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Advanced Geometry II–Symmetric spaces

Exercises 6–Lie groups– Hand in by May 24 May 10, 2019

6.1. For $X \in \mathrm{Mat}(n,n,\mathbb{C})$, consider the limit

$$\sum_{k=1}^{\infty} \frac{1}{k!} X^k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} X^k$$

(a) Prove that for every X the limit exists in $\mathrm{Mat}(n,n,\mathbb{C})$. We will denote it e^X. (Hint: choose a suitable norm on $\mathrm{Mat}(n,n,\mathbb{C})$, prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!}|X|^k$ exists in $\mathbb{R}$, and from this deduce the same result for matrices.)

(b) Prove that for X, Y commuting matrices, $e^{X+Y} = e^X e^Y$.

(c) Prove that for all $X \in \mathrm{Mat}(n,n,\mathbb{C})$, $e^X \in GL(n,\mathbb{C})$.

(d) Prove that for every $X \in \mathrm{Mat}(n,n,\mathbb{C})$, the map

$$\mathbb{R} \ni t \to e^{tX} \in GL(n,\mathbb{C})$$

is a one-parameter subgroup of $GL(n,\mathbb{C})$.

(e) Prove that the map

$$\mathrm{Mat}(n,n,\mathbb{C}) \ni X \to e^X \in GL(n,\mathbb{C})$$

is locally invertible around 0.

(f) Prove that the above map is the exponential map of the Lie group $GL(n,\mathbb{C})$.

6.2. Compute the killing form for $SL(n,\mathbb{R})$ and prove that $SL(n,\mathbb{R})$ is semisimple. Is $GL(n,\mathbb{R})$ semisimple?

6.3. Let G be a compact Lie group.

(a) Show that G admits a bi-invariant Riemannian metric.
Hint: Find a bijection between bi-invariant Riemannian metrics and $\mathrm{Ad}(G)$-invariant scalar products on $\mathfrak{g}$. Then construct one such scalar product.

(b) Show that for any bi-invariant metric, the one parameter subgroups are geodesics.
Hint: Use Kozul's formula to show that $\nabla_X Y = \frac{1}{2}[X,Y]$, where X, Y are left invariant vector fields.

(c) How many bi-invariant metrics are there on the torus?