In Memoriam Ernst Steinitz (1871-1928)

by Peter Roquette

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In 1910, in volume 137 of Crelle's Journal there appeared a paper with the title

> *Algebraische Theorie der Körper* (Algebraic Theory of Fields).

The author was Ernst Steinitz. Let us use this occasion of a centenary to recall the impact which Steinitz's paper had upon the mathematicians of the time, and its role in the development of today's algebra. Bourbaki [Bou60] states that this paper has been

> *. . . un travail fondamental qui peut être considéré comme ayant donné naissance à la conception actuelle de l'Algèbre.*
>
> . . . a fundamental work which may be considered as the origin of today's concept of algebra.

One of the first eager readers of Steinitz's paper was Emmy Noether. At the time when the paper appeared she was still living in her hometown Erlangen, during what may be called the period of her apprenticeship, studying the highlights of contemporary mathematics of the time. Her guide and mentor in this period was Ernst Fischer. We can almost be sure that Steinitz's paper was the object of extensive discussions between Fischer and Emmy Noether.[1]

Steinitz's ideas contributed essentially to the shaping of Emmy Noether's concept of mathematics and in particular of algebra. During the next decade every one of Noether's papers (except those on mathematical physics) contains a reference to Steinitz [Ste10].

[1] The mathematical letters between Fischer and Noether are preserved in the archive of the University of Erlangen. – Fischer's name is still remembered today from the "Riesz-Fischer Theorem" in the theory of Hilbert spaces.

Later in Göttingen, when she had started her own "*completely original mathematical path*"[2] she took Steinitz's results as "well known" and she used them as the basis for her work on fields and rings – in particular in her project to reformulate classical algebraic geometry in terms of abstract commutative algebra. She urged her students and her fellow mathematicians to study the classical papers which she considered to be the roots of abstract algebra – among them was invariably the 1910 paper by Steinitz. Van der Waerden reports in [vdW75]:

> *When I came to Göttingen in 1924, a new world opened up before me. I learned from Emmy Noether that the tools by which my questions could be handled had already been developed by Dedekind and Weber, by Hilbert, Lasker and Macaulay, by Steinitz and by Emmy Noether herself. She told me that I had to study the fundamental paper of E. Steinitz ...*

Van der Waerden's textbook "*Moderne Algebra*" [vdW30], which was based on lectures by Emmy Noether and Artin, contains in his first volume a whole chapter about Steinitz's theory. Van der Waerden says in [vdW75]:

> *In earlier treatises, number fields, and fields of algebraic functions were usually treated in separate chapters, and finite fields in still another chapter. The first to give a unified treatment, starting with an abstract definition of "field", was E. Steinitz in his 1910 paper. In my Chapter 5, called "Körpertheorie", I essentially followed Steinitz ...*

Van der Waerden's "*Moderne Algebra*" was widely read and translated into many languages; in this way Steinitz's ideas became known worldwide as part of the basics of contemporary algebra, and they found their way into the syllabus of beginner courses. Almost all the notions and facts about fields which we teach our students in such a course, are contained in Steinitz's paper.

But what are those notions and facts? Let us point out first that the main point of Steinitz's paper was his abstract approach. As Purkert says in his essay on the genesis of abstract field theory [Pur73]:

> *Hier wurde erstmalig eine abstrakte algebraische Struktur auf der Grundlage ihres Axiomensystems zum Gegenstand der Untersuchung gemacht. Dieses formalagebraische Denken einerseits, die*

[2] Quoted from Alexandrov's obituary for Emmy Noether [Ale83].

> *Verbindung mit der Mengenlehre andererseits – das sind die Charakterzüge der modernen strukturellen Algebra.*
>
> This was the first time that an abstract algebraic structure was studied on the basis of its system of axioms. On the one hand the formal algebraic thinking, on the other hand the connection to set theory – these are the characteristics of the modern algebra of structures.

