University Bonn — SS 2018/19	Maria Beatrice Pozzetti
Advanced Geometry II–Symmetric spaces	
Exercises 12–Flats and parallel sets	June 27, 2019

12.1. Let $(\mathfrak{g}, \Theta)$ be an OSLA. Let $X \in \mathfrak{p}$ be a vector.

(a) Show that $\mathfrak{k} \cap Z_{\mathfrak{g}}(X)$ is compactly embedded in $Z_{\mathfrak{g}}(X)$, and therefore $Z_{\mathfrak{g}}(X)$ is an OSLA.

(b) In the case $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R})$ describe all the possible isomorphism classes of $Z_{\mathfrak{g}}(X)$, for different elements X in $\mathfrak{p}$.

(c) In the case $\mathfrak{g} = \mathfrak{so}(p, q)$, with $q - p \geq 3$ show that $Z_{\mathfrak{g}}(X)$ is never abelian.

12.2. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$, endowed with the usual Cartan involution Θ. Choose a maximal abelian subalgbra $\mathfrak{a} \subset \mathfrak{p}$.

(a) What is the set Σ of roots? As α varies in Σ, what is the dimension of the rootspace $\mathfrak{g}_\alpha$?

(b) Explicitely describe a Weyl chamber $\overline{\mathfrak{a}^+}$. What are the positive roots for your choice?

12.3. Let M be a Riemannian symmetric space of non-compact type.

(a) Two geodesics $\gamma_1, \gamma_2 : \mathbb{R} \to M$ are parallel if $d(\gamma_1(t), \gamma_2(t))$ is constant. Let $P(\gamma)$ be the union of all geodesics parallel to γ. Show that $P(\gamma)$ is a totally geodesic submanifold of M.

(b) Consider now the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ associated to the point $\gamma(0)$ and let $X = \dot{\gamma}(0) \in \mathfrak{p}$. Show that the Lie triple system associated to $P(\gamma)$ is $\mathfrak{p} \cap Z_{\mathfrak{g}}(X)$.

(c) Let $\mathcal{X}$ denote the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. What are the isometry types of the possible parallel sets of geodesics in $\mathcal{X}$?