Amalgams and fixed points

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References

[Serre74] J.P. Serre. "Amalgames et points fixes". *Proceedings of the Second International Conference on the Theory of Groups*, 1974.

[Serre77] J.P. Serre. *Trees*, 1974.

Contents

1 Amalgam

We recall the definition of an amalgamated product of two groups.

Definition. Let A, G_1 and G_2 be groups with $G_1 \neq A \neq G_2$. Let $i_1 : A \hookrightarrow G_1$, $i_2 : A \hookrightarrow G_2$ be *injective* homomorphisms (we thus identify A with its image under these morphisms). We have the following diagram in **Grp**:

$$\begin{tikzcd} & G_2 \\ G_1 & A \arrow[l, "i_1"'] \arrow[u, "i_2"'] \end{tikzcd}$$

Taking the colimit (direct limit) results in the pushout

$$\begin{tikzcd} H & & \\ & G_1 *_A G_2 \arrow[ul, dashed, "\exists!"] & G_2 \arrow[l] \arrow[ull, bend right] \\ & G_1 \arrow[u] \arrow[uul, bend left] & A \arrow[l] \arrow[u] \end{tikzcd}$$

which we denote as the **amalgam** $G_1 *_A G_2$ of G_1 and G_2 over A.

Remark. We may write G as the quotient of the free product $G_1 * G_2$ (the coproduct in **Grp**) with the normal subgroup generated by the relations $i_1(a)i_2(a)^{-1}$.

We wish to address the following:

Question. *When is a group G **not** an amalgam?*

2 The fixed point property for groups acting on trees

Let G be a group which acts on a tree X. As before, we assume that this groups acts **without inversion**, that is

$$gy \neq \overline{y} \qquad \forall g \in G,\ y \in \text{edge}(X).$$

This allows to again define $G\backslash X$ in an obvious way. ($x \sim y$ if x and y are in the same orbit Gx).

Claim. Let X^G be the set of points fixed by the action $G \curvearrowright X$. If X^G is non-empty, it is a subtree of X.

Proof. If P and Q are fixed by G, the geodesic $P-Q$ joining these points (in X) is also fixed and therefore contained in X^G. The graph X^G is therefore connected; since any connected subgraph of a tree is a tree, the claim follows. $\square$

Definition. We say that a group G has **property (FA)**[1] if the following holds:

$$X^G \neq \emptyset \text{ for **any** tree on which } G \text{ acts.}$$

Under certain conditions, this is equivalent to the group G *not* being an amalgam. More precisely, the following holds:

Theorem 2.1. *Let G be a denumerable group. G has property (FA) if and only the following conditions are satisfied:*

1. *G is not an amalgam.*
2. *G has no quotient isomorphic to* $\mathbb{Z}$.
3. *G is finitely generated.*

Remark. G is denumerable (as a set) if and only it as a denumerable generating set. Therefore every finitely generated group is denumerable, but not vice-versa.

Corollary 2.1. $SL_2(\mathbb{Z})$ *does not have property (FA).*

Proof. The claim follows since $SL_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and isomorphic to the amalgam $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$. $\square$

Proof of Theorem 2.1, "⇒". We first show that *if G satisfies property (FA)*, that the three given conditions are satisfied.

1. Assume $G = G_1 *_A G_2$ with $G_1 \neq A \neq G_2$, and that G has a fixed point for any tree X upon which it acts.

   By talk 4, there is a (unique up to isomorphism) tree X upon which G acts, with fundamental domain a segment PQ, where $G_P = G_1$ and $G_P = G_2$ as stabilisers. By Remark A.2, any stabilisers of vertices in X are conjugates of either G_P or G_Q. By assumption, $G \neq G_P$ and $G \neq G_Q$; the same must hold for these conjugates. Contradiction.

2. Assume there exists $H \trianglelefteq G$ with

   $$G/H \cong \mathbb{Z} = \langle 1 \rangle .$$

   Then G/H acts freely on the Cayley graph $\Gamma(\mathbb{Z}, \{1\})$, a doubly infinite chain, by left multiplication, i.e. it has no fixed points.

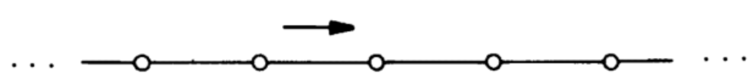

Figure 1: Action on a double infinite chain

[1]Presumably, "F" stands for "fixe" and "A" for "arbre". The property was originally introduced under the name *"La propriété de point fixe pour les groupes opérant sur les arbres"* in [Serre74].

3. Since G is denumerable, it is the union of an increasing sequence

$$G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq \cdots$$

of finitely generated subgroups.

