Coxeter groups with (FA)

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Abstract

Recall that a group has property (FA) when for any action (without inversion) on a tree there is a global fixed point. Last time we have discussed some criteria for a group to be (FA); in particular, if the group is denumerable, it cannot be an amalgam. Today we look at groups with a *presentation*.

Definition. Let S be a set, $\mathcal{F}(S)$ the free group over S, and $R = (r_j)_{j \in J}$ a family of words in $\mathcal{F}(S)$. A presentation is then defined by $\langle S \mid R \rangle \coloneqq \mathcal{F}(S) / \langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the smallest normal subgroup containing R.

It is known that every group G is isomorphic to a group $\langle S \mid R \rangle$. Our goal for today is to derive from a group's presentation if it has property (FA). In particular, we will show that every *2-spherical* Coxeter group – sometimes referred to as *abstract reflection group* – has property (FA). To achieve this, we first look at *automorphisms of a tree* and their fixed points.

References

[Serre] J. Serre. *Trees*, 1977.

[Thomas] A. Thomas. *Geometric and Topological Aspects of Coxeter Groups and Buildings*, 2018.

Contents

1 Fixed points of an automorphism of a tree

We formulate some needed terms for the study of automorphisms.

1.1 Distance and subtrees

Recall that a **geodesic** is a path c without backtracking. We know that if X is a tree, its geodesics are *unique* and *injective*.

Let $P-Q$ denote the geodesic joining two vertices P and Q of a tree, where P and Q lie in subtrees T_1 resp. T_2.

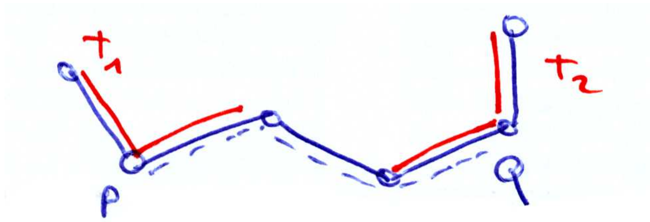

Figure 1: A tree X with subtrees T_1, T_2

Definition. The **distance** between two subgraphs is defined as

$$\mathrm{dist}(T_1, T_2) = \inf_{P \in T_1, Q \in T_2} \ell(P, q)$$

where ℓ is the length of the geodesic joining P and Q.

Remark. If there is no path (that is, T_1 and T_2 lie in different connected components), we set the distance to ∞. Since trees are connected, this case does not apply to us. If the distance is 0, T_1 and T_2 meet at some vertex.

If X is a tree, we can make the following statements about distance.

Lemma 1. *Let T_1 and T_2 be disjoint subtrees of a tree X, with* $\mathrm{dist}(T_1, T_2) = n$.

1. *There is a unique pair (P_1, P_n) in $vert(T_1) \times vert(T_2)$ such that*

$$\ell(P_1, P_n) = n.$$

2. *If $Q_1 \in vert(T_1)$ and $Q_2 \in vert(T_2)$, then*

$$\ell(Q_1, Q_2) = \ell(Q_1, P_1) + n + \ell(P_n, Q_2). \tag{1}$$

3. *Every subtree T of X with $T \cap T_1 \neq \emptyset$ and $T \cap T_2 \neq \emptyset$ contains the geodesic $P_1 - P_n$ (with P_1, P_n as in 1.)*

We say that $P_1 - P_n$ is the geodesic **joining** T_1 and T_2.

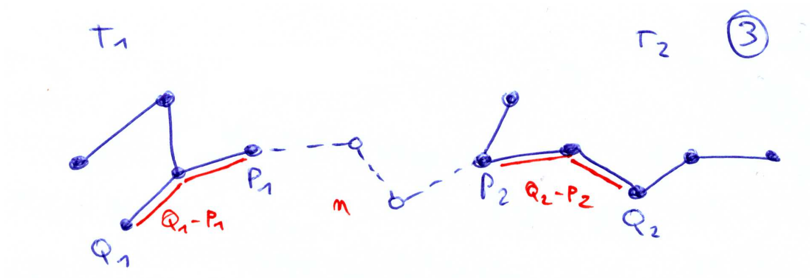

Figure 2: Juxtaposing of geodesics

Proof. Let $P_1 \in \mathrm{vert}(T_1)$ and $P_n \in \mathrm{vert}(T_2)$ such that $\ell(P_1, P_n) = n$. (Such a pair exists by definition, but it is not unique a-priori.) We first prove the formula (1).

