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# On the Finiteness Theorem of Siegel and Mahler Concerning Diophantine Equations

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In this paper we present a new proof, involving so-called nonstandard arguments, of Siegel's classical theorem on diophantine equations: Any irreducible algebraic equation $f(x, y) = 0$ of genus $g > 0$ admits only finitely many integral solutions. We also include Mahler's generalization of this theorem, namely the following: Instead of solutions in integers, we are considering solutions in rationals, but with the provision that their denominators should be divisible only by such primes which belong to a given finite set. Then again, the above equation admits only finitely many such solutions. From general nonstandard theory, we need the definition and the existence of enlargements of an algebraic number field. The idea of proof is to compare the natural arithmetic in such an enlargement, with the functional arithmetic in the function field defined by the above equation.

## 1. INTRODUCTION

We work over a given algebraic number field $K$ of finite degree. We consider a plane algebraic curve $\Gamma$, defined by an irreducible algebraic equation

$$f(x, y) = 0,$$

whose coefficients are contained in $K$. In his classical paper [22], Siegel has proved the following theorem.

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*If $\Gamma$ has genus $g > 0$, then there are only finitely many points $(x, y)$ on $\Gamma$, whose coordinates are algebraic integers in $K$.*

Mahler [11] has extended this result by considering also those points $(x, y)$ in $\Gamma$ whose coordinates admit finitely many prime divisors in their denominators. More precisely, let

$$\mathfrak{S} = \{\mathfrak{p}_1, ..., \mathfrak{p}_s\}$$

be a finite set of prime divisors of the field $K$, and let us consider those algebraic numbers in $K$ whose denominators are divisible by primes from $\mathfrak{S}$ only. For brevity, these numbers will be called the *quasi-integers* in $K$, with respect to $\mathfrak{S}$ as the set of admissible denominatorial prime divisors. It is clear that every integer in the ordinary sense is also a quasi-integer. Now, the above theorem remains true if, in this theorem, the notion of integer is replaced by the notion of quasiinteger, with respect to a given finite set $\mathfrak{S}$ of prime divisors of $K$.

We shall refer to this statement as the *Siegel–Mahler Theorem*. Mahler [11] gave a proof for curves of genus $g = 1$, and he conjectured that the theorem is still true for curves of higher genus. This conjecture was verified by Lang [7] and, again, by LeVeque [9].

In the present paper we shall give a nonstandard proof of the Siegel–Mahler Theorem, i.e., a proof which uses the methods of nonstandard analysis or, as we should rather say in our case, of *nonstandard arithmetic*. Our main purpose is to exhibit the usefulness of these nonstandard methods in dealing with problems on diophantine equations.

From general nonstandard theory we shall use the *existence* of enlargements of our field $K$, for a higher order language [14]. For the convenience of the reader we shall explain the definition and the basic properties of such enlargements in Section 2. We shall try to keep these explanations self-contained, so that this paper can be followed also by those readers who are not yet acquainted with nonstandard arguments. Such a reader might perhaps prefer to look through Section 2 before proceeding further in this section, where we are now going to state a nonstandard version of the Siegel–Mahler Theorem.

$^*K$ denotes an enlargement of the field $K$, fixed throughout the following discussion. $^*K$ is a certain field extension of $K$ whose basic properties are explained in Sections 2 and 3; in particular, let us note that $K$ is algebraically closed in $^*K$. The elements of $K$ are called standard, while the elements of $^*K$ not in $K$ are called nonstandard. A point $(x, y) \in {}^*K \times {}^*K$ is nonstandard if not both its coordinates $x$ and $y$ are standard.

Now let us assume that, contrary to the assertion of the Siegel–Mahler Theorem, there are infinitely many points on $\Gamma$ whose coordinates are quasi-integers in $K$, with respect to a given finite set $\mathfrak{S}$ of prime divisors of $K$. Then, according to the enlargement principle (see Section 2), there exists a nonstandard point $(x, y)$ on $\Gamma$ with the same property; i.e., the coordinates $x$ and $y$ are quasi-integers in $^*K$ with respect to the same finite set $\mathfrak{S}$ of prime divisors. By construction, $\mathfrak{S}$ consists solely of standard prime divisors; hence, we conclude that *the denominators of $x$ and $y$ are not divisible by any nonstandard prime divisor of $^*K$.*

As we have noted above, the field $K$ is algebraically closed in $^*K$. Hence, since the point $(x, y)$ is nonstandard, at least one of its coordinates is transcendental over $K$. We conclude that $(x, y)$ is a *generic* point over $K$ of the curve $\Gamma$. Therefore, the field $F = K(x, y)$ is $K$-isomorphic to the field of $K$-rational functions on $\Gamma$. We may identify $F$ with this function field; thus, the inclusion $F \subset {}^*K$ might be interpreted *as a representation of the functions in $F$ by means of (nonstandard) algebraic numbers.* The genus of $F$ in the sense of algebraic function fields equals the genus $g$ of the curve $\Gamma$.

By construction, $F$ is generated by the two functions $x$ and $y$ which, when regarded as algebraic numbers in $^*K$, do not admit any nonstandard prime divisor in their denominators. *We assert that this is impossible if $g > 0$*; if this assertion is proved, then we obtain a contradiction to our assumption about infinitely many quasi-integral points on $\Gamma$, and the theorem of Siegel–Mahler follows.

Thus we are faced with proving the following theorem, referring to a given algebraic number field $K$ and its enlargement $^*K$.

THEOREM 1.1. *Let $F$ be an algebraic function field of one variable over $K$, and assume that $F$ is embedded into $^*K$, so that $K \subset F \subset {}^*K$.*

*If $F$ has genus $g > 0$, then every nonconstant function $x \in F$ admits at least one nonstandard prime divisor of $^*K$ in its denominator.*

We have shown above that the Siegel–Mahler Theorem is an immediate consequence of Theorem 1.1. It would be equally easy to show that, conversely, our Theorem 1.1 follows from the Siegel–Mahler Theorem. Thus 1.1 can be regarded as the nonstandard equivalent of the Siegel–Mahler Theorem.

There is another version of Theorem 1.1 which refers to the prime divisors of the function field $F \mid K$, instead of to the nonconstant functions. Here, the notion of prime divisor of $F \mid K$ is to be understood in the usual sense; these prime divisors correspond to the nontrivial valuations of $F$ over $K$. We shall refer to these prime divisors of $F \mid K$ as the *functional* *prime divisors*, in contrast to the *arithmetical prime divisors* of $^*K$ which belong to the nontrivial internal valuations of $^*K$.

Now every *nonstandard* arithmetical prime divisor $\mathfrak{p}$ of $^*K$ is trivial on $K$ (see Section 3). Hence, $\mathfrak{p}$ induces in $F$ a valuation which, if it is not entirely trivial on $F$, belongs to some functional prime divisor $P$ of $F \mid K$. We say that $P$ is *induced* by $\mathfrak{p}$, and we write $\mathfrak{p} \mid P$. There arises the question whether every functional prime divisor of $F$ is induced by some nonstandard arithmetical prime divisor of $^*K$.

If this is the case, then the assertion of Theorem 1.1 follows immediately. Namely, according to the general theory of algebraic function fields, every nonconstant function $x \in F$ has at least one pole, say $P$, which is a functional prime divisor in the above sense. If we know that $P$ is induced by a nonstandard arithmetical prime $\mathfrak{p}$ of $^*K$, then we have found a $\mathfrak{p}$ which induces a pole of $x$ and hence divides the denominator of $x$.

Conversely, it follows from Theorem 1.1 that every functional prime divisor $P$ of $F \mid K$ is induced by some nonstandard arithmetical prime divisor of $^*K$. This is because, by the Riemann-Roch Theorem, there exists a function $x \in F$ admitting $P$ as its *only* pole. Applying Theorem 1.1 to this function, we find a nonstandard prime divisor $\mathfrak{p}$ of $^*K$ which divides the denominator of $x$ and hence induces in $F$ a pole of $x$. Since $P$ is the only pole of $x$, we see that $\mathfrak{p} \mid P$.

The foregoing arguments show that the following theorem is an equivalent version of Theorem 1.1.

THEOREM 1.2. *In the same situation as in Theorem* 1.1, *we assume again that* $g > 0$. *Then every functional prime divisor* $P$ *of* $F \mid K$ *is induced by some nonstandard, arithmetical prime divisor* $\mathfrak{p}$ *of* $^*K$.

If the hypothesis $g > 0$ is not satisfied, then Theorems 1.1 and 1.2 need not be true. For instance, consider the rational function field $F = K(x)$, where $x$ is some nonstandard integer in $^*K$. Then $g = 0$. The denominator of $x$ in $^*K$ does not contain any (nonarchimedean) prime divisor at all, standard or nonstandard; hence, Theorem 1.1 is not true for this function $x \in F$. Theorem 1.2 is not true for the pole of $x$ in $F$.

However, our methods of proof will also yield some information in the case $g = 0$. We shall show that in this case there are *at most two* functional prime divisors of $F \mid K$ which are exceptional in the sense of Theorem 1.2, i.e., which are not induced by any nonstandard arithmetical prime. This will yield a parametrization of those functions $x \in F$ which do not have any nonstandard arithmetical prime in their denominators. These results can be regarded as the nonstandard version of Siegel's parametrization of curves of genus 0 which admit infinitely many integral points [22]. For details, we refer to Section 8 below.

## 2. General Remarks on Enlargements

Nonstandard methods are based on the fact that every mathematical structure $M$ admits what is called an *enlargement*. Such an enlargement $^*M$ is an extension of $M$ such that the following properties hold, which we shall state as "principles."

The first of these principles expresses the fact that $^*M$ is a *model* of $M$.

2.1. Principle of Permanence. *Every mathematical statement about $M$ has an interpretation in $^*M$, and this interpretation is true in $^*M$ if and only if the original statement is true in $M$.*

Let us explain this in more detail. Mathematical statements about $M$ are envisaged as being expressed in a formal language of higher order predicate calculus over $M$. This language contains names for all individuals[1] in $M$, as well as for all entities of higher type in $M$ (e.g., sets of individuals, relations between individuals, relations between sets, etc.). Starting from these and from a sufficient supply of variables, every sentence in this language is built up in finitely many steps, according to the well-known rules of predicate calculus, with the help of the logical connectives and quantifiers. Quantification is permitted not only with respect to individuals but also with respect to entities of any given type.

In most cases, of course, mathematical statements about $M$ are *not* explicitly expressed in this formal language. Instead, one usually prefers to use the more informal common language to describe mathematical statements, provided it is clear that a translation into the formal language exists. For instance, the Siegel-Mahler Theorem in Section 1 can be regarded as a mathematical statement about $K$ in the above sense.

The basic property of nonstandard models, as given above, refers to the concept of "interpretation" of mathematical statements about $M$, in the enlargement $^*M$. This interpretation is meant in a special way, namely according to the following provisions.

(i) The interpretation of the logical connectives ("and," "or," "not," "implies") is the usual one.

(ii) The name of any individual in $M$ is the same in $^*M$, and quantification with respect to individuals ("*there exists* a number," "*for all* numbers") has its usual meaning in $^*M$.

(iii) The names of other entities in $M$ (sets, relations, relations between sets, etc.) also denote corresponding entities in $^*M$ which are then called

[1] The individuals of a mathematical structure are the elements of its underlying set. If $M$ is an algebraic number field, then the individuals of $M$ are the algebraic numbers of that field.

*standard* entities. However, quantification with respect to these entities ("*there exists* a relation," "*for all* sets," etc.) does not refer to the class of all entities of that type in $^*M$ but to a certain subclass of them whose members are called *internal* entities (sets, relations, etc.). Among these are the standard entities.[2]

As an example, let us consider the case $M = \mathbf{N}$, the natural numbers, and Peano's principle of mathematical induction which implies that every nonempty, bounded subset of **N** has a maximal element. This statement contains a quantifier with respect to sets. According to (iii), its interpretation in $^*\mathbf{N}$ refers to *internal* sets only. Hence, every nonempty bounded internal subset of $^*\mathbf{N}$ contains a maximal element; this is a true statement in $^*\mathbf{N}$ since it is the interpretation of a true statement in **N**. The reader should note that there are nonempty, bounded subsets of $^*\mathbf{N}$, necessarily external, which do *not* contain a maximal element; for instance, **N** is bounded in $^*\mathbf{N}$, every *nonstandard* number $x \in {}^*\mathbf{N}$ being a bound of **N**. (This last argument uses the fact that $^*\mathbf{N}$ is a proper extension of **N**, i.e., that there are nonstandard numbers in $^*\mathbf{N}$ (see 2.4 below).)

It should be observed that the notions of "standard" and "internal" entities belong to the definition of enlargement (in much the same way as the notion of "open" sets belong to the definition of topological space). More precisely, an enlargement $^*M$ of $M$ is defined to be a higher order structure, extending $M$, in which certain entities are distinguished as being "standard" or "internal" respectively, and such that the basic principles 2.1–2.5 hold.

The next principle is concerned with *relations* in the structure $M$. Let $R$ denote such a relation, say an $n$-ary relation between individuals. According to (iii), $R$ also denotes a certain standard $n$-ary relation in the enlargement $^*M$. Let $a_1, \ldots, a_n$ be $n$ individuals of $M$, and consider the statement that $R(a_1, \ldots, a_n)$ holds. According to the principle of permanence, this statement is true in $^*M$ if and only if it is true in $M$. In other words, the new relation $R$ in $^*M$ is an extension of the original relation $R$ in $M$. Thus we obtain the following principle which, as we have seen, is actually a consequence of 2.1 but which we prefer to state separately as a convenient reference.

2.2. Extension Principle. *Any relation in $M$ extends naturally and uniquely to a standard relation of the same type in $^*M$. The extended relation in $^*M$ is usually denoted by the same symbol as the original relation*

[2] The terminology of "internal" and "standard" is also applied to individuals. Namely, every individual in $^*M$ is internal; the individuals in $M$ and only those are standard. With this terminology, (ii) can be regarded as a special case of (iii).

*in $M$. Every property of the original relation which is expressible in the language of $M$ does also hold for the extended standard relation, provided it is interpreted in $^*M$ as explained above.*

This principle holds not only for relations between individuals but also for relations between entities of higher type. We have used it already in Section 1 in the case $M = K$, an algebraic number field, where we have said that *$^*K$ is a field extension of $K$.* In order to see this, consider the two ternary relations which represent addition ($a + b = c$) and multiplication ($a \cdot b = c$) in the field $K$. These relations extend to certain standard ternary relations in $^*K$, also denoted in the same way as addition and multiplication. Now, the original relations in $K$ satisfy all the axioms which make $K$ into a field. Therefore, these field axioms are also satisfied by their standard extensions. This shows that, indeed, $^*K$ is a field extension of $K$. As to the nature of this extension, we have already noted in Section 1 that *$K$ is algebraically closed in $^*K$.* This is an immediate consequence of the principle of permanence. Namely, let $g(X) \in K[X]$ be any polynomial; we have to show that $g(X)$ has a root in $^*K$ if and only if it has a root in $K$. In fact, the statement that $g(X)$ has a root belongs to the language of $K$ and hence it is true in $K$ if and only if it is true in $^*K$.

We observe that principles 2.1 and 2.2 do not imply that $^*M$ is a proper extension of $M$; they are trivially valid in $M$ instead of $^*M$. In contrast, the next principle asserts that $^*M$ is sufficiently large such as to contain certain nonstandard entities; this explains its name of "enlargement."

We consider binary relations in $M$. Let $R$ denote such a binary relation, say a relation between individuals. Let $a$ be an individual in $M$. If there exists $b$ in $M$ such that $R(a, b)$ holds, then $a$ is said to be in the *left domain* of $R$. The relation $R$ is said to be *concurrent* in $M$ if the following holds: Given finitely many elements $a_1, \ldots, a_n$ in the left domain of $R$, then there exists $b$ in $M$ such that the relations $R(a_i, b)$ hold simultaneously for $1 \leqslant i \leqslant n$.

2.3 ENLARGEMENT PRINCIPLE FOR BINARY RELATIONS. *If the relation $R$ is concurrent in $M$, then there exists $x$ in $^*M$ such that $R(a, x)$ holds simultaneously for all $a$ in $M$ which are contained in the left domain of $R$.*

In other words, the (possibly infinite) system of conditions $R(a, x)$ can be solved in the enlargement $^*M$, provided any finite subsystem can be solved in $M$ already. Of course, in writing $R(a, x)$ we have to interpret $R$ as a binary relation in $^*M$ according to 2.2.

If $M$ is *infinite*, then the enlargement principle guarantees the existence of *nonstandard* individuals in $M$. For, consider the relation of inequality $a \neq b$ between individuals of $M$, the left and right domain of this relation being $M$ itself. Since $M$ is infinite, this relation is concurrent and we conclude from 2.3 the existence of $x \in {}^*M$ which is different from every $a \in M$; i.e., $x$ is nonstandard.

More generally, let $S$ be any set in $M$, say a set of individuals. According to (iii), $S$ determines a certain standard set ${}^*S$ in ${}^*M$ whose characteristic properties are identical with the characteristic properties of $S$ in $M$, if these are interpreted in ${}^*M$. From this it is clear that ${}^*S$ is an *extension* of $S$; more precisely, $S$ consists exactly of the standard elements which are contained in ${}^*S$. There arises the question whether ${}^*S$ is a *proper* extension of $S$, i.e., whether ${}^*S$ contains nonstandard elements. If $S$ is finite, say with $n$ elements, then ${}^*S$ coincides with $S$. This is because the statement that $S$ consists of $n$ elements belongs to the language of $M$ and hence remains true in its interpretation in ${}^*M$, which says that ${}^*S$ consists of $n$ elements. On the other hand, if $S$ is infinite, then we may consider the relation of inequality $a \neq b$, *restricted to the set* $S$; similarly as above, we deduce from 2.3 that there exists a nonstandard element in ${}^*S$. Thus we have the following.

2.4. Enlargement Principle for Sets. *Every set $S$ in $M$ determines naturally and uniquely a certain standard set ${}^*S$ in ${}^*M$; if $S$ is described by a sentence in the language of $M$, then this same sentence, if interpreted in ${}^*M$, yields a description of ${}^*S$. The original set $S$ consists precisely of the standard elements which belong to ${}^*S$. If $S$ is infinite, and only in this case, does ${}^*S$ contain nonstandard elements; i.e., ${}^*S$ is then a proper extension of $S$.*

We have used 2.4 already in Section 1 in order to deduce the existence of certain nonstandard points on our curve $\Gamma$.

Enlargement principles 2.3 and 2.4 are valid not only for relations and sets of individuals but also for relations and sets of entities of higher type. Perhaps a further comment is necessary to explain the situation of $S$ with respect to ${}^*S$, if $S$ is a set of entities of higher type (e.g., a set of relations or functions, etc.) In this case, the elements $a \in S$ are entities of higher type in $M$, and *a priori* they are *not* contained in the structure ${}^*M$. However, every entity extends naturally and uniquely to a standard entity of ${}^*S$; if we assign to every $a \in S$ its corresponding standard entity, then we obtain an *injection* from $S$ into ${}^*S$. It is with respect to this injection that ${}^*S$ is to be regarded as an extension of $S$, in the context of 2.4. Often it will be convenient, during the investigation of a given set $S$, to identify the elements of $S$ with their corresponding standard entities, so that $S$ becomes a *subset* of ${}^*S$, viz., the subset of standard entities in ${}^*S$. For instance, we shall not distinguish between the prime divisors of the number field $K$ and their corresponding standard extensions to ${}^*K$.

It should be noted that every infinite set $S$ in $M$, if regarded as a subset of $^*S$ in the way as explained above, is necessarily *external.* To see this, consider a denumerable infinite subset of $S$, which may be identified with the set N of natural numbers. There exists a *surjective* map $\varphi: S \to \mathbf{N}$. In view of extension principle 2.2, $\varphi$ extends uniquely to a standard map $\varphi : {}^*S \to {}^*\mathbf{N}$. As a standard map, $\varphi : {}^*S \to {}^*\mathbf{N}$ is *a fortiori* internal and hence maps internal subsets of $^*S$ onto internal subsets of $^*\mathbf{N}$. Thus, if $S$ were internal, then $\varphi(S) = \mathbf{N}$ would be internal too. But we have already remarked above that N is an external subset of $^*\mathbf{N}$, as a consequence of Peano's principle of mathematical induction. Thus we have the following.

