University Bonn — SS 2018/19 Maria Beatrice Pozzetti

Advanced Geometry II–Symmetric spaces

Exercises 9–Cartan decomposition– Not to hand in May 30, 2019

9.1. Consider the symmetric space

$$S_n = \{M \in \mathrm{Sym}^+(n,\mathbb{R}) \mid \det(M) = 1\}$$

associated to the Riemannian symmetric pair $(SL(n,R), SO(n))$.

(a) Compute the Cartan decomposition associated to the point

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in S_n,$$

where $\lambda_1 > \ldots > \lambda_n$ and $\lambda_1 \cdot \cdots \cdot \lambda_n = 1$.

(b) Denote by Θ^Λ (resp. Θ^{Id}) the Cartan involutions associated to the points Λ (resp. Id), and by $\mathfrak{p}^\Lambda$ (resp. $\mathfrak{p}^{\mathrm{Id}}$) the eigenspaces for -1. What is $\mathfrak{p}^{\mathrm{Id}} \cap \mathfrak{p}^\Lambda$?

(c) In general given two points $x, y \in S_n$ what are the possible dimensions of $\mathfrak{p}^x \cap \mathfrak{p}^y$? *Hint:* show that for every pair (x, y) there exists real numbers $\lambda_1 \geq \ldots \lambda_n$ with $\lambda_1 \ldots \lambda_n = 1$ and g in $\mathrm{SL}(n,\mathbb{R})$ such that $(g{\cdot}\mathrm{Id}, g{\cdot}\Lambda) = (x, y)$.

9.2. Let $M = G/K$ be a globally symmetric space and $p \in M$. Denote by $s_p : M \to M$ the geodesic symmetry about $p \in M$. Recall that $\sigma : G \ni g \to s_p g s_p \in G$, and $\pi : G \ni g \to gp \in M$. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated decomposition. Denote by $\exp : \mathfrak{g} \to G$ the exponential map in the sense of Lie groups and by $\mathrm{Exp}_p : T_pM \to M$ the exponential map in the sense of Riemannian manifolds. Recall from class that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{D_e\pi|_{\mathfrak{p}}} & T_pM \\ \downarrow & & \downarrow \\ G & \xrightarrow{\pi} & M \end{array}$$

where the arrow $\mathfrak{p} \to G$ is given by $\exp|_{\mathfrak{p}}$ and the arrow $T_pM \to M$ is given by Exp_p. Make this explicit for $M = \mathbb{R}^2, \mathbb{S}^2, \mathbb{H}^2$.

9.3. Give an example of an irreducible orthogonal symmetric Lie algebra of compact type and one of an irreducible OSLA of non-compact type. Is $(\mathfrak{so}(2,2), \Theta)$ irreducible? Recall that

$$\mathfrak{so}(2,2) = \{g \in \mathfrak{sl}(4,\mathbb{R}) | X^T S_{2,2} + S_{2,2} X = 0\}$$

with $S_{2,2} = \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_2 \end{pmatrix}$ and $\Theta(X) = S_{2,2} X S_{2,2}$.