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An introduction to Deuring's theory of constant reductions

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Abstract. This note presents a brief introduction to Deuring's theory of reduction of function fields. We assume that the reader is familiar with the elementary theory of function fields (or curves) and their divisors, up to the Theorem of Riemann-Roch. One of the new features here is the definition and use of the notion of "inert" extensions of valued fields, which works for extensions of arbitrary degree of transcendency. We study the behavior of the genus of a function field under reduction (section 4). In the case of "good" reduction, where the genus does not change, we obtain Deuring's reduction map for divisors (section 5). Any function field admits good reduction with respect to almost all valuations of the base field (section 6). In a follow-up paper it will be shown that the valuation for good reduction, if it exists, is unique.

Introduction

In this note, a "**function field**" is a finitely generated field extension $F|K$ of transcendence degree 1. Thus there exists an element $x \in F$ which is transcendent over K, and F appears as a finite algebraic extension of the field $K(x)$ of rational functions. We assume that the reader is familiar with the elementary theory of function fields and their divisors, up to the Theorem of Riemann-Roch.

A "**valued function field**" is a function field equipped with a valuation v and the property that the residue field $\overline{F}$ is transcendental over the residue field $\overline{K}$ of K. Thus there exists an element $x \in F$ such that its residue $\overline{x}$ is transcendental over $\overline{K}$. We shall see in section 1 that this condition uniquely characterizes the valuation on the rational subfield $K(x) \subset F$, with residue field $\overline{K}(\overline{x})$. The residue field $\overline{F}$ of F is a finite extension of $\overline{K}(\overline{x})$ and so $\overline{F}|\overline{K}$ is also a function field. It is called the "**reduction**" of $F|K$ modulo the given valuation v.

In general we have $[\overline{F} : \overline{K}(\overline{x})] \leq [F : K(x)]$. The reduction is called "**regular**" if there exists $x \in F$ such that

$$[\overline{F} : \overline{K}(\overline{x})] = [F : K(x)]\,.$$

The main object of this note is to study regular reduction of function fields. In particular we will be interested in the corresponding reduction of divisors, as well as the behavior of the genus under reduction. The theory of reduction of function fields has been developed by Deuring in the year 1942, but for discrete valuations only; see [**1**].

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Notation. v is to be a (non-trivial) valuation in the sense of Krull; there is no restriction concerning its value group or its residue field. Given the valuation v of F we denote by $\mathcal{O}_F$ its valuation ring, and similarly $\mathcal{O}_K$ its valuation ring in K. Thus we have $\mathcal{O}_K = K \cap \mathcal{O}_F$. Similarly let $\mathcal{M}_F, \mathcal{M}_K$ be the maximal ideals of $\mathcal{O}_F$ and $\mathcal{O}_K$ respectively, so that $\mathcal{M}_K = K \cap \mathcal{M}_F$. For the residue fields we use the notations already explained above: $\overline{F} = \mathcal{O}_F/\mathcal{M}_F$ and $\overline{K} = \mathcal{O}_K/\mathcal{M}_K$. If $x \in \mathcal{O}_F$ then $\overline{x}$ is its residue in $\overline{F}$. If $x \in F$ and $x \notin \mathcal{O}_F$ then, as usual, we write $\overline{x} = \infty$. For any $0 \neq x \in F$ we denote by (x) its principal divisor. $(x)_\infty$ is the pole divisor of x and $(x)_0$ its zero divisor, so that $(x) = (x)_0 - (x)_\infty$.

1. The Gauss valuation

Let K be a field equipped with a valuation v. Consider the rational function field $F = K(x)$ over K. There are many ways to extend the valuation v from K to F. Some of them, called "Gauss valuations", are particularly easy to describe.

Theorem 1 (Gauss Theorem). *Let x be a generator of the rational function field $F = K(x)$. There is one and only one extension of v to F for which $\overline{x}$ is transcendental over $\overline{K}$. The value $v(f)$ of a polynomial*

$$f = a_o + a_1 x + \cdots + a_n x^n \in K[x]$$

is given by

$$v(f) = \min_{0 \le i \le n} v(a_i)\,. \tag{1}$$

The value group and residue field are:

$$v(F) = v(K) \qquad \textit{and} \qquad \overline{F} = \overline{K}(\overline{x})\,.$$

Proof. If such an extension of v to F exists then its valuation ring $\mathcal{O}_F$ contains the polynomial ring $\mathcal{O}_K[x]$, and its reduction map $\mathcal{O}_F \to \overline{F}$ induces a homomorphism

$$\mathcal{O}_K[x] \to \overline{K}[\overline{x}] \tag{2}$$

of the respective polynomial rings. The homomorphism is obtained coefficientwise, i.e. the image of a polynomial

$$f = f(x) = a_o + a_1 x + \cdots + a_n x^n \in \mathcal{O}_K[x] \tag{3}$$

is given by

$$\overline{f} = \overline{f}(\overline{x}) = \overline{a}_o + \overline{a}_1 \overline{x} + \cdots + \overline{a}_n \overline{x}^n \in \overline{K}[\overline{x}]. \tag{4}$$

This homomorphism extends naturally and uniquely to its ring of quotients, which consists of all quotients fg^{-1} with $f, g \in \mathcal{O}_K[x]$ and $\overline{g} \neq 0$. We shall see below that this ring of quotients equals $\mathcal{O}_F$. This gives uniqueness.

As to the existence, consider the rational function field of one variable over $\overline{K}$ and one of its generators; denote this generator by $\overline{x}$. Define the homomorphism (2) coefficient wise by (4). Extend that homomorphism to its ring of quotients as above. We claim that this ring of quotients is a valuation ring. Let us denote this ring of quotients already by $\mathcal{O}_F$, so that we have a homomorphism

$$\mathcal{O}_F \to \overline{K}(\overline{x})\,. \tag{5}$$

We are now going to show that, indeed, $\mathcal{O}_F$ is a valuation ring of F. To this end we will verify the characteristic property of a valuation ring: for $h \in F$ we have to show:

$$h \notin \mathcal{O}_F \Longrightarrow h^{-1} \in \mathcal{O}_F\,. \tag{6}$$

We write $h = fg^{-1}$ with $f, g \in K[x]$. We define the function $f \mapsto v(f)$ on $K[x]$ by formula (1). It will turn out that this function is a valuation but for the moment we just have to know that it is a function defined on $K[x]$ with values in the value group $v(K)$.

Let $a \in K^\times$ be an element whose value $v(a) = v(f)$. For instance, a may be a coefficient a_i of f in (3) for which the minimum on the right hand side of (1) is assumed. We write

$$f = a \cdot f_o \qquad \text{with} \qquad v(f_o) = 0\,. \tag{7}$$

The polynomial f_o with $v(f_o) = 0$ is called *primitive*. Thus every polynomial can be normalized by a factor from K to become primitive. The condition $v(f_o) = 0$ is equivalent to $\overline{f}_o \neq 0$.

Similarly we write $g(x) = b \cdot g_o(x)$ with $b \in K$ and g_o primitive. Thus

$$h = fg^{-1} = c \cdot f_o g_o^{-1} \qquad \text{with} \qquad c = ab^{-1} \in K^\times\,. \tag{8}$$

Now, if $v(c) \geq 0$ then $c \in \mathcal{O}_K$ and thus $c \cdot f_o \in \mathcal{O}_K[x]$; hence h is contained in the ring of quotients $\mathcal{O}_F$ as defined above. Consequently, if $h \notin \mathcal{O}_F$ then $v(c) < 0$; this implies $c^{-1} \in \mathcal{O}_K$ and therefore

$$h^{-1} = c^{-1} \cdot g_o f_o^{-1} \in \mathcal{O}_F\,.$$

This establishes our contention (6) and so, indeed, $\mathcal{O}_F$ is a valuation ring.