Hence, when we now describe the content of Steinitz's paper we have to keep in mind that not only are those definitions and theorems important, but also the fact that they were obtained in the *abstract axiomatic setting*, notwithstanding the fact that some of them had already appeared in earlier treatises for fields of special kinds (fields of numbers, of functions, and finite fields). Steinitz's paper was the first systematic investigation of the structure of abstract fields.[3]

In his preface Steinitz states that he wishes

> *eine Übersicht über alle möglichen Körpertypen zu gewinnen und ihre Beziehungen untereinander in ihren Grundzügen festzustellen.*
>
> to obtain a general overview of all possible types of fields and to determine their relations with each other.

In any abstract field, Steinitz showed that there is a unique smallest subfield which he called the **prime field**; this is either infinite (and then isomorphic to the rationals) or its cardinality is a prime number p (and then it is isomorphic to the integers modulo p). Accordingly he defined the **characteristic** of a field to be either 0 or p respectively. Any integral domain determines a unique **field of quotients**; this is a frequently used method to construct fields. For any field K and an irreducible polynomial $f(X) \in K[X]$ there is an **extension field** L **containing a root** ϑ of $f(X)$. If L is minimal with this property then L is uniquely determined up to K-isomorphism. Steinitz discovered that in prime characteristic an irreducible polynomial may have

[3]Actually, an earlier article by Heinrich Weber [Web93] also works in the framework of abstract field theory. Steinitz mentions that in his article. But he points out that Weber's investigation is directed to Galois theory only while his (Steinitz) goal is the systematic investigation of the structure of all fields. As said in [Kle99]: "While Weber *defined* fields abstractly, Steinitz *studied* them abstractly."

multiple roots, and so one has to distinguish between **separable** and **inseparable** algebraic extensions.[4] Galois theory in this abstract setting holds for separable extensions only. Steinitz defined the notion of **transcendence degree** and he showed that every field can be obtained as an **algebraic extension of a purely transcendental field**, i.e., a field of rational functions over the prime field. Finally, he constructed for every field an **algebraic closure** and showed that it is unique up to isomorphism – this is probably Steinitz's most important result. This theorem on the algebraic closure of any field is sometimes considered to be the proper **Fundamental Theorem of Algebra**.

All the above has been included by van der Waerden in the first volume of his textbook "*Moderne Algebra*". This appeared in 1930. The second volume, appearing one year later, contains the beginning of modern algebraic geometry in the framework of commutative algebra. In his Heidelberg lecture [vdW97] van der Waerden gives a lively account of how he discovered the algebraic definition of "**generic point**" and "**dimension**" of an algebraic variety – and in this connection he again refers to Steinitz's paper. He says about it:

> *Die Wichtigkeit dieser Arbeit kann man garnicht überschätzen, das Erscheinen dieser Arbeit war ein Wendepunkt in der Geschichte der Algebra des 20. Jahrhunderts.*
>
> One cannot overestimate the importance of this paper. The appearance of this paper marks a turning point in the history of algebra of the 20th century.

Immediately after the publication of van der Waerden's book it was rated, by the referee in "*Jahrbuch für die Fortschritte der Mathematik*", to be the

> *Standardwerk der modernen Algebra in der ganzen mathematischen Welt.*
>
> standard treatise of modern algebra in the whole mathematical world.

Van der Waerden's was not the first textbook in which Steinitz's theory was incorporated. Two such textbooks had appeared earlier. One of them was

[4] Steinitz speaks of exensions of the *first kind* and *second kind* respectively. The terminology *separable* and *inseparable* appears in van der Waerden's "*Moderne Algebra*". It seems probable that this terminology had been created by Artin. For, van der Waerden reports in [vdW75] that he took lecture notes of Artin's course on algebra in the summer of 1926. He says: "*In the theory of fields Artin mainly followed Steinitz, and I just worked out my notes ... the presentation given in my book is Artin's.*"

