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Advanced Geometry II–Symmetric spaces

Exercises 1–Grassmannians April 5, 2019

1.1. Let $\mathrm{Gr}_k(\mathbb{R}^n)$, $k < n$, be the set of all k-dimensional linear subspaces of $\mathbb{R}^n$ (the Grassmannian).

Given $V \in \mathrm{Gr}_{n-k}(\mathbb{R}^n)$, consider the set

$$\mathcal{U}_V = \{U \in \mathrm{Gr}(k, \mathbb{R}^n) \mid U \cap V = (0)\}$$

(a) Choose $U_0 \in \mathcal{U}_V$, and notice that $U_0 \oplus V = \mathbb{R}^n$.

(b) Define a bijection

$$\phi_{V,U_0} : \mathcal{U}_V \to \mathrm{Hom}(U_0, V) = \mathbb{R}^{k\times(n-k)}.$$

(c) Prove that the set of pairs $\{(\mathcal{U}_V, \phi_{V,U_0})\}$ is an atlas for a smooth structure on $\mathrm{Gr}_k(\mathbb{R}^n)$.

1.2. We consider the following Riemannian metric on $\mathrm{Gr}_k(\mathbb{R}^n)$. Choose a scalar product on $\mathbb{R}^n$. For every $U \in \mathrm{Gr}_k(\mathbb{R}^n)$, recall from Exercise 1.1 that $\mathcal{U}_{U^\perp}$ is a neighborhood of U identified, via $\phi_{U^\perp,U}$, with $\mathrm{Hom}(U, U^\perp)$. Hence the tangent space $T_U\mathrm{Gr}_k(\mathbb{R}^n)$ is identified, via $d\phi_{U,U^\perp}$ with $T_0\mathrm{Hom}(U, U^\perp) = \mathrm{Hom}(U, U^\perp)$, where 0 denotes the zero map. The scalar product on U and $U^\perp$ induces a scalar product g_U on $\mathrm{Hom}(U, U^\perp) = U^* \otimes U^\perp$. Show that

(a) The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ is homogeneous, i.e. given two elements $U, V \in \mathrm{Gr}_k(\mathbb{R}^n)$, there is an isometry $f \in \mathrm{Isom}(\mathrm{Gr}_k(\mathbb{R}^n))$ with $f(U) = V$.

(b) The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ is a symmetric space, i.e. for every $U \in \mathrm{Gr}_k(\mathbb{R}^n)$, there is an isometry $s_U \in \mathrm{Isom}(\mathrm{Gr}_k(\mathbb{R}^n))$ with $s_U(U) = U$,

$$(ds_U)_U = -\mathrm{Id} : T_U\mathrm{Gr}_k(\mathbb{R}^n) \to T_U\mathrm{Gr}_k(\mathbb{R}^n).$$

1.3. Choose a scalar product on $\mathbb{R}^n$, as in Exercise 1.3. This gives a Riemannian metric on $\mathrm{Gr}_k(\mathbb{R}^n)$. Choose an orthonormal family of vectors $v_1, \ldots, v_n$.

(a) Let $V_m \subset \mathrm{Span}\{v_1, \ldots, v_{k-1}\}^\perp$ an m-dimensional subspace, where $2 \le m \le n - k + 1$. Show that

$$P_m = \{\mathrm{Span}\{v_1, \ldots, v_{k-1}, v\} \mid v \in V_m \setminus (0)\}$$

is a submanifold isometric to $\mathbb{RP}^{m-1}$.

(b) Let $m \le \min(k, n-k)$, and let V_i, for $1 \le i \le m$, be a sequence of 2-dimensional, mutually orthogonal, subspaces of $\mathrm{Span}\{v_1, \ldots, v_{k-m}\}^\perp$. Show that

$$T_m = \{\mathrm{Span}\{v_1, \ldots, v_{k-m}, w_1, \ldots, w_m\} \mid w_i \in V_i \setminus (0)\}$$

is a submanifold isometric to a flat m-torus.