By construction, $K \cap \mathcal{O}_F = \mathcal{O}_K$. Thus the valuation of F belonging to $\mathcal{O}_F$ is an extension of the valuation v in K. Next we show that this valuation of F is given by (1).

First we claim that the function on $K[x]$ defined by (1) is multiplicative:

$$v(fg) = v(f) + v(g) \qquad \text{for} \qquad f, g \in K[x]\,. \tag{9}$$

(This statement is usually called **Gauss' Lemma**.) To see this, write $f = a \cdot f_o$ as in (7), and similarly $g = b \cdot g_o$, so that

$$fg = ab \cdot f_o g_o\,.$$

Here $\overline{f}_o \neq 0$ and $\overline{g}_o \neq 0$ and thus $\overline{f_o g_o} = \overline{f}_o \overline{g}_o \neq 0$ (because the polynomial ring $\overline{K}[\overline{x}]$ has no zero divisors). Consequently $v(f_o g_o) = 0$ and thus

$$v(fg) = v(ab) = v(a) + v(b) = v(f) + v(g)$$

as contended.

The formula (1) defines the function v on the polynomial ring $K[x]$. We now extend this function to the whole field F by multiplicativity: If $h = fg^{-1} \in F$ with $f, g \in K[x]$ we put $v(h) = v(f) - v(g)$. Because of (9) this is well defined, and it gives a multiplicative function from $F^\times$ to $v(K)$.

We have seen above that any $h \in F$ may be written in the form (8), and so $v(h) = v(c)$. We have also seen that $v(c) \geq 0$ if and only if $h \in \mathcal{O}_F$. We conclude:

$$v(h) \geq 0 \Longleftrightarrow h \in \mathcal{O}_F.$$

This implies by multiplicativity the triangle inequality:

$$v(f+g) \geq \min\left(v(f), v(g)\right) . \tag{10}$$

To see this, assume $v(g) \leq v(f)$ and write

$$f+g = (h+1)g \qquad \text{with} \qquad h = fg^{-1}.$$

By assumption $v(h) \geq 0$, hence $h \in \mathcal{O}_F$ and therefore $h+1 \in \mathcal{O}_F$. Hence $v(h+1) \geq 0$ and $v(f+g) \geq v(g)$.

We have shown that the function v on F is a valuation and that $\mathcal{O}_F$ is its valuation ring. The statements in Theorem 1 concerning value group and residue field are immediate from (1) and (5). □

By construction, the Gauss valuation depends on the choice of the generator x of F. If we wish to indicate this then we speak of the x-Gauss valuation of $K(x)$, and we write v_x. If y is another generator, then the question arises under which condition $v_y = v_x$.

Now, if $K(y) = K(x)$ then

$$y = \frac{ax+b}{cx+d} \qquad \text{with} \qquad a,b,c,d \in K \quad \text{and} \quad ad-bc \neq 0 \tag{11}$$

Note that the coefficients $a,b,c,d \in K$ are not uniquely determined by y. They are determined up to a common factor in $K^\times$.

Proposition 2. *We have $v_y = v_x$ if and only if the coefficients a,b,c,d in (11) are in $\mathcal{O}_K$ and $v_x(ad-bc) = 0$ – after division by a suitable common factor in $K^\times$ if necessary.*

Proof. Let the $\overline{y}$ denote the residue of y with respect to the valuation v_x. If $v_y = v_x$ then

$$\overline{y} \neq 0, \infty \qquad \text{and} \qquad \overline{K}(\overline{y}) = \overline{K}(\overline{x}). \tag{12}$$

The first condition implies $0 = v_x(y) = v_x(ax+b) - v_x(cx+d)$. We take $u \in K$ such that $v_x(u) = v_x(ax+b) = v_x(cx+d)$; after dividing a,b,c,d by u we may assume that $v_x(ax+b) = 0 = v_x(cx+d)$. We then have

$$\overline{y} = \frac{\overline{a}\,\overline{x} + \overline{b}}{\overline{c}\,\overline{x} + \overline{d}}.$$

The second condition in (12) implies $\overline{a}\,\overline{d} - \overline{b}\,\overline{c} \neq 0$ which is to say that $v_x(ad-bc) = 0$. □

Remark 3. Theorem 1 and the above proof remain valid without change for rational function fields $F = K(x)$ in several variables $x = (x_1, x_2, \ldots)$. In this general situation the condition that "$\overline{x}$ is transcendental over $\overline{K}$" has to be replaced by: "$\overline{x}$ is algebraically free over $\overline{K}$".

When we try to treat power series fields $K((x))$ in the same way then there arises the problem that in formula (1) there are infinitely many coefficients a_i involved and hence the minimum on the right hand side may not exist. Hence one has to restrict the discussion to the case of the field $K((x))_o$ of "convergent" power series, i.e.,

$$f(x) = \sum_{0 \leq i < \infty} a_i x^i \qquad \text{with} \qquad \lim_{i \to \infty} a_i = 0$$

where the limit is meant in the sense of the v-adic valuation of K; this means $v(a_i) \to \infty$. In this situation Theorem 1 and its proof remain valid.

2. Regular reduction

Now let $F|K,v$ be an arbitrary valued function field, not necessarily rational. By definition, there exists $x \in F$ such that $\overline{x}$ is transcendental over $\overline{K}$. From Theorem 1 we see that the valuation in the subfield $K(x) \subset F$ coincides with the Gauss valuation with respect to x. Thus, in order to investigate valued function fields, we have to study the extensions of the Gauss valuation of $K(x)$ to the finite extension F.

General valuation theory tells us that there are only finitely many such extensions. If these are $v_1, \ldots, v_r$ with ramification degrees e_i and residue degrees f_i then

$$\sum_{1 \le i \le r} e_i f_i \le [F : K(x)]. \tag{13}$$

If, say, $v_1 = v$ is the given valuation of $F|K$ then $f_1 = [\overline{F} : \overline{K}(\overline{x})]$ and we conclude

$$[\overline{F} : \overline{K}(\overline{x})] \le [F : K(x)]\,.$$

As said in the introduction, here we are mainly interested in the case when

$$[\overline{F} : \overline{K}(\overline{x})] = [F : K(x)]\,. \tag{14}$$

In this case $\overline{F}|\overline{K}$ is said to be a **regular reduction** of $F|K$, and x is called a **regular element**. From (13) we conclude that in the regular case, v is the *only* valuation of F which extends the Gauss valuation v_x of $K(x)$. And v is *unramified* over $K(x)$. Since the Gauss valuation v_x of $K(x)$ is unramified over K (compare (1)) we conclude that $F|K$ is an unramified field extension, provided we have regular reduction.