Haupt's two-volume "*Algebra*" which appeared in 1929 [Hau29]. This book too had been written under the influence of Emmy Noether. Otto Haupt held a professorship in Erlangen since 1921. In his reminiscences [Hau88][5] he wrote:

> *Ich kam nach Erlangen als „klassisch" gebildeter Mathematiker, noch völlig unberührt von den damals aufkommenden, als „modern" bezeichneten neuen Ideen in der Mathematik. In dieser Verfassung machte ich die Bekanntschaft der ... eifrigen Propagandistin der modernen Algebra Emmy Noether. Auf gemeinsamen Spaziergängen erzählte E. N. uns von ihren algebraischen Arbeiten. Ich verstand nicht viel von ihren Erzählungen und fragte E. N. wie ich zu einem besseren Verständnis kommen könne. Sie verwies mich als beste Einführung auf die 1910 erschienene Crellearbeit von Steinitz.*
>
> I arrived in Erlangen as a "classically" educated mathematician, still untouched by the so-called "modern" ideas which emerged at that time. In these circumstances I got to know Emmy Noether[6] , ... the eager propagandist for modern algebra. On joint walks she told us about her algebraic work. I did not comprehend much of what she told us, and I asked her how to get to a better understanding. She recommended the Crelle paper of Steinitz which had appeared in 1910.

We can picture the situation: Emmy Noether being an ardent walker, striding speedily up the *Rathausberg* near Erlangen and fervently persuading Haupt, who tries to keep pace with her, not only to understand modern algebra but also to write a textbook on it. Which he did.

For one year, 1929-1930, Haupt's book was the only source for outsiders to learn about the "modern" ideas of algebra. Even in far away Yale, a young student, Saunders Mac Lane, was told by his teacher Oystein Ore to

> *read the monograph by Steinitz and the textbook on algebra by Otto Haupt...*

As Mac Lane reports [ML81], this happened in the year 1929, one year before van der Waerden's "*Moderne Algebra*" appeared. Today Haupt's book

[5] written when Haupt was 100 years old.

[6] At that time Emmy Noether was living in Göttingen but she used to visit her home town Erlangen from time to time.

is almost forgotten, although it contains some more material of Steinitz's field theory than van der Waerden's, in particular concerning inseparability phenomena. The fact that van der Waerden's book was more popular than Haupt's in terms of both its number of editions and translations is probably due to the style of writing. Although sometimes it is pretended that in mathematics only the facts are important, it is a common experience that even in mathematics the style of writing counts.

Let us turn back to the year 1910 when Steinitz's paper [Ste10] had appeared. Besides Emmy Noether there was another mathematician who was keenly interested in this paper, namely Kurt Hensel in Marburg. For Steinitz had mentioned Hensel's p-adic fields in a footnote of the introduction:

> *Zu diesen allgemeinen Untersuchungen wurde ich besonders durch Hensels Theorie der algebraischen Zahlen angeregt, in welcher der Körper der p-adischen Zahlen den Ausgangspunkt bildet, ein Körper, der weder den Funktionenkörpern noch den Zahlkörpern im gewöhnlichen Sinne beizurechnen ist.*
>
> I was inspired to these general investigations by Hensel's "Theory of Algebraic Numbers", in which the field of p-adic numbers is used as the starting point. Such a field cannot be counted among the function fields or number fields in the ordinary sense.

Here, Steinitz refers to Hensel's book [Hen08] which had appeared in 1908.

In the year 1912 Hensel attended the 5-th International Congress of Mathematicians (ICM) in Cambridge, England. There he met the Hungarian mathematician Josef Kürschak who gave a talk with the title:

> *Über Limesbildung und allgemeine Körpertheorie.*
>
> On the concept of limit and general field theory.

Kürschak worked in the framework of Steinitz's abstract field theory. He defined what today is called a **valuation** of a field K, as a map $a \mapsto |a|$ from K to the real numbers, with the standard properties. He showed that Cantor's method, which Cantor had used for the construction of the reals by means of Cauchy sequences of rationals, works for any valuation of an abstract field in the sense of Steinitz. In this way he constructed the **completion** of a valued field. As an application this gives a construction of Hensel's p-adic field as the completion of the p-adic valuation of the rationals. This method is standard today.

