On Artin's and Herbrand's Lemma

Recently, among the Hasse papers which are kept at the University Library in Göttingen, I found a sheet of paper carrying Hasse's remark "Roquette, December 1949". It appears that the paper contained a presentation of Herbrand's Lemma which I as a student had given that year in Hasse's Number Theory seminar at the Humboldt University in Berlin. Hasse seems to have liked it since he had kept it in his files through all these years. Perhaps it may be of general interest to see that the proof of Herbrand's Lemma, which nowadays is usually embedded into algebraic cohomology, can be given just in form of a diagram, understandable without text at least by those who are familiar with Hasse diagrams. The original paper is handwritten and contains the diagrams only. I have now transcribed it with TeX and added some comments how to read those diagrams.

Addendum (September 9, 2014): Same for the generalized Herbrand Lemma (page 4).

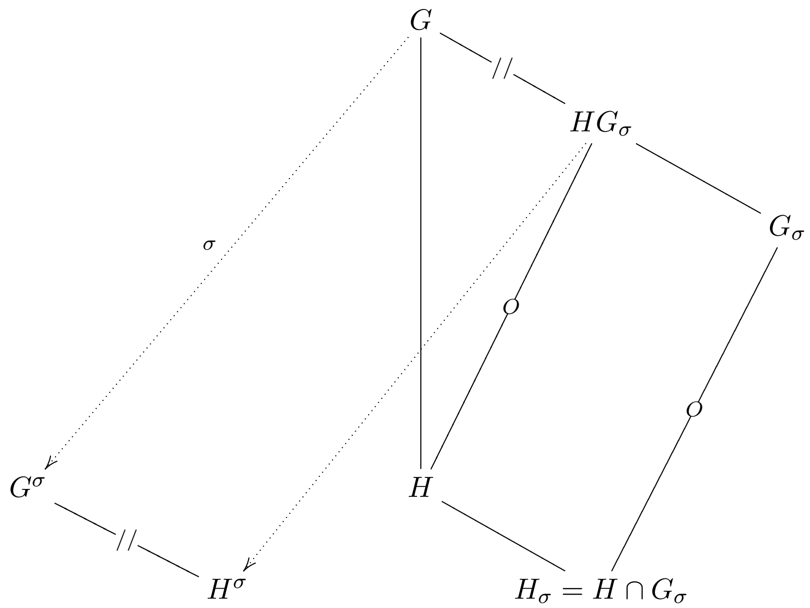
Consider the following situation:

G, H	an abelian group and a subgroup,
σ	a homomorphism of G into some group.
G^σ, H^σ	the image under σ of G and H respectively,
G_σ, H_σ	the kernel of σ in G and H respectively.

Artin's Lemma: *If the group index* $(G : H)$ *is finite then*

$$\boldsymbol{(G : H) = (G^\sigma : H^\sigma)(G_\sigma : H_\sigma)}\,.$$

Proof:



Such a diagram[1] illustrates the situation of the lemma to such degree that it does not need additional text to explain the proof – at least for those who are familiar with the elementary rules of Emmy Noether's algebra: One of her isomorphism theorems is that the factor groups $G_\sigma/H\cap G_\sigma$ and HG_σ/H are canonically isomorphic; this is indicated in the diagram by drawing the lines representing those factor groups parallel and of the equal length. Another theorem is that HG_σ is the foreimage of H^σ, and that G/HG_σ is isomorphic to G^σ/H^σ by means of σ. Finally, the group index is multiplicative in the sense that $(G : H) = (G : HG_\sigma)(HG_\sigma : H)$. All this is tacitly indicated in the drawing.

[1]Diagrams of this kind are called "Hasse diagrams".

Herbrand's Lemma refers to the situation of Artin's lemma with the following additional stipulations:

Now there are given two homomorphisms σ, τ of G which map G and H into itself respectively, i.e., G is a σ,τ-module and H a submodule. It is assumed that

$$G^\tau \subset G_\sigma \text{ and } G^\sigma \subset G_\tau\,, \quad \text{i.e.,} \quad \sigma\tau = \tau\sigma = \text{trivial map.}$$

