University Bonn — SS 2018/19 Maria Beatrice Pozzetti

Advanced Geometry II–Symmetric spaces

Exercises 7–Compact Lie groups– Not to hand in May 14, 2019

7.1. Consider the following subsets of the space of matrices:

$$O(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid A^T A = AA^T = I\}$$
$$U(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{C}) \mid \bar{A}^T A = A\bar{A}^T = I\}$$

(a) Prove that these subsets are groups for matrix multiplication.

(b) Prove that they are compact submanifolds of $\mathrm{Mat}(n, n, \mathbb{R})$, respectively $\mathrm{Mat}(n, n, \mathbb{C})$, and compute their dimension.

(c) Show that some of them are connected and some are disconnected.

(d) Find the tangent space at the identity, more precisely prove that:

$$T_I O(n) = \mathfrak{o}(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid A^T + A = 0\}$$
$$T_I U(n) = \mathfrak{u}(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{C}) \mid \bar{A}^T + A = 0\}$$

(e) Prove that the Lie bracket is given by the matrix commutator $[A, B] = AB - BA$.

(f) Compute the Killing form of the two groups, and verify that it is negative definite.

7.2. Let G be a compact Lie group endowed wlth a bi-invariant Riemannian metric. Define explicitely the geodesic involution at a point $h \in G$ and verify that it is isometric

7.3. Consider the space $\mathbb{R}^n$ with scalar product $\langle \cdot, \cdot \rangle$. Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace, and let $V^\perp$ be its orthogonal. We denote by $O(k) \times O(n-k)$ the subgroup of $O(n)$ that preserves the orthogonal decomposition $V \oplus V^\perp$, where the factor $O(k)$ is acting on V and the factor $O(n-k)$ is acting on $V^\perp$.

Consider the group $G = SO(n)$ and the subgroups $K_1 = SO(k) \times SO(n-k)$, $K_2 = S(O(k) \times O(n-k))$, where $S(H)$ denotes all the elements of the group H with determinant 1.

(a) Prove that (G, K_1) and (G, K_2) are Riemannian symmetric pairs with the same involutive automorphism.

(b) Find a natural covering map $G/K_1 \to G/K_2$.

(c) Show that the group K_2/K_1 has a natural action on G/K_1, that realizes it as the group of deck transformations of the covering above.

(d) What are G/K_1 and G/K_2? Can you find a geometric meaning of G/K_1 and G/K_2 that relates them to objects coming from linear algebra?