Advanced Geometry II–Symmetric spaces

Exercises 8–Riemannian Symmetric pairs– Hand in by June 5th May 30, 2019

8.1. (a) Give an expression for two distinct involutive automorphisms $\sigma_1, \sigma_2 : SO(n) \to SO(n)$ such that the groups G_{σ_i} are non trivial and such that the quotients G/G_{σ_i} are isomorphic. What are the groups G_{σ_i} in your example?

(b) Find two non-isomorphic simply connected symmetric spaces M_1, M_2 having $\mathrm{Iso}^0(M_i) = SO(n)$. For each one of them, give one example of an involutive automorphism σ_i, such that $G/G_{\sigma_i} = M_i$.

8.2. Consider the vector space $\mathbb{C}^{n+1}$ with the standard positive definite Hermitian form:

$$h(x, y) = \sum_{i=0}^{n} \bar{x}_i y_i$$

Denote by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ the usual projection into the projective space. You can identify the tangent space at every point of $\mathbb{C}^{n+1}$ with the vector space $\mathbb{C}^{n+1}$.

(a) Show that, $d\pi_v$ induces an isomorphism of the orthogonal space at v for the form h with the tangent space at $[v] = \pi(v)$.

(b) Define a Riemannian metric on $\mathbb{CP}^n$ using the real part of the restriction of h to these orthogonal subspaces. This metric is the Fubini-Study metric.

(c) Show that $\mathbb{CP}^n$ with the Fubini-Study metric is a Riemannian symmetric space by finding the associated Riemannian symmetric pair.

8.3. Consider $\mathbb{C}^n$ with the standard positive definite Hermitian form h. We identify it with the pair $(\mathbb{R}^{2n}, J)$, where J is a linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $J^2 = -\mathrm{Id}$. The real part of h gives a scalar product $<,>$ on $\mathbb{R}^{2n}$. A totally real subspace is a real vector subspace V of dimension n such that V is orthogonal to $J(V)$ with respect to $<,>$. A linear map A is complex antilinear if $A(J(v)) = -J(A(v))$.

(a) Verify that, for the scalar product $<,>$ on $\mathbb{R}^{2n}$, the map J is orthogonal.

(b) Prove that the map that associates to a complex anti-linear reflection its set of fixed points is a bijection between the set of complex anti-linear reflections and the set of totally real subspaces.

(c) The set $TR(n)$ of totally real subspaces is a subset of the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^{2n})$. Prove that it is a totally geodesic submanifold. (Hint: show that $TR(n)$ is the subset of fixed points of an isometry).

(d) Verify that the real span of an unitary basis is a totally real subspace, and that an orthogonal basis of a totally real subspace is an unitary basis.

(e) Show that $TR(n)$ with the metric induced by the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^{2n})$ has a structure of Riemannian symmetric space by finding an associated Riemannian symmetric pair.