On convergent power series

We consider the following situation:

K	a field equipped with a non-archimedean absolute value $\lvert\cdot\rvert$ which is assumed to be *complete*
$K[[T]]$	the ring of formal power series in one variable T over K
$K((T))$	the field of quotients of $K[[T]]$
$K[[T]]_r$	for $r > 0$, the subring of those power series which are convergent in the disc $\lvert t\rvert \leq r$ [1])
$K[[T]]_0$	$= \bigcup_{r>0} K[[T]]_r$ the "ring of convergent power series" over K
$K((T))_0$	its field of quotients

Our aim is to present a proof of the following

Theorem 1 *$K((T))_0$ is algebraically closed in $K((T))$. Moreover, $K((T))$ is a regular field extension of $K((T))_0$ and, hence, $K((T))_0$ is existentially closed in $K((T))$.*

Remark: In an earlier version of this manuscript dated 13 May 1996, the second part of the above theorem, concerning the separability of $K((T))$ over $K((T))_0$, had been proved only under the additional hypothesis that K is of finite degree of inseparability. The general proof as given here is due to F.V. Kuhlmann.

We denote by $\mathrm{ord}(z)$ the initial degree of the power series $z \in K[[T]]$. This defines a discrete, additively written valuation of $K((T))$ over K for which $K((T))$ is *complete*, hence henselian. We shall see below that

$$K[[T]] \cap K((T))_0 = K[[T]]_0 \tag{1}$$

which means that $K[[T]]_0$ is the valuation ring in $K((T))_0$ belonging to the initial degree valuation. First we are going to show:

Proposition 2 *$K((T))_0$ is Henselian with respect to the initial degree valuation. Consequently $K((T))_0$ is separably algebraically closed within $K((T))$.*

[1]) Here and in the following, $r > 0$ will always denote a real number in the value group of K.

Proof: We consider a polynomial

$$f(Y) = a_0 + a_1 Y + \cdots + a_k Y^k \tag{2}$$

with coefficients $a_i \in K[[T]]_0$. We assume that

$$f(0)(0) = 0 \quad \text{and} \qquad f'(0)(0) \neq 0 \tag{3}$$

and have to show that there exists $y \in K[[T]]_0$ such that

$$f(y) = 0 \quad \text{and} \qquad y(0) = 0\,. \tag{4}$$

Observe that $f(0) = a_0$ and $f'(0) = a_1$ are convergent power series; thus $f(0)(0) = a_0(0)$ and $f'(0)(0) = a_1(0)$ denote the constant coefficients of those power series.

The usual procedure to construct such zero y of $f(Y)$ is by means of **Newton's iteration method**: Starting with $y_0 = 0$ as initial value we define successively

$$y_{n+1} = y_n - f'(y_n)^{-1} f(y_n) \qquad (n = 0, 1, 2, \ldots) \tag{5}$$

It is shown that $y = \lim_{n\to\infty} y_n$ exists and $f'(y) \neq 0$; then it is seen from the definition (5) that y is a zero of $f(Y)$. Moreover, it is seen that $y(0) = 0$.

If in the above, the notion of "limit" is understood in the sense of the initial degree valuation $\mathrm{ord}(\,\cdot\,)$ then one can only deduce that the limit is contained in the ord-completion of $K[[T]]_0$, but this is $K[[T]]$ and hence this argument does not lead to any information about the limit belonging to $K[[T]]_0$. Therefore, we have to regard the y_n as a convergent sequence with respect some other suitable valuation, as follows.

