L' = K(y')$ where y, y' correspond under the isomorphism $L \cong L'$ and hence $\wp(y) = \wp(y') \in K$. By Lemma 3 we have $y = x + c + w$ and $y' = x' + c' + w'$ with suitable $x \in V$, $x' \in V'$, $c, c' \in K$. From (30) we obtain:

$$\begin{aligned} x^2 + \wp(w) &\equiv x'^2 + \wp(w') && \mod \wp(K) \\ x^2 + x'^2 &\equiv \wp(w) + \wp(w') && \mod \wp(K) && (32) \\ x^2 + x'^2 &= \wp(d) + \wp(w + w') && \text{with } \ d \in K\,. && (33) \end{aligned}$$

hence

$$x^2 + x'^2 + \wp(d + w + w') = 0\,.$$

Here, $\wp(d+w+w') = N(d+w+w')$; see (31). It follows that the quadratic space $V \perp V' \perp K(w+w')$ is isotropic.

Conversely, assume that $V \perp V' \perp K(w+w')$ is isotropic. We have to show that there exists a common separable quadratic splitting field of A and A'. There exists a nontrivial relation of the form

$$x^2 + x'^2 + N(z) = 0 \qquad \text{with} \quad x \in V,\ x' \in V',\ z \in K(w+w')\,.$$

We write $z = c_0 + c_1(w+w')$ and use formula (31) for the norm. It follows

$$x^2 + x'^2 + c_0^2 + c_0c_1 + c_1^2\wp(w+w') = 0 \tag{34}$$

Suppose first that $c_1 \neq 0$. After dividing by c_1^2 on both sides in (34) and changing notation we may assume $c_1 = 1$. Hence

$$x^2 + x'^2 + \wp(c_0 + w + w') = 0$$

which gives

$$x^2 + \wp(w) \equiv x'^2 + \wp(w') \bmod \wp(K)$$

and using (30):

$$\wp(x+w) \equiv \wp(x'+w') \mod \wp(K)\,.$$

It follows that the quadratic Artin-Schreier extensions $K(x+w) \subset A$ and $K(x'+w') \subset A'$ are isomorphic.

If $c_1 = 0$ then (34) shows that $V \perp V' \perp K$ is isotropic. From Lemma 2 we infer that A and A' have a common *inseparable* splitting field. We have already mentioned Draxl's lemma which says that then there is also a common *separable* splitting field. For a simple proof of Draxl's lemma see the next section 13.4.

□

Proof of Baeza's theorem part (ii) (see page 27) :

We assume that every regular quadratic space of dimension > 4 is isotropic. We claim that every two non-split quaternions A, A' over K have a common inseparable quadratic splitting field.

We write $A = C(V)$ as the Clifford algebra of a 2-dimensional regular quadratic space V as in (23), (24), and similarly $A' = C(V')$. As above we consider the separable quadratic extension $K(w+w') \subset A \otimes A'$ as a quadratic space with respect to the norm. The 6-dimensional space

$V \perp V' \perp K(w + w')$ is isotropic (by assumption) and hence Lemma 4 shows that A and A' have a common separable quadratic splitting field L. But we are looking for a common *inseparable* quadratic splitting field; this will be established as follows.

The common separable quadratic splitting field L can be embedded into A and into A'; this yields isomorphic separable quadratic subfields in A and in A'. After changing notation we now identify these two fields, so that A, A' appear as crossed products of the same separable quadratic field L. Writing $L = K(w)$ with $\wp(w) = c \in K$ we have $A = (a, c]$ and $A' = (a', c]$ with certain $a, a' \in K$. We refer to our discussion on page 17. As explained there, A, A' appear now as the Clifford algebras of the quadratic spaces $L^{(a)}$ and $L^{(a')}$ respectively, which are scaled quadratic spaces of the space L with respect to the norm.

Now, by hypothesis the 6-dimensional space

$$L^{(a)} \perp L^{(a')} \perp L$$

is isotropic. Thus there is a nontrivial relation of the form

$$aN(z) + a'N(z') + N(y) = 0 \qquad \text{with} \qquad z,\, z',\, y \in L\,. \tag{35}$$

If $y \neq 0$ then after dividing by $N(y)$ and changing notation we obtain

$$aN(z) + a'N(z') + 1 = 0 \qquad \text{with} \qquad z, z' \in K(w)\,.$$

This shows that $L^{(a)} \perp L^{(a')} \perp K$ is isotropic. Applying Lemma 2 we see that there exists a common inseparable quadratic splitting field of A and A'.

