Compactification of moduli spaces of representations

GEAR Log Cabin Workshop

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These are the notes of the GEAR Log Cabin workshop *Compactification of Moduli spaces of Representations* organized in Montana in June 2017.

During this workshop, we studied the different compactifications of moduli spaces of representations of surface groups into reductive Lie groups. In the first part, we focused on the different tools to compactify SL_2 character variety. More precisely, we studied the algebraic compactification of Morgan-Shalen, Thurston's compactification, limit of grafting and the bordification using Higgs bundles. In the second part, we focused on the current developement of these tools for higher rank Lie groups. In particular, we studied tropicalization of coordinates, compactifications by action on buildings and harmonic maps to building.

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Chapter 1

Lie theory and symmetric spaces

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1.1 Introduction

The goal of these notes is to recall some of the important properties of semisimple Lie groups and symmetric spaces, with an accent on symmetric spaces of non-compact type. To be concise, most of the proofs are only sketched when not omitted and some notions (like the root space decomposition or the classification of semisimple Lie algebras) are not discussed. The author used the notes [4], part 1, 3 and 4 of [5], the first six chapters of [2], the first lecture of [3] and the first three chapters of [1].

1.2 Lie groups and Lie algebras

1.2.1 First definitions

Here we briefly recall what Lie groups and Lie algebras are.

Definition 1.2.1.

- A *real Lie group* (resp. *complex*) is a real analytic manifold (resp. complex) with a group structure such that $(x, y) \mapsto xy^{-1}$ is real analytic (resp. holomorphic).
- A *morphism of Lie groups* is an analytic group morphism $f : G_1 \to G_2$ between two Lie groups.
- A *Lie subgroup* of a Lie group G is a subgroup H of G equipped with a Lie group structure such that the inclusion is a morphism of Lie group which is an immersion.
- The *Lie algebra* of a Lie group is its tangent space at the identity.

If G is a Lie group, its Lie algebra (often denoted $\mathfrak{g}$) is, for now, just a vector space. It is real if G is real and complex if G is complex. Note that every complex Lie group can be seen as a real Lie group as well.

Example 1.2.2. Here are some examples of common Lie groups together with their Lie algebras. Let $K = \mathbb{R}$ or $K = \mathbb{C}$

- $GL_n(K)$, $\mathfrak{gl}_n(K) := \{u : K^n \to K^n,\ u \text{ linear}\}$.
- $SL_n(K) := \{M \in GL_n(K), \det M = 1\}$, $\mathfrak{sl}_n(K) := \{u \in \mathfrak{gl}_n(K), \operatorname{tr} u = 0\}$.
- $SO_n(\mathbb{C}) := \{M \in SL_n(\mathbb{C}), {}^{t}MM = I_n\}$,
  $\mathfrak{so}_n(\mathbb{C}) := \{u \in \mathfrak{sl}_n(\mathbb{C}), {}^{t}u + u = 0\}$.
- $SO(n) := \{M \in SL_n(\mathbb{R}), {}^{t}MM = I_n\}$, $\mathfrak{so}(n) := \{u \in \mathfrak{sl}_n(\mathbb{R}), {}^{t}u + u = 0\}$.
- $SU(n) := \{M \in SL_n(\mathbb{C}), {}^{t}\overline{M}M = I_n\}$, $\mathfrak{su}(n) := \{u \in \mathfrak{sl}_n(\mathbb{C}), {}^{t}\overline{u} + u = 0\}$.
- $Sp_{2n}(K) := \{M \in GL_{2n}(K), {}^{t}MJ_nM = J_n\}$,
  $\mathfrak{sp}_{2n} := \{u \in \mathfrak{gl}_{2n}(K), {}^{t}uJ_n + J_nu = 0\}$, where
$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$
- $B_n(K) := \{M \in SL_n(K),\ M$ is upper triangular with 1's along the diagonal$\}$, $\mathfrak{b}_n(K) := \{u \in \mathfrak{sl}_n K,\ M$ is upper triangular with 0's along the diagonal$\}$.

  All these Lie groups and algebras are called *linear*, as the Lie groups (resp. algebras) here are all Lie subgroups (resp. Lie subalgebras) of $GL_n(K)$ (resp. $\mathfrak{gl}_n(K)$).

Definition 1.2.3.
- A *real Lie algebra* (resp. *complex*) is a real (resp. complex) vector space (here of finite dimension) equipped with a bilinear operation $[\cdot,\cdot]$ from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$, called *Lie bracket*, such that
  - $\forall X, Y \in \mathfrak{g}, [X, Y] = -[Y, X]$;
  - $\forall X, Y, Z \in \mathfrak{g},\ [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$, called the *Jacobi identity*.
- A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ which is closed under Lie bracket.
- A *morphism of Lie algebra* is a linear map $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ between Lie algebras such that $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{g}_1$.

Example 1.2.4. Let M be a smooth manifold. Let $\Gamma(TM)$ be the vector space of vector fields on M. Then $\Gamma(TM)$ is a Lie algebra for the bracket of vector field.

In particular, if G is a Lie group, $\Gamma(TG)$ is a Lie algebra. Recall that G acts on itself by left-transaltion $L_g : h \mapsto gh$. A vector field X on G is said *left-invariant* if $L_g^* X = X$ for all $g \in G$. The Lie algebra of left-invariant vector fields ${}^G\Gamma(TG)$ is a Lie subalgebra of $\Gamma(TG)$.

Proposition 1.2.5. *Let G be a Lie group. Then the map $X \in \mathfrak{g} \mapsto \tilde{X} \in {}^G\Gamma(TG)$ where $\tilde{X}(g) = T_e L_g(X)$ with $L_g(h) = gh$ is a linear isomorphism.*

Proof. The converse map is $V \mapsto V(e)$, and if V is a left-invariant vector field, we have $V(g) = L_{g*}V(e)$. □

Here we define a Lie algebra structure on the tangent space at the identity of a Lie group, which will turn out to be isomorphic to the Lie algebra of left-invariant vector field.

Let G be a Lie group and T_eG his tangent space at the identity. For $g \in G$, let $C_g : G \to G, h \mapsto ghg^{-1}$ be the conjugation by g.

Let $\mathrm{Ad} : G \to GL(T_eG)$ be defined by $\mathrm{Ad}g = T_eC_g : T_eG \to T_eG$. It is a Lie group representation called the *adjoint representation* of G.

Let $\mathrm{ad} = T_e\mathrm{Ad} : T_eG \to \mathfrak{gl}(T_eG)$. Now for every X and Y in T_eG, let $[X,Y] := \mathrm{ad}X(Y)$.

Example 1.2.6. The Lie bracket on the Lie algebra $\mathfrak{gl}_n(K)$ is given by $[u,v] = \mathrm{ad}u(v) = u \circ v - v \circ u$.

Proposition 1.2.7.

1. *Let G be a Lie group and $\mathfrak{g} = T_eG$ its tangent space at the identity equipped with $[\cdot,\cdot] : (X,Y) \mapsto \mathrm{ad}X(Y)$. Then $\mathfrak{g}$ is a Lie algebra, called the* Lie algebra of G *and* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a Lie algebra representation.*

2. *With this Lie bracket on $\mathfrak{g}$, the linear map $X \mapsto \tilde{X}$ is a Lie algebra isomorphism from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra of left-invariant vector fields ${}^G\Gamma(TG)$ on G.*

3. *Let $f : G_1 \to G_2$ be a Lie groups morphism. Then $T_ef : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebras morphism.*

Proof. To prove (1) it is enough to show the Jacobi identity for $(X,Y) \mapsto \mathrm{ad}X(Y)$, which follows from the Jacobi identity of the vector field and the fact that the linear isomorphism between $\mathfrak{g}$ and ${}^G\Gamma(TG)$ preserves Lie brackets (which is (2)). For (3), note that for all $g \in G$, $f \circ C_g = C_{f(g)} \circ g$. Differentiating two times gives

$$T_ef \circ (\mathrm{ad}X) = (\mathrm{ad}T_efX) \circ T_ef$$

for all $X \in \mathfrak{g}$, which is the desired formula if taken in Y. The reader will find a down to earth proof of (2) in proposition 1.6 of [5]. □

1.2.2 Lie subgroups and Lie subalgebras

For a Lie group G, let X be a vector in its Lie algebra $\mathfrak{g}$. The vector X defines $\tilde{X}$ a left-invariant vector field. If $\gamma_t = \gamma_t^X$ is the flow line of $\tilde{X}$ passing through e the identity of G, then γ_t is defined for all $t \in \mathbb{R}$ and $t \mapsto \gamma_t$ is a morphism of Lie groups between $\mathbb{R}$ and G. We call such morphisms *one parameter subgroups.*

Proposition 1.2.8. *The map $X \mapsto \gamma_t^X$ is a bijection from $\mathfrak{g}$ to the set of one parameter subgroups of G.*

Proof. This is an application of the Picard-Lindelöf theorem. Note that the flow line $\gamma_t = \gamma_t^X$ is indeed a Lie group morphism, because $s \mapsto \gamma_s \gamma_t$ and $s \mapsto \gamma_{s+t}$ are flow lines of $\tilde{X}$ with the same initial condition. □

The application $\mathfrak{g} \to G, X \mapsto \gamma_1^X$ is called the *exponential* and we write $\exp(X) = \gamma_1^X$.

Note that $\gamma_1^{tX} = \gamma_t^X$. Also, if G is a linear Lie group, then the exponential is the exponentiation of matrices.

Theorem 1.2.9. *The set of connected Lie subgroups of G is in bijection with the set of Lie subalgebras of $\mathfrak{g}$.*

Proof. Given a Lie subgroup H of G, the inclusion map induces an injective Lie algebra morphism of $\mathfrak{h} \to \mathfrak{g}$. For the converse, we use Frobenius theorem. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. The left action of G on itself allows us to construct a left invariant subbundle $\underline{\mathfrak{h}}$ of TG with $\underline{\mathfrak{h}}_e = \mathfrak{h}$, which is closed under Lie bracket because $\mathfrak{h}$ is a Lie algebra and by left-invariance of $\underline{\mathfrak{h}}$, it is enough to check the closure at $\underline{\mathfrak{h}}_e = \mathfrak{h}$. By Frobenius theorem, we get an immersed submanifold H of G containing e whose tangent space is $\underline{\mathfrak{h}}$. Now if $g \in H$, $\underline{\mathfrak{h}}$ is invariant by L_g, thus $L_g(H) = H$ and H is indeed a subgroup. □

In particular, to every Lie group G we may associate the *adjoint group of* G, denoted $Int(\mathfrak{g})$, defined as the connected subgroup of $GL(\mathfrak{g})$ whose Lie algebra is the image of $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, or equivalently as the image of $\mathrm{Ad} : G \to GL(\mathfrak{g})$ if G is connected.

The following theorem, due to E. Cartan, gives a criterion on whether a Lie subgroup is imbedded or not. See theorem 2.10, chapter 2 of [2].

Theorem 1.2.10. *An immersed subgroup of a Lie group is imbedded if and only if it is closed. Moreover, if G is a real Lie group, a subgroup of G is an imbedded Lie subgroup if and only if it is closed.*

We end this section by quoting two theorems illustrating the importance of the linear case.

Theorem 1.2.11. (Ado's) *Every Lie algebra over $K = \mathbb{R}, \mathbb{C}$ can be embedded as a Lie subalgebra of $\mathfrak{gl}_n(K)$;*

(Lie's Third) *Any Lie algebra is the Lie algebra of a Lie group.*

Proof. Ado's theorem is rather simple to prove for semisimple Lie algebras as the adjoint representation gives the wanted embedding. It is trickier in the general case.

Lie's third theorem follows from Ado's theorem and the correspondance between Lie subgroups and Lie subalgebras. □

1.2.3 Semisimple Lie algebras

We will see later that the Lie algebra of the isometry group of a symmetric space is semisimple (if it has no Euclidean factors). Using the properties of semisimple Lie algebras, one may classify them. This classification leads to the classification of semisimple Lie groups and symmetric spaces. We won't say more about these classifications and this part is mainly composed of standard definitions.

Definition 1.2.12. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

- An *ideal* $\mathfrak{i} \subset \mathfrak{g}$ is a linear subspace such that for all $X \in \mathfrak{i}$, for all $Y \in \mathfrak{g}$, $[X,Y] \in \mathfrak{i}$.
- $\mathfrak{g}$ is *abelian* if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.
- If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras, the *product Lie algebra* is $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ equipped with $[(X_1,X_2),(Y_1,Y_2)] := ([X_1,Y_1]_1,[X_2,Y_2]_2)$.
- $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and has no non-trivial ideals.
- $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ is the product of simple ideals.
- If $\mathfrak{g}$ is real, it is *compact* if $Int(\mathfrak{g})$ is compact.
- G is *simple (semisimple)* if $\mathfrak{g}$ is simple (semisimple).

Example 1.2.13.
1. $\mathfrak{sl}_n(K)$, $\mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_n(K)$, $\mathfrak{so}(n)$, $\mathfrak{su}(n)$ are simple.
2. $\mathfrak{gl}_n(K)$ is not semisimple because the subspace of scalar matrices is a non-trivial abelian ideal, while any ideals in a semisimple Lie algebra is a product of simple ideals.
3. $\mathfrak{t}_n(K)$ is not semisimple because the subspace generated by $E_{1,n}$ is a non-trivial abelian ideal.
4. $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$ are compact.

1.2.4 The Killing form

Let $\mathfrak{g}$ be a Lie algebra. If $X \in \mathfrak{g}$, we have $\mathrm{ad}X \in \mathfrak{gl}(\mathfrak{g})$ and we define the Killing form of $\mathfrak{g}$ to be the bilinear form given by:

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$$

for all $X, Y \in \mathfrak{g}$, where tr is the trace on $\mathfrak{gl}(\mathfrak{g})$.

Example 1.2.14.

- On $\mathfrak{gl}_n(K)$, we have $B(X,Y) = 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y)$.
- On $\mathfrak{sl}_n(K)$, $B(X,Y) = 2n\mathrm{tr}(XY)$.
- On $\mathfrak{so}(n)$, $B(X,Y) = (n-2)\mathrm{tr}(XY)$.

The following properties of the Killing form are straightforward:

Proposition 1.2.15.

1. *The Killing form is symmetric.*
2. *If $\alpha : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra isomorphism, then $B(\alpha(X), \alpha(Y)) = B(X,Y)$.*
3. *$B(\mathrm{ad}X(Y), Z) = -B(Y, \mathrm{ad}X(Z))$ for all $X, Y, Z \in \mathfrak{g}$.*
4. *If $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ then $B_{\mathfrak{g}} = B_{\mathfrak{g}_1} + \cdots + B_{\mathfrak{g}_n}$.*
5. *If $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then $B_{\mathfrak{g}}|_{\mathfrak{i}\times\mathfrak{i}} = B_{\mathfrak{i}}$.*

The Killing form detects whether a Lie algebra $\mathfrak{g}$ is semisimple or not, and if $\mathfrak{g}$ is semisimple its Killing form detects if it is compact.

Proposition 1.2.16.

1. *$\mathfrak{g}$ is semisimple if and only if $B_{\mathfrak{g}}$ is non-degenerate.*
2. *If $\mathfrak{g}$ is a real and semisimple, $\mathfrak{g}$ is compact if and only if $B_{\mathfrak{g}}$ is negative definite.*

Proof. Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{i}^\perp$ the orthogonal of $\mathfrak{i}$ for the Killing form B is also an ideal. In particular, $\mathfrak{g}^\perp$ is an ideal, and if $\mathfrak{g}$ is simple then $\mathfrak{g}^\perp$ must be $\mathfrak{g}$ or $\{0\}$, but in the first case, $\mathfrak{g}$ is abelian. Thus $\mathfrak{g}^\perp = \{0\}$ and B is non-degenerate. The same holds if $\mathfrak{g}$ is only semisimple by proposition 1.2.15 (4) and (5). For the converse, assume B is non-degenerate and pick any non trivial ideal $\mathfrak{a} \subset \mathfrak{g}$. We then have $\mathfrak{g} = \mathfrak{a} \times \mathfrak{a}^\perp$ and the Killing forms of $\mathfrak{a}$ and $\mathfrak{a}^\perp$, being restrictions of B, are again non-degenerate by proposition 1.2.15. By iteration we can write $\mathfrak{g}$ as a product of simple ideals, which ends the proof of (1).

Now, suppose $\mathfrak{g}$ is a semisimple Lie algebra with definite negative Killing form B. By propostion 1.2.15, $t \mapsto B(\exp(t\mathrm{ad}X)Y, \exp(t\mathrm{ad}X)Z)$ is constant, so $Int(\mathfrak{g})$ preserves B and is a closed subgroup of the orthogonal group of $-B$. Conversely, if $Int(\mathfrak{g})$ is compact in $GL(\mathfrak{g})$, it preserves a scalar product Q. Let $X \in \mathfrak{g}$ and $(a_{i,j})$ be the matrix of $\mathrm{ad}X$ in a Q-orthonormal basis of $\mathfrak{g}$. Then $(a_{i,j})$ is skew-symmetric and

$$B(X,X) = \sum_{i,j} a_{i,j}a_{j,i} = -\sum_{i,j} a_{i,j} \leqslant 0$$

and equality holds only if X lies in the center of $\mathfrak{g}$ which is trivial by semisimplicity. □

1.3 Symmetric spaces

We quickly recall a useful lemma, which holds for all riemannian manifold and is proven in [2], chapter 1.

Lemma 1.3.1. *Let M be a connected riemannian manifold, φ and ψ two isometries of M. Suppose there exists $p \in M$ such that $\varphi(p) = \psi(p)$ and $T_p\varphi = T_p\psi$. Then $\varphi = \psi$.*

A *symmetric space* is a connected simply connected real analytic riemannian manifold M such that for every $p \in M$, there exists an involutive isometry s_p of M fixing x with $T_p s_p = -\mathrm{id}_{T_pM}$.

Let $G = I_0(M)$ be the connected component of the identity of the isometry group of M (equipped with the compact-open topology).

A symmetric space M is complete, and the action of G on M is transitive.

Theorem 1.3.2. *Let M be a symmetric space. Then $G := I_0(M)$ is a (real) Lie group. Moreover, if $p \in M$, the stabilizer K of p for the G-action on M is a compact Lie subgroup of G.*

Proof. First we show that K is a compact subgroup of the topological group G. Consider the continuous representation $k \mapsto T_pk$, which is well defined because $k(p) = p$. It is an isomorphism (by virtues of lemma 1.3.1) onto a closed subgroup of the orthogonal group of the Euclidean space T_pM, and thus K is compact. Moreover it has the structure of a Lie group as a closed subgroup of a real Lie group.

Now we need to show that M has a Lie group structure. Let $p \in M$ be any point, $r > 0$ and $B_r(p)$ the ball of radius r around p in M. We are going to construct a subset S of G which will be identified with $B_r(p)$. If $q \in B_r(p)$ and q' is the midpoint between p and q (assume r small enough so that q' is unique), let $\psi(q) := s_{q'}s_p$ which is an isometry mapping p to q. Now the map ψ maps homeomorphically $B_r(p)$ onto a subset S of G, and S inherits the analytic structure from $B_r(p)$. If $\varphi_p(g) = g \cdot p$, $\varphi_p^{-1}(B_r(p))$ identifies with $S \times K$ through $(s,k) \mapsto s \circ k \cdot p$ and the later is then an open subset of G carrying an analytic structure. For the details and the fact that the transition maps are analytic, we refere to [2] lemma 3.2 chapter 4. □

In particular, the orbital map $\varphi_p : g \in G \mapsto g \cdot p \in M$ induces a diffeomorphism

$$G/K \xrightarrow{\sim} M,$$

and $T_e\varphi_p : \mathfrak{g} \to T_pM$ is onto with kernel the Lie algebra of K.

1.3.1 Cartan involutions and Cartan decomposition

Let M be a symmetric space, $p \in M$, K the stabilizer of p and s_p the geodesic isometry at p. The application $\sigma_p : G \to G$ defined by $g \mapsto s_p \circ g \circ s_p$ is an involutive group automorphism, thus $\theta_p := T_e\sigma_p : \mathfrak{g} \to \mathfrak{g}$ is an involutive Lie algebra automorphism, called *Cartan involution* (at p).

Let $\mathfrak{k} := \{X \in \mathfrak{g} : \theta_p X = X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g} : \theta_p X = -X\}$ be the eigenspaces of θ_p. They have the following properties:

Proposition 1.3.3. *1. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and this direct sum is $\mathrm{Ad}(K)$-invariant.*

2. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

3. $\mathfrak{k}$ is the Lie algebra of K and $T_e\varphi_p|_{\mathfrak{p}} : \mathfrak{p} \to T_pM$ is an isomorphism.

4. $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal for B.

5. The geodesics of M passing through p at $t = 0$ are of the form $t \mapsto \exp(tX) \cdot p$ where $X \in \mathfrak{p}$.

Proof. To prove (1), see that the direct sum follows from $\theta_p^2 = \mathrm{id}$. If $k \in K$, we have $k s_p k^{-1} = s_{kp} = s_p$ (by lemma 1.3.1) which implies $C_k \circ \sigma_p = \sigma_p \circ C_k$ and taking the differential at e yields $(\mathrm{Ad}k) \circ \theta_p = \theta_p \circ \mathrm{Ad}k$, from which the invariance of $\mathfrak{p}$ and $\mathfrak{k}$ follows.

Since θ_p is a Lie algebra automorphism, (2) is obvious.

For (3), let $G^{\sigma_p} := \{g \in G : \sigma_p(g) = g\}$ and $G_0^{\sigma_p}$ the connected component of e in G^{σ_p}. Both are closed subgroups of G and thus are Lie embedded subgroups. It is classical to check that $G_0^{\sigma_p} \subset K \subset G^{\sigma_p}$, so all these groups have the same Lie algebra, say $\mathfrak{l}$. Moreover, the exponential map of G satisfies

$$\sigma_p(\exp tX) = \exp(t\theta_p(X))$$

for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Consequently $\mathfrak{l} = \mathfrak{k}$.

For (4), observe that if $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, $B(X, Y) = B(\theta_p X, \theta_p Y) = -B(X, Y)$.

One can prove that for all $X \in \mathfrak{p}$, $\exp(tX) \cdot p = \gamma(t)$ where γ is the unique unit speed geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = T_e\varphi_p X$, which implies (5). □

In particular, $\mathfrak{k}$ is a compact subalgebra of $\mathfrak{g}$ while $\mathfrak{p}$ is a subalgebra if and only if it is abelian.

1.3.2 De Rham decomposition

A symmetric space is said *irreducible* if it is non-empty and not isometric to the product of two non-trivial symmetric spaces.

Theorem 1.3.4. *Let M be a symmetric space. There exist $k, l \in \mathbb{N}$ and $M_1, \ldots, M_l$ irreducible symmetric spaces non isometric to $\mathbb{R}$ such that M is isometric to the product $\mathbb{R}^k \times M_1 \times \cdots \times M_l$.*

Moreover, the couple (k, l) is unique, and the M_i are unique up to isometries and permutations. In particular,

$$I_0(M) = I_0(\mathbb{R}^k) \times I_0(M_1) \times \cdots \times I_0(M_l).$$

Proof. While the existence is trivial, the unicity is not. One may show it using Lie algebra theory like in the chapter 5 of [2], or riemannian geometry like in the part 4 of [5]. □

The following theorem, which we admit, illustrates the importance of the semisimple condition.

Theorem 1.3.5. *If M is a symmetric space with no Euclidean factor, then $I_0(M)$ is semisimple.*

If all the factors in the De Rham decomposition of M are compact, then M is said to be of *compact type*. If all the factors are neither compact nor Euclidean, M is said to be of *non-compact type*.

Theorem 1.3.6.

1. *M is of non-compact type if and only if M has non-positive curvature and no Euclidean factor.*
2. *M is of compact type if and only if M has non-negative curvature and no Euclidean factor.*

Note that since $M \simeq G/K$ with K compact, G is compact if and only if M is of compact type.

1.4 Flats, Weyl chambers

Let M be a symmetric space of the non-compact type, $G = I_0(M)$, $\mathfrak{g}$ its Lie algebra, $p \in M$ and $\theta := \theta_p$ the associated Cartan involution. As before, let $\mathfrak{k}$ and $\mathfrak{p}$ be the eigenspaces of θ for $+1$ and -1 respectively.

1.4.1 Cartan subalgebras and Weyl chambers in $\mathfrak{g}$

A *Cartan subspace* of $\mathfrak{p}$ is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$.

The following proposition is a routine computation involving the definition of a Cartan subspace.

Proposition 1.4.1. *Let $X \in \mathfrak{p}$. The following conditions are equivalent:*

1. *the subspace $\mathfrak{z}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is abelian where $\mathfrak{z}_{\mathfrak{g}}(X) := \{Y \in \mathfrak{g} : [X, Y] = 0\}$;*
2. *the vector X lies in a unique Cartan subspace.*

A vector verifying these conditions is said *regular*, otherwise it is *singular*.

For a fixed Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$, the connected components of the set of regular elements in $\mathfrak{a}$ are called *Weyl chambers* of $\mathfrak{a}$.

Proposition 1.4.2. *Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two Cartan subspaces of $\mathfrak{p}$ and $\mathfrak{C}_i$ a Weyl chamber of $\mathfrak{a}_i$. There exists $k \in K$ such that*

$$\mathrm{Ad}k(\mathfrak{a}_1) = \mathfrak{a}_2 \text{ and } \mathrm{Ad}k(\mathfrak{C}_1) = \mathfrak{C}_2.$$

Proof. For $i = 1, 2$ let $X_i \in \mathfrak{C}_i$. Since X_i is regular, $\mathfrak{z}_{\mathfrak{g}}(X_i) \cap \mathfrak{p} = \mathfrak{a}_i$ (see proposition 1.4.1) and $\mathfrak{a}_i$ is the unique Cartan subspace containing X_i. The idea is to consider the function $f : K \to \mathbb{R}$ defined by $k \mapsto B(\mathrm{Ad}k(X_1), X_2)$. Since K is compact, f has a minimum, say k_0. Then one can show that $\mathrm{Ad}k_0(\mathfrak{a}_1) = \mathfrak{a}_2$ and $\mathrm{Ad}k_0(\mathfrak{C}_1) = \mathfrak{C}_2$. See corollaire 4.18 of [5] for details. □

1.4.2 Flats and Weyl chambers in M

A *flat* in M is a totally geodesic submanifold of M isometric to some $\mathbb{R}^k$ where $k \in \mathbb{N}$. It is *maximal* if it is maximal for the inclusion. The *rank* of M is the maximal dimension of a flat.

A geodesic of M is a flat so any geodesic is contained in a maximal flat. A geodesic is said *regular* if it is contained in a unique maximal flat, it is *singular* otherwise.

An *(open) Weyl chamber* of M is a connected component of the set of y in a maximal flat pointed at x such that $y \neq x$ and the unique geodesic from x to y is regular.

Proposition 1.4.3.

1. *The map $\mathfrak{a} \mapsto \exp(\mathfrak{a}) \cdot p$ is a bijection from the set of Cartan subspace of $\mathfrak{p}$ to the set of maximal flats containing p.*
2. *The geodesic $\exp(tX) \cdot p$ is regular if and only if X is regular.*
3. *Let $\mathfrak{a}$ be a Cartan subspace and $F = \exp(\mathfrak{a}) \cdot p$ the associated pointed flat. Then the Weyl chambers of (F, p) are the $\exp(\mathfrak{C}) \cdot p$ where $\mathfrak{C}$ is a Weyl chamber of $\mathfrak{a}$.*
4. *The group G acts transitively on the set of pointed maximal flats of M and on the set of Weyl chambers of M.*
5. *The rank of M is equal to the dimension of a maximal flat of M and to the dimension of a Cartan subspace.*

Proof. (1) is due to the formula for the curvature of a symmetric space: we may identify the tangent space of M at p with $\mathfrak{p}$ through $T_e\varphi_p$ where $\varphi_p(g) = g \cdot p$. Now if $X, Y, Z \in T_pM$, the curvature tensor is given by $R(X,Y)Z = -[[X,Y],Z]$ (see theorem 4.2 in chapter 4 of [2]). The rest follows from proposition 1.4.2. □

1.5 Gromov boundary of a symmetric space

Let M be a symmetric space of the non-compact type. Two subsets of M are said *asymptotic* if their Hausdorff distance is finite, or equivalently if each is contained in a bounded neighborhood of the other.

The *Gromov boundary* of M is the space $\partial_\infty M$ of equivalence classes of geodesic rays (isometries $[0,+\infty[\to M$) for the relation "to be asymptotic".

If $p \in M$, the map {geodesic rays starting from p} $\to \partial_\infty M$ which takes the geodesic ray to its equivalence class is a bijection, and thus the boundary at infinity of M is in bijection with the unit tangent bundle at p. The topology of $T_p^1 M$ induces a topology on $\partial_\infty M$ which does not depend on $p \in M$ and makes $\partial_\infty M$ into a sphere of dimension $\dim M - 1$.

It is well known that in the hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, any two points at infinity may be joined by a geodesic. While this is no longer true in higher rank, we have the following generalization, which is proposition 2.21.14 of [1], and where we define $\partial_\infty F \subset \partial_\infty M$ to be the set of equivalence classes $\gamma(\infty)$ where γ is a geodesic of a flat F.

Proposition 1.5.1. *Let $\zeta, \xi \in \partial_\infty M$. There exists a flat F of M such that $\zeta, \xi \in \partial_\infty F$.*

1.5.1 Cone topology

We define a topology on $\overline{M} := M \cup \partial_\infty M$ such that its restriction to M is the topology from the manifold structure and its restriction to $\partial_\infty M$ is the previously defined topology. First define for any two non-zero tangent vectors X, Y at p the *angle* between X and Y to be the unique $\angle_p(X,Y) \in [0,\pi]$ such that

$$\cos(\angle_p(X,Y)) = \frac{g(X,Y)}{\|X\| \cdot \|Y\|},$$

where g is the riemannian metric on M. For any two points $z_1, z_2 \in \overline{M}$, let $\angle_p(z_1,z_2) = \angle_p(\dot{c}_1(0), \dot{c}_2(0))$ where c_i is the unique geodesic from p to z_i.

For $p \in M$, $\xi \in \partial_\infty M$ and $\varepsilon > 0$, let

$$C_x(\xi,\varepsilon) = \{y \in \overline{M} : y \neq x \text{ and } \angle_x(\xi,y) < \varepsilon\}.$$

The *cone topology* on $\overline{M}$ is the topology generated by the open sets of M and the the cones $C_x(\xi,\varepsilon)$. With this topology, $\overline{M}$ is homeomorphic to a closed Euclidean ball of dimension $\dim M$, the image of M in $\overline{M}$ is the corresponding open ball and the image of $\partial_\infty M$ is the sphere.

This topology does not reflects the geometry of M (see example 1.5.2) and we will define the Tits metric on $\partial_\infty M$ which contains more information.

1.5.2 Tits metric on $\partial_\infty M$

The Tits metric is a metric on $\partial_\infty M$ reflecting the topology of M and the configuration of the maximal flats in M. We first define the *angle metric*.

Let $\zeta, \xi \in \partial_\infty M$ be two points at infinity and let

$$\angle(\zeta,\xi) := \sup_{p \in M} \angle_p(\zeta,\xi).$$

We leave it to the reader to show that $\angle$ is a distance on $\partial_\infty M$.

Example 1.5.2.

- If $M = \mathbb{R}^n$ with the Euclidean metric, then $\angle$ is the angle between two lines, that is the spherical distance on $\partial_\infty \mathbb{R}^n = \mathbb{S}^{n-1}$.
- If $M = \mathbb{H}^n_\mathbb{R}$ is the real hyperbolic space, then $\angle(\zeta, \xi) = \pi$ whenever $\zeta \neq \xi$, for there exists a geodesic ζ and ξ as extremities.

To define the Tits metric, we need the notion of interior metric of a metric, which we quickly recall: if (X, d) is a metric space and $c : [0,1] \to X$ is a curve, the *lenght* of c is

$$L(c) := \sup \sum_{i=0}^{k} d(c(t_i), c(t_{i+1})),$$

where the sup is taken over all subdivision $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{k+1} = 1$ of $[0,1]$. The *interior metric* d_i of d is

$$d_i(x, y) := \inf L(c),$$

where the inf is taken over all curves from x to y. Note that if there is no curve from x to y, then $d_i(x, y) = \infty$. The interior metric of a metric is always a metric, except that it may take ∞ as a value.

The *Tits metric on* $\partial_\infty M$ is the interior metric associated to $\angle$, and is denoted d_{Tits}.

Example 1.5.3.

- If $M = \mathbb{R}^n$, we have $\angle = d_{\text{Tits}}$.
- If $M = \mathbb{H}^n_\mathbb{R}$, we have $d_{\text{Tits}}(\zeta, \xi) = \infty$ whenever $\zeta \neq \xi$. In particular, the induces topology on the boundary of M is discrete and thus is not the same as the cone topology.

Proposition 1.5.4. *Let $\zeta, \xi \in \partial_\infty M$.*

1. *The open sets of the cone topology are open for the Tits topology.*
2. *If ζ and ξ cannot be joined by a geodesic of M, then $d_{\text{Tits}}(\zeta, \xi) \leqslant \pi$.*
3. *Let F be a flat in M. Then the cone topology and the Tits topology coincide on $\partial_\infty F$.*
4. *If the rank of M is 1, the Tits topology is discrete.*
5. *If the rank of M is greater than 1, we have $d_{\text{Tits}} \leqslant \pi$.*

Proof. For (1), observe that $d_{\text{Tits}} \geqslant \angle_p$ for all $p \in M$, so a sequence in $\partial_\infty M$ converging in the Tits topology converges in the cone topology. To prove (2), suppose we have $\omega \in \partial_\infty M$ such that $\angle(\zeta, \omega) = \angle(\xi, \omega) = 1/2\angle(\zeta, \xi)$. Iterating this and using the completeness of $\angle$ (see [3], lemma 4.5), we obtain a path of lenght $\angle(\zeta, \xi)$ between ζ and ξ, and $d_{\text{Tits}}(\zeta, \xi) = \angle_i(\zeta, \xi) = \angle(\zeta, \xi)$.

