University Bonn — SS 2018/19 Maria Beatrice Pozzetti

Advanced Geometry II–Symmetric spaces

Exercises 11–Lie triple systems duality June 21, 2019

11.1. Consider the subgroup $\mathrm{SO}(p,q) < \mathrm{SL}(n,\mathbb{R})$.

(a) Choose a point $x \in S_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ such that $\mathrm{SO}(p,q)\cdot x$ is a copy of the positive Grassmannian embedded as a totally geodesic subspace of S_n.

(b) Compute the associated Lie triple system.

(c) Choose a point $y \in S_n \setminus \mathrm{SO}(p,q)\cdot x$. What is the stabilizer of y in $\mathrm{SO}(p,q)$?

11.2. (a) Show that if the symmetric space $\mathcal{X}$ splits as the Riemannian product $\mathcal{X}_1 \times \mathcal{X}_2$, then, for every Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g} = \mathrm{Lie}(\mathrm{Isom}^0(\mathcal{X}))$ associated to a point $x \in \mathcal{X}$, there are Lie triple systems $\mathfrak{n}_1, \mathfrak{n}_2 < \mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$.

(b) In $\mathfrak{sp}(2n,\mathbb{R})$ denote by $\mathfrak{n}_1$ (resp $\mathfrak{n}_2$) the vector subspaces

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \mid A \text{ symmetric} \right\}$$

$$\mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \mid A \text{ symmetric} \right\}.$$

Show that they are Lie triple systems and there is a Cartan decomposition such that $\mathfrak{p} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$.

(c) Is the simply connected symmetric space associated to $\mathfrak{sp}(2n,\mathbb{R})$ irreducible?

11.3. Let $\mathbb{H}$ denote the algebra of quaternions, $\mathbb{H}_+$ the vector space of quaternions with vanishing real part, Q the multiplicative group of quaternions of norm one.

(a) Show that Q is a twofold cover of $\mathrm{SO}(3)$ *Hint*: define a norm preserving action of Q on $\mathbb{H}_+$

(b) Show that $Q \times Q$ is a two-fold cover of $\mathrm{SO}(4)$ *Hint*: study the action

$$\begin{array}{rcl} (Q \times Q) \times \mathbb{H} & \to & \mathbb{H} \\ ((x,y),X) & \mapsto & xXy^{-1}. \end{array}$$

(c) Consider the Lie algebra automorphisms

$$\begin{array}{rrcl} s_0: & \mathfrak{so}(3)\times\mathfrak{so}(3) & \to & \mathfrak{so}(3)\times\mathfrak{so}(3) \\ & (X,Y) & \mapsto & (Y,X) \end{array} \qquad \begin{array}{rrcl} \sigma_{1,3}: & \mathfrak{so}(4) & \to & \mathfrak{so}(4) \\ & X & \mapsto & \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id}_3 \end{pmatrix} X \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id}_3 \end{pmatrix} \end{array}$$

Show that the orthogonal symmetric Lie algebras $(\mathfrak{so}(3)\times\mathfrak{so}(3), s_0)$ and $(\mathfrak{so}(4), \sigma_{1,3})$ are isomorphic. *Hint:* consider the derivative of the homomorphism constructed in (b).

(d) Compute the orthogonal symmetric Lie algebras dual to $(\mathfrak{so}(3)\times\mathfrak{so}(3), s_0)$ and $(\mathfrak{so}(4), \sigma_{1,3})$ respectively