If $y = 0$ then from (35) we infer that $L^{(a)} \perp L^{(a')}$ is isotropic, hence $L^{(a)} \perp L^{(a')} \perp K$ is isotropic too and again Lemma 2 applies. $\square$

13.4 Draxl's Lemma

Lemma 5 (Draxl)**.** *Let A and A' be two nonsplitting quaternion algebras over K. If A, A' have a common inseparable quadratic splitting field then they also have a common separable quadratic splitting field.*

Proof:

We represent $A = C(V)$ as the Clifford algebra of a binary quadratic space V, so that

$$A = V + K(w) \;\text{ with }\; V = Ku + Kv, \;\; uv + vu = 1, \;\; w = uv \tag{36}$$

as explained in section 6. Similarly for A':

$$A' = V' + K(w') \text{ with } V' = Ku' + Kv', \quad u'v' + v'u' = 1, \quad w' = u'v'. \quad (37)$$

Let $L \subset A$ and $L' \subset A'$ be isomorphic inseparable quadratic subfields of A and A' respectively. We may write $L = K(t)$ and $L' = K(t')$ where t, t' correspond to each other under the isomorphism $L \cong L'$, hence $t^2 = t'^2 \in K$.

Choose $s \in V + K$ which does not commute with t; then $0 \neq ts + st = d \in K$. Dividing s by d we may assume $ts + st = 1$. Thus $Kt + Ks$ is also a quadratic space with the Clifford algebra A. Changing notation, we write again u, v in place of t, s now have the situation (36) with u being the chosen generator of the field $L = K(u) \subset A$.

We do the same with L' and A', and now (after changing notation) we have the situation (37) with u' being the chosen generator of $L' = K(u')$ which corresponds to u. Thus

$$u^2 = u'^2 \in K.$$

We have to find isomorphic separable quadratic subfields $K(y) \subset A$ and $K(y') \subset A'$. To this end we put according to Lemma 3:

$$y = cu + v + w \qquad \text{and} \qquad y' = cu' + w'$$

where the coefficient $c \in K$ will be determined below. Using (30) we compute (congruences are mod $\wp(K)$):

$$\begin{aligned} \wp(y) &\equiv (cu + v)^2 + \wp(w) \\ &\equiv c^2 a + b + c + \wp(w)\,; \\ \wp(y') &\equiv c^2 a + \wp(w')\,. \end{aligned}$$

We see that

$$\wp(y) \equiv \wp(y') \iff c \equiv b + \wp(w) + \wp(w')\,.$$

If c is chosen that way then the separable quadratic Artin-Schreier fields $K(y) \subset A$ and $K(y') \subset A'$ are isomorphic.
□

References

[AJ95] B. Aravire and B. Jacob. p-Algebras over maximally complete fields. *Proc. Symp. Pure Math.*, 58(2):27–49, 1995.

[AJ96] R. Aravire and B. Jacob. Versions of Springer´s theorem for quadratic forms in characteristic 2. *Am. J. Math.*, 118:235–261, 1996.

[Alb38] A.A. Albert. Quadratic null formes over a function field. *Ann. Math. (2)*, 39:494–505, 1938.

[Arf39] C. Arf. Untersuchungen über reinverzweigte Erweiterungen diskret bewerteter perfekter Körper. *J. Reine Angew. Math.*, 181:1–44, 1939.

[Arf41] C. Arf. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. I. *J. Reine Angew. Math.*, 183:148–167, 1941.

[Arf43] C. Arf. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. II. *Re. Fac. Sci. Univ. Istanbul (A)*, 8:297–327, 1943.

[Arf90] C. Arf. *The Collected Papers.* Turkish Mathematical Society, 1990. 422 pp.

[Bae82] R. Baeza. Comparing u-invariants of fields in characteristic 2. *Bol. Soc. Bras. Mat.*, 13(1):105–114, 1982.

[BHN32] R. Brauer, H. Hasse, and E. Noether. Beweis eines Hauptsatzes in der Theorie der Algebren. *J. Reine Angew. Math.*, 167:399–404, 1932.

[Dic01] L. E. Dickson. *Linear groups: With an exposition of the Galois field theory.* Dover, New York, 1901. XVI, 312 pp. Reprint from the original edition 1901.

[Dra75] P.K. Draxl. Über gemeinsame quadratische Zerfällungskörper von Quaternionenalgebren. *Nachr. Akad. Wiss. Göttingen. Math. Phys. Kl. 3.F.*, 16:251–259, 1975.

[EKM08] R. Elman, N. Karpenko, and A. Merkurjev. *The algebraic and geometric theory of quadratic forms.*, volume 56 of *Colloquium Publications.* American Mathematical Society, Providence, RI, USA, 2008. 435 pp.

[FR08] G. Frei and P. Roquette, editors. *Emil Artin and Helmut Hasse Their correspondence 1923-1934. With an introduction in English.* Universitäts–Verlag, Göttingen, 2008. 497 pp.

[Fre85] G. Frei. Helmut Hasse (1898-1979). *Expositiones Math.*, 3:55–69, 1985.

[Has23] H. Hasse. Über die Äquivalenz quadratischer Formen im Körper der rationalen Zahlen. *J. Reine Angew. Math.*, 152:205–224, 1923.

[Has24] H. Hasse. Äquivalenz quadratischer Formen in einem beliebigen algebraischen Zahlkörper. *J. Reine Angew. Math.*, 153:158–162, 1924.

[Has36a] H. Hasse. Zur Theorie der abstrakten elliptischen Funktionenkörper. I. die Struktur der Divisorenklassen endlicher Ordnung. *J. Reine Angew. Math.*, 175:55–62, 1936.

