§4 Trees and Amalgams

SEMINAR GROUPS ACTING ON TREES – SUMMER 2018

In talk §3 we have been introduced to the connection of free groups and free group-actions. Today we want to have a look at non-free group actions and see that given the right circumstances there is a close relation to amalgamated products of groups.

But first we want to see an example for a non-free group action, the action of the $SL_2(\mathbb{Z})$ on the so called *Farey Tree*, to get a feeling for this kind of action. We will make some interesting observations, that are gonna give us, together with the theorem on trees and amalgams, an astounding result on the $SL_2(\mathbb{Z})$.

1 Definition of the Farey Graph

Definition 1.1. We define the FAREY GRAPH F' in the following way:

$$\operatorname{Vert} F' = \left.\left\{ (n,m) \in \mathbb{Z}^2 \mid (n,m) \text{ is primitive} \right\}\right/_{\{\pm\}} .$$

We say that two vertices $\pm(p,q)$, $\pm(r,s) \in \operatorname{Vert} F'$ are connected if and only if

$$\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1 .$$

Remark. This is not the definition of a tree in Serre's sense. We will later on consider Serre's definition after introducing the Farey tree.

Definition 1.2. We define a group action of $G = \mathrm{SL}_2(\mathbb{Z})$ on the Farey Graph for $\pm(p,q) \in \operatorname{Vert} F'$ and $A \in G$ by

$$A \cdot \pm(p,q) = \pm A \cdot \begin{pmatrix} p \\ q \end{pmatrix} .$$

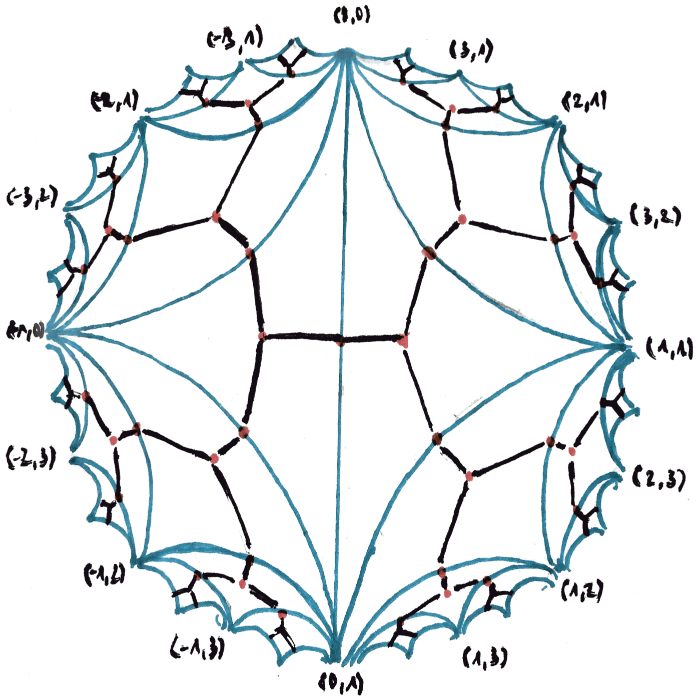


Figure 1: The Farey Tree superimposed on the Farey Graph. The points are denoted as the positive representatives to make the image easier to overview.

2 The Farey Complex and the Farey Tree

We observe that the Farey Graph consists of boundaries of triangles (triples of vertices pairwise joint by an edge). By "gluing in" all possible triangles we obtain the FAREY COMPLEX (we can formally realize this by the notion of quotient space).

Furthermore we observe that the definition of our group action carries on to the Farey Complex (the boundary of a triangle is taken to the boundary of a triangle).

Definition 2.1. We can now define the FAREY TREE F with the following set of edges and vertices.

$$\text{Vert}\, F : \text{the union of all possible boundaries and sides of triangles}$$

We say two vertices are connected if and only if there is an containment relation between a triangle and a side.

Again the action of $\mathrm{SL}_2(\mathbb{Z})$ carries over and maps triangles on triangles and sides on sides. We especially observe that this action is not free ($-\mathbb{1}$ fixes the whole tree).

Remark. We can view this tree as a tree in Serre's sense by adding the inverse of each edge to our set of edges. We define the set of positive edges as the ones starting in a vertex corresponding to a side and terminating in a vertex corresponding to a triangle. (This works thanks to the definition of edges of the Farey Tree).