To show the existence of such an ω, choose $p \in M$ and let $\alpha, \gamma : [0, \infty[\to M$ be the unique unit speed rays from p to ζ, ξ. For $j \in \mathbb{N}$, let p_j be the unique point on the geodesic segment from $\alpha(j)$ to $\gamma(j)$ which is at minimal distance from p. Since there is no geodesic from ζ to ξ, the geodesic segment between $\alpha(j)$ and $\gamma(j)$ does not accumulate on a geodesic, and thus the sequence p_j has no accumulation point in M. By compacity of $\overline{M}$ for the cone topology, we may assume that p_j converges in the cone topology to a point $\omega \in \partial_\infty M$. We refere to lemma 4.7 of [3] for the rest of the proof of (2).

The reader may find proofs of (3) in [1]. The rest follows from 1.5.1 and (3). □

Remark 1.5.5. Note that by proposition 1.5.1, for symmetric spaces of rank > 1, we have $d_{\mathrm{Tits}} = \angle$. Yet this construction of the Tits metric allows one to construct it on the Gromov boundary of any CAT(0) space. See [3].

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Chapter 2

Compactifications of Teichmüller space

Sara Maloni

2.1 Introduction

Thse notes are based on a talk given at the workshop "Compactifications of moduli spaces of representations" organised by Brian Collier, Giuseppe Martone and Jérémy Toulisse. I was supposed to give a review of the different compactifications of Teichmüller space $\mathcal{T}(\Sigma)$. The organisers suggested as a reference the article [13] written by Ohshika. Ohshika's paper starts with the definition of Thurston's compactification, which can probably be considered the most 'important' one since the action of the mapping class group $\mathrm{Mod}(\Sigma)$ extends continuously to the boundary.

In these notes I decided to use a chronological order and start with the definition of the Teichmüller compactification in Section 3.1, followed by the Bers compactification in Section 3.2, the Thurston compactification in Section 2.4, the Gardiner–Masur compactification in Section 2.5 and I will finish with a brief overview of horofunction compactification in Section 2.6. Since I assumed the participants were more familiar with the Thurston's compactification of Teichmüller space, I decided to focus more on the Teichmüller and on the Bers compactifications. These notes just wants to summarize what I discussed during the talk. I tried to include the references so that the reader could look for the appropriate articles or books where the definitions and results are discussed in more details. Since the theme is so vast, I decided to try to give a brief overview of the many different points of view rather than choosing a favourite direction and presenting all the details.

I want to thank the organisers for all their work and all the participants for the fun and fruitful week we spent in the beautiful mountains in Montana.

2.2 Teichmüller compactification

For this section we will mostly follow Kerckhoff [9].

2.2.1 A crash course on quadratic differentials

In order to state Teichmüller's theory, we need to recall some facts about holomorphic quadratic differentials. A holomorphic quadratic differential q on $X \in \mathcal{T}(\Sigma)$ can be expressed locally as $q = \theta(z)dz^2$, with θ holomorphic. Let $\mathcal{Q}(X)$ be the set of all holomorphic quadratic differential on X. It defines two foliations —the *horizontal* [resp. *vertical*] *foliations*— as the sets of paths whose tangent vectors evaluate under q to positive [resp. negative] real numbers. For example, if we have the holomorphic quadratic differential $q = dz^2$ on $\mathbb{C}$, given $z \in \mathbb{C}$ and $v \in T_z\mathbb{C}$, we have $q((z,v)) = v^2$; so the horizontal [resp. vertical] foliations correspond to the union of horizontal [resp. vertical] lines. Every point of X has natural coordinates $w = x + iy$ so that $q = dw^2 (= \theta(z)dz^2)$. Also, there exists a metric g_q associated with q which can be expressed locally as $|\theta(z)|^{\frac{1}{2}}|dz|$ and which defines a singular flat structure on X. For a review of the theory of holomorphic quadratic differentials, you can see Section 11.1 of Farb and Margalit [5].

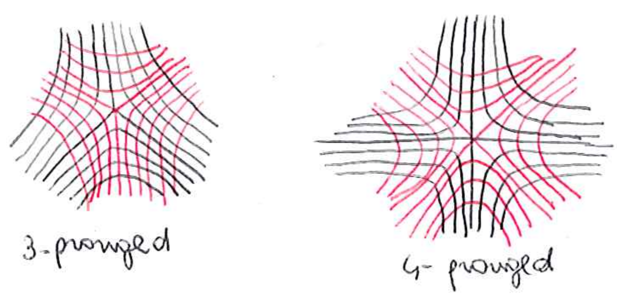

Figure 2.1: 3-pronged and 4-pronged singularities.

In the proof of Kerckhoff's Theorem which we will discuss in the next section, we need some results about Jenkins-Strebel differentials. Let $\mathcal{S} = \mathcal{S}(\Sigma)$ be the set of free homotopy classes of simple closed curve on Σ and let $\mathcal{CYL}$ be the set of foliations of Σ such that the complement of the critial leaves is a set of p cylinders $C_1, \dots, C_p$ with core curve σ_i (and where $1 \leq p \leq 3g-3$). The quadratic differentials q such that $F_q^{hor} \in \mathcal{CYL}$ are called *Jenkins-Strebel (or JS) differentials* since Jenkins [8] in 1957 and Strebel [14] in 1966 studied them extensively. Let All leaves of C_i are freely homotopic to $\sigma_i \in \mathcal{S}(\Sigma)$. Let A_i be the homotopy class of arcs in the cylinders C_i connecting the 2 boundary components. Then the measure class of a foliation $F \in \mathcal{CYL}$ is completely determined by σ_i and $i(A_i, F)$. Let $F \in \mathcal{CYL}$ be the horizontal foliation of a quadratic differential q, the q induces a flat metric on the C_i and the heights h_i and lengths ℓ_i of C_i are given by $h_i = i(A_i, F) = i(A_i, F_q^{hor})$ and $\ell_i = i(\sigma_i, F_q^{vert})$. For each $1 \leq i \leq p$, let $m_i = \frac{h_i}{\ell_i}$. We say that two foliations $F = F_q^{hor}$ and $F' = F_{q'}^{hor}$ in $\mathcal{CYL}$ are *modularly equivalent* if and only if $\sigma_i = \sigma_i'$ for all $1 \leq i \leq p$ and there exists a constant $C > 0$ such that $Cm_i = m_i'$ for all $1 \leq i \leq p$. Remember that, on the other hand, F and F' are said to be *projectively equivalent* if and only if there exists a constant $C > 0$ such that for all simple closed curves γ we have $Ci(\gamma, F) = i(gamma, F')$.

Theorem 2.2.1 (Jenkins [8], Strebel [14]). *Given $X \in \mathcal{T}(\Sigma)$, we have*

- *In $\mathcal{Q}^1(X)$ there exists a unique JS differential in each projective equivalence class.*
- *There exists a unique JS differential in each modular class.*

Note that the relationship between these two classes depends on the point $X \in \mathcal{T}(\Sigma)$.

2.2.2 Definition of Teichmüller compactification

Let's start with defining the *Teichmüller distance* on Teichmüller space. Recall that given a closed orientable surfaces of genus $g \geq 2$, the *Teichmüller space* $\mathcal{T}(\sigma)$ is the set of marked Riemann surfaces X diffeomorphic to Σ via $g\colon \Sigma \to X$, up to isotopy, that is $(X, g) \sim (Y, h) \iff g \circ h^{-1}\colon Y \to X$ is isotopic to a holomorphic diffeomorphism.

Let X be a Riemann surface. A *quadrilater* Q in X is an embedded closed disk with 4 distinguished points on its boundary. Note that Q is conformally equivalent to a rectangle (unique up to scaling). So, given a quadrilater Q in X, we can define its modulus as the ratio of the length and the width: $m(Q) = \frac{\ell(Q)}{w(Q)}$. Now, if we have a homeomorphism $f\colon X \to X'$ between Riemann surfaces, then quadrilaters are preserved under f, so we can define

$$K_f = \sup_{Q \text{ quadrilater in } X} \frac{m(f(Q))}{m(Q)}$$

. If $K_f < \infty$, then f is called *K–quasi-conformal.* Notice that there are many equivalent way to define the quasi-conformality constant K_f. For example, another definition uses the notion of $K_f(p)$, dilatation of f at $p \in X$, which is the eccentricity of the ellipse obtained as the image under f of the unit tangent circle at p. Then one can define $K_f = \sup_{p \in X} K_f(p)$.

Given two different points $(X, g), (Y, h) \in \mathcal{T}(\Sigma)$, Teichmüller studied the problem of minimising $K(F)$ over all $F\colon Y \to X$, $F \simeq g \circ h^{-1}$. Grötzsch solved this problem for rectangles and in that case the minimising map is the natural affine map between them. Teichmüller generalised Grötzsch's idea.

Now for $K \geq 1$, we can define the *(K, q)–stretch map* $f_{K,q}$ on X which, in term of the natural coordinates w, can be expressed as

$$\begin{cases} x \mapsto K^{\frac{1}{2}} x \\ y \mapsto K^{-\frac{1}{2}} y. \end{cases}$$

So $f_{K,\theta}$ defines a new point $f_{K,\theta}(X) = X' \in \mathcal{T}(\Sigma)$ and a K–quasi-conformal map between $X \to X'$.

Theorem 2.2.2 (Teichmüller)**.** *For any two points X and X' there is a (K,q)–stretch map $f_{K,q}\colon X \to X'$ such that $K_{f_{K,q}} < K_f$ for any other quasi-conformal map $f\colon X \to X'$. In addition, K is unique and q is unique up to multiplication by a scalar $\alpha > 0$.*

The map $f_{K,q}$ is called the *Teichmüller map* from X to X'. Now, given $X, X' \in \mathcal{T}(\Sigma)$, we can define *Teichmüller distance* as

$$d_{Teich}(X, X') = \frac{1}{2}\log K,$$

where $f_{K,q}$ is the Teichmüller map from X to X'.

Even if this metric is not Riemannian, one still has the property that for any two distinct points $X, X' \in \mathcal{T}(\Sigma)$, there exists a unique Teichmüller geodesic which can be described as the points $f_{t,q}(X)$, where $t \in [1, K]$. It you consider $t \to \infty$, you can then define the ray $r(q)$ in the direction of the quadratic differential q. Note that $r(q) = r(\alpha q)$ for $\alpha > 0$. It makes sense to define the projective quivalence class $[q] \in \mathcal{Q}^1(X) = \mathcal{Q}(X)/\mathbb{R}_{>0}$. Note that you can embed $\mathcal{Q}^1(X)$ in $\mathcal{Q}(X)$ as the set of holomorphic quadratic differential q on X such that the associated metric g_q has unit area.

Now, using Riemann-Roch Theorem, one can see that $\mathcal{Q}(X_0)$ has the structure of real vector space of dimension $6g - 6$ for any $X_0 \in \mathcal{T}(\Sigma)$. For any (K,q)–stretch map $f_{K,q}$ we denote $X_{K,q} = f_{K,q}(X_0)$. Note that Teichmüller Theorem tells us that $X_{K,q} = X_{K,q'}$ if and only if $[q] = [q'] \in \mathcal{Q}^1(X)$. In addition $X_{1,q} = X_0$ for all $q \in \mathcal{Q}(X)$. We can then consider the polar coordinates $(k,q) \in [0,1) \times \mathcal{Q}^1(X_0)$ on the (open) ball $\mathbb{B}^{6g-6}$ and define the map

$$\Omega_{X_0}\colon \mathbb{B}^{6g-6} \to \mathcal{T}(\Sigma)$$

by $\Omega_{X_0}(k,q) = f_{K,q}(X_0)$ where $K = \frac{1+k}{1-k}$.

Theorem 2.2.3 (Teichmüller)**.** *Ω_{X_0} is a homeomorphism.*

You can then define the *Teichmüller compactification* of Teichmüller space by extending Ω_{X_0} to be a homeomorphismof the closed ball $\overline{\mathbb{B}}^{6g-6}$. Note that $\partial_{Teich}(\mathcal{T}(\Sigma)) \equiv \mathbb{S}^{6g-7}$, so $\partial_{Teich}(\mathcal{T}(\Sigma)) \equiv \partial_{Thur}(\mathcal{T}(\Sigma))$.

2.2.3 Kerckhoff's Theorem

The *mapping class group* $\mathrm{Mod}(\Sigma)$ can be defined as the set of orientation preserving homeomorphisms of Σ up toisotopy. See Farb-Margalit [5] for learning more about this group. The reason why we introduce it here is because $\mathrm{Mod}(\Sigma)$ acts on $\mathcal{T}(\Sigma)$ sending Teichmüller rays to Teichmüller rays, but in general elements of $\mathrm{Mod}(\Sigma)$ do not fix the base point X_0. So a natural question is the following:

Question 1. Does the $\mathrm{Mod}(\Sigma)$–action extend continuously to $\overline{\mathcal{T}(\Sigma)}^{Teich}$? Equivalently, given $\varphi \in \mathrm{Mod}(\Sigma)$ are rays based at X_0 compatible with rays based at $\varphi(X_0)$?

Kerckhoff [9] answered this question in 1978 negatively.

Theorem 2.2.4 (Kerckhoff [9]). *There is no continuous extension of the* Mod(Σ)–*action to the Teichmüller compactification of* $\mathcal{T}(\Sigma)$.

For the proof, let $\varphi \in \mathrm{Mod}(\Sigma)$ such that $X = \varphi(\Sigma) \neq X_0$. Note that

$$\Phi^{-1}\colon \mathcal{Q}^1(X) \to \mathcal{Q}^1(X_0)$$

is a homeomorphism. Also given $q \in \mathcal{Q}^1(X_0)$, we denote $r_0(q)$ the Teichmüller ray based at X_0 and 'in the direction' of q. We define then

$$P_0\colon \mathcal{Q}^1(X_0) \to \partial_{Teich}(\mathcal{T}(\Sigma))$$

by sending $q \in \mathcal{Q}^1(X_0)$ to the endpoint of the ray $r_0(q)$. Now the closure $\overline{r_0(q)}$ of the Teichmüller ray $r_0(q)$ is exactly given by $\overline{r_0(q)} = r_0(q) \cup P_0(q)$. Consider the closure of a ray $\overline{r_X(q)}$ based at X. If $\overline{r_X(q)} \smallsetminus r_X(q)$ consists of a single point, we say that the ray $r_X(q)$ *converges.*

Suppose by contradiction that the Mod(Σ)–action extends continuously to $\overline{\mathcal{T}(\Sigma)}^{Teich}$. If φ is a homeomorphism of $\overline{\mathcal{T}(\Sigma)}^{Teich}$, then all rays $r_X(q)$ converge and

$$P_X\colon \mathcal{Q}^1(X) \to \partial_{Teich}(\mathcal{T}(\Sigma))$$

defined by $P_X(q) = \overline{r_X(q)} \smallsetminus r_X(q)$ is a homeomorphism. We will instead prove that P_X is discontinuous.

Recall that one of Thurston's Theorems prove that $\mathcal{S} \times \mathbb{R}_{\geq 0}$ embeds in the set of measured foliations $\mathcal{MF}(\Sigma)$ and the image is dense, and similarly the image of $\mathcal{S}$ is dense in $\mathcal{PMF}(\Sigma)$. So, given $\gamma \in \mathcal{S}$, we have a unique point $[\gamma] \in \mathcal{PMF}(\Sigma)$ and so there exists a unique JS differential in $\mathcal{Q}^1(X)$, which we denote $[\gamma]_X$. With abuse of notation we will denote $[\gamma]_X$ also the ray from X associated to $[\gamma]_X$.

We can now recall two results that Kerckhoff proved and that we will need in this proof. Let $\gamma \in \mathcal{S}$ and let σ be a pants decomposition of Σ.

Proposition 2.2.5 (Kerckhoff [9]).

- *For every* $X \in \mathcal{T}(\Sigma)$ *and for every* $\gamma \in \mathcal{S}(\Sigma)$, $[\gamma]_X$ *is asymptotic to* $[\gamma]_{X_0}$.
- *For every* $X \in \mathcal{T}(\Sigma)$ *and for every* σ *pants decomposition of* Σ, $[\sigma]_X$ *is asymptotic to* $[\sigma]_{X_0}$ *iff* $[\sigma]_X$ *is modularly equivalent to* $[\sigma]_{X_0}$.

Now, let

$$PF_X\colon \mathcal{Q}^1(X) \to \mathcal{PMF}(\Sigma)$$

be defined by $PF_X(q) = [F_q^{hor}] \in \mathcal{PMF}(\Sigma)$. This map is continuous, injective and surjective [Exercise!]. Let $\gamma \in \mathcal{S}$ be a curve intersecting all the pants curves in σ, let D_σ be the Dehn-twist about σ and let $\gamma^n = D_\gamma^n(\gamma)$, then $[\gamma^n] \in \mathcal{PMF}(\Sigma)$ converges to $[\sigma]$ with equal measure on each curve. If we consider the pre-images under PF_X, we have $PF_X^{-1}[\gamma^n] \to PF_X^{-1}[\sigma] = [\sigma]_X^e$, where $[\sigma]_X^e$ is the JS differential with core curves σ and cylinders with equal heights, but with lengths of σ_i that varies according to X. So if you choose $\varphi \in \mathrm{Mod}(\Sigma)$ such that for $X = \varphi(X_0)$ and X_0 the JS diffeerentials $PF_X^{-1}[\sigma]$ and $PF_{X_0}^{-1}[\sigma]$ are not modularly equivalent, then:

- By Proposition 2.2.5 (i), $[\gamma^n]_X$ and $[\gamma^n]_{X_0}$ converge for all n.
- By Proposition 2.2.5 (ii), $[\sigma]^e_X$ and $[\sigma]^e_{X_0}$ do not converge (but $[\sigma]^e_X$ is asymptotic to $[\sigma]^\mu_{X_0}$ with $\mu \neq e$).

This proves that P_X is discontinuous, since $[\gamma^n]_X \to [\sigma]^e_X$ but $P([\gamma^n]_X) = P([\gamma^n]_{X_0}) \to P([\sigma]^e_{X_0}) \neq P([\sigma]^\mu_{X_0}) = P([\sigma]^e_X)$, and so concludes the proof.

Kerckhoff noticed that if we consider the quotient of the Teichmüller boundary obtained by collapsing holomorphic quadratic differentials with same horizontal foliations after forgetting the transverse measures, than the action of the mapping class group extends continuously to this space.

2.3 Bers compactification

2.3.1 Definition of Bers compactification — a crash course of quasi-Fuchsian groups

A discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is called *Fuchsian*, while a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ is called *Kleinian.* Any Kleinian group Γ will acts properly discontinuosly on $\mathbb{H}^3$, but there will be accumulation points in $\mathbb{CP}^1$. We can then define:

- the *limit set* Λ_Γ as the set of accumulation points in $\mathbb{CP}^1$ for the action of Γ on $\mathbb{H}^3$ (or on $\mathbb{CP}^1$);
- the *domain of discontinuity* Ω_Γ as the set of points in $\mathbb{CP}^1$ where Γ acts properly discontinuosly.

One can then prove that $\mathbb{CP}^1 = \Lambda_\Gamma \sqcup \Omega_\Gamma$.

We say that a Kleinian group Γ is *quasi-Fuchsian* if Λ_Γ is topologically a circle or, equivalently, if Ω_Γ splits into two open Γ–invariant disks Ω^+ and Ω^-. We define $\mathcal{QF}(\Sigma)$ as the set of representations $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{C})$ such that $\Gamma_\rho = \rho(\pi_1(\Sigma))$ is quasi-Fuchsian, up to conjugation. We also have $M_\rho = \mathbb{H}^3/\Gamma_\rho$ is homeomorphic to $\Sigma \times (0,1)$ and its compactification $M_\rho = (\mathbb{H}^3 \cup \Omega_{\Gamma_\rho})/\Gamma_\rho$ is homeomorphic to $\Sigma \times [0,1]$.

Bers' Simultaneous Uniformization Theorem [3] tells us that the complex structures of Ω^+/Γ_ρ and Ω^-/Γ_ρ uniquely determines the 3–manifold M_ρ. So it defines a homeomorphism

$$QF\colon \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathcal{QF}(\Sigma).$$

Given $X_0 \in \mathcal{T}(\Sigma)$ we can define an embedding $B_{X_0}\colon \mathcal{T}(\Sigma) \to \mathcal{QF}(\Sigma)$ by $B_{X_0}(X) = QF(X, X_0)$ and the image of this map is called a *Bers slices over* X_0. Similarly one can also define $B^{X_0}\colon \mathcal{T}(\Sigma) \to \mathcal{QF}(\Sigma)$ by $B^{X_0}(X) = QF(X_0, X)$. There are other important slides of $\mathcal{QF}(\Sigma)$, like the Maskit or the Earle slice, but we will not have time to discuss them here. Note also that we defined $\mathcal{QF}(\Sigma)$ as a set of representations, but with abuse of notation we will also refer to its elements to be 'groups' by considering the image groups.

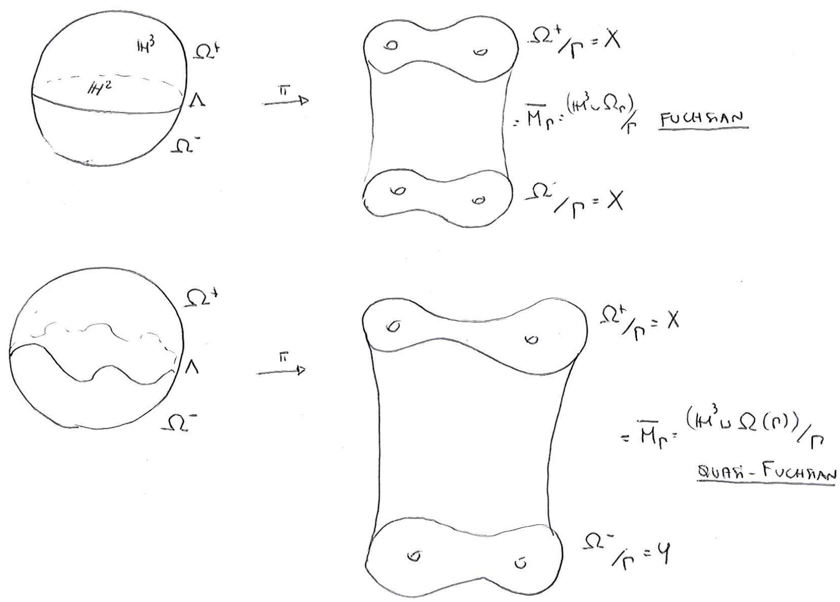

Figure 2.2: Fuchsian and quasi-Fuchsian manifold.

Bers proved the following result. Remember that $\mathrm{AH}(\Sigma)$ is the set of discrete and faithful representations $\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{C})$ with the algebraic convergence topology. The Density Theorem tells us that $\mathrm{AH}(\Sigma)$ corresponds to the closure of $\mathcal{QF}(\Sigma)$. The proof uses the fact that the translation length of an element in a quasi-Fuchsian group is bounded above by the lengths in the upper and lower conformal structures. On the other hand, a similar bound can be also proved using Sullivan's theorem.

Theorem 2.3.1 (Bers [4])**.** *The Bers slice is relatively compact (that is, its closure is compact) in* $\mathrm{AH}(\Sigma)$*. In addition, the Kleinian groups in the boundary of a Bers slice are b-groups (boundary groups) and each group has a unique invariant component in the domain of discontinuity*

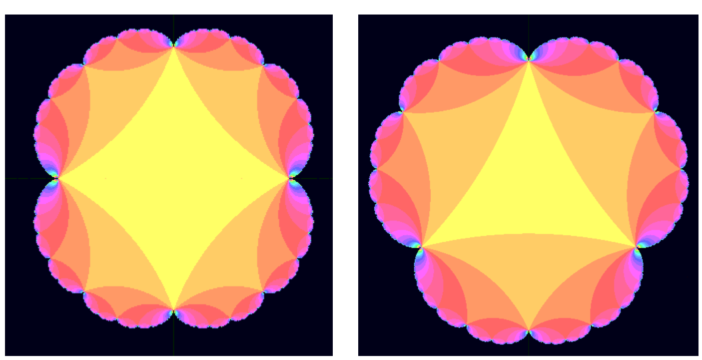

Figure 2.3: A Bers slice with base surface an 'square torus' (left) and an 'exagonal torus' (right).[Picture courtesy of Y. Yamashita]

You can den define the *Bers compactification* of Teichmüller space as the closure of the image of B_{X_0} in $\mathrm{AH}(\Sigma)$. As noted above, one can adapt all the discussion to the image of B^{X_0}. Its boundary is called the *Bers boundary* of Teichmüller space and is denoted $\partial_{Bers}(\mathcal{T}(\Sigma))$. The Kleinian groups Γ in the boundary of a Bers slice $B_{X_0}(\mathcal{T}(\Sigma))$ are *b-groups* (boundary groups) and for those groups there exists only one Γ–invariant component Ω^- in Ω_Γ which is the one such that $\Omega^-/\Gamma \cong X_0$. In particular, the b-groups in the boundary of a Bers slice are of 3 types:

- *cusp groups* or *geometrically finite group* and in that case $(\Omega_\Gamma \smallsetminus \Omega^-)/\Gamma$ is a surface obtained from Σ by pinching some curves.
- *singly degenerate group* and in that case $\Omega_\Gamma = \Omega^-$
- *partially degenerate group* and in that case $(\Omega_\Gamma \smallsetminus \Omega^-)/\Gamma$ is a strict subsurface of Σ with possibly some pinched curves.

So, if $\Omega_\Gamma = \Omega^-$, then Γ is singly degenerate, while if $\Omega^- \subsetneq \Omega_\Gamma$, then $(\Omega_\Gamma \smallsetminus \Omega^-)/\Gamma \in \mathcal{T}(F)$, where F is a subsurface of Σ possibly disconnected and ∂F corresponds to a parabolic element. Let P be the union of such parabolic elements. By Margulis Lemma we can consider disjoint (open) neighborhoods of the $\mathbb{Z}$–cusps, and denote it $N_\varepsilon(P)$. Each cusp neighborhoods has a core curve homotopic to one of the boundary components of F. Let $M_0 = \overline{M}_\Gamma \smallsetminus N_\varepsilon(P)$. The *ends* of M_0 face X_0, a component of F or a component of $\Sigma \smallsetminus F$.

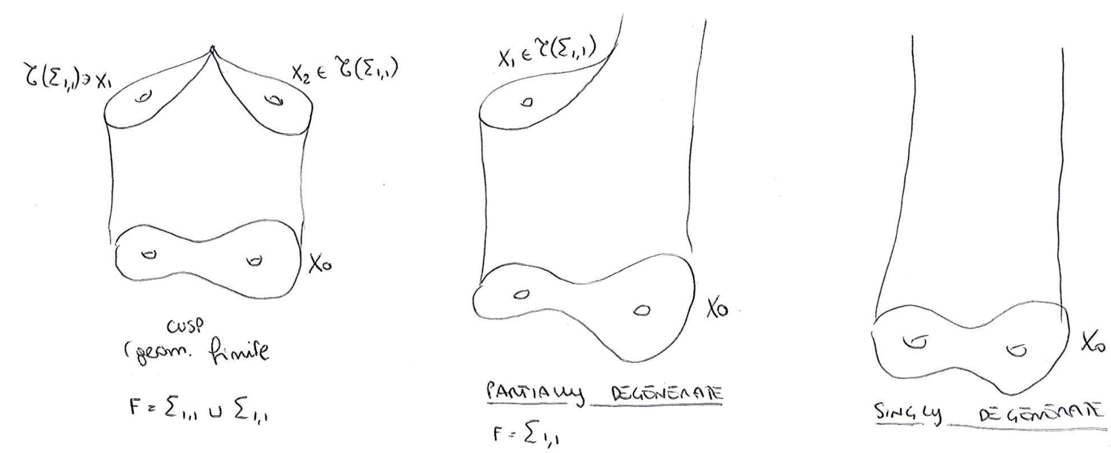

Figure 2.4: Types of b-groups in the boundary of a Bers slice.

An interesting addendum will be the discussion of ending laminations and the Ending Lamination Theorem, see ... for references.

2.3.2 Geometric limits

We will now discuss geometric convergence and some examples because we will need it in the proof of Kerckhoff–Thurston's Theorem in the next section.

Let G_n be a sequence of Kleinian groups. We say that G_n converges to H *geometrically* (and H is called the *geometric limit* of G_n) if

- $\forall h \in H$, $\exists g_n \in G_n$ such that $g_n \to h$; and
- $\forall$ convergent subsequences $g_{n_k} \in G_{n_k}$ with $g_{n_k} \to \widehat{g}$, then $\widehat{g} \in H$.

Jørgensen was one of the first people to study such limits, but we will present an example from Thurston [15]. As a first example we discuss a famous example due to . Let $\rho_n\colon \mathbb{Z} \to \mathrm{PSL}(2,\mathbb{C})$ be the representations defined by $\rho_n(1) = \begin{bmatrix} \exp(\omega_n) & n\sinh(\omega_n) \\ 0 & \exp(-\omega_n) \end{bmatrix}$, where $\omega_n = \frac{1}{n^2} + \pi\frac{i}{n}$. The elements $\rho_n(1)$ are loxodromic elements with axis the geodesic between $a_n = -\frac{n}{2} \in \mathbb{C}$ and ∞. So given $x \in \mathbb{H}^3$ and $n \in \mathbb{N}$, the element $\rho_n(1)$ moves x around the cone centered at the axis of $\rho_n(1)$. See the picture. On the other hand, as $n \to \infty$ the axis moves farther and farther from x. Also notes that $p_n(n)$ translates x vertically and $p_n(1)$ and $p_n(n)$ translate x by roughly the same amount. So, since $\rho_n(n) \to \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we have that:

- ρ_n converges algebraically to $\langle \begin{bmatrix} 1 & \pi i \\ 0 & 1 \end{bmatrix} \rangle$; and
- ρ_n converges geometrically to $\langle \begin{bmatrix} 1 & \pi i \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$.

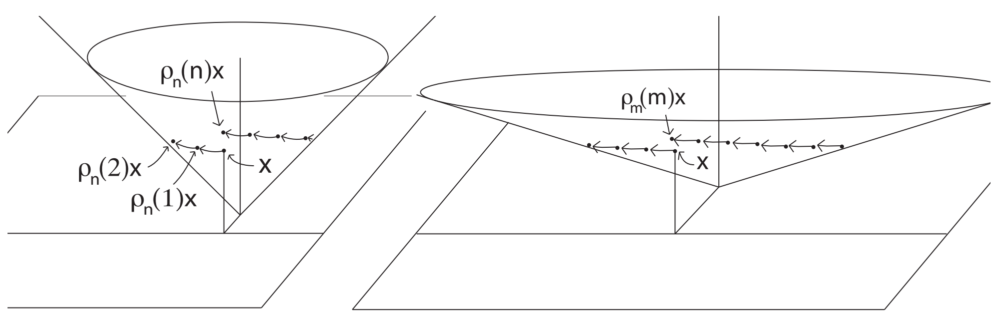

Figure 2.5: Cones around the axis of $\rho_n(1)$ on which the $\rho_n(1)$–orbits of the point x lie. [Picture courtesy of J. Brock.]

2.3.3 Kerckhoff–Thurston's Theorem

Theorem 2.3.2 (Kerckhoff–Thurston [10]). *There is no continuous extension of the* $\mathrm{Mod}(\Sigma)$*–action to the Bers compactification of* $\mathcal{T}(\Sigma)$.

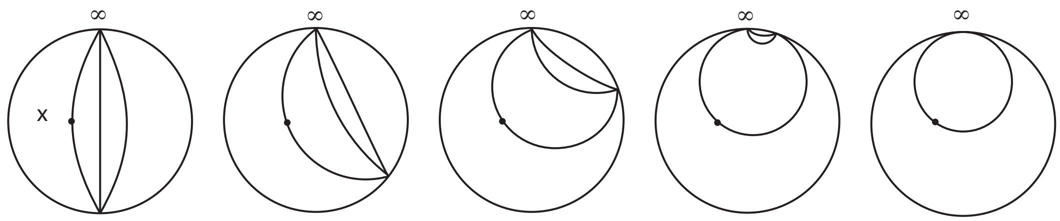

Figure 2.6: Limit of the axis of $\rho_n(1)$. [Picture courtesy of J. Brock.]

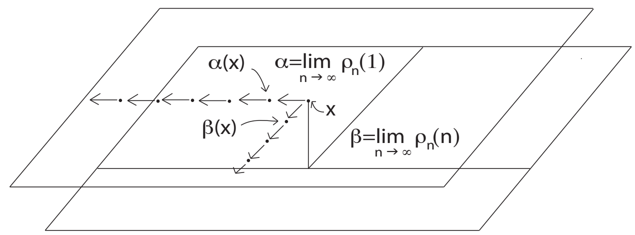

Figure 2.7: Elements in the geometric limit. [Picture courtesy of J. Brock.]

Note that the Mod(Σ)–action extends continuously to the Bers compactification of $\mathcal{T}(\Sigma)$ if and only if there exists a continuous extension of the natural identification between two Bers slices $B^{X_0}(\mathcal{T}(\Sigma))$ and $B^{\varphi(X_0)}(\mathcal{T}(\Sigma))$ for all $\varphi \in \mathrm{Mod}(\Sigma)$.