By definition, $K[[T]]_0$ is the union of the rings $K[[T]]_r$ for $r \to 0$. A power series

$$z = c_0 + c_1 T + c_2 T^2 + \cdots \in K[[T]]_0 \tag{6}$$

is convergent in the disc $|t| \leq r$ if and only if

$$\lim_{\nu\to\infty} |c_\nu| r^\nu = 0\,.$$

If this is the case then

$$\|z\|_r = \max_\nu |c_\nu| r^\nu \tag{7}$$

does exist. If the residue class field of K is infinite then it is well known that $\|z\|_r$ is the norm of uniform convergence in the disc, i.e.,

$$\|z\|_r = \max_{|t|\leq r} |z(t)|\,.$$

In any case, whether the residue field of K is infinite or not, $\|\cdot\|_r$ is an absolute value of $K[[T]]_r$, and $K[[T]]_r$ is *complete* with respect to this absolute value. $\|\cdot\|_r$ is briefly called the *functional norm* on the r-disc.

This shows that Proposition 2 is an immediate consequence of:

Proposition 3 *If r is sufficiently small then all the y_n of Newton's sequence (5) are contained in $K[[T]]_r$ and the sequence y_n converges with respect to the functional r-norm $\|\cdot\|_r$. Hence its r-norm limit $y = \lim_{n\to\infty} y_n$ is contained in $K[[T]]_r$. Moreover, we have $y(0) = 0$.*

This proposition is well known from elementary analysis under the name of "Theorem of implicit functions".[2]) Nevertheless let us present a proof here.

Before starting with the proof proper let us note:

Lemma 4 *Let $z \in K[[T]]_r$ and suppose $\|z-1\|_r < 1$. Then $z^{-1} \in K[[T]]_r$ and $\|z\|_r = \|z^{-1}\|_r = 1$.*

Proof: We put $z = 1 - u$, then $\|u\|_r < 1$ and hence the geometric series

$$(1-u)^{-1} = \sum_{0\le\nu<\infty} u^\nu$$

converges with respect to $\|\cdot\|_r$; we conclude that $z^{-1} = (1-u)^{-1} \in K[[T]]_r$. The relations $\|z\|_r = 1$ and $\|z^{-1}\|_r = 1$ are implied by the non-archimedean property of the valuation $\|\cdot\|_r$. □

Corollary 5 *If $z \in K[[T]]_0$ and $z(0) \neq 0$ then $z^{-1} \in K[[T]]_0$.*

Proof: We put $z = z(0)\cdot z^*$ and see that the invertibility of z is equivalent to the invertibility of z^*. Writing again z instead of z^* we may assume from the start that $z(0) = 1$:

$$z = 1 + c_1T + c_2T^2 + \cdots = 1 - u\,.$$

z and hence u are convergent in some disc $|t| \le r_0$. For $r \le r_0$ we have

$$\|u\|_r = \max_{\nu\ge1} |c_\nu| r^\nu \le \frac{r}{r_0}\cdot\max_{\nu\ge1}|c_\nu| r_0^\nu \le \frac{r}{r_0}\cdot\|z\|_{r_0}\,.$$

Hence from Lemma 4 we conclude:

$$r < \frac{r_0}{\|z\|_{r_0}} \Rightarrow \|u\|_r < 1 \Rightarrow z^{-1} \in K[[T]]_r\,.$$

In other words: if r is sufficiently small then z^{-1} belongs to $K[[T]]_r$ and hence to $K[[T]]_0$. □

[2]) In elementary analysis the underlying absolute value is archimedean and hence the ordinary proof differs from ours in those places where we use the non-archimedean property of the valuation. But of course, the idea of proof is the same in both cases, archimedean and non-archimedean, and it is possible to give a unified proof.

Corollary 5 shows that every $z \in K[[T]]_0$ can be uniquely written as

$$z = T^k z_0$$

where $k = \mathrm{ord}(z)$ and z_0 is a unit in $K[[T]]_0$. This implies the relation (1) already stated above.

Proof of Proposition 3: For simplicity we may assume that

$$f'(0)(0) = 1\,. \tag{8}$$

If this should not be the case then multiplication of the coefficients of $f(Y)$ with the constant $c = f(0)(0)^{-1}$ will achieve this.