2.5. PRINCIPLE OF EXTERNITY. *Every infinite set S which consists of only standard elements is necessarily external in $^*M$. In other words, every infinite internal set in $^*M$ contains a nonstandard element.*

The enlargements $^*M$ of a given structure $M$ are not unique. There are many ways of realizing enlargements, e.g., by ultrapower methods. Such explicit methods might perhaps yield a more graphic illustration of the notions of "standard" and "internal" entities, but it does not make any difference in our arguments. This is why we prefer to define enlargements by their relevant properties 2.1–2.5 only, without reference to their possible modes of construction.

Assume that, for a given higher order structure $M$, we have chosen a definite enlargement $^*M$. Let $S$ be a set in $M$, either a set of individuals or a set of entities of higher type. We know that $S$ determines naturally and uniquely a certain standard set $^*S$ in $^*M$, consisting of internal entities of the same type as the entities in $S$. Now, let us regard $S$ not only as a set, but also as a higher order substructure of $M$ and, similarly, $^*S$ as a substructure of $^*M$ (with the notions of "internal" and "standard" as those which are induced by $^*M$). Then it is easy to see that $^*S$ is an enlargement of $S$. Thus, *our fixed enlargement $^*M$ of $M$ contains naturally an enlargement $^*S$ for every substructure $S$ of $M$.* In other words, we have a well-defined functor

$$S \mapsto {}^*S$$

from substructures of $M$ to substructures of $^*M$ such that $^*S$ is an enlargement of $S$. By the basic property 2.1 of $^*M$, this functor is faithful, not only with respect to the inclusion relation $S \subset S'$ between substructures, but also with respect to all those relations between substructures which can be expressed in the language of $M$.

In view of this it will be convenient, during the discussions of this paper, to adopt the following viewpoint. We work in a fixed universe $M$, in the sense that $M$ as a higher order structure contains all mathematical structures which are of interest in number theory, at least in the usual treatments and in any case in this paper. More precisely, $M$ should contain all algebraic number fields and their completions with respect to their various valuations. We choose a fixed enlargement $^*M$ of $M$. This being settled, we regard every structure $S$ occurring in our arguments as a substructure of $M$ and, therefore, using the remark above, its enlargement $^*S$ is *uniquely defined* as a substructure of $^*M$. In this way, we have eliminated all ambiguity concerning the choice of enlargements, and the star symbol will have a well-defined meaning throughout our discussion.

As to the possible choice of the universe $M$, we may take $M = \mathbf{N}$; more precisely, $M$ is to be the full higher order structure based on the set $\mathbf{N}$ of natural numbers. The substructures of this universe are precisely those which can be described in the language of the natural numbers. Thus, $M$ contains the integers $\mathbf{Z}$ (which can be described as pairs of natural numbers—more precisely, as certain equivalence classes of such pairs). It follows that $M$ contains the rational numbers $\mathbf{Q}$ (pairs of integers) and the real numbers $\mathbf{R}$ as well as the $p$-adic numbers $\mathbf{Q}_p$ (sequences of rationals). If $K$ is our algebraic number field of finite degree and if $u_1, \ldots, u_n$ is a basis of $K$, then we can describe the elements of $K$ by means of their coordinates with respect to this basis, i.e., as $n$-tuples of rational numbers. Thus $K$ is also contained in $M$.

The above explanations should give the reader a general idea of the underlying theory and the nature of our arguments pertaining to nonstandard arithmetic. He may consult [14] for a more systematic treatment of this theory, including an existence proof for enlargements (based on the compactness theorem of algebraic logic). We also refer the reader to Section 3, where he will find an opportunity for exercises in applying the general principles of this section, which will perhaps help him understand nonstandard methods.

## 3. Prime Divisors and Divisors in the Enlargement of an Algebraic Number Field

As before, $K$ denotes an algebraic number field of finite degree. $^*K$ is its enlargement as explained in Section 2. In this section we are going to discuss the arithmetic properties of $^*K$, as far as they are relevant for our purpose. Also, we want to fix our notations which we are going to use in this paper.

The arithmetic structure of the field $K$ can be described by means of its *prime divisors*, which are defined in terms of valuations. More precisely, a prime divisor $\mathfrak{p}$ of $K$ is defined to be a class of nontrivial valuations of $K$, with respect to the ordinary equivalence relation for valuations. There are two types of prime divisors of $K$: namely, the archimedean primes, which correspond to the archimedean valuations, and the non-archimedean primes, which correspond to the nonarchimedean valuations.

Let $\mathfrak{p}$ be a nonarchimedean prime divisor of $K$. Among the valuations belonging to $\mathfrak{p}$ there is exactly one which is normalized such that its value group is $\mathbf{Z}$, the additive group of integers. This valuation is denoted by $v_\mathfrak{p}$ and is called the $\mathfrak{p}$-*adic ordinal function* of $K$. In addition to this ordinal function, we also consider the normalized *absolute value*, defined by the formula

$$\| x \|_\mathfrak{p} = N\mathfrak{p}^{-v_\mathfrak{p}(x)}; \tag{3.1}$$

this is a multiplicative valuation belonging to $\mathfrak{p}$. As usual, $N\mathfrak{p}$ denotes the *norm* of $\mathfrak{p}$, i.e., the number of elements in the $\mathfrak{p}$-adic residue field.

Now let $\mathfrak{p}$ be an archimedean prime divisor of $K$. Among all the valuations belonging to $\mathfrak{p}$ there is exactly one which induces in $\mathbf{Q}$ the ordinary absolute value; this valuation is denoted by $| x |_\mathfrak{p}$. We also consider the normalized absolute value, defined by the formula

$$\| x \|_\mathfrak{p} = \begin{cases} | x |_\mathfrak{p} & \text{if } \mathfrak{p} \text{ is real,} \\ | x |_\mathfrak{p}^2 & \text{if } \mathfrak{p} \text{ is complex.} \end{cases} \tag{3.2}$$

As usual, $\mathfrak{p}$ is called real or complex according to whether the $\mathfrak{p}$-adic completion of $K$ is isomorphic to the field $\mathbf{R}$ of real numbers or to the field $\mathbf{C}$ of complex numbers.

The absolute values of $K$ are normalized in such a way that the following product formula holds for every nonzero element $x \in K$:

$$\prod_\mathfrak{p} \| x \|_\mathfrak{p} = 1. \tag{3.3}$$

Here, $\mathfrak{p}$ ranges over all prime divisors of $K$, archimedean or nonarchimedean. The product in (3.3) is essentially a finite product. For, given $0 \neq x \in K$, there are only finitely many prime divisors $\mathfrak{p}$ for which $\| x \|_\mathfrak{p} \neq 1$.

Sometimes in the literature, the term of prime divisor is restricted to denote nonarchimedean primes only. We shall not follow this terminology since we have to take the archimedean primes also into consideration, and it seems more natural to put them on equal footing with the nonarchimedean ones. This is a well-established procedure in number theory. In order to obtain unified formulas, we extend the above notations to include the case when $\mathfrak{p}$ is archimedean:

$$v_{\mathfrak{p}}(x) = -\log \| x \|_{\mathfrak{p}},$$

$$N\mathfrak{p} = e \text{ (base of the logarithm).}$$

With these notations, (3.1) holds also for archimedean primes.

Now, let $V$ be the set of all prime divisors of $K$. According to the general principles explained in Section 2, its enlargement $*V$ is to be interpreted as the set of all internal prime divisors of $*K$. Here, the notion of internal prime divisor is to be defined as an equivalence class of nontrivial internal valuations of $*K$. As we know, every true statement in $K$ concerning prime divisors yields a true statement in $*K$ which concerns internal prime divisors by means of its interpretation. Therefore, the following statements hold.

There are two types of internal prime divisors of $*K$: archimedean and nonarchimedean. Let $\mathfrak{p}$ be a nonarchimedean internal prime divisor. Among all the (internal) valuations of $*K$ which belong to $\mathfrak{p}$ there is exactly one which is normalized such that its value group is $*\mathbf{Z}$, the additive group of standard or nonstandard integers. This valuation is denoted by $v_{\mathfrak{p}}$ and is called the $\mathfrak{p}$-*adic ordinal function* of $*K$. Thus, for $0 \neq x \in *K$, its $\mathfrak{p}$-adic ordinal $v_{\mathfrak{p}}(x)$ is a standard or nonstandard integer. $v_{\mathfrak{p}}$ is an additive valuation of $*K$ in the sense of Krull, with the extra condition that $v_{\mathfrak{p}}$ is *internal,* a notion which is inherent in the structure of enlargement. The $\mathfrak{p}$-adic residue field of $*K$ is not necessarily finite; however, it is *starfinite* in the following sense. There is an element $N \in *\mathbf{N}$ and an internal bijection from the residue field onto the initial interval $1 \leqslant \nu \leqslant N$ in $*\mathbf{N}$. It is clear that this notion of "starfinite" is the interpretation of the ordinary notion of "finite." The above number $N \in *\mathbf{N}$ is uniquely determined and is called the *norm* of $\mathfrak{p}$: notation $N\mathfrak{p}$. In addition to the ordinal function, we also consider the normalized $\mathfrak{p}$-*adic absolute value,* defined by formula (3.1). This formula is now interpreted in $*K$. That is, $x$ is an element in $*K$ and $\| x \|_{\mathfrak{p}}$ is a nonnegative element in $*\mathbf{Q}$.

Now let $\mathfrak{p}$ be an archimedean internal prime divisor of $*K$. Among all the valuations belonging to $\mathfrak{p}$ there is exactly one which induces in $*\mathbf{Q}$ the ordinary standard absolute value. This valuation is denoted by $| x |_{\mathfrak{p}}$. We also consider the normalized absolute value $\| x \|_{\mathfrak{p}}$, defined by formula (3.2), which is now to be interpreted in $*K$. The prime $\mathfrak{p}$ is called real or complex according to whether the $\mathfrak{p}$-*adic* completion of $*K$ is internally isomorphic to $*\mathbf{R}$ or to $*\mathbf{C}$.

Now, *product formula* (3.3) holds in $*K$, since it is the interpretation of a true formula in $K$. In this interpretation, $x$ denotes any nonzero element in $^*K$ and $\mathfrak{p}$ ranges over all internal prime divisors of $^*K$. The product in (3.3) is essentially starfinite, for, given $0 \neq x \in {}^*K$, the set of $\mathfrak{p} \in {}^*V$ with $\| x \|_\mathfrak{p} \neq 1$ is starfinite.

Perhaps it is useful to insert a few general remarks about starfinite products. Let $A$ be any internal abelian group, written multiplicatively, and let $a_i$ be an internal sequence in $A$ with starfinite support. That is, the index $i$ ranges over an internal index set $I$, the map $i \to a_i$ is an internal map from $I$ to $A$, and the set of those $i$ for which $a_i \neq 1$ is starfinite. Under these conditions, the product $\prod_{i \in I} a_i$ is well defined as an element in $A$. This definition is the interpretation of the obvious definition of finite products in the usual sense. Starfinite products satisfy all the rules which are satisfied by finite products, as long as these rules can be expressed in the language of $M$. To be sure, the number of "factors" in a starfinite product need not be finite, and thus it is not a product in the sense of ordinary algebra. Nevertheless, the starfinite product is a certain operator which is inherent in the structure of enlargement. The notation and the name "product" are justified since it satisfies the usual rules for finite products. (The situation is much the same as in ordinary analysis, where one considers infinite "products", these being not products in the algebraic sense but defined by limit operations.) It goes without saying that similar remarks apply to starfinite sums, etc., in abelian groups which are additively written.

Let us continue with the discussion of prime divisors. According to the general extension principle 2.2, every prime divisor $\mathfrak{p}$ of $K$ extends naturally and uniquely to a standard prime divisor of $^*K$. This standard extension is denoted with the same symbol $\mathfrak{p}$, and it enjoys the same properties as the original prime divisor of $K$, as long as these properties are expressed in the language of $K$. For example, both norms $N\mathfrak{p}$ coincide, regardless of whether we look at $\mathfrak{p}$ as a prime divisor of $K$ or as a standard prime divisor of $^*K$. Also, if $a \in K$, then its absolute value $\| a \|_\mathfrak{p}$ is same whether $\mathfrak{p}$ is regarded as a prime of $K$ or as a standard prime in $^*K$.[3]

Due to this notational convention, every prime $\mathfrak{p} \in V$ now appears also as a standard prime in $^*V$. That is, the set $V$ appears as a subset of $^*V$, viz., the set of standard primes. (Note that $V$ is an *external* subset of $^*V$.) Since there are infinitely many prime divisors of $K$, we know that $V$ is an infinite set and, hence, due to enlargement principle 2.4, we conclude that $^*V$ is a proper extension of $V$. In other words, *there are nonstandard prime divisors of* $^*K$. The following lemma gives some of their fundamental properties.

[3] If $\mathfrak{p}$ is a prime divisor of $K$, then it can be shown that its standard extension is a $\mathfrak{p}$-extension in the sense of [19].

LEMMA 3.1. *Every nonstandard prime divisor* $\mathfrak{p}$ *of* $^*K$ *is trivial on* $K$. *In particular,* $\mathfrak{p}$ *is nonarchimedean. The norm* $N\mathfrak{p}$ *is infinitely large. If* $x \in {}^*K$ *is such that* $\| x \|_{\mathfrak{p}} > 1$, *then* $\| x \|_{\mathfrak{p}}$ *is infinitely large.*

An element in $^*\mathbf{R}$ is called infinitely large if it is greater than any standard real number.

*Proof.* Let $a$ be a nonzero element in $K$. The set $S$ of those $\mathfrak{q} \in V$ for which $\| a \|_{\mathfrak{q}} \neq 1$ is finite. Hence, this set is not enlarged in $^*V$, due to enlargement principle 2.4. That is, if $\mathfrak{q}$ is any internal prime divisor in $^*V$ for which $\| a \|_{\mathfrak{q}} \neq 1$, then $\mathfrak{q} \in S$. In particular, since $S \subset V$, we see that every such $\mathfrak{q}$ is standard.

Now, since $\mathfrak{p}$ is assumed to be nonstandard, it follows that $\| a \|_{\mathfrak{p}} = 1$. This being true for any $0 \neq a \in K$, we see that $\mathfrak{p}$ is trivial on $K$. As archimedean valuations are nontrivial on $\mathbf{Q}$, hence nontrivial on $K$, it follows that $\mathfrak{p}$ is nonarchimedean.

Since $\mathfrak{p}$ is trivial on $K$, we see that $K$ is isomorphically contained in the $\mathfrak{p}$-adic residue field of $^*K$. In particular, the $\mathfrak{p}$-adic residue field of $^*K$ is infinite. On the other hand, we know from the above that this residue field is starfinite, and that there exists an internal bijection from it to the initial intervall $1 \leqslant \nu \leqslant N\mathfrak{p}$ in $^*\mathbf{N}$. We conclude that $N\mathfrak{p}$ is nonstandard and, in fact, infinitely large.

If $\| x \|_{\mathfrak{p}} > 1$, then $v_{\mathfrak{p}}(x) < 0$. Now, $v_{\mathfrak{p}}(x)$ is an element in $^*\mathbf{Z}$. Therefore, from $v_{\mathfrak{p}}(x) < 0$ we conclude $v_{\mathfrak{p}}(x) \leqslant -1$. (This conclusion is valid in $^*\mathbf{Z}$ since it is valid in $\mathbf{Z}$.) In view of (3.1) we obtain $\| x \|_{\mathfrak{p}} \geqslant N\mathfrak{p}$; hence $\| x \|_{\mathfrak{p}}$ is infinitely large too. Q.E.D.

Now let $c \geqslant 1$ be a standard real number. Consider those elements $x \in K$ which are contained in the "parallelotope," given by the conditions

$$\| x \|_{\mathfrak{p}} \leqslant c \tag{3.4}$$

for all $\mathfrak{p} \in V$. A well-known theorem of algebraic number theory says that the set of these $x \in K$ is *finite*. Hence, in view of 2.4, this set is not enlarged in $^*K$. Therefore, if an element $x \in {}^*K$ satisfies the conditions (3.4) for all $\mathfrak{p} \in {}^*V$, then $x$ is contained in $K$ already; i.e., $x$ is standard. In other words, if the element $x \in {}^*K$ is *nonstandard*, then the conditions (3.4) are not all satisfied; i.e., there exists at least one prime $\mathfrak{p} \in {}^*V$ such that

$$\| x \|_{\mathfrak{p}} > c.$$

This prime $\mathfrak{p} \in {}^*V$ might depend on the choice of $c \in \mathbf{R}$. However, we claim there is a prime $\mathfrak{p} \in {}^*V$ such that $\| x \|_{\mathfrak{p}} > c$ holds simultaneously for all standard $c \in \mathbf{R}$. To show this, we distinguish two cases: Let $\mathfrak{S}_x$ denote the set of those $\mathfrak{p} \in {}^*V$ for which $\| x \|_{\mathfrak{p}} > 1$.

*Case* 1. $\mathfrak{S}_x$ *is finite.* In this case, we put successively $c = 1, 2, 3,...$ and, by the above remark, find a sequence $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3,..., \in {}^*V$ such that $\| x \|_{\mathfrak{p}_n} > n$ for $n \in \mathbf{N}$. Every member $\mathfrak{p}_n$ of this sequence is contained in the finite set $\mathfrak{S}_x$. Hence, there is an infinite subsequence which is constant; i.e., there is $\mathfrak{p} \in \mathfrak{S}_x$ such that $\mathfrak{p} = \mathfrak{p}_n$ for infinitely many $n \in \mathbf{N}$. By construction, this prime $\mathfrak{p}$ satisfies $\| x \|_{\mathfrak{p}} > n$ for infinitely many (and hence all) standard natural numbers $n$. In other words, $\| x \|_{\mathfrak{p}}$ is infinitely large.

*Case* 2. $\mathfrak{S}_x$ *is infinite.* We remark that, according to its definition, $\mathfrak{S}_x$ is an *internal* set. We know from 2.5 that every infinite internal set contains a nonstandard element. Thus, there is a nonstandard prime $\mathfrak{p}$ which lies in $\mathfrak{S}_x$, i.e., for which $\| x \|_{\mathfrak{p}} > 1$. Lemma 3.1 now shows that $\| x \|_{\mathfrak{p}}$ is infinitely large.

We have proved the following.

LEMMA 3.2. *Let* $x \in {}^*K$ *be nonstandard. Then there exists a prime divisor* $\mathfrak{p} \in {}^*V$ *such that* $\| x \|_{\mathfrak{p}}$ *is infinitely large.*

We shall often prefer to work with the *logarithmic values*

$$w_{\mathfrak{p}}(x) = -\log \| x \|_{\mathfrak{p}} = v_{\mathfrak{p}}(x) \log(N\mathfrak{p}). \tag{3.5}$$

To say that $\| x \|_{\mathfrak{p}}$ is infinitely large is equivalent to saying that $w_{\mathfrak{p}}(x)$ is infinitely small, i.e., less than every (positive or negative) standard real number.

If $\mathfrak{p}$ is nonarchimedean, then $w_{\mathfrak{p}}$ is an additive valuation of the field ${}^*K$ in the sense of Krull; it differs from the normalized ordinal function $v_{\mathfrak{p}}$ by the factor $\log(N\mathfrak{p})$ only. Hence, in the nonarchimedean case, we have the following rules which express the properties of an additive valuation:

$$w_{\mathfrak{p}}(xy) = w_{\mathfrak{p}}(x) + w_{\mathfrak{p}}(y),$$
$$w_{\mathfrak{p}}(x + y) \geqslant \min(w_{\mathfrak{p}}(x), w_{\mathfrak{p}}(y)).$$

If $\mathfrak{p}$ is archimedean, then we still have the first of these rules, which expresses the fact that $w_{\mathfrak{p}} : {}^*K \to {}^*\mathbf{R}$ is a homomorphism of the multiplicative group of ${}^*K$ into the additive group of ${}^*\mathbf{R}$. The second rule has to be modified in the archimedean case, namely as follows.