Moreover, Kürschak showed that the algebraic closure (in the sense of Steinitz) of a complete field carries a unique valuation extending the valuation of the base field. In other words, he proved that a complete field is **Henselian** in contemporary terminology. In this abstract setting Kürschak used and proved what is now called Hensel's Lemma. It seems remarkable that his proof works at the same time for both archimedean and nonarchimedean valuations.

Kurt Hensel was impressed by this and he took Kürschak's paper for publication in Crelle's Journal of which he (Hensel) was the chief editor. The paper [Kür13] appeared in volume 142 as kind of follow-up to Steinitz's paper [Ste10] which had appeared in volume 137.

Kürschak did not publish any further paper on valuation theory. But there was a young mathematician in Marburg with Hensel who was eager to take over and to extend, generalize and simplify the Steinitz–Kürschak theory. This was Alexander Ostrowski.[7]

Ostrowski was a brilliant young mathematician. He had started his studies in Kiev but was sent by his academic teacher Grave to Germany since, being of Jewish origin, Ostrowski seemed not to have much chance in the academic world in Russia at that time. Ostrowski arrived in Marburg in 1911 when he was 18 years old. He soon was engrossed in valuation theory in the sense of Steinitz–Kürschak. His contributions during the years 1913-1918 shaped valuation theory into essentially the form we use today. One of the interested readers of Ostrowski's papers was Emmy Noether, who in the year 1916 started a correspondence with him.[8]

As a somewhat bizarre story let us mention that Ostrowski, being a citizen of Russia, had been interned in Marburg during World War I when Germany was at war with Russia. But Hensel could persuade the authorities to permit Ostrowski to use the University Library for his studies during the day. Hence for more than three years Ostrowski sat daily in the reading room of Marburg University Library. It was there where he wrote his seminal papers on valuation theory which were published in Crelle's Journal, Acta Mathematica and Mathematische Annalen. He also wrote his big monograph on valuation theory [Ost34] which was completed in 1916 but appeared only in 1934 – not as a book but in three parts in the journal "*Mathematische Zeitschrift*". Van der Waerden says in [vdW97]:

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[7] Hensel himself, although interested in Kürschak's work, stayed on the traditional side. He continued to use his own construction of p-adic fields, based on p-adic power series.

[8] For more details, see [Roq02].

> *Ostrowski setzte der Bewertungstheorie mit seinen großen Abhandlungen in der Mathematischen Zeitschrift, Band 39, die Krone auf. Die Darstellung dieser Theorie im zweiten Band meiner Algebra beruht ganz auf dem Werk von Ostrowski.*
>
> Ostrowski crowned valuation theory with his great papers in the *Mathematische Zeitschrift*, volume 39. The presentation of this theory in the second volume of my Algebra is completely based on Ostrowski.[9]

After the war, in the year 1918, Ostrowski left Marburg for Göttingen. He did not publish any other paper on valuation theory. (But in his paper [Ost33] on Dirichlet series he used the results of Steinitz–Kürschak and himself.)

It was some years later, in 1921, that the young student Helmut Hasse arrived in Marburg from Göttingen in order to study p-adic fields. There Hasse learned about the work of Steinitz–Kürschak–Ostrowski. In the course of many years Hasse became, in the words of van der Waerden [vdW75],

> *Hensel's best and a great propagandist of p-adic methods.*

In the fall of 1922 Hasse went to the University of Kiel as *Privatdozent*; he stayed there until 1925. Since 1920 Steinitz had held a professorship in Kiel; thus Hasse and Steinitz were at the same university during those years. I did not find any information about whether Hasse and Steinitz had mathematical discussions or joint work during this time. Steinitz was known as "*der große Schweiger*"[10] (the great silent man) – which meant that it was not easy to come into close contact with him. But certainly both had met. Frei [Fre77] reports that there was only one office in the mathematics department, and this had to be shared by all staff: by the two professors Steinitz and Toeplitz and the two *Privatdozenten* Hasse and Robert Schmidt.