It is important to note, that this way a edge is never mapped to its inverse. Therefore $SL_2(\mathbb{Z})$ acts on the Farey Tree without inversion.

3 Some Observations on the Group Action

We make some interesting observations on the group action of the $\mathrm{SL}_2(\mathbb{Z})$ on the Farey Tree:

(i) Non trivial vertex and edge stabilizer groups:

a) The vertex corresponding to the side $\{\pm(1,0), \pm(0,1)\}$ is fixed by

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}/_{4\mathbb{Z}} \, .$$

b) The vertex corresponding to the triangle $\{\pm(0,1), \pm(1,0), \pm(-1,1)\}$ is fixed by

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}/_{6\mathbb{Z}} \, .$$

c) The edge connecting these two vertices is fixed by

$$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong \mathbb{Z}/_{2\mathbb{Z}}$$

(ii) The group actions divides the tree into two orbits: triangles and edges. We furthermore note that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the set of edges.

The stabilizer group of an arbitrary triangle (respectively side of a triangle) corresponds to a coset of the group in (b) (respectively (a)).

4 The Connection between Trees and Amalgamation

In this section we make the convention that each group acting on a graph acts without inversion.

Definition 4.1.

(i) Let G be a group acting on a graph X. A FUNDAMENTAL DOMAIN of ${}^{X}/_{G}$ is a subgraph $T \subseteq X$ such that $T \cong {}^{X}/_{G}$.

(ii) A graph isomorphic to Path_1 is called a SEGMENT.

We make the following interesting observations on the relationship of a certain kind of tree and the group actions of amalgamated products of groups:

Theorem 4.2. let G be a group acting on a graph X and let T be a segment of X consisting of an edge γ with $o(\gamma) = P$ and $t(\gamma) = Q$ such that T is a fundamental domain of X modulo G. Let G_P, G_Q and G_γ denote the stabilizer groups of the vertices and edge of T.

Then the following statements are equivalent

(i) X is a tree

(ii) the homomorphism $G_P *_{G_\gamma} G_Q \to G$ induces by the inclusions $G_P \hookrightarrow G$ and $G_Q \hookrightarrow G$ is an isomorphism.

Remark. As G_γ stabilizes both P and Q, G_γ is indeed a subgroup of both G_P and G_Q.

With this theorem and our former observations on the group action of the $\mathrm{SL}_2(\mathbb{Z})$ on the Farey Tree we can directly follow this corollary on the structure of $SL_2(\mathbb{Z})$.

Corollary 4.3. It holds true

$$SL_2(\mathbb{Z}) \cong {}^{\mathbb{Z}}/_{4\mathbb{Z}} *_{\mathbb{Z}/2\mathbb{Z}} {}^{\mathbb{Z}}/_{6\mathbb{Z}} \ .$$

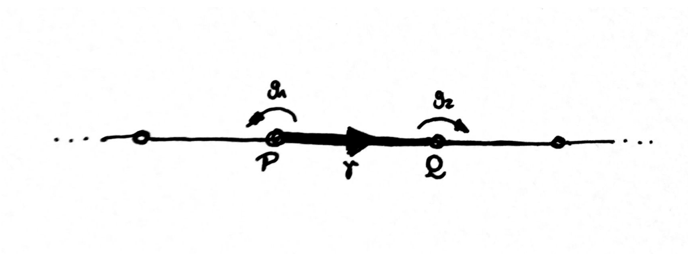
We therefore obtain the following representation:

$$SL_2(\mathbb{Z}) = \langle a, b \mid a^4 = 1, b^6 = 1, a^2 = b^3 \rangle$$

Theorem 4.4. Let $G = G_1 *_A G_2$ be an amalgam of two groups over a common subgroup A. Then there is a tree X (and only one up to isomorphism) on which G acts, with fundamental domain T a segment of X consisting of an edge γ with $o(\gamma) = P$ and $t(\gamma) = Q$, the vertices and edges of which have $G_P = G_1$, $G_Q = G_2$ and $G_\gamma = A$ as stabilizer groups.

Example 4.5. Here are some Examples of trees associated with the amalgam of two groups:

(i) The infinite Dihedral Group $\mathbb{D}_\infty : G_1 = {}^{\mathbb{Z}}/_{2\mathbb{Z}} = G_2, A = \{1\}$



(ii) $G = {}^{\mathbb{Z}}/_{3\mathbb{Z}} * {}^{\mathbb{Z}}/_{4\mathbb{Z}}$