Recall that $\| x \|_{\mathfrak{p}} = | x |_{\mathfrak{p}}$ if $\mathfrak{p}$ is real, and $\| x \|_{\mathfrak{p}} = | x |_{\mathfrak{p}}^2$ if $\mathfrak{p}$ is complex. Now, the ordinary absolute value $| x |_{\mathfrak{p}}$ satisfies

$$| x + y |_{\mathfrak{p}} \leqslant | x |_{\mathfrak{p}} + | y |_{\mathfrak{p}} \leqslant 2 \max(| x |_{\mathfrak{p}}, | y |_{\mathfrak{p}}).$$

If we square this relation, then the factor 2 is replaced by 4. Hence in any case, real or complex, we have

$$\| x + y \|_{\mathfrak{p}} \leqslant 4 \max(\| x \|_{\mathfrak{p}}, \| y \|_{\mathfrak{p}})$$

and therefore

$$w_{\mathfrak{p}}(x + y) \geqslant -\log(4) + \min(w_{\mathfrak{p}}(x), w_{\mathfrak{p}}(y)).$$

Compared with the corresponding rule in the nonarchimedean case, the additional term $-\log(4)$ appears here. This term, although not negligible, is nevertheless standard, hence finite, and therefore it vanishes if we consider the infinitary orders of magnitude only. Let us explain this in more detail.

A real number $a \in {}^*\mathbf{R}$ is called *finite* if there exists a positive standard number $c \in \mathbf{R}$ such that

$$-c \leqslant a \leqslant c.$$

In particular, every standard real number is finite. If the above inequalities hold for every standard $c > 0$, then $a$ is *infinitesimal.* Every finite number $a$ is infinitely close to a standard number ${}^\circ a$, which is to say that $a = {}^\circ a + h$ with infinitesimal $h$. Namely, ${}^\circ a$ is the standard real number which represents the Dedekind cut in $\mathbf{R}$ determinded by $a$.

The finite numbers form an additive subgroup of ${}^*\mathbf{R}$, which we denote by $\mathbf{R}_{\text{fin}}$.[4] If two numbers $a, b \in {}^*\mathbf{R}$ differ by a finite number only, then $a, b$ are said to be of the same order of magnitude; notation: $a \doteq b$.[5] This means that $a$ and $b$ determine the same residue class in the factor group

$$\dot{\mathbf{R}} = {}^*\mathbf{R}/\mathbf{R}_{\text{fin}}.$$

Note that $\dot{\mathbf{R}}$ carries naturally an order relation, which it inherits from ${}^*\mathbf{R}$ such that the natural projection ${}^*\mathbf{R} \to \dot{\mathbf{R}}$ is order preserving. If $a, b \in {}^*\mathbf{R}$, then we write

$$a \dot{\leqslant} b$$

in order to indicate that the order of magnitude of $a$ is less or equal to the order of magnitude of $b$. Explicitly, this means that there is a finite number $c \in \mathbf{R}_{\text{fin}}$ such that $b - a \geqslant c$. It is easily verified that this indeed defines an order relation in the factor group $\dot{\mathbf{R}}$; this is due to the fact that, by definition, $\mathbf{R}_{\text{fin}}$ is an *isolated* subgroup of ${}^*\mathbf{R}$. (That is, if $c, d \in \mathbf{R}_{\text{fin}}$, then $\mathbf{R}_{\text{fin}}$ contains every $u \in {}^*\mathbf{R}$ which lies between $c$ and $d$.)

[4] On other occasions [17], the group of finite numbers has been denoted by $\mathbf{R}_0$. However, we wish to reserve the index "0" for another purpose, namely for the group of divisors of size or degree 0 (see below). This is why we use the index "fin" to denote the group of finite elements, not only with respect to $\mathbf{R}$, but also with respect to other groups in due course.

[5] This is the notation which has been proposed by Hasse [5] in this context.

This being said, let us return to our above formula involving $-\log(4)$. Since this is a standard number, its order of magnitude vanishes. We conclude that for every prime $\mathfrak{p} \in {}^*V$ the following formulas hold:

$$w_\mathfrak{p}(xy) \doteq w_\mathfrak{p}(x) + w_\mathfrak{p}(y),$$
$$w_\mathfrak{p}(x + y) \dot{\geqslant} \min(w_\mathfrak{p}(x), w_\mathfrak{p}(y)).$$

These formulas say that the map

$$\dot{w}_\mathfrak{p} : {}^*K \to \dot{\mathbf{R}},$$

which is obtained from $w_\mathfrak{p} : {}^*K \to {}^*\mathbf{R}$ by applying the projection ${}^*\mathbf{R} \to \dot{\mathbf{R}}$, is a *valuation in the sense of Krull.* This valuation is trivial on $K$. Namely, if $0 \neq x \in K$, then $w_\mathfrak{p}(x) = -\log \| x \|_\mathfrak{p}$ is standard and hence finite; therefore, we have $w_\mathfrak{p}(x) \doteq 0$. (This holds also if $\mathfrak{p}$ is nonstandard, since then $w_\mathfrak{p}(x) = 0$ in view of Lemma 3.1.) Thus we have seen the following.

*Every prime divisor $\mathfrak{p} \in {}^*V$ defines naturally a Krull valuation $\dot{w}_\mathfrak{p}$ of ${}^*K$, which is trivial on $K$ and whose values are contained in the group $\dot{\mathbf{R}}$. By definition, $\dot{w}_\mathfrak{p}(x)$ is the order of magnitude of the logarithmic value $w_\mathfrak{p}(x) = -\log \| x \|_\mathfrak{p}$.*

If $\mathfrak{p}$ is standard (archimedean or not), then it can be shown that the value group of $\dot{w}_\mathfrak{p}$ is the full group $\dot{\mathbf{R}}$; its residue field is isomorphic to the $\mathfrak{p}$-adic completion of $K$. If $\mathfrak{p}$ is nonstandard, then the value group of $\dot{w}_\mathfrak{p}$ is a proper subgroup of $\dot{\mathbf{R}}$; the valuation $\dot{w}_\mathfrak{p}$ is in fact equivalent to the original valuation $w_\mathfrak{p}$, and both have isomorphic value groups and residue fields. Since we shall not make use of these facts explicitly, we leave the proofs to the reader.

If $x \in {}^*K$ is nonstandard, then by Lemma 3.2 there exists at least one prime $\mathfrak{p} \in {}^*V$ such that $w_\mathfrak{p}(x) \dot{<} 0$; this implies in particular that $\dot{w}_\mathfrak{p}$ does not vanish on $x$. Thus we see that $K$ is the exact field of constants with respect to the set of valuations $\dot{w}_\mathfrak{p}$. Therefore, these valuations may be used to build a divisor theory which describes ${}^*K$ relative to $K$ as its ground field. The situation is much the same as in the corresponding case with function fields, where there is also a field of constants. In the rest of this section, we shall develop this divisor theory of ${}^*K$.

First, let us discuss the ordinary notion of divisor in the algebraic number field $K$. This notion is defined as usual, with the provision, however, that the archimedean prime divisors should also be included. This leads to the following definition: The divisor group $\mathfrak{D}$ of $K$ is the direct sum

$$\mathfrak{D} = \mathfrak{D}' \oplus \mathfrak{D}'',$$

where $\mathfrak{D}'$ is the free **R**-module generated by the archimedean primes, and $\mathfrak{D}''$ the free **Z**-module generated by the non-archimedean primes. This definition implies that every divisor $\mathfrak{a} \in \mathfrak{D}$ has a unique representation in the form

$$\mathfrak{a} = \sum_{\mathfrak{p}} \alpha_{\mathfrak{p}} \cdot \mathfrak{p}, \tag{3.6}$$

where $\mathfrak{p}$ ranges over the primes in $V$, and the coefficients $\alpha_{\mathfrak{p}}$ satisfy the following conditions:

(i) $\alpha_{\mathfrak{p}} \in \mathbf{R}$ if $\mathfrak{p}$ is archimedean;
(ii) $\alpha_{\mathfrak{p}} \in \mathbf{Z}$ if $\mathfrak{p}$ is nonarchimedean;
(iii) $\alpha_{\mathfrak{p}} \neq 0$ for finitely many $\mathfrak{p} \in V$ only.

Thus, the divisor group $\mathfrak{D}$ can be represented as the group of all functions $\alpha : V \to \mathbf{R}$ satisfying these conditions (i)–(iii).

Every nonzero element $x \in K$ determines a divisor $(x) \in \mathfrak{D}$, namely its *principal* divisor, which is defined by

$$(x) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(x) \cdot \mathfrak{p}. \tag{3.7}$$

The map $x \mapsto (x)$ yields a homomorphism $K \to \mathfrak{D}$ from the multiplicative group of $K$ to the additive group $\mathfrak{D}$. Its kernel consists precisely of the roots of unity which are contained in $K$; in particular, this kernel is finite. Its cokernel $\mathfrak{C} = \mathfrak{D}/K$ is called the *divisor class group* of $K$, whose structure we shall discuss later.

Now let us interpret the above notions in the enlargement. $^*\mathfrak{D}$ is the group of all internal divisors. The internal prime divisors $\mathfrak{p} \in {}^*V$ are contained in $^*\mathfrak{D}$, and every $\mathfrak{a} \in {}^*\mathfrak{D}$ admits a unique representation in form (3.6), $\mathfrak{p}$ ranging over the primes in $^*V$. The coefficients $\alpha_{\mathfrak{p}}$ in (3.6) satisfy the conditions (i)–(iii) which now have to be interpreted in the enlargement ; more precisely,

*(i) $\alpha_{\mathfrak{p}} \in {}^*\mathbf{R}$ if $\mathfrak{p}$ is archimedean;
*(ii) $\alpha_{\mathfrak{p}} \in {}^*\mathbf{Z}$ if $\mathfrak{p}$ is nonarchimedean;
*(iii) the set of those $\mathfrak{p} \in {}^*V$ for which $\alpha_{\mathfrak{p}} \neq 0$ is starfinite.

Moreover, $\alpha_{\mathfrak{p}}$ depends internally on $\mathfrak{p}$, which is to say that the function $\mathfrak{p} \mapsto \alpha_{\mathfrak{p}}$ from $^*V$ to $^*\mathbf{R}$ is internal. In other words, the group $^*\mathfrak{D}$ can be represented as the group of all internal functions $\alpha : {}^*V \to {}^*\mathbf{R}$ satisfying *(i)–*(iii).

Again we have a direct sum decomposition

$$^*\mathfrak{D} = {}^*\mathfrak{D}' \oplus {}^*\mathfrak{D}'',$$

where $^*\mathfrak{D}'$ is the archimedean component, and $^*\mathfrak{D}''$ the nonarchimedean component of $^*\mathfrak{D}$.

The principal divisor map $K \to \mathfrak{D}$ has a standard extension

$$^*K \to {}^*\mathfrak{D},$$

which is described by formula (3.7). Since the kernel of $K \to \mathfrak{D}$ is finite, it is not enlarged in $^*K$. That is, the kernel of $^*K \to {}^*\mathfrak{D}$ is finite, and it consists of the roots of unity in $K$. The corresponding cokernel

$$^*\mathfrak{C} = {}^*\mathfrak{D}/{}^*K$$

is the group of internal divisor classes.

As a matter of notation, the coefficients $\alpha_\mathfrak{p}$ in (3.6) will be denoted by $v_\mathfrak{p}(\mathfrak{a})$; they are called the $\mathfrak{p}$-adic ordinals of the divisor $\mathfrak{a}$. According to (3.7), this notation is coherent with the corresponding notation $v_\mathfrak{p}(x)$ for $x \in {}^*K$.

We also define the $\mathfrak{p}$-adic absolute value of a divisor $\mathfrak{a} \in {}^*\mathfrak{D}$ by the formula

$$\| \mathfrak{a} \|_\mathfrak{p} = N\mathfrak{p}^{-v_\mathfrak{p}(\mathfrak{a})},$$

in analogy to (3.1). Most often we shall work with its logarithm

$$w_\mathfrak{p}(\mathfrak{a}) = -\log \| \mathfrak{a} \|_\mathfrak{p} = v_\mathfrak{p}(\mathfrak{a}) \log(N\mathfrak{p}).$$

in analogy to (3.5). Recall that for archimedean $\mathfrak{p}$ we have defined $N\mathfrak{p}$ such that $\log(N\mathfrak{p}) = 1$, so that $w_\mathfrak{p}(\mathfrak{a}) = v_\mathfrak{p}(\mathfrak{a})$ in this case.

Every divisor $\mathfrak{a} \in {}^*\mathfrak{D}$ is uniquely determined by its logarithmic absolute values $w_\mathfrak{p}(\mathfrak{a})$, and the divisorial relations are reflected in corresponding relations between those values. For instance, the addition of divisors $\mathfrak{a} + \mathfrak{b}$ is given by

$$w_\mathfrak{p}(\mathfrak{a} + \mathfrak{b}) = w_\mathfrak{p}(\mathfrak{a}) + w_\mathfrak{p}(\mathfrak{b})$$

for all $\mathfrak{p} \in {}^*V$. The order relation $\mathfrak{a} \leqslant \mathfrak{b}$ is given by

$$w_\mathfrak{p}(\mathfrak{a}) \leqslant w_\mathfrak{p}(\mathfrak{b})$$

for all $\mathfrak{p}$. If $\mathfrak{a} \leqslant \mathfrak{b}$, then $\mathfrak{a}$ is said to *divide* $\mathfrak{b}$; this terminology takes its motivation from the arithmetic background. Accordingly, $\min(\mathfrak{a}, \mathfrak{b})$ is the greatest common divisor, and $\max(\mathfrak{a}, \mathfrak{b})$ the least common multiple of $\mathfrak{a}$ and $\mathfrak{b}$.

The number

$$\sigma(\mathfrak{a}) = \sum_\mathfrak{p} w_\mathfrak{p}(\mathfrak{a}) = \sum_\mathfrak{p} v_\mathfrak{p}(\mathfrak{a}) \log(N\mathfrak{p}) \tag{3.8}$$

is called the (additive) *size* of the divisor $\mathfrak{a}$. This formula is either to be read in $\mathfrak{D}$, in which case the size yields a homomorphism $\sigma : \mathfrak{D} \to \mathbf{R}$, or else we have to interpret the formula in $^*\mathfrak{D}$ (the sum being starfinite), in which case we obtain a homomorphism $\sigma$: $^*\mathfrak{D} \to {}^*\mathbf{R}$ which is the standard extension of the former. In any case, $\sigma$ is surjective. The kernel of $\sigma$ is denoted by $\mathfrak{D}_0$ resp. $^*\mathfrak{D}_0$. In view of the product formula (3.3), which can also be read as a sum formula

$$\sum_{\mathfrak{p}} w_{\mathfrak{p}}(x) = 0,$$

we see that principal divisors are contained in $\mathfrak{D}_0$ resp. $^*\mathfrak{D}_0$. Let $\mathfrak{C}_0 = \mathfrak{D}_0/K$ resp. $^*\mathfrak{C}_0 = {}^*\mathfrak{D}_0/{}^*K$ denote the corresponding divisor class groups.

We consider $\mathfrak{D}$ as a subgroup of $^*\mathfrak{D}$, viz., the subgroup of all *standard* divisors. A divisor $\mathfrak{a} \in {}^*\mathfrak{D}$ is called *finite* if there is a standard divisor $\mathfrak{c} > 0$ such that $-\mathfrak{c} \leqslant \mathfrak{a} \leqslant \mathfrak{c}$. In particular, standard divisors are finite. If the above inequalities hold for every standard $\mathfrak{c} > 0$, then $\mathfrak{a}$ is said to be *infinitesimal*; this implies that $w_{\mathfrak{p}}(\mathfrak{a}) = 0$ for nonarchimedean $\mathfrak{p}$, while $w_{\mathfrak{p}}(\mathfrak{a})$ is an infinitesimal real number for archimedean $\mathfrak{p}$. Every finite divisor $\mathfrak{a}$ is infinitely close to a standard divisor ${}^\circ\mathfrak{a}$, which is to say that $\mathfrak{a} = {}^\circ\mathfrak{a} + \mathfrak{y}$ with $\mathfrak{y}$ infinitesimal.

The finite divisors in $^*\mathfrak{D}$ form an isolated subgroup $\mathfrak{D}_{\mathrm{fin}}$ and the factor group

$$\dot{\mathfrak{D}} = {}^*\mathfrak{D}/\mathfrak{D}_{\mathrm{fin}}$$

is the *group of divisorial orders of magnitude.* As in the case with real numbers, we write $\mathfrak{a} \doteq \mathfrak{b}$ in order to indicate that $\mathfrak{a}$ and $\mathfrak{b}$ are of the same order of magnitude; this means that $\mathfrak{a}$ and $\mathfrak{b}$ determine the same residue class in $\dot{\mathfrak{D}}$. Also, $\mathfrak{a} \mathrel{\dot{\leqslant}} \mathfrak{b}$ means that there exists $\mathfrak{c} \in \mathfrak{D}_{\mathrm{fin}}$ such that $\mathfrak{b} - \mathfrak{a} \geqslant \mathfrak{c}$; this defines an order relation in $\dot{\mathfrak{D}}$ such that the natural projection $^*\mathfrak{D} \to \dot{\mathfrak{D}}$ is order preserving. Moreover, the operations max and min are preserved by this projection.

The group $\dot{\mathfrak{D}}$ will play a central role in our considerations. One may regard $\dot{\mathfrak{D}}$ as consisting of the same elements as $^*\mathfrak{D}$, namely internal divisors, but with the equality sign $=$ being replaced by the sign $\doteq$, which indicates the same order of magnitude. In this sense the following lemma is, in fact, a statement concerning $\dot{\mathfrak{D}}$.

LEMMA 3.3. *Let $\mathfrak{a}, \mathfrak{b}$ denote internal divisors in $^*\mathfrak{D}$. If $\mathfrak{a} \mathrel{\dot{\leqslant}} \mathfrak{b}$, then*

$$w_{\mathfrak{p}}(\mathfrak{a}) \mathrel{\dot{\leqslant}} w_{\mathfrak{p}}(\mathfrak{b}) \qquad \textit{for each } \mathfrak{p} \in {}^*V,$$

*and conversely. In particular, it follows that the relation $\mathfrak{a} \doteq \mathfrak{b}$ is equivalent to*

$$w_{\mathfrak{p}}(\mathfrak{a}) \doteq w_{\mathfrak{p}}(\mathfrak{b}) \qquad \textit{for each } \mathfrak{p} \in {}^*V.$$

*Proof.* If $\mathfrak{a} \mathrel{\dot{\leqslant}} b$, then there exists a standard divisor $\mathfrak{c}$ such that

$$\mathfrak{b} - \mathfrak{a} \geqslant \mathfrak{c}.$$

It follows $w_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) = w_\mathfrak{p}(\mathfrak{b}) - w_\mathfrak{p}(\mathfrak{a}) \geqslant w_\mathfrak{p}(\mathfrak{c})$. Since $\mathfrak{c}$ is standard, its $\mathfrak{p}$-adic logarithmic value $w_\mathfrak{p}(\mathfrak{c})$ is a standard real number. Therefore, $w_\mathfrak{p}(\mathfrak{a}) \mathrel{\dot{\leqslant}} w_\mathfrak{p}(\mathfrak{b})$.

Conversely, assume that $w_\mathfrak{p}(\mathfrak{a}) \mathrel{\dot{\leqslant}} w_\mathfrak{p}(\mathfrak{b})$ for every $\mathfrak{p} \in {}^*V$. This means there is a standard real number $\gamma_\mathfrak{p}$ such that $w_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) \geqslant \gamma_\mathfrak{p}$. By definition, we have $w_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) = v_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) \log(N\mathfrak{p})$. If $\mathfrak{p}$ is nonstandard, then $N\mathfrak{p}$ is infinitely large (Lemma 3.1) and so is $\log(N\mathfrak{p})$. Therefore, if the ordinal $v_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) \in {}^*\mathbf{Z}$ would be $< 0$, then $v_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) \log(N\mathfrak{p})$ would be infinitely small and hence $< \gamma_\mathfrak{p}$, contradicting our assumption. We conclude that $v_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) \geqslant 0$ and hence $w_\mathfrak{p}(b - \mathfrak{a}) \geqslant 0$; this holds for every nonstandard prime $\mathfrak{p}$.