Definition. Let X be a graph with vertex set:

$$\operatorname{vert}(X) = \bigsqcup_{G_n < G} {}^{G}\!/\!_{G_n}$$

with two vertices P, Q joined by an edge iff $P \in {}^{G}\!/\!_{G_n}$ and $Q \in {}^{G}\!/\!_{G_{n+1}}$ correspond under the canonical homomorphism:

$$\begin{aligned} \pi : {}^{G}\!/\!_{G_n} &\longrightarrow {}^{G}\!/\!_{G_{n+1}} \\ gG_n &\longmapsto gG_{n+1} \end{aligned}$$

Claim. X is a tree.

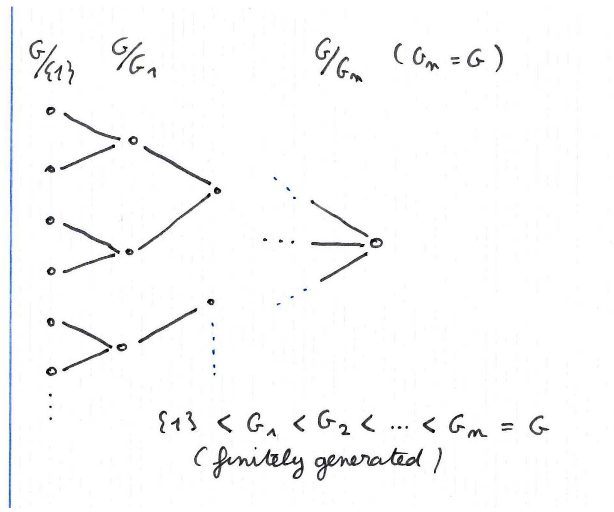

Figure 2: The tree X

Proof. Since the cosets in ${}^{G}\!/\!_{G_n}$ form a partition of G, and since $G_n \subseteq G_{n+1}$ for each $n \in \mathbb{N}$, π is a well defined and surjective map. Therefore X has no circuits and is connected, respectively. $\square$

Let G act on X via:

$$g.(hG_n) = (gh)G_n$$

Note that g fixes hG_n if $g \in hG_nh^{-1}$. If G has property (FA), there is a vertex P of X invariant under the action of G. Let $P \in {}^{G}\!/\!_{G_n}$, that is

$$P = hG_n \quad \text{for some } h \in G.$$

For any other coset gG_n we have:

$$gG_n = g.(h^{-1}h)G_n = (gh^{-1}).hG_n = hG_n \qquad \forall g \in G.$$

where the last equality holds because $gh^{-1} \in G$. Therefore $G = G_n$, i.e. G is finitely generated.

Remark 2.1. Let G be a denumerable group which is not necessarily (FA). Let G acts on the tree X as above.

- For each $g \in G$ there exists some vertex P such that $gP = P$:

  Since G is denumerable, it is the countable union of finitely generated sets G_n. For each $g \in G$, this implies there is an $n \in \mathbb{N}$ such that $g \in G_n$. Then g fixes G_n (as a coset!).

- In general, there is no P fixed by the whole G! (Trivial case: G is finitely generated, then ${}^{G}\!/_{G} = \{G\}$ is a vertex, which is fixed by all $g \in G$)

□

We now wish to prove the opposite implication of the theorem. We will require the following corollary to the *Structure Theorem*:

Corollary 2.2. *Let G be a group which acts without inversion on a tree X. Let R be the normal subgroup of G generated by the G_P, $P \in vert(X)$. Then ${}^{G}\!/_{R}$ can be identified with the fundamental group of the graph $Y = G\backslash X$.*

Proof.

- Since X is a tree, by the structure theorem the map $\pi_1(G, Y, T) \to G$ is an isomorphism, where $Y = G\backslash X$ and T is a maximal tree of Y.
- By the remark to [Serre77, 5.1], the quotient ${}^{\pi_1(G,Y,T)}\!/_{R}$ is the fundamental group $\pi_1(Y, T)$ of the graph Y. Since T is simply connected, this is isomorphic to $\pi_1(Y, P_0)$ for some $P_0 \in T$.

It follows that ${}^{G}\!/_{R}$ is isomorphic to $\pi_1(Y, P_0) \cong \pi_1(Y)$.

□

Remark. R is a normal subgroup since it contains all conjugates of G_P, cf. Remark A.2.

Proof of Theorem 2.1, "⇐". Suppose that G has properties 1, 2 and 3 and acts on some tree X. We wish to find a fixed point of X by the action of G.