For simplicity, in the proof we will focus on a special case: the genus 2 surface $\Sigma = \Sigma_2$. This example can be modified for the general case. Let $\gamma \in \Sigma$ be the separating curve, see the picture. Let $D_\gamma \in \mathrm{Mod}(\Sigma)$ be the Dehn-twist around the curve γ and let $(D_\gamma)_\star\colon \pi_1(\Sigma) \to \pi_1(\Sigma)$ be the associated map. Let $X_0 \in \mathcal{T}(\Sigma)$, and consider the representation $\rho_n\colon \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{C})$ and $\Gamma_n = \rho_n(\pi_1(\Sigma))$ defined by $QF(X_0, (D_\gamma^n)_\star(X_0))$. Then sequence ρ_n converges algebraically $\rho_n \xrightarrow{\text{alg.}} \rho_\infty$ to a geometrically finite b-group $\Gamma_A = \rho_\infty(\pi_1(\Sigma)) \in \partial(B^{X_0})$ with parabolic locus $P = \{\gamma\}$ and lower conformal structures in $\mathcal{T}(\Sigma \smallsetminus \{\gamma\})$. Kerckhoff and Thurston proved that Γ_n converge geometrically to Γ_G, that is $\Gamma_n \xrightarrow{\text{geom.}} \Gamma_G$. Let $M_G = \mathbb{H}^3/\Gamma_G$ and Kerckhoff and Thurston calculated its topology.

Theorem 2.3.3 (Kerckhoff–Thurston [10])**.** $M_G \cong \Sigma \times \mathbb{R} \smallsetminus (\{\gamma\} \times \{0\})$.

We will now describe how to understand these limit manifold. We will show that in the geometric limit there is a rank–2 cusp with tubolar neighborhood $N_\varepsilon(\gamma)$ such that $\pi_1(N_\varepsilon(\gamma)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be generators of $\pi_1(\Sigma)$. Then $(D_\gamma)_\star$ fixes α_1, β_1 but α_2, β_2 are conjugated by γ.

By precomposing the representations ρ_n by $(D_\gamma)_\star^{-1}$, we can define the representations $\rho_n' = \rho_n \circ (D_\gamma)_\star^{-1}$, which correspond to the groups $QF((D_\gamma)_\star^{-1}X_0, X_0)$ and so belong to the Bers slice $B_{X_0}(\mathcal{T}(\Sigma))$, which is also relatively compact in AH(Σ). On

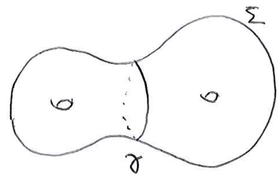

Figure 2.8: The surface $\Sigma = \Sigma_2$ and the curve $\gamma \subset \Sigma$.

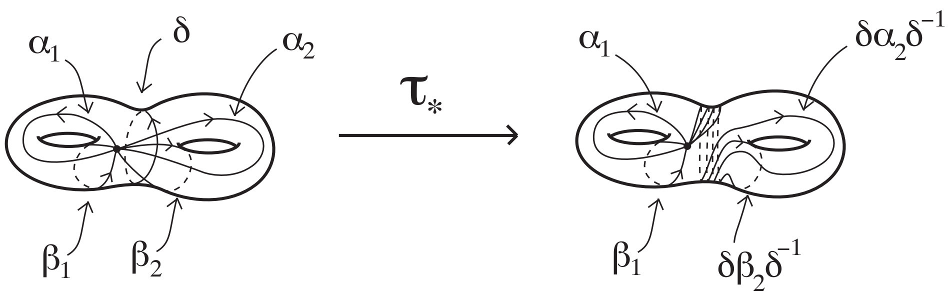

Figure 2.9: The elements $\alpha_1, \beta_1, \alpha_2, \beta_2$ on $\Sigma = \Sigma_2$ and their image under the Dehn twist $(D_\gamma)_\star$.

the other hand, by precomposing by $(D_\gamma)_\star^{-1}$ we do not change the Kleinian groups (but only their marking), that is $\rho_n(\pi_1(\Sigma)) = \rho_n'(\pi_1(\Sigma))$, so the geometric limit of ρ_n and ρ_n' coincides to Γ_G. Recall that $\gamma^n = D_\gamma^n(\gamma)$.

In Γ_G we have the following elements:

- $\lim \rho_n(\alpha_1) = \lim \rho_n'(\alpha_1)$;
- $\lim \rho_n(\beta_1) = \lim \rho_n'(\beta_1)$;
- $\lim \rho_n(\alpha_2)$;
- $\lim \rho_n(\beta_2)$;
- $\lim \rho_n'(\alpha_2) = \lim(\rho_n(\gamma^n)\rho_n(\alpha_2)\rho_n(\gamma^{-n}))$;
- $\lim \rho_n'(\beta_2) = \lim(\rho_n(\gamma^n)\rho_n(\beta_2)\rho_n(\gamma^{-n}))$;
- $\lim \rho_n(\gamma)$.

Exercise 1. Prove the following facts:

1. $\rho_n(\gamma^n) \to \overline{\gamma} \in \mathrm{PSL}(2,\mathbb{C})$ (so $\overline{\gamma} \in \Gamma_G$.
2. $\overline{\gamma} \notin \Gamma_A$.

3. $\langle \overline{\gamma}, \rho_\infty(\gamma) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$.

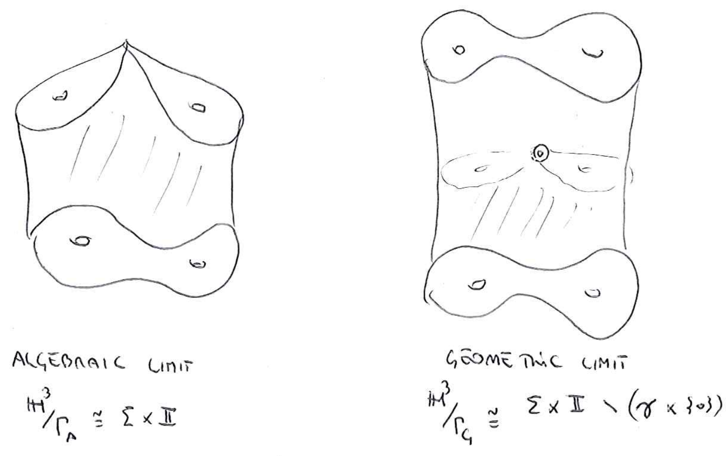

Figure 2.10: The quotient $\mathbb{H}^3/\Gamma_A$ and $\mathbb{H}^3/\Gamma_G$.

This defines a map

$$r \colon \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma_{1,1}) \times \mathcal{T}(\Sigma_{1,1})$$

defined by sending $(X_0, Y_0) \in \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ onto the lower structures of $QF(X_0, (D_\gamma)^n_\star Y_0)$. If the $\mathrm{Mod}(\Sigma)$–action extends continuously to the boundary of the Bers slice $B^{X_0}(\mathcal{T}(\Sigma))$, then the base changing between the Bers slices by mapping classes $\varphi \in \mathrm{Mod}(\Sigma)$ should extends continuosly to the boundary. So r should descends to

$$\overline{r} \colon (\mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)) \times \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma_{1,1}) \times \mathcal{T}(\Sigma_{1,1})$$

. Now we claim the following result. For details of the proof you can see Section 4 of Kerckhoff–Thurston. This finishes the proof of Theorem 2.3.2.

Claim 2.3.4. *There exists an element* $\varphi \in \mathrm{Mod}(\Sigma)$ *such that* $r(X_0, \cdot) \colon \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma_{1,1}) \times \mathcal{T}(\Sigma_{1,1})$ *and* $r(\varphi_\star(X_0), \cdot) \colon \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma_{1,1}) \times \mathcal{T}(\Sigma_{1,1})$ *are different.*

Note that, similarly to the case of Teichmüller compactification descussed before, the problems in the non-extendability of the action of the mapping class group seem to depend on the existence of the deformation spaces $\mathcal{T}(\Sigma_{1,1}) \times \mathcal{T}(\Sigma_{1,1})$, so Thurston conjectured that if one considers the quotient space obtained by collapsing each deformation space, then the $\mathrm{Mod}(\Sigma)$–action extends continuously, and Ohshika proved this conjecture.

2.4 Thurston compactification

For the definition of Thurston's compactification I decided to summarize Leininger's approach, which followed mostly Fathi–L There are a lot of different but equivalent approaches using:

- Earthquakes;
- Gromov limit and $\pi_1(\Sigma)$–actions on $\mathbb{R}$–trees;
- Bonahon's theory of geodesic currents;
- Gromov's theory of horofunction compactification (see Section 2.6);
- holomorphic quadratic differentials, as Hubbard and Masur discuss.

We will only give a sketch of the ideas behind this compactification and we will refer to Leininger's [11] or the references listed there for the details and the proofs. This compactification is particularly important because the Mod(Σ)–action on $\mathcal{T}(\Sigma)$ will extend continuosly to the compactification.

Recall that given $X \in \mathcal{T}(\Sigma)$, the length function $\ell_X \colon \mathcal{S}(\Sigma) \to \mathbb{R}_+$ is defined by $\ell_X(\alpha) = 2\cosh^{-1}(\frac{|\mathrm{tr}(\rho_X(\alpha))|}{2})$ where $\alpha \in \mathcal{S}(\Sigma)$ and ρ_X is the holonomy of X. This defines a map

$$\ell \colon \mathcal{T}(\Sigma) \to (\mathbb{R}_+)^{\mathcal{S}}$$

where $\ell(X) = \ell_X$.

Theorem 2.4.1. *The map ℓ is a proper embedding. In fact, the*

The proof of this result uses Fenchel–Nielsen coordinates, see Section 2 in Leininger In sections 3, 4 and 5, Chris introduces in details the space of measured foliations, of measured laminations and their relationship, which we recall here for completeness:

Theorem 2.4.2 (Thurston)**.** *There is an $\mathbb{R}_+$–invariant homeomorphism, called the straightening map*

$$\mathrm{Str} \colon \mathcal{MF}(\Sigma) \to \mathrm{ML}(\Sigma)$$

which is the identity on the image of $\mathbb{R}_+ \times \mathcal{S}(\Sigma)$.

Thurston also defined the map

$$i_{ML} \colon \mathrm{ML}(\Sigma) \to (\mathbb{R}_+)^{\mathcal{S}}$$

by $i_{ML}(\lambda) = i(\lambda, \cdot) \colon \mathcal{S}(\Sigma) \to \mathbb{R}_+$ where i is the geometric intersection number. We can also projectivize the map to a map

$$i_{PML} \colon \mathrm{PML}(\Sigma) \to \mathbb{P}((\mathbb{R}_+)^{\mathcal{S}}).$$

Thuston proved the following very important result. Let $\pi \colon ((\mathbb{R}_+)^{\mathcal{S}} \smallsetminus \{0\}) \to \mathbb{P}((\mathbb{R}_+)^{\mathcal{S}})$.

Theorem 2.4.3 (Thurston)**.** *The image of $\pi \circ \ell$ is relatively compact in $\mathbb{P}((\mathbb{R}_+)^{\mathcal{S}})$. Its boundary coincides with the image of i_{PML} (which is homeomorphic to a sphere $\mathbb{S}^{6g-7}$) and the closure $\overline{\pi \circ \ell(\mathcal{T}(\Sigma))} = \pi \circ \ell(\mathcal{T}(\Sigma)) \cup i_{PML}(\mathrm{PML}(\Sigma))$ is homeomorphic to a closed ball $\overline{\mathbb{D}^{6g-6}}$.*

The *Thurston compactification* $\overline{\mathcal{T}(\Sigma)}^{Thu}$ of Teichmüller space is then defined to be $\overline{\pi \circ \ell(\mathcal{T}(\Sigma))}$ and *Thurston boundary* is $\partial_{Thu}(\mathcal{T}(\Sigma)) = i_{PML}(\mathrm{PML}(\Sigma))$.

Since the action of $\mathrm{Mod}(\Sigma)$–action on $\mathbb{P}((\mathbb{R}_+)^{\mathcal{S}})$ is continuous and compatible with the action on $\mathcal{T}(\Sigma)$, we have the following result.

Corollary 2.4.4. *The* $\mathrm{Mod}(\Sigma)$*–action on* $\mathcal{T}(\Sigma)$ *extends continuously to the Thurston compactification* $\overline{\mathcal{T}(\Sigma)}^{Thu}$.

A nice application of these results is the classification of elements of $\mathrm{Mod}(\Sigma)$ as periodic, reducible or pseudo-Anosov. See Leininger [11].

2.5 Gardiner–Masur compactification

Given a simple closed curve $\gamma \subset \Sigma$ and given $X \in \mathcal{T}(\Sigma)$, we define the *extremal length* $\mathrm{Ext}_X(\gamma) = \frac{1}{\mathrm{mod}(\gamma)}$, where $\mathrm{mod}(\gamma)$ is the supremum of the moduli of the cylinders embedded in X with core curve homotopic to γ.

Proposition 2.5.1 (Kerckhoff)**.** $d_{Teich}(X, X') = \frac{1}{2}\log(\sup_{\alpha \in \mathcal{S}(\Sigma)} \frac{\mathrm{Ext}_X(\alpha)}{\mathrm{Ext}_Y(\alpha)})$.

Gardiner and Masur studied the map

$$\Phi_{GM} \colon \mathcal{T}(\Sigma) \to \mathbb{P}(\mathbb{R}_+^{\mathcal{S}})$$

defined by $\Phi_{GM}(X)(\alpha) = (\mathrm{Ext}_X(\alpha))^{\frac{1}{2}}$ and proved the following result.

Theorem 2.5.2 (Gardiner-Masur)**.** *Φ_{GM} is an embedding and the image is relatively compact.*

So we define the *Gardiner-Masur compactification* of Teichmüller space $\overline{\mathcal{T}\Sigma}^{GM}$ as the closure of the image of Φ_{GM}, and the *Gardiner-Masur boundary* as $\partial_{GM}(\mathcal{T}(\Sigma)) = \overline{\mathcal{T}\Sigma}^{GM} \smallsetminus \Phi_{GM}(\mathcal{T}(\Sigma))$.

Theorem 2.5.3 (Gardiner-Masur [6])**.** *$\mathcal{PMF}(\Sigma) \subset \partial_{GM}(\mathcal{T}(\Sigma))$. In addition, if $\Sigma \neq \Sigma_{1,1}, \Sigma_{0,4}$, then $\mathcal{PMF}(\Sigma) \subsetneq \partial_{GM}(\mathcal{T}(\Sigma))$.*

Using Borsuk-Ulam, one can see the following.

Corollary 2.5.4. *If $\Sigma \neq \Sigma_{1,1}, \Sigma_{0,4}$, then $\partial_{GM}(\mathcal{T}(\Sigma)) \ncong \mathbb{S}^{6g-7}$.*

Proposition 2.5.5. *If $\Sigma = \Sigma_{1,1}, \Sigma_{0,4}$, then $\partial_{GM}(\mathcal{T}(\Sigma)) = \mathcal{PMF}(\Sigma) \cong \mathbb{S}^{6g-7}$.*

Some very interesting questions that are still open, as far as I know:

Question 2.

1. What is the topology of $\overline{\mathcal{T}\Sigma}^{GM}$ and $\partial_{GM}(\mathcal{T}(\Sigma))$?

2. Which geometric objects do the points in $\partial_{GM}(\mathcal{T}(\Sigma))$ represent?

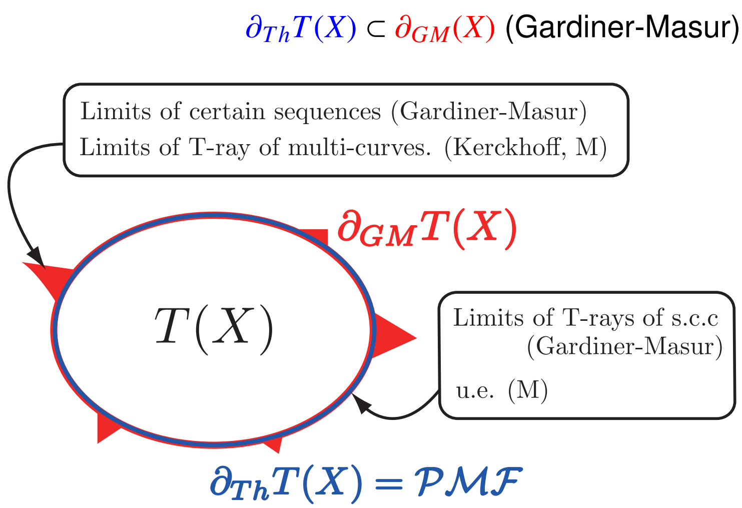

Figure 2.11: The surface $\Sigma = \Sigma_2$ and the curve $\gamma \subset \Sigma$.[Picture courtesy of H. Miyachi.]

2.6 Horofunction compactification (á la Gromov)

Let (M,d) be a locally compact geodesic metric space. Fix a basepoint $x_0 \in M$ and for all $z \in M$ let $\Phi_z\colon M \to \mathbb{R}$ be defined by $\Phi_z(x) = d(x,z) - d(x,x_0)$. Let $\mathcal{C}(M)$ be the set of continuous functions $f\colon M \to \mathbb{R}$ endowed with the locally uniform convergence topology. Gromov [7] proved that

$$\Phi\colon M \to \mathcal{C}(M)$$

defined by $\Phi(z) = \Phi_z$ is a proper embedding, and the closure of the image is called the *horofunction compactification* and the elements of $\partial_{horo}(M,d) = \overline{\Phi(M)} \smallsetminus \Phi(M)$ are called *horofunctions*. Note that if you chhose a different basepoints $x_0' \in M$, then the new function defined at a point $z \in M$ is related to the old one by $\Phi_z'(x) = \Phi_z(x) - \Phi_z(x_0')$, so the new horofunction boundary obtained with the new basepoint is homeomorphic to the old one. Note also that the action of the isometry group of (M,d) extends continuously to an action by homeomorphisms on the horofunction boundary.

We have the following results. Recall that the *Thurston distance* on Teichmüller space is defined as follows: let $X, Y \in \mathcal{T}(\Sigma)$, then $d_{Thur}(X,Y) = \log \sup_{\alpha\in\mathcal{S}} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)}$. Note that since this is a non-symmetric metric, the definition above needs to be slighyly modified, but we refer the reader to Ballmann [2].

Theorem 2.6.1 (Walsh [16])**.** *The horofuction compactification of $(\mathcal{T}(\Sigma), d_{Thur})$ is homeomorphic to Thurston's compactification of Teichmüller space.*

Theorem 2.6.2 (Liu–Su [12])**.** *The horofuction compactification of $(\mathcal{T}(\Sigma), d_{Teich})$ is homeomorphic to Gardiner–Masur compactification of Teichmüller space.*

Theorem 2.6.3 (Alessandrini–Liu–Papadopoulos–Su [1]). *The horofunction compactification of the arc metric on $\mathcal{T}(\Sigma)$ is homeomorphic to Thurston's compactification of Teichmüller space.*

As a corollary of Theorem 2.6.2, Miyachi observed the following result. Recall that the *extended mapping class group* is $\mathrm{Mod}^{\pm}(\Sigma) \cong \mathrm{Out}(\pi_1(\Sigma))$.

Proposition 2.6.4 (Miyachi). *Theaction of* $\mathrm{Mod}^{\pm}(\Sigma)$ *on* $\mathcal{T}(\Sigma)$ *extends continuously to the Gardiner–Masur compactification of Teichmüller space.*

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Chapter 3

Morgan-Shalen compactification

Léo Bénard

3.1 Introduction

This text arises from the notes of a talk I gave at the Workshop "Compactification of Moduli Spaces" in Montana. It is my pleasure to thank the organizers Brian Collier, Giuseppe Martone and Jeremy Toulisse for inviting me to this great experience. I'm going to talk about the so-called "Morgan-Shalen" compactification of character varieties. Let us give an overview of the construction : here Γ will be a finitely generated group, and we will denote by $X(\Gamma)$ its *space of characters*. For $X \subset X(\Gamma)$ a subvariety, the construction starts by associating to an *ideal point* a *valuation* $v : \mathbb{C}(X)^* \to \Lambda$, where Λ is an abelian ordered group. This valuation measures "how fast traces go to infinity" through this ideal point. From this valuation, following the seminal construction from Bass and Serre, they construct a Λ-*tree* endowed with a Γ-action by isometries, determined by traces valuation. The first motivation was the following : in the early eighties, Marc Culler and Peter Shalen used this construction to obtain important results in 3 dimensional topology. Here $\Gamma = \pi_1(M)$ is the fundamental group of a 3 manifold M, and X is a curve of characters. The smooth projective model $\widehat{X}$ of X can be seen as the set of valuations $v : \mathbb{C}(X)^* \to \mathbb{Z}$, hence any ideal point $x \in \widehat{X} \setminus X$ provides such a valuation v, and a simplicial tree T_v, with a Γ-action. Then one can construct a Γ-equivariant map $f = \tilde{M} \to T_v$, and pull-back the set of mid-points of the edges of T_v, say E. One obtains a surface $f^{-1}\{E\} = \tilde{S} \subset \tilde{M}$ that is Γ-invariant, hence a surface $S \subset M$. Up to slight modifications, one can prove furthermore that $\pi_1(S) \hookrightarrow \pi_1(M)$, and one says that S is dual to T_v, see Figure 3.1.

This construction led to great new results in topology of 3-manifolds, for instance a proof of the weak Neuwirth conjecture that can be formulated as follows :

Theorem 3.1.1 (Culler-Shalen '83)**.** *For M a 3 manifold with boundary a torus, there is a separating surface $S \subset M$, with $\pi_1(S) \hookrightarrow \pi_1(M)$ and $\partial S \neq \varnothing$, which is not parallel to ∂M.*

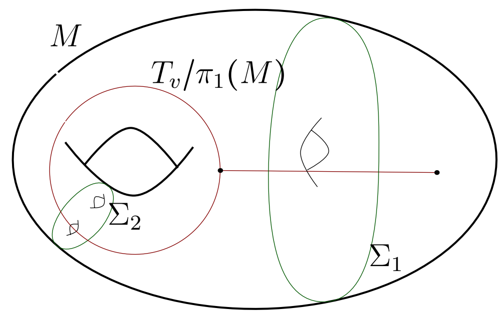

Figure 3.1: The surface $\Sigma = \Sigma_1 \cup \Sigma_2 \subset M$ in green, and the dual graph $T_v/\pi_1(M)$ in red.

What I'm gonna describe here is how those techniques applied to give a compactification of the whole set of characters. When $\Gamma = \pi_1(S)$ is a surface group, the Teichmüller space $\mathcal{T}_S$ embeds as a real subvariety of $X(\Gamma)$, and some ideal points of Teichmüller space can again provide a simplicial tree with Γ-action, and in the same way a system of curves dual to this tree. But some other ideal points give rise to more complicated objects, namely Λ-trees, that are dual to Thurston's mesured laminations.

Let's enter into the construction. In the first section I will recall what is needed on character varieties theory, and present a nice way to compactify them, independent of a choice of coordinates. Then in the second section I will relate this construction with valuations, and describe some particular kind of trees, namely $\mathbb{R}$-trees, that will be seen to come into the picture. We will see that the natural action of Γ on this tree has no fixed point, which is the basic step to allow the geometric construction described above.

The references to follow along the lecture are [MS87, Otal], and omitted proofs in this text can be found there.

3.2 Character varieties and compactification

3.2.1 Character varieties

Definition 3.2.1. Let Γ be a group generated by $\{\gamma_1, ..., \gamma_n\}$, then its *representation variety* is defined by $R(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. It is an affine complex variety, embedded by $R(\Gamma) \hookrightarrow \mathrm{SL}_2(\mathbb{C}) \subset \mathbb{C}^{4n}$, $\rho \mapsto (\rho(\gamma_1), ..., \rho(\gamma_n))$. Its functions algebra is

$$\mathbb{C}[R(\Gamma)] = \mathbb{C}[X_\gamma^{i,j}, \gamma \in \Gamma, i, j = 1, 2]/(X_\gamma X_\delta - X_{\gamma\delta}, X_e - 1, \det(X_\gamma) - 1)$$

where $X_\gamma = \begin{pmatrix} X_\gamma^{1,1} & X_\gamma^{1,2} \\ X_\gamma^{2,1} & X_\gamma^{2,2} \end{pmatrix}$.

The group $\mathrm{SL}_2(\mathbb{C})$ is acting by conjugation on $R(\Gamma)$, however the quotient $R(\Gamma)/\mathrm{SL}_2(\mathbb{C})$ is not an algebraic variety, nor Hausdorff in general.

Example 3.2.2. Let $\Gamma = \mathbb{Z}$, define $\rho_1, \rho_2 \in R(\mathbb{Z})$ by $\rho_1(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\rho_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ The one parameter subgroup $\left\{ M_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}_{t \in \mathbb{C}^*}$ acts on ρ_2 by $M_t.\rho_2(1) = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}$ hence $\rho_1 \in \overline{\mathrm{SL}_2(\mathbb{C}).\rho_2}$. In particular it can't be separated from ρ_2 in the quotient despite ρ_1 and ρ_2 are not conjugated.

Definition 3.2.3. We define the *character variety* $X(\Gamma)$ to be the biggest Hausdorff quotient of ${R(\Gamma)}\diagup{\mathrm{SL}_2(\mathbb{C})}$. By classical G.I.T. arguments, it is an affine algebraic variety with functions algebra $\mathbb{C}[X(\Gamma)] = \mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2(\mathbb{C})}$, the sub-algebra of invariants.

Theorem 3.2.4 (Procesi '87)**.** *The map from* $\mathbb{C}[Y_\gamma, \gamma \in G]/(Y_e - 2, Y_\gamma Y_\delta - Y_{\gamma\delta} - Y_{\gamma\delta^{-1}})$ *onto* $\mathbb{C}[X(\Gamma)]$ *that sends* Y_γ *to* $X_\gamma^{1,1} + X_\gamma^{2,2}$ *is an isomorphism of algebras.*

The Y_γ's are called *trace functions*, they map the character $[\rho]$ of a representation to the complex number $\mathrm{Tr}(\rho(\gamma))$.

Remark 3.2.5.

1. All those varieties are defined over $\mathbb{Q}$
2. In the sequel we will assume that the function rings have no nilpotent elements, up to quotient them by their nilradical.
3. Those are algebraic sets, but not necessarily *irreducible* algebraic sets, nevertheless we call them varieties.

Let $X \subset X(\Gamma)$ an irreducible component, then $\mathbb{C}[X]$ is a domain, and we denote by $\mathbb{C}(X) = \mathrm{Frac}(\mathbb{C}[X])$ its fraction field. There is an obvious *tautological representation* $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C}[R(\Gamma)])$ that maps γ to X_γ. In fact, a theorem of Saito allows us to define it directly on K : component by component $\rho_X : \Gamma \to \mathrm{SL}_2(K)$ where K is a degree two field extension of $\mathbb{C}(X)$. In the sequel we will forget this technical point and assume that $\rho_X : \Gamma \to \mathrm{SL}_2(\mathbb{C}(X))$.

Example 3.2.6. For $\Gamma = \mathbb{Z}$ again, any representation is equivalent to ρ_λ, where $\rho_\lambda(1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for some λ in $\mathbb{C}^*$, hence $X(\mathbb{Z}) \simeq \mathbb{C}$ is parametrized by $t = \lambda + \lambda^{-1}$ and $\rho_\lambda : \mathbb{Z} \to \mathrm{SL}_2(\mathbb{C}(\lambda))$ is the tautological representation with entries in a quadratic extension of $\mathbb{C}(t)$. Observe that the representation $\rho_t : \mathbb{Z} \to \mathrm{SL}_2(\mathbb{C}(t))$, $1 \mapsto \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}$ is a tautological representation with entries in the function field $\mathbb{C}(t)$, but it has no reasons to exist in general.

3.2.2 Compactification

For any embedded affine variety $X \subset \mathbb{C}^N$, with $\mathbb{C}[X] = \mathbb{C}[X_1, ..., X_N] \big/ (f_1, ..., f_k)$, a compactification is given by $\widehat{X} = \{f_1^H = ... = f_k^H = 0\} \subset \mathbb{CP}^N$, where f_I^H is the homogenized polynomial from f_i. Unfortunately, it depends on the choice of an embedding of X in $\mathbb{C}^N$. As we have seen in the previous section, there is no canonical finite system of coordinates for $X(\Gamma)$. The following is an analogue of Thurston's compactification of Teichmüller space by curves lengths. Let $\mathcal{C}$ denotes the set of conjugacy class of elements in Γ. It is a countable set. We define the "positive projective space" $\pi : [0, +\infty[^{\mathcal{C}} \setminus \{0\} \to \mathcal{P} = [0, +\infty[^{\mathcal{C}} \setminus \{0\} \big/]0, +\infty[$, and

$$\begin{aligned} \Theta_0 : X &\to [0, +\infty[^{\mathcal{C}} \\ [\rho] &\mapsto (\log(|\mathrm{Tr}(\rho(\gamma))| + 2))_{\gamma \in \mathcal{C}} \end{aligned}$$

Finally, we define $\Theta : X \to \mathcal{P}$ to be $\pi \circ \Theta_0$.

Proposition 3.2.7. *The closure of* $\Theta(X)$ *in* $\mathcal{P}$ *is compact.*

Proof. Pick $Y_{\gamma_1}, ..., Y_{\gamma_m}$ a finite set of generators of $\mathbb{C}[X]$, then for any $Y_\gamma, \gamma \in \mathcal{C}$ there exists a constant c_γ such that for any $[\rho] \in X$

$$\log(|Y_\gamma[\rho]| + 2) \leq c_\gamma \max_{\gamma_i} \log(|Y_{\gamma_i}[\rho]| + 2)$$

hence $\tilde{\Theta}_0 = \frac{\Theta_0}{\max \log(|Y_{\gamma_i}|+2)}$ has image included in $[0, c_\gamma]^{\mathcal{C}}$ that is compact, hence its closure is compact. As $0 \notin \overline{\Theta_0(X)}$ then $\overline{\Theta(X)}$ is compact. □

In general Θ has no reason to be one-one, but if we denote by $\widehat{X}$ the one point compactification $X \cup \{\infty\}$, then the map

$$\begin{aligned} \widehat{\Theta} : X &\to \widehat{X} \times \mathcal{P} \\ x &\mapsto (x, \Theta(x)) \end{aligned}$$

is then injective, and has compact closure $\bar{X}$. We will denote by $B(X) = \bar{X} \setminus X$ its boundary.

3.3 Valuations and trees

3.3.1 Valuations

Let Λ be an abelian ordered group, we will always think of Λ as a subgroup of some $\mathbb{R}^n$ with the lexicographic order. Then Λ can be filtered by convex subgroups $\Lambda_0 = \{0\} \subset \Lambda_1 \subset ... \subset \Lambda_n = \Lambda$ such that each $\Lambda_i / \Lambda_{i-1}$ is a subgroup of $\mathbb{R}$. For any element $\lambda \in \Lambda$, we define its *height* $h(\lambda)$ to be the smallest index i such that $\lambda \in \Lambda_i$ but not in

Λ_{i-1}. For any λ, λ' negative elements of Λ, one can define $\lambda/\lambda' \in [0, +\infty]$ as follows : if $h(\lambda) < h(\lambda')$, then $\lambda/\lambda' = +\infty$; if $h(\lambda) > h(\lambda')$ then $\lambda/\lambda' = 0$ and if $h(\lambda) = h(\lambda')$, then $\lambda/\lambda' \in \Lambda_h/\Lambda_{h-1}$ is its value after embed Λ_h/Λ_{h-1} into $\mathbb{R}$. This embedding turns λ and λ' into real numbers, defined up to scale, but the ratio is well-defined.

Definition 3.3.1. Let K be an extension of $\mathbb{Q}$ (have in mind $K = \mathbb{Q}(X)$ for X a variety defined over $\mathbb{Q}$). A $\mathbb{Q}$-*valuation* $v : K^* \to \Lambda_v$ is a surjective homomorphism such that $v(a+b) \geq \min(v(a), v(b))$ for all $a, b \in K^*$, and that $v(\mathbb{Q}) = 0$. The *valuation ring* is $\mathcal{O}_v = \{f \in K^*, v(f) \geq 0\}$ and its unique maximal ideal is $\mathfrak{m}_v = \{f \in K^*, v(f) > 0\}$. Notice that the pair $(\mathcal{O}_v, \mathfrak{m}_v)$ in K determines v, with $\Lambda_v = K^*/(\mathcal{O}_v \setminus \mathfrak{m}_v)$.

In the spirit of the case of curves, we want to see valuations as "limit points" of sequences in X.

Definition 3.3.2.

- A point $x \in X$ is $\mathbb{Q}$-generic if it is not contained in any subvariety of X defined over $\mathbb{Q}$.
- A $\mathbb{Q}$-valuating sequence $(x_i) \in X$ is a sequence of $\mathbb{Q}$-generic points such that for all $f \in \mathbb{Q}(X)$, $\lim f(x_i)$ exists in $\mathbb{C} \cup \{\infty\}$.

Remark 3.3.3. The $\mathbb{Q}$-genericity ensures that for all $f \in \mathbb{Q}(X)$, for all i, $f(x_i)$ exists in $\mathbb{C}$.

Proposition 3.3.4. *A $\mathbb{Q}$-valuating sequence (x_i) defines a valuation v by $\mathcal{O}_v = \{f \in \mathbb{Q}(X), \lim f(x_i) \in \mathbb{C}\}$ and $\mathfrak{m}_v = \{f \in \mathbb{Q}(X), \lim f(x_i) = 0\}$. Moreover any valuation on $\mathbb{Q}(X)$ can be obtained in this way.*

Proposition 3.3.5. *For any valuating sequence (x_i) in X, let v be the associated valuation, then for all $f, g \in \mathbb{Q}(X)$ with $v(f) \leq 0, v(g) < 0$, then $\lim \frac{\log|f(x_i)|}{\log|g(x_i)|} = \frac{v(f)}{v(g)} \in [0, +\infty]$*

Proof.