Now let $S$ denote the set of those $\mathfrak{p} \in {}^*V$ for which $w_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}) < 0$. Then $S$ is an *internal* set (since $\mathfrak{a}$ and $\mathfrak{b}$ are internal divisors). We have just seen that $S$ does not contain any nonstandard divisor. Therefore, we infer from 2.5 that $S$ consists of finitely many prime divisors only. Every $\mathfrak{p} \in S$ is standard, and hence $\log(N\mathfrak{p})$ is a standard positive number. It follows that the numbers $\gamma_\mathfrak{p}/\log(N\mathfrak{p})$ for $\mathfrak{p} \in S$ are standard. Let $c$ be a standard lower bound for these finitely many numbers; we may assume $c \in \mathbf{Z}$. Then the divisor

$$\mathfrak{c} = c \sum_{\mathfrak{p} \in S} \mathfrak{p}$$

is standard, and it satisfies $\mathfrak{c} \leqslant \mathfrak{b} - \mathfrak{a}$.

Namely, for $\mathfrak{p} \in S$ we have, by construction,

$$w_\mathfrak{p}(\mathfrak{c}) = c \log(N\mathfrak{p}) \leqslant \gamma_\mathfrak{p} \leqslant w_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}).$$

For $\mathfrak{p} \notin S$ we have

$$w_\mathfrak{p}(\mathfrak{c}) = 0 \leqslant w_\mathfrak{p}(\mathfrak{b} - \mathfrak{a}),$$

by definition of $S$.

We have thus found a standard divisor $\mathfrak{c}$ such that $\mathfrak{b} - \mathfrak{a} \geqslant \mathfrak{c}$; this shows that $\mathfrak{a} \mathrel{\dot{\leqslant}} \mathfrak{b}$. Q.E.D.

If $\mathfrak{a} \in {}^*\mathfrak{D}$, let us denote by $\dot{w}_\mathfrak{p}(\mathfrak{a})$ the order of magnitude of $w_\mathfrak{p}(\mathfrak{a})$. Thus $\dot{w}_\mathfrak{p}(\mathfrak{a}) \in \dot{\mathbf{R}}$. According to Lemma 3.3, $\dot{w}_\mathfrak{p}(\mathfrak{a})$ depends on the order of magnitude of $\mathfrak{a}$ only. In other words, if we regard $\mathfrak{a}$ as an element in $\dot{\mathfrak{D}}$, then $\dot{w}_\mathfrak{p}(\mathfrak{a})$ is still well defined as an element in $\dot{\mathbf{R}}$. If $\mathfrak{p}$ ranges over the primes in ${}^*V$, then we obtain a function $\mathfrak{p} \mapsto \dot{w}_\mathfrak{p}(\mathfrak{a})$ from ${}^*V$ to $\dot{\mathbf{R}}$; Lemma 3.3 shows that $\mathfrak{a} \in \dot{\mathfrak{D}}$ is uniquely determined by this function. In this way, we see that the group $\dot{\mathfrak{D}}$ can be represented faithfully as a certain group of functions from $*V$ to $\dot{\mathbf{R}}$.[6]

Now let $x$ be a nonzero element in $*K$. Consider the principal divisor $(x) \in *\mathfrak{D}$, and its image in $\dot{\mathfrak{D}}$. The function representing this image is $\mathfrak{p} \mapsto \dot{w}_\mathfrak{p}(x)$, where $\dot{w}_\mathfrak{p}$ means the Krull valuation of $*K$ over $K$ as introduced above; i.e., $\dot{w}_\mathfrak{p}(x)$ is the order of magnitude of $w_\mathfrak{p}(x) = -\log \| x \|_\mathfrak{p}$. Hence, *if we regard* $(x)$ *as an element in* $\dot{\mathfrak{D}}$, *then this element comprises the information about the values of* $x$ *at the Krull valuation* $\dot{w}_\mathfrak{p}$ *simultaneously for all* $\mathfrak{p} \in *V$. In view of this situation, the element $(x) \in \dot{\mathfrak{D}}$ is to be regarded as the "principal divisor" of $x$ with respect to the set of Krull valuations $\dot{w}_\mathfrak{p}$. If we assign to every $x \in *K$ its principal divisor $(x)$ in $\dot{\mathfrak{D}}$, then we obtain the "principal divisor map" $*K \to \dot{\mathfrak{D}}$ belonging to the valuations $\dot{w}_\mathfrak{p}$. (By definition, this map consists of first applying the internal principal divisor map $*K \to *\mathfrak{D}$, and then projecting $*\mathfrak{D}$ onto $\dot{\mathfrak{D}}$.) In this connection, the elements in $\dot{\mathfrak{D}}$ will be called "divisors," and $\dot{\mathfrak{D}}$ the corresponding "divisor group."

If $x$ is standard, then $(x)$ is standard too and hence $(x) \doteq 0$. On the other hand, if $x$ is nonstandard, then $(x) \not\doteq 0$ in view of Lemma 3.2. Hence, Lemma 3.2 can be regarded as describing the kernel of the principal divisor map $*K \to \dot{\mathfrak{D}}$, namely, this kernel is the multiplicative group of $K$. That is, the sequence $1 \to K \to *K \to \dot{\mathfrak{D}}$ is exact. We are now going to describe the *image* of the principal divisor map $*K \to \dot{\mathfrak{D}}$. Let us consider the size $\sigma : *\mathfrak{D} \to *\mathbf{R}$ as defined in (3.8). If $\mathfrak{a} \doteq \mathfrak{b}$, then $\sigma(\mathfrak{a}) \doteq \sigma(\mathfrak{b})$. Thus $\sigma$ defines a map $\sigma : \dot{\mathfrak{D}} \to \dot{\mathbf{R}}$ which is surjective since the original $\sigma$ is surjective. Let $\dot{\mathfrak{D}}_0$ denote its kernel; it consists of those internal divisors $\mathfrak{a}$ whose size $\sigma(\mathfrak{a})$ is finite. (More precisely, $\dot{\mathfrak{D}}_0$ consists of the orders of magnitude of those divisors.) For any such $\mathfrak{a}$, we can find a divisor $\mathfrak{a}_0$ such that $\mathfrak{a}_0 \doteq \mathfrak{a}$ and $\sigma(\mathfrak{a}_0) = 0$. To see this, let $\sigma(\mathfrak{a}) = c$; this is a certain finite real number. Let $\mathfrak{p}$ be an archimedean prime; then $c\mathfrak{p}$ is a finite divisor and $\sigma(c\mathfrak{p}) = c$. Therefore, the divisor $\mathfrak{a}_0 = \mathfrak{a} - c\mathfrak{p}$ solves our problem.

We have thus shown that every element in $\dot{\mathfrak{D}}_0$ can be represented by a divisor of size 0, i.e., by a divisor in $*\mathfrak{D}_0$. In other words $\dot{\mathfrak{D}}_0$ *can be described as being the image of* $*\mathfrak{D}_0$ *in* $\dot{\mathfrak{D}}$, *consisting of the orders of magnitude of divisors with vanishing size.*

As a consequence of product formula (3.3), we have seen above that the size function $\sigma$ vanishes on principal divisors. It follows that the image of the principal divisor map $*K \to \dot{\mathfrak{D}}$ is contained in $\dot{\mathfrak{D}}_0$.

[6] There arises the problem of characterizing those functions from $*V$ to $\dot{\mathbf{R}}$ which represent elements in $\dot{\mathfrak{D}}$. The exact condition for this is obtained by saying that this function must originate from an internal divisor as described above, and then by representing this internal divisor as an internal function from $*V$ to $\mathbf{R}$ satisfying *(i)–*(iii). We leave the explicit formulation of this condition to the reader.

THEOREM 3.4. *Every divisor in* $\dot{\mathfrak{D}}_0$ *is principal; i.e.,* $\dot{\mathfrak{D}}_0$ *is the image of the principal divisor map* $^*K \to \dot{\mathfrak{D}}$. *Consequently, the following sequence is exact, exhibiting kernel and cokernel of the principal divisor map*:

$$1 \to K \to {}^*K \to \dot{\mathfrak{D}} \xrightarrow{\sigma} \dot{\mathbf{R}} \to 0.$$

*Proof.* Let $\mathfrak{a} \in {}^*\mathfrak{D}_0$; we have to show that its image in $\dot{\mathfrak{D}}_0$ is principal. This means that there is an element $x \in {}^*K$ such that $\mathfrak{a} \doteq (x)$, i.e., $\mathfrak{a} = (x) + \mathfrak{b}$ with some finite divisor $\mathfrak{b}$. As usual, we use the symbol $\sim$ to denote the divisor equivalence with respect to principal divisors; thus our contention is that $\mathfrak{a} \sim \mathfrak{b}$ for some finite divisor $\mathfrak{b}$. We shall exhibit a certain *standard* divisor $\mathfrak{c} \geqslant 0$ such that the following statement holds.

*Every divisor* $\mathfrak{a} \in {}^*\mathfrak{D}_0$ *is equivalent to some divisor* $\mathfrak{b} \in {}^*\mathfrak{D}_0$ *which satisfies* $-\mathfrak{c} \leqslant \mathfrak{b} \leqslant \mathfrak{c}$.

Since $\mathfrak{c}$ is standard, these inequalities indeed show that $\mathfrak{b}$ is finite.

Now, the above statement is the interpretation in $^*\mathfrak{D}$ of a statement in $\mathfrak{D}$, and hence it is true in $^*\mathfrak{D}$ if and only if the original statement is true in $\mathfrak{D}$. Thus it suffices to prove the original statement; this reads as follows.

*Every divisor* $\mathfrak{a} \in \mathfrak{D}_0$ *is equivalent to some divisor* $\mathfrak{b} \in \mathfrak{D}_0$ *which satisfies* $-\mathfrak{c} \leqslant \mathfrak{b} \leqslant \mathfrak{c}$.

Of course, this statement makes sense only if we have specified the divisor $\mathfrak{c}$. Rather than do this here, we leave this specification until the end of proof.

According to the definition of $\mathfrak{D}$, we have a direct sum decomposition $\mathfrak{D} = \mathfrak{D}' \oplus \mathfrak{D}''$, where $\mathfrak{D}'$ is the archimedean part and $\mathfrak{D}''$ the nonarchimedean part of the divisor group. If we consider only the nonarchimedean primes and their divisors, disregarding the archimedeans, then this means applying the projection $\mathfrak{D} \to \mathfrak{D}''$ which has kernel $\mathfrak{D}'$. The resulting principal divisor map $K \to \mathfrak{D} \to \mathfrak{D}''$ leads to a factor group $\mathfrak{C}'' = \mathfrak{D}''/K$; this is the nonarchimedean part of the divisor class group $\mathfrak{C}$. It is well known that $\mathfrak{C}''$ is finite, its order being the class number $h$ of $K$.

If the projection $\mathfrak{D} \to \mathfrak{D}''$ is restricted to $\mathfrak{D}_0$, then it is still surjective. To see this, let $\mathfrak{a}'' \in \mathfrak{D}''$ and put $\sigma(\mathfrak{a}'') = c$. Let $\mathfrak{p}$ be an archimedean prime. Then the divisor $\mathfrak{a}'' - c\mathfrak{p}$ has size 0, hence is in $\mathfrak{D}_0$, and it is projected onto $\mathfrak{a}''$ in $\mathfrak{D}''$.

Because of the surjectivity of $\mathfrak{D}_0 \to \mathfrak{D}''$, we can find divisors $\mathfrak{c}_1, \dots, \mathfrak{c}_h \in \mathfrak{D}_0$ whose images in $\mathfrak{D}''$ represent the different classes in $\mathfrak{C}''$. Let $\mathfrak{c}_1'', \dots, \mathfrak{c}_h''$ denote these images. Now let $\mathfrak{a} \in \mathfrak{D}_0$. Its image in $\mathfrak{D}''$ is equivalent (modulo principal divisors in $\mathfrak{D}''$) to one $\mathfrak{c}_j''$ with $1 \leqslant j \leqslant h$. In $\mathfrak{D}$ this means an equivalence $\mathfrak{a} \sim \mathfrak{c}_j + \mathfrak{a}'$ where $\mathfrak{a}'$ has the image 0 in $\mathfrak{D}''$; i.e., $\mathfrak{a}' \in \mathfrak{D}' \cap \mathfrak{D}_0$.

It remains to discuss the divisor $\mathfrak{a}'$. Let us put $\mathfrak{D}_0' = \mathfrak{D}' \cap \mathfrak{D}_0$. This group is the kernel in $\mathfrak{D}'$ of the size map $\sigma : \mathfrak{D}' \to \mathbf{R}$. By definition, $\mathfrak{D}'$ is the free $\mathbf{R}$-module generated by the archimedean primes; i.e., $\mathfrak{D}'$ is an $r$-dimensional real vector space, $r$ being the number of archimedean primes of $K$. Since the size $\sigma : \mathfrak{D}' \to \mathbf{R}$ is $\mathbf{R}$-linear, it follows that $\mathfrak{D}_0'$ is an $(r-1)$-dimensional hyperplane in $\mathfrak{D}'$. This hyperplane contains those principal divisors which are contained in $\mathfrak{D}'$, i.e., which have vanishing components at the nonarchimedean primes. These are precisely the principal divisors of the *units* $u \in K$. By the Dirichlet unit theorem, the group of units is finitely generated of rank $r - 1$. Moreover, if $u_1, ..., u_{r-1}$ are $\mathbf{Z}$-independent units, then their principal divisors $(u_1), ..., (u_{r-1})$ form an $\mathbf{R}$-basis of $\mathfrak{D}_0'$. In fact, this last statement is equivalent to the nonvanishing of the regulator of the field $K$; note that the principal divisor $(u)$ has components $v_\mathfrak{p}(u) = -\log \| u \|_\mathfrak{p}$ at the archimedean primes $\mathfrak{p}$.

It follows from the above that every divisor $\mathfrak{a}' \in \mathfrak{D}_0'$ has a unique representation of the form

$$\mathfrak{a}' = \sum_{1 \leqslant i \leqslant r-1} \lambda_i(u_i)$$

with real coefficients $\lambda_i$. Let $n_i$ denote the largest integer in $\mathbf{Z}$ which is $\leqslant \lambda_i$, so that

$$\lambda_i = n_i + \rho_i \quad \text{with } 0 \leqslant \rho_i < 1.$$

Then

$$\begin{aligned} \mathfrak{a}' &= \sum_{1 \leqslant i \leqslant r-1} n_i(u_i) + \sum_{1 \leqslant i \leqslant r-1} \rho_i(u_i) \\ &= (u) + \mathfrak{b}' \sim \mathfrak{b}', \end{aligned}$$

where $u = u_1^{n_1} \cdots u_{r-1}^{n_{r-1}}$ is a unit, and where we have put $\mathfrak{b}' = \sum \rho_i(u_i)$. Since the coefficients $\rho_i$ are bounded, the divisor $\mathfrak{b}'$ is contained in a bounded region. That is, we can find a divisor $\mathfrak{c}' \geqslant 0$, independent from the $\rho_i$, such that $-\mathfrak{c}' \leqslant \mathfrak{b}' \leqslant \mathfrak{c}'$. Explicitly, we may take

$$\mathfrak{c}' = \sum_{1 \leqslant i \leqslant r-1} \max(0, (u_i), -(u_i)).$$

We have shown that every $\mathfrak{a}' \in \mathfrak{D}_0'$ is equivalent to some $\mathfrak{b}'$ such that $-\mathfrak{c}' \leqslant \mathfrak{b}' \leqslant \mathfrak{c}'$. Hence, by what we have seen above, every $\mathfrak{a} \in \mathfrak{D}_0$ is equivalent to some $\mathfrak{c}_j + \mathfrak{b}'$ where $1 \leqslant j \leqslant h$ and $-\mathfrak{c}' \leqslant \mathfrak{b}' \leqslant \mathfrak{c}'$. Now we put

$$\mathfrak{c} = \max(0, \pm\mathfrak{c}_1, ..., \pm\mathfrak{c}_h) + \mathfrak{c}'.$$

Then the divisor $\mathfrak{b} = \mathfrak{c}_j + \mathfrak{b}'$ is equivalent to $\mathfrak{a}$ and satisfies $-\mathfrak{c} \leqslant \mathfrak{b} \leqslant \mathfrak{c}$.

Q.E.D.

*Remarks.* We could have shortened our above proof by observing that the divisor class group $\mathfrak{C}_0$ is compact, and then using the nonstandard characterization of compactness: Every class in $^*\mathfrak{C}_0$ is near-standard, hence finite (see, e.g., [14]). We have preferred the proof as given above since this exhibits explicitly its sources, namely the Dirichlet unit theorem together with the theorem about the finiteness of the class number $h$. In fact, it is easily seen that these two theorems are equivalent to our Theorem 3.4.

We would like to point out the similarity between Theorem 3.4 and the similar statement for rational function fields over $K$. In the latter case too, every divisor of degree 0 is principal. In this respect, the field extension $^*K \mid K$ behaves like the rational function field; this indicates that $^*K \mid K$ should be regarded in some sense as a function field of a *simply connected* space. In fact, our discussion in this paper will show that $^*K$, in relation to a function field which it contains, looks very much like the field of the *universal covering space*. It would be desirable to investigate this situation and to explain this similarity which, as for now, appears to be purely formal only.

## 4. Function Fields Embedded into the Enlargement of an Algebraic Number Field

According to Section 1, we consider the following situation: $F \mid K$ is a function field of one variable which is embedded in $^*K$; i.e., we have $K \subset F \subset {}^*K$. Our aim is to investigate the divisor theory of $F \mid K$ in relation to that of $^*K \mid K$.

As mentioned in Section 1 already, the prime divisors of $F \mid K$ will be called "functional" in order to indicate that they belong to $F$ as a function field over $K$. By contrast, the internal prime divisors of $^*K$ as introduced in Section 3 will be called "arithmetical" in order to indicate that they are connected with the arithmetic properties of the field $^*K$. This terminology—functional versus arithmetical—will also be used for divisors instead of prime divisors, etc.

By definition, a functional prime divisor $P$ is an equivalence class of nontrivial valuations of $F$ which are trivial on $K$. There are no archimedean functional prime divisors. Among all valuations of $F$ which belong to the functional prime $P$, there is exactly one which is normalized such that its value group is $\mathbf{Z}$. Again, this is called the $P$-adic *ordinal function* of $F$; notation $v_P$. The $P$-adic residue field of $F$ is an extension of $K$ of finite degree: notation $\deg(P)$. If we put

$$w_P(x) = v_P(x) \deg(P)$$

then we have the *sum formula*

$$\sum_P w_P(x) = 0, \tag{4.1}$$

which expresses the fact that the element $0 \neq x \in F$ has as many poles as it has zeros. In this formula, $P$ ranges over all prime divisors of $F \mid K$; the sum is essentially a *finite* sum, since $w_P(x) \neq 0$ for finitely many $P$ only.

The group $D$ of *functional divisors* of $F$ is defined to be the free $\mathbf{Z}$-module generated by the functional primes $P$. Thus every functional divisor $A \in D$ admits a unique representation

$$A = \sum_P \alpha_P \cdot P,$$

where the $\alpha_P$ are integers, only finitely many of them being $\neq 0$. We define

$$v_P(A) = \alpha_P$$

and

$$w_P(A) = v_P(A) \deg(P).$$

The number

$$\deg(A) = \sum_P w_P(A)$$

is called the degree of $A$. This defines a homomorphism deg: $D \to \mathbf{Z}$ whose kernel is denoted by $D_0$, the functional divisor group of degree 0.