Let $T = G\backslash X$. By the above corollary, $\pi_1(T) \cong {}^{G}\!/_{R}$. Since $\pi_1(T)$ is a free group by [Serre77, 5.1], but G has no quotient isomorphic to $\mathbb{Z}$ by hypothesis 2, $\pi_1(T) = \{1\}$ must hold. Therefore, T is a tree. By [Serre77, 3.1] we can lift T to a subtree of X.

$$\begin{tikzcd} & \tilde{T} \subset X \arrow[d] \\ T = {}^{G}\!/_{X} \arrow[ur] \arrow[r, "\mathrm{id}"'] & T \end{tikzcd}$$

This subtree $\tilde{T}$ gives us the fundamental domain required to apply [Serre77, 4.5, Theorem 10]. , so that

$$G \cong G_T = \varinjlim(G, T),$$

limit of the tree of groups defined by the groups G_P and G_y, fixing $P \in \mathrm{vert}(T)$ and $y \in \mathrm{edge}(T)$, respectively. By repeatedly adjoining a vertex P and geometric edge $\{y, \overline{y}\}$, we have

$$G = \bigcup_{T' \underset{\text{finite}}{\subset} T} \varinjlim(G, T')$$

the union of the groups $G_{T'} = \varinjlim(G, T')$, with T' finite. Since G is finitely generated, there is a T' such that $G = G_{T'}$; choose a minimal T' with this property.

Case 1. If T' reduces to a single vertex P, we have $G = G_P$ and G has a fixed point.

Case 2. If not, T' has a terminal vertex P, and

$$T'' = T' - \{P\}$$

is a tree by Proposition A.1. If y denotes the unique edge which joins P to T'', there holds: (compare Remark A.3)

$$G = G_{T'} = G_{T''} *_A G_P, \qquad \text{where } A = G_y.$$

Since T' was minimal we have $G_{T''} \neq G$ and $G_P \neq G$. This implies G is an amalgam, which is absurd by hypothesis 1.

□

3 Examples of property (FA)

Proposition 3.1. *Let G be a group with property (FA). If G is a subgroup of an amalgam $G_1 *_A G_2$, then G is a subgroup of a conjugate of G_1 or of G_2.*

Proof. Let $G < G_1 *_A G_2$. By talk 4, $G_1 *_A G_2$ acts on (unique up to isomorphism) tree X with fundamental domain $T \subset X$, and stabilisers:

$$\underset{P}{\bullet} \longrightarrow \underset{Q}{\bullet}, \quad G_P = G_1, \quad G_Q = G_2.$$

Since G has property (FA), the action $G \curvearrowright X$ has a fixed point. Denote wlog. this fixed point as $P_0 := gP$, where $g \in G_1 *_A G_2$. Let $h \in G$ be arbitrary. There holds:

$$hP_0 = hgP = gP \quad \Leftrightarrow \quad g^{-1}hgP = P.$$

That is, $g^{-1}hg \in G_1$ resp. $h \in gG_1g^{-1}$. ◻

Example 3.1. A *finitely generated* **torsion** group has property (FA).

Proof. Recall that a group is a torsion group if every element has finite order. Such groups have no quotient isomorphic to $\mathbb{Z}$.

By assumption and Theorem 2.1, it suffices to check they are *not* an amalgam $G_1 *_A G_2$. Let $s_1 \in G_1 - A$ and $s_2 \in G_2 - A$ be (non-trivial) right coset representatives of $A\backslash G_1$ and $A\backslash G_2$, respectively. Then the element

$$g = f(a)f_1(s_1)f_2(s_2)$$

is cyclically reduced, and is thus of infinite order; a contradiction. ◻

Example 3.2. If G has property (FA), so has every quotient of G.

Proof. Let $H \trianglelefteq G$, and assume by contradiction that ${}^{G}\!/\!_{H}$ acts on a tree X with $X^{G/H} \neq \emptyset$. Let

$$\pi : G \to {}^{G}\!/\!_{H}, \qquad g \mapsto gh$$

denote the canonical homomorphism. We have the composition:

$$G \overset{\pi}{\twoheadrightarrow} {}^{G}\!/\!_{H} \overset{h}{\to} \mathrm{Mor}(X).$$

If $x \in X$ is fixed by the action of G (this point exists due property FA), $h \circ \pi(g)(x) = x$ for all $g \in G$. Consequently, x is fixed for all $gH \in {}^{G}\!/\!_{H}$; a contradiction. ◻

Example 3.3. Let H be a normal subgroup of G. If H and ${}^{G}\!/\!_{H}$ have property (FA), then so has G.