- If $v(f) = 0$, then $\lim \log|f(x_i)| \in \mathbb{C}^*$, hence both terms are zero.
- If $v(f) < 0$, it is enough to prove that $\lim \frac{\log|f(x_i)|}{\log|g(x_i)|} \leq \frac{v(f)}{v(g)}$. Pick any rational number $\frac{p}{q}$ with p, q positive integers, such that $\frac{v(f)}{v(g)} < \frac{p}{q}$, then one have $qv(f) > pv(g)$ and $v(\frac{g^p}{f^q}) = pv(g) - qv(f) < 0$. Hence $\frac{g(x_i)^p}{f(x_i)^q} \to \infty$ and consequently $p \log|g(x_i)| > q \log|f(x_i)|$ for i big enough. Finally $\frac{\log|f(x_i)|}{\log|g(x_i)|} \leq \frac{p}{q}$ and the conclusion follows.

□

Definition 3.3.6. A valuation v on $\mathbb{Q}(X)$ is *supported at infinity* if $\mathbb{C}[X]$ is not included in $\mathcal{O}_v$. In an equivalent manner, if the valuating sequence defining v is unbounded in X.

Remark 3.3.7. For any valuation v on $\mathbb{Q}(X)$, supported at infinity, there is a $f \in \mathbb{Q}(X)$ with $v(f)$ negative and of minimal height h. Then we can define a valuation $\bar{v} : \mathbb{Q}(X)^* \to \bar{\Lambda}$ where $\bar{\Lambda} = \Lambda_h/\Lambda_{h-1}$.

Theorem 3.3.8. *For any $b \in B(X)$, there is a valuation v, supported at infinity, such that*

$$[b_\gamma]_{\gamma\in\mathcal{C}} = [-\min(\bar{v}(Y_\gamma), 0)]_{\gamma\in\mathcal{C}}$$

Proof. Any $b \in B(X)$ is a limit of some $\Theta(x_i)$, for some sequence (x_i). One can assume that (x_i) is a valuating sequence, whence defines a valuation v, that is supported at infinity. Then by definition of Θ, $b = \left[\lim \frac{\log(|Y_\gamma(x_i)|+2)}{\log(|f(x_i)|+2)}\right]_{\gamma\in\mathcal{C}}$. As soon as $v(Y_\gamma) \leq 0$, the latter is equal to $\frac{v(Y_\gamma)}{v(f)}$. □

3.3.2 The tree

Definition 3.3.9. A $\mathbb{R}$-tree is a metric space T such that

1. For all $x, y \in T$, there is an unique closed segment (isometric to a closed interval) linking x to y, denoted by $x.y$.

2. If $x.y \cap y.z = \{y\}$, then $x.y \cup y.z$ is the closed segment $x.z$.

For any valuation $v : \mathbb{Q}(X)^* \to \Lambda \subset \mathbb{R}$, one can construct an $\mathbb{R}$-tree T as follows. Let $F = \mathbb{Q}(X)$ and $V \simeq F^2$ be a two dimensional F-vector space. A lattice $L \subset V$ is a free $\mathcal{O}_v$-module of rank two such that it spans V over F. We will say that L is equivalent to L' if there is an $\alpha \in F^*$ such that $L' = \alpha L$, and we define the set of equivalence classes $[L]$ to be T. We need to construct a distance for T being a tree.

Pick two lattices L, L', up to equivalence one can assume that $L' \subset L$, that is if (e_1, e_2) is a basis of L, and (f_1, f_2) a basis of L', there exists $a, b, c, d \in \mathcal{O}_v$ such that $f_1 = ae_1 + be_2, f_2 = ce_1 + de_2$. One can assume $v(a) = \min(v(a), v(b), v(c), v(d))$, hence a divides b, c and d in $\mathcal{O}_v$ and one can write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \begin{pmatrix} a & 0 \\ 0 & d - \frac{bc}{a} \end{pmatrix} B$ with $A, B \in \mathrm{SL}_2(\mathcal{O}_v)$. Hence $L_0 = L'/a$ has a basis $(e_1, \frac{ad-bc}{a^2} e_2)$. We denote $\beta = \frac{ad-bd}{a^2}$, then ${}^{L}\!/_{L'} \simeq {}^{\mathcal{O}_v}\!/_{(\beta)}$. We say that L' is *cocyclic* to L if $L' \subset L$ and ${}^{L}\!/_{L'} \simeq {}^{\mathcal{O}_v}\!/_{(\beta)}$ for some $\beta \in \mathcal{O}_v$. In this case, L' is unique in its equivalence class, and $v(\beta)$ only depends on $[L]$ and $[L']$. Hence we define the distance to be $d([L], [L']) = v(\beta) = v(ad - bc) - 2\min(v(a), v(b), v(c), v(d))$.

Proposition 3.3.10. *The map $d : T \times T \to \mathbb{R}$ is a distance, and T is a $\mathbb{R}$-tree.*

First step of the proof. The only step we give here is the fact that any two points $x, y \in T$ can be linked by a segment. Let L, L' be representative of x and y such that $L' \subset L$ is cocyclic, hence there is a basis (e_1, e_2) of L such that $(e_1, \beta e_2)$ is a basis of L' and $d(x, y) = v(\beta)$. Then for any real number $z \in [0, v(\beta)] \cap \Lambda$, pick an element $\gamma_z \in \mathcal{O}_v$ with $v(\gamma_z) = z$, and define the lattice L_z with basis $(e_1, \gamma_z e_2)$. It defines a segment joining x to y. □

Remark 3.3.11. In fact, the tree T is a Λ-tree in some sense analogous to the definition above. As Λ is a subgroup of $\mathbb{R}$, there is an unique "completion procedure" that turns a Λ-tree into an $\mathbb{R}$-tree, see [MS87, Theorem II.1.9]. For sake of shortness and clarity, we prefer to avoid this technical issue, but we stress out that the reader have to be careful with the statement of Proposition 3.3.10.

The group $\mathrm{SL}_2(F)$ acts on T by isometries. If $g \in \mathrm{SL}_2(F)$ fixes a point $x \in T$, then one can assume up to conjugation that x is the standard lattice $L = \mathcal{O}_v^2$, that $g = A\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}B$ with $A, B \in \mathrm{SL}_2(\mathcal{O}_v)$. Moreover, as $\det g = 1, b = a^{-1}$ hence $g.L = a\mathcal{O}_v \oplus a^{-1}\mathcal{O}_v$. But x is fixed implies that $g.L = \alpha L$ for some $\alpha \in F^*$, and in any case $g.L \subset L$ or $L \subset g.L$, hence $a, a^{-1} \in \mathcal{O}_v$ and we conclude that $a \in \mathcal{O}_v^*$, that is, $g \in \mathrm{SL}_2(\mathcal{O}_v)$. We have almost proved the following :

Proposition 3.3.12. *An element $g \in \mathrm{SL}_2(F)$ has a fixed point when acting on the tree T_v iff $\mathrm{Tr} g \in \mathcal{O}_v$.*

Proof. We have already seen that g has a fixed point iff it is conjugated to a matrix in $\mathrm{SL}_2(\mathcal{O}_v)$. On the other hand, if $\mathrm{Tr} g \in \mathcal{O}_v$, then for any element $e \in F$ that is not an eigenvector for g, in the basis $(e, g.e)$ one express g as the matrix $\begin{pmatrix} 0 & 1 \\ -1 & \mathrm{Tr} g \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_v)$, and the proof is complete. □

From this result we want to deduce the property that the action of Γ through the tautological representation on the tree T_v constructed from a valuation supported at infinity has no global fixed point. The crucial result is the following tree-theoretical lemma :

Lemma 3.3.13. *A finitely generated group Γ acting on a $\mathbb{R}$-tree has a global fixed point iff every $g \in \Gamma$ has a fixed point.*

Proof. The key is to prove that any $g, h \in \Gamma$ have a common fixed point. We proceed as follows : let x a fixed point of g, and y a fixed point of h. The intersection $x.y \cap hx.y$ is a closed segment $y'.y$, and y' is fixed by h. In the same way, the intersection $x.y \cap x.gy = x.x'$, and x' is fixed by g. Now if $x' = y'$, we are done, and if not, we can prove that the isometry hg^{-1} acts as a translation along the segment $gy'.y'$, see Figure 3.2. A classification of the isometries of $\mathbb{R}$-trees shows that such a translation has no fixed points, a contradiction.

The second step is to use this result, by induction on the number of generators, to prove that the whole group has a fixed point. □

We easily deduce the following group-theoretical result :

Corollary 3.3.14. *A finitely generated subgroup Γ of $\mathrm{SL}_2(F)$ is such that for every $\gamma \in \Gamma, \mathrm{Tr}\gamma \in \mathcal{O}_v$ iff Γ is conjugated in $\mathrm{GL}_2(F)$ to a subgroup of $\mathrm{SL}_2(\mathcal{O}_v)$.*

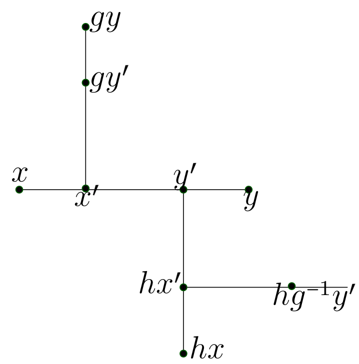

Figure 3.2:

Let us recall what we've obtained : from a finitely generated group Γ, we construct from an unbounded sequence in its space of character $X(\Gamma)$ a valuation $v : \mathbb{Q}(X)^* \to \mathbb{R}$ corresponding to this ideal point, and then a $\mathbb{R}$-tree T_v with an action of Γ without global fixed point. In fact, such a tree and an action is determined by the knowledge of the *length* l_γ of each element $\gamma \in \Gamma$, that is $l_\gamma = \inf_{x \in T_v} d(x, \gamma x)$.

We conclude with the following proposition, that ensures that this length depends only on the valuation $v(Y_\gamma)$, hence on the ideal point in X.

Proposition 3.3.15. *For any $\gamma \in \Gamma$, the length $l_\gamma = \min_{x \in T_v} d(x, \gamma x)$ and is equal to $-2\min(0, v(Y_\gamma))$.*

Proof. The fact that the length is in fact a minimum is obtained by the classification of $\mathbb{R}$-tree's isometries. Let us prove that for all $x \in T_v$, $d(x, \gamma x) \geq -2v(\mathrm{Tr}\rho(\gamma))$. One can assume as usual that x is the standard lattice $\mathcal{O}_v^2$, hence $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F)$ and $d(x, \gamma x) = -2\min(v(a), v(b), v(c), v(d)) \geq -2\min(v(a), v(d)) \geq -2v(\mathrm{Tr}\rho(\gamma))$. On the other hand, one can show that there exists x_0 such that $d(x_0, \gamma x_0) = -2\min(0, v(\mathrm{Tr}\rho(\gamma)))$, by taking for x_0 a basis where $\rho(\gamma) = \begin{pmatrix} 0 & 1 \\ -1 & \mathrm{Tr}\rho(\gamma) \end{pmatrix}$. □

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Chapter 4

Buildings and limits of symmetric spaces

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This note is the lecture note for my 2.5-hour introductionary talk in the GEAR log cabin workshop "Workshop on Compactifications of moduli spaces of representations" at Montana in June 11-18, 2017. I want to thank Brian Collier, Giuseppe Martone and Jeremy Toulisse for their excellent organization work and offering me this opportunity.
The materials in this note follows from [1], the paper by B. Kleiner and B. Leeb, "Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings".

4.1 Space with nonpositive curvature

4.1.1 Definition $CAT(k)$

Let $k \in \mathbb{R}$, let M_k^2 be the two dimensional model space with constant curvature k; let $D(k) = Diam(M_k^2)$.

Definition 4.1.1. A complete metric space $(X, |.|)$ is a $CAT(k)$ space if
1. Every pair $x_1, x_2 \in X$ with $|x_1x_2| < D(k)$ is joined by a geodesic segment;
2. Triangle or Distance Comparison: every geodesic triangle in X with perimeter $< 2D(k)$ is at best as thin as the corresponding triangle in M_k^2. More precisely, for each geodesic triangle $\triangle$ in X with sides $\sigma_1, \sigma_2, \sigma_3$ with $Perimeter(\triangle) = |\sigma_1| + |\sigma_2| + |\sigma_3| < 2D(k)$, we construct a comparison triangle $\tilde{\triangle}$ in M_k^2 with sides $\tilde{\sigma}_i$ satisfying $|\tilde{\sigma}_i| = |\sigma_i|$. Every point x on $\triangle$ corresponds to a unique point $\tilde{x}$ on $\tilde{\triangle}$ which divides the corresponding side in the same ratio. We require that for all $x_1, x_2 \in \triangle$ we have $|x_1x_2| \le |\tilde{x_1}\tilde{x_2}|$.

X is not necessarily locally compact.
A $CAT(0)$-space is also called a Hadamard space.

4.1.2 Angles and the space of directions of a $CAT(k)$ space

Use $\tilde{\angle}_v(x,y)$ denote the angle of the comparison triangle at the vertex $\tilde{v}$. If $x' \in \overline{vx}, y' \in \overline{vy}$, then $\tilde{\angle}_v(x',y') \leq \tilde{\angle}_v(x,y)$.

From this monotonicity, $\lim_{x',y' \to v} \tilde{\angle}_v(x',y')$ exists, and we denote it by $\angle_v(x,y)$.

Observe that $\angle_v(x,y) = \lim_{x' \to v} \tilde{\angle}_v(x',y) \leq \tilde{\angle}_v(x,y)$.

Proposition 4.1.2. *(1) $\angle_v(x_1,x_2) = \pi$ iff $\overline{x_2 v x_1}$ is a geodesic segment.*
(2) (triangle inequality) $\angle_v(x_1,x_3) \leq \angle_v(x_1,x_2) + \angle_v(x_2,x_3)$.
(3) (triangle filling lemma)
If $\angle_v(x,y) = \tilde{\angle}_v(x,y)$, then $\triangle(v,x,y)$ coincides with the comparison triangle.
(4) Let σ_1, σ_2 be asymptotic rays, then

$$\angle_x(y,\sigma_1) + \angle_y(x,\sigma_2) \leq \pi.$$

If equality holds, it bounds a half flat strip.

Space I: $\Sigma_x X$.

The condition that two geodesic segments with initial point $v \in X$ have angle zero at v is an equivalence relation. Let

$$\Sigma_x^* X := \{\text{equivalent classes of geodesic segements at } v\}.$$

The angle defines a metric on $\Sigma_v^* X$, and we let $\Sigma_v X$ be the completion of $\Sigma_v^* X$ with respect to this metric. We call elements of $\Sigma_v X$ directions at v, and $\vec{vx}$ denotes the direction represented by $\overline{vx}$.

Space II: $\partial_\infty X$

Let X be a Hadamard space. Two geodesic rays are asymptotic if they remain bounded from each other, i.e. their Hausdorff distance is finite. Asymptoticity is an equivalence condition. Let

$$\partial_\infty X := \{\text{equivalent classes of asymptotic rays}\}.$$

For any $x \in X$ and any ray $\xi \in \partial_\infty X$, there is a unique ray $\overline{x\xi}$ starting from x which represents ξ. The pointed Hausdorff topology on rays emanating from $x \in X$ induces a topology on $\partial_\infty X$. This topology does not depend on x and is called the cone topology on $\partial_\infty X$. $\partial_\infty X$ with the cone topology is called the geometric boundary. The cone topology naturally extends to $X \cup \partial_\infty X$.

Define the angle between two geodesics $\overline{vx}, \overline{vy}$ at $v \in X$ by using the monotonicity of comparison angles $\tilde{\angle}_v(x',y')$ as $x',y' \to v$.

Definition 4.1.3. (The Tits metric) Consider a pair of rays $\overline{v\xi}, \overline{v\eta}$, and define their Tits angle (or angle at infinity) by

$$\angle_{Tits}(\xi,\eta) := \lim_{x' \to \xi, y' \to \eta} \tilde{\angle}_v(x',y')$$

where $x' \in \overline{v\xi}, y' \in \overline{v\eta}$.

Proposition 4.1.4. *(1)* $\angle_{Tits}$ *defines a metric on* $\partial_\infty X$ *which is independent of the basepoint* v *chosen. Call the metric space* $\partial_{Tits}X := (\partial_\infty X, \angle_{Tits})$ *the Tits boundary of* X *and* $\angle_{Tits}$ *the Tits (angle) metric.*
(2)

$$\angle_{Tits}(\xi,\eta) = \lim_{t\to\infty} \angle_{\gamma(t)}(\xi,\eta)$$

for any geodesic ray $\gamma : \mathbb{R}^+ \to X$ *asymptotic to* ξ *or* $\angle_{Tits}(\xi,\eta) = \sup_{x\in X} \angle_x(\xi,\eta)$.
(3) The Tits boundary of a Hadamard space is a $CAT(1)$ *space.*
(4) If X *is complete, then* $(\partial_\infty X, \angle_{Tits})$ *is complete.*

4.2 Ultralimits and asymptotic cones

4.2.1 Ultrafilters and ultralimits

Definition 4.2.1. A non-principal ultrafilter is a finitely additive probability measure ω on the subsets of $\mathbb{N}$ such that
1. $\omega(S) = 0$ or 1 for every $S \subset \mathbb{N}$.
2. $\omega(S) = 0$ for every finite subset $S \subset \mathbb{N}$.

Given a compact metric space X and a map $a : \mathbb{N} \to \mathbb{R}$, there is a unique element $\omega - \lim a \in X$ such that for every neighborhood U of $\omega - \lim a$, $a^{-1}(U) \subset \mathbb{N}$ has full measure. In particular, given any bounded sequence $a : \mathbb{N} \to \mathbb{R}$, $\omega - \lim a$ (or a_ω) is a limit point selected by ω.

(Existence: suppose there exists a neighborhood U_p of each p such that $\omega(U_p) = 0$. Since X is compact, there are finitely many U_p's covering X. But $\omega(X) = \sum \omega(U_p) = 0$. Contradiction. Uniqueness: Suppose p, q both have the property. Since X is Hausdorff, there are two disjoint neighborhoods U_p of p and U_q of q satisfying $\omega(U_p) = 1, \omega(U_q) = 1$. Contradiction.)

4.2.2 Ultralimits of sequences of pointed metric spaces

Let $(X_i, d_i, *_i)$ be a sequence of metric spaces with basepoints $*$. Consider

$$X_\infty = \{x \in \Pi_{i\in\mathbb{N}} X_i | d_i(x_i, *_i) \text{is bounded}\}.$$

Since $d_i(x_i, y_i)$ is a bounded sequence, we may define $\tilde{d}_\omega : X_\infty \times X_\infty \to \mathbb{R}$ by $\tilde{d}_\omega(x,y) = \omega - \lim d_i(x_i, y_i)$. Here $\tilde{d}_\omega$ is a pseudo-distance. We define the ultralimits of the sequence $(X_i, d_i, *_i)$ to be the quotient metric space (X_ω, d_ω). Here $x_\omega \in X_\omega$ denotes the element corresponding to $x = (x_i) \in X_\infty$ and $*_\omega := (*_\omega)$ is the basepoint of (X_ω, d_ω).

Proposition 4.2.2. *(1) If* $(X_i, d_i, *_i)$ *is a sequence of pointed metric spaces, then* $(X_\omega, d_\omega, *_\omega)$ *is complete.*
(2) If for each i, $f_i : X_i \to Y_i$ *is a* (L, C)*-quasi-isometry with* $d_i(f_i(*_i), *_i)$ *bounded then* f_i *induce an* (L, C) *-quasi-isometry* $f_\omega : X_\omega \to Y_\omega$.
(3) If $(X_i, d_i, *_i)$ *is a* $CAT(k)$ *space for each* i*, then so is* $(X_\omega, d_\omega, *_\omega)$.

Example 4.2.3. (1) Let $(X, d_X), (Y, d_Y)$ be two distinct compact metric spaces. Let (X_n, d_n) be such that for $n \in A_1$, $(X_n, d_n) = (X, d_X)$; for $n \in A_1^c$, $(X_n, d_n) = (Y, d_Y)$. So one of A_1, A_1^c has $\omega-$measure 1 and the other is 0. So $\lim_\omega (X_n, d_n) = (X, d_X)$ if $\omega(A_1) = 1$; $\lim_\omega (X_n, d_n) = (Y, d_Y)$ if $\omega(A_1) = 0$. So the ultralimit can depend on the ultrafilters.
(2) If $(X_i, d_i, *_i)$ form a Hausdorff precompact family of pointed metric spaces, then $(X_\omega, d_\omega, *_\omega)$ is a limit point of the sequence with repsect to the pointed Hausdorff topology.

4.2.3 Asymptotic cones

Let X be a metric space and let $*_n \in X$ be a sequence of basepoints. We define the asymptotic cone $Cone(X)$ of X with respect to the non-principal ultrafilter ω, the sequence of scale factors λ_n with $\omega - lim \lambda_n = \infty$ and basepoints $*_n$, as the ultralimit of the sequence of rescaled spaces $(X_n, d_n, *_n) := (X, \frac{1}{\lambda_n} d, *_n)$.

Remark 4.2.4. When the sequence $*_n = *$ is constant, then $Cone(X)$ does not depend on the basepoint $*$ and has a canonical basepoint $*_\omega$ which is, represented by any sequence $(x_n) \subset X$ satisfying $\omega - lim \frac{1}{\lambda_n} \cdot d(x_n, *) = 0$, by any constant sequence (x_n).

Proposition 4.2.5. *(1) If X is a geodesic metric space, then $Cone(X)$ is a geodesic metric space.*
(2) If X is a Hadamard space, then $Cone(X)$ is a Hadamard space.
(3) If X is a $CAT(k)$-space for some $k < 0$, then $Cone(X)$ is a metric tree.
(4) A (L, C)-quasi-isometry of metric spaces $\Phi : X \to Y$ induces a bilipschitz map $Cone(\varphi) : Cone(X) \to Cone(Y)$ of asymptotic cones.
(5) $Cone(\mathbb{R}^m, d) = (\mathbb{R}^m, d)$.

Assume now that X is a Hadamard space. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of k-flats in X and suppose that $\omega - lim \frac{1}{\lambda_n} d(F_n, *) < \infty$. Then the ultralimit of the embeddings of pointed metric spaces

$$(F_n, \frac{1}{\lambda_n} d_{F_n}, \pi_{F_n}(*)) \hookrightarrow (X, \frac{1}{\lambda_n} d_X, \pi_{F_n}(*))$$

is a k-flat $\mathbb{R}^k \to Cone(X)$ in the asymptotic cone.

4.3 Spherical buildings

A spherical building is a $CAT(1)$ space equipped with extra structure.

4.3.1 Spherical Coxeter complexes

Let S be a Euclidean unit sphere. If $W \subset Isom(S)$ is a finite subgroup generated by reflections (an involutive isometry whose fixed point set is a subspace of codimension one, its wall), we call the pair (S, W) a spherical Coxeter complex and W its Weyl group.

The finite collection of walls belonging to reflections in W divide S into isometric open convex sets. The closure of any of these sets is called a chamber, and is a fundamental domain for the action of W. Chambers are convex spherical polyhedron., i.e. finite intersections of hemispheres. A face of a chamber is an intersection of the chamber with same walls.

A regular point in S is an interior point of chamber. The regular points form a dense subset. The orbit space

$$\triangle_{mod} := S/W$$

with the orbital distance metric is a spherical polyhedron isometric to each chamber.

The quotient map $\theta = \theta_S : S \to \triangle_{mod}$ is 1-Lipschitz and its restriction to each chamber is distance preserving. For $\delta, \delta' \in \triangle_{mod}$, we set

$$D(\delta, \delta') := \{d_S(x, x') | x, x' \in S, \theta x = \delta, \theta x' = \delta'\}$$

and $D^+(\delta) := D(\delta, \delta)/\{0\}$.

4.3.2 Definition of spherical buildings

Let (S, W) be a spherical Coxeter complex. A spherical buildings modelled on (S, W) is a $CAT(1)$-space B together with a collection $\mathcal{A}$ of isometric embeddings $\iota : S \to B$, called charts, which satisfies properties SB1-2 described below and which is closed under precomposition with isometries in W. An apartment in B is the image of a chart $\iota : S \to B$; ι is a chart of the apartment $\iota(S)$ and $\mathcal{A}$ is called the atlas of the spherical building.

SB1: Plenty of apartments. Any two points in B are contained in a common apartment.

Let ι_{A_1}, ι_{A_2} be charts for apartments A_1, A_2, and let $C = A_1 \cap A_2,$, $C' = \iota_{A_2}^{-1}(C) \subset S$. The charts ι_{A_i} are W-compatible if $\iota_{A_1}^{-1} \circ \iota_{A_2}|_C$ is the restriction of an isometry in W.

SB2: Compatible apartments. The charts are W-compatible.

Remark 4.3.1. The axioms yield a well-defined 1-Lipschitz anisotropy map

$$\theta_B : B \to S/W =: \triangle_{mod}$$

satisfying the discreteness condition: $d_B(x_1, x_2) \in D(\theta_B(x_1), \theta_B(x_2)), \quad \forall x_1, x_2 \in B.$

Remark 4.3.2. Any two faces of with a common interior point coincide. Consequently, the intersection of faces in B is a face in B.

4.3.3 Recognize the spherical building structure

We can recognize the spherical building structure on a $CAT(1)$ space B using this anisotropy map θ_B by an easy criterion.

Proposition 4.3.3. *Let (S, W) be a spherical Coxeter complex. Let B be a $CAT(1)$-space of diameter π equipped with a 1-Lipschitz anisotropy map $\theta_B : B \to \triangle_{mod}$ satisfying the discreteness condition. Suppose moreover that each point and each pair of antipodal regular points is contained in a subset isometric to S (two points are antipodal if their distance is π.) Then there is a unique atlas $\mathcal{A}$ of charts $\iota : S \to B$ forming a spherical building structure on B modelled on (S, W), with associated anisotropy map θ_B.*

4.3.4 Reducing to a thick building structure

The spherical building is called thick if each wall belongs to at least three half-apartments.

A reduction of the spherical building structure on B consists of a reflection subgroup $W' \subset W$ and a subset $\mathcal{A}' \subset \mathcal{A}$ which defines a spherical building structure modelled on (S, W'). The $triangle_{mod}$-direction map factors as $\theta_B = \pi \circ \pi'_B$ where $\pi : S/W' \to S/W$ and $\theta'_B : B \to S/W' =: \triangle'_{mod}$.

Proposition 4.3.4. *Let B be a spherical building modelled on (S, W) with anisotropy polyhedron $\triangle_{mod} = S/W$. Then there exists a reduction $(W', \mathcal{A}')$ which is a thick building structure on B. W' is unique up to conjugacy in W; $\mathcal{A}'$ is determined by W'. In particular, the thick reduction is unique up to equivalence.*

4.4 Euclidean buildings

4.4.1 Eucldiean Coxeter complexes

Let E be a finite dimensional Euclidean space. Its Tits boundary is a round sphere and there is a canonical homomorphism

$$\rho : Isom(E) \to Isom(\partial_{Tits} E)$$

which assigns to each affine isometry its rotational part. We call a subgroup $W_{aff} \subset Isom(E)$ an affine Weyl group if it is generated by reflections and if $W = \rho(W_{aff}) \subset Isom(\partial_{Tits} E)$ is finite. The pair (E, W_{aff}) is said to be a Euclidean Coxeter complex and

$$\partial_{Tits}(E, W_{aff}) := (\partial_{Tits} E, W)$$

is called its spherical Coxeter complex at infinity. Its anisotropy polyhedron is the spherical polyhedron

$$\triangle_{mod} := (\partial_{Tits} E)/W.$$

An oriented geodesic segment $\overline{xy}$ in a E determines a points in $\partial_{Tits} E$ and we call its projection to $\triangle_{mod}$ the $\triangle_{mod}$-directions of $\overline{xy}$. A wall is a hyperplane which is the fixed points set of a reflection in W_{aff} and singular subspaces are defined as intersections of walls. A half-space bounded by a wall is called singular or a half-apartment.

4.4.2 The Euclidean buildings axioms

Let (E, W_{aff}) be a Euclidean Coxeter complex. A Euclidean building modelled on (E, W_{aff}) is a Hadamard space X endowed with the structure as follows:

EB1: Direction.

To each nontrivial oriented segment $\overline{xy} \subset X$ is assigned a $\triangle_{mod}(= \partial_{Tits}E/W)$-direction $\theta(\overline{xy}) \in \triangle_{mod}$ and it satisfies that

$$d(\theta(\overline{xy}), \theta(\overline{xz})) \leq \tilde{\angle}_x(y, z).$$

EB 2: Angle rigidity.

The angle between two geodesic segments $\overline{xy}$ and $\overline{xz}$ lies in the finite set $D(\theta(\overline{xy}), \theta(\overline{xz}))$. (Recall the given $\delta_1, \delta_2 \in \triangle_{mod}$, $D(\delta_1, \delta_2)$ is the finite set of possible distances between the Weyl group orbits $\theta^{-1}_{\partial_{Tits}E}(\delta_1)$ and $\theta^{-1}_{\partial_{Tits}E}(\delta_2)$.)

We assume that there is given a collection $\mathcal{A}$ of isometric embeddings $\iota : E \to X$ which preserve $\triangle_{mod}$-directions and which is closed under precomposition with isometries in W_{aff}. These isometric embeddings are called charts, their images are apartments, and $\mathcal{A}$ is called the atlas of the Euclidean building.

EB3: Plenty of apartments.

Each segment, ray and geodesic is contained in an apartment. The Euclidean coordinate chart ι_A for an apartment A is well-defined up to precomposition with an isometry $\alpha \in \rho^{-1}(W)$. Two charts ι_{A_1}, ι_{A_2} for apartments A_1, A_2 are said to be compatible if $\iota_{A_1}^{-1} \circ \iota_{A_2}$ is the restriction of an isometry in W_{aff}. This holds automatically when $W_{aff} = \rho^{-1}(W)$.

EB4: Compatibility of apartments.

The Euclidean coordinate charts for the apartments in X are compatible.

Remark 4.4.1. This definition looks somewhat different from Tits' original definition. But it is proved later by Ann Parreau that the definitions are equivalent.

4.4.3 Associated spherical building structure of a Euclidean building structure

Some immediate consequences of the axiom EB1 are as follows.

Lemma 4.4.2. *Let x, y, z be points in X.*

1. *If y lies on $\overline{xz}$, then $\theta(\overline{xz}) = \theta(\overline{xy}) = \theta(\overline{yz})$.*

2. *If If $\vec{xy}, \vec{xz} \in \Sigma_x X$ coincide, then $\theta(\overline{xy}) = \theta(\overline{xz})$.*

3. *Asymptotic geodesic rays in X have the same $\triangle_{mod}$-direction.*

We call a segment, ray or geodesic in X regular if its $\triangle_{mod}$-direction is an interior point of $\triangle_{mod}$.

1. The Tits boundary $\partial_\infty X$.

First it is a $CAT(1)$-space. By Lemma 4.4.2, there is a well-defined $\triangle_{mod}$-direction map

$$\theta_{\partial_{Tits}X} : \partial_{Tits}X \to \triangle_{mod}$$

which is 1-Lipschitz by EB1.

Proposition 4.4.3. *$\partial_{Tits}X$ carries a spherical building structure modelled on the spherical Coxeter complex $(\partial_{Tits}E, W)$ with the $\triangle_{mod}$-direction map.*

Proof. We verify the conditions of recognizing spherical building structure are satisfied. Axiom EB2 implies $\theta_{\partial_{Tits}X}$ satisfies the discreteness condition. If A is a Euclidean apartment in X, then $\partial_{Tits}A$ is a standard sphere in $\partial_{Tits}X$.

Clearly, any point $\xi \in \partial_{Tits}X$ lies in a standard sphere. It remains to check that any two points $\xi_1, \xi_2 \in \partial_{Tits}X$ with $d_{Tits}(\xi_1, \xi_2) = \pi$ are ideal points of a geodesic in X. To see this, pick $p \in X$, note that $<_z (\xi_1, \xi_2)$ increases as z moves along $\overline{p\xi_1}$ towards ξ_1. By EB2, $<_z (\xi_1, \xi_2)$ assumes only finitely many values, so when z is sufficiently far out, we have

$$<_z (\xi_1, \xi_2) = <_{Tits} (\xi_1, \xi_2) = \pi,$$

and the rays $\overline{z\xi_i}$ fit together to form a geodesic with ideal endpoints ξ_1 and ξ_2.

□

□

2. The space of directions $\Sigma_x X$.

First $\Sigma_x X$ is a $CAT(1)$-space. By Lemma 4.4.2, there is a well-defined 1-Lipschitz map from the space of germs of segments in a point $x \in X$:

$$\theta_{\Sigma_x X} : \Sigma_x^* X \to \triangle_{mod}.$$

Proposition 4.4.4. *$\Sigma_x X$ carries a spherical building structure modelled on the spherical Coxeter complex $(\partial_{Tits}E, W)$ with the $\triangle_{mod}$-direction map $\theta_{\Sigma_x X}$, where A is an apartment in X.*

By EB2, $\theta_{\Sigma_x X}$ satisfies the discreteness condition.

Lemma 4.4.5. *$\Sigma_x^* X$ is complete, so $\Sigma_x^* X = \Sigma_x X$.*

4.4.4 Reducing to a thick Euclidean building structure

The Euclidean building X is called thick if each wall bounds at least 3 half-apartments with disjoint interiors.