There exists $r_0 > 0$ such that all the coefficients $a_0, a_1, \ldots, a_k$ of $f(Y)$ are convergent in the disc $|t| \le r_0$. Let us put

$$M = \max_{0 \le i \le k} \|a_i\|_{r_0}\,.$$

It follows from (8) that $\|a_1\|_{r_0} = \|f'(0)\|_{r_0} \ge 1$; hence $M \ge 1$.

In the following proof we choose a radius $r < r_0$ which is "sufficiently small"; the precise condition for r will be seen in the course of our arguments. We fix such r and put

$$I_r = \left\{\; y \in K[[T]]_r \;:\; \|y\|_r < M^{-1} \;\right\}\,.$$

This is an ideal in the ring $\mathcal{O}_r = \{\; y \in K[[T]]_r \;:\; \|y\|_r \le 1 \;\}$. Observe that I_r is *closed* with respect to the valuation $\|\cdot\|_r$, hence complete.

The proof of Proposition 3 *will now be presented as a succession of observations* (i)–(iii) *with the conclusion* (iv).

Observation (i) *If $y \in I_r$ then $f(y) \in I_r$. Thus f maps the ideal I_r into itself.*

To see this we write

$$f(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_k y^k\,. \tag{9}$$

According to (3) the power series function $a_0 = a_0(t) = f(0)(t)$ vanishes for $t = 0$; hence by continuity we have $|a_0(t)| \to 0$ if $t \to 0$. Thus *if r is sufficiently small* we have

$$\|a_0\|_r < M^{-1}.\ ^{3)}$$

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[3]) Explicitly: write $a_0(T) = \sum_{\nu \ge 1} a_{0,\nu} T^\nu$ and use the estimate $|a_{0,\nu}| r^\nu \le M r_0^{-\nu} \cdot r^\nu \le M r_0^{-1} r$ for $\nu \ge 1$ to see that the condition $r < r_0 M^{-2}$ does suffice.

As to the other terms in (9) we compute for $i \geq 2$

$$\|a_i y^i\|_r \leq \|a_i\|_r \|y\|_r^i \leq \|a_i\|_{r_0} \|y\|_r^i \leq M \cdot M^{-i} \leq M^{-1}\,.$$

For $i = 1$ we have $a_1 = f'(0)$ and the same estimate holds because, due to the next observation, we have $\|f'(0)\|_r = 1$. Hence the estimate $\|a_i\|_r \|y\|_r^i \leq M^{-1}$ holds for all $i = 0, 1, 2, \ldots, k$ and so

$$\|f(y)\|_r < M^{-1}$$

as contended. □

Observation (ii) *If $y \in I_r$ then $f'(y)^{-1} \in K[[T]]_r$ and $\|f'(y)^{-1}\|_r = 1$.*

To see this we write

$$f'(y) - 1 = (a_1 - 1) + 2a_2 y + \cdots + k a_k y^{k-1}\,. \tag{10}$$

According to (8) the power series function $a_1(t) - 1 = f'(0)(t) - 1$ vanishes for $t = 0$; hence again by continuity, we have

$$\|a_1 - 1\|_r < 1$$

if r is sufficiently small.[4]) For the remaining terms in (10) we have

$$\|i a_i y^{i-1}\|_r \leq \|a_i\|_r \|y\|_r^{i-1} \leq \|a_i\|_{r_0} \|y\|_r^{i-1} \leq M \cdot M^{-(i-1)} < 1$$

since $i \geq 2$. It follows

$$\|f'(y) - 1\|_r < 1$$

and we apply Lemma 4.□

Next we introduce the **Newton operator**

$$\mathcal{N}(y) = y - f'(y)^{-1} f(y)\,. \tag{11}$$

From observations (i) and (ii) we see that $\mathcal{N}$ maps the ideal I_r into itself.