2. Consider the (unique) geodesic $P_1 - P_n$ in X. The inner vertices $P_2, \ldots, P_{n-1}$ do not belong to either T_1 or T_2; otherwise, the geodesic would not have minimal length.

   Let now $Q_1 \in T_1$, $Q_2 \in T_2$ with geodesics $Q_1 - P_1$ resp. $Q_2 - P_n$. By the above, the juxtaposed path

   $$Q_1 - Q_2 \coloneqq (Q_1 - P_1, P_1 - P_n, P_n - Q_2)$$

   is without backtracking, which implies (1).

1. Let (Q_1, Q_2) be a pair of vertices such that $\ell(Q_1, Q_2) = n$. By (1),

   $$\ell(Q_1, P_1) = \ell(P_n, Q_2) = 0,$$

   that is, $Q_1 = P_1$ resp. $P_2 = Q_2$.

3. Let T be a subtree meeting both T_1 and T_2. Let $Q_1 \in T_1$ and $Q_2 \in T_2$, and let $\gamma \coloneqq Q_1 - Q_2$ be the unique geodesic joining Q_1 and Q_2 in X; by 2. this geodesic contains $P_1 - P_n$.

   Let γ_T be the geodesic joining Q_1 and Q_2 in the subtree T. As X contains no circuits, the paths γ and γ_T must coincide, and the claim follows.

□

1.2 Automorphisms with fixed points

Last time, we have formulated fixed points for an action $G \curvearrowright \mathrm{Mor}(X)$. In this auxiliary section, it suffices to look at general automorphisms in X, without necessary involvement of a group action.

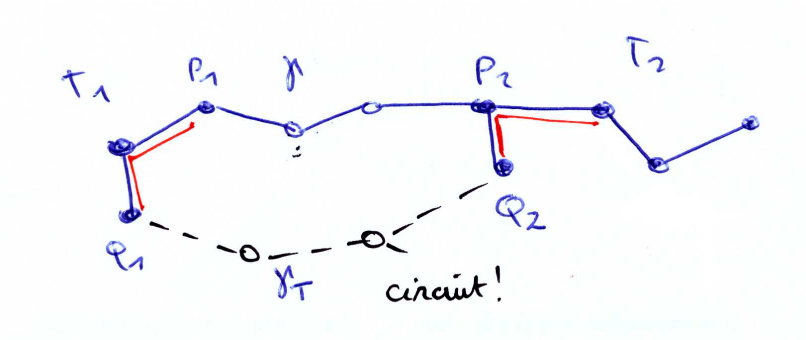

Figure 3: A tree X may contain no circuits

Definition. Let s be an automorphism *without inversion* of a tree X, that is $sy \neq \overline{y}$ for all $y \in \text{edge}(X)$. We say that s has a **fixed point** if the subgraph $X^s \subset X$ of points fixed by s is non-empty, in which case it is a tree. [As before for X^G, if s fixes two points in a tree, it must fix the geodesic joining them, implying connectivity.]

If an automorphism s has *some* fixed point, the following proposition will allow us to construct a fixed point, by considering the geodesic joining any point P in the tree with its image under s.

Proposition 2. *Let s be an automorphism without inversion of a tree X. Suppose s has a fixed point. Let $P \in vert(X)$, and let* $\text{dist}(P, X^s) = n$. *Let $P - P'$ be the geodesic joining P to X^s. (see Lemma 1.)*

Then the geodesic $P - sP$ is obtained by juxtaposing the geodesics $P - P'$ and $P' - sP = s(P' - P)$.

Remark. The last equality $P' - sP = s(P' - P)$ holds by definition of a graph morphism.