Suppose for example that $F = K(x, y)$ and there is an irreducible equation of the form

$$\Phi(x, y) = 0 \tag{15}$$

where

$$\Phi(X, Y) = Y^n + f_{n-1}(X)Y^{n-1} + \cdots + f_o(X) \in K[X, Y] \tag{16}$$

with $f_i(X) \in \mathcal{O}_K[X]$. (This implies $n = [F : K(x)]$.) Then y is integral over $\mathcal{O}_K[x]$ and hence $\overline{y} \neq \infty$. The equation

$$\overline{\Phi}(\overline{x}, \overline{y}) = 0 \tag{17}$$

is obtained by reducing the coefficients of $\Phi(X, Y)$ modulo v, where

$$\overline{\Phi}(X, Y) = Y^n + \overline{f}_{n-1}(X)Y^{n-1} + \cdots + \overline{f}_o(X) \in \overline{K}[X, Y] \tag{18}$$

Now, *if* $\overline{\Phi}(X, Y)$ is an *irreducible* polynomial over $\overline{K}$ then we have

$$n = [\overline{K}(\overline{x}, \overline{y}) : \overline{K}(\overline{x})] \le [\overline{F} : \overline{K}(\overline{x})] \le [F : K(x)] = n$$

and we see that (14) holds, i.e., $F|K$ has regular reduction. Thus, the irreducibility of the reduced polynomial $\overline{\Phi}(X, Y)$ implies regularity.

However, this sufficient criterion is not necessary. If $\overline{\Phi}(X, Y)$ is reducible there may be other generators of F over K whose defining equation stays irreducible after reduction.

But if we start from an arbitrary function field $\overline{E}|\overline{K}$ over the residue field of K, then it is possible to construct a "regular lift", i.e., a function field $F|K$ with a valuation extending the given valuation v on K, such that the residue field $\overline{F} = \overline{E}$, and that F admits a regular element.

Indeed: for example we assume $\overline{E} = \overline{K}(\overline{x}, \overline{y})$ with a defining irreducible equation (17) of the form (18). For each non-zero coefficient of $\overline{\Phi}(X, Y)$ we choose a foreimage in K and and obtain a polynomial $\Phi[X, Y] \in K[X, Y]$, of the same degree in Y, which is irreducible since its reduction $\overline{\Phi}(X, Y) \in \overline{K}[X, Y]$ is irreducible. Now let $F = K(x, y)$ be the function field over K defined by the irreducible equation $\Phi(x, y) = 0$. By construction, the substitution $(x, y) \mapsto (\overline{x}, \overline{y})$ defines a homomorphism $K[x, y] \to \overline{K}[\overline{x}, \overline{y}]$ extending the residue map $K \to \overline{K}$ modulo v. That homomorphism can be extended to a place of $F = K(x, y)$, i.e., to the residue map of a valuation of F, again denoted by v, extending the given valuation v of K. In this way $F|K$ becomes a valued function field and, we claim, it has regular reduction, the residue field being the given field $\overline{E}$. Indeed: On the one hand, since x is mapped onto $\overline{x}$ which is transcendental over $\overline{K}$ it follows that in $K(x)$ the valuation v is the Gauss valuation v_x. (See Theorem 1.) The residue field of v_x is $\overline{K}(\overline{x})$. Now by construction, $[F : K(x)] = n$ and the residue field of F contains $\overline{F}(\overline{x}, \overline{y}) = \overline{E}$ which is of degree n over $\overline{K}(\overline{x})$. In view of (13) we conclude that, indeed, $\overline{F} = \overline{E}$, and that x is a regular element in F.

In the situation discussed above it is said that $F|K$ arises by **lifting** the given function field $\overline{E}|\overline{K}$. Note that the lifting process as described above is *not unique*. There are several liftings, in fact infinitely many, of any given function field $\overline{E}|\overline{K}$.

3. The inertia theorem

The notion of an "inert" extension of valued fields had been introduced by Hilbert. But he considered extensions $L|K$ of *finite degree* only. [1] According to Hilbert, $L|K$ is called "inert" if $[\overline{L} : \overline{K}] = [L : K]$. Hence the definition of "regular element" in (14) can be phrased by saying that F is inert over $K(x)$, with respect to the Gauss valuation of $K(x)$.

Hilbert's definition, however, does not make sense for valued field extensions $F|K$ of infinite degree. In that case we have to consider finite dimensional K-submodules $M \subset F$. We denote by $\overline{M}$ the reduction of M. More precisely, if $\mathcal{O}_F$ denotes the valuation ring of F then $\overline{M}$ is the image of $\mathcal{O}_F \cap M$ under the residue map. $\overline{M}$ is a $\overline{K}$-module and we note:

LEMMA 4. *For any finite K-module $M \subset F$ we have $\dim_{\overline{K}} \overline{M} \leq \dim_K M$. More generally, if $N \subset M$ are K-submodules of F and if $\dim_K M/N$ is finite then $\dim_{\overline{K}} \overline{M}/\overline{N} \leq \dim_K M/N$.*

PROOF. Let $\overline{u}_1, \ldots, \overline{u}_r \in \overline{M}$ be $\overline{K}$-linearly independent modulo $\overline{N}$, and let the $u_i \in M$ be foreimages of the $\overline{u}_i$. We claim that the u_i are K-linearly independent modulo N. For, assume that there would be a nontrivial linear relation

$$a_1 u_1 + \cdots + a_r u_r \equiv 0 \bmod N \tag{19}$$

[1] In Hilbert's situation $L|K$ was an extension of finite number fields, and the valuation belonged to a prime ideal of the ring of integers.

with $a_i \in K$. Let $a \in K$ such that $v(a) = \min_i v(a_i)$. After dividing the above relation by a and changing notation we may assume

$$\min_{1\le i\le r} v(a_i) = 0\,.$$

In other words, not all $\overline{a}_i = 0$. Applying the residue homomorphism to (19) we obtain a nontrivial linear relation

$$\overline{a}_1\,\overline{u}_1 + \cdots + \overline{a}_r\,\overline{u}_r \equiv 0 \bmod \overline{N}$$

contradicting the $\overline{K}$-linear independence of the $\overline{u}_i$ modulo $\overline{N}$. □

Corollary 5. *Suppose $M \subset F$ is finite dimensional over K. If $\dim \overline{M} = \dim M$ then for every K-submodule $N \subset M$ we also have $\dim \overline{N} = \dim N$.*[2]

Proof. Let $m, n, \overline{m}, \overline{n}$ be the dimensions of $M, N, \overline{M}, \overline{N}$. By assumption we have $\overline{m} = m$. By Lemma 4 we have $\overline{m} - \overline{n} \le m - n$ and so

$$n \le m - (\overline{m} - \overline{n}) = \overline{n} \le n$$

hence $\overline{n} = n$. □

Corollary 5 permits to generalize Hilbert's notion of "inert" field extensions to valued field extensions $F|K$ of infinite degree and, more generally, for K-submodules $M \subset F$. As follows:

Definition 1. Let $F|K$ be an extension of valued fields and $M \subset F$ any K-module, of finite or infinite dimension. M is said to be **inert** if for every finite-dimensional K-module $N \subset M$ we have $\dim \overline{N} = \dim N$.

We are now able to formulate the Inertia Theorem which is announced in the title of this section. It will be the basis of all that follows.

Theorem 6 (Inertia Theorem). *Let $F|K$ be a valued function field with regular reduction $\overline{F}|\overline{K}$. Then $F|K$ is inert in the sense as defined above.*

Remark 7. The definition of "regular" reduction as given in the foregoing section says that $F|K(x)$ is inert (in the sense of Hilbert). The above Theorem now asserts that this implies the whole function field $F|K$ to be inert.

The proof of the inertia Theorem rests on certain general properties of the notion of inertia. Let us state these properties in two Lemmas. In these Lemmas $F|K$ may be an arbitrary extension of valued fields.