Proof. Assume by contradiction that G acts on a tree X with $X^G \neq \emptyset$. Let X^H be the subtree fixed by $H \curvearrowright X$. Since H has property (FA), $X^H \neq \emptyset$. Since ${}^{G}\!/\!_{H}$ has property (FA), the subtree $X^{G/H}$ fixed by ${}^{G}\!/\!_{H} \curvearrowright X^H$ is non-empty; a contradiction. ◻

Fact 3.1. *If a subgroup G' has finite index in a group G, then the core*

$$\mathrm{Core}(G') = \bigcap_{g \in G} g^{-1}G'g$$

is a normal subgroup with finite index in G.

Fact 3.2. *Any action of a finite group on a (non-empty) tree has a global fixed point.*

Example 3.4. Let G' be a subgroup of finite index in G. If G acts on a tree X and if $X^{G'} \neq \emptyset$, then $X^G \neq \emptyset$.

Proof. Let H the core of G' in G; in particular, H is normal. By assumption

$$\emptyset \neq X^{G'} \subset X^H$$

where the inclusion holds because of $H < G'$. As in Example 3.3, consider the action ${}^{G}\!/\!_{H} \curvearrowright X^H$. The group ${}^{G}\!/\!_{H}$ is finite by Fact 3.1, and has a fixed point by Fact 3.2. It follows that $X^G \neq \emptyset$. ◻

Remark. Put differently, a group *virtually* having property (FA) is equivalent to the group having this property. This does *not* imply that **all** subgroups of finite index have property (FA).

For example, the *Schwartz group* has a subgroup of finite index which is isomorphic to a surface group; such a group has a quotient isomorphic to $\mathbb{Z}$, and does therefore not satisfy property (FA).

A Preliminaries

A.1 Graphs

Definition A.1. Let n be an integer ≥ 0. Consider the oriented graph Path_n.

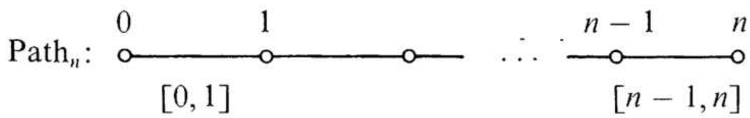

Figure 3: Path_n

It has $n+1$ vertices $0, 1, \ldots, n$ and the orientation given by the n edges $[i, i+1]$, $0 \leq i < n$ with $o([i, i+1]) = i$ and $t([i, i+1]) = i+1$. A path (of length n) in a graph Γ is a morphism of Path_n into Γ.

Remark A.1. A pair of the form $(y_i, y_{i+1}) = (y_i, \overline{y}_i)$ in the path is called a **backtracking**. If there is a path from P to Q in Γ, then there is one without backtracking.

Definition A.2. A graph is said to be **connected** if any two vertices are the extremities of at least one path. The maximal connected subgraphs (under the relation of inclusion) are called the **connected components** of the graph.

Definition A.3. Let Γ be a graph and let $X = \mathrm{vert}\,\Gamma$, $Y = \mathrm{edge}\,\Gamma$. Let P be a vertex and let Y_P be the set of edges y such that $P = t(y)$. The cardinal n of Y_P is called the **index** of P.

- If $n = 0$ one says that P is **isolated**; if Γ is connected this is not possible unless $X = \{P\}$, $Y = \emptyset$.
- If $n \leq 1$ one says that P is a **terminal vertex** (or a **pending vertex**).

We let $\Gamma - P$ denote the subgraph of Γ with vertex set $X - \{P\}$ and edge set $Y - (Y_P \cup \overline{Y}_P)$.

Proposition A.1. *Let P be a non-isolated terminal vertex of a graph Γ.*

1. *Γ is connected if and only if $\Gamma - P$ is connected.*
2. *Every circuit of Γ is contained in $\Gamma - P$.*
3. *Γ is a tree if and only if $\Gamma - P$ is a tree.*

A.2 Cyclically reduced elements

Suppose we are given a group A, a family of groups $(G_i)_{i \in I}$ and, for each $i \in I$, an injective homomorphism $A \to G_i$. We identify A with its image in each of the G_i. We denote by $*_A G_i$ the direct limit of the family (A, G_i) with respect to these homomorphisms, and call it the **sum** of the G_i with A **amalgamated**.

Definition A.4 (Reduced word)**.** For all $i \in I$ choose a set S_i of *right coset representatives* of $G_i \backslash A$, and assume $1 \in S_i$. The map $(a, s) \mapsto as$ is then a bijection of $A \times S_i$ onto G_i mapping $A \times (S_i - \{1\})$ onto $G_i - A$.