Observation (iii) *If $y \in I_r$ then $\|f(\mathcal{N}y)\|_r \leq M \cdot \|f(y)\|_r^2$.*

To obtain this estimate we use *Taylor's formula* which we write in the form

$$f(Y + Z) = f(Y) + f'(Y)Z + f^{(2)}(Y)Z^2 + \cdots + f^{(k)}(Y)Z^k \tag{12}$$

where Y, Z are independent variables. The polynomials $f^{(\nu)}(Y)$ are the modified higher derivatives of $f(Y)$ in the sense of Hasse and F.K.Schmidt; they are defined by the formula

$$f^{(\nu)}(Y) = \sum_{\nu \leq i \leq k} \binom{i}{\nu} a_i Y^{i-\nu}\,. \tag{13}$$

[4]) Similar explicit verification as in (i). This time the condition $r < r_0 M^{-1}$ will suffice.

For the coefficients we have the estimate

$$\left\| \binom{i}{\nu} a_i \right\|_r \leq \|a_i\|_r \leq \|a_i\|_{r_0} \leq M$$

and hence for $y \in I_r$ (since $\|y\|_r < M^{-1} < 1$)

$$\|f^{(\nu)}(y)\|_r \leq M\,. \tag{14}$$

Now, in Taylor's formula we substitute $Y \mapsto y$ and $Z \mapsto -f'(y)^{-1}f(y)$ and observe that, due to the definition of the Newton operator, the first two terms cancel and Taylor's formula starts with the quadratic term:

$$f(\mathcal{N}y) = f^{(2)}(y)\left(f'(y)^{-1}f(y)\right)^2 + \cdots + f^{(k)}(y)\left(f'(y)^{-1}f(y)\right)^k\,.$$

For each term on the right hand side, we deduce using (14) and observation (ii):

$$\left\|f^{(\nu)}(y)\left(f'(y)^{-1}f(y)\right)^\nu\right\|_r \leq M \cdot \|f(y)\|_r^\nu \leq M \cdot \|f(y)\|_r^2$$

the last inequality because $\nu \geq 2$ and $\|f(y)\|_r < 1$ by (i). □

Conclusion (iv):

Consider the Newton sequence y_n defined inductively by

$$y_{n+1} = \mathcal{N}(y_n)$$

as in (5), with the initial term $y_0 = 0$. From observation (iii) we obtain by induction

$$\|f(y_n)\|_r \leq M^{2^n-1}\|f(y_0)\|_r^{2^n} \qquad (n = 1, 2, 3, \ldots)$$

Moreover the definition (5) of the Newton sequence shows that

$$\|y_{n+1} - y_n\|_r = \|f'(y_n)^{-1}f(y_n)\|_r \leq \|f(y_n)\|_r \leq M^{2^n-1}\|f(y_0)\|_r^{2^n}$$

where we have used observation (ii). Since $M\|f(y_0)\|_r < 1$ by observation (i), these formulas show us, firstly, that the Newton sequence y_n is a Cauchy sequence with respect to the r-norm $\|\cdot\|_r$; hence the limit $y = \lim_{n\to\infty} y_n$ exists in I_r. Secondly, we see that $f(y_n)$ converges to 0; hence $f(y) = 0$.

Since $y_0 = 0$ we see that $f(y_0)(0) = f(0)(0) = 0$ due to the hypothesis (3). From (5) we conclude $y_1(0) = 0$. By induction one verifies that

$$f(y_n)(0) = f(0)(0) = 0 \quad \text{and} \qquad y_{n+1}(0) = 0 \qquad (n = 1, 2, 3, \ldots)$$

Hence for the limit $y = \lim_{n\to\infty} y_n$ we also have $y(0) = 0$.