If $M \subset F$ is a K-module of finite dimension and $\dim \overline{M} = \dim M$ then the proof of Lemma 4 (for $N = 0$) shows that there exists a K-basis $u_1, \ldots, u_m$ of M such that their residues $\overline{u}_1, \ldots, \overline{u}_m$ form a $\overline{K}$-basis of $\overline{M}$. If $z \in M$ is represented as a linear combination

$$z = \sum_{1\le i\le m} a_i u_i \qquad \text{with} \qquad a_i \in K$$

then

$$v(z) = \min_{1\le i\le m} v(a_i)\,. \tag{20}$$

[2] Here and in the following, to simplify notation we shall write $\dim M$ and $\dim \overline{M}$, thus omitting the reference to K and $\overline{K}$ respectively, whenever it is clear from the context which base field is referred to.

Such a basis is called a **valuation basis** of M. Hence $\dim M = \dim \overline{M}$ if and only if M admits a valuation basis over K.

The notion of "valuation basis" can also be defined for infinite dimensional modules M, as a K-basis u_i such that the $\overline{u}_i$ form a $\overline{K}$-basis of $\overline{M}$ (where i ranges over a suitable index set I). We do not claim that every infinite inert K-module $M \subset F$ admits a valuation basis. However:

LEMMA 8. *Let M be any K-submodule of F, of finite or infinite dimension. If M admits a valuation basis u_i $(i \in I)$ then M is inert.*

PROOF. Every finite dimensional submodule $N \subset M$ is contained in a finite dimensional submodule of the form $M_1 = \sum_{i \in I_1} K u_i$ with a suitable finite index set $I_1 \subset I$. Since M_1 is inert (because it admits a valuation basis) we may apply Corollary 5 to conclude that N is inert. □

LEMMA 9. *Suppose $R \subset F$ is a K-algebra and F is the field of quotients of R. If R is inert over K then so is F.*

PROOF. Let $M \subset F$ be a finite dimensional K-submodule. We choose a K-basis of M. Every basis element is a quotient of two elements from R. We can choose a common denominator $z \in R$ for these finitely many quotients (because R is an algebra, hence closed with respect to products). Hence $zM \subset R$, and therefore zM is inert. Since $v(z) \in v(K)$ we may assume $v(z) = 0$, after dividing z by a suitable element $a \in K$. It follows $\overline{zM} = \overline{z}\overline{M}$ and therefore

$$\dim \overline{M} = \dim \overline{z}\overline{M} = \dim \overline{zM} = \dim zM = \dim M .$$

This being the case for every finite-dimensional module $M \subset F$, it follows that F is inert. □

COROLLARY 10. *Let $F = K(x)$ be a rational function field, equipped with the Gauss valuation with respect to x. Then F is inert over K.*

For, F is the field of quotients of the polynomial ring $R = K[x]$ and this admits the powers x^i $(i = 0, 1, 2, \ldots)$ as a valuation basis in view of (1).

REMARK 11. The corollary holds also for function fields of several variables and for the field of convergent power series; see Remark 3 at the end of section 1.

Now consider again an arbitrary valued field extension $F|K$.

LEMMA 12. *Let $K \subset F_o \subset F$. If $F_o|K$ is inert and $F|F_o$ is inert then $F|K$ is inert.*

PROOF. Let $M \subset F$ be a finite dimensional K-module. Then $F_o M$ is a finite dimensional F_o-module. By the hypothesis of the Theorem, $F_o M$ is inert over F_o. Hence there exists a valuation-F_o-basis u_i of $F_o M$.

Every element of a K-basis of M can be written as a linear combination of the u_i with coefficients in F_o. Let $M_o \subset F_o$ denote the K-module generated by those finitely many coefficients. M_o is inert over K and so there exists a valuation-K-basis w_j of M_o. By construction, M is contained in the K-module M^* generated by the $w_j u_i$. These elements form a valuation basis of M^*. For, the $\overline{u}_i$ are $\overline{F}_o$-linearly independent and the $\overline{w}_j \in \overline{F}_o$ are $\overline{K}$-linearly independent. Hence M^* is inert over K and so is M as a K-submodule of M^*. □

Proof of the Inertia Theorem. Let $x \in F$ be a regular element. Then the subfield $K(x) \subset F$ is valued by the Gauss valuation and, hence, it is inert over K by corollary 10. Since $[F : K(x)] = [\overline{F} : \overline{K}(\overline{x})]$ we see that F is inert over $K(x)$. Hence $F|K$ is inert by Lemma 12. □

4. Reduction of genus

In dealing with the genus of a function field $F|K$ it is usually assumed that K is relatively algebraically closed in F. Sometimes this is expressed by saying that K is the "exact field of constants" of F. Then the genus g can be defined as the number occurring in the Riemann-Roch Theorem, as follows: Every divisor A of $F|K$ defines a K-module consisting of all elements $y \in F^\times$ whose pole divisor divides A. This module is denoted by $\mathcal{L}(A)$. Its K-dimension is finite and denoted by $\dim A$. The **Riemann-Roch Theorem** implies that $\dim A$ is related to the degree of A by the formula

$$\dim A \geq \deg A + (1-g)\,. \tag{21}$$

More precisely, we have

$$\dim A = \deg A + (1-g) + \dim(W-A) \tag{22}$$

where W is a canonical divisor of the function field $F|K$. Here we do not need to go into the details of definition of canonical divisor. The only fact which we have to know is that

$$\dim W = g \qquad \text{and} \qquad \deg W = 2g-2\,,$$

which is immediate from (22). Since divisors of negative degree have vanishing dimension we conclude

$$\deg A > 2g-2 \qquad \Longrightarrow \qquad \dim A = \deg A + (1-g) \tag{23}$$

Since $\dim A \leq 1 + \deg A$ (proof by induction on $\deg A$) we also have

$$\dim A > 2g-1 \qquad \Longrightarrow \qquad \dim A = \deg A + (1-g) \tag{24}$$

The number $\chi = 1-g$ which appears in these formulas is called the **arithmetic genus** of the function field. Sometimes g is then called the "geometric genus" of $F|K$.

The above formulas hold if K is relatively algebraically closed in F. But we do not wish to make this assumption, simply because it is not necessary for the general theory. So let K' denote the algebraic closure of K in F and $d = [K' : K]$ its degree. K' is called the "field of constants" of F and d the "degree of constants". For a divisor A of $F|K$ its degree over K is d times the degree over K', and same for $\dim A$. Hence, if the arithmetic genus χ of $F|K$ is now defined to be $d(1-g)$ then the Riemann-Roch Theorem (22) for $F|K$ appears in the form

$$\dim A = \deg A + \chi + \dim(W-A) \tag{25}$$

without the assumption that K is relatively algebraically closed in F. If in (25) we set $A = 0$ and then $A = W$ we obtain [3]

$$\dim W = -\chi + d \qquad \text{and} \qquad \deg W = -2\chi\,. \tag{26}$$

Note that $\chi < 0$ except if $g = 1$ when $\chi = 0$, or when $g = 0$ in which case $\chi = d > 0$.

[3] Observe that for the zero divisor $A = 0$ we have $\deg(0) = 0$ and $\dim(0) = d$.

The formulas (23) and (24) now appear in the form:

$$\deg A > -2\chi \quad \Longrightarrow \quad \dim A = \deg A + \chi \tag{27}$$

$$\dim A > -2\chi + d \quad \Longrightarrow \quad \dim A = \deg A + \chi\,. \tag{28}$$

In order to simplify our formulas we shall write $A \gg 0$ if $A > 0$ and $\deg A$ and $\dim A$ are sufficiently large, so that we have in particular

$$A \gg 0 \quad \Longrightarrow \quad \dim A = \deg A + \chi\,. \tag{29}$$

If $A > 0$ then for a sufficiently large integer $t > 0$ we have $tA \gg 0$.

