University Bonn — SS 2018/19	Maria Beatrice Pozzetti
Advanced Geometry II–Symmetric spaces	
Exercises 4–Symmetric spaces– To be handed in by May 10	April 25, 2019

4.1. Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere, c an arbitrary parallel of latitude on $\mathbb{S}^2$ and v a tangent vector to $\mathbb{S}^2$ at a point of c.

a) Describe geometrically the parallel transport of v along c.
Hint: Consider the cone C tangent to $\mathbb{S}^2$ along c, and show that the parallel transport of v along c is the same, whether taken relative to $\mathbb{S}^2$ or to C.

b) Is there an isometry $f : \mathbb{S}^2 \to \mathbb{S}^2$ realizing the parallel transport along c? Is it a transvection?

4.2. Consider the action of the group $\mathbb{Z}^n$ on $\mathbb{R}^n$ by integral translations:

$$\mathbb{Z}^n \times \mathbb{R}^n \ni ((m_1, \dots, m_n), (x_1, \dots, x_n)) \longrightarrow (x_1 + m_1, \dots, x_n + m_n) \in \mathbb{R}^n.$$

Notice that the translations are isometries for the standard Riemannian metric of $\mathbb{R}^n$. The cubical torus is the quotient $\mathbb{R}^n/\mathbb{Z}^n$, with its induced Riemannian metric. Prove that the cubical torus is a symmetric space. What is its group of isometries?

4.3. Let M be either the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$. For $\mathbb{H}^n$, you can use any model you know, after shortly describing it and its group of isometries. Let $p \in M$ be a point. In each of the two cases

(a) Write the geodesic symmetry at p.

(b) Find two transvections T, S with $T(p) = S(p)$ and $T \neq S$.