Let X be a Euclidean building modelled on the Euclidean Coxeter complex (E, W_{aff}), with atlas $\mathcal{A}$. A reduction of the Euclidean building structure is a subgroup $W'_{aff} \subset W_{aff}$ together with a compatible subset $\mathcal{A}' \subset \mathcal{A}$ forming an atlas for a Euclidean building structure modelled on (E, W'_{aff}).

Remark 4.4.6. In contrast to the spherical building case, the affine Weyl group of a Euclidean building does not necessarily have a canonical reduction with respect to which it becomes thick. For example, a metric tree with variable edge lengths does not admit a thick Euclidean building structure. However, there is always a canonical minimal reduction, and this is thick when it has no tree factors.

Proposition 4.4.7. *Let X be a Euclidean building modelled on (E, W_{aff}). Then there is a unique minimal reduction $W'_{aff} \subset W_{aff}$ so that (X, E, W'_{aff}) splits as a product ΠX_i where each X_i is either a thick irreducible Euclidean building or a 1-dimensional Euclidean building.*

4.5 Asymptotic cones of symmetric spaces and Euclidean buildings

We show that asymptotic cones of symmetric spaces and ultralimits of sequences of Euclidean buildings (of bounded rank) are Euclidean buildings.

EB1-4 behave well with respect to ultralimits.

4.5.1 Ultralimits of Euclidean buildings are Euclidean buildings

Theorem 4.5.1. *Let $X_n, n \in \mathbb{N}$ be Euclidean buildings with the same anisotropy polyhedron $\triangle_{mod}$. Then, for any sequence of basepoint $*_n \in X_n$, the ultralimits $(X_\omega, *_\omega) = \omega - \lim(X_n, *_n)$ admits a Euclidean building structure with anisotropy polyhedron $\triangle_{mod}$.*

Proof. X_ω is a Hadamard space. A Euclidean building structure on X_ω consists of an assignment of $\triangle_{mod}$-directions for segments EB1+2 and of an atlas of compatible charts for apartments EB3+4. We assume that X has no Euclidean de Rham factor.

EB1: We can assign a $\triangle_{mod}$-direction to an oriented geodesic segment in X_ω as follows. A segment $\overline{x_\omega y_\omega}$ arises as ultralimit of a sequence of segments $\overline{x_n y_n}$ in X, and we define the direction as

$$\theta(\overline{x_\omega y_\omega}) := \omega - \lim \theta(\overline{x_n y_n}) \in \triangle_{mod}$$

The ultralimit exists because $\triangle_{mod}$ is compact. The inequality in (EB1) passes to the ultralimit:

$$d_{\triangle_{mod}}(\omega - \lim \theta(\overline{x_n y_n}), \omega - \lim \theta(\overline{x_n z_n})) \leq \tilde{\angle}_{x_\omega}(y_\omega, z_\omega)$$

This implies that $\theta(\overline{x_\omega y_\omega})$ is well-defined and

$$d_{\triangle_{mod}}(\theta(\overline{x_\omega y_\omega}), \theta(\overline{x_\omega z_\omega})) \leq \tilde{\angle}_{x_\omega}(y_\omega, z_\omega)$$

The axiom EB1 holds.

EB2: Since geodesics are extendible in X_ω, it suffices to show:

Lemma 4.5.2. *If $x_\omega \in X_\omega$ and $\xi_\omega, \eta_\omega \in \partial_{Tits} X_\omega$, then $\sphericalangle_{x_\omega} (\xi_\omega, \eta_\omega)$ is contained in $D := D(\theta(\overline{x_\omega \xi_\omega}), \theta(\overline{x_\omega \eta_\omega}))$.*

Proof. The rays $\overline{x_\omega \xi_\omega}$ and $\overline{x_\omega \eta_\omega}$ are ultralimits of sequences of rays $\overline{x_n \xi_n}$ and $\overline{x_n \eta_n}$ in X_n and we can choose $\xi_n, \eta_n \in \partial_{Tits} X_n$ so that $\theta(\xi_n) = \theta(\overline{x_\omega \xi_\omega})$ and $\theta(\eta_n) = \theta(\overline{x_\omega \eta_\omega})$. let $\rho_n : [0, \infty) \to X_n$ be a unit speed parametrization for the geodesic ray $\overline{x_n \xi_n}$. The angle $\sphericalangle_{\rho_n(t)} (\xi_n, \eta_n)$ is non-decreasing and continuous from the right in t. Since X_n satisfies EB2, the angle $\sphericalangle_{\rho_n(t)} (\xi_n, \eta_n)$ takes values in the finite set $D = D(\theta(\xi_n), \theta(\eta_n))$. For $d \in D$, set

$$t_n(d) := \min\{t \geq 0 | \sphericalangle_{\rho_n(t)} (\xi_n, \eta_n) \geq d\} \in [0, \infty)$$

and $t_\omega(d) = \omega - \lim t_n(d)$. Then there exist $d_0 \in D, T > 0$ such that $t_\omega(d_0) = 0, 2T < t_\omega(d)$ for all $d > d_0$. (d_0 is the jump, D is a finite set.)

Define $x'_n = \rho_n(t_n(d_0))$ and then for ω−all n, $x'_n = \rho_n(t_n(d_0)) = \rho_n(0) = x_n$. Define $x''_n = \rho_n(T)$ and then for ω−all n, $x''_n = \rho_n(T) \neq \rho_n(0) = x_n$ and $\sphericalangle_{x''_n} (\xi_n, \eta_n) = d_0$.

Then x'_n, x''_n satisfy for ω-all n:

$$x'_\omega := \omega - \lim x'_n = x_\omega, \quad x''_\omega \neq x_\omega.$$

Also for ω-all n, $\sphericalangle_{x'_n} (x''_n, \eta_n) = \sphericalangle_{x''_n} (\xi_n, \eta_n) = d_0$. This implies that the ideal triangle $\triangle(x'_n, x''_n, \eta_n)$ has angle sum π. By a version of the triangle filling lemma for ideal triangles in Hadamard spaces, $\triangle(x'_n, x''_n, \eta_n)$ can be filled in by a semi-infinite flat strip S_n. The ultralimit $\omega - \lim\ S_n$ is a semi-infinite flat strip filling in the ideal triangle $\triangle(x_\omega, x''_\omega, \eta_\omega)$. Therefore

$$\sphericalangle_{x_\omega} (\xi_\omega, \eta_\omega) = \omega - \lim \sphericalangle_{x'_n} (\xi_n, \eta_n) = d_0 \in D.$$

□

□

EB3: After enlarging the affine Weyl groups of the model Coxeter complexes of the buildings X_n, we may assume that the X_n are modelled on the same Euclidean Coxeter complex (E, W_{aff}) whose affine Weyl group W_{aff} contains the full translation subgroup of $Isom(E)$, i.e. $\rho^{-1}(W) = W_{aff}$. The atlas A_n for the building structures on X_n gives rises to an atlas for a building structure on X_ω as follows: If $\iota_n \in \mathcal{A}_n$ are charts for apartments in X_n so that $\omega - \lim d(\iota_n(e), *_n) < \infty$ for each point $e \in E$, then $\iota_\omega (:= \omega - \lim \iota_n) : E \to X_\omega$ is an isometric embedding which parametrizes a flat in X_ω. The collection $\mathcal{A}_\omega$ of all such embeddings ι_ω satisfies axiom EB3.

Axiom EB4 holds trivially, because coordinate changes $\iota_\omega \circ \iota'^{-1}_\omega$ between charts $\iota_\omega, \iota'_\omega \in A_\omega$ are $\triangle_{mod}$-directions preserving isometries between convex subsets of E and such isometries are induced by isometries in $\rho^{-1}(W) = W_{aff}$. Hence $\mathcal{A}_\omega$ is an atlas for a Euclidean building structure on X_ω with model Coxeter complex (E, W_{aff}). □

□

4.5.2 Asymptotic cones of symmetric spaces are Euclidean buildings

Let X be a symmetric space of noncompact type. A k-flat in X is a totally geodesic submanifold isometric to $\mathbb{E}^k$. The group $G = Isom^0(X)$, acts on X and acts transitively on the family of maximal flats. Any two maximal flats in X have the same dimension, called the rank of X. We will call the maximal flats apartments. Pick an apartment E in X and let W_{aff} be the quotient of the set-wise stabiliser $Stab_G(E)$ by the point-wise stabiliser $Fix_G(E)$. Then W_{aff} can be identified with a subgroup of $Isom(E)$. This subgroup is generated by reflections at hyperplanes and contains the full translation group. We call (E, W_{aff}) the Euclidean Coxeter complex associated to X.

Consider the collection of all isometric embeddings $\iota : E \to X$ so that W_{aff} is identified with $Stab_G(\iota(E))/Norm_G(\iota(E))$. The induced isometric embeddings $\partial_{Tits}\iota : \partial_{Tits}E \to \partial_{Tits}X$ form an atlas for a thick spherical Coxeter complex

$$(\partial_{Tits}E, W) = \partial_{Tits}(E, W_{aff})$$

together with the anisotropy map $\theta_{\partial_{Tits}X} : \partial_{Tits}X \to \triangle_{mod} := \partial_{Tits}E/W$. ($W$ is isometric to the Weyl group of the symmetric space X.)

Composing the anisotropy map $\theta_{\partial_{Tits}X}$ with the map $SX \to \partial_{Tits}X$ which assigns to every unit vector v the ideal endpoint of the geodesic ray $t \mapsto exp(tv)$ one obtains a natural map $\theta : SX \to \triangle_{mod}$. Let S_pX be the unit sphere at $p \in X$, equipped with the angular metric, and let G_p be the isotropy group of p. Then θ induces a canonical isometry

$$S_p/G_p \cong \triangle_{mod}$$

The quotient map $S_pX \to \triangle_{mod}$ is 1-Lipschitz and, for any $x, y \in X$ we have

$$d_{\triangle_{mod}}(\theta(\overline{px}), \theta(\overline{py})) \leq \angle_p(x, y) \leq \tilde{\angle}_p(x, y).$$

Theorem 4.5.3. *Let X be a non-empty symmetric space with associated Euclidean Coxeter complex (E, W_{aff}). Then , for any sequence of basepoints $\star_n \in X$ and scale factors λ_n with $\omega - limit \lambda_n = 0$, the asymptotic cone $X_\omega = \omega - limit(\lambda_n X, \star_n)$ is a thick Euclidean building modelled on (E, W_{aff}).*

In fact, EB1, EB3, EB4 are also satisfied by symmetric spaces, i.e. the existence of $\triangle_{mod}$-directions and an apartment atlas, pass directly to ultralimits. However, unlike Euclidean buildings, symmetric spaces do not satisfy the angle rigidity axiom EB2. The verification of EB2 for ultralimits of symmetric spaces is the only technical point, as opposed to the building case. Symmetric spaces satisfy angle rigidity merely at ∞; their Tits boundaries are spherical buildings. Intuitively speaking, the rescaling process involved in forming ultralimits pulls back spherical building structure from infinity to the space of directions.

Proof. EB1: Let $\triangle_{mod}$ be the anisotropy polyhedron for (E, W_{aff}). The construction of $\triangle_{mod}$-directions for segments in X_ω as in the building case. Easy to check EB1 holds

as in building case.
EB2: Need to show the following lemma.

Lemma 4.5.4. *If $p \in X_\omega$ and $x_1, x_2 \in X_\omega - \{p\}$, then $\angle_p(x_1,x_2) \in D(\theta(\overline{px_1}), \theta(\overline{px_2}))$.*

Proof. If $z'_k \in \overline{px_1} - p$ and $z'_k \to p$, then $\angle_{z'_k}(x_1,x_2) \to \angle_p(x_1,x_2)$ and $\angle_{z'_k}(p,x_2) \to \pi - \angle_p(x_1,x_2)$. Then we can find $x'_{1k} \in \overline{z'_k x_1}$, $x'_{2k} \in \overline{z'_k x_2}$, and $p'_k \in \overline{z'_k p}$ such that

$$\tilde{\angle}_{z'_k}(x'_{1k}, x'_{2k}) \to \angle_p(x_1,x_2), \tilde{\angle}_{z'_k}(p'_k, x'_{2k}) \to \pi - \angle_p(x_1,x_2)$$

and

$$\theta(\overline{z'_k x'_{2k}}) = \theta(\overline{z'_k x_2}) \to \theta(\overline{px_2}).$$

Since geodesic segments in X_ω are ultralimits of geodesic segments in $\lambda_n X$, we can find sequences $p_k, x_{1k}, x_{2k}, z_k \in X$ such that $z_k \in \overline{p_k x_{1k}}$,

$$\tilde{\angle}_{z_k}(x_{1k}, x_{2k}) \to \angle_p(x_1,x_2), \tilde{\angle}_{z_k}(p_k, x_{2k}) \to \pi - \angle_p(x_1,x_2)$$

$$\theta(\overline{z_k x_{2k}}) \to \theta(\overline{px_2}), \theta(\overline{p_k x_{1k}}) \to \theta(\overline{px_1})$$

and finally $|z_k x_{1k}|, |z_k x_{2k}|, |z_k p_k| \to \infty$. (Using the scaling factors $\lambda_n \to 0$.)

Applying a sequence of elements $g_k \in G = Isom^0(X)$ we may assume in addition that z_k is the constant point o. Hence the sequences of segments $\overline{ox_{1k}}, \overline{ox_{2k}}, \overline{op_k}$ subconverges to rays $\overline{o\xi_1}, \overline{o\xi_2}, \overline{o\eta}$ respectively, which satisfy the following properties:
1. $\theta_{\partial_{Tits}X}(\xi_i) = \theta(\overline{o\xi_i}) = \theta(\overline{px_i})$
2. $\angle_{Tits}(\xi_1,\xi_2) \leq \angle_p(x_1,x_2), \angle_{Tits}(\eta,\xi_2) \leq \pi - \angle_p(x_1,x_2)$ by the lower semicontinuity of the Tits metric with respect to the cone topology (Proposition 4.1.4 (2)).
3. $\overline{o\xi_1} \cup \overline{o\eta}$ is a geodesic, so $\angle_{Tits}(\xi_1,\eta) = \pi$.
We conclude that

$$\angle_p(x_1,x_2) = \angle_{Tits}(\xi_1,\xi_2) \in D(\theta(\xi_1), \theta(\xi_2)) = D(\theta(\overline{px_1}), \theta(\overline{px_2})).$$

Note here we use the discreteness condition of the map $\theta_{\partial_{Tits}}$, i.e. the spherical building structure of $\partial_{Tits}X$. □

□

EB3 and EB4: The Euclidean Coxeter complex (E, W_{aff}) is invariant under rescaling, because $W_{aff} \subset Isom(E)$ contains all translations. Apartments in X_ω and their charts arise as ultralimits of sequences of apartments and charts in X. And axioms EB3+4 follow as in the building case. □

□

4.6 Main results

An (L, C) quasi-isometry is a map $\Phi : X \to X'$ between metric spaces such that for all $x_1, x_2 \in X$ we have

$$L^{-1}d(x_1, x_2) - C \leq d(\Phi(x_1), \Phi(x_2)) \leq Ld(x_1, x_2) + C$$

and

$$d(x', Im(\Phi)) < C$$

for all $x' \in X'$.

Gromov initiated a systematic study of global geometry of groups in terms of quasi-isometries. One problem concerns the rigidity of symmetric spacs of higher rank under quai-isometries, a conjecture due to Margulis in late 70', which is proved in this paper by Kleiner and Leeb and stated as follows.

Theorem 4.6.1 (Rigidity)**.** *Let X, X' be as in previous theorem. Assume that X is either a nonflat irreducible symmetric space of noncompact type of rank ≥ 2, or a thick irreducible Euclidean building of rank ≥ 2 with cocompact affine Weyl group and Moufang Tits boundary. Then any (L, C) quasi-isometry $\Phi : X \to X'$ lies at distance $< D$ from a homothety $\Phi_0 : X \to X'$ where $D = D(L, C)$.*

More generally, they also consider

Theorem 4.6.2. *(Splitting) For $1 \leq i \leq k, 1 \leq j \leq k'$, let each X_i , X'_j be either a nonflat irreducible symmetric space of noncompact type or an irreducible thick Euclidean Tits building with cocompact affine Weyl group. Let $X = \mathbb{E}^n \times \Pi_{i=1}^k X_i$, $X' = \mathbb{E}^{n'} \times \Pi_{k=1}^{k'} X'_j$ be metric products. Then for every (L, C), there exist $\overline{L}, \overline{C}, \overline{D}$ such that if $\Phi : X \to X'$ is an (L,C) quasi-isometry, then $n = n', k = k'$, and after re-indexing the factors of X', there are $(\overline{L}, \overline{C})$ quasi-isometry $\Phi_i : X_i \to X'_i$ so that $d(p' \circ \Phi, \Pi\Phi_i \circ p) < \overline{D}$, where $p : X \to \Pi_{i=1}^k X_i$, and $p' : X' \to \Pi_{i=1}^k X'_i$ are the projections.*

An immediate consequence of these two theorems and [2] is the classification of symmetric spaces up to quasi-isometries:

Corollary 4.6.3. *Let X, X' be symmetric spaces of noncompact type. If X, X' are quasi-isometric, then they become isometric after the metrics on their de Rham factors are suitably renormalized.*

Idea of proving these two theorems:

Step 1: Choose an ultrafilter ω and scale metrics on X, X' by λ_i, then (L, C)-quasi-isometries becomes $(L, \lambda_i C)$-quasi-isometries. Let $\lambda_i \to 0$, take a limit of $\Phi : \lambda_i X \to \lambda_i X'$, we obtain a $(L, 0)$-quasi-isometry (i.e. a bilipschitz homeomorphism) $\Phi_\omega : X_\omega \to X'_\omega$ between two thick Euclidean buildings.

Step 2: To study the topology of the Euclidean buildings X_ω, X'_ω. Let X, X' be Euclidean buildings, then any homeomorphism $\psi : X \to X'$ carries apartments to apartments.

Lemma 4.6.4. *(Splitting for homeomorphisms of Euclidean buildings) Let Y_i, Y_i' be thick irreducible Euclidean buildings with topologically transitive affine Weyl group. Let $Y = \mathbb{E}^n \times \Pi_{i=1}^k Y_i$, $Y_i' = \mathbb{E}^{n'} \times \Pi_{j=1}^{k'} Y_j'$. If $\psi : Y \to Y'$ is a homeomorphism, then $n = n', k = k'$ and after re-indexing factors there are homeomorphisms $\psi_i : Y_i \to Y_i'$ so that $p' \circ \psi = \Pi \psi_i \circ p$.*

Lemma 4.6.5. *(Rigidity for homeomorphisms of Euclidean buildings) let Y be an irreducible thick Euclidean building with topologically transitive affine Weyl group and rank ≥ 2. Then any homeomorphism from Y to a Euclidean building is a homothety.*

Remark 4.6.6. In contrast of this, homeomorphism of rank 1 Euclidean buildings with non-discrete affine Weyl group (i.e. $\mathbb{R}$-trees) can be quite arbitrary: there are examples of $\mathbb{R}$-trees T for which every homeomorphism $A \to A$ of an apartment $A \subset T$ can be extended to a homeomorphism of T.

Step 3: We deduce the main theorems from their topological analogs. By using a scaling argument, we show that if $\Phi : X \to X'$ is an (L, C)-quasi-isometry, then the image of a maximal flat in X under Φ lies within uniform Hausdorff distance of a maximal flat in X'; the Hausdorff distance can be bounded uniformly by (L, C). Then follow similar steps in the proof of Mostow rigidity to prove the result.

In the case of the splitting theorem we use this to deduce that the quasi-isometry respects the product structure, and in the case of rigidity theorem we use this to show Φ induces a well-defined homeomorphism $\partial\Phi : \partial X \to \partial X'$ an isometry of Tits metrics. Using Tits' work [4] (also in the proof of the Mostow strong rigidity in [2]), $\partial\Phi$ is also induced by an isometry $\Phi_0 : X \to X'$, and $d(\Phi, \Phi_0)$ is bounded uniformly by (L, C).

Remark 4.6.7. In the proof of the Mostow strong rigidity, we push things out to infinity to get an isomorphism of the spherical Tits buildings. Here, we pull in infinity towards a basepoint to get the asymptotic cone and the symmetric space becomes Euclidean buildings. Both procedures ignore geometry at finite and turn a quasi-isometry into a more precise map between two more rigid spaces.

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Chapter 5

Compactification via affine buildings

Beatrice Pozzetti

In her thesis [6] A. Parreau constructed a compactification of the character variety $\mathbb{X}(\Gamma, G)$ of a finitely generated group Γ in a reductive Lie group G, whose boundary points correspond to (Weyl-chamber valued) marked length spectra of actions on affine buildings $\mathcal{B}$. The building $\mathcal{B}$ appearing in the construction is the asymptotic cone of the symmetric space $\mathcal{X} = G/K$ associated to G, but can also be interpreted as the affine $\mathbb{R}$-building associated to the algebraic group $G({}^\omega\mathbb{R}_\sigma)$, where ${}^\omega\mathbb{R}_\sigma$ is the Robinson field, a non-Archimedean, real closed field. The algebraic interpretation of the asymptotic cone $\mathrm{Cone}(\mathcal{X})$ originally suggested by B. Leeb and proven by A. Parreau also allows to define a non-standard symmetric space $\mathcal{X}({}^\omega\mathbb{R}_\sigma)$ endowed with a $G({}^\omega\mathbb{R}_\sigma)$ invariant multiplicative pseudodistance whose metric quotient is $G({}^\omega\mathbb{R}_\sigma)$-equivariantly identified with the building $\mathcal{B}$ [4]. The interplay between $\mathcal{X}({}^\omega\mathbb{R}_\sigma)$ and $\mathcal{B}$ played a major role in the work of M. Burger and the author [1] on the study of the actions arising in the boundary of maximal representations.

Purpose of the notes is to outline A. Parreau's constructions from [6], with emphasis on the details relevant for [1]. In Section 5.1 we define the Robinson field ${}^\omega\mathbb{R}_\sigma$ and describe the main ingredient in the proof of the identification $\mathcal{B} \cong \mathrm{Cone}(\mathcal{X})$, more detail can be found in [6, Section 3] and [1, Section 10]. In Section 5.2 we outline the main steps allowing to compactify the character variety $\mathbb{X}(\Gamma, G)$.

5.1 An algebraic perspective on asymptotic cones of symmetric spaces

5.1.1 The hyperreals

Let ω be a non-principal ultrafilter: a non-atomic $\{0,1\}$-valued finitely additive measure on $\mathbb{N}$. One should think of ω as a magic wand, allowing to compatibly pick an accumulation point for any bounded sequence. Non-principal ultrafilters exists (this is an easy application of the axiom of choice).

The choice of an ultrafilter allows to turn infinite products of fields into fields. We will denote by ${}^\omega\mathbb{R}$ the hyperreal: this is the quotient of the product of countably many copies of $\mathbb{R}$ modulo the equivalence relation given by $(x_n) \equiv (y_n)$ if and only if $\omega(\{n|\, x_n = y_n\}) = 1$:

$$^\omega\mathbb{R} = \prod_{n\in\mathbb{N}} \mathbb{R}/_{(x_n)\equiv(y_n)}$$

It is easy to verify that ${}^\omega\mathbb{R}$ is indeed a field and that the order on $\mathbb{R}$ induces a total order on ${}^\omega\mathbb{R}$ whose positive elements are the squares in ${}^\omega\mathbb{R}$. In particular ${}^\omega\mathbb{R}$ is a real closed field containing $\mathbb{R}$ as a subfield (included as equivalence class of constant sequences). The algebraic closure ${}^\omega\mathbb{C}$ of ${}^\omega\mathbb{R}$ is a degree two extension of ${}^\omega\mathbb{R}$ that can be described with the same construction.

It is a deep result of Erdös, Gillman and Heinrika [2] that, assuming the continuum hypothesis, the isomorphism type or ${}^\omega\mathbb{R}$ doesnt depend on the ultrafilter ω. Notice, instead, that since ${}^\omega\mathbb{C}$ is an algebraically closed field of the cardinality of the continuum, ${}^\omega\mathbb{C}$ is always abstractly isomorphic to the field of complex numbers $\mathbb{C}$.

For any real vector space V the ultraproduct ${}^\omega V$ is an ${}^\omega\mathbb{R}$ vector space that can be canonically identified with $V\otimes{}^\omega\mathbb{R}$. Similar we have natural identifications ${}^\omega\mathrm{End}(V)\cong \mathrm{End}({}^\omega V)$ inducing an identification ${}^\omega\mathrm{GL}(V)\cong\mathrm{GL}({}^\omega V)$. Moreover, denoting by $\mathrm{Gr}_k(V)$ the Grassmannian of k dimensional vector subspaces of V, we have an equivariant identification ${}^\omega\mathrm{Gr}_k(V)\cong\mathrm{Gr}_k({}^\omega V)$.

5.1.2 The Robinson field

We can now construct the Robinson field [7], a quotient of a subring of ${}^\omega\mathbb{R}$. An element $\tau\in{}^\omega\mathbb{R}$ is an *infinitesimal* if $\tau < 1/n$ for every $n\in\mathbb{N}$.

Example 5.1.1. Given a diverging sequence $(\lambda_n)_{n\in\mathbb{N}}$ we denote by σ the class in ${}^\omega\mathbb{R}$ of the sequence $(e^{-\lambda_n})_{n\in\mathbb{N}}$. It is easy to verify that σ is indeed an infinitesimal.

For any infinitesimal τ, we will denote by $\mathcal{O}_\tau\subset{}^\omega\mathbb{R}$ the subring of elements comparable with τ:

$$\mathcal{O}_\tau = \{x\in{}^\omega\mathbb{R}|\ \exists k\in\mathbb{N} \text{ with } |x| < \sigma^{-k}\}$$

Example 5.1.2. If σ is as in Example 5.1.1,

$$\begin{aligned}\mathcal{O}_\sigma &= \{x\in{}^\omega\mathbb{R}|\ \exists k\in\mathbb{N} \text{ with } |x_n| < e^{-k\lambda_n} \text{for } \omega\text{-almost every } n\}\\ &= \{x\in{}^\omega\mathbb{R}|\ \exists k\in\mathbb{N} \text{ with } |x_n|^{1/\lambda_n} \text{ is } \omega\text{-bounded}\}\end{aligned}$$

One immediately verifies that $\mathcal{O}_\tau$ is a ring with maximal ideal

$$\mathcal{I}_\tau = \{x\in{}^\omega\mathbb{R}|\ \forall k\in\mathbb{N} \text{ it holds } |x| < \sigma^k\}.$$

The ideal $\mathcal{I}_\tau$ consists precisely of the elements of $\mathcal{O}_\tau$ whose inverse doesn't belong to $\mathcal{O}_\tau$, and the quotient ${}^\omega\mathbb{R}_\tau := \mathcal{O}_\tau/\mathcal{I}_\tau$ is the Robinson field assoicated to the infinitesimal τ. In the sequel we will denote by ${}^\omega\mathbb{R}_\sigma$ the Robinson field associated to an infinitesimal of the form described in Example 5.1.1. In this case the Robinson field ${}^{\omega}\mathbb{R}_\sigma$ is the ultralimit of the sequence of pointed metric spaces $(\mathbb{R}, |\cdot|^{1/\lambda_n}, 0)$:

$${}^{\omega}\mathbb{R}_\sigma = \{(a_k) \in \mathbb{R}^{\mathbb{N}} |\ |a_k|^{1/\lambda_k} \text{ bounded }\}/_{\lim_\omega |a_k - b_k|^{1/\lambda_k} = 0}.$$

Here $|\cdot|$ denotes the absolute value.

Remark 5.1.3. Assuming the continuum hypothesis, Thornton proved [8] the field ${}^{\omega}\mathbb{R}_\sigma$ does not depend on ω nor on σ, on the other hand, assuming the negation of the continuum hypothesis Kramer, Shelah, Tent and Thomas showed that there is an uncountable set of non principal ultrafilters whose associated Robinson fields are pairwise not isomorphic [4].

It will be important for the construction of affine buildings the fact that ${}^{\omega}\mathbb{R}_\sigma$ admits a valuation

$$\begin{array}{rccc} v_\sigma : & {}^{\omega}\mathbb{R}_\sigma{}^* & \to & \mathbb{R} \\ & x & \mapsto & \sup\{c \in \mathbb{R} |\ x < \sigma^c\} \end{array}$$

and hence a norm $\|x\| = e^{-v_\sigma(x)}$. For every $r \in \mathbb{R}$ we have $\|r\| = 1$, and in particular the norm $\|\cdot\|$ is non-Archimedean. We will denote by $\mathcal{U}_\sigma$ the valuation ring

$$\mathcal{U}_\sigma = \{x \in {}^{\omega}\mathbb{R}_\sigma | v_\sigma \geq 0\}.$$

In order to give an explicit description of the ${}^{\omega}\mathbb{R}_\sigma$-points of vector spaces and algebraic groups we need to fix an auxiliary scalar product: let $\|\cdot\|_\omega : {}^{\omega}V \to {}^{\omega}\mathbb{R}$ be a quadratic form, and set

$${}^{\omega}V(\mathcal{O}_\sigma) := \{v \in {}^{\omega}V | \|v\|_\omega \in \mathcal{O}_\sigma\}$$
$${}^{\omega}V(\mathcal{I}_\sigma) := \{v \in {}^{\omega}V | \|v\|_\omega \in \mathcal{I}_\sigma\}$$

The quotient ${}^{\omega}V_\sigma :={}^{\omega} V(\mathcal{O}_\sigma)/{}^{\omega}V(\mathcal{I}_\sigma)$ is an ${}^{\omega}\mathbb{R}_\sigma$ vector space (which is identified with $V \otimes {}^{\omega}\mathbb{R}_\sigma$ provided if $\|\cdot\|_\omega$ is induced by a norm on V). Moreover, if we endow $\text{End}({}^{\omega}V)$ with the norm induced by $\|\cdot\|_\omega$ we get a natural identification ${}^{\omega}\text{End}(V)_\sigma \cong \text{End}({}^{\omega}V_\sigma)$.

Furthermore it is possible to define (cfr. [1, Section 5]) a map

$$\begin{array}{rccc} p : & \text{Gr}_k({}^{\omega}V) & \to & \text{Gr}_k({}^{\omega}V_\sigma) \\ & W & \mapsto & W(\mathcal{O}_\sigma)/W(\mathcal{I}_\sigma). \end{array}$$

However the map p crucially depends on the choice of the quadratic form $\|\cdot\|_\omega$ and doesn't, in general, preserve transversality.

5.1.3 Affine buildings associated with algebraic groups

Semisimple algebraic groups over fields admitting a real valuation act on affine buildings. We will be only concerned with real closed fields:

Theorem 5.1.4 (Parreau [5], Kramer-Tent [3]). *Let G be a semisimple algebraic group, $\mathbf{F}$ a non-Archimedean, real closed valued field. Denoting by $\mathcal{U} < \mathbf{F}$ the valuation subring, the quotient space $G(\mathbf{F})/G(\mathcal{U})$ can be $G(\mathbf{F})$-equivariantly identified with the vertex set of an affine building.*

We will refer to the notes of Q. Li in this workshop [?] for a definition of an affine building, but we will sketch the construction of the affine building associated to $\mathrm{SL}(n, {}^\omega\mathbb{R}_\sigma)$ following Parreau [6].

We denote by ${}^\omega V_\sigma$ an n-dimensional ${}^\omega\mathbb{R}_\sigma$ vector space on which $\mathrm{SL}(n, {}^\omega\mathbb{R}_\sigma)$ acts linearly. A *norm* $\eta : {}^\omega V_\sigma \to \mathbb{R}^+$ is a function with the following properties:

- $\eta(v) = 0$ if and only if $v = 0$,
- $\eta(av) = \|a\|\eta(v)$,
- $\eta(v + w) \leq \max\{\eta(v), \eta(w)\}$.

A norm is *adapted* to a basis $\{e_1, \dots, e_n\}$ of ${}^\omega V_\sigma$ if

$$\eta(\sum a_i e_i) = \max \eta(e_i)\|a_i\|.$$

Furthermore the norm η *has determinant one* if for one (and ence every) adapted basis $e_1, \dots, e_n$ it holds $\prod \eta(e_i) = 1$.

Remark 5.1.5. If η is a good norm adapted to $\{e_1, \dots, e_n\}$ then for every $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$, the norm η' defined by $\eta'(\sum a_i e_i) = \max_i\{\alpha_i \eta(e_i)\|a_i\|\}$ is also adapted to $\{e_1, \dots, e_n\}$.

Example 5.1.6. Assume, for simplicity, that $n = 2$. If η is adapted to e_1, e_2 and $\eta(e_1) > \eta(e_2)$ then η is also adapted to $e_1 + e_2, e_2$. Indeed we need to verify that $\eta(a(e_1 + e_2) + (b - a)e_2)$ is equal to $\max\{\|a\|\eta(e_1 + e_2), \|b - a\|\eta(e_2)\}$. On the one hand we have

$$\begin{array}{rl} & \max\{\|a\|\eta(e_1 + e_2), \|b - a\|\eta(e_2)\} \\ \leq & \max\{\|a\|\eta(e_1), \|b\|\eta(e_2)\} \end{array}.$$

On the other hand, since η is adapted to e_1, e_2 we know that

$$\eta(a(e_1 + e_2) + (b - a)e_2) = \max\{\|a\|\eta(e_1), \|b\|\eta(e_2)\}.$$

Since $\|b - a\| = \|b\|$ if $\|b\| \geq \|a\|$ the result follows.