Now we consider a valued function field $F|K$ with regular reduction $\overline{F}|\overline{K}$. Then we have the same formulas as above for divisors $\overline{A}$ of $\overline{F}|\overline{K}$, where $\deg \overline{A}$ and $\dim \overline{A}$ are to be understood over $\overline{K}$: [4]

$$\overline{A} \gg 0 \quad \Longrightarrow \quad \dim \overline{A} = \deg \overline{A} + \overline{\chi}\,. \tag{30}$$

Here, $\overline{\chi}$ denotes the arithmetic genus of $\overline{F}|\overline{K}$, and the condition $\overline{A} \gg 0$ refers to the bounds given by $\overline{F}|\overline{K}$.

In general, if we consider at the same time both $F|K$ and its reduction $\overline{F}|\overline{K}$ then the condition $A \gg 0$ refers to the above degree conditions for both $F|K$ and $\overline{F}|\overline{K}$.

Our aim is to compare the arithmetic genus χ of $F|K$ with the arithmetic genus $\overline{\chi}$ of the reduced field $\overline{F}|\overline{K}$.

THEOREM 13 (Genus Reduction Theorem). *Let $F|K$ be a valued function field with regular reduction $\overline{F}|\overline{K}$. Let χ be the arithmetic genus of $F|K$ and $\overline{\chi}$ the arithmetic genus of $\overline{F}|\overline{K}$. Then $\chi \le \overline{\chi}$. Specifically, let $x \in F$ be a regular element and let R_x denote the integral closure of $K[x]$ in F. Similarly $R_{\overline{x}}$ denotes the integral closure of $\overline{K}[\overline{x}]$ in $\overline{F}$. Then*

$$\overline{\chi} - \chi = \dim R_{\overline{x}}/\overline{R_x}. \tag{31}$$

Note that since $\dim R_{\overline{x}}/\overline{R_x}$ is finite it follows that $R_{\overline{x}}$ is the integral closure of $\overline{R_x}$ in $\overline{F}$. In particular, we have $\overline{\chi} = \chi$ if and only if $\overline{R_x}$ is integrally closed.

REMARK 14. By definition we have $\chi = d(1-g)$ where g is the geometric genus of $F|K$ and d is the degree of constants. Similarly $\overline{\chi} = \overline{d}(1-\overline{g})$. In general we have $\overline{d} \ge d$. If the reduction preserves the degree of constants, i.e., if $\overline{d} = d$ then the statement $\chi \le \overline{\chi}$ means $g \ge \overline{g}$.

PROOF OF THEOREM 13. For a divisor $A \gg 0$ of $F|K$ define the divisor $\overline{A}$ of $\overline{F}|\overline{K}$ by

$$\overline{A} := \{\sup(\overline{y})_\infty \mid 0 \ne \overline{y} \in \overline{\mathcal{L}(A)}\ \}. \tag{32}$$

Let us call $\overline{A}$ the divisor "associated to A".[5] By definition we have

$$\overline{\mathcal{L}(A)} \subset \mathcal{L}(\overline{A}) \qquad \text{hence} \qquad \dim A \le \dim \overline{A} \tag{33}$$

where we have used the Inertia Theorem. In particular it follows $\overline{A} \gg 0$.

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[4] More precisely we should have written $\deg_{\overline{F}|\overline{K}}(\overline{A})$ and $\dim_{\overline{F}|\overline{K}}(\overline{A})$. But for simplicity of notation we skip the index $\overline{F}|\overline{K}$ if it is clear from the context which function field is referred to. Similarly we write $\mathcal{L}(\overline{A})$ instead of $\mathcal{L}_{\overline{F}|\overline{K}}(\overline{A})$ etc.

[5] Until further notice $\overline{A}$ is defined for $A \gg 0$ only. But see Theorem 17 below.

If there exists $\overline{y} \in \overline{\mathcal{L}(A)}$ with $(\overline{y})_\infty = \overline{A}$ let $y \in \mathcal{L}(A)$ be a foreimage. We compute:

$$\deg \overline{A} = \deg(\overline{y})_\infty = [\overline{F} : \overline{K}(\overline{y})] \leq [F : K(y)] = \deg(y)_\infty \leq \deg A\,, \tag{34}$$

the last relation since $y \in \mathcal{L}(A)$ and hence $(y)_\infty \leq A$.

If the base field $\overline{K}$ of $\overline{F}$ is infinite then indeed there exists such $\overline{y} \in \overline{\mathcal{L}(A)}$. (See Lemma 15 below.) If $\overline{K}$ is finite we have to use some other argument conclude that

$$\deg \overline{A} \leq \deg A\,. \tag{35}$$

(See Lemma 16 below.) From (29) it follows in view of (33) and (35)

$$\begin{aligned} \overline{\chi} - \chi &= (\dim \overline{A} - \deg \overline{A}) - (\dim A - \deg A) \\ &= (\dim \overline{A} - \dim A) + (\deg A - \deg \overline{A}) \geq 0\,. \end{aligned} \tag{36}$$

Now let $A = (x)_\infty$ be the pole divisor of a regular element x of $F|K$. After replacing x by x^t with sufficiently large t we may assume $A \gg 0$. We have $\deg(\overline{x})_\infty = \deg A$. Since $\overline{x} \in \overline{\mathcal{L}(A)}$ it follows $\deg A \leq \deg \overline{A}$, hence in view of (35)

$$\deg \overline{A} = \deg A \tag{37}$$

and therefore by (36)

$$\overline{\chi} - \chi = \dim \overline{A} - \dim A = \dim \mathcal{L}(\overline{A})/\overline{\mathcal{L}(A)}\,. \tag{38}$$

If we replace x by x^t with $t > 0$ then A has to be replaced by tA and we obtain in the same way

$$\overline{\chi} - \chi = \dim \mathcal{L}(\overline{tA})/\overline{\mathcal{L}(tA)}\,. \tag{39}$$

Here we have $\overline{tA} = t\overline{A}$. This can be seen as follows: We have

$$\begin{aligned} \mathcal{L}(A)^t &\subset \mathcal{L}(tA)\,, \\ \overline{\mathcal{L}(A)}^t &\subset \overline{\mathcal{L}(tA)}\,, \\ t\overline{A} &\leq \overline{tA} && \text{by definition (32)}\,, \\ t\overline{A} &= \overline{tA}\,, \end{aligned}$$

the last relation holds since both sides have the same degree by (37) when applied to tA. Note that x^t is also regular, and it has pole divisor tA.

Now, R_x consists of all elements y whose pole divisor $(y)_\infty$ is composed by prime divisors contained in A. Thus

$$R_x = \bigcup_{t>0} \mathcal{L}(tA) \qquad \text{hence} \qquad \overline{R_x} = \bigcup_{t>0} \overline{\mathcal{L}(tA)}\,.$$

Similarly

$$R_{\overline{x}} = \bigcup_{t>0} \mathcal{L}(t\overline{A})\,,$$

We conclude from (39) for $t \to \infty$ that $\overline{\chi} - \chi = \dim R_{\overline{x}}/\overline{R_x}$ as announced in the Genus Theorem.

We now turn to the two lemmas which we have used in the foregoing proof.

LEMMA 15. *Let $F|K$ be a function field and $M \neq 0$ a finitely generated K-module in F. Define the divisor*

$$M_\infty := \{\sup(y)_\infty \mid 0 \neq y \in M\,\}. \tag{40}$$

If K is infinite then there exists $y \in M$ such that $(y)_\infty = M_\infty$.