Let $\boldsymbol{i} = (i_1, \ldots, i_n)$ be a sequence of elements of I (with $n \geq 0$) satisfying the following condition:

$$i_m \neq i_{m+1} \qquad \text{for } 1 \leq m \leq n-1. \qquad (T)$$

A **reduced word of type $\boldsymbol{i}$** is any family

$$m = (a; s_1, \ldots, s_n)$$

where $a \in A$, $s_1 \in S_{i_1}, \ldots, s_n \in S_{i_n}$ and $s_j \neq 1$ for all j.

We denote by f (resp. f_i) the canonical homorphism of A (resp. G_i) into the group $G = *_A G_i$.

Theorem A.1. *For all $g \in G$ there is a sequence $\boldsymbol{i}$ satisfying (T) and a reduced word $m = (a; s_1, \ldots, s_n)$ of type $\boldsymbol{i}$ such that*

$$g = f(a) f_{i_1}(s_1) \cdots f_{i_n}(s_n).$$

Furthermore, $\boldsymbol{i}$ and m are unique.

Definition A.5. The integer n is called the **length** of g. An element g of length ≥ 2 is called **cyclically reduced** if its type $\boldsymbol{i} = (i_1, \ldots, i_n)$ is such that $i_1 \neq i_n$.

Proposition A.2. *Let G denote the group $*_A G_i$.*

1. *Every element g of G is conjugate to a cyclically reduced element, or an element of one of the G_i.*

2. *Every cyclically reduced element is of infinite order.*

A.3 Graph of groups

Definition A.6. Let G be a group acting on a graph X without inversion. A **fundamental domain** of $G\backslash X$ is a *subgraph* T of X such that $T \to G\backslash X$ is an isomorphism.

Proposition A.3. *Let $G = G_1 *_A G_2$ be an amalgam of two groups. Then there is a tree X (and only one, up to isomorphism) on which G acts, with fundamental domain a segment*

$$T = \overset{P}{\bullet} \xrightarrow{y} \overset{Q}{\bullet},$$

the vertices of which have $G_P = G_1$, $G_Q = G_2$ and $G_y = A$ as their respective stabilizers.

Remark A.2. Let gP be a point in the graph X, with $g \in G$, $P \in \mathrm{vert}(G\backslash X)$ and stabilizer $G_P = G_1$ as in the above proposition. Then gP is fixed by a conjugate of G_1:

$$(gg_1g^{-1})gP = gg_1(g^{-1}g)P = gg_1P = gP, \quad g_1 \in G_1.$$

Similarly, any vertex gQ is fixed by a conjugate of G_2.

Definition A.7. A graph isomorphic to $\mathrm{Path}_1 = \overset{0}{\bullet} \longrightarrow \overset{1}{\bullet}$ is called a *segment*.

Definition A.8. A **graph of groups** (G, T) consists of a graph T, a group G_P for each $P \in \mathrm{vert}\ T$, and a group G_y for each $y \in \mathrm{edge}\ T$, together with a monomorphism

$$G_y \to G_{t(y)}, \quad a \mapsto a^y;$$

one requires in addition that $G_{\overline{y}} = G_y$. In the case where T is a *tree*, we say that (G, T) is a *tree of groups.*

Example A.1. Take T to be a segment $\overset{P}{\bullet} \xrightarrow{y} \overset{Q}{\bullet}$. Then G_T is equal to $G_P *_{G_y} G_Q$.

Remark A.3. Suppose that T is obtained by adjoining a vertex P and a geometric edge $\{y, \overline{y}\}$ to a tree T'. In other words, P is a terminal vertex of T, and $T' = T - P$. We then have

$$G_T = G_{T'} *_{G_y} P, \quad \text{whence } G_{T'} = \varinjlim(G, T')$$

Definition A.9. Let T be a maximal tree of Y. The **fundamental group** $\pi_1(G, Y, T)$ of (G, Y) at T is defined as:

$$F(G,Y) \big/ \langle\langle y \rangle\rangle, \qquad y \in \mathrm{edge}(T)$$

Thus, if g_y denotes the image of y in $\pi_1(G, Y, T)$, the group $\pi_1(G, Y, T)$ is generated by the groups G_P ($P \in$ *vert* Y) and the elements g_y ($y \in \mathrm{edge}\ Y$) subject to the relations:

$$g_y a^y g_y^{-1} = a^{\overline{y}}, \quad g_{\overline{y}} = g_y^{-1}, \quad \text{if } y \in \mathrm{edge}\ Y, \quad a \in G_y$$

$$g_y = 1 \quad \text{if } y \in \mathrm{edge}\ T.$$

In particular, we have $a^y = a^{\overline{y}}$ if $y \in \mathrm{edge}\ T$, $a \in G_y$.