It is possible to verify that for every pair (μ, η) of adaptable norms of determinant one there exists a common adapted basis $\{e_1, \dots, e_n\}$, and that the function

$$d(\eta, \mu) = \sqrt{\sum \left|\log \frac{\mu}{\eta}(e_i)\right|^2}$$

doesn't depend on the choice of the common adapted basis $e_1, \dots, e_n$ and induces a CAT(0) distance on the set $\mathcal{B}$ of adapted norms of determinant one. In particular, for every basis $e_1, \dots, e_n$ of ${}^\omega V_\sigma$, the set of norms of determinant one that are adapted to $e_1, \dots, e_n$ forms an isometrically embedded copy of $\mathbb{R}^{n-1}$ inside $\mathcal{B}$.

Using linear algebra one can verify that the distance d, and the system of apartments corresponding to the bases of ${}^\omega V_\sigma$ endowes the set $\mathcal{B}$ with the structure of an affine $\mathbb{R}$-building (cfr [6, Section 3]). Furthermore, generalizing Example 5.1.6, it is possible to check that $\mathrm{SL}_n(\mathcal{U})$ coincides with the stabilizer of the standard norm adapted to the standard basis.

Given the explicit model for the affine building associated to $\mathrm{SL}_n({}^\omega\mathbb{R}_\sigma)$ it is not too hard to verify that this coincides with the asymptotic cone of the symmetric space associated to the sequence of scales (λ_n) (see [?] for a definition of the asymptotic cone):

Theorem 5.1.7 ([6, Theorem 3.21]). *Let $\mathcal{X} = \mathrm{SL}_n(\mathbb{R})/O(n)$ be the symmetric space associated to $\mathrm{SL}(n,\mathbb{R})$. Then there is a natural identification*

$$\mathrm{Cone}_\omega(\mathcal{X}, d/\lambda_n, x) \cong \mathcal{B}$$

Recall that a model of the symmetric space associated to $\mathrm{SL}(n,\mathbb{R})$ is given by positive definite symmetric matrices of determinant one. To any such matrix X_k one can associate a norm η_k on $\mathbb{R}^n$. The isomorphism of Theorem 5.1.7 can then be given by the explicit formula

$$[\eta_k] \mapsto \lim_\omega \eta_k^{1/\lambda_k}.$$

Observe that the sequence η_k defines a point in the asymptotic cone $\mathrm{Cone}_\omega(\mathcal{X}, d/\lambda_n, x)$ if and only if

$$\lim_\omega \frac{d(\eta_k, \eta_0)}{\lambda_k} < \infty$$

which is in turn equivalent to the fact that

$$\lim_\omega \frac{\log \eta_k(e_i)}{\lambda_k} < \infty$$

so that the right hand side is well defined. We refer to [6, Section 6] for a proof of the fact that it indeed induces an adapted norm of determinant one on ${}^\omega\mathbb{R}_\sigma{}^n$, and that the map induces an isometry of the metric spaces.

5.2 Marked length spectrum compactifications of character varieties

Let Γ be a finitely generated group, G a semisimple Lie group of noncompact type (that we identify, up to passing to a finite quotient of a finite index subgroup, with a subgroup of $\mathrm{SL}(n,\mathbb{R})$ for some suitable n). We denote by $\mathcal{X} = G/K$ the symmetric space associated to G. We furthermore denote by $\mathbb{X}(\Gamma, G)$ the character variety, namely the biggest Hausdorff quotient of the representation variety $R(\Gamma, G)$ for the action of G by conjugation.

We denote by $\overline{\mathfrak{C}}^+$ a model closed Weyl chamber in a maximal flat of $\mathcal{X}$ and by $\delta : \mathcal{X} \times \mathcal{X} \to \overline{\mathfrak{C}}^+$ the natural projection. For any isometry g in G it is possible to define its Weyl chamber valued translation distance $\nu(g) \in \overline{\mathfrak{C}}^+$ as the vector of minimal length in the closure of the set $\{\delta(x, gx) | \, x \in \mathcal{X}\}$. This is well defined and coincides with the Jordan projection of g (cfr. [6, Section 4]). The main result of [6] is then the following:

Theorem 5.2.1 ([6, Theorem 1]). *The continuous map*

$$\mathbb{P}\nu : \mathbb{X}(\Gamma, G) \setminus \nu^{-1}(0) \to \mathbb{P}(\overline{\mathfrak{C}}^{+})^{\Gamma}$$

induces a compactification of $\mathbb{X}(\Gamma, G)$ *whose boundary points* $\partial_\infty \mathbb{X} \subset \mathbb{P}(\overline{\mathfrak{C}}^{+})^{\Gamma}$ *have the form* $[\nu \circ \rho]$ *where* ρ *is an action of* Γ *on an asymptotic cone of* $\mathcal{X}$*.*

Remark 5.2.2. It is possible to construct examples in which the action ρ is not uniquely determined by its translation length.

In order to give a short outline of the proof of Theorem 5.2.1 recall that given an action $\rho : \Gamma \to \mathrm{Isom}(\mathcal{X})$ of the group Γ generated by its finite part S, the displacement ℓ_ρ of ρ is given by

$$\ell_\rho = \inf_{s \in S} \sqrt{\sum_{s \in S} d(x, \rho(s)x)^2}.$$

If now $\rho_k : \Gamma \to G$ is a sequence of representations, and we let x_k be a point in $\mathcal{X}$ whose ρ_k displacement is not bigger that $\ell_\rho + 1$, it is easy to verify that the sequence ρ_k induces an isometric action

$$\rho_\omega : \Gamma \to \mathrm{Cone}\left(\mathcal{X}, \frac{d}{\ell_{\rho_k}}, x_k\right).$$

The action ρ_ω is the action ρ from the statement of the theorem, and has the desired properties since the Weyl Chamber valued translation distance is asymptotically continuous. We refer the reader to [6, Section 4] for the nice proof of this last statement which uses nice geometric arguments from CAT(0) geometry.

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Chapter 6

Positive spaces

Tengren Zhang

6.1 Introduction.

In this note, we introduce the notion of a positive variety, and explain how one can compactify them. We then give a careful description of two important examples of positive varieties, namely the Fock-Goncharov $\mathcal{A}$ and $\mathcal{X}$ moduli spaces. The material covered here is not original; most of the material here is from [1] and [2]. This note is the accompanying lecture notes for a lecture given at the "Workshop on Compactifications of Moduli spaces of Representations" organized in 2017.

6.2 Positive varieties and their compactifications

In this section, we will define the notion of an abstract positive variety $\mathcal{P}$, and construct an algebraic compactification of $\mathcal{P}(\mathbb{R}^+)$.

6.2.1 Semifields

To define a positive variety, we first need the notion of a semifield.

Definition 6.2.1. A *semifield* is a triple $(K, +, \cdot)$, where K is a set, $(K, +)$ is an abelian semigroup, $(K \setminus \{0\}, \cdot)$ is an abelian group, and $(a + b) \cdot c = a \cdot c + b \cdot c$.

In the above definition, $0 \in K$ denotes the additive identity in the semigroup $(K, +)$, which might or might not exist. We now give some examples of commonly used semifields.

Example 6.2.2.

1. Any field is a semifield with 0.

2. $\mathbb{R}_+$ and $\mathbb{Q}_+$ are semifields without 0, and $\mathbb{R}_+ \cup \{0\}$ and $\mathbb{Q}_+\{0\}$ are semifields with 0.

3. Let $\mathbb{Z}(X_1, \dots, X_d)$ denote the field of rational functions in the variables $X_1, \dots, X_d$, with integer coeffecients. We say that $f \in \mathbb{Z}(X_1, \dots, X_d)$ is *positive* if $f = \frac{f_1}{f_2}$ for some non-zero $f_1, f_2 \in \mathbb{Z}[X_1, \dots, X_d]$ with non-negative coeffecients. Then

$$\mathbb{Z}(X_1, \dots, X_d)_+ := \{f \in \mathbb{Z}(X_1, \dots, X_d) : f \text{ is positive}\}$$

equipped with the usual addition and multiplication is a semifield without 0.

4. Let $\mathbb{Z}^t := (\mathbb{Z}, \oplus, \odot)$ and $\mathbb{R}^t := (\mathbb{R}, \oplus, \odot)$, where the addition $\oplus$ is given by $x \oplus y := \max\{x, y\}$ and the multiplication $\odot$ is given by $x \odot y = x + y$ (here, + is the usual addition on $\mathbb{Z}$ or $\mathbb{R}$). Then $\mathbb{Z}^t$ and $\mathbb{R}^t$ are semi-fields without 0, commonly known as the *semi-field of tropical integers* and the *semi-field of tropical real numbers* respectively.

6.2.2 Positive varieties

We will now define the notion of a positive variety over a semifield K (with or without 0).

Definition 6.2.3. Let X be a set, K be a semifield, and $d \in \mathbb{Z}^+$. A *K^d-positive atlas* of X is a collection of maps $\{\varphi_\alpha : K^d \dashrightarrow X\}_{\alpha \in A}$ so that

- every $x \in X$ lies in $\varphi_\alpha(K_d)$ for some $\alpha \in A$,
- for all $\alpha, \beta \in A$,
$$\varphi_\alpha^{-1} \circ \varphi_\beta : K^d \dashrightarrow K^d$$
is a birational map given coordinate-wise by algebraic expressions in $\mathbb{Z}(X_1, \dots, X_d)_+$.

In the above definition, the broken arrow $\dashrightarrow$ denotes a map that is defined on a Zariski-open subset of K^d. When the charts are implicit, we will often write $\mathcal{P}(K) = X$. Any map in a positive atlas is called a *positive chart*. The pair

$$\mathcal{P}(K) := (X, \{\varphi_\alpha\}_{\alpha \in A})$$

as a *positive variety* over K. The collection of formal algebraic expressions $\mathcal{P} := \{\varphi_\alpha^{-1} \circ \varphi_\beta\}_{\alpha,\beta \in A} \subset \mathbb{Z}(X_1, \dots, X_d)_+$ is called an *abstract positive variety.*

For any $\alpha \in A$, let $K_\alpha^d := \{p \in K^d : \varphi_\alpha(p) \in K^d \text{ for some } \alpha \in A\}$. Then note that X can be abstractly given by

$$X = \left(\bigcup_{\alpha \in A} K_\alpha^d\right) \Big/ \sim,$$

where $p, q \in \bigcup_{\alpha \in A} K_\alpha^d$ are equivalent under $\sim$ if $p \in K_\alpha^d$, $q \in K_\beta^d$, and $\varphi_\alpha \circ \varphi_\beta^{-1}(q) = p$. Also, observe that any collection of formal algebraic expressions $f_1, \dots, f_d \in \mathbb{Z}(X_1, \dots, X_d)_+$

defines a birational map $K^d \dashrightarrow K^d$. As such, specifying the positive variety $\mathcal{P}(K)$ over K is equivalent to specifying the semifield K and the abstract positive variety $\mathcal{P}$.

Let $\mathcal{P} = \{\varphi_\alpha^{-1} \circ \varphi_\beta\}_{\alpha,\beta\in A}$ be an abstract positive variety. If $K = \mathbb{R}$, then the fact that $\varphi_\alpha^{-1} \circ \varphi_\beta \in \mathbb{Z}(X_1, \dots, X_d)_+$ implies $\varphi_\alpha(\mathbb{R}_+^d) = \varphi_\beta(\mathbb{R}_+^d)$. Thus, we can define the notion of a *positive point* in $\mathcal{P}(\mathbb{R})$ to be a point in $\varphi_\alpha(\mathbb{R}_+^d)$ for some (equiv. all) $\alpha \in A$. Observe that the set of positive points in $\mathcal{P}(\mathbb{R}_+)$ is itself naturally a positive variety

$$\mathcal{P}(\mathbb{R}_+) := \left(\bigcup_{\alpha\in A} \varphi_\alpha(\mathbb{R}_+^d), \{\varphi_\alpha\}_{\alpha\in A} \right)$$

over the semifield $\mathbb{R}_+$. Furthermore, since $\varphi_\alpha^{-1} \circ \varphi_\beta$ is a homeomorphism from $\mathbb{R}_+^d$ to itself for any $\alpha, \beta \in A$, we can topologize $\mathcal{P}(\mathbb{R}_+)$ by identifying it with $\mathbb{R}_+^d$ via some chart.

The main examples of positive spaces are the Fock-Goncharov $\mathcal{X}$ and $\mathcal{A}$ moduli spaces that we will define in the next section. Before doing so, we will explain a general procedure to compactify any positive variety over the semifield $\mathbb{R}_+$ that is due to Fock-Goncharov.

6.2.3 Compactifying $\mathcal{P}(\mathbb{R}^+)$

Let $\mathcal{P}$ be any abstract positive variety. First, note that for all $\varphi_\alpha^{-1} \circ \varphi_\beta \in \mathcal{P}$, the map

$$\begin{array}{rcl} \mathbb{Z}(X_1, \dots, X_d)_+ & \to & \mathbb{Z}(X_1, \dots, X_d)_+ \\ f & \mapsto & f \circ \varphi_\alpha^{-1} \circ \varphi_\beta \end{array}$$

is a bijection. This implies that the space of *positive rational functions* on $\mathcal{P}(\mathbb{R}_+)$,

$$\mathcal{Q}_+(\mathcal{P}(\mathbb{R}_+)) := \{ f \circ \varphi_\alpha^{-1} : f \in \mathbb{Z}(X_1, \dots, X_d)_+ \}$$

is well-defined, i.e. it does not depend on the choice of φ_α.

It is well-known that $\mathbb{Z}(X_1, \dots, X_d)_+$ is countably generated as a module over the semifield $\mathbb{Q}_+$, so the same is true for $\mathcal{Q}_+(\mathcal{P}(\mathbb{R}_+))$. Let $\mathcal{B} = \{f_1, f_2, \dots\}$ be a countable generating set for $\mathcal{Q}_+(\mathcal{P}(\mathbb{R}_+))$, and define

$$\begin{array}{rcl} \iota : \mathcal{P}(\mathbb{R}_+) & \to & \mathbb{R}^{\mathcal{B}} \\ x & \mapsto & (\log \circ f(x))_{f\in\mathcal{B}} \end{array}$$

It is clear that ι is injective because the coordinate functions of any chart of $\mathcal{P}(\mathbb{R}_+)$ lie in $\mathcal{Q}_+(\mathcal{P}(\mathbb{R}_+))$. Furthermore, since the squares of the coordinate functions of any chart of $\mathcal{P}(\mathbb{R}_+)$ also lies in $\mathcal{Q}_+(\mathcal{P}(\mathbb{R}_+))$, we see that ι descends to an injective map

$$\iota : \mathcal{P}(\mathbb{R}_+) \to \mathbb{P}(\mathbb{R}^{\mathcal{B}}) := \mathbb{R}^{\mathcal{B}} / (\mathbb{R} \setminus \{0\}).$$

If we equip $\mathbb{R}^{\mathcal{B}}$ with the weak topology, then $\mathbb{P}(\mathbb{R}^{\mathcal{B}})$ is compact. We can thus construct the following compactification of $\mathcal{P}(\mathbb{R}^+)$.

Definition 6.2.4. The closure of $\iota(\mathcal{P}(\mathbb{R}_+))$ in $\mathbb{P}(\mathbb{R}^{\mathcal{B}})$ is the *tropical compactification* of $\iota(\mathcal{P}(\mathbb{R}_+))$. Denote its boundary by $\partial\mathcal{P}(\mathbb{R}_+)$.

Since $\mathcal{P}(\mathbb{R}_+)$ is locally compact, this implies that ι is a homeomorphism onto its image. Hence, the tropical compactification of $\iota(\mathcal{P}(\mathbb{R}_+))$ is indeed a compactification of $\iota(\mathcal{P}(\mathbb{R}_+))$. One can also verify that the tropical compactification of $\iota(\mathcal{P}(\mathbb{R}_+))$ does not depend on the choice of $\mathcal{B}$.

6.3 Fock-Goncharov $\mathcal{A}$ and $\mathcal{X}$ moduli spaces.

In this section, we will define the Fock-Goncharov $\mathcal{A}$ and $\mathcal{X}$ moduli spaces, and describe the positive atlases that make both of these spaces positive varieties.

6.3.1 Flags and Affine flags

We begin by explaining the notion of flags and affine flags. Suppose that K is a field and V is a m-dimensional real vector space.

Definition 6.3.1.

1. A *flag* F of V is a nested sequence of subspaces of V, one of each dimension. For any $l = 1, \dots, m-1$, let $F^{(l)}$ denote the l-th dimensional subspace of V given by F. Also, denote the space of flags in V by $\mathcal{F}(V)$.

2. An *affine flag* AF of V is a flag F equipped with a tuple
$$\omega_{AF} = (\omega_{AF}^{(1)}, \dots, \omega_{AF}^{(m-1)}),$$
where $\omega_{AF}^{(l)} \in \bigwedge^l F^{(l)}$ for all $l = 1, \dots, m-1$. We refer to F as the *underlying flag* of the affine flag AF. Denote the space of affine flags in V by $\mathcal{AF}(V)$.

We say that a triple of flags $(F, G, H) \in \mathcal{F}(V)^3$ is *transverse* if for all $i, j, k \in \{0, \dots, m-1\}$ so that $i + j + k = m$, we have that
$$F^{(i)} + G^{(j)} + H^{(k)} = V.$$
Similarly, we say that a triple of affine flags $(AF, AG, AH) \in \mathcal{AF}(V)^3$ is *transverse* if the underlying triple of flags $(F, G, H) \in \mathcal{F}(V)^3$ is transverse.

Note that $\mathrm{PGL}(V)$ acts naturally on $\mathcal{F}(V)$, while $\mathrm{SL}(V)$ acts naturally on $\mathcal{AF}(V)$. For any flag $F \in \mathcal{F}(V)$, we say that a basis $\{f_1, \dots, f_m\}$ of V is *associated* to F if $F^{(l)} = \mathrm{Span}_K(f_1, \dots, f_l)$ for all $l = 1, \dots, m-1$.

Let $(F, G, H) \in \mathcal{F}(V)^3$ and let
$$\{f_1, \dots, f_m\}, \ \ \{g_1, \dots, g_m\}, \ \text{ and } \ \{h_1, \dots, h_m\}$$

be bases of V associated to F, G and H respectively. Then for all $i,j,k\in\{0,\dots,m-1\}$ so that $i+j+k=m$, we can define

$$F^{(i)}\wedge G^{(j)}\wedge H^{(k)} := f_1\wedge\dots\wedge f_i\wedge g_1\wedge\dots\wedge g_j\wedge h_1\wedge\dots\wedge h_k\in\bigwedge^m V.$$

Note that this is an abuse of notation because $F^{(i)}\wedge G^{(j)}\wedge H^{(k)}$ depends not only on $F^{(i)}$, $G^{(j)}$, $H^{(k)}$, but also on the choice of associated bases to F,G,H. However, if (F,G,H) happens to be a transverse triple of flags, then $F^{(i)}\wedge G^{(j)}\wedge H^{(k)}\neq 0$ for any choice of associated bases. If we further choose a linear identification $\bigwedge^m V\simeq\mathbb{R}$, then $F^{(i)}\wedge G^{(j)}\wedge H^{(k)}\in\mathbb{R}$.

Using this, we can define a family of projective invariants of a transverse triple of flags. For every transverse triple $(F,G,H)\in\mathcal{F}(V)^3$, and any $i,j,k\in\{1,\dots,m-2\}$ so that $i+j+k=m$, we can define the *triple ratio*

$$T_{i,j,k}(F,G,H)=\frac{F^{(i+1)}\wedge G^{(j)}\wedge H^{(k-1)}}{F^{(i+1)}\wedge G^{(j-1)}\wedge H^{(j)}}\cdot\frac{F^{(i-1)}\wedge G^{(j+1)}\wedge H^{(k)}}{F^{(i)}\wedge G^{(j+1)}\wedge H^{(k-1)}}\cdot\frac{F^{(i)}\wedge G^{(j-1)}\wedge H^{(k+1)}}{F^{(i-1)}\wedge G^{(j)}\wedge H^{(k+1)}}.$$

One can verify that $T_{i,j,k}(F,G,H)$ does not depend on the choice of associated bases for (F,G,H), nor on the choice of identification $\bigwedge^m V\simeq\mathbb{R}$. Furthermore, it is easy to see that $T_{i,j,k}(F,G,H)=T_{i,j,k}(g\cdot F,g\cdot G,g\cdot H)$ for all $g\in\mathrm{PGL}(V)$. Hence, the triple ratios are indeed $\mathbb{R}$-valued projective invariants of transverse triples of flags. The following symmetry also holds:

$$T_{i,j,k}(F,G,H)=T_{j,k,i}(G,H,F).$$

Similarly, if $F,G,H_1,H_2\in\mathcal{F}(V)$ so that (F,G,H_l) is a transverse triple for both $l=1,2$, we can define the *cross ratio*

$$C_i(F,H_1,H_2,G)=\frac{F^{(i)}\wedge G^{(n-i-1)}\wedge H_2^{(1)}}{F^{(i)}\wedge G^{(n-i-1)}\wedge H_1^{(1)}}\cdot\frac{F^{(i-1)}\wedge G^{(n-i)}\wedge H_1^{(1)}}{F^{(i-1)}\wedge G^{(n-i)}\wedge H_2^{(1)}}$$

for all $i=1,\dots,m-1$. Just like the triple ratio, the cross ratio is a projective invariant that does not depend on the choice of associated bases for F,G,H_1,H_2, nor on the choice of identification $\bigwedge^m V\simeq\mathbb{R}$. The following symmetry also holds:

$$C_i(F,H_1,H_2,G)=C_{n-i}(G,H_2,H_1,F).$$

To define $\mathbb{R}$-valued invariants for affine flags, one needs to further choose a linear identification $\bigwedge^m V\simeq\mathbb{R}$. Unlike the case of flags, the invariants we define for affine flags depend on the choice of this linear identification. For any triple of affine flags $(AF,AG,AH)\in\mathcal{F}(V)^3$ and any $i,j,k\in\{0,\dots,m-1\}$ so that $i+j+k=m$, we can define

$$t_{i,j,k}(AF,AG,AH):=\omega_F^{(i)}\wedge\omega_F^{(j)}\wedge\omega_F^{(k)}\in\bigwedge^m V\simeq\mathbb{R}.$$

It is easy to see that $t_{i,j,k}(AF,AG,AH)=t_{i,j,k}(g\cdot AF,g\cdot AG,g\cdot AH)$ for all $g\in\mathrm{SL}(V)$. However, unlike the triple ratio, observe that $t_{i,j,k}(AF,AG,AH)=t_{j,k,i}(AG,AH,AF)$ does not necessarily hold when m is even (for example, take $i=1,j=1,k=m-2$). We will see later that this lack of rotational symmetry is the reason we have to define decorated representations using flat bundles over T^1S instead of flat bundles over S.

6.3.2 Decorated and framed representations

Let S be an oriented, connected, compact surface with non-empty boundary, such that each boundary component has finitely many (possibly no) marked points. Let g be the genus of S, let r be the number of boundary components of S, and let $\{p_1, \dots, p_n\} \subset \partial S$ denote the set of marked points. We will also assume that if $S = \mathbb{D}^2$, then $n \geq 3$, and if $S = S^1 \times [0,1]$, then each boundary component of S has at least one marked point.

Definition 6.3.2. Let $i : \partial S \setminus \{p_1, \dots, p_n\} \to S$ be the obvious inclusion, and let $j : i^*(T^1S) \to T^1S$ be the obvious inclusion.

1. A *framed representation* on S is the pair $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a flat $\mathcal{F}(V)$-bundle over S and β is a flat section of $i^*(\mathcal{L})$.

2. A *decorated representation* on S is the pair $(\mathcal{M}, \alpha)$, where $\mathcal{M}$ is a flat $\mathcal{AF}(V)$-bundle over T^1S so that the holonomy about the circle fiber of the projection $T^1S \to S$ is
$$s = \begin{cases} \mathrm{Id} & \text{if } \dim(V) \text{ is odd} \\ -\mathrm{Id} & \text{if } \dim(V) \text{ is even} \end{cases},$$
and α is a flat section of $j^*(\mathcal{M})$.

Note that when $\dim(V)$ is odd, then $s = \mathrm{Id}$, so $\mathcal{M}$ induces a flat $\mathcal{AF}(V)$-bundle $\mathcal{M}'$ over S, and α induces a flat section of α' of $i^*(\mathcal{M}')$. For general n, we need to define a framed representation using a bundle over T^1S instead of over S in order to ensure the moduli space of decorated representations has a positive structure. We will see this later.

Suppose that a boundary component c of S has no marked points. Then for every framed representation $[\mathcal{L}, \beta]$ (resp. decorated representation $[\mathcal{M}, \alpha]$) on S and every point $p \in c$, the holonomy based at p about c fixes $\beta(p)$ (resp. $\alpha(p)$). In particular, the holonomy about c for any decorated representation on S is unipotent.

Definition 6.3.3.

1. Two framed representations $(\mathcal{L}, \beta)$ and $(\bar{\mathcal{L}}, \bar{\beta})$ are *isomorphic* if there is a $\mathrm{PGL}(V)$-bundle isomorphism $\varphi : \mathcal{L} \to \bar{\mathcal{L}}$ so that $\bar{\beta} = \varphi \circ \beta$.

2. Two framed representations $(\mathcal{M}, \alpha)$ and $(\bar{\mathcal{M}}, \bar{\alpha})$ are *isomorphic* if there is a $\mathrm{SL}(V)$-bundle isomorphism $\varphi : \mathcal{M} \to \bar{\mathcal{M}}$ so that $\bar{\alpha} = \varphi \circ \alpha$.

With this notion of equivalence, we can define $\mathcal{X}_{\mathrm{PGL}(V),S}(\mathbb{R})$ to be the set of isomorphism classes of framed representations on S. Similarly, $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ is the set of isomorphism classes of decorated representations on S. These are commonly known as the Fock-Goncharov $\mathcal{X}$ and $\mathcal{A}$ moduli spaces respectively. It is a theorem of Fock-Goncharov that these two spaces are examples of positive varieties.

Theorem 6.3.4 (Fock-Goncharov)**.** *$\mathcal{X}_{\mathrm{PGL}(V),S}(\mathbb{R})$ and $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ are positive varieties over $\mathbb{R}$.*

As a consequence of this theorem and the discussion in Section 6.2.2, it now makes sense to define $\mathcal{X}_{\mathrm{PGL}(V),S}(K)$ and $\mathcal{A}_{\mathrm{SL}(V),S}(K)$ for any semifield K, and these are positive varieties over K as well. In Section 6.3.4 and Section 6.3.5, we will describe the charts that make $\mathcal{X}_{\mathrm{PGL}(V),S}(\mathbb{R})$ and $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ positive varieties over $\mathbb{R}$.

6.3.3 Triangulations on S

To describe the positive charts of $\mathcal{X}_{\mathrm{PGL}(V),S}(\mathbb{R})$ and $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$, we first need the notion of a triangulation on S. To define this, we treat every boundary component of S without any marked points as a puncture on S. More formally, we consider the surface $S/\sim$, where $p\sim q$ if and only if p and q both lie in a common boundary component of S that does not contain any marked points. The image of such a boundary component of S under the quotient map $S\to S/\sim$ is a *puncture*.

Definition 6.3.5. A *triangulation* of $S/\sim$ is a maximal collection of simple curves in $S/\sim$ so that

- The endpoints of these curves are either a puncture of $S/\sim$ or a marked point in $\partial(S/\sim)$.
- No two of these curves intersect.
- No two of these curves are homotopic relative endpoints.

Two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of $S/\sim$ are *equivalent* if every curve in $\mathcal{T}_1$ is homotopic relative endpoints to a curve in $\mathcal{T}_2$. Denote the set of equivalence classes of triangulations of $S/\sim$ by $\Delta(S)$. Our assumptions on S ensure that $S/\sim$ has at least one equivalence class of triangulations. Also, observe that any triangulation of S has $6g-6+3r+2n$ edges, and cuts S into $4g-4+2r+n$ triangles.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulations of $S/\sim$, let $e\in\mathcal{T}$, and let T_1, T_2 be the two triangles of $\mathcal{T}$ that share e as a common edge. Also, let x,y be the endpoints of e, and for $i=1,2$, let w_i be the vertex of T_i that is neither x not y. Then let e' be a simple curve with endpoints w_1 and w_2 that intersects the curves in $\mathcal{T}$ only at e. We say that $\mathcal{T}'$ is related to $\mathcal{T}$ by a *flip* about e if $\mathcal{T}'=(\mathcal{T}\cup\{e'\})\smallsetminus\{e\}$.

Observe that for any triangulation $\mathcal{T}$ of $S/\sim$, if $\mathcal{T}'$ and $\mathcal{T}''$ are related to $\mathcal{T}$ by a flip about e, then $[\mathcal{T}']=[\mathcal{T}'']$. As such, $\Delta(S)$ is naturally the vertex set of a graph, where two equivalence classes of triangulations $[\mathcal{T}]$ and $[\mathcal{T}']$ are adjacent if some (equiv. any) representatives $\mathcal{T}\in[\mathcal{T}]$ and $\mathcal{T}'\in[\mathcal{T}']$ are related by a flip. It is well-known that this graph is connected, i.e. it is possible to get from any triangulation to any other by a finite sequence of flips.

In the next two sections, we will see that the set of charts for $\mathcal{X}_{\mathrm{PGL}(V),S}(\mathbb{R})$ and $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ are parameterized by $\Delta(S)$. To do so, we first make the following observation. Let $q:S\to S/\sim$ be the quotient map. Then for every $[\mathcal{T}]\in\Delta(S)$, there is a representative $\mathcal{T}$ in $[\mathcal{T}]$ so that for every edge $e\in\mathcal{T}$, there is a curve $e'\subset S$ that intersects every boundary component with no marked points at most once, and $q(e')=e$. We refer to e' as the *lift* of e, and refer to $\mathcal{T}$ as a *good* representative of $[\mathcal{T}]$.

6.3.4 $\mathcal{X}_{\mathrm{PGL}(V),S}(\mathbb{R})$ is a positive variety over $\mathbb{R}$

Let $(\mathcal{L},\beta)$ be any framed representation on S. For any oriented curve e in S with p and q as its backward and forward endpoints respectively, let $P_{e,\mathcal{L}}:\mathcal{L}_p\to\mathcal{L}_q$ denote the parallel transport along e induced by the flat structure on $\mathcal{L}$.

For any $[\mathcal{T}]\in\Delta(S)$, we will define a chart $K^d \longrightarrow \mathcal{X}_{\mathrm{PGL}(V),S}$, where

$$d=(6g-6+3r+n)\cdot(m-1)+(4g-4+2r+n)\cdot\frac{(m-1)(m-2)}{2}.$$

Choose a good representative $\mathcal{T}$ of $[\mathcal{T}]$. Using this, we define two families of invariants on $\mathcal{X}_{\mathrm{PGL}(V),S}$.

The first family of invariants are associated to the triangles given by the traingulation $\mathcal{T}$ of $S/\sim$. The orientation on S induces a natural counter-clockwise orientation on the boundary of every triangle in $S/\sim$ given by $\mathcal{T}$. Let e_1,e_2,e_3 be the edges of such a triangle oriented according to the counter-clockwise orientation on its boundary, so that the forward endpoint of e_l agrees with the backward endpoint of e_{l+1} for all $l=1,2,3$ (arithmetic in the subscripts are done modulo 3). Then let $e_1',e_2',e_3'\subset S$ be the lifts of e_1,e_2,e_3 respectively. If the forward endpoint b_l of e_l' agrees with the backward endpoint a_{l+1} of e_{l+1}', then define $e_l'':=e_l'$. On the other hand, if they do not agree, then b_l and a_{l+1} both lie in a boundary component $c\subset S$ with no marked points. The orientation of S also induces an orientation on c so that S lies to the left of c. Let $e_{l,l+1}\subset c$ be the oriented curve whose forward and backward endpoints are a_{l+1} and b_l respectively, and so that the orientation of $e_{l,l+1}$ agrees with the orientation of c. In this case, let e_l'' denote the oriented curve that is the concatenation $e_l'\cdot e_{l,l+1}$.

In either case, note that e_1'', e_2'' and e_3'' are oriented curves in S so that the forward endpoint of e_l'' and the backward endpoint of e_{l+1}'' are both a_{l+1} for $l=1,2,3$. Then for every $i_1,i_2,i_3\in\{1,\dots,m-2\}$ so that $i_1+i_2+i_3=m$, define the *triangle invariant*

$$\begin{aligned} T^{i_1,i_2,i_3}_{a_1,a_2,a_3}:\mathcal{X}_{\mathrm{PGL}(V),S} &\longrightarrow K\\ [\mathcal{L},\beta] &\mapsto T_{i_1,i_2,i_3}\Big(P_{e_1'',\mathcal{L}}\big(\beta(a_1)\big),\beta(a_2),P^{-1}_{e_2'',\mathcal{L}}\big(\beta(a_3)\big)\Big).\end{aligned}$$

Note that $T^{i_1,i_2,i_3}_{a_1,a_2,a_3}$ is not defined at every point in $\mathcal{X}_{\mathrm{PGL}(V),S}$; it is only defined when $\Big(P_{e_1'',\mathcal{L}}\big(\beta(a_1)\big),\beta(a_2),P^{-1}_{e_2'',\mathcal{L}}\big(\beta(a_3)\big)\Big)$ is a transverse triple. However, this is generically true. Furthermore, by the symmetry of the triple ratio, it is clear that $T^{i_1,i_2,i_3}_{a_1,a_2,a_3}=T^{i_2,i_3,i_1}_{a_2,a_3,a_1}=T^{i_3,i_1,i_2}_{a_3,a_1,a_2}$. Since there are $\frac{(m-1)(m-2)}{2}$ partitions of m into three positive numbers, this assigns $\frac{(m-1)(m-2)}{2}$ different triangle invariants to every triangle given by $\mathcal{T}$. Since $\mathcal{T}$ determines $4g-4+2r+n$ triangles, all the triangles given by $\mathcal{T}$ together determine $(4g-4+2r+n)\cdot\frac{(m-1)(m-2)}{2}$ such invariants.

