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Advanced Geometry II–Symmetric spaces

Exercises 10–Curvature and duality– Hand in by June 21th June 5, 2019

10.1. Let $\mathcal{X}$ be a Riemannian symmetric space. Show that if the associated OSLA $\mathfrak{g}$ is of compact type, then $\mathcal{X}$ is a compact manifold.

10.2. Let L be a compact Lie group with a bi-invariant metric.

(a) Given left invariant vector fields $X, Y, Z \in \mathfrak{l}$ we know that $\nabla_X Y = \frac{1}{2}[X, Y]$ (cfr. Exercise 6.3). Use this fact to compute explicitely the Riemann tensor.

(b) Consider now L is a symmetric space. We showed in class that forall $X, Y, Z \in \mathfrak{p} = T_e L$ it holds

$$R(X, Y)Z = [[X, Y], Z].$$

Check that these two results are compatible.

10.3. Consider the symmetric space $S_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ from Exercise 9.1. Find a 2-plane in $T_p S_n$ where the sectional curvature is maximal and a 2-plane where the sectional curvature is minimal.

10.4. Consider the Lie algebra $\mathfrak{g} = \mathfrak{su}(p, q)$.

(a) What is the complexified Lie algebra?

(b) Construct a symmetric space M such that the Lie algebra of $\mathrm{Isom}_0(M)$ is $\mathfrak{g}$.

(c) Construct the compact dual of M.