Proposition 3 is proved. □

Accordingly, we now know that $K((T))_0$ is henselian. Since it is dense in $K((T))$, it is well known that the *regularity* of $K((T))$ over $K((T))_0$ is necessary and sufficient for $K((T))_0$ to be existentially closed in $K((T))$. See e.g., F. V. Kuhlmann's thesis [K]. It remains to prove:

Proposition 6 *Let $p = char(K) > 0$. The field $K((T))$ is a separable extension of $K((T))_0$. This means that $K((T))$ is linearly disjoint to $(K((T))_0)^{1/p}$ over $K((T))_0$.*

Let us remark that

$$(K[[T]])^{1/p} = K^{1/p}[[T^{1/p}]] \,.$$

If a power series $z \in K[[T]]$ converges in a disc with radius $r > 0$ then its p-th root $z^{1/p} \in K^{1/p}[[T^{1/p}]]$, as a power series in the variable $T^{1/p}$, converges in the disc with radius $r^{1/p}$. And conversely. Thus

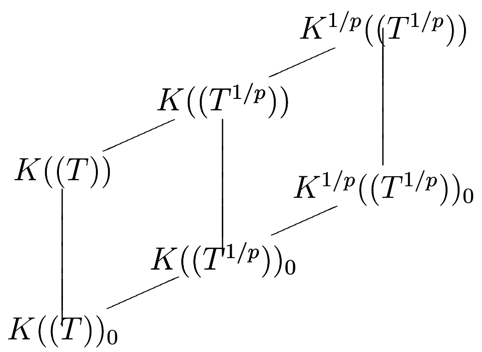
$$(K[[T]]_r)^{1/p} = K^{1/p}[[T^{1/p}]]_{r^{1/p}} \,.$$

For $r \to 0$ we obtain

$$(K[[T]]_0)^{1/p} = K^{1/p}[[T^{1/p}]]_0 \,.$$

The same relation holds for the respective quotient fields, i.e., we may replace the double square brackets by double ordinary brackets.

Now consider the following diagram:



The next lemma shows that linear disjointness holds in the left lower portion of the diagram:

Lemma 7 *$K((T))$ and $K((T^{1/p}))_0$ are linearly disjoint over $K((T))_0$. In fact, the p elements $T^{j/p}$, $0 \leq j \leq p-1$, form a basis of $K((T^{1/p}))$ over $K((T))$, and also a basis of $K((T^{1/p}))_0$ over $K((T))_0$.*

Proof: According to the euclidean algorithm each integer n has a *unique* representation of the form

$$n = j + pm \quad \text{with} \qquad 0 \leq j \leq p-1 \tag{15}$$

and m integer. Hence $T^{n/p} = T^{j/p}T^m$ and we obtain the direct sum decomposition

$$K[[T^{1/p}]] = \sum_{0\le j\le p-1} T^{j/p}K[[T]]\,,$$

which shows that the $T^{j/p}$ form a basis of $K[[T^{1/p}]]$ over $K[[T]]$, and hence also for the respective quotient fields. Explicitly, if

$$z = \sum_n b_n T^{n/p}$$

is a power series in $T^{1/p}$ with coefficients $b_n \in K$, and if we express each exponent n as in (15) then

$$z = \sum_{0\le j\le p-1} T^{j/p} z_j$$

where

$$z_j = \sum_m c_{j,m}T^m \quad \text{with} \qquad c_{j,m} = b_{pm+j}\,.$$

If $z \in K[[T^{1/p}]]_r$ then $\lim_{n\to\infty} |b_n| r^{n/p} = 0$. For each fixed j, the subsequence $|b_{pm+j}|r^{m+(j/p)}$ also tends to 0; multiplication by $r^{-j/p}$ shows $\lim_{m\to\infty} |c_{j,m}|r^m = 0$, hence z_j converges in the disc $|t| \le r$, which is to say that $z_j \in K[[T]]_r$. The arguments can be reversed, and therefore