In the academic year 1924/25 Hasse gave a course on "*Höhere Algebra*". In the second part he presented field theory in the form of Steinitz's paper [Ste10]. Hasse's notes from this course became the basis for his two-volume textbook "*Höhere Algebra*" (Higher Algebra). The book appeared in the "Göschen" textbook series, which at that time was well known among the German-speaking mathematicians. Its second part appeared 1927. (This was

[9] In the same paragraph in [vdW97] van der Waerden says that valuation theory was started by Rychlik. But there seems to have been some mix-up. As said above, valuation theory had been inititiated by Kürschak in 1912 on the basis of Steinitz's paper. Rychlik has given some contributions to it, beginning in 1919, and he cites Kürschak and Ostrowski.

[10] Quoted from Haupt's reminiscences [Hau88].

two years before Haupt's book mentioned above.) The referee (R. Brauer) of this part stated:

> *Der zweite Band behandelt die Theorie der Gleichungen höheren Grades, er schließt sich an die Arbeit von Steinitz (1910) an ...*
>
> The second volume covers the theory of equations of higher degree. Its concept follows that of Steinitz's paper (1910) ...

In the correspondence between Hasse and Emmy Noether [LR06] one can see that the latter sent Hasse some advice for his algebra book. But this concerned certain details of proof only, e.g., the primitive element theorem. As said above, Hasse devised the concept of the book during his years in Kiel, independent of Emmy Noether.

Hasse's estimate of Steinitz's paper can be seen from a footnote in the second volume of his algebra textbook where he said:

> *In diese grundlegende Originalarbeit zur Körpertheorie sollte jeder Algebraiker einmal hineingesehen haben.*
>
> Every algebraist should have read at least once this basic original paper on field theory.

In order to facilitate this, Hasse had Steinitz's paper reprinted in book form, together with comments and an appendix by Baer [Ste30]. This happened in the years 1928/29 while Hasse held a professorship at the University of Halle. Reinhold Baer was *Privatdozent* there. In the preface of the book the editors Baer and Hasse praise the text as a

> *klassisch schöne, formvollendete und in allen Einzelheiten durchgeführte Darstellung...*
>
> classically beautiful, perfectly structured exposition taking care of every detail...

And they continue:

> *... auch heute noch ist die Steinitzsche Arbeit eine vortreffliche, ja geradezu unentbehrliche Einführung für jeden, der sich auf dem Gebiet der neueren Algebra eingehenden Studien hingeben will.*
>
> ... still today Steinitz's paper is an excellent and in fact indispensable introduction for everybody who wishes to study modern algebra more extensively.

The appendix by Baer contains a detailed presentation of Galois theory, which was not explicitly covered by Steinitz.

The comments by the two editors concern those proofs in Steinitz's paper which use Zermelo's well-ordering theorem and the principle of transfinite induction. These were necessary to treat infinite algebraic extensions, or fields with infinite degree of transcendency over their prime field. Steinitz was well aware that this depends on the axiom of choice which at that time was not generally accepted. He said:

> *Das Auswahlprinzip erscheint auch unvermeidlich, wenn man den Beweis der Existenz einer algebraisch abgeschlossenen Erweiterung für jeden beliebigen Körper führen will ... Noch stehen viele Mathematiker dem Auswahlprinzip ablehnened gegenüber. Mit der zunehmenenden Erkenntnis, dass es Fragen der Mathematik gibt, die ohne dieses Prinzip nicht entschieden werden können, dürfte der Widerstand gegen dasselbe mehr und mehr schwinden...*
>
> Also, the axiom of choice seems to be unavoidable if one wishes to prove the existence of an algebraically closed extension of an arbitrary field[11] ... Many mathematicians still object to the use of the axiom of choice. This resistance against using the axiom of choice will dwindle with the realization that there are mathematical problems which cannot be decided without this axiom ...