Every $0 \neq x \in F$ determines a *principal divisor*

$$[x] = \sum_P v_P(x)P,$$

where we use brackets in order to distinguish this functional principal divisor in $D$ from the "arithmetical" principal divisor $(x) \in {}^*\mathfrak{D}$ as introduced in Section 3.

The kernel of the principal divisor map $F \to D$ is the multiplicative group of the field of constants $K$. That is, the sequence $1 \to K \to F \to D$ is exact. In view of the sum formula above, the image of $F \to D$ is contained in $D_0$. The groups $C = D/F$ and $C_0 = D_0/F$ are the functional divisor class group and the functional divisor class group of degree 0, respectively.

This being said, we now start to investigate the connection between functional and arithmetical primes. Let $\mathfrak{p} \in {}^*V$ be an arithmetical prime. In (3.5) we have defined its logarithmic absolute value $w_\mathfrak{p}(x) = -\log \| x \|_\mathfrak{p}$. In general, this function is neither a valuation of ${}^*K$ (namely if $\mathfrak{p}$ is archimedean) nor is it trivial on $K$ (if $\mathfrak{p}$ is standard). However, we have seen in Section 3 that the modified function $\dot{w}_{\mathfrak{p}}(x)$, which measures the order of magnitude of $w_{\mathfrak{p}}(x)$, is a valuation which is trivial on $K$. Hence, the restriction of $\dot{w}_{\mathfrak{p}}$ to the subfield $F$, if it is not entirely trivial on $F$, yields a valuation of $F \mid K$ which belongs to one of its functional prime divisors, say $P$. If this is the case, then $\mathfrak{p}$ is said to be *effective* on $F$, or on $P$; we also say that $P$ is *induced* by $\mathfrak{p}$, notation: $\mathfrak{p} \mid P$. In using this symbol one has always to keep in mind that not the logarithmic value $w_{\mathfrak{p}}$ itself, but only its modified valuation $\dot{w}_{\mathfrak{p}}$ induces in $F$ a valuation which is equivalent to $w_P$. Explicitly, the condition that $\dot{w}_{\mathfrak{p}}$ and $w_P$ are equivalent on $F$ means that there is a real number $\rho_{\mathfrak{p}} \gtrdot 0$ such that

$$w_{\mathfrak{p}}(x) \doteq \rho_{\mathfrak{p}} w_P(x) \tag{4.2}$$

for all $x \in F$. The order of magnitude of $\rho_{\mathfrak{p}}$ is uniquely determined by this relation. Let $\pi \in F$ be a uniformizing variable at $P$; i.e., $v_P(\pi) = 1$. If we put

$$e_{\mathfrak{p}} = v_{\mathfrak{p}}(\pi), \qquad f_{\mathfrak{p}} = \log(N\mathfrak{p})/\deg(P)$$

then we obtain from (4.2)

$$\rho_{\mathfrak{p}} \doteq e_{\mathfrak{p}} f_{\mathfrak{p}} .$$

It is clear that $e_{\mathfrak{p}}$ may be regarded as the $\mathfrak{p}$-adic ramification index, and $f_{\mathfrak{p}}$ as the $\mathfrak{p}$-adic residue degree of $^*K$ relative to $F$. However, we shall not consider these but work mainly with the invariant $\rho_{\mathfrak{p}}$ itself, which is to be regarded as kind of $\mathfrak{p}$-adic relative degree of $^*K$ over $F$.

Our first result is the following.

LEMMA 4.1. *Every functional prime divisor $P$ of $F$ is induced by some arithmetical prime divisor $\mathfrak{p}$ of $^*K$.*

*Proof.* By the theorem of Riemann-Roch there exists $x \in F$ which admits $P$ as its only pole. That is, we have $w_P(x) < 0$, and $P$ is the only functional prime of $F$ with this property. Since $x \notin K$, we infer from Lemma 3.2 that there is an arithmetical prime $\mathfrak{p}$ such that $w_{\mathfrak{p}}(x) \lessdot 0$. This inequality shows, first, that $\mathfrak{p}$ is effective on $F$ and, secondly, that $\mathfrak{p}$ induces in $F$ a functional prime which is a pole of $x$. Since there is only one pole of $x$, namely $P$, we conclude $\mathfrak{p} \mid P$. Q.E.D.

Lemma 4.1 can be found in [17] already. It shows that, in a sense, the functional divisor theory of $F$ is induced by the arithmetical divisor theory of $^*K$. This is not literally true, however, since we have pointed out above already that the valuations $w_{\mathfrak{p}}$ have to be modified first before they can be said to induce a valuation in $F$. Accordingly, in formula (4.2) and in the next formulas to come, we see the sign $\doteq$ appear where in the ordinary theory one would expect the equality sign $=$. Apart from this difference, we shall develop our results in such a way that it is in complete analogy to the ordinary theory of extensions of valued fields.

Our first task is to construct what is ordinarily called the *conorm*, which in our case is an injection $i : D \to \dot{\mathfrak{D}}$ from the functional divisor group $D$ of $F$ into the arithmetical divisor group $\dot{\mathfrak{D}}$ of $^*K$. (The appearance of $\dot{\mathfrak{D}}$ instead of $^*\mathfrak{D}$ signifies that we have to use $\doteq$ instead of $=$, as has just been explained.) If $A \in D$, then we would like its image $iA \in \dot{\mathfrak{D}}$ to have the following $\mathfrak{p}$-adic values:

$$w_{\mathfrak{p}}(iA) \doteq \begin{cases} \rho_{\mathfrak{p}} w_P(A) & \text{if } \mathfrak{p} \mid P, \\ 0 & \text{if } \mathfrak{p} \text{ is not effective on } F. \end{cases} \tag{4.3}$$

It is not yet clear that such a divisor $iA \in \dot{\mathfrak{D}}$ exists; if it does exist, however, then $iA$ is uniquely determined in view of Lemma 3.3. The following lemma solves the existence problem and more.

LEMMA 4.2. *For every functional divisor $A \in D$ there exists an arithmetical divisor $iA \in {}^*\mathfrak{D}$ satisfying* (4.3). *This divisor is uniquely determined in its order of magnitude; the resulting map*

$$i : D \to \dot{\mathfrak{D}}$$

*is an injective homomorphism which has the following properties:*

*strong order preservation,*

$$A \leqslant B \Leftrightarrow iA \mathrel{\dot{\leqslant}} iB;$$

*maximum preservation,*

$$i \max(A, B) \doteq \max(iA, iB);$$

*minimum preservation,*

$$i \min(A, B) \doteq \min(iA, iB);$$

*principal divisor preservation,*

$$i[x] \doteq (x).$$

*Proof.* First, we prove the existence of $iA \in {}^*\mathfrak{D}$ satisfying (4.3). If $A = [x]$ is the principal divisor of some $x \in F$, then we can put $iA = (x)$; Eqs. (4.2) guarantee the validity of (4.3) in this case. In general, we shall try to represent $A$ in some way by means of principal divisors; then we shall use the same representation in $^*\mathfrak{D}$ to define $iA$.

To start with, we may assume without loss that $A \geqslant 0$. Namely, if this is not the case, then we write $A = B - C$, where $B = \max(0, A) \geqslant 0$ and $C = \max(0, -A) \geqslant 0$. If we know the existence of $iB$ and $iC$, then we can put $iA = iB - iC$; note that conditions (4.3) are additive in character.

So let us assume $A \geqslant 0$; i.e., $w_P(A) \geqslant 0$ for every functional prime divisor $P$. There are only finitely many $P$ with $w_P(A) > 0$. Using the approximation theorem for valuations, we can find a nonzero element $x \in F$ such that

$$w_P(x) = w_P(A) \quad \text{if } w_P(A) > 0.$$

Again, there are only finitely many $P$ with $w_P(x) > 0$; thus we find $0 \neq y \in F$ such that

$$\begin{aligned} w_P(y) &= w_P(A) && \text{if } \; w_P(A) > 0, \\ w_P(y) &= 0 && \text{if } \; w_P(A) = 0 \;\text{ and }\; w_P(x) > 0. \end{aligned}$$

Now we have

$$\min(w_P(x), w_P(y)) \begin{cases} = w_P(A) & \text{if } w_P(A) > 0, \\ = 0 & \text{if } w_P(A) = 0 \text{ and } w_P(x) > 0, \\ \leqslant 0 & \text{if } w_P(x) \leqslant 0. \end{cases}$$

It follows that

$$\max(0, \min(w_P(x), w_P(y))) = w_P(A)$$

for *every* functional prime divisor $P$, which is to say that

$$\max(0, \min([x], [y])) = A.$$

This is the representation of $A$ by means of principal divisors, as announced above. Now we put

$$iA = \max(0, \min((x), (y))),$$

which is the same expression in $*\mathfrak{D}$ as $A$ in $\mathfrak{D}$. For every arithmetical prime $\mathfrak{p}$, we have

$$w_{\mathfrak{p}}(iA) = \max(0, \min(w_{\mathfrak{p}}(x), w_{\mathfrak{p}}(y))).$$

Now, if $\mathfrak{p}$ is effective on $P$, then using (4.2) for $x$ and $y$ we obtain

$$w_{\mathfrak{p}}(iA) \doteq \rho_{\mathfrak{p}} \max(0, \min(w_P(x), w_P(y))) = \rho_{\mathfrak{p}}\, w_P(A).$$

On the other hand, if $\mathfrak{p}$ is not effective on $F$, then $w_{\mathfrak{p}}(x) \doteq 0 \doteq w_{\mathfrak{p}}(y)$ and it follows that $w_{\mathfrak{p}}(iA) \doteq 0$. We have proved that (4.3) holds.

As already said above, we have shown in Lemma 3.3 that any divisor in $\dot{\mathfrak{D}}$ is uniquely determined by its $\mathfrak{p}$-adic values in $\dot{R}$. Hence, $iA$ is uniquely determined in $\dot{\mathfrak{D}}$ by conditions (4.3), irrespective of the choice of the elements $x, y \in F$ used above to construct $iA$. Thus we obtain a map $i : D \to \dot{\mathfrak{D}}$. It is clear that this map is a homomorphism, since the divisor addition in $D$ resp. $\dot{\mathfrak{D}}$ is faithfully reflected in the addition of the local $P$-adic resp. $\mathfrak{p}$-adic values. (We have used this remark above already by saying that conditions (4.3) are additive in character.) For similar reasons, it is clear that $i : D \to \dot{\mathfrak{D}}$ preserves the order relation of divisors, as well as the operations max and min. Formula (4.2) shows that if $A = [x]$, then $iA \doteq (x)$. It remains to prove the following strong order preservation property which at the same time yields the injectivity:

$$iA \mathrel{\dot{\leqslant}} iB \Rightarrow A \leqslant B.$$

Let $P$ be a functional prime divisor. By Lemma 4.1 there exists an arithmetical prime divisor $\mathfrak{p}$ which is effective on $P$. Since $iA \mathrel{\dot{\leqslant}} iB$, we have $w_{\mathfrak{p}}(iA) \mathrel{\dot{\leqslant}} w_{\mathfrak{p}}(iB)$; in view of (4.3) this implies

$$\rho_{\mathfrak{p}} w_P(A) \mathrel{\dot{\leqslant}} \rho_{\mathfrak{p}} w_P(B),$$

which is to say that

$$\rho_{\mathfrak{p}} w_P(A) \leqslant \rho_{\mathfrak{p}} w_P(B) + c,$$

where $c$ is a finite number. It follows that

$$w_P(A) \leqslant w_P(B) + h,$$

where $h = c/\rho_{\mathfrak{p}}$.

We know from (4.2) that $\rho_{\mathfrak{p}} \mathrel{\dot{>}} 0$; i.e., $\rho_{\mathfrak{p}}$ is infinitely large. Since $c$ is finite, it follows that $h$ is infinitesimal; in particular, we have $h < 1$ and therefore $w_P(A) < w_P(B) + 1$. Since $w_P(A)$ and $w_P(B)$ are (standard) integers, we have $w_P(A) \leqslant w_P(B)$.

Here $P$ is an arbitrary functional prime divisor. Hence $A \leqslant B$. Q.E.D.

*Remark.* The injection $i : D \to \dot{\mathfrak{D}}$ is the nonstandard counterpart of Weil's representation of functional divisors as so-called distributions [23–25]. In this sense, our Lemma 4.2 may be compared to what is usually called the "theorem of decomposition" for Weil distributions.

Lemma 4.2 says that the map $i : D \to \dot{\mathfrak{D}}$ is injective and faithful with respect to all the relevant relations between divisors. We have already said above that this map should be regarded as an analog to the conorm mapping, in the ordinary theory of extensions of valued fields. Now, in that theory it is quite common to *identify* the divisor group of the subfield with its image under the conorm. We shall follow this procedure also in our situation.

Henceforth we identify $D$ with its image $iD \subset \dot{\mathfrak{D}}$ *whenever this is convenient and no misunderstanding seems possible.* Accordingly, we shall regard functional divisors $A \in D$ as arithmetical internal divisors, with the provision, however, that the equality sign is replaced by $\doteq$, which measures the order of magnitude only.

Due to this identification convention, our formulas will become more lucid, leaving out the redundant symbol $i$. For instance, the $\mathfrak{p}$-adic logarithmic value of $iA$ is now to be regarded as the $\mathfrak{p}$-adic logarithmic value of $A$ itself, and to be denoted by $w_\mathfrak{p}(A)$. Formula (4.3) now reads

$$w_\mathfrak{p}(A) \doteq \begin{cases} \rho_\mathfrak{p} w_P(A) & \text{if } \mathfrak{p} \mid P \\ 0 & \text{if } \mathfrak{p} \text{ is not effective on } F. \end{cases} \tag{4.4}$$

This is in complete analogy to (4.2). The principal divisor property of Lemma 4.2 reads

$$[x] \doteq (x) \qquad (x \in F), \tag{4.5}$$

showing that the two principal divisors of $x$ coincide in their order of magnitude.

Every functional divisor $A \in D$, if regarded as an element in $\dot{\mathfrak{D}}$, has a *size* $\sigma(A)$; according to Section 3 this is an element in $\dot{\mathbf{R}}$, i.e., a real number whose order of magnitude is uniquely determined by $A$. We thus obtain the size map $\sigma : D \to \dot{\mathbf{R}}$ which is induced by the ordinary size map $\sigma : {}^*\mathfrak{D} \to {}^*\mathbf{R}$ in the manner as described, via the embedding $D \subset \dot{\mathfrak{D}}$. On the other hand, we have the functional degree map deg: $D \to \mathbf{Z}$. There arises the question as to the connection between these two invariants $\sigma$ and deg on $D$.

Again, we use the ordinary theory of extensions of valued fields as a guide line and motivation. In that theory, the degrees of divisors of the ground field are multiplied by a fixed number $\rho$, if these divisors are regarded in the extension field by means of the conorm embedding. In fact, $\rho$ equals the degree of the field extension. Now, in our situation the field extension ${}^*K$ over $F$ is not finite; nevertheless, we can ask whether there is such a number $\rho$ as above. Of course, the role of the degree in the extension field ${}^*K$ is taken by the size $\sigma$. Hence we ask, more precisely, is there a number $\rho \mathrel{\dot{>}} 0$ such that, for every functional divisor $A \in D$, we have

$$\sigma(A) \doteq \rho \deg(A)?$$

Now this is not true in general, the reason being the nonarchimedean type of the ordering of $\dot{\mathbf{R}}$. Nevertheless, we shall show that quite a similar statement does hold, namely:

$$\frac{\sigma(A)}{\rho} \simeq \deg(A), \tag{4.6}$$

where the symbol $\simeq$ means "infinitely close." Let us first explain this relation; afterwards we shall state our main theorem.

Let $x, y \in {}^*\mathbf{R}$. We say that $x$ is *infinitely close* to $y$ if $x = y + h$ with infinitesimal $h \in {}^*\mathbf{R}$. Notation: $x \simeq y$. It is clear that every infinitesimal number is finite; hence, $x \simeq y$ implies $x \doteq y$. As in the case of finite numbers, the infinitesimal numbers form an *isolated* subgroup $\mathbf{R}_{\text{inf}}$ of ${}^*\mathbf{R}$; this implies that the factor group ${}^*\mathbf{R}/\mathbf{R}_{\text{inf}}$ inherits naturally its order relation from ${}^*\mathbf{R}$. Accordingly, we shall write $x \lesssim y$ in order to say that $x \leqslant y + h$ with infinitesimal $h$.

By definition, both relations $x \dot{\leqslant} y$ and $x \lesssim y$ are of additive character; they are in general not coherent with respect to multiplication. Their behavior with respect to multiplication can be described by saying that the finite numbers $\mathbf{R}_{\text{fin}}$ form (not only an additive group but) a *valuation ring of* ${}^*\mathbf{R}$, and that $\mathbf{R}_{\text{inf}}$ is the *maximal ideal* of that valuation ring. For later references, we shall state the following lemma.

LEMMA 4.3. *Let $\rho \in {}^*\mathbf{R}$ be infinitely large. If $x \dot{\leqslant} y$ then $x/\rho \lesssim y/\rho$. In particular, if $x \doteq y$ then $x/\rho \simeq y/\rho$. In other words, the map $x \mapsto x/\rho$ is an order-preserving homomorphism from the additive group $\dot{\mathbf{R}} = {}^*\mathbf{R}/\mathbf{R}_{\text{fin}}$ onto ${}^*\mathbf{R}/\mathbf{R}_{\text{inf}}$, the real numbers modulo infinitesimals.*

*Proof.* If $x \dot{\leqslant} y$, then $x \leqslant y + c$, where $c$ is a finite number. It follows that $x/\rho \leqslant y/\rho + h$, where $h = c/\rho$. Since by assumption $\rho$ is infinitely large and $c$ is finite, it follows that $h$ is infinitesimal. Hence, $x/\rho \lesssim y/\rho$. Q.E.D.

If $A \in D$, then we know that $\sigma(A)$ is determined modulo finite numbers. From Lemma 4.3 we infer that the ratio $\sigma(A)/\rho$ is determined modulo infinitesimals. Thus we see that statement (4.6) is at least meaningful, and it is the best what we can expect. The following theorem says that it is true.

THEOREM 4.4. *There exists an infinitely large number $\rho \in {}^*\mathbf{R}$ such that*

$$\frac{\sigma(A)}{\rho} \simeq \deg(A)$$

*for all functional divisors $A \in D$. That is, the size on $D$ is proportional to the degree, up to infinitesimals.*

*The number $\rho$ is uniquely determined up to infinitesimals, in the following multiplicative sense: If $\lambda \in {}^*\mathbf{R}$ is another such number, then $\rho/\lambda \simeq 1$.*

*Proof.* Let us start by stating the formal properties of the size which are responsible for the validity of Theorem 4.4.

(i) *The size $\sigma : D \to \dot{\mathbf{R}}$ is an order-preserving homomorphism which vanishes on principal divisors of $D$.*

Namely, these properties are inherited from the original size function $\sigma : {}^*\mathfrak{D} \to {}^*\mathbf{R}$, via the inclusion $D \subset \dot{\mathfrak{D}}$. As to the vanishing on principal divisors, we mean of course *functional* principal divisors $[x]$ where $x \in F$. However, due to formula (4.5), these may be identified with their arithmetical principal divisors $(x)$; we know that the original size function $\sigma : {}^*\mathfrak{D} \to {}^*\mathbf{R}$ vanishes on those $(x)$, as a consequence of product formula (3.3).

(ii) *The size $\sigma : D \to \dot{\mathbf{R}}$ does not vanish identically. In fact, for every $A > 0$ we have $\sigma(A) \dot{>} 0$.*

For, if $A > 0$ in $D$, then $A \dot{>} 0$ in $\dot{\mathfrak{D}}$. Thus we have to show, for every internal divisor, $\mathfrak{a} \dot{>} 0 \Rightarrow \sigma(\mathfrak{a}) \dot{>} 0$. By Lemma 3.3, if $\mathfrak{a} \dot{>} 0$, then there exists $\mathfrak{p} \in {}^*V$ such that $w_\mathfrak{p}(\mathfrak{a}) \dot{>} 0$. Therefore, it suffices to prove that $\mathfrak{a} \dot{\geqslant} 0 \Rightarrow \sigma(\mathfrak{a}) \dot{\geqslant} w_\mathfrak{p}(\mathfrak{a})$ for any internal divisor $\mathfrak{a}$ and any internal prime $\mathfrak{p}$. Now this rule in $\dot{\mathfrak{D}}$ is clearly inherited from the corresponding rule in ${}^*\mathfrak{D}$, namely $\mathfrak{a} \geqslant 0 \Rightarrow \sigma(\mathfrak{a}) \geqslant w_\mathfrak{p}(\mathfrak{a})$. But this is trivially true in view of definition (3.8) of the size.