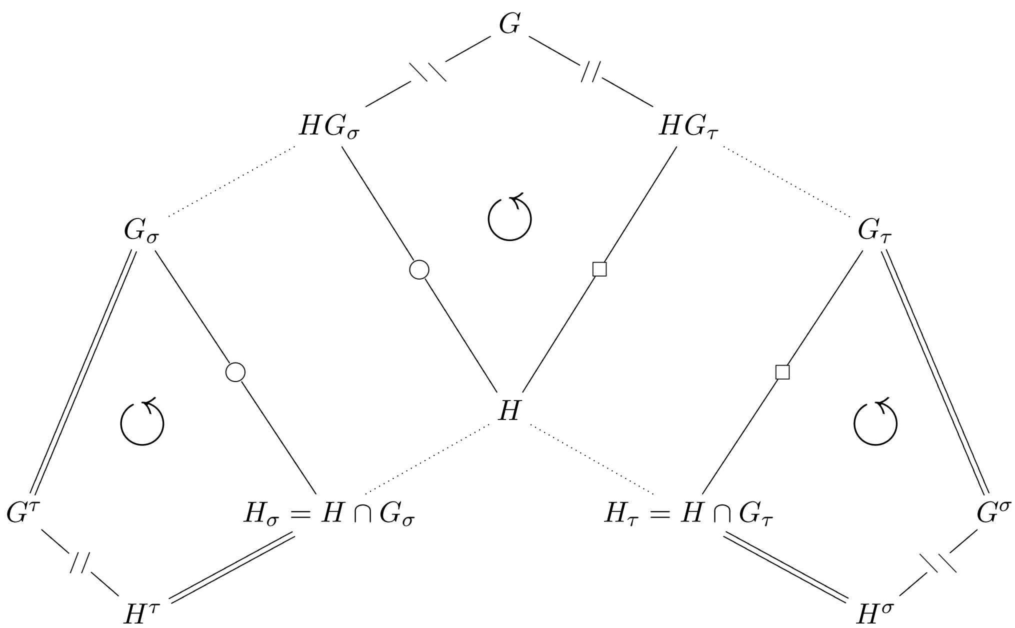
Defintion: $q(G) := \dfrac{(G_\sigma : G^\tau)}{(G_\tau : G^\sigma)}$ is the "Herbrand quotient" of G with respect to σ,τ. It is defined whenever the two group indices are finite. Similarly for H.

Herbrand's Lemma: *If the group index $(G : H)$ is finite then*

$$\boldsymbol{q(G) = q(H)}$$

in the sense that if one of the two Herbrand quotients is defined then the other is defined too. In other words: *The Herbrand quotient does not change if G is replaced by a subgroup H of finite index.*

Proof:



This diagram is to be read by successively moving counterclockwise along the three polygons (this is indicated by ↺), thereby multiplying step by step all the the various group indices or their inverses, according to whether the direction of the move is down or up. The result is 1 for each polygon because of the multiplicativity of the group index. The various isomorphisms indicated in the diagram imply that the corresponding factors cancel since they are passed in different directions (for instance, $(G : HG_\sigma)(G^\sigma : H^\sigma)^{-1} = 1$). There remain only the four indices shown by double lines, hence

$$(G_\sigma : G^\tau)(H_\sigma : H^\tau)^{-1}(H_\tau : H^\sigma)(G_\tau : G^\sigma)^{-1} = 1$$

which is the assertion. Note that all group indices in question are finite by assumption.

ADDENDUM (September 2014): What if $(G:H)$ is not finite? We claim:

$$\boldsymbol{q(G) = q(H)q(G/H)}$$

if these Herbrand quotients are defined. (It is sufficient that two of them are defined, then the third is defined too.)
The corresponding diagram arises from the former diagram by introducing the groups $\overline{G}_\sigma, \overline{G}^\sigma$ (where $\overline{G} := G/H$) and similarly for τ. The group $\overline{G}_\sigma$ conists of the residue classes of those elements $a \in G$ for which $a^\sigma \in H$. The map σ defines an isomorphism $\overline{G}_\sigma / HG_\sigma \approx G^\sigma \cap H/H^\sigma$, and similarly for τ. We have indicated these isomorphisms by $\backslash\backslash$ and $//$. The diagram has to be read similarly as the former diagram. Now there are 6 triangles along which one has to move counterclockwise, multiplying the corresponding group indices or their inverses respectively. (These indices are finite by assumption.) The double lines correspond to the terms appearing in one of the Herbrand quotients $q(G), q(H), q(\overline{G})$. The other terms cancel according to the various isomorphisms shown in the diagram. We obtain

$$(G_\sigma : G^\tau)(H_\sigma : H^\tau)^{-1}(\overline{G}_\sigma : \overline{G}^\tau)^{-1}(\overline{G}_\tau : \overline{G}^\sigma)(G_\tau : G^\sigma)^{-1}(H_\tau : H^\sigma) = 1 \qquad \text{which is the assertion.}$$