University Bonn — SS 2018/19 Maria Beatrice Pozzetti

Advanced Geometry II–Symmetric spaces

Exercises 2–Revision –To be handed in by April 26 April 11, 2019

2.1. Let $p : \bar{M} \to M$ be a covering map, and g be a Riemannian metric on M.

(a) Show that there exists a unique Riemannian metric $\bar{g}$ on $\bar{M}$ such that p is a local isometry. Such metric is called the *covering metric*.

(b) Show that (M, g) is complete if and only if $(M, \bar{g})$ is complete.

2.2. Let M, N be Riemannian manifolds, N connected, and $h, k : N \to M$ be local isometries. Assume there is a point $p \in N$ with $h(p) = k(p)$ and $d_p h = d_p k$. Show that $h = k$.

2.3. Let M be a Riemannian manifold of constant sectional curvature K, and let $\gamma : [0, \ell] \to M$ be a geodesic parametrized by arc-lenght, let γ' be its derivative and let J be a Jacobi field along γ, normal to γ'.

(a) Prove that the Jacobi equation can be written as

$$\frac{D^2 J}{dt^2} + KJ = 0.$$

Hint: you can use without proving it, that in a manifold of constant sectional curvature it holds;

$$g(R(X,Y)W, Z) = K\left(g(X,W)g(Y,Z) - g(Y,W)g(X,Z)\right).$$

(b) Let $w(t)$ be a parallel field along γ with $\langle \gamma'(t), w(t) \rangle = 0$ and $|w(t)| = 1$. Prove that the solution of the Jacobi equation with initial conditions $J(0) = 0$, $J'(0) = w(0)$ are given by:

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t), & \text{if } K > 0 \\ t w(t), & \text{if } K = 0 \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t), & \text{if } K < 0 \end{cases}$$

(c) Find all the pairs of conjugate points on the sphere $\mathbb{S}^n$. What is their multiplicity?

2.4. Let ∇ be the Levi-Civita connection on the Riemannian manifold (M, g), and R its curvature tensor. Prove that $\nabla R = 0$ if and only if for every smooth curve $c : I \to M$ and for every three parallel vector fields X, Y, Z along c, the vector field $R(X,Y)Z$ is a parallel vector field along c.