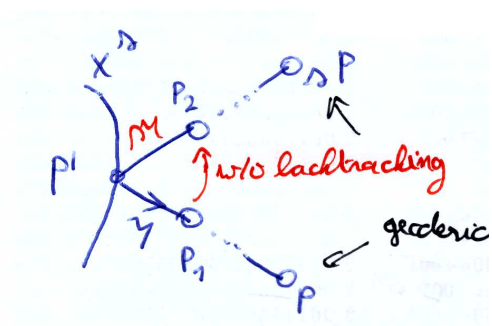

Figure 4: The geodesic $P - sP$

Proof. If $P \in X^s$, then $sP = P$ and the statement is clear. Let therefore $P \notin X^s$, that is, $\operatorname{dist}(P, X^s) \geq 1$.

Let $y \notin X^s$ denote the edge of the geodesic $P - P'$ with origin $P' \in X^s$. There holds $sy \neq y$; otherwise, $y \in X^s$. It follows that the path obtained by juxtaposing $P' - sP$ and $P - P'$ is without backtracking; by uniqueness, this path is the geodesic $s - sP$. □

Corollary. $\ell(P, sP) = 2n$.

Proof. By Proposition 2, $\ell(P, sP) = \ell(P - P') + \underbrace{\ell(P' - sP)}_{=n\,(s \text{ aut.})}$. □

Corollary. *The midpoint of $P - sP$ is fixed by s.*

Proof. By Proposition 2, the midpoint of $P - sP$ is given by $P' \in X^s$. □

2 Groups with fixed points

We now have all required tools to derive property (FA) from a group presentation. We formulate our main result:

Proposition 3. *Consider the group $G = \langle a_i, b_j \rangle$ with subgroups $A = \langle (a_i)_{i \in I} \rangle$, $B = \langle (b_j)_{j \in J} \rangle$. Assume that G acts on a tree X so that $X^A \neq \emptyset$, $X^B \neq \emptyset$, and so that for all pairs (i, j) the automorphism has $a_i b_j$ has a fixed point.*

Then $X^G \neq \emptyset$, that is, G has a fixed point.

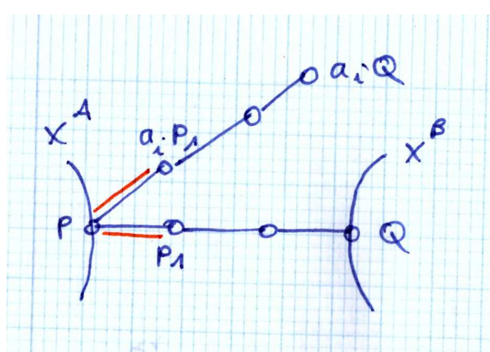

Figure 5:

Proof. By definition, $X^G = X^A \cap X^B$. Assume that X^A and X^B are disjoint, and let $P - Q$ denote the geodesic joining them, with $P \in X^A$, $Q \in X^B$. (cf. Lemma 1).

Let P_1 be the vertex of $P - Q$ at distance 1 from P. We have $P_1 \notin X^A$; it follows there is an index i such that $a_i P_1 \neq P_1$. The path obtained by juxtaposing is therefore without backtracking; it is the geodesic $Q - a_i Q$ with midpoint P.

As $Q \in X^B$, we have $b_j Q = Q$ for all j, in particular $a_i Q = a_i b_j Q$. Since $a_i b_j$ has a fixed point (by assumption), Proposition 2 shows that the midpoint $P \in X^A$ of $Q - a_i b_j Q$ is fixed by $a_i b_j$, that is $a_i b_j P = P$ resp. $b_j P = a_i^{-1} P = P$. This shows that P is fixed by all b_j; a contradiction. □

We have the following simple consequence:

Corollary. *Let a, b and c be three automorphisms of a tree such that $abc = 1$. If a, b, c have fixed points, then they have a common fixed point.*

Proof. Let $P \in X^c$. Then $ab \underbrace{cP}_{=P} = P = abP$ by assumption; it follows that ab has a fixed point. By Proposition 3, a, b, ab have a common fixed point Q. As $\cancel{ab}cQ = Q = \cancel{ab}Q$, this point is also fixed by c. □

Proposition 2 is somewhat unwieldy when considering group presentations $\langle S \mid R \rangle$. We therefore formulate:

Corollary. *Suppose that $G = \langle s_1, \ldots, s_n \rangle$ is a finitely generated group, and that s_j and $s_i s_j$ have fixed pionts for all $i, j \in \{1, \ldots, m\}$, $i \neq j$. Then G has a fixed point.*

Proof. Proceed by induction on m, applying Proposition 2. □

3 Coxeter groups

Definition. Let I be a **finite** index set. A **Coxeter group** is a group W with presentation

$$W \cong \left\langle S \mid s_i^2 = 1 \ \forall i \in I, \ (s_i s_j)^{m_{ij}} = 1 \ \forall i \neq j \right\rangle$$

where $m_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ and $m_{ij} = m_{ji}$ for all $i, j \in I$. S is called the **Coxeter generating set**. $m_{ij} = \infty$ denotes that $s_i s_j$ is of infinite order.