This being said, we now start with the proof of Theorem 4.4. Let $g$ denote the genus of the function field $F \mid K$. If $\deg(A) \geqslant g$, then the theorem of Riemann–Roch shows that there is a positive divisor $A' \geqslant 0$ which is equivalent to $A$, in the sense that $A - A' = [x]$ is principal. From (i) it follows that $\sigma(A) \doteq \sigma(A') \dot{\geqslant} 0$.

If $\deg(A) > 0$, then there exists a natural number $n \in \mathbf{N}$ such that $\deg(nA) \geqslant g$; we conclude that $\sigma(nA) = n\sigma(A) \dot{\geqslant} 0$ and hence $\sigma(A) \dot{\geqslant} 0$. Applying this to $A - B$ instead of $A$, we obtain

$$\deg(A) > \deg(B) \Rightarrow \sigma(A) \dot{\geqslant} \sigma(B).$$

We replace $A$, $B$ by $nA$ resp. $mB$, where $n, m \in \mathbf{Z}$. Thus,

$$n \deg(A) > m \deg(B) \Rightarrow n\sigma(A) \dot{\geqslant} m\sigma(B) \qquad (*)$$

This statement remains true if $m, n$ denote arbitrary rational numbers in $\mathbf{Q}$. For, this may be reduced to the case of integers by multiplication with the least common denominator.

Now we choose a fixed positive divisor $B > 0$. Then $\deg(B) > 0$. We know from property (ii) above that $\sigma(B) \dot{>} 0$. Let $\rho \in {}^*\mathbf{R}$ be such that $\rho \doteq \sigma(B)/\deg(B)$; then $\rho$ is infinitely large. In (*) we take $n = 1$ and $m = r/\deg(B)$, where $r \in \mathbf{Q}$; we obtain

$$\deg(A) > r \Rightarrow \sigma(A) \dot{>} r\rho \Rightarrow \sigma(A)/\rho \gtrsim r,$$

the last conclusion in view of Lemma 4.3. This statement is true for *every* rational number $r < \deg(A)$. If $r$ tends to $\deg(A)$, we see that

$$\frac{\sigma(A)}{\rho} \gtrsim \deg(A).$$

On the other hand, formula (*) also holds if $A$ and $B$ are interchanged. We get, similarly,

$$\frac{\sigma(A)}{\rho} \lesssim \deg(A);$$

hence,

$$\frac{\sigma(A)}{\rho} \simeq \deg(A).$$

It remains to show the uniqueness property of $\rho$. Quite generally, if $x$ and $y$ are real numbers and $x$ is not infinitesimal, then $x \simeq y \Rightarrow y/x \simeq 1$. For, we have $y = x + h$ with infinitesimal $h$, and hence $y/x = 1 + k$ where $k = h/x$ is again infinitesimal, due to the fact that $x$ itself is not infinitesimal.

Now, if $\lambda$ is as in Theorem 4.4, choose any divisor $A$ of positive degree; then the relations

$$\frac{\sigma(A)}{\rho} \simeq \deg(A) \simeq \frac{\sigma(A)}{\lambda}$$

show that neither of the two quotients in infinitesimal; hence division leads to $\rho/\lambda \simeq 1$ since $\sigma(A)$ cancels. Q.E.D.

*Remark* 4.5. It is clear from our proof that Theorem 4.4 holds, not only for the size, but for an arbitrary function satisfying (i) and (ii). In other words, we have the following statement of Artin–Whaples type [1].

*Let $\varphi : D \to \dot{\mathbf{R}}$ be any nontrivial order-preserving homomorphism which vanishes on principal divisors. Then $\varphi$ is proportional to the degree, up to infinitesimals. That is, there exists an infinitely large $\rho$ such that $\varphi(A)/\rho \simeq \deg(A)$ for every $A \in D$.*

COROLLARY 4.6. *Let $A, B \in D$. If $\deg(A) > 0$, then $\sigma(A) \gtrsim 0$ and*

$$\frac{\sigma(B)}{\sigma(A)} \simeq \frac{\deg(B)}{\deg(A)}.$$

*That is, the size quotient is infinitely close to the degree quotient.*

This follows immediately from Theorem 4.4 since the factor $\rho$ cancels out. Note that the size quotient $\sigma(B)/\sigma(A)$ is well defined up to infinitesimals.

Now let us consider a nonconstant element $x \in F$, and let us take for $A$ the divisor of poles of $x$ in $D$. That is,

$$A = -\min(0, [x]) = \max(0, -[x]).$$

It is well known that $\deg(A) = [F : K(x)]$, the right-hand side denoting the field degree of $F$ over the rational function field generated by $x$. On the other hand, if we consider $A$ as an internal divisor, we have

$$A \doteq \max(0, -(x)),$$

which is the *denominator* of $x$ in the arithmetic sense (including the archimedean primes). To compute the size of this denominator, we notice that

$$w_{\mathfrak{p}} \max(0, -(x)) = \max(0, -w_{\mathfrak{p}}(x)) = \log \max(1, \| x \|_{\mathfrak{p}}).$$

If we put

$$H(x) = \prod_{\mathfrak{p}} \max(1, \| x \|_{\mathfrak{p}}),$$

then $H$ is the *height function* as introduced by Hasse [4], and we have $\sigma(A) \doteq \log H(x)$. We conclude the following.

COROLLARY 4.7. *Let $x \in F$ be nonconstant. For every divisor $B \in D$ we have*

$$\frac{\sigma(B)}{\log H(x)} \simeq \frac{\deg(B)}{[F : K(x)]}.$$

In particular, taking for $B$ the pole divisor of another nonconstant $y \in F$, we obtain the next corollary.

COROLLARY 4.8. *For nonconstant $x, y \in F$ we have*

$$\frac{\log H(y)}{\log H(x)} \simeq \frac{[F : K(y)]}{[F : K(x)]}.$$

*That is, the logarithmic height quotient is infinitely close to the degree quotient.*

This last formula has been obtained in [17] already. It can be regarded as the nonstandard equivalent of what is called the first basic inequality of Siegel [22].

## 5. Exceptional Divisors

Let $P$ be a functional prime divisor of $F$. We know from Lemma 4.1 that there is at least one arithmetical prime $\mathfrak{p}$ of $^*K$ which is effective on $P$, i.e., $\mathfrak{p} \mid P$. The contention of Theorem 1.2 is that among these arithmetical primes $\mathfrak{p}$ there exists a *nonstandard* one provided the genus $g$ of $F$ is positive. Thus we have to study those functional primes $P$ which do not admit a nonstandard $\mathfrak{p}$ with $\mathfrak{p} \mid P$. These primes $P$ are called *exceptional*; the contention is that exceptional primes exist in the case $g = 0$ only.

It will be convenient to extend the notion of "exceptional" to divisors instead of prime divisors: namely, a functional divisor $A \in D$ is called *exceptional* if

$$A = P_1 + P_2 + \cdots + P_r,$$

where the $P_i$ are distinct exceptional prime divisors. This definition implies that exceptional divisors are positive and without multiple components.

Our first aim is to obtain an estimate for the degree of an exceptional divisor $A$; this will at the same time give an upper bound for the number $r$ of exceptional prime divisors of $F$ (if there are any). As we know from Theorem 4.4, the degree is intimately connected with the size; this leads us to study the size of an exceptional divisor $A$.

An arithmetical prime $\mathfrak{p}$ is said to be effective on $A$ if $\mathfrak{p}$ is effective on some component of $A$. Notation: $\mathfrak{p} \mid A$.

Lemma 5.1. *Assume $A \in D$ is an exceptional divisor of $F$. There are only finitely many arithmetical primes $\mathfrak{p}$ of $^*K$ which are effective on $A$. These primes $\mathfrak{p} \mid A$ are characterized by the condition $w_\mathfrak{p}(A) \mathrel{\dot{>}} 0$, and we have*

$$\sigma(A) \doteq \sum_{\mathfrak{p} \mid A} w_\mathfrak{p}(A).$$

Notice that the sum on the right-hand side has finitely many terms only.

*Proof.* If $\mathfrak{p} \mid A$, then there is some component $P$ of $A$ such that $\mathfrak{p} \mid P$. Using (4.4), we conclude that

$$w_\mathfrak{p}(A) \doteq \rho_\mathfrak{p} w_P(A) \mathrel{\dot{\geqslant}} \rho_\mathfrak{p} \mathrel{\dot{>}} 0.$$

Conversely, assume that $w_\mathfrak{p}(A) \mathrel{\dot{>}} 0$. Then (4.4) shows, first of all, that $\mathfrak{p}$ is effective on $F$; i.e., there is some functional prime $P$ such that $\mathfrak{p} \mid P$.

Moreover, (4.4) shows that $w_P(A) > 0$ for this prime $P$; i.e. $P$ is a component of $A$ and hence $\mathfrak{p} \mid A$. We have shown that $\mathfrak{p} \mid A$ if and only if $w_\mathfrak{p}(A) \dot{>} 0$, which is one of the contentions of Lemma 5.1.

In the above arguments, we had to regard $A$ as an element of $\dot{\mathfrak{D}}$, according to the embedding $D \subset \dot{\mathfrak{D}}$ as explained in Section 4. Now let $\mathfrak{a} \in {}^*\mathfrak{D}$ be any internal divisor which represents $A$, i.e., such that $\mathfrak{a} \doteq A$. We then have $w_\mathfrak{p}(\mathfrak{a}) \doteq w_\mathfrak{p}(A)$ for every $\mathfrak{p}$, which means that the (standard or nonstandard) real number $w_\mathfrak{p}(\mathfrak{a})$ represents $w_\mathfrak{p}(A) \in \dot{\mathbf{R}}$. Let $S$ denote the set of those arithmetical primes $\mathfrak{p}$ for which $w_\mathfrak{p}(\mathfrak{a}) > 0$; then $S$ is internal and, by what we have shown above, $S$ contains every $\mathfrak{p}$ which is effective on $A$. Moreover, if $\mathfrak{p} \in S$ is not effective on $A$, then $w_\mathfrak{p}(\mathfrak{a}) \doteq 0$.

We claim that $S$ does not contain any nonstandard prime. For, assume $\mathfrak{p} \in S$ would be nonstandard. Then $\mathfrak{p} \nmid A$ (since $A$ is exceptional) and hence $w_\mathfrak{p}(\mathfrak{a}) \doteq 0$. That is, the real number $w_\mathfrak{p}(\mathfrak{a}) = v_\mathfrak{p}(\mathfrak{a}) \log(N\mathfrak{p})$ would be positive and finite. But this contradicts the fact that $N\mathfrak{p}$ is infinitely large (Lemma 3.1).

Now, since $S$ is internal and does not contain any nonstandard prime, it follows that $S$ is *finite* (see Section 2). In particular, there are only finitely many $\mathfrak{p}$ which are effective on $A$.

Let us put

$$\mathfrak{a}' = \sum_{\mathfrak{p} \mid A} v_\mathfrak{p}(\mathfrak{a})\mathfrak{p}.$$

This sum contains finitely many terms only, and thus $\mathfrak{a}'$ is a well-defined internal divisor. By construction, $\mathfrak{a}'$ coincides with $\mathfrak{a}$ at the primes $\mathfrak{p} \mid A$, and $\mathfrak{a}'$ vanishes at the other primes; hence,

$$\begin{aligned} w_\mathfrak{p}(\mathfrak{a}') &\doteq w_\mathfrak{p}(A) \dot{>} 0 && \text{if} \quad \mathfrak{p} \mid A \\ w_\mathfrak{p}(\mathfrak{a}') &= 0 \doteq w_\mathfrak{p}(A) && \text{if} \quad \mathfrak{p} \nmid A. \end{aligned}$$

We conclude that $\mathfrak{a}' \doteq A$; i.e., $\mathfrak{a}'$ is also a representative of $A$ in ${}^*\mathfrak{D}$. Hence,

$$\sigma(A) \doteq \sigma(\mathfrak{a}') = \sum_{\mathfrak{p} \mid A} w_\mathfrak{p}(\mathfrak{a}) \doteq \sum_{\mathfrak{p} \mid A} w_\mathfrak{p}(A).$$

Q.E.D.

COROLLARY 5.2. *Let $A$ be exceptional as in Lemma* 5.1; *in addition, we assume that every component of $A$ is of degree* 1. *Given any nonconstant element $x \in F$ which is $A$-integral, there are elements $a_\mathfrak{p} \in K$ (for $\mathfrak{p} \mid A$) such that*

$$\sigma(A) \dot{\leqslant} \sum_{\mathfrak{p} \mid A} w_\mathfrak{p}(x - a_\mathfrak{p}).$$

An element $x \in F$ is called $A$-integral if none of the poles of $x$ is a component of $A$, i.e., if $w_P(x) \geqslant 0$ for every component $P$ of $A$.

*Proof.* If $\mathfrak{p} \mid A$, let $P$ denote the component of $A$ on which $\mathfrak{p}$ is effective. Let $a_\mathfrak{p}$ denote the $P$-adic residue of $x$; since $\deg(P) = 1$, we know that $a_\mathfrak{p} \in K$. By construction, $x - a_\mathfrak{p}$ has a zero at $P$, i.e., $v_P(x - a_\mathfrak{p}) \geqslant 1$. On the other hand, $P$ is a *simple* component of $A$ (see the above definition of exceptional divisors). This implies $v_P(A) = 1 \leqslant v_P(x - a_\mathfrak{p})$; hence, $w_P(A) \leqslant w_P(x - a_\mathfrak{p})$ and therefore, in view of (4.4),

$$w_\mathfrak{p}(A) \mathrel{\dot{\leqslant}} w_\mathfrak{p}(x - a_\mathfrak{p}).$$

Now apply Lemma 5.1. Q.E.D.

Our problem of estimating the sizes of exceptional divisors is now reduced to estimating finite sums, such as appear on the right-hand side of the formula of Corollary 5.2. To this end, we use the well-known theorem of Roth. Let us briefly recall its content.

The theorem of Roth concerns the following situation in the number field $K$ : *Let $S$ be a finite set of prime divisors of $K$, let $a_\mathfrak{p}$ be elements in $K$* belonging to the primes $\mathfrak{p} \in S$, and let $\kappa > 2$ be a real number in $\mathbf{R}$. Referring to these data we have the theorem below.

THEOREM OF ROTH. *There are only finitely many elements $x \in K$ which satisfy the approximation conditions*

$$\prod_{\mathfrak{p} \in S} \| x - a_\mathfrak{p} \|_\mathfrak{p} \leqslant \frac{1}{H(x)^\kappa}. \tag{5.1}$$

Actually, the theorem of Roth in its usual formulation is somewhat more general, since the $a_\mathfrak{p}$ may be arbitrary algebraic numbers, not necessarily contained in $K$. Roth [21] considered the case $K = \mathbf{Q}$, and $S$ consisting of one prime only, namely the archimedean prime of $\mathbf{Q}$. The case of several primes was settled by Ridout [13], in the case $K = \mathbf{Q}$. A proof in the general case, for an arbitrary number field $K$, can be found in the book of Lang [7].

By the theorem of Roth, the set of elements $x \in K$ satisfying (5.1) is finite; hence it is not enlarged in $^*K$. That is, if $x \in {}^*K$ satisfies (5.1), then $x$ is already contained in $K$; i.e., $x$ is standard. In other words, if $x \in {}^*K$ is nonstandard, then $x$ does not satisfy (5.1). Thus we obtain the following statement in $^*K$ which, as we have seen, is the nonstandard version of Roth's theorem. Let $S$ be a finite set of standard prime divisors, let $a_\mathfrak{p}$ be standard elements in $K$, belonging to the primes $\mathfrak{p} \in S$, and let $\kappa > 2$ be a standard real number.

In this situation we have the following proposition.

PROPOSITION 5.3. *For every nonstandard* $x \in {}^*K$,

$$\prod_{\mathfrak{p}\in S} \| x - a_{\mathfrak{p}} \|_{\mathfrak{p}} > \frac{1}{H(x)^{\kappa}}\,.$$

Taking the logarithm of both sides, this is equivalent to the additive inequality

$$\sum_{\mathfrak{p}\in S} w_{\mathfrak{p}}(x - a_{\mathfrak{p}}) < \kappa \log H(x). \tag{5.2}$$

Now, if we consider the situation of Corollary 5.2 and use (5.2) for the finite set of those $\mathfrak{p}$ which are effective on the exceptional divisor $A$, then we find

$$\sigma(A) \mathrel{\dot{\leqslant}} \kappa \log H(x)$$

for every standard number $\kappa > 2$. Since $\log H(x)$ is infinitely large (Lemma 3.1), we conclude, in view of Lemma 4.3, that

$$\frac{\sigma(A)}{\log H(x)} \lesssim \kappa.$$

Since $\kappa > 2$ is arbitrary standard, it follows that

$$\frac{\sigma(A)}{\log H(x)} \lesssim 2.$$

On the other hand, we know from Corollary 4.7 that

$$\frac{\sigma(A)}{\log H(x)} \simeq \frac{\deg(A)}{[F:K(x)]}\,.$$

Therefore,

$$\frac{\deg(A)}{[F:K(x)]} \lesssim 2$$

and hence

$$\deg(A) \leqslant 2\,[F:K(x)], \tag{5.3}$$

since both sides are standard integers.

We have proved formula (5.3) for every exceptional divisor $A$ and every nonconstant $x \in F$, under the additional assumptions of Corollary 5.2: namely, (i) $x$ is $A$-integral; (ii) every component of $A$ is of degree 1. But we claim that these additional assumptions are unnecessary for the validity of (5.3). That is, (5.3) holds for every exceptional divisor $A$ and every nonconstant $x \in F$, without further conditions.

In order to eliminate (i), we observe that formula (5.3) depends on the rational function field $K(x)$ only and not on the choice of its generator $x$. Hence, if $x$ should not be $A$-integral, then we choose another generator $y$ of the same field $K(x)$, such that $y$ is $A$-integral; the validity of (5.3) for $y$ then implies its validity for $x$, since $K(x) = K(y)$. For instance, we may choose $y = 1/(x - c)$, where $c \in K$ is selected such that $c$ is different from the finitely many $P$-adic residues of $x$, for every component $P$ of $A$ which is not a pole of $x$.

In order to eliminate condition (ii), we use the method of constant field extension. If $K'$ is a finite algebraic field extension of $K$, then let $F' = FK'$ denote the corresponding constant field extension of $F$. The divisor group $D$ of $F$ is naturally embedded into the divisor group $D'$ of $F'$; it is well known that this embedding is degree preserving. That is, if we consider $A$ as a divisor of $D'$, then its degree (over the new constant field $K'$) equals the degree of $A$ when considered in $D$. Moreover, for every nonconstant element $x \in F$ we have $[F : K(x)] = [F' : K'(x)]$. Hence, the validity of (5.3) in $F' \mid K'$ implies its validity in $F \mid K$. Now, it is well known that $K'$ can be chosen in such a way that every component of $A$ splits in $D'$ into primes of degree 1; such a field $K'$ is called a "splitting field" of $A$. Thus we know that (5.3) holds for $A$ in $F' \mid K'$ and therefore also in $F \mid K$, provided we can show that $A$, being exceptional in $F$, remains exceptional in $F'$. This can be shown as follows.