But those proofs in Steinitz's paper which use the principle of transfinite induction were somewhat long-winded and therefore the comments of Baer and Hasse try to simplify and streamline Steinitz's arguments. Today we would prefer to use Zorn's Lemma instead; this would lead to a still greater simplification combined with a considerable shortening of Steinitz's paper (which originally had 134 pages). But Zorn's Lemma was not yet formulated in 1929.

The Hasse–Baer edition of Steinitz's paper was reprinted 1950 by the Chelsea Publishing Company in New York, in its series reprinting classical treatises. (In 1997 the American Mathematical Society acquired Chelsea. This title is not listed anymore as being available.) I had bought a copy myself in the 1950s and followed Hasse's advice to read this classic work.

[11] But note that Banaschewski proved in 1992 that the existence and uniqueness of the algebraic closure can be derived from the Boolean Ultrafilter Theorem already, which is weaker than the axiom of choice [Ban92].

Let us return to the year 1924. In that year the *Hamburger Abhandlungen* published the paper by Artin and Schreier: "*Algebraische Konstruktion reeller Körper*" (Algebraic construction of real fields). This paper is in some sense a companion to Kürschak's: whereas Kürschak developed the notion of valued field on the basis of Steinitz's abstract field theory, Artin and Schreier do the same for the notion of ordered field. They construct the real closure of an ordered field with the help of Zermelo's well ordering theorem – similar to what Steinitz had done for the construction of the algebraic closure. In fact, they cite Steinitz in connection with some details of this proof. And in the introduction the authors say:

> *E. Steinitz hat durch seine „Algebraische Theorie der Körper" weite Teile der Algebra einer abstrakten Behandlungsweise erschlossen; seiner bahnbrechenden Untersuchung ist zum großen Teil die starke Entwicklung zu danken, die seither die moderne Algebra genommen hat ...*
>
> E. Steinitz, through his "Algebraic Theory of Fields", has opened up large parts of algebra to an abstract treatment; since then, thanks to his groundbreaking work, modern algebra has seen a strong revival ...

Nowadays we do not often find such enthusiastic references to Steinitz [Ste10] in current mathematical papers. The reason for this is that the main ideas and results of Steinitz have become a matter of course, not the least through the early textbooks by Hasse, Haupt and van der Waerden mentioned above.

At the end of his introduction Steinitz says in his paper of 1910:

> *Der vorliegende Aufsatz behandelt nur die Grundzüge einer allgemeinen Körpertheorie. Weitergehende Untersuchungen sowie Anwendungen auf Geometrie, Zahlen- und Funktionentheorie beabsichtige ich, in einigen weiteren Anhandlungen folgen zu lassen.*
>
> The present article covers the foundations of a general field theory only. I am planning to follow-up with more advanced investigations, and with applications to Geometry, Number Theory and the Theory of Functions.

But Steinitz did not publish anything in this direction, and also in his literary estate nothing of this kind was found. We do not know why he did not later write what he had announced. In any case, we observe that his 1910 paper has exerted a great influence upon the delopment of modern algebra during the next decades. As evidence for this we may cite the following papers which are to be considered as immediate follow-ups to [Ste10], and have each opened up a long line of development:

- **Analysis, including p-adic analysis:** Kürschak–Ostrowski on valuation theory [Kür13], [Ost34];
- **Number Theory:** Emmy Noether on Dedekind domains in abstract fields [Noe26];
- **Algebraic Geometry:** Emmy Noether on primary decomposition of ideals in a Noetherian ring [Noe21], and van der Waerden on the foundations of algebraic geometry [vdW75];
- **Real Algebra:** Artin and Schreier on real fields [AS27].

This, of course, is not a complete list.[12] For a biography of Steinitz we refer to the *Dictionary of Scientific Biography.*

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[12] Compare with [Pur73] and [Kle99].

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