Remark. It is a priori not clear what the order of elements $s_i s_j$, $m_{ij} \neq \infty$ is; all we know that the order divides m_{ij}. We may prove that the presentation is "honest" using a faithful linear representation

$$\rho : W \to GL_n(\mathbb{R}),$$

the **Tits representation** (by Jacques Tits).

Example. The dihedral groups D_{2m} (isometry groups of a regular polygon) give a simple class of examples for Coxeter groups. For $m = 3$, we have the representation:

$$D_6 = \left\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 \right\rangle.$$

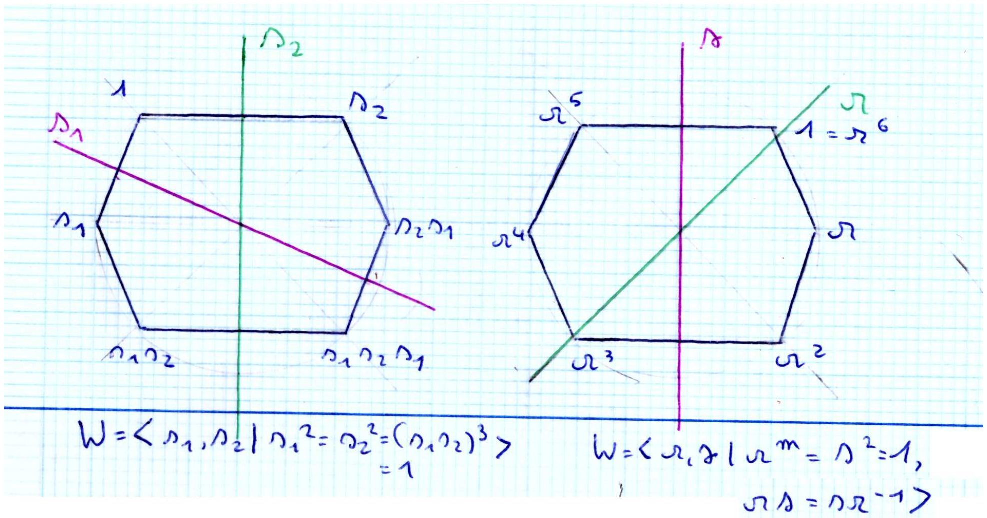

Figure 6: Presentations of the dihedral group D_6.

By replacing an axis of reflection with an axis of rotation, we get:

$$G = \left\langle r, s \mid r^m = s^2 = 1,\ rs = sr^{-1} \right\rangle$$

While the group G is isomorphic to D_6, it is no longer a Coxeter group by the definition above.

Remark. More generally, different Coxeter generating sets may generate the same group W (up to isomorphism). For example, D_{12} has the presentations:

$$\begin{aligned} W &= \left\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^6 = 1 \right\rangle, \\ &= \left\langle t_1, t_2, t_3 \mid t_1^2 = t_2^2 = (t_1 t_2)^3 = (t_2 t_3)^2 = (t_3 t_1)^2 = 1 \right\rangle. \end{aligned}$$

with **Coxeter diagrams** $\bullet \overset{6}{\text{———}} \bullet$ resp. $\bullet \text{———} \bullet \qquad \bullet$. (The fact that these graphs are not isomorphic leads to the notion of *rigidity*.)

Definition. A Coxeter group (W, S) is **2-spherical** if $m_{ij} \neq \infty$ for each (i, j).

We now formulate our final proposition.

Proposition 4. *Let (W, S) be a 2-spherical Coxeter group. Then W has property (FA).*

Proof. By definition, each generator and pair $s_i s_j$ are of finite order, and thus have a fixed point when acting on a tree. By Proposition 3, W has property (FA). $\square$