The notion of "exceptional" refers to the embedding $F \subset {}^*K$ of $F$ into the enlargement ${}^*K$ of $K$. The constant field extension $F' = FK'$ is imbedded into the field compositum ${}^*KK'$. Now, it is easily verified that this field compositum equals the enlargement ${}^*K'$ of the field $K'$. Namely, let $u_1, \ldots, u_n$ be a basis of $K'$ over $K$. The statement that these elements form a basis of $K'$ over $K$ remains true in the enlargement, which shows that the $u_1, \ldots, u_n$ form a basis of ${}^*K'$ over ${}^*K$. Hence, ${}^*K' = {}^*KK'$, and this field compositum is linearly disjoint over $K$. In particular, it follows that the constant field extension $F' = FK'$ is naturally embedded into ${}^*K'$; it is with respect to this embedding $F' \subset {}^*K'$ that we claim $A$ to be exceptional. In fact, let $\mathfrak{p}'$ be any arithmetical prime of ${}^*K'$ which is effective on $A$; we have to show that $\mathfrak{p}'$ is standard. Let $P'$ denote the functional prime of $F'$ which is induced by $\mathfrak{p}'$; then $P'$ is a component of $A$. Let $P$ be the prime divisor of $F$ which is induced by $P'$; then $P$ is a component of $A$ in $F \mid K$. Moreover, let $\mathfrak{p}$ denote the arithmetical prime induced by $\mathfrak{p}'$ in ${}^*K$. We then have the situation indicated in Fig. 1, a diagram of fields and corresponding primes. It is clear by construction that $\mathfrak{p}$ is effective on $P$; since $P$ is a component of $A$ and $A$ is exceptional in $F$, it follows that $\mathfrak{p}$ is standard. In particular, $\mathfrak{p}$ is nontrivial on $K$ and hence $\mathfrak{p}'$, which is an extension of $\mathfrak{p}$, is nontrivial on $K'$. Consequently, $\mathfrak{p}'$ is a standard prime

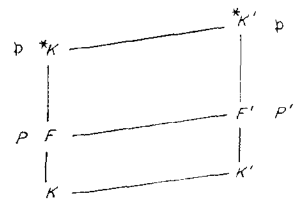


FIGURE 1

of $^*K'$, since nonstandard primes are trivial on the ground field $K'$ (Lemma 3.1).

We also observe that the constant extension $F' = FK'$ of $F$ is *unramified*, as is well known from the general theory of function fields. Hence, since $A$ has no multiple components as a divisor of $F \mid K$, the same is true for $A$ in $F' \mid K'$. (Recall that the condition for $A$ to be exceptional included the fact that $A$ is without multiple components.)

The above discussion shows that every exceptional divisor $A$ of $F \mid K$ remains exceptional in $F' \mid K'$. Hence, if we choose $K'$ as a splitting field of $A$, we infer from the above that (5.3) is true over $K'$, hence also over $K$.

We have now proved (5.3) in the general case, without the additional assumptions (i) and (ii).

Let us denote by $d$ the *minimal degree* of $F$ over a rational subfield; i.e.,

$$d = \min_{\substack{x \in F \\ x \notin K}} [F : K(x)]. \tag{5.4}$$

This is a certain invariant of the function field $F \mid K$. From (5.3) we infer that $\deg(A) \leqslant 2d$ for every exceptional divisor $A$ of $F$. In particular, we see that the number of components of $A$ is bounded by $2d$. It follows that there are only finitely many exceptional prime divisors and their number is $\leqslant 2d$.

Let us gather our results into the following.

THEOREM 5.4. *There are only finitely many functional prime divisors $P_1, \ldots, P_r$ of $F$ which are exceptional in $^*K$. If we put*

$$A = P_1 + \cdots + P_r,$$

*then*

$$\deg(A) \leqslant 2d$$

*where the invariant $d$ of $F \mid K$ is given by* (5.4).

Theorem 5.4 can be regarded as the nonstandard equivalent of the so-called second fundamental inequality of Siegel [22].

COROLLARY 5.5. *$\deg(A) \leqslant 2g + 2$, where $g$ is the genus of $F \mid K$.*

*Proof.* We have to verify that $d \leqslant g + 1$. In fact, if $B$ is any functional divisor of $F$ of degree $g + 1$ then, by the theorem of Riemann-Roch, we have $\dim(B) \geqslant 2$ and hence there are at least two different positive divisors $B', B'' \geqslant 0$ which are equivalent to $B$ (modulo principal divisors). We have $B' - B'' = [x]$, where $x \in F$ is nonconstant, and $[F : K(x)] \leqslant \deg(B'') = \deg(B) = g + 1$. Q.E.D.

*Remark* 5.6. In the argument just given, we have used the fact that there is a functional divisor of degree $g + 1$. In fact, *every* standard integer is the degree of some functional divisor of $F$. To show this, it suffices to exhibit at least one functional divisor of degree 1. Now, *every algebraic function field $F \mid K$ imbedded into $^*K$ has infinitely many prime divisors $P$ of degree* 1. This fact belongs to the fundamentals of nonstandard arithmetic and has been proved in [16] already. The argument is as follows.

Let $x_1 \in F$ be nonconstant, and let $R$ denote the integral closure of $K[x_1]$ in $F$. Then $R$ is a finitely generated $K$-algebra, say $R = K[x_1, ..., x_m]$. Let

$$f_j(x_1, ..., x_m) = 0 \qquad (1 \leqslant j \leqslant r)$$

be a system of defining relations of the $x_1, ..., x_m$ over $K$. Since these equations have a nonstandard solution—namely, $x_1, ..., x_m$—it follows from 2.4 that there are infinitely many solutions in $K$. That is, there are infinitely many $m$-tuples $a_1, ..., a_m \in K \times \cdots \times K$ such that

$$f_j(a_1, ..., a_m) = 0 \qquad (1 \leqslant j \leqslant r).$$

Every such solution defines a $K$-homomorphism $R \to K$ which maps $x_j$ onto $a_j$ $(1 \leqslant j \leqslant r)$; the kernel of this is a maximal ideal of $R$. *Thus, $R$ has infinitely many maximal ideals $M$ such that $R/M = K$.* On the other hand, we know from the general theory of function fields that $R$ is a Dedekind ring; its maximal ideals $M$ are in one-to-one correspondence to those prime divisors $P$ of $F$ whose valuation ring contains $R$ (this means that $P$ should not be among the poles of $x_1$). In this correspondence, the valuation ring of $P$ equals the quotient ring of $R$ with respect to $M$, and the residue field of $P$ is isomorphic to $R/M$. We conclude that there are infinitely many functional prime divisors $P$ of $F$ whose residue field is $K$, i.e., which have degree 1.

The converse is also true: If an abstract function field $F \mid K$ of one variable has infinitely many prime divisors of degree 1, then there is a $K$-isomorphism from $F$ into $^*K$. Again, this follows from 2.4 by reversing the above arguments; note that a nonstandard solution of the above equations is necessarily *generic over $K$*. See also Section 1.

## 6. Unramified Extensions: Elliptic and Hyperelliptic Function Fields

We conserve the notations and assumptions of the foregoing section. Let $A$ be an exceptional divisor of $F$; i.e.,

$$A = P_1 + \cdots + P_r,$$

where the $P_i$ are exceptional prime divisors, mutually distinct. The inequalities of Theorems 5.4 and 5.5 show that

$$\deg(A) \leqslant 2d \leqslant 2g + 2.$$

If $g = 0$, then it follows that $\deg(A) \leqslant 2$; we shall see in Section 8 that every possibility within these limits can indeed be realized by a suitable function field of genus 0. We now assume $g > 0$ and proceed to prove that in fact $A = 0$; i.e., there are no exceptional primes in $F$. We start from the above estimate and try to improve it such that finally $\deg(A) < 1$; then it will follow that $A = 0$. In order to obtain the desired improved estimate, we shall study *unramified extension fields of $F$ in $^*K$*; the application of Theorem 5.4 to such extension fields will lead to the desired result.

Let $E$ be an extension field of $F$ which is contained in $^*K$, so that $K \subset F \subset E \subset {}^*K$. We assume that the field degree $[E : F]$ is finite, which implies that $E$ is an algebraic function field of one variable with $K$ as its field of constants.

Lemma 6.1. *A functional prime $P$ of $F$ is exceptional if and only if every one of its extensions to $E$ is exceptional.*

*Proof.* If $P$ is exceptional, then by definition $P$ is induced by *standard* primes $\mathfrak{p} \in {}^*V$ only. Now, if $Q$ is an extension of $P$ to $E$, then every $\mathfrak{p} \in {}^*V$ which induces $Q$ on $E$ will induce $P$ on $F$. Hence, $\mathfrak{p}$ is standard, showing that $Q$ is exceptional.

Conversely, assume that every extension of $P$ to $E$ is exceptional. Every prime $\mathfrak{p} \in {}^*V$ inducing $P$ on $F$ will induce on $E$ some prime $Q$ which is an extension of $P$. Hence $\mathfrak{p}$ is standard, showing that $P$ is exceptional. Q.E.D.

Now let $Q_1, \dots, Q_s$ be those functional primes of $E$ which appear as extensions of one of the exceptional primes $P_1, \dots, P_r$ appearing in $A$. Lemma 6.1 shows that every $Q_j$ is an exceptional prime of $E$. If $A$ is considered as a divisor of $E$, then it has the form $A = e_1Q_1 + \cdots + e_sQ_s$, where $e_j$ denotes the ramification index of $Q_j$ over $F$.

Now assume that $E$ is unramified over $F$. Then $A = Q_1 + \cdots + Q_s$, and we conclude that $A$ is an exceptional divisor of $E$. (Recall that the definition of exceptional divisors implies that every component should be simple.) In other words, the divisor $A$ remains exceptional in $E$. Therefore, Theorem 5.4 applied to $E$ yields $\deg_E(A) \leqslant 2d_E$, where $d_E$ denotes the field invariant appearing in Theorem 5; i.e., $d_E$ is the minimum of the numbers $[E : K(y)]$ with $y \in E$. Moreover, $\deg_E(A)$ denotes the degree of $A$ if considered as a divisor of $E$; it is well known that

$$\deg_E(A) = [E : F] \cdot \deg_F(A).$$

That is, if a divisor of $F$ is regarded as a divisor of $E$, then its degree is multiplied by $[E : F]$. Combining these observations, we obtain the following corollary.

COROLLARY 6.2. *Assume $E$ to be unramified over $F$. Then every exceptional divisor $A$ of $F$ remains exceptional in $E$. The degree of $A$ in $F$ is estimated as*

$$\deg(A) \leqslant \frac{2d_E}{[E : F]}.$$

In particular, if $E$ is constructed such that

$$d_E < \tfrac{1}{2}[E : F], \tag{6.1}$$

then we conclude that $\deg(A) = 0$; i.e., $A = 0$.

In view of this we are now going to construct unramified extensions $E$ of $F$ in $^*K$, such that $[E : F]$ is sufficiently large. This construction will be quite explicit and elementary if $d_F = 2$, i.e., if $F$ is a quadratic extension of a rational function field. This case will be discussed in this section, whereas the general case will be found in Section 7.

Let us add one more preliminary remark, concerning constant field extensions. Let $K'$ be a finite algebraic extension field of $K$, and let $F' = FK'$ denote the corresponding constant field extension. This is a subfield of the field compositum $^*KK'$ which, as we have seen in Section 5, coincides with the enlargement $^*K'$ of $K'$. Thus we have the situation $K' \subset F' \subset {}^*K'$. The exceptional divisor $A = P_1 + \cdots + P_r$ of $F$ remains exceptional in $F'$, as has been shown in Section 5. Since $F'$ is a constant field extension of $F$, the degree of $A$ in $F'$ coincides with the degree of $A$ in $F$. Also, $F'$ has the same genus as $F$. These remarks show that, in order to prove that $F$ has no exceptional prime divisors, it is permissible to replace $F$ by a constant field extension $F'$ and to prove that $F'$ has no exceptional divisors.

This simple remark will allow us to apply a suitable constant field extension before starting with our constructions; this will simplify our discussion considerably. (However, it should be remarked that these constant field extensions are not really necessary; it is possible to use constructions which are rational over $K$.)

This being said, we now proceed with the discussion of quadratic function fields. Thus, we assume that $F$ is given as a quadratic extension of a rational subfield $K(x)$, i.e., $[F : K(x)] = 2$. There is a generator $y$ of $F$ over $K(x)$ such that $y^2 = f(x)$, where $f(x) \in K[x]$ is a polynomial without multiple roots. It is well known that the genus $g$ of $F$ is computed by means of the degree $m = \deg f(x)$ in the form

$$g = \begin{cases} \frac{1}{2}(m-1) & \text{if } m \text{ is odd,} \\ \frac{1}{2}(m-2) & \text{if } m \text{ is even.} \end{cases}$$

Our assumption $g > 0$ thus means that $m \geqslant 3$. If $m = 3$ or $4$, then $g = 1$; i.e., $F$ is elliptic. If $m \geqslant 5$, then $F$ is hyperelliptic.

After applying a suitable constant field extension we may assume that the polynomial $f(x)$ has at least two roots $a, b \in K$, so that

$$y^2 = (x-a)(y-b)\,g(x),$$

where $g(x) \in K[x]$ is of degree $m-2$. Let $P_a$ be a functional prime at which $x - a$ has a zero; then it follows from this equation that $P_a$ is a *double zero* of $x - a$, i.e., $v_{P_a}(x-a) = 2$, and that there is no other zero of $x - a$ in $F$. (In other words, $P_a$ is ramified over $K(x)$.) This shows that the principal divisor of $x - a$ has the form $[x-a] = 2P_a - X_\infty$, where $X_\infty$ is the pole divisor of $x$. Similarly, $[x-b] = 2P_b - X_\infty$, where $P_b \neq P_a$ since $a \neq b$. Hence, the element

$$z = \frac{x-a}{x-b} \tag{6.2}$$

has the principal divisor

$$[z] = 2P_a - 2P_b, \tag{6.3}$$

which is seen to be divisible by 2; i.e., $[z]$ is twice some functional divisor of $F$. As a consequence of this property of $[z]$, we now claim the following.

*There is a constant* $0 \neq c \in K$ *such that* $\sqrt{cz} \in {}^*K$.

*Proof.* According to Section 4, we may regard $P_a$ and $P_b$ as divisors in $\dot{\mathfrak{D}}$, and then we have $(z) \doteq 2P_a - 2P_b$. Since the size $\sigma$ vanishes on principal divisors, we conclude $0 \doteq \sigma(2P_a - 2P_b) \doteq 2\sigma(P_a - P_b)$. Let us recall that the sign $\doteq$ stands for the relation of equality in the group $\dot{\mathbf{R}}$; since this group is totally ordered, it does not admit any torsion and therefore

$$\sigma(P_a - P_b) \doteq 0.$$

Thus the divisor $P_a - P_b$ of $\mathfrak{D}$ has vanishing size; we conclude from Theorem 3.4 that this divisor is principal in $\mathfrak{D}$. That is, there is an element $t \in {}^*K$ such that $(t) \doteq P_a - P_b$. We have $(t^2) = 2(t) \doteq 2P_a - 2P_b \doteq (z)$ and therefore, in view of Theorem 3.4, $t^2 = cz$ with some $c \in K$. Q.E.D.

It follows from (6.2) that $z$ and, hence, $cz$ generate the same field as $x$; i.e., $K(x) = K(z) = K(cz)$. Therefore, since $t = \sqrt{cz}$, we obtain

$$K(x) \subset K(t) \text{ and } [K(t) : K(x)] = 2.$$

On the other hand, we know that $[F : K(x)] = 2$. If $t$ would be contained in $F$, then $F = K(t)$, contrary to our assumption that $F$ has genus $g > 0$. Hence, $t$ is a quadratic irrationality over $F$; if we put $E = F(t)$, then $[E : F] = 2$. Thus we have constructed a certain quadratic extension field $E$ of $F$ inside ${}^*K$. We claim that $E$ is *unramified* over $F$. In fact, $E$ is generated over $F$ by the quadratic radical $t = \sqrt{cz}$; hence, every prime of $F$ which is ramified in $E$ appears in the principal divisor $[cz] = [z]$ with an odd multiplicity. But there is no such prime; we have seen in (6.3) that every prime appearing in $[z]$ has multiplicity 2. Therefore, there is no prime ramified in $E$.

In our construction of this unramified extension field $E$, we had assumed that $F$ should be quadratic, i.e., quadratic over a rational subfield. Now, this property is inherited by the field $E$. Namely, we have $E = F(t) = K(x, y, t) = K(y, t)$ and $y^2 \in K(x) \subset K(t)$. Therefore, $[E : K(t)] = 2$; i.e., $E$ is quadratic too. Hence, this construction may be repeated and applied to $E$ instead of $F$, and so on. After $n$ steps we have the following situation.

After applying a suitable constant extension, we can construct an unramified extension $E^{(n)}$ of $F$ inside ${}^*K$ such that

$$[E^{(n)} : F] = 2^n \quad \text{and} \quad d_{E^{(n)}} = 2.$$

*If $n = 3$, we see that condition (6.1) is satisfied; hence, $F$ does not have any exceptional prime. Theorem 1.2 is proved for quadratic function fields of genus $> 0$, i.e., for elliptic and hyperelliptic fields.*

## 7. Unramified Extensions in the General Case

Now we drop our assumption that $F$ is elliptic or hyperelliptic, and we consider the case of an arbitrary function field of genus $g > 0$. Again, our aim is to construct unramified extensions $E$ of $F$ within ${}^*K$, which have large degree $[E : F]$. More precisely, the degree should be large compared with $d_E$, namely such that the inequality (6.1) holds:

$$d_E < \tfrac{1}{8}[E : F].$$

Let $n$ denote a standard natural number, fixed through the following discussion. Our constructions will be based on the following.

LEMMA 7.1. *Let $T$ be a functional divisor of $F$ such that $nT$ is principal in $F$. Then there exists $t \in {}^*K$ such that $T \doteq (t)$. This element $t$ is an nth radical over $F$; i.e., $t^n \in F$. The extension $F(t)$ is unramified over $F$.*

*Proof.* Because of the hypothesis, there exists $u \in F$ such that $nT = [u]$, the functional principal divisor of $u$. According to Section 4, we may regard $T$ as a divisor in $\dot{\mathfrak{D}}$, and then we have $nT \doteq (u)$. We apply the size map $\sigma : \dot{\mathfrak{D}} \to \dot{\mathbf{R}}$ and recall that $\sigma$ vanishes on principal divisors; it follows that $\sigma(nT) \doteq n\sigma(T) \doteq 0$. Therefore, $\sigma(T) \doteq 0$, because the group $\dot{\mathbf{R}}$ is totally ordered and hence does not admit any torsion. Now we apply Theorem 3.4, which shows that every divisor in $\dot{\mathfrak{D}}$ with vanishing size is principal. We conclude that $T \doteq (t)$ with some $t \in {}^*K$. It follows that $(u) \doteq nT \doteq (t^n)$ and therefore $t^n = cu$ with some $c \in K$. Hence $t$ is an $n$th radical over $F$. If a functional prime $P$ of $F$ does not appear in the principal divisor $[u] = [cu]$, then $P$ is unramified in $E = F(t)$. This follows immediately from the fact that the polynomial $X^n - cu$, which admits $t$ as a root, has the discriminant $\pm n(cu)^{n-1}$, whose divisor does not contain $P$. On the other hand, if $P$ does appear in $[u]$, then $v_P(u) = nv_P(T)$ is divisible by $n$; if we choose $x \in F$ such that $v_P(x) = v_P(T)$, then $P$ does not appear in the principal divisor of $u' = ux^{-n}$. If we replace $t$ by $t' = tx^{-1}$, which also generates $E$ over $F$, then $t'^n = cu'$; i.e., $t'$ is an $n$th radical of $cu'$. The above argument can be applied to $u'$ and $t'$ instead of $u$ and $t$, showing again that $P$ is not ramified in $E$. Thus, $E$ is unramified over $F$. Q.E.D.