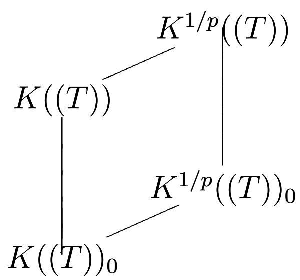
$$K[[T^{1/p}]]_r = \sum_{0\le j\le p-1} T^{j/p}K[[T]]_r\,.$$

For $r \longrightarrow 0$ we conclude

$$K[[T^{1/p}]]_0 = \sum_{0\le j\le p-1} T^{j/p}K[[T]]_0\,.$$

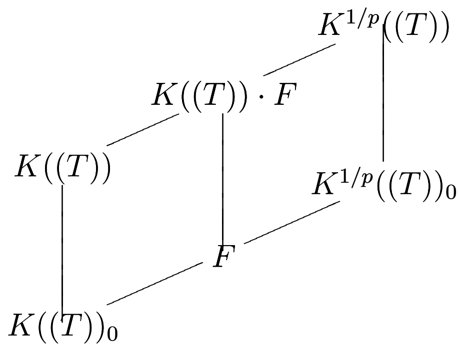
Thus $1, T^{1/p}, \dots, T^{(p-1)/p}$ is a basis of $K[[T^{1/p}]]_0$ over $K[[T]]_0$, and the same holds for the respective quotient fields. $\square$

It remains to show linear disjointness for the right upper portion of the above diagram. That portion refers to $T^{1/p}$ as indeterminate. For the purpose of the following proof, let us write T again instead of $T^{1/p}$ in order to comply with the notations used elsewhere in this manuscript. Thus we have to consider the following diagram:



In this situation we claim:

Lemma 8 *The fields $K((T))$ and $K^{1/p}((T))_0$ are linearly disjoint over $K((T))_0$. In fact: Let $F \subset K^{1/p}((T))_0$ be a subextension of finite degree over $K((T))_0$; let $u_1, \ldots, u_m$ be a K-basis of the residue field L of F and $\widetilde{u}_1, \ldots, \widetilde{u}_m$ be foreimages of the u_i in F. Then the $\widetilde{u}_i$ form a basis of F over $K((T))_0$ which at the same time is a basis of $K((T)) \cdot F$ over $K((T))$.*



Remark: In the above lemma the notion of "residue field" refers, of course, to the initial degree valuation of the power series field. The residue field of $K((T))$ is K, and K is also the residue field of $K((T))_0$.

If K is of finite inseparability degree then $[K^{1/p} : K]$ is finite, hence in the above Lemma we can take $F = K^{1/p}((T))$ and $L = K^{1/p}$. Moreover, we can take $\widetilde{u}_i = u_i$. In this case the following proof is identical with the proof given in our earlier version of this manuscript. It has been Kuhlmann's idea how to modify that proof in order to deal also with fields of infinite degree of inseparability.

Proof of Lemma 8: Since the residues u_i of the $\widetilde{u}_i$ are linearly independent over K by construction, it follows that the $\widetilde{u}_i$ are linearly independent over $K((T))$. In fact, for every linear combination

$$z = \sum_{1 \le i \le m} z_i \widetilde{u}_i$$

with $z_i \in K((T))$ we have

$$\operatorname{ord}(z) = \min_{1 \le i \le m} \operatorname{ord}(z_i) \tag{16}$$

and this implies the linear independency of the $\widetilde{u}_i$ over $K((T))$. Hence it remains to show that the $\widetilde{u}_i$ form a *basis* of F over $K((T))_0$ [5]) which is to say that

$$[F : K((T))_0] = [L : K]\,.$$

In general, the residue field extension is of degree $\leq$ than the degree of the original field extension, and equality implies that the field extension has no defect.

Thus we have to show that the valued field F, as an extension of $K((T))_0$, has no defect. [6])

We proceed by induction with respect to the field degree $[F : K((T))_0]$. If the degree is 1 then there is nothing to prove. Now we assume that F is already known to be defectless over $K((T))$, and we consider a proper simple extension $F' = F(z) \subset K^{1/p}((T))_0$. We can assume that $z \in K^{1/p}[[T]]_0$, i.e., that z is a convergent power series with coefficients in $K^{1/p}$.