Proof. The assertion is trivial if $M_\infty = 0$. Otherwise, decompose M_∞ into prime divisors:

$$M_\infty = \sum_i a_i P_i$$

with coefficients $a_i \geq 1$. For each i there exists $y_i \in M$ with $v_{P_i}(y_i) = -a_i$, according to the definition (40). (v_{P_i} denotes the valuation of $F|K$ belonging to P_i.) Let $N_i \subset M$ be the submodule consisting of those $y \in M$ with $v_{P_i}(y) > -a_i$. Then $y_i \notin N_i$, hence N_i is a proper submodule of M. Now, if K is infinite there exists $y \in M$ which is not contained in any of these finitely many proper submodules N_i. (Proof by induction on the dimension of M.) □

We apply Lemma 15 to the function field $\overline{F}|\overline{K}$ and the $\overline{K}$-module $\overline{\mathcal{L}(A)}$. We obtain an element $\overline{y} \in \overline{\mathcal{L}(A)}$ with $(\overline{y})_\infty = \overline{A}$, as we have used in the computation (34) – provided the base field $\overline{K}$ is infinite.

If $\overline{K}$ is finite we have to argue by means of base field extensions. Again, as in Lemma 15 we formulate the next Lemma for any function field $F|K$, but then we will have to apply it for $\overline{F}|\overline{K}$ in order to obtain (35).

In the situation of Lemma 15 let $K' = K(z)$ be a rational function field with a new variable z. Consider the base field extension $F' = FK'$, i.e., the field of quotients of the integral domain $F \otimes_K K'$. The divisor group of $F|K$ embeds naturally into the divisor group of $F'|K'$:

$$\mathrm{Div}(F|K) \subset \mathrm{Div}(F'|K'). \tag{41}$$

Under this embedding the degree of divisors is stable, i.e., for $A \in \mathrm{Div}(F|K)$ we have $\deg_{F|K}(A) = \deg_{F'|K'}(A)$.

Lemma 16. *Let $0 \neq M$ be a finite dimensional submodule of $F|K$ and $M' = M \otimes_K K'$. Then $M_\infty = M'_\infty$ according to the embedding* (41). *Hence $\deg_{F|K} M_\infty = \deg_{F'|K'} M'_\infty$.*

Proof. Since $M \subset M'$ we have $M_\infty \leq M'_\infty$. On the other hand, M' consists of the elements y' of the form $y' = \sum_j c_j y_j$ with $c_j \in K'$ and $y_j \in M$. It follows $(y')_\infty \leq \sup_j (y_j)_\infty \leq M_\infty$, hence $M'_\infty \leq M_\infty$. □

We apply this Lemma in the proof of Theorem 13 as follows: We consider the function field $F' = F(z)$ over $K' = K(z)$. We extend the given valuation of F to F' by means of the Gauss valuation with respect to z. Then $F'|K'$ becomes a valued function field with reduction $\overline{F'} = \overline{F}(\overline{z})$ over $\overline{K'} = \overline{K}(\overline{z})$. For any divisor A we have

$$\mathcal{L}_{F'|K'}(A) = \mathcal{L}_{F|K}(A) \otimes_K K'\,.$$

(This is well known from the general theory of base field extensions.) It follows

$$\overline{\mathcal{L}_{F'|K'}(A)} = \overline{\mathcal{L}_{F|K}(A)} \otimes_{\overline{K}} \overline{K'}\,.$$

We apply Lemma 16 to $\overline{F}|\overline{K}$ and the module $\overline{\mathcal{L}_{F|K}(A)}$ and conclude that the divisor $\overline{A} = \overline{\mathcal{L}_{F|K}(A)}_\infty$ of $\overline{F}|\overline{K}$ (defined by (32)) coincides with the divisor which is defined similarly for $\overline{F'}|\overline{K'}$. Since $\overline{K'}$ is infinite we obtain

$$\deg_{\overline{F}|\overline{K}}(\overline{A}) = \deg_{\overline{F'}|\overline{K'}}(\overline{A}) \leq \deg_{\overline{F'}|\overline{K'}}(A) = \deg_{F|K}(A)$$

as required for (35). □

5. Good reduction

$F|K$ denotes a valued function field with regular reduction $\overline{F}|\overline{K}$, arithmetic genus χ and reduced arithmetic genus $\overline{\chi}$.

DEFINITION 2. $F|K$ is said to have **good reduction** if: $\chi = \overline{\chi}$.

Formula(36) shows that in the case of good reduction we have for any divisor $A \gg 0$:

$$\deg \overline{A} = \deg A \qquad \text{and} \qquad \dim \overline{A} = \dim A\,. \tag{42}$$

This implies

$$\overline{\mathcal{L}(A)} = \mathcal{L}(\overline{A}) \tag{43}$$

in view of (33). Here $\overline{A}$ is the divisor which we have defined in (32) for $A \gg 0$, and which we had called there "associated to A". The following Theorem shows that this definition can be extended to all divisors of $F|K$, leading to a homomorphism of the divisor groups.

THEOREM 17 (Divisor Reduction Theorem). *Suppose the valued function field $F|K$ admits good reduction. Then there exists one and only one homomorphism* $\mathrm{Div}(F|K) \to \mathrm{Div}(\overline{F}|\overline{K})$ *of the divisor groups which preserves the degree, the order relation and principal divisors.*

More precisely, the homomorphism $\mathrm{Div}(F|K) \to \mathrm{Div}(\overline{F}|\overline{K})$ *has the following properties, where $\overline{A}$ denotes the image of $A \in \mathrm{Div}(F|K)$:*

$$\deg \overline{A} = \deg A \tag{44}$$

$$A \leq B \implies \overline{A} \leq \overline{B} \tag{45}$$

$$\overline{(x)} = (\overline{x}) \qquad\qquad (\text{if } \overline{x} \neq 0, \infty)\,. \tag{46}$$

PROOF. For any divisor $A \gg 0$ of $F|K$ we define $\overline{A}$ as in (32). If $B \gg 0$ is another divisor then we conclude successively:

$$\begin{aligned} \mathcal{L}(A) \cdot \mathcal{L}(B) &\subset \mathcal{L}(A+B) \\ \overline{\mathcal{L}(A)} \cdot \overline{\mathcal{L}(B)} &\subset \overline{\mathcal{L}(A+B)} \\ \overline{A} + \overline{B} &\leq \overline{A+B} \qquad \text{by definition (32)} \end{aligned}$$

$$\overline{A} + \overline{B} = \overline{A+B}\,, \tag{47}$$

the last conclusion holds since both sides have the same degree in view of (42). This holds for $A, B \gg 0$. Now, if A is an arbitrary divisor of $F|K$ then we write

$$A = B - C \qquad \text{with} \qquad B, C \gg 0 \tag{48}$$

and define:

$$\overline{A} := \overline{B} - \overline{C}\,. \tag{49}$$

Because of (47) this is well defined, i.e., does not depend on the choice of $B, C \gg 0$, and it leads to a homomorphism of the divisor groups.. By (42) the degree formula (44) holds if $A \gg 0$, hence it remains true for arbitrary A because of (49).

If $A > 0$ then $tA \gg 0$ for sufficiently large t, hence $\overline{tA} = t\overline{A} \gg 0$ and therefore $\overline{A} > 0$, i.e. the map $A \mapsto \overline{A}$ preserves the order relation for divisors.