In the following, $C_n$ denotes the so-called $n$th *division group* of $C$, consisting of those divisor classes of $C$ which are annihilated by $n$. The hypothesis of Lemma 7.1 states that $T$ should represent some class in $C_n$. Now let us consider *all* $n$th radicals $t \in {}^*K$ such that $(t) \doteq T$ with some functional divisor $T \in D$ representing a class in $C_n$. It is clear these $t \in {}^*K$ form a multiplicative group $W_n$ containing all elements $\neq 0$ of $F$. If we assign to each $t \in W_n$ the class of its corresponding divisor $T \in D$, then we obtain a homomorphism $W_n \to C_n$ which, in view of Lemma 7.1, is surjective. Clearly, the kernel of this homomorphism is the multiplicative group of $F$; hence, we have an *isomorphism* $W_n/F = C_n$. The field $F(W_n)$ is unramified over $F$, since it is generated by the unramified extensions $F(t)$ for $t \in W_n$. If $K$ contains the $n$th roots of unity, then Kummer theory shows that $F(W_n)$, being generated by $n$th radicals, is abelian of exponent $n$ over $F$. Moreover, it follows from Kummer theory that $[F(W_n) : F]$ equals the order of the radical factor group $W_n/F$; hence (since $C_n$ is finite), $[F(W_n) : F] = | C_n |$, where the right-hand side denotes the group order of $C_n$.

Still assuming $K$ to contain the $n$th roots of unity, we claim that $F(W_n)$ is the *maximal* extension of $F$ within $^*K$, which is unramified and abelian of exponent $n$. By Kummer theory, any such extension is generated by $n$th radicals; thus, we have to show that each of these radicals is contained in $W_n$. So let $t \in {}^*K$ be an $n$th radical over $F$, and assume that $F(t)$ is unramified over $F$. Let us put $t^n = u \in F$. Since $F(t)$ is unramified over $F$, it follows $v_P(u) \equiv 0 \bmod n$ for every functional prime $P$ of $F$. Hence the principal divisor $[u]$ is divisible by $n$ in the functional divisor group $D$; i.e., $[u] = nT$ with some $T \in D$. This shows first that the class of $T$ is annihilated by $n$; i.e., this class is in $C_n$. Secondly, we have $nT \doteq (u) \doteq (t^n)$ and therefore $T \doteq (t)$. Thus $t \in W_n$.

We have shown the following corollary.

COROLLARY 7.2. *Assume that the nth roots of unity are contained in K. Then all the nth radicals t of Lemma* 7.1 *generate the maximal extension of F within *K, which is unramified and abelian of exponent n. The degree of this maximal extension equals* $| C_n |$, *the order of the nth division class group of F over K.*

If $K'$ is an algebraic extension of $K$, then we denote by $F' = FK'$ the corresponding constant field extension of $F$. Also, $C'$ is the divisor class group of $F'$ and $C_n'$, its $n$th division group. The inclusion $F \subset F'$ defines a natural map $C \to C'$ which is injective (since $F'$ is a constant field extension of $F$). Hence we may regard $C$ as a subgroup of $C'$, and then we have $C_n = C \cap C_n'$. If we take for $K'$ the algebraic closure of $K$, then it is known from the general theory of algebraic function fields that $| C_n' | = n^{2g}$. It follows that $| C_n | \leqslant n^{2g}$. If $| C_n | = n^{2g}$, then we say that all the $n$th division classes of $F$ are rational over $K$ (more precisely, the $n$th division classes of all constant field extensions of $F$ are rational over $K$). It is well known that this implies the $n$th roots of unity to lie in $K$. Hence, we obtain the following.

COROLLARY 7.3. *Assume that all the nth division classes of F are rational over K. Let* $E_n$ *denote the maximal extension of F within *K, which is unramified and abelian of exponent n. Then* $[E_n : F] = n^{2g}$.

This field $E_n$ is called the *nth division field* of $F$ within $^*K$. So far, it is only defined if the $n$th division classes of $F$ are rational over $K$. If $K'$ is a finite algebraic extension of $K$ then, clearly, $E_n' = E_nK'$ is the $n$th division field of $F'$ within $^*K'$.

In order to define $E_n$ in the general case, with no assumptions about the rationality of division classes, we introduce the notion of semiabelian extensions. Let $E$ be a finite extension of $F$ within $*K$. Then $E$ is called *semiabelian of exponent n* over $F$ if there exists a finite extension $K' \supset K$ such that $E'$ is abelian of exponent $n$ over $F'$. (Here, $E' = EK'$ and $F' = FK'$.) Assume this to be the case and, in addition, that $E$ is unramified over $F$. Then $E'$ is unramified over $F'$. In view of Corollary 7.2 (applied to $F'$) we conclude

$$[E : F] = [E' : F'] \leqslant | C_n' | \leqslant n^{2g}.$$

This relation holds for every extension $E$ of $F$ with the following properties:

(i) $E$ is contained in $*K$;

(ii) $E$ over $F$ is unramified;

(iii) $E$ over $F$ is semiabelian of exponent $n$.

Each of these properties is preserved under field composita. Thus there exists a *maximal* extension of $F$ with these properties. Let $E_n$ denote this maximal extension. Then again,

$$[E_n : F] \leqslant n^{2g}. \tag{7.1}$$

If the $n$th division classes of $F$ are rational over $K$, then we conclude that this field $E_n$ coincides with the field $E_n$ of Corollary 7.3; in particular, it follows that equality holds in (7.1). We claim that this is true in any case.

THEOREM 7.4. *Let $E_n$ denote the maximal extension of $F$ within $*K$, which is unramified and semiabelian of exponent n. Then $[E_n : F] = n^{2g}$. We shall call $E_n$ the nth division field of F within $*K$.*

*Proof.* Let $K'$ denote a finite Galois extension of $K$ such that all the $n$th division classes of $F$ are rational over $K'$. Consider the constant extension $F' = FK'$, which is imbedded into $*K'$, and let $E_n'$ be the $n$th division field of $F'$ within $*K'$. We know from the above that $[E_n' : F'] = n^{2g}$, and that $E_n'$ can be characterized as being the maximal extension of $F'$ in $*K'$ which is unramified and abelian of exponent $n$. This characterization shows, in particular, that every automorphism of $*K'$ which maps $F'$ onto itself does also map $E_n'$ onto itself.

Now let $G$ be the Galois group of $K'$ over $K$. Every automorphism of $G$ has a standard extension to $*K'$, and hence $G$ appears now as the Galois group of $*K'$ over $*K$. The field $F' = FK'$ is mapped under $G$ onto itself (since this is true for each component $F$ and $K'$). Therefore, by what we have said above, $G$ maps $E_n'$ onto itself. Hence, $G$ induces in $E_n'$ a certain

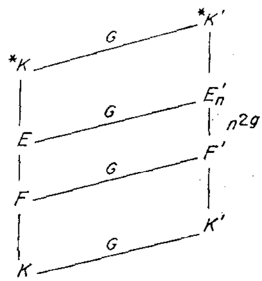


FIGURE 2

group of automorphisms. Let $E$ denote the field of fixed elements in $E_n'$. This field $E$ has the following properties (shown in Fig. 2):

(i) *$E$ is contained in* $^*K$. For, since $^*K$ is the field of fixed elements in $^*K'$, we have $E = {}^*K \cap E_n'$.

(ii) *$E$ is unramified over $F$.* For, we know that $E \subset E_n'$. But $E_n'$ is unramified over $F'$ (by definition of $E_n'$) and $F'$ is unramified over $F$ (as a constant extension). Hence $E_n'$ and therefore $E$ too is unramified over $F$.

(iii) $EK' = E_n'$. This follows from Galois theory since every nontrivial automorphism of $G$ moves the elements of $K'$ and hence those of $EK'$. Now, since $E_n'$ is abelian and of exponent $n$ over $F'$, it follows that *$E'$ over $F'$ is semiabelian and of exponent $n$.*

From (i)–(iii) it follows that $E \subset E_n$. Hence, from (7.1),

$$[E : F] \leqslant [E_n : F] \leqslant n^{2g}.$$

On the other hand, we infer from (iii) that

$$[E : F] = [EK' : FK'] = [E_n' : F'] = n^{2g}.$$

Thus we conclude that $E = E_n$ and $[E_n : F] = n^{2g}$. Q.E.D.

Theorem 7.4 shows the existence of unramified extensions $E_n$ of $F$ within $^*K$, of arbitrarily large degree. Now we are interested in the degree invariants

$$d_n = d_{E_n} = \min_{\substack{y \in E_n \\ y \notin K}} [E_n : K(y)]$$

of these fields. As above, we write $d = d_F$.

LEMMA 7.5. *There exists a constant $\gamma$, depending on the genus of $F$ only, such that $d_n \leqslant \gamma d n^{2g-2}$.*

A proof of Lemma 7.2 has been given by Siegel [22], with $\gamma = g^3$. Siegel used the analytic theory of theta functions in his proof. There is also an algebraic proof available, using Deuring's theory of correspondences of algebraic function fields and the inequality of Castelnuovo-Severi [18]. Since this proof has nothing to do with nonstandard methods, we have preferred to exclude it from the present paper and to publish it separately [20].

Putting Theorem 7.4 and Lemma 7.5 together, we obtain

$$d_n \leqslant \frac{\gamma d}{n^2} [E_n : F].$$

If $n$ is sufficiently large, we conclude that inequality (6.1) holds:

$$d_n \leqslant \tfrac{1}{2}[E_n : F].$$

As was explained in Section 6, this shows that there are no exceptional primes of $F$. Theorem 1.2 is proved.

## 8. The Case $g = 0$

Although our main interest in this paper is concerned with function fields of higher genus, let us briefly review our results obtained in the case of genus zero. Thus, in this section let us assume that $g = 0$; in view of Remark 5.6 this implies that $F = K(t)$ is a rational function field.

Let $P_1, \ldots, P_r$ denote the exceptional prime divisors of $F$. Our aim is to describe these prime divisors, as well as the ring $R$ of exceptional elements of $F$. Here an element $x \in F$ is said to be exceptional if its functional pole divisor is composed of exceptional primes only; this is equivalent to saying that, if $x$ is regarded as an element of $^*K$, then its denominator is not divisible by any nonstandard prime. (That is, Theorem 1.1 fails to hold for $x$.) It is clear that the exceptional elements form a subring $R$ of $F$, containing $K$.

According to Theorem 5.4, the divisor $A = P_1 + P_2 + \cdots + P_r$ has degree $\leqslant 2$. Hence, there are only the following four cases possible.

*Case* 0: $\deg(A) = 0$. In this case, there are no exceptional prime divisors, and $R = K$.

*Case* 1: $\deg(A) = 1$. We have $r = 1$. The only exceptional prime divisor $P$ of $F$ has degree 1. We can choose the generator $t$ of $F \mid K$ such that $P$ is the only pole of $t$, and of order 1. If an element $x \in F = K(t)$ has no pole except $P$, then $x$ is a polynomial in $K[t]$, and conversely. Hence, $R = K[t]$.

*Case* 2: $\deg(A) = 2$ and $r = 2$. There are two exceptional prime divisors $P_1$ and $P_2$, each of degree 1. We can choose the generator $t$ of $F \mid K$ such that $P_1$ is the pole and $P_2$ the zero of $t$, both of order 1. If an element $x \in F = K(t)$ has no pole except $P_1$ and $P_2$, then $x \in K[t, t^{-1}]$, and conversely. Hence $R = K[t, t^{-1}]$.

*Case* 2a: $\deg(A) = 2$ and $r = 1$. There is only one exceptional prime divisor $P$; it has degree 2. If $t$ is any generator of $F \mid K$, then $P$ is the zero of some quadratic irreducible polynomial $\varphi(t) \in K[t]$. After a suitable linear transformation of $t$, we may assume $\varphi(t)$ of the form $\varphi(t) = t^2 - a$, where $a \in K$, but $\sqrt{a} \notin K$. If an element $x \in F = K(t)$ has no pole except $P$, then $x$ is of the form $x = \sum_{0 \leqslant i \leqslant n} h_i \varphi^{-i}$, where the $h_i \in K[t]$ are *linear* polynomials. This shows that $x$ is representable as a polynomial in $\varphi^{-1}$ and $t\varphi^{-1}$. Hence $R = K[\varphi^{-1}, t\varphi^{-1}]$.

Case 2a reduces to Case 2 over the quadratic extension $K' = K(\sqrt{a})$. For, the prime $P$ splits in $F' = FK'$ into two functional primes $P_1'$ and $P_2'$, each of degree 1, which are the pole and the zero of the generator

$$t' = \frac{t + \sqrt{a}}{t - \sqrt{a}}$$

of $F' \mid K'$. Therefore, $R$ is contained in the ring $R' = K'[t', t'^{-1}]$, and $R = F \cap R'$. Let

$$\tau\colon \sqrt{a} \mapsto -\sqrt{a}$$

denote the nontrivial automorphism of $K' \mid K$. Extended to an automorphism of $F' \mid F$, we see that $\tau$ sends $t'$ into $t'^{-1}$. Also, $R$ is the ring of fixed elements under $\tau$. This gives the description

$$R = \mathrm{Fix}^{(\tau)}\, K'[t', t'^{-1}],$$

where

$$K' = K(\sqrt{a}) \qquad \text{and} \qquad \tau\colon \begin{Bmatrix} \sqrt{a} \mapsto -\sqrt{a} \\ t' \mapsto t'^{-1} \end{Bmatrix}.$$

The foregoing discussion gives a *classification* of the subfields $F \subset {}^*K$ of genus zero, with respect to their exceptional behavior. It is complemented by the following *existence statement*: Each of the four cases 0–2a is realized by some subfield $F \subset {}^*K$ of genus zero. In Case 2a we may prescribe the quadratic irrationality $\sqrt{a}$ entering into its description.

To prove this, one has to exhibit nonstandard elements $t_i \in {}^*K$ such that the function field $F_i = K(t_i)$ satisfies case $i$. We shall only state the mode of construction of these elements, leaving to the reader the straightforward verification that the conditions in the respective cases are indeed satisfied.

First, let us deal with Case 2. Let $u \in K$ be any element which is neither zero nor a root of unity, so that $u^n \neq u^m$ for all $n, m \in \mathbf{N}$. Then we put $t_2 = u^z$, where $z$ is an *infinitely large* natural number in $^*\mathbf{N}$. It is easily seen that $t_2$ is nonstandard and that $F_2 = K(t_2)$ satisfies Case 2, the exceptional prime divisors being the pole and the zero of $t_2$. Now we put

$$t_1 = t_2 + t_2^{-1},$$
$$t_0 = t_1 + t_1^{-1}.$$

The fields $F_1 = K(t_1)$ and $F_0 = K(t_0)$ then satisfy Case 1 and Case 0, respectively.

Finally, in case 2a we work over the quadratic extension $K' = K(\sqrt{a})$. We first put $t_2' = u'^z$, the same construction as $t_2$ above, but now we require in addition that $\tau u' = u'^{-1}$, $\tau$ being the nontrivial automorphism of $K' \mid K$. Such an element $u' \in K'$ can be found, for instance, in the form

$$u' = \frac{u + \sqrt{a}}{u - \sqrt{a}},$$

where $u \in K$ is chosen such that $u'$ is not a root of unity. $\tau$ extends to a standard automorphism of $^*K'$ over $^*K$, and we have $\tau t_2' = t_2'^{-1}$.

Finally, we put

$$t_{2a} = \sqrt{a}\,\frac{t_2' + 1}{t_2' - 1}.$$

Then $\tau t_{2a} = t_{2a}$; i.e., $t_{2a} \in {}^*K$. It is readily verified that the field $F_{2a} = K(t_{2a})$ satisfies Case 2a, the exceptional prime of degree 2 being the zero of $\varphi = t_{2a}^2 - a$.

The above results can be used to obtain Siegel's classification of those curves of genus zero, which are exceptional in the sense that the Siegel–Mahler theorem does not hold for them. Namely, let $\Gamma : f(x, y) = 0$ be such a curve over $K$; there are infinitely many $K$-rational points on $\Gamma$ whose denominatorial prime divisors all belong to some finite set $\mathfrak{S}$. As explained in Section 1, we then have a *nonstandard* point $(x, y)$ on $\Gamma$ whose denominatorial primes belong to $\mathfrak{S}$, hence are standard. That is, the elements $x, y$ are exceptional in the function field $F = K(x, y) \subset {}^*K$. According to the above classification, we therefore have one of the following *birational parametrizations* of our curve $\Gamma$.

*Case* 1. $x = \Phi(t)$, $y = \Psi(t)$, where $\Phi$, $\Psi$ are polynomials over $K$.

*Case* 2. $x = \Phi(t)$, $y = \Psi(t)$, where $\Phi$, $\Psi$ are finite Laurent series over $K$.

*Case* 2a. $x = \Phi(t')$, $y = \Psi(t')$, where $\Phi$, $\Psi$ are finite Laurent series over the field $K' = K(\sqrt{a})$, satisfying the condition $\Phi(t') = \Phi^{\tau}(t'^{-1})$ and similarly for $\Psi$. Here, $\tau$ denotes the nontrivial automorphism of $K' \mid K$, and $\Phi^{\tau}$ is obtained from $\Phi$ by applying $\tau$ to its coefficients. These statements do not refer any more to an embedding of the function field of $\Gamma$ into $^*K$. They are geometric in nature, showing how the points on $\Gamma$ are parametrized by one parameter $t$.

These parametrizations are not only necessary, but also sufficient for $\Gamma$ to be exceptional. Namely, the substitutions $t = t_1$ (resp. $t = t_2$ resp. $t' = t_2'$) yield a nonstandard point $(x, y)$ of $\Gamma$ such that $x$ and $y$ are exceptional in $^*K$. Let $\mathfrak{S}$ denote the set of those internal primes of $^*K$ which appear in the denominator of $x$ or of $y$. We know that $\mathfrak{S}$ is internal and does not contain any nonstandard prime; hence, $\mathfrak{S}$ is a *finite* set of standard primes. Since there is a nonstandard point whose denominatorial primes all belong to $\mathfrak{S}$, there are infinitely many standard points with this property. Hence $\Gamma$ is exceptional.

## EPILOGUE[7]

In the summer of 1973, Abraham Robinson visited Heidelberg where he gave a lecture on algebraic function fields and nonstandard arithmetic [16, 17]. He expounded his ideas on embedding function fields into nonstandard models of their ground fields, or into finite extensions of such models. If the ground field is an algebraic number field, then this yields a representation of algebraic functions by (nonstandard) algebraic numbers. In this way it should be possible to explain the arithmetic structure of function fields directly, using well-established nonstandard principles only, by means of the arithmetic structure carried by the ground field.

In my opinion, these ideas of Abraham Robinson are of far-reaching importance, providing us with a new viewpoint and guideline towards our understanding of diophantine problems. It seems wothwhile to put these ideas to a test in order to verify their usefulness and applicability in connection with explicit diophantine problems. Perhaps a good test in this sense would be the explanation, in nonstandard terms, of Weil's theory of distributions and, closely connected with it, the theorems of Mordell and Weil and of Siegel and Mahler. The preceding paper is meant to be

[7] By Peter Roquette.

just such a test and, as we believe, a successful one. Although we deal explicitly with the Siegel–Mahler theorem only, it will be clear to anyone familiar with the subject that Weil's theory of distributions also can be explained in this context.

The paper was written as a result of several weeks of close collaboration with Abraham Robinson at Yale. He completed the first draft by his own hand in November 1973. His severe illness and tragic death prevented him from participating in the discussion of the following versions. Hence, although I want to make it clear that the basic ideas are Robinson's (compare also [16] and [17]), I have to take full responsibility for the form of presentation of the subject. In particular, this refers to Sections 2 and 3, which are of introductory nature. We had in mind to provide an introduction, however short, for those readers who are not acquainted with nonstandard methods but want to understand the basic ideas of our proof. Therefore, these two sections have been added. I hope they serve their purpose: to interest number theorists in nonstandard arithmetic. (By the way, the use of enlargements is not really necessary; the whole proof can be carried out in an ordinary nonstandard model.)

In Robinson's first draft, there was also a section on effectiveness. There, it was pointed out that his methods yield a "relative" effective procedure, relative to the bounds provided by Roth's theorem. This section has been excluded from the present paper; it is planned to publish it separately under the name of Abraham Robinson.

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