The p-th power z^p is in $K[[T]]_0$ and hence in F. Thus $[F' : F] = p$. For the residue fields we have $[L' : L] \leq p$. Since $L'|L$ is purely inseparable we conclude that $[L' : L] = p$ or $= 1$. In the first case F' is defectless over F. By induction assumption F is defectless over $K((T))_0$ and it follows that F' is defectless over $K((T))_0$.

In the second case F' has the same residue field as F, hence F' is an *immediate* extension of F. We have to show that this case does not occur.

As an immediate extension, F' is contained in the *maximal* immediate extension of F, i.e., in the completion $\widehat{F}$ of F. [7]) This completion contains $K((T))$ which is the completion of $K((T))_0$. Thus

$$\widehat{F} \supset K((T)) \cdot F = \sum_{1 \leq i \leq m} K((T))\widetilde{u}_i$$

(where we have used the induction assumption). Now the right hand side, being a finite dimensional vector space over the complete field $K((T))$, is also complete and therefore

$$\widehat{F} = \sum_{1 \leq i \leq m} K((T))\widetilde{u}_i\,.$$

Consequently, since $z \in F' \subset \widehat{F}$ we conclude that there is a linear represen-

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[5]) In the terminology of [K] the $\widetilde{u}_i$ form a *valuation basis* of F over $K((T))_0$.

[6]) It seems that by slight modification of our arguments it would be possible to show that any finite extension F of $K((T))_0$ is defectless, i.e., that $K((T))_0$ is a defectless field.

[7]) Observe that the valuation of F is *discrete*, being induced by the initial degree valuation of $K^{1/p}((T))$.

tation of the form

$$z = \sum_{1 \le i \le m} a_i \widetilde{u}_i \tag{17}$$

with coefficients $a_i \in K((T))$. From (16) we infer that $a_i \in K[[T]]$.

Since z is convergent this implies, we claim, that the coefficients a_i are also convergent, hence they are contained in $K[[T]]_0$ and therefore $z \in F$, a contradiction.

Thus we have to prove the following lemma which, actually, is the heart of Kuhlmann's proof of Proposition 6. For clarity, let us review the situation of the lemma.

K	a complete valued field of characteristic p
K'	a valued algebraic field extension of K [8])
u_i	for $1 \le i \le m$, finitely many elements in K' which are linearly independent over K
$\widetilde{u}_i$	$= \widetilde{u}_i(T)$ power series in $K'[[T]]$ such that $\widetilde{u}_i(0) = u_i$
z	$= z(T)$ a power series in $K'[[T]]$ which can be written as a linear combination of the form (17) with coefficients $a_i \in K[[T]]$

In this situation we have:

Lemma 9 *If the power series $z(T)$ and the $\widetilde{u}_i(T)$ are convergent, then the coefficients $a_i(T)$ are convergent too.*

Proof: (i) The proof will based on the following fact concerning the vector space

$$L = \sum_{1 \le i \le m} K u_i$$

spanned by the u_i. For any $x \in L$ we write $x = x_1 u_1 + \cdots + x_m u_m$ with $x_i \in K$ and put

$$\mu(x) = \max(|x_1|, \ldots, |x_m|) .$$

The function $x \to \mu(x)$ is a *vector space norm* of L over K, i.e., it satisfies the relations

$$\begin{aligned} \mu(x + x') &\le \max(\mu(x), \mu(x')) \\ \mu(cx) &= |c| \cdot \mu(x) \qquad \text{for } c \in K \\ \mu(x) = 0 &\Leftrightarrow \mu = 0 \end{aligned}$$

The valuation $x \to |x|$ is also a vector space norm of L. Now since K is complete, it is well known that *any two vector space norms on L define the*

8) For the following lemma it is not necessary that K' is purely inseparable over K.

same topology. Explicitly, there is a constant $M > 0$ (depending on the given basis u_i of L) such that

$$\mu(x) \leq M \cdot |x| \,. \tag{18}$$

See e.g. our old paper [R].