The second family of invariants are known as cross ratios, and are associated to the edges of the triangulation $\mathcal{T}$ that do not lie in the boundary of $S/\sim$. Let e be such an edge, and let x, y be the vertices of e. Also, let T_1 and T_2 be the two triangles of $\mathcal{T}$ that share e as a common edge, so that if z_i is the vertex of T_i that is neither x nor

y, then $x < z_1 < y < z_2 < x$ in the clockwise cyclic order on the boundary of $T_1 \cup T_2$ induced by the orientation on S. For $l = 1,2$, let e_l be the edge of T_l with vertices z_l and x, oriented from z_l to x, and let e_0 denote the edge e, oriented from y to x. For $l = 0,1,2$, let $e_l' \subset S$ be the (oriented) lift of e_l, and let a_l and b_l be the backward and forward endpoints of e_l' respectively.

Define $e_0'' := e_0'$. Also, for $l = 1,2$, if $b_l = b_0$, then let $e_l'' := e_l'$. On the other hand, if $b_l \neq b_0$, then b_l and b_0 both lie in a boundary component c of S that has no marked points. Let $\bar{e}_l$ be the curve in c that does not contain b_{3-l}, and whose backward and forward endpoints are b_l and b_0 respectively. In this case, define e_l'' to be the (oriented) concatenation $e_l' \cdot \bar{e}_l$. Then for every $i \in \{1,\dots,m-1\}$, define the *edge invariant*

$$\begin{array}{rcl} C^{i,m-i}_{b_0,a_0} : \mathcal{X}_{\mathrm{PGL}(V),S} & \longrightarrow & K \\ {[\mathcal{L},\beta]} & \mapsto & -C_i\Big(\beta(b_0), P_{e_1'',\mathcal{L}}\big(\beta(a_1)\big), P_{e_2'',\mathcal{L}}\big(\beta(a_2)\big), P_{e_0'',\mathcal{L}}\big(\beta(a_0)\big)\Big) \end{array}$$

Just like the triangle invariants, the edge invariants are not defined at every point in $\mathcal{X}_{\mathrm{PGL}(V),S}$, but they are defined for on a generic points. By the symmetry of the cross ratio, it is easy to see that $C^{i,m-i}_{b_0,a_0} = C^{m-i,i}_{a_0,b_0}$. Hence, every edge of $\mathcal{T}$ that does not lie in the boundary of $S/\sim$ determines $m-1$ edge invariants, and there are $6g-6+3r+n$ such edges. Together, these edges determine $(6g-6+3r+n)\cdot(m-1)$ edge invariants.

The triangle invariants and the edge invariants determine a map

$$\Phi_{[\mathcal{T}]} : \mathcal{X}_{\mathrm{PGL}(V),S} \longrightarrow K^d$$

which is defined on a generic set of points in $\mathcal{X}_{\mathrm{PGL}(V),S}$. It is easy to see that this map does not depend on the choice of good representative $\mathcal{T}$ of $[\mathcal{T}]$, nor on the choice of representative $(\mathcal{L},\beta)$ of $[\mathcal{L},\beta]$. Furthermore, Fock-Goncharov proved that this map is generically invertible, so we have a chart

$$\Phi^{-1}_{[\mathcal{T}]} : K^d \longrightarrow \mathcal{X}_{\mathrm{PGL}(V),S}.$$

To prove that $\mathcal{X}_{\mathrm{PGL}(V),S}$ is a positive variety, it is now sufficient to show that if $[\mathcal{T}]$ and $[\mathcal{T}']$ are related by a flip, then $\Phi_{[\mathcal{T}]} \circ \Phi^{-1}_{[\mathcal{T}']} : K^d \longrightarrow K^d$ is given coordinate-wise by an expression in $\mathbb{Z}(X_1,\dots,X_d)_+$. This was explicitly computed by Fock-Goncharov.

6.3.5 $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ is a positive variety over $\mathbb{R}$

Just like the case of $\mathcal{X}_{\mathrm{PGL}(V),S}(\mathbb{R})$, we will define a positive chart $K^d \longrightarrow \mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ for every $[\mathcal{T}] \in \Delta(S)$, where

$$d = (6g-6+3r+2n)\cdot(m-1) + (4g-4+2r+n)\cdot\frac{(m-1)(m-2)}{2}.$$

Choose an identification $\bigwedge^m V \simeq \mathbb{R}$ so that $t_{i,j,k}(AF,AG,AH) \in \mathbb{R}$ is well-defined for any triple of affine flags $(AF,AG,AH) \in \mathcal{F}(V)^3$. Also, for any piecewise smooth curve $\widehat{e}$ in

T^1S with backward and forward endpoints p and q respectively, and for any decorated representation $(\mathcal{M},\alpha)$ on S, let $P_{\widehat{e},\mathcal{M}}:\mathcal{M}_p\to\mathcal{M}_q$ be the parallel transport along $\widehat{e}$ induced by the flat structure on $\mathcal{M}$.

If e is a piecewise smooth, closed, oriented curve in S, we first define a piecewise smooth curve $\widehat{e}$ in T^1S defined in the following way. Let e be the cyclic concatenation $e_1\cdot e_2\cdot\dots\cdot e_k$, where $e_l:[0,1]\to S$ is a smooth (oriented) curve for all $l=1,\dots,k$. Then let $\widehat{e_l}$ be the smooth curve in T^1S defined by $\widehat{e_l}(t)=(e_l(t),e_l'(t))$. Also, at the point $p=e_l(1)=e_{l+1}(0)$, let $\theta_{l,l+1}(t)\in T^1_pS$ be the clockwise rotation from $e_l'(1)=:\theta_{l,l+1}(0)$ to $e_{l+1}'(0)=:\theta_{l,l+1}(1)$. Using this, define $\widehat{e}_{l,l+1}$ to be the smooth curve defined by $\widehat{e}_{l,l+1}(t)=(e_l(1),\theta_{l,l+1}(t))$, and define $\widehat{e}$ to be the cyclic concatenation

$$\widehat{e}=\widehat{e}_1\cdot\widehat{e}_{1,2}\cdot\widehat{e}_2\cdot\widehat{e}_{2,3}\cdots\widehat{e}_k\cdot\widehat{e}_{k,1}.$$

Also, for any $i\neq j\in\{1,\dots,k\}$, let $e_i\cdot\widehat{e_{i+1}\cdots}e_j$ be the concatenation

$$e_i\cdot\widehat{e_{i+1}\cdots}e_j:=\widehat{e}_i\cdot\widehat{e}_{i,i+1}\cdots\widehat{e}_j\cdot\widehat{e}_{j,j+1}.$$

Choose a good representative $\mathcal{T}$ of $[\mathcal{T}]$. Then for any triangle in $S/\sim$ given by $\mathcal{T}$, let $e_1'',e_2'',e_3''\subset S$ be the oriented curves as defined in the third paragraph of Section 6.3.4. If we apply the construction in the above paragraph to the piecewise smooth, closed, oriented curve $e_1''\cdot e_2''\cdot e_3''$ in S, then we obtain three piecewise smooth curves $\widetilde{e}_1''$, $\widetilde{e}_2''$, $\widetilde{e}_3''$ in T^1S so that $e_1''\cdot\widehat{e_2''\cdot}e_3''$ is the cyclic concatenation $\widetilde{e}_1''\cdot\widetilde{e}_2''\cdot\widetilde{e}_3''$.

For $l=1,2,3$, let a_{l+1} be common endpoint of $\widetilde{e}_l''$ and $\widetilde{e}_{l+1}''$. Then for every $i_1,i_2,i_3\in\{1,\dots,m-2\}$ so that $i_1+i_2+i_3=m$, define

$$\begin{array}{rcl} t^{i_1,i_2,i_3}_{a_1,a_2,a_3}:\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R}) & \longrightarrow & K\\ {[\mathcal{M},\alpha]} & \mapsto & t_{i_1,i_2,i_3}\Big(P_{\widetilde{e}_1'',\mathcal{M}}\big(\alpha(a_1)\big),\alpha(a_2),P^{-1}_{\widetilde{e}_2'',\mathcal{M}}\big(\alpha(a_3)\big)\Big).\end{array}$$

Observe that if m is odd, then $s=\mathrm{Id}$ (see Definition 6.3.2), and either i_1 or i_2+i_3 is even. This means that

$$\begin{array}{rl} & t_{i_1,i_2,i_3}\Big(P_{\widetilde{e}_1'',\mathcal{M}}\big(\alpha(a_1)\big),\alpha(a_2),P^{-1}_{\widetilde{e}_2'',\mathcal{M}}\big(\alpha(a_3)\big)\Big)\\ = & t_{i_1,i_2,i_3}\Big(P_{\widetilde{e}_2'',\mathcal{M}}\circ P_{\widetilde{e}_1'',\mathcal{M}}\big(\alpha(a_1)\big),P_{\widetilde{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big),\alpha(a_3)\Big)\\ = & t_{i_1,i_2,i_3}\Big(P^{-1}_{\widetilde{e}_3'',\mathcal{M}}\big(\alpha(a_1)\big),P_{\widetilde{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big),\alpha(a_3)\Big)\\ = & t_{i_2,i_3,i_1}\Big(P_{\widetilde{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big),\alpha(a_3),P^{-1}_{\widetilde{e}_3'',\mathcal{M}}\big(\alpha(a_1)\big)\Big).\end{array}$$

On the other hand, if m is even, then $s = -\mathrm{Id}$, so

$$\begin{aligned}
& t_{i_1,i_2,i_3}\Big(P_{\widehat{e}_1'',\mathcal{M}}\big(\alpha(a_1)\big), \alpha(a_2), P^{-1}_{\widehat{e}_2'',\mathcal{M}}\big(\alpha(a_3)\big)\Big) \\
= \; & t_{i_1,i_2,i_3}\Big(P_{\widehat{e}_2'',\mathcal{M}} \circ P_{\widehat{e}_1'',\mathcal{M}}\big(\alpha(a_1)\big), P_{\widehat{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big), \alpha(a_3)\Big) \\
= \; & t_{i_1,i_2,i_3}\Big(s \circ P^{-1}_{\widehat{e}_3'',\mathcal{M}}\big(\alpha(a_1)\big), P_{\widehat{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big), \alpha(a_3)\Big) \\
= \; & \begin{cases} t_{i_1,i_2,i_3}\Big(P^{-1}_{\widehat{e}_3'',\mathcal{M}}\big(\alpha(a_1)\big), P_{\widehat{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big), \alpha(a_3)\Big) & \text{if } i_1 \text{ and } i_2 + i_3 \text{ is even,} \\ -t_{i_1,i_2,i_3}\Big(P^{-1}_{\widehat{e}_3'',\mathcal{M}}\big(\alpha(a_1)\big), P_{\widehat{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big), \alpha(a_3)\Big) & \text{if } i_1 \text{ and } i_2 + i_3 \text{ are odd.} \end{cases} \\
= \; & t_{i_1,i_2,i_3}\Big(P_{\widehat{e}_2'',\mathcal{M}}\big(\alpha(a_2)\big), \alpha(a_3), P^{-1}_{\widehat{e}_3'',\mathcal{M}}\big(\alpha(a_1)\big)\Big).
\end{aligned}$$

In either case, this proves that $t^{i_1,i_2,i_3}_{a_1,a_2,a_3} = t^{i_2,i_3,i_1}_{a_2,a_3,a_1} = t^{i_3,i_1,i_2}_{a_3,a_1,a_2}$. Hence, this assigns $\frac{(m-1)(m-2)}{2}$ invariants of $\mathcal{A}_{\mathrm{PGL}(V),S}(\mathbb{R})$ to every triangle in $S/\sim$ given by $\mathcal{T}$. Since $\mathcal{T}$ gives $4g-4+2r+n$ triangles, we have a total of $(4g-4+2r+n)\cdot\frac{(m-1)(m-2)}{2}$ such invariants.

Also, for every $i \in \{1, \dots, m-1\}$, define

$$\begin{aligned}
s^{i,m-i}_{a_1,a_2} : \mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R}) & \longrightarrow K \\
[\mathcal{M}, \alpha] & \mapsto t_{i,m-i,0}\Big(P_{\widehat{e}_1'',\mathcal{M}}\big(\alpha(a_1)\big), \alpha(a_2), -\Big).
\end{aligned}$$

Note that $t_{i,m-i,0}(F,G,H)$ does not depend H, so the map above is well-defined. Furthermore, if m is odd, then one of i or $m-i$ is even and $s = \mathrm{Id}$, so

$$\begin{aligned}
s^{i,m-i}_{a_1,a_2}[\mathcal{M},\alpha] &= t_{i,m-i,0}\Big(P_{\widehat{e}_1'',\mathcal{M}}\big(\alpha(a_1)\big), \alpha(a_2), -\Big) \\
&= t_{i,m-i,0}\Big(\alpha(a_1), P^{-1}_{\widehat{e}_1'',\mathcal{M}}\big(\alpha(a_2)\big), -\Big) \\
&= t_{i,m-i,0}\Big(\alpha(a_1), P_{\widehat{e}_1''^{-1},\mathcal{M}}\big(\alpha(a_2)\big), -\Big) \\
&= t_{m-i,i,0}\Big(P_{\widehat{e}_1''^{-1},\mathcal{M}}\big(\alpha(a_2)\big), \alpha(a_1), -\Big) \\
&= s^{m-i,i}_{a_2,a_1}[\mathcal{M},\alpha].
\end{aligned}$$

On the other hand, if m is even, then $s = -\mathrm{Id}$ and

$$\begin{aligned}
s^{i,m-i}_{a_1,a_2}[\mathcal{M},\alpha] &= t_{i,m-i,0}\Big(P_{\widehat{e}''_1,\mathcal{M}}\big(\alpha(a_1)\big),\alpha(a_2),-\Big)\\
&= t_{i,m-i,0}\Big(\alpha(a_1),P^{-1}_{\widehat{e}''_1,\mathcal{M}}\big(\alpha(a_2)\big),-\Big)\\
&= t_{i,m-i,0}\Big(\alpha(a_1),s\circ P_{\widehat{e}''^{-1}_1,\mathcal{M}}\big(\alpha(a_2)\big),-\Big)\\
&= \begin{cases} t_{i,m-i,0}\Big(\alpha(a_1),P_{\widehat{e}''^{-1}_1,\mathcal{M}}\big(\alpha(a_2)\big),-\Big) & \text{if } i \text{ is even,}\\ -t_{i,m-i,0}\Big(\alpha(a_1),P_{\widehat{e}''^{-1}_1,\mathcal{M}}\big(\alpha(a_2)\big),-\Big) & \text{if } i \text{ is odd.}\end{cases}\\
&= t_{m-i,i,0}\Big(P_{\widehat{e}''^{-1}_1,\mathcal{M}}\big(\alpha(a_2)\big),\alpha(a_1),-\Big)\\
&= s^{m-i,i}_{a_2,a_1}[\mathcal{M},\alpha].
\end{aligned}$$

Hence, this assigns $m-1$ invariants of $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ to every edge in $\mathcal{T}$. Since $\mathcal{T}$ gives has $6g-6+3r+2n$ edges, we have a total of $(6g-6+3r+2n)\cdot(m-1)$ such invariants. Together, these invariants define a map

$$\Psi_{[\mathcal{T}]}:\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})\dashrightarrow K^d$$

which is defined on a generic set of points in $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$. It is easy to see that this map does not depend on the choice of good representative $\mathcal{T}$ of $[\mathcal{T}]$, nor on the choice of representative $(\mathcal{M},\alpha)$ of $[\mathcal{M},\alpha]$. Furthermore, Fock-Goncharov proved that this map is generically invertible, so we have a chart

$$\Psi^{-1}_{[\mathcal{T}]}:K^d\dashrightarrow\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R}).$$

To prove that $\mathcal{A}_{\mathrm{SL}(V),S}(\mathbb{R})$ is a positive variety over $\mathbb{R}$, it is now sufficient to show that if $[\mathcal{T}]$ and $[\mathcal{T}']$ are related by a flip, then $\Phi_{[\mathcal{T}]}\circ\Phi^{-1}_{[\mathcal{T}']}:K^d\dashrightarrow K^d$ is given coordinate-wise by a rational function with positive integer coeffecients. This was explicitly computed by Fock-Goncharov.

Bibliography

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Chapter 7

Degeneration of projective structures

Xin Nie

7.1 Introduction

7.1.1 Augmented Teichmüller space

Fix $g \geq 2$ and let S be an oriented closed surface of genus g. Put

$$C(S) = \left\{ \sigma = \{c_1, \cdots, c_k\} \;\middle|\; \begin{array}{c} 0 \leq k \leq 3g-3,\ c_i\text{'s are distinct non-trivial} \\ \text{homotopy classes of simple loops on } S \end{array} \right\},$$

The *augmented Teichmüller space* $\overline{\mathcal{T}}_g$ is a bordification of the Teichmüller space $\mathcal{T}_g$ constructed by adding certain "degenerate hyperbolic structures". As a set,

$$\overline{\mathcal{T}}_g = \bigsqcup_{\sigma \in C(S)} \mathcal{T}_g^\sigma,$$

where $\mathcal{T}_g^\sigma$ is the stratum consisting of complete hyperbolic structures of finite area on the complement of a multi-curve represented by σ (more prescisely, one should introduce markings and take the quotient by isotopy group as in the definition of $\mathcal{T}_g$). Note that $\mathcal{T}_g^\varnothing = \mathcal{T}_g$ by convention. A natural topology on $\overline{\mathcal{T}}_g$ is defined so that a sequence in $\mathcal{T}_g$ obtained by pinching a hyperbolic surface along closed geodesics, as shown in the following picture, converges to to one in $\mathcal{T}_g^\sigma$ with σ.

7.1.2 Noded Riemann surfaces, regular cubic differentials

The Deligne-Mumford moduli space

$$\overline{\mathcal{M}_g} := \overline{\mathcal{T}}_g/\mathrm{MCG}_g = \bigsqcup_{\sigma \in C(S)/\mathrm{MCG}(S)} \mathcal{M}_g^\sigma.$$

is a compact complex orbifold, compactifying $\mathcal{M}_g = \mathcal{T}_g/\mathrm{MCG}_g$. Each stratum $\mathcal{M}_g^\sigma$ consists of "noded Riemann surfaces" of a given topological type.

Remark. A nice way to image $\overline{\mathcal{T}}_g$ and $\overline{\mathcal{M}}_g$ is to think of their $g = 1$ analogues:

$$\begin{aligned}\mathcal{T}_g &\longleftrightarrow \mathbb{H}\\ \overline{\mathcal{T}}_g &\longleftrightarrow \mathbb{H} \cup \mathbb{Q}\\ \mathrm{MCG}_g &\longleftrightarrow \mathrm{PSL}(2,\mathbb{Z})\\ \mathcal{M}_g &\longleftrightarrow \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})\\ \overline{\mathcal{M}}_g &\longleftrightarrow (\mathbb{H} \cup \mathbb{Q})/\mathrm{PSL}(2,\mathbb{Z})\end{aligned}$$

The moduli space $\mathbb{C}_g$ of holomorphic cubic differentials on S is a holomorphic vector bundle over $\mathcal{T}_g$, the fibre over $[\Sigma] \in \mathcal{T}_g$ (where Σ is a Riemann surface) being $H^0(\Sigma, K^3)$. $\mathbf{V}_g := \mathbb{C}_g/\mathrm{MCG}_g$ is an orbifold vector bundle over $\mathcal{M}_g$ and can be extended to an orbifold vector bundle over $\overline{\mathcal{M}}_g$ such that the fibre over $[\Sigma]$ (where Σ is a noded Riemann surface) is the space of *regular cubic differentials*, *i.e.*holomorphic cubic differentials U on $\Sigma^\circ := \Sigma \setminus \mathrm{Nodes}_\Sigma$ such that U has removable singularities or poles of order at most 3 at the punctures and the residues at the two punctures across each node are opposite.

7.1.3 Bulging deformation of convex projective structures

A convex projective structure on S is by definition a $(\mathrm{SL}(3,\mathbb{R}), \mathbb{RP}^2)$-structure (in terms of homogeneous (G, X)-structures) whose developing map is a homeomorphism from $\widetilde{S}$ to a properly convex open set (*i.e.*a bounded convex open set in some affine chart). The *Goldman space* $\mathcal{G}_g = \mathcal{G}(S)$ is the moduli space of convex projective structures.

In the Klein model, the hyperbolic plane is a disk $\Omega \subset \mathbb{RP}^2$ and the orientation-preserving hyperbolic isometry group is the subgroup $\mathrm{Aut}(\Omega) \cong \mathrm{SO}(2,1)$ of $\mathrm{SL}(3,\mathbb{R})$ preserving Ω. Therefore, $\mathcal{T}_g$ is naturally included in $\mathcal{G}_g$ as a submanifold.

The Labourie-Loftin correspondence is a canonical bijection $\mathcal{G}_g \cong \mathbb{C}_g$ under which $\mathcal{T}_g \subset \mathcal{G}_g \cong \mathbb{C}_g$ is the zero section the vector bundle $\mathbb{C}_g$. More details will be given later.

Let $\Omega \subset \mathbb{RP}^2$ be a disk and $\pi_1(S) \cong \Gamma = A *_\gamma B \subset \mathrm{Aut}(\Omega) \subset \mathrm{SL}(3,\mathbb{R})$ be such that Ω/Γ is a hyperbolic surface (homeomorphic to S), where A and B are fundamental groups of the two components of S cut along a separating loop c and

$$\gamma = \begin{pmatrix}\lambda & & \\ & \lambda^{-1} & \\ & & 1\end{pmatrix}$$

is the holonomy of c. Put

$$\begin{aligned}\Gamma_t &:= A *_\gamma \begin{pmatrix}t & & \\ & t & \\ & & t^{-2}\end{pmatrix} B \begin{pmatrix}t^{-1} & & \\ & t^{-1} & \\ & & t^{2}\end{pmatrix}\\ \approx \Gamma'_t &:= \begin{pmatrix}t^{-1} & & \\ & t^{-1} & \\ & & t^{2}\end{pmatrix} A \begin{pmatrix}t & & \\ & t & \\ & & t^{-2}\end{pmatrix} *_\gamma B \subset \mathrm{SL}(3,\mathbb{R})\end{aligned}$$

There are properly convex sets $\Omega_t, \Omega'_t \subset \mathbb{RP}^2$ preserved by Γ_t, Γ'_t, giving the same convex projective surface $X_t = \Omega_t/\Gamma \cong \Omega'_t/\Gamma'_t$. As $t \to +\infty$, we see two convex projective surfaces at the limit:

- Ω_∞/A is a convex projective surface with "straight end";
- Ω'_∞/B is a convex projective surface with "triangular end".

Question 1. How to interpret the limit of X_t as $t \to +\infty$?

Question 2. How to interpret the limit of the point in $\mathbb{C}_g$ corresponding to X_t (under the Labourie-Loftin) as $t \to +\infty$?

7.1.4 Outline of results of Loftin

•Defined an *augmented Goldman space* $\overline{\mathcal{G}}_g = \bigsqcup_{\sigma \in C(S)} \mathcal{G}_g^\sigma$ as a bordification of $\mathcal{G}_g = \mathcal{G}_g^\varnothing$, such that the convex projective structure $X_\infty := \Omega_\infty/A \sqcup \Omega'_\infty/B$ on the surface $S \smallsetminus c$ is in the stratum $\mathcal{G}_g^c$ and is the limit of $X_t \in \mathcal{G}_g$. Elements of $\overline{\mathcal{G}}_g$ are "regular convex projective structures", see definitions below.

•Extended the Labourie-Loftin correspondence to a bijection between regular cubic differentials and regular convex projective structures ($\Rightarrow$ bijection $\overline{\mathbf{V}}_g \cong \overline{\mathcal{G}}_g/\mathrm{MCG}_g$), with a formula relating residue of cubic differential and eigenvalue of holonomies of convex projective structure.

•Proved that $\overline{\mathbf{V}}_g \cong \overline{\mathcal{G}}_g/\mathrm{MCG}_g$ is homeomorphism. In particular, this implies that if $(\Sigma_t, U_t) \in \mathbf{V}_g$ corresponds to X_t then Σ_t converges to a noded Riemann surface Σ_∞ which is topologically S with c pinched to a point, and U_t converges to a regular cubic differential U_∞ on Σ_∞.

7.2 Definition of $\overline{\mathcal{G}}_g$

7.2.1 Goldman space

For a (not necessarily closed) connected surface S is defined as the topological space

$$\mathcal{G}(S) := \left\{ (\rho, \Omega) \middle| \begin{array}{l} \Omega \subset \mathbb{RP}^2 \text{ properly convex,} \\ \rho \in \mathrm{Hom}(\pi_1(S), \mathrm{SL}(3,\mathbb{R})),\ \Omega/\rho(\pi_1(S)) \cong S \end{array} \right\} / \mathrm{SL}(3,\mathbb{R})$$

$$\subset (\mathrm{Hom}(\pi_1(S), \mathrm{SL}(3,\mathbb{R})) \times \mathfrak{C})/\mathrm{SL}(3,\mathbb{R})$$

where $\mathfrak{C}$ is the space of properly convex sets with Hausdorff topology and $\mathcal{G}(S)$ is equipped with the subspace and quotient topology.

More generally, when S is disconnected, its Goldman space $\mathcal{G}(S)$ is defined as the product of the Goldman spaces of the connected components.

7.2.2 Classification of automorphisms of convex sets

All elements in $\mathrm{SL}(3,\mathbb{R})$ that preserve some properly convex open set $\Omega \subset \mathbb{RP}^2$ other than a triangle without fixed points in Ω are classified as follows:

•**Parabolic** ones, *i.e.*conjugate to $\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$;

•**Hyperbolic** ones, *i.e.*conjugate to $\begin{pmatrix} a & 1 & \\ & a & \\ & & a^{-2} \end{pmatrix}$, $a > 0$;

•**Quasi-hyerbolic** ones, *i.e.*conjugate to $\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$, $a > b > c > 0$.

7.2.3 Some particular types of ends of open convex projective surfaces

An end of a convex projective surface is called a

•**Cusp** if the holonomy is parabolic;

•**Quasi-hyperbolic end** if the holonomy is quasi-hyperbolic;

•**Straight hyperbolic end** if the holonomy is hyperbolic and the end is the quotient of an open set $U \subset \Omega$ whose boundary meets $\partial\Omega$ at a line segment joining the attracting and repelling fixed points of the holonomy;

•**Triangular hyperbolic end** if the holonomy is hyperbolic and the end is the quotient of an open set $U \subset \Omega$ whose boundary meets $\partial\Omega$ at two line segments: one joining the attracting fixed point and unstable fixed point, the other joining the attracting fixed point and unstable fixed point.

7.2.4 Pulling maps

Given a connected surface S and a simple loop c on S, the pulling map

$$\mathsf{Pull}_c : \mathcal{G}(S) \to \mathcal{G}(S \smallsetminus c)$$

is defined as follows. Given a element $X = [\rho, \Omega] \in \mathcal{G}(S)$,

•if c is a separating loop, so that $\pi_1(S) = \pi_1(S_1) *_\gamma \pi_1(S_2)$ is an amalgamated product, where S_1 and S_2 are the two components of $S \smallsetminus c$, we define

$$\mathsf{Pull}_c(X) := \left(\left[\rho|_{\pi_1(S_1)}, \Omega)\right], \left[\rho|_{\pi_1(S_2)}, \Omega)\right]\right) \in \mathcal{G}(S_1) \times \mathcal{G}(S_2) = \mathcal{G}(S \smallsetminus c);$$

•if c is a non-separating loop, so that $\pi_1(S) = \pi_1(S \smallsetminus c) *_\gamma$ is an HNN extension, we define

$$\mathsf{Pull}_c(X) := \left(\rho|_{\pi_1(S \smallsetminus c)}, \Omega\right) = \mathcal{G}(S \smallsetminus c).$$

Roughly speaking, $\mathsf{Pull}_c(X)$ is defined by taking the same developing image as X and restricting the holonomy representation.

Remark 7.2.1.
1. $\mathsf{Pull}_c(X)$ is different from the convex projective structure given by plainly restricting the $(\mathrm{SL}(3,\mathbb{R}), \mathbb{RP}^2)$-charts of X to $S \smallsetminus c$.

2. Pull_c is not surjective since the Dehn twist of $X \in \mathcal{G}(S \smallsetminus \sigma)$ along c produces points in $\mathcal{G}(S \smallsetminus \sigma)$ whose images under Pull_c are the same as that of X.

More generally, for any $\sigma < \tau \in C(S)$, the pulling map $\mathsf{Pull}_{\sigma,\tau} : \mathcal{G}(S \smallsetminus \sigma) \to \mathcal{G}(S \smallsetminus \tau)$ is defined as the composition of pulling maps with respect to each loop in τ but not in σ.

7.2.5 Separated necks

Now let S be a closed orientable surface S of genus ≥ 2. Fix $\sigma \in C(S)$ and a convex projective structure $X \in \mathcal{G}(S \smallsetminus \sigma)$.

•a loop c in σ is called a *regular separated neck* of a convex projective structure if the two ends of X across c belong to one of the following cases

- both are cusps;

- both are quasi-hyperbolic ends, with holonomies inverse to each other;

- one is a straight hyperbolic end and the other triangular hyperbolic end, with holonomies inverse to each other.

•c is called a *trivial separated neck* of X if X is in the image of the pulling map

$$\mathsf{Pull}_c : \mathcal{G}(S \smallsetminus (\sigma - c)) \to \mathcal{G}(S \smallsetminus \sigma).$$

•Let $\mathcal{G}^\sigma(S) \subset \mathcal{G}(S \smallsetminus \sigma)$ be the subspace consisting of convex projective structures such that each $c \in \sigma$ is a regular or trivial separated neck.

•Let $\mathcal{G}^\sigma_{\mathrm{reg}}(S) \subset \mathcal{G}(S \smallsetminus \sigma)$ be the subspace consisting of convex projective structures such that each $c \in \sigma$ is a regular separated neck.

7.2.6 The augmented Goldman space

Theorem 7.2.2 (Theorem 3 in [Lof15]). *Let S be a surface whose components have negative Euler characteristics. Then $X \in \mathcal{G}(S \smallsetminus c)$ is in the closure of the image of*

$$\mathsf{Pull}_c : \mathcal{G}(S) \to \mathcal{G}(S \smallsetminus c)$$

if and only if c is a regular or trivial separated neck of X.

An immediate corollary is that if S is a closed surface then $\mathsf{Pull}_c : \mathcal{G}(S) \to \mathcal{G}^c(S)$ has dense image and the complement of the image is $\mathcal{G}^c_{\mathrm{reg}}(S)$. Thus one can define a topology on

$$\mathcal{G}(S) \sqcup \mathcal{G}^c_{\mathrm{reg}}(S)$$

so that a sequence $(X_n) \subset \mathcal{G}(S)$ converges to $X \in \mathcal{G}^c_{\mathrm{reg}}(S)$ if and only if $\mathsf{Pull}_c(X_n)$ converges to X in $\mathcal{G}^c(S)$ – this is exactly the coarsest topology to make the natural map $\mathcal{G}(S) \sqcup \mathcal{G}^c_{\mathrm{reg}}(S) \to \mathcal{G}^c(S)$ (given by pulling and inclusion) continuous.

Therefore, $\mathcal{G}(S) \sqcup \mathcal{G}^c_{\mathrm{reg}}(S)$ is already a bordification of $\mathcal{G}(S)$ where one can make sense of the limit of the bulging deformation sequence X_t discussed earlier: the convex projective surfaces $X_\infty = \Omega_\infty/A \sqcup \Omega'_\infty/B$ is exactly the limit of $\mathsf{Pull}_c(X_t)$ in $\mathcal{G}^c(S) \subset \mathcal{G}(S \smallsetminus c)$.

The idea of the augmented Goldman space $\overline{\mathcal{G}}_g$ as defined by Loftin is to take this bordification further:

Definition 7.2.3. The augmented Goldman space is defined as

$$\overline{\mathcal{G}}_g = \overline{\mathcal{G}}(S) := \bigsqcup_{\sigma \in C(S)} \mathcal{G}^\sigma_{\mathrm{reg}}(S)$$

equipped with the coarsest topology such that for any $\tau \in C(S)$, the map

$$\overline{\mathcal{G}}_g(S) \supset \bigsqcup_{\sigma \le \tau} \mathcal{G}^\sigma_{\mathrm{reg}}(S) \to \mathcal{G}(S \smallsetminus \tau)$$

defined by pulling is continuous.

Note that the space of degenerate hyperbolic structures $\mathcal{T}^\sigma(S)$ discussed at the beginning is contained in $\mathcal{G}^\sigma(S)$. Thus the augmented Goldman space $\overline{\mathcal{G}}_g$ naturally contains the augmented Teichmüller space $\overline{\mathcal{T}}_g$.

7.3 Identification between strata of $\overline{\mathcal{G}}_g/\mathrm{MCG}_g$ and $\overline{\mathbf{V}}_g$

Fundamental theorem of surface theory: a twisted immersion of a surface S into the Euclidean space $\mathbb{E}^3$ is determined up to isometry by a pair of 2-tensors on S satisfying some PDE. Namely, first and second fundamental forms and Gauss-Codazzi equations. Here a "twisted immersion" means an immersion of the universal cover $\widetilde{S}$ together with a representation $\pi_1(\Sigma) \to \mathrm{Isom}(\mathbb{E}^3)$ for which the immersion is equivariant.

Affine differential geometry: a twisted locally strictly convex immersion of S into the affine space $\mathbb{A}^3$ is determined up to special affine transformations by a 2-tensor (affine second fundamental form, a.k.a. Blaschke metric) and an affine connection (Blaschke connection) satisfying some PDE. There is an affine invariant canonical transverse vector field $\boldsymbol{n}$ (affine normal) and the affine second fundamental form measures the variation of $\boldsymbol{n}$ along the surface.