If $\overline{x} \neq 0, \infty$ then we write $(x) = B - A$ with $A, B \gg 0$ and have

$$\begin{aligned}
(x) + A &= B \\
x \cdot \mathcal{L}(A) &= \mathcal{L}(B) \\
\overline{x} \cdot \overline{\mathcal{L}(A)} &= \overline{\mathcal{L}(B)} \\
\overline{x} \cdot \mathcal{L}(\overline{A}) &= \mathcal{L}(\overline{B}) \\
(\overline{x}) + \overline{A} &= \overline{B} \\
(\overline{x}) = \overline{B} - \overline{A} &= \overline{B - A} = \overline{(x)}\,.
\end{aligned}$$

This proves the existence of the divisor reduction map. To see that it is unique, assume that there is another such map $A \mapsto A^*$. It suffices to show that $A^* = \overline{A}$ for $A \gg 0$.

If $0 \neq x \in \mathcal{L}(A))$ then $(x) \geq -A$. From (45) and (46) (applied to A^*) it follows $(\overline{x}) \geq -A^*$, i.e., $(\overline{x})_\infty \leq A^*$. In view of (32) it follows $\overline{A} \leq A^*$, hence $\overline{A} = A^*$ since both have the same degree by (44). □

Corollary 18. *Suppose $F|K$ admits good reduction. If W is a canonical divisor of $F|K$ then its image $\overline{W}$ is a canonical divisor of $\overline{F}|\overline{K}$.*

Proof. Until now we have used the Riemann-Roch Theorem in the short version (29) only, which applies to divisors $A \gg 0$. But now, since $W \not\gg 0$ we have to go back to its original version (25). For a canonical divisor W we have seen in (26) that

$$\dim W = -\chi + d \qquad \text{and} \qquad \deg W = -2\chi\,. \tag{50}$$

Canonical divisors are characterized by these values of $\dim(\cdot)$ and $\deg(\cdot)$. For, if A is any divisor with $\deg A = -2\chi$ then by the Riemann-Roch Theorem:

$$\dim(A) = -2\chi + \chi + \dim(W - A)$$

The divisor $W - A$ is of degree 0. A divisor of degree 0 has dimension 0 except if it is a principal divisor in which case its dimension is d. Now, if $W - A$ is a principal divisor then A belongs to the same divisor class as W, i.e., A is in the canonical class. We conclude:

$$\deg A = -2\chi \implies \dim(A) = \begin{cases} -\chi + d \text{ if } A \text{ is a canonical divisor} \\ -\chi \quad \text{if } A \text{ is not canonical.} \end{cases}$$

Now, for the reduced function field $\overline{F}|\overline{K}$ we have the same situation, with d to be replaced by $\overline{d}$, the degree of the relative algebraic closure of $\overline{K}$ in $\overline{F}$. From (44) we know that $\deg \overline{W} = \deg W = -2\chi = -2\overline{\chi}$. Thus, if $\overline{W}$ would not be canonical then $\dim \overline{W} = -\overline{\chi} = -\chi$. But this is not the case, as can be seen as follows: $\mathcal{L}(W)$ consists of those $x \in F$ for which $(x) \geq -W$. Hence for $\overline{x} \in \overline{\mathcal{L}(W)}$ we have $(\overline{x}) \geq -\overline{W}$ by (45) and (46). It follows $\overline{\mathcal{L}(W)} \subset \mathcal{L}(\overline{W})$. Using the Inertia Theorem it follows $\dim W = -\chi + d \leq \dim(\overline{W})$, hence $\dim(\overline{W}) > -\chi$. □

Remark 19. If the reduction preserves the degree of constants, i.e., if $d = \overline{d}$, then from (50), applied to W and $\overline{W}$, we see that $\dim W = \dim \overline{W}$. Hence $\overline{\mathcal{L}(W)} = \mathcal{L}(\overline{W})$ in this case.

PROPOSITION 20 (Regular elements). *Let $F|K$ be a valued function field with good reduction. For every divisor $A > 0$ and sufficiently large $t > 0$ the divisor tA is the pole divisor of a regular element.*

PROOF. First let us choose t such that $tA \gg 0$. Changing notation, we write again A instead of tA. In view of (43) we have $\overline{\mathcal{L}(A)} = \mathcal{L}(\overline{A})$. If $\overline{K}$ is infinite then we have seen in the proof of the Genus Theorem that there exists $\overline{y} \in \overline{\mathcal{L}(A)}$ with $(\overline{y})_\infty = \overline{A}$. (See Lemma 15.) Let $y \in \mathcal{L}(A)$ be a foreimage. We apply the computation (34) and conclude since $\deg \overline{A} = \deg A$:

$$[\overline{F} : \overline{K}(\overline{y})] = [F : K(y)] \, .$$

Hence y is regular and $(y)_\infty = A$. If $\overline{K}$ is finite then we apply the following Lemma to $\overline{F}|\overline{K}$.

LEMMA 21. *Let $F|K$ be a function field and $A \gg 0$ a divisor. For every $t \geq 2$ there exists $y \in \mathcal{L}(tA)$ with denominator $(y)_\infty = tA$.*

PROOF. Decompose A into prime divisors:

$$A = \sum_{1 \leq i \leq r} a_i P_i$$

with multiplicities $a_i \geq 1$. We define $B_i := A + (ta_i - 1)P_i \geq A$. Then $B_i \gg 0$. Put $C_i := B_i + P_i = A + ta_i P_i$. Then also $C_i \gg 0$ and therefore

$$\dim(C_i) - \dim(B_i) = \deg(C_i) - \deg(B_i) = \deg P_i > 0 \, .$$

Therefore there exists $y_i \in \mathcal{L}(C_i)\, , \, y_i \notin \mathcal{L}(B_i)$ $(1 \leq i \leq r)$. We have

$$v_{P_i}(y_i) = -ta_i \qquad \text{and} \qquad v_{P_j}(y_i) \geq -a_j > -ta_j \quad \text{for} \quad j \neq i \, .$$

(v_{P_i}, v_{P_j} are the valuations belonging to the prime divisors P_i, P_j respectively.) The sum $y = \sum_{1 \leq i \leq r} y_i$ has $v_{P_i}(y) = -ta_i$ for $1 \leq i \leq r$, hence $(y)_\infty = tA$. □

COROLLARY 22. *Suppose $F|K$ has good reduction. Let $R \subset F$ be any subring containing K and with quotient field F. If R is integrally closed then $\overline{R}$ is integrally closed.*

PROOF. Let $x \in R$ be a transcendental element and $A = (x)_\infty$. Since R is integrally closed it contains R_x, the integral closure of $K[x]$. The latter consists of all elements $y \in F$ whose denominator $(y)_\infty$ contains those primes only which are contained in A. Let t be sufficiently large so that tA, by Proposition 20, is the denominator of a regular element, say z. Then $R_z = R_x \subset R$. By the Genus Theorem the reduction $\overline{R_z}$ is integrally closed, hence a Dedekind ring. Every overring of a Dedekind ring in its quotient field is again a Dedekind ring. Hence $\overline{R}$ is integrally closed in $\overline{F}$. □

6. Almost all reductions

Consider a function field $F|K$. This time we assume that the base field K is algebraically closed. Our aim is the following

THEOREM 23. *For almost all valuations v of K there exists an extension of v to F such that $F|K$ has good reduction with respect to v.*

Here, the expression "almost all" means that there are finitely many non-zero elements $c_1, \ldots, c_r \in K$ (depending on F) such that the contention holds for those valuations v of K for which $\overline{c}_i \neq \infty$ for $i = 1, \ldots, r$.

Proof. Let $x \in F$ be transcendental over K and $n = [F : K(x)]$. Let $F = K(x, y)$ and

$$\Phi(x, y) = y^n + f_{n-1}(x)y^{n-1} + \cdots + f_0(x) = 0 \tag{51}$$

the corresponding irreducible equation; we may assume that the coefficients $f_i(x) \in K[x]$. Thus $[F : K(x)] = n$.