[Has36b] H. Hasse. Zur Theorie der abstrakten elliptischen Funktionenkörper. II. Automorphismen und Meromorphismen. Das Additionstheorem. *J. Reine Angew. Math.*, 175:69–88, 1936.

[Has36c] H. Hasse. Zur Theorie der abstrakten elliptischen Funktionenkörper. III. die Struktur des Meromorphismenrings. *J. Reine Angew. Math.*, 175:193–207, 1936.

[Ker00] I. Kersten. Biography of Ernst Witt. *Contemp. Math.*, 272:155–171, 2000.

[Kne54] M. Kneser. Bestimmung des Zentrums der Cliffordschen Algebren einer quadratischen Form über einem Körper der Charakteristik 2. *J. Reine Angew. Math.*, 193:123–125, 1954.

[Knu80] M.-A. Knus. *Quadratic and Hermitian Forms over Rings.* Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, Heidelberg, New York, 1980. VII, 520 pp.

[Knu93] M.-A. Knus. Sur la forme d'Albert et le produit tensoriel de deux algèbres de quaternions. *Bull. Soc. Math. Belg.*, 45:333–337, 1993.

[KW54] W. Klingenberg and E. Witt. Über die Arfsche Invariante quadratischer Formen mod 2-. *J. Reine Angew. Math.*, 193:121–122, 1954.

[Lam02] T.Y. Lam. On the linkage of quaternion algebras. *Bull. Belg. Math. Soc.*, 9:415–418, 2002.

[Lan04] R. Langlands. Benim tanidigim Cahit Arf (Recollections of a year in Turkey with Cahit Arf) (turkish). *Matematik Dünyasi*, 2004(winter number), 2004.

[Lor08] F. Lorenz. *Algebra. Volume II: Fields with structure, algebras and advanced topics. Transl. from the German by Silvio Levy. With the collaboration of the translator.* Universitext. Springer, New York, 2008. 336 pp.

[LR03] F. Lorenz and P. Roquette. The theory of Grunwald-Wang in the setting of valuation theory. In Franz-Viktor et al. Kuhlmann, editor, *Valuation theory and its applications, vol II. Proceedings of the international conference and workshop, University of Saskatchewan, Saskatoon, Canada, July 28–August 11, 1999.*, volume 33 of *Fields Inst. Commun.*, pages 175–212, Providence, RI, 2003. American Mathematical Society.

[LR10] F. Lorenz and P. Roquette. On the Arf invariant in historical perspective. *Mathematische Semesterberichte*, 57:73–102, 2010.

[LR11] F. Lorenz and P. Roquette. On the Arf invariant in historical perspective. part 2. *Mathematische Semesterberichte*, 59, 2011.

[Roh98] H. Rohrbach. Helmut Hasse and Crelle's Journal. *J. Reine Angew. Math.*, 500:5–13, 1998.

[Roq00] P. Roquette. On the history of Artin's L-functions and conductors. Seven letters from Artin to Hasse in the year 1930. *Mitt. Math. Ges. Hamburg*, 19*:5–50, 2000.

[Roq05] P. Roquette. *The Brauer-Hasse-Noether Theorem in historical perspective.*, volume 15 of *Schriftenreihe der Heidelberger Akademie der Wissenschaften.* Springer–Verlag, Berlin, Heidelberg, New York, 2005. I, 77 pp.

[Sch36] H. L. Schmid. Zyklische algebraische Funktionenkörper vom Grade p^n über endlichem Konstantenkörper der Charakteristik p. *J. reine angew. Math.*, 175:108–123, 1936.

[Seg03] S. L. Segal. *Mathematicians under the Nazis.* Princeton University Press, Princeton, NJ, 2003. xxii, 530 pp.

[Ser08] A.S. Sertöz. A Scientific Biography of Cahit Arf (1910-1997). unpublished manuscript, 2008.

[Tei36a] O. Teichmüller. p-Algebren. *Deutsche Math.*, 1:362–388, 1936.

[Tei36b] O. Teichmüller. Verschränkte Produkte mit Normalringen. *Deutsche Math.*, 1:92–102, 1936.

[Tit93] J. Tits. Sur les produits de deux algèbres de quaternions. *Bull. Soc. Math. Belg.*, 45:329–331, 1993.

[Wit34] E. Witt. Riemann–Rochscher Satz und ζ–Funktion im Hyperkomplexen. *Math. Ann.*, 110:12–28, 1934.

[Wit37a] E. Witt. Theorie der quadratischen Formen in beliebigen Körpern. *J. Reine Angew. Math.*, 176:31–44, 1937.

[Wit37b] E. Witt. Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p. *J. Reine Angew. Math.*, 176:126–140, 1937.

[Wit54] E. Witt. Über eine Invariante quadratischer Formen modulo 2. *J. Reine Angew. Math.*, 193:119–120, 1954.

[Wit98] E. Witt. *Collected papers – Gesammelte Abhandlungen. Ed. by Ina Kersten. With an essay by Günter Harder on Witt vectors.* Springer, Berlin, 1998. xvi, 420 pp.