Hyperbolic affine spheres. A hyperbolic affine sphere in $\mathbb{A}^3$ is by definition a strictly convex surface whose affine normals have a common endpoint (the center of the affine sphere). A twisted hyperbolic affine spherical immersion $f : \widetilde{S} \to \mathbb{A}^3$ is determined up to special affine transformations by the Blaschke metric g and a holomorphic cubic differential U satisfying "Wang's equation"

$$\kappa_g = -1 + \|U\|_g^2$$

U is essentially the difference between the Blaschke connection and the Levi-Civita connection of g. f is a proper embedding if and only if g is complete. In this case, if we identify $\mathbb{A}^3 \cong \mathbb{R}^3$ by letting the center of the affine sphere be the origin, then $C = \mathbb{R}_{\geq 0} \cdot f(\widetilde{S})$ is a properly convex cone and $f(\tilde{S})$ is asymptotic to ∂C. Explicit examples:

- $C =$ light cone, $f(\widetilde{S}) =$ hyperboloid;
- $C = \mathbb{R}^3_{\geq 0}$, $f(\widetilde{S}) = \{(x,y,z) \mid xyz = 1\}$.

Theorem 7.3.1 (Cheng-Yau)**.** *In every properly convex cone $C = \mathbb{R}^3$ there exists a unique embedded hyperbolic affine sphere $\Sigma \subset C$ asymptotic to the boundary of C.*

Corollary 7.3.2. *For an oriented surface S, there is a canonical identification*

$$\mathcal{G}(S) \cong \mathcal{W}(S) := \{(g,U) \text{ satisfying Wang's equation, } g \text{ complete}\}/\text{Isotopies}$$

Let $\mathbb{C}(S)$ be the space of pairs $(\boldsymbol{J}, U)$ where $\boldsymbol{J}$ is a conformal structure on S and U holomorphic cubic differential on $(S, \boldsymbol{J})$, modulo isotopies. There is a natural map $\mathcal{W}(S)$

Given $S = \overline{S} \smallsetminus P$ where $\overline{S}$ is an oriented closed surface and P a finite set of punctures, denote

•$\mathcal{G}_0(S)$ = the subspace of $\mathcal{G}(S)$ consisting of convex projective structure whose ends are among the types described in Section 7.2.3.

•$\mathbb{C}_0(S)$ = the subspace of $\mathbb{C}(S)$ consisting of those $(\boldsymbol{J}, U)$ such that each $p \in P$ is conformally a cusp under $\boldsymbol{J}$ and is a pole of order at most 3 for U;

•$\mathcal{W}_0(S)$ = the subspace of $\mathcal{W}(S)$ consisting of those (g, U) satisfying

- $(\boldsymbol{J}, U) \in \mathbb{C}_0(S)$ for the conformal structure $\boldsymbol{J}$ of g;
- for each $p \in P$ we have

$$\lim_{x \to p} \kappa_g(x) = \begin{cases} -1 & \text{if } p \text{ is a removable singularity or pole of order} \leq 2, \\ 0 & \text{if } p \text{ is a pole of order 3.} \end{cases}$$

Remark. Since (g, U) satisfies Wang's equation and $\|U\|_g^2 = \left(\frac{g_U}{g}\right)^{\frac{2}{3}}$ (where g_U is the flat metric given by U (with conic singularity at the zeros) and $\frac{g_U}{g}$ is the conformal ratio between g_U and g), the last condition is equivalent to

$$\lim_{x \to p} \frac{g_U}{g}(x) = \begin{cases} 0 & \text{if } p \text{ is a removable singularity or pole of order} \leq 2, \\ 1 & \text{if } p \text{ is a pole of order 3.} \end{cases}$$

Theorem 7.3.3 (Labourie, Loftin, Benoist-Hulin)**.** *Let S be an oriented closed surface or a punctured surface as above, with negative Euler characteristic. Then*

1. *The canonical identification $\mathcal{G}(S) \cong \mathcal{W}(S)$ restricts to a bijection $\mathcal{G}_0(S) \cong \mathcal{W}_0(S)$.*

2. *Given a convex projective surface in $\mathcal{G}_0(S)$ corresponds to $(g, U) \in \mathcal{W}_0(S)$, for each $p \in P$, letting R be the residue of U at R, we have*

   - *p is a cusp $\Leftrightarrow$ $R = 0$ (i.e.p is a removable singularity or pole of order ≤ 2);*
   - *p is a quasi-hyperbolic end $\Leftrightarrow$ $R \in \boldsymbol{i}\mathbb{R}^*$;*
   - *p is a straight hyperbolic end $\Leftrightarrow$ $\mathsf{Re}(R) > 0$;*
   - *p is a triangular hyperbolic end $\Leftrightarrow$ $\mathsf{Re}(R) < 0$.*

   Moreover, the eigenvalues of the holonomy around p are $e^{-4\pi\mu_i}$ where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are the imaginary parts of the three cubic roots of $R/2$, respectively.

3. *For any $(\boldsymbol{J}, U)$ there is a unique g conformal to $\boldsymbol{J}$ such that $(g, U) \in \mathcal{W}_0(S)$. Or equivalently, the natural map $\mathcal{W}_0(S) \to \mathbb{C}_0(S)$ ($(g, U) \to (\boldsymbol{J}, U)$ where $\boldsymbol{J}$ is the conformal structure underlying $\boldsymbol{J}$) is bijective.*

This theorem gives an identification $\mathcal{G}_0(S) \cong \mathbb{C}_0(S)$ and a correspondence between types of ends of a convex projective structure and its corresponding cubic differential. By definition of the strata $\mathcal{G}^\sigma(S)$, if we let $\mathbb{C}_{\text{reg}}(S \setminus \sigma)$ denote the moduli space of regular cubic differentials over all noded Riemann surfaces obtained from S by pinching each loop in σ to a point, then the above theorem yields an bijection

$$\mathcal{G}^\sigma(S) \cong \mathbb{C}_{\text{reg}}(S \setminus \sigma).$$

The quotient of $\mathbb{C}_{\text{reg}}(S \setminus \sigma)$ by the mapping class group $\text{MCG}(S \setminus \sigma)$ is the piece of the orbifold holomorphic vector bundle $\overline{\mathbf{V}}_g \to \overline{\mathcal{M}}_g$ over the stratum $\mathcal{M}^\sigma$ of $\mathcal{M}$. Thus we get

Corollary 7.3.4. *There is a canonical bijection between the spaces*

$$\overline{\mathcal{G}}_g/\text{MCG}_g = \bigsqcup_{\sigma \in C(S)/\text{MCG}_g} \mathcal{G}^\sigma(S)/\text{MCG}(S \setminus \sigma),$$

$$\overline{\mathbf{V}}_g = \bigsqcup_{\sigma \in C(S)/\text{MCG}_g} \mathbb{C}_{\text{reg}}(S \setminus \sigma)/\text{MCG}(S \setminus \sigma)$$

which identifies each stratum of both spaces.

The main result of Loftin [Lof15] can now be summarized as follows.

Theorem 7.3.5. *The bijection in the above corollary is a homeomorphism.*

Proofs of the continuity for both directions of the dijection are rather technical and won't be discussed here.

Bibliography

[Lof15] J. Loftin, *Convex $\mathbb{RP}^2$ Structures and Cubic Differentials under Neck Separation*, arXiv:1506.03895 (2015).

Chapter 8

Bordification of the moduli space of Higgs bundles

Laura Fredrickson

The two listed references for this talk are:

- R. Mazzeo, J. Swoboda, H. Weiss, and F. Witt, "Limiting configurations for solutions of Hitchin's equations."
- T. Mochizuki, "Asymptotic behavior of certain families of harmonic bundles on Riemann surfaces." `1508.05997`

In this talk, I will discuss Mazzeo-Swoboda-Weiss-Witt's compactification of the Hitchin moduli space.

8.1 Background: Hitchin Moduli Space

Let Σ be a compact Riemann surface of genus ≥ 2. Our underlying gauge group will be $G = SU(2)$ ($G_{\mathbb{C}} = SL(2, \mathbb{C})$). Consequently, let $E \to \Sigma$ be a complex vector bundle of rank two and degree zero with fixed determinant line bundle $\mathrm{Det}(E)$. We'll restrict our attention to automorphisms of E which fix $\mathrm{Det}(E)$, i.e $\mathrm{Aut}(E) = \mathrm{SL}(E)$. Let $\mathcal{M} = \mathcal{M}(\Sigma, E)$ be the associated Hitchin moduli space.

8.1.1 Non-abelian Hodge correspondence

As a hyperkähler manifold, the Hitchin moduli space has a $\zeta \in \mathbb{CP}^1$-worth of complex structures. The Hitchin moduli space has different avatars in these different complex structures.

In the $\zeta = 0$ complex structure $\mathcal{M}_{\zeta=0}$ is the moduli space of stable Higgs bundles $(\bar{\partial}_E, \varphi)$.

Definition 8.1.1. A Higgs bundle on $E \to \Sigma$ is a pair $(\bar{\partial}_E, \varphi)$ consisting of a holomorphic structure $\bar{\partial}_E$ on E and Higgs field $\varphi \in H^0(\Sigma, \mathrm{End}E \otimes K_\Sigma)$.

Locally, in a frame where $\bar{\partial}_E = \bar{\partial}$,

$$\varphi = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} \mathrm{d}z,$$

where $a(z), b(z), c(z)$ are holomorphic functions. The Higgs field is traceless because the determinant line bundle of $\mathrm{Det}(E)$ is fixed, i.e. $\mathrm{Aut}(E) = SL(E) \Rightarrow \mathrm{End}(E) = \mathfrak{sl}(E)$.

The Hitchin moduli space can then be identified as the triples $(\bar{\partial}_E, \varphi, h)$ consisting of a Higgs bundle $(\bar{\partial}_E, \varphi)$ and hermitian metric h on E such that

$$F_{D(\bar{\partial}_E, h)} + [\varphi, \varphi^{*_h}] = 0, \tag{8.1}$$

where $D(\bar{\partial}_E, h)$ is the Chern connection, i.e. the unique connection compatible with $\bar{\partial}_E$ (i.e. $D^{0,1} = \bar{\partial}_E$) and h (i.e. D is h-unitary).

In the $\zeta \in \mathbb{C}^\times$ complex structure, $\mathcal{M}_\zeta$ is the moduli space of irreducible flat $SL(2,\mathbb{C})$-connections ∇. (This can be identified with the character variety. By taking the holonomy of ∇, we get a representation $\pi_1(\Sigma) \to SL(2,\mathbb{C})$.)

From this perspective the Hitchin moduli space consists of pairs (∇, h') consisting of a flat $SL(2,\mathbb{C})$-connection and distinguished hermitian metric h' on E.

The non-abelian Hodge correspondence gives a map between Higgs bundles and flat connections. Given a Higgs bundle $(\bar{\partial}_E, \varphi) \in \mathcal{M}_{\zeta=0}$, the corresponding flat $SL(2,\mathbb{C})$-connection in $\mathcal{M}_{\zeta \in \mathbb{C}^\times}$ is

$$\nabla_\zeta = \zeta^{-1}\varphi + D_{(\bar{\partial}_E, h)} + \zeta \varphi^{*_h}. \tag{8.2}$$

(It is typical to take $\zeta = 1$.)

8.1.2 Hitchin fibration

The compactification that I will describe is most natural from the Higgs bundle perspective, so we now consider the geometry of the moduli space of Higgs bundles. Viewed as the moduli space of Higgs bundles, the Hitchin moduli space is a complex integrable system. This manifests is a particularly nice fibration, called the Hitchin fibration, of $\mathcal{M}$ over a complex vector space $\mathcal{B}$ of dimension $\dim \mathcal{B} = \frac{1}{2}\dim \mathcal{M}$.

$$\begin{array}{rccc} \mathrm{Hit}: & \mathcal{M} & \to & \mathcal{B} = H^0(\Sigma, K_\Sigma^2) \\ & (\bar{\partial}_E, \varphi) & \mapsto & -\mathrm{Det}(\varphi) \end{array} \tag{8.3}$$

As shown in as shown in Figure 8.4, the generic fibers are compact complex tori of complex dimension $\frac{1}{2}\dim_{\mathbb{C}} \mathcal{M}$. The torus fibers degenerate over some locus $\mathcal{B}_{\text{sing}}$ of complex codimension one.

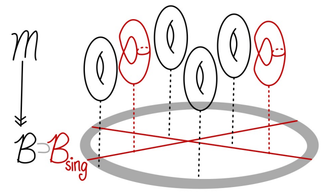

Figure 8.1: *The Hitchin fibration.*

The base $\mathcal{B}$ can be interpretted in a number of ways. Fundamentally, Hit maps the Higgs field φ to its eigenvalues $\lambda_1, \lambda_2 = -\lambda_1$. These can be encoded in the characteristic polynomial $\text{char}_\varphi(\lambda) = \lambda^2 - \text{Det}(\varphi)$, thus the identification of $\mathcal{B}$ with the space of holomorphic quadratic differentials $H^0(\Sigma, K_\Sigma^2) \ni -\text{Det}(\varphi)$. The eigenvalues can also be geometrically encoded in the spectral cover, some ramified $2:1$ cover $\tilde{\Sigma}_{2:1}$ of Σ

$$\begin{array}{ccc} \tilde{\Sigma}_{2:1} & \subset & K_\Sigma \\ & \searrow & \downarrow \\ & & \Sigma. \end{array} \tag{8.4}$$

where each sheet represents a different K_Σ-valued eigenvalue of φ. The spectral cover is ramified over Z, the zeros of $\text{Det}(\varphi)$. Counted with multiplicity, there are $4g-4$ zeros.

Just as the point in the base encodes the eigenvalues of φ, the point in the torus fiber encodes the eigenspaces of φ. Let $\mathcal{E} = (E, \bar{\partial}_E)$. At each point $x \in \Sigma - Z$, there are two 1-dimensional subspaces of $\mathcal{E}_x$ which are respectively the eigenspaces. For generic $(\bar{\partial}_E, \varphi)$, these fit together into a holomorphic line bundle $\mathcal{L} \to \tilde{\Sigma}_{2:1}$–even extending over the branch points where the eigenvalues are not distinct. Since we've taken $G_\mathbb{C} = SL(2,\mathbb{C})$ rather than $GL(2,\mathbb{C})$, the torus fiber is not the full Jacobian variety $\text{Jac}(\tilde{\Sigma}_{2:1})$, but rather the Prym subvariety.

We added the word "generic" in the above discussion because there is a complex codimension one locus where the fibers are degerate tori. Where is this? This is where $\text{Det}(\varphi) = \lambda_1\lambda_2$ has a non-simple zero, or equivalently, where the order of ramification between λ_1 and λ_2 is not simple, or equivalently, where $\tilde{\Sigma}_{2:1}$ is not smooth. Let $\mathcal{B}' = \mathcal{B} - \mathcal{B}_{\text{sing}}$, and $\mathcal{M}' = \text{Hit}^{-1}(\mathcal{B}')$. Mazzeo-Swoboda-Weiss-Witt's result holds on $\mathcal{M}'$.

8.1.3 Overview of Mazzeo-Swoboda-Weiss-Witt's paper

Given a Higgs bundle $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$, consider the ray $(\bar{\partial}_E, t\varphi)$ of Higgs bundles and associated ray $(\bar{\partial}_E, t\varphi, h_t)$ of solutions of Hitchin's equation. Hitchin's equations are equations for a hermtiian metric h_t. Mazzeo-Swoboda-Weiss-Witt's goal is to "construct" h_t for $t >> 0$, i.e. "near the ends" of the Hitchin moduli space. Note that this is hard! Hitchin's equations are a coupled system of non-linear PDE.

Their strategy is as follows:

1. Construct h_∞ "at the end." This h_∞ will be singular at $p \in Z$.

2. Construct a smooth family h_t^{model} of model solutions on the unit disk in $\mathbb{C}$.

3. Around each point $p \in Z$, glue in the smooth solution h_t^{model} so that the new glued metric h_t^{approx} agrees with h_t^{model} inside $B_{\frac{1}{2}}(p)$ and agrees with h_∞ outside $\cup_{p \in Z} B_1(p)$. *Note: The glued hermitian metric fails to solve Hitchin's equations on the gluing annuli.*

4. Lastly, pertub to an actual solution of Hitchin's equations using an implicit function theorem type argument. The claim is that there is a unique nearby solution of Hitchin's equations.

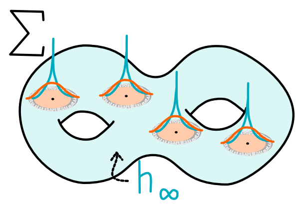

Figure 8.2: *Glued solution.*

The first part of their strategy is the most relevant tho this workshop. Consequently, the bulk of the talk will be about this limiting metric h_∞. The talk is arranged as follows:

Section 2. We'll give a description of Mazzeo-Swoboda-Weiss-Witt's limiting solutions h_∞.

Section 3. We'll compare Mazzeo-Swoboda-Weiss-Witt's compactification and the Morgan-Shalen-compactification.

Section 4. We'll discuss the difficulty in extending Mazzeo-Swoboda-Weiss-Witt construction to the entire ends of the moduli space. We'll discuss Mochizuki's results about the non-generic case where $\mathrm{Det}(\varphi)$ has a non-simple zero.

8.2 Limiting configuration h_∞

Given a Higgs bundle $(\bar\partial_E, t\varphi)$, Hitchin's equations are equations for a hermitian metic h_t. The hermitian metric h_t solves Hitchin's equations if

$$F_{D(\bar\partial_E, h_t)} + t^2[\varphi, \varphi^{*_{h_t}}] = 0. \tag{8.5}$$

What can we say about h_∞?

The first thing is that h_∞ solves "the decoupled Hitchin's equations"

$$F_{D(\bar\partial_E, h_\infty)} = 0 \qquad\qquad [\varphi, \varphi^{*_{h_\infty}}] = 0. \tag{8.6}$$

Mochizuki proved the "asymptotic decoupling" of Hitchin's equations— for any rank $G = SU(n)$ and for all (rather than *generic*) Higgs bundles.

Theorem 8.2.1. *(Mochizuki '15) Let $(\bar\partial_E, t\varphi, h_t)$ be a family of solution of Hitchin's equations. On a compact subset $\overline{U} \subset \Sigma - Z$, there exist constants $c_0, \varepsilon_0 > 0$ such that at any point in $\overline{U}$*

$$\| [\varphi, \varphi^{*_{h_t}}] \|_{g_\Sigma, h_t} \le c_0 \mathrm{e}^{-\varepsilon_0 t}. \tag{8.7}$$

Now, we actually describe the solutions of Hitchin's equations

8.2.1 Construction of h_∞

. The guiding slogan is: "Solutions of Hitchin's equations abelianize at the ends of $\mathcal{M}$, i.e. come from push forward of data on $\mathcal{L} \to \tilde\Sigma_{2:1}$." The exact way that we get h_∞ is summarized in Figure 8.3. We walk through this diagram for the rest of the section.

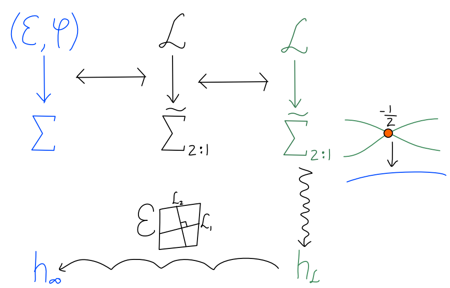

Figure 8.3: *Diagram for Section 8.2.1*

We've already described the correspondence between $(\mathcal{E}, \varphi) \to \Sigma$ and spectral $\mathcal{L} \to \tilde\Sigma_{2:1}$. Now, we do something that's simultaneously trivial and a little deep: we view $\mathcal{L} \to \tilde\Sigma_{2:1}$ as a holomorphic *parabolic* line bundle with weights $-\frac{1}{2}$ at $\tilde p_i \in \tilde Z$. This is trivial because all we are doing is marking this points, and assigning some numbers to then. This is a little deep because with this choice, the parabolic degree of $\mathcal{L}$ is equal to the degree of $\mathcal{E}$, which is zero.

Now, on $\mathcal{L} \to \tilde\Sigma_{2:1}$, there is a hermitin metric $h_{\mathcal{L}}$ which is Hermitian-Einstein $(F_{D(\bar\partial_{\mathcal{L}}, h_{\mathcal{L}})} = 0)$ and is "adapted to the parabolic structure" in the sense that near $\tilde p_i$, $h_{\mathcal{L}} \sim |w|^{2 \cdot -\frac{1}{2}}$. (Here w is a holomorphic coordinate centered at p_i.) Note this metric is singular.

Lastly, we obatin h_∞ from pushing forward this metric. In a given fiber of $\mathcal{E}$, we declare the two eigenspaces of the Higgs field to be orthogonal. Inside of each eigenspace, we use the metric $h_{\mathcal{L}}$.

Near each $p \in Z$ let z be a holomorphic corodiante centered at $p \in Z$ such that $-\operatorname{Det}(\varphi) = z \mathrm{d}z^2$. Then, we can find a gauge where

$$\bar\partial_E = \bar\partial \qquad t\varphi = t \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \mathrm{d}z \qquad h_\infty = \begin{pmatrix} |z|^{1/2} & \\ & |z|^{-1/2} \end{pmatrix}. \tag{8.8}$$

8.2.2 h_∞ as a limit

So far, we've described a metric h_∞. To prove that h_t converges to h_∞, we have to know something about h_t.

Mazzeo-Swoboda-Weiss-Witt explicitly construct a hermitian metric h_t^{approx}, which approximately solves Hitchin's equations. (Later, they show that h_t^{approx} is close to h_t.)

On the disks, the model solution h_t^{model} is

$$\bar\partial_E = \bar\partial \qquad t\varphi = t\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \mathrm{d}z \qquad h_t^{\text{model}} = \begin{pmatrix} |z|^{1/2}\mathrm{e}^{u_t} & \\ & |z|^{-1/2}\mathrm{e}^{-u_t} \end{pmatrix}, \tag{8.9}$$

where $u_t(|z)$ is a solution of Painlevé III with boundary conditions $u_t \sim -\frac{1}{2}\log|z|$ near $|z| = 0$ (so $|z|^{1/2}\mathrm{e}^{u_t}$ is not singular) and $\lim_{|z|\to\infty} u_t(|z|) = 0$.

The approximate solution interpolates between h_t^{model} (on each of the disks of radius $\frac{1}{2}$ around $p \in Z$) and h_∞ (on the complement of the unit disks around $p \in Z$). To write down h_t^{approx} we just add a cut-off function χ to the above expression.

$$\bar\partial_E = \bar\partial \qquad t\varphi = t\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \mathrm{d}z \qquad h_t^{\text{approx}} = \begin{pmatrix} |z|^{1/2}\mathrm{e}^{\chi u_t} & \\ & |z|^{-1/2}\mathrm{e}^{-\chi u_t} \end{pmatrix} \tag{8.10}$$

Lastly, they prove that h_t is close to h_t^{approx}. Define γ_t by $h_t = h_t^{\text{approx}}\mathrm{e}^{\gamma_t}$. (This is schematically true, but the relationship between h_t, h_t^{approx} and γ_t in their paper is more complicated in form but not in meaning than this because they work in a unitary gauge.)

Theorem 8.2.2. *(Mazzeo-Swoboda-Weiss-Witt '14) There exists $m > 0$ such that for t sufficiently large $\|\gamma_t\|_{L^{2,2}}(i\mathfrak{su}(E)) \le t^{-m}$*

Their theorem is summarized by the following picture:

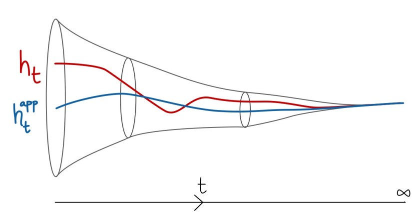

Figure 8.4: *Picture of Theorem 8.2.2*

The upshot of this is that h_t and h_t^{approx} are close. Since we know that h_t^{approx} converges to h_∞, we also know that h_t also converges:

$$\lim_{t\to\infty} h_t = \lim_{t\infty} h_t^{\text{approx}} = h_\infty. \tag{8.11}$$

8.3 Daskalapoulos-Dostoglou-Wentworth interpretation of MS compactification

Mazzeo-Swoboda-Weiss-Witt (partially) compactify the Hitchin moduli space by adding a point at infinity to each ray $(\mathcal{E}, t\varphi, h_t)$ lying over $\mathcal{B}' \subset \mathcal{B}$. Because their compactification avoids the degenerate tori, it is not a full compactification.

To place Mazzeo-Swoboda-Weiss-Witt's compactification in context, we begin by reviewing Morgan-Shalen's compactification of the the character variety.

Morgan-Shalen gave a compactification of the $SL(2,\mathbb{C})$-character variety. The boundary points in the Morgan-Shalen compactification correspond to elements of the space of projective classes of length functions

Remark 8.3.1. To see why projective classes of length functions might appear in the compactification of the $SL(2,\mathbb{C})$ character variety, observe that it is natural to associate a length function to a representation. In particular, given a representation $\rho : \pi_1(\Sigma) \to SL(2,\mathbb{C}) \curvearrowright \mathbb{H}^3$, we can associate a length function $\ell_\rho : \pi_1(\Sigma) \to \mathbb{R}^+$ where for $\gamma \in \pi_1(\Sigma)$

$$\ell_\rho(\gamma) = \inf_{x \in \mathbb{H}^3} \{\mathrm{dist}_{\mathbb{H}^3}(x, \rho(\gamma)x)\}. \tag{8.12}$$

on $\pi_1(\Sigma)$ with the weak topology (note this is a slightly non-standard length function).

Daskalapoulos-Dostoglou-Wentworth give an interpretation of the Morgan-Shalen compactification of the $SL(2,\mathbb{C})$ character variety of $\pi_1(\Sigma)$ in terms of a natural compactification of the moduli space of Higgs bundles $\mathcal{M}^{\mathrm{Higgs}}$ via the Hitchin map $\mathrm{Hit} : \mathcal{M}^{\mathrm{Higgs}} \to \mathcal{B} \cong H^0(\Sigma, K^2)$. In their compactification of the Higgs bundle moduli space, they think of the space of holomorphic quadratic differentials as an open $(6g-6)$-dimensional ball rather than a $(6g-6)$-dimensional vector space. Then, they add the sphere at ∞, which they call the space of "normalized holomorphic quadratic differentials." Having compactified the Hitchin base $\mathcal{B}$ by adding the sphere at infinity $\partial\overline{\mathcal{B}}$, they compactify the Hitchin moduli space by adding the same boundary $\partial\overline{\mathcal{B}}$. Note that in this compactification the torus fibers are crushed! (This is not be the case in Mazzeo-Swoboda-Weiss-Witt's compactification.)

How does Daskalapoulos-Dostoglou-Wentworth's compactification of $\mathcal{M}^{\mathrm{Higgs}}$ compare with the Morgan-Shalen compactification of $\chi(\pi_1(\Sigma))$? The non-abelian Hodge correspondence gives a map from the character variety to the Higgs bundle moduli space

$$\mathrm{NAHC} : \chi(\pi_1(\Sigma)) \to \mathcal{M}^{\mathrm{Higgs}}(\Sigma). \tag{8.13}$$

though solutions of Hitchin's equations. Note that $\chi(\pi_1(\Sigma))$ depends on the topology of Σ while $\mathcal{M}^{\mathrm{Higgs}}$ depends on the complex structure on Σ. Consequently, we get a different map NAHC for every complex structure on Σ. In order to compare the compactifications, Daskalapoulos-Dostoglou-Wentworth extend NAHC to the compactifications

$$\overline{\mathrm{NAHC}} : \overline{\chi(\pi_1(\Sigma))} \to \overline{\mathcal{M}^{\mathrm{Higgs}}(\Sigma)}. \tag{8.14}$$

and prove

Theorem 8.3.2. *The map* $\overline{\mathrm{NAHC}}$ *is continuous and surjective. Restricted to the compactification of the discrete, faithful representations,* $\overline{\mathcal{D}(\pi_1(\Sigma))}$*, it is a homeomorphism onto its image.*

In contrast, Mazzeo-Swoboda-Weiss-Witt's compactification does not collapse the torus fibers. This is desirable from the perspective of the natural L^2-metric g_{L^2} on the Hitchin moduli space. In this metric, torus fibers have roughly constant size along each ray—as Mazzeo-Swoboda-Weiss-Witt recently proved, and we now briefly discuss.

There is a metric g_{sf} called the "semi-flat" metric on $\mathcal{M}'$ in which all the torus fibers over some fixed ray $tq_2 \in \mathcal{B}'$ have the same metric. (This metric is flat on the torus fibers, hence the name "semi-flat.") The metric g_{sf} is hyperkähler, but badly singular as one approaches $\mathcal{B}_{\mathrm{sing}}$. Mazzeo-Swoboda-Weiss-Witt prove that difference between the actual hyperkähler metric g_{L^2} on $\mathcal{M}$ and g_{sf} on $\mathcal{M}'$ approaches zero near the ends. They prove that this difference is a sum of exponentially-decaying terms and less-understood polynomially-decaying terms.

Theorem 8.3.3. *(Mazzeo-Swoboda-Weiss-Witt '17)*

$$g_{L^2} = g_{\mathrm{sf}} + \sum_{j=1}^{\infty} t^{-2j/3} G_j + O(\mathrm{e}^{-\beta t}) \tag{8.15}$$

I've lumped the exponentially-decaying terms together in $O(\mathrm{e}^{-\beta t})$. However, these exponentially decaying terms can be precisely expressed in terms of Donaldson-Thomas invariants, as conjectured by Gaiotto-Moore-Neitzke.

Because g_{L^2} is close to g_{sf}, we say that the torus fibers have roughly constant size along the ray.

8.4 Non-generic ends

For most of the talk, we've actually focused on Mazzeo-Swoboda-Weiss-Witt's construction of h_∞ "at the generic ends" rather than Mazzeo-Swoboda-Weiss-Witt's approximation of h_t by h_t^{approx} "near the generic ends." Extending Mazzeo-Swoboda-Weiss-Witt's construction of h_t^{approx} to the full moduli space is difficult and ongoing work. However, Mochizuki has given a description of h_∞ even for non-generic ends. Mochizuki considers rays $(\bar{\partial}_E, t\varphi, h_t)$ of Higgs bundles together with hermitian metrics and seeks to describe h_∞.

Before turning to Mochizuki's description of h_∞, let's make a rough guess at the description of h_∞, generalizing the description in 8.2.1. We still want h_∞ to come as the pushforward of the Hermitian-Einstein metric on some holomorphic parabolic line bundle over the associated spectral cover. However, now $\tilde{\Sigma}_{2:1}$ is not smooth and we have to think carefully about how to assign something like parabolic weights to the rank-1 torsion-free sheaf $\mathcal{L} \to \tilde{\Sigma}_{2:1}$. These issues are non-trivial.

Mochizuki's description of h_∞ is a little difficult to unpack because he only considers the case where the spectral curve of the Higgs bundle is reducible to two components, i.e. $\mathrm{Det}(\varphi)$ has only zeros of even order. Any case can be reduced to this case. E.g. if the spectral cover for $(\mathcal{E},\varphi) \to \Sigma$ is irreducible, then the spectral cover for $(\pi^*\mathcal{E},\pi^*\varphi) \to \tilde{\Sigma}_{2:1}$ is reducible; we remember that the new base curve $\tilde{\Sigma}_{2:1}$ is itself a spectral cover by keeping track of the map $\rho : \tilde{\Sigma}_{2:1} \to \tilde{\Sigma}_{2:1}$ which exchanges the two sheets.

When the spectral curve is reducible to two components, then globally we can label one eigenvalue of φ as λ_1 and the other as λ_2. Inside $\pi^*\mathcal{E} \to \tilde{\Sigma}_{2:1}$, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the associated λ_1 and λ_2-eigenspaces of $\pi^*\varphi$. In order to assign parabolic weights at each of the ramification points $p \in Z$, we need to keep track of the following data:

- Globally, we need the degrees of the line bundles, $\deg(\mathcal{L}_i)$.
- At each $p \in Z$, we need to keep track of $m_p \in \mathbb{Z}^{>0}, \ell_p \in \mathbb{Z}^{\geq 0}$ which parameterize the basic local models of the Higgs field near p. Roughly, the local shape[1] of φ near p is

$$\varphi = \begin{pmatrix} z^m & 0 \\ z^{m-\ell} & -z^m \end{pmatrix} \mathrm{d}z. \tag{8.16}$$

From this data Mochizuki is able to define the parabolic weight $\alpha_p - \frac{\ell_p}{2}$ of $\mathcal{L}_1$ at p, and corresponding parabolic weight $\alpha_p + \frac{\ell_p}{2}$ of $\mathcal{L}_2$ at p. (Note: if $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2)$, then $\alpha_p{=}0$.) The limiting metric h_∞ comes from the pushforward of the Hermitian-Einstein metrics on the parabolic line bundles $\mathcal{L}_1, \mathcal{L}_2$.

Working backwards from the above general case, we see that if $\mathrm{Det}(\varphi)$ has any odd-order zero, $\tilde{\Sigma}_{2:1}$ is irreducible. In the lifted story, $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2) = 0$, hence we automatically know that the parabolic weight at p on $\mathcal{L}_i$ is determined by ℓ_p. At an odd-order zero ℓ_p must be odd. Roughly, the local shape of φ near p is

$$\varphi = \begin{pmatrix} 0 & z^k \\ z^{k-\ell} & 0 \end{pmatrix} \mathrm{d}z. \tag{8.17}$$

[1] Mochizuki Section 3.2