Let v be any valuation of K. We use the so-called

Theorem 24 (Theorem of Bertini-Noether). *Let $\Phi[X, Y] \in K[X, Y]$ be any irreducible polynomial. Then for almost all valuations v of K the reduced polynomial $\overline{\Phi}(X, Y) \in \overline{K}[X, Y]$ is defined and is irreducible.*

A proof of the Bertini-Noether Theorem can be found in the book "Field Arithmetic" by Jarden and Fried, third edition (2008), page 184; see [**3**]. Recall that here we assume the base field K to be algebraically closed. Without this assumption the Bertini-Noether Theorem is valid for *absolutely irreducible* polynomials, i.e., polynomials which stay irreducible over the algebraic closure of K.

When we say that "$\overline{\Phi}(X, Y)$ is defined" then this means that for all coefficients $c \neq 0$ appearing in $\Phi[X, Y]$ we have $\overline{c} \neq \infty$.

We apply this to the polynomial (51) and obtain for almost all v an irreducible polynomial equation

$$\overline{\Phi}(\overline{x}, \overline{y}) = \overline{y}^n + \overline{f}_{n-1}(\overline{x})\overline{y}^{n-1} + \cdots + \overline{f}_0(\overline{x}) = 0$$

which defines a function field $\overline{F}|\overline{K}$ with $[\overline{F} : \overline{K}(\overline{x})] = n$. For those v the map $x, y \mapsto \overline{x}, \overline{y}$ defines a valuation of $F|K$ extending the given valuation v of K, having x as a regular element. $F|K$ is inert with respect to v. In particular we have $\overline{\chi} \geq \chi$ for the genera of $\overline{F}|\overline{K}$ and $F|K$. Let V denote the set of those valuations v. Our claim is that for almost all $v \in V$ we have $\overline{\chi} = \chi$.

Using Theorem 13 we have to show that $\overline{R_x}$ is integrally closed for almost all $v \in V$. It is well known that R_x is a finite $K[x, y]$-module. In fact, there exists one element $z \neq 0$ such that $R_x = K[x, y, z]$. For almost all v we have $\overline{z} \neq 0, \infty$. For, writing

$$z = \sum_{0 \leq i \leq r-1} \frac{f_i(x)}{g_i(x)} y^i$$

with polynomials $f_i(x), g_i(x) \in K[x]$ then for almost all v, every nonzero coefficient c of these polynomials has $\overline{c} \neq 0, \infty$, so that

$$\overline{z} = \sum_{0 \leq i \leq r-1} \frac{\overline{f}_i(\overline{x})}{\overline{g}_i(\overline{x})} \overline{y}^i .$$

Consider the affine curve Γ over K with generic point (x, y, z). The fact that R_x is integrally closed can be expressed in geometric terms by saying that every K-rational point (a, b, c) of Γ is simple. From algebraic geometry we will use the following well known necessary and sufficient criterion for a point (a, b, c) on Γ to be simple.

Let $\varphi_i(X, Y, Z)$ be a defining set of polynomials for Γ, i.e., the φ_i generate the ideal in the polynomial ring $K[X, Y, Z]$ which is the kernel of the homomorphism given by $(X, Y, Z) \mapsto (x, y, z)$. The index i ranges over a finite index set, say $1 \le i \le r$.

Consider the matrix of partial derivatives

$$\Delta(X, Y, Z) := \begin{pmatrix} \partial_X \varphi_i(X, Y, Z) \\ \partial_Y \varphi_i(X, y, Z) \\ \partial_Z \varphi_i(X, Y, Z) \end{pmatrix} \qquad (i = 1, \dots r)$$

which determines the tangent space. It has 3 rows and r columns. A point (a, b, c) of Γ is simple if and only if the matrix $\Delta(a, b, c)$ has rank 2, i.e., there is at least one 2×2 non-vanishing subdeterminant. If this is to hold for all K-rational points (a, b, c) of Γ then the ideal generated by all 2×2 subdeterminants $d_\nu(x, y, z)$ of $\Delta(x, y, z)$ is not contained in any maximal ideal of $K[x, y, z] = R_x$, i.e., there exists a relation of the form

$$1 = \sum_\nu g_\nu(x, y, z)\, d_\nu(x, y, z)$$

with polynomials $g_\nu(X, Y, Z) \in K[X, Y, Z]$. (The index ν ranges over a suitable finite index set.) Now, for almost all v every nonzero coefficient c appearing in these polynomials has $\overline{c} \neq 0, \infty$. Hence

$$1 = \sum_\nu \overline{g}_\nu(\overline{x}, \overline{y}, \overline{z})\, \overline{d}_\nu(\overline{x}, \overline{y}, \overline{z})$$

for these v. This implies that each K-rational point $\overline{a}, \overline{b}, \overline{c}$ of the reduced curve $\overline{\Gamma}$ is simple. The reduced curve $\overline{\Gamma}$ is defined by the generic point $(\overline{x}, \overline{y}, \overline{z})$. Since $\overline{K}[\overline{x}, \overline{y}, \overline{z}] = \overline{R_x}$ we conclude that $\overline{R_x}$ is integrally closed. □

COROLLARY 25. *The statement of the theorem holds also if the base field K is arbitrary infinite, provided the function field $F|K$ is conservative, i.e., the genus of $F|K$ coincides with the genus of every constant field extension.*

PROOF. Take K' to be the algebraic closure of K and let $F' = FK'$ the corresponding constant field extension. By Theorem 23 almost all valuations $v_{K'}$ of K' can be extended to a valuation $v_{F'}$ such as to give good reduction of $F'|K'$, i.e., $\chi_{F'} = \chi_{\overline{F'}}$. On the other hand, $v_{F'}$ induces in F a valuation v_F and in K a valuation v_K. In this way $F|K$ becomes a valued function with regular reduction, and from the Genus Reduction Theorem 13 we see that $\chi_F \le \chi_{\overline{F}}$. Since $F|K$ is conservative we have $\chi_F = \chi_{F'}$. Together we conclude $\chi_{\overline{F}} \ge \chi_{\overline{F'}}$. Since $F' = FK'$ we have $\overline{F'} = \overline{F}\,\overline{K'}$, i.e., $\overline{F'}|\overline{K'}$ is a constant field extension of $\overline{F}|\overline{K}$. From the general theory it is well known that the arithmetic genus χ can only increase after constant field extensions. (The geometric genus g can only decrease.) It follows $\chi_{\overline{F}} = \chi_{\overline{F'}} = \chi_F$. Hence v_F defines good reduction on $F|K$.

This holds "for almost all valuations $v_{K'}$". By definition this means that there is finite set $S_{K'} \subset K'$ such that the contention holds for those $v_{K'}$ for which $v_{K'}(c) \neq \infty$ for all $c \in S_{K'}$. For every such c let $f_c(X)$ denote its irreducible polynomial over K. Let $S_K \subset K$ denote the set of coefficients of all those polynomials $f_c(X)$. If for every $a \in S_K$ we have $v_K(a) \neq \infty$ then $v_{K'}(c) \neq \infty$ for every $c \in K'$ and every extension $v_{K'}$ of v_K to K'. Hence, as shown above, v_K admits an extension v_F to $F|K$ with good reduction. □

REMARK 26. *A sufficient criterium for conservativity of a function field $F|K$ is that the base field K is perfect and is relatively algebraically closed in F.* As a reference we mention Deuring's Lectures on the theory of algebraic functions, published 1973; see[**2**].

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