(ii) This being said, we now start with the proof of Lemma 9. After replacing z by $z - z(0)$ we may assume that $z(0) = 0$. Since the power series $z(T)$ converges in some neighborhood of zero in K' we conclude by continuity that $\lim_{r \to 0} \|z\|_r = 0$. Hence for r sufficiently small we have that

$$\|z\|_r \leq M^{-1}. \tag{19}$$

For each i we write

$$\widetilde{u}_i = u_i + z_i(T)$$

where $z_i(0) = 0$. Again, since $z_i(T)$ converges we have

$$\|z_i\|_r \leq M^{-1} \qquad (1 \leq i \leq m) \tag{20}$$

if r is sufficiently small.

With r chosen this way, we claim that each $\|a_i\|_r$ exists and $\|a_i\|_r \leq 1$. To see this, let us write

$$\begin{aligned} z(T) &= \sum_{n>0} c_n T^n \\ z_i(T) &= \sum_{n>0} c_{i,n} T^n \\ a_i(T) &= \sum_{n \geq 0} a_{i,n} T^n. \end{aligned}$$

with coefficients $c_n, c_{i,n} \in K'$ and $a_{i,n} \in K$.

Observe that $a_{i,0} = a_i(0) = 0$; this follows from

$$0 = z(0) = \sum_{1 \leq i \leq m} a_i(0) u_i$$

and the linear independency of the u_i over K. Our contention is that

$$|a_{i,n}| r^n \leq 1 \qquad (1 \leq i \leq m) \tag{21}$$

if r is sufficiently small in the sense of (19), (20). This can be seen by induction. For $n = 0$ we have $a_{i,n} = 0$ and there is nothing to prove. Now suppose $n > 0$. Comparing the coefficients of T^n in (17) we obtain

$$c_n = \sum_{1 \leq i \leq m} a_{i,n} u_i + \sum_{1 \leq i \leq m} \sum_{0 \leq \nu < n} a_{i,\nu} c_{i,n-\nu} \,.$$

In the second sum on the right hand side only those coefficients $a_{i,\nu}$ do appear for which the induction assumption applies. We conclude

$$\begin{aligned} \left|\sum_{1\le i\le m} a_{i,n}u_i\right| r^n &\le \max\left(|c_n|r^n, \max_{1\le i\le m}\max_{0\le\nu<n}(|a_{i,\nu}|r^\nu|c_{i,n-\nu}|r^{n-\nu})\right) \\ &\le \max\left(\|z\|_r, \max_{1\le i\le m}(1\cdot\|z_i|_r)\right) \\ &\le M^{-1} \end{aligned}$$

where we have used (19) and (20), besides of the induction assumption. On the other hand, in view of (18) we have for each i:

$$|a_{i,n}| \le M\cdot\left|\sum_{1\le i\le m} a_{i,n}u_i\right|$$

and multiplication with r^n yields (21) for the exponent n.

We have seen that for sufficiently small radius $r>0$ we have

$$\|a_i\|_r = \max_{n\ge0}|a_{i,n}|r^n \le 1 \qquad (1\le i\le m)\,.$$

We choose one such radius, say r_0. Then for $r<r_0$ (and r in the value group of K) we compute as usual

$$|a_{i,n}|r^n \le \|a_i\|_{r_0}\cdot\left(\frac{r}{r_0}\right)^n \le \left(\frac{r}{r_0}\right)^n$$

and this converges to 0 for $n\to\infty$. In other words: If $r<r_0$ then the power series $a_i(T)$ converges in the disc $|t|\le r$.

Lemma 9 is proved, and so is Theorem 1.

References:

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