Konvexgeometrie

Unten findet ihr eine Beschreibung der einzelnen Vorträge. Die Beschreibungen sind auf Englisch, da die meisten Quellen auch auf Englisch sind. Wenn es nicht explizit geschrieben wird, beziehen sich die Kapitel auf das Buch [Web], das in der Bibliothek von Mathematikon im Apparat vom nächsten Semester zu finden ist. Andere interessante Quellen findet ihr unten im Abschnitt "Further Readings". Wenn ihr Probleme habt, Zugang zu einer der Quellen zu haben (zum Beispiel seid ihr nicht in Heidelberg), schreibt mir bitte eine Email. Wir folgen den Syllabus von JProf. Dr. Gabriele Benedetti (Heidelberg, WiSe 20/21)

1 Einführung in die konvexen Mengen: konvexe Hülle und Satz von Gauss-Lucas (Proseminar)

Speakers: Jonas Biba, Simon Weiß

In this talk, you will introduce operation on sets and recall some known facts about affine sets and transformations. Then, you will define convex sets and the notion of convex hull. As a nice application you prove the Theorem of Gauss–Lucas about roots of complex polynomials. If you are short on time, you could leave out some of the part referring to Chapter 1 (which should be familiar from Linear Algebra), and only recall them orally when you use them.

Roadmap: From Chapter 1 you should recall the main definitions (translate, sum of sets and of multiplication of a set by a scalar [Ch.1.1], flat and segment [Ch.1.2], affine hull, affine combination, affine dependent/independent points [Ch. 1.3], affine transformation[Ch.1.5]) possibly with some examples/counterexamples. You should then focus on Chapter 2.1-2, and define convex sets, convex combination, convex hull, you should state and prove Theorem 2.2.2 characterizing convex hulls in terms of convex combinations and Theorem 2.2.9 (Gauss–Lucas Theorem): The roots of the derivative of a non-constant complex polynomial belong to the convex hull of the set of roots of the polynomial itself. Observe that the theorem is false for real polynomials. Time permitting you could include some other results from Chapter 2, such as Theorems 2.1.4 and/or 2.1.5.

2 Der Satz von Caratheodory und seine Korollare: Radon, Helly, Shapley-Folkman (Proseminar)

Speakers: David Barth, Benno Wendland

In the last talk, we saw that every element in the convex hull of a set A in $\mathbb{R}^n$ can be expressed as convex combination of points of A (Theorem 2.2.2). How many points do we need at most in the convex combination? Caratheodory's theorem says: $n + 1$. You will use this fact to prove a number of surprising results for convex sets: Radon's theorem (a set of at least $n+2$ points can be partitioned in two sets whose convex hulls intersect) and Helly's Theorem (a family of convex sets such that any $n+1$ of them intersect has a common intersection point). This has also application to economics via the Shapley–Folkman Theorem: The arithmetic mean of a large number of sets contained in the unit ball is approximately convex.

Roadmap: Recall Theorem 2.2.2, prove Theorems 2.2.4, 2.2.5 and 2.2.6, 7.1.1, 7.1.2. If time permits, discuss Theorem 7.1.3 alternatively prove [Ber, Proposition 5.7.1] (Shapley-Folkman Theorem). Deduce the following corollary: Let $Q_1, \ldots, Q_d$ be subsets of the unit ball of $\mathbb{R}^n$ and consider their arithmetic mean $Q := \frac{1}{d}(Q_1 + \ldots + Q_d)$. Then every point in the convex hull of Q is at distance at most n/d to a point in Q. Hence, this distance goes to zero as d goes to infinity.

3 Topologische Eigenschaften von konvexen Mengen (Proseminar)

Speakers: Lukas Hemberger, Anne Kollmar

In this talk, you will discuss some important topological properties of convex sets. You will prove that a convex set A always has non-empty interior $\mathrm{ri}(A)$ in the affine space generated by it and give a geometric characterization of $\mathrm{ri}(A)$. You will introduce the notion of distance between sets and show that for every closed convex set C and every point x in $\mathbb{R}^n$ there is a unique point y on C with minimal distance from x. Fun fact: A theorem of Motzkin (which we do not cover in the seminar) says that the closed subsets of $\mathbb{R}^n$ having this property are exactly the convex ones!

Roadmap: Definition of relative interior of a set and of relative boundary [Page 37]. Give a couple of examples. Definition of affine basis. Theorem 1.3.9 and Corollary 1.3.10 with proof. Baricentric coordinates. Theorem 2.3.1 with proof. Lemma 2.3.3 and Theorem 2.3.4 with proof. Theorem 2.3.6 and 2.3.8 with proof. Corollary 2.3.10 without proof. Definition of distance to a set and proof of Lipschitz condition (page 45). Theorem 1.9.1 and 1.9.4 with proof. Theorem 2.4.1 with proof.

4 Trennung von konvexen Mengen und Stützebenen (Seminar)

Speakers: Jannik Simianer, Laura Wamsler

In this talk, you will show that two convex sets can be separated by a hyperplane exactly when their relative interiors are disjoint. Two interesting consequences of this fact: Closed convex sets are the intersection of all the closed half spaces containing them. Every convex set has a non-trivial supporting hyperplane at every point in the relative boundary. This gives us back the intuitive picture that convex regions in $\mathbb{R}^2$ lies on one side of the lines tangent to their boundary. A fact that we do not cover in the seminar: Existence of supporting hyperplanes characterizes convex sets. In other words, if a set with non-empty interior admits a supporting hyperplane at each of its boundary points, then it is convex!

Roadmap: Corollary 2.4.2 with proof. Corollary 2.4.3 without proof (draw a picture). Theorem 2.4.4 with proof. Notions of separation. Theorem 2.4.6 with proof. Corollary 2.4.8 with proof. Lemma 2.4.9 with proof. Theorem 2.4.10 with proof. Corollary 2.4.11 with proof. Definition of (non-trivial) support hyperplane. Theorem 2.4.12 with proof. If there is time, Example 2.4.13.

5 Extrempunkte und der Satz von Krein-Milman (Seminar)

Speakers: Celine Lißmannm, Valentin Sumser

For familiar convex sets like cubes or pyramids, we intuitively know, what a k-dimensional face is. You will generalize this intuition to arbitrary convex set and see how one can reconstruct a closed convex set by taking the convex hull of its primitive faces. You will show then, that primitive faces are either flats or half-flats. Thus, you will derive a very important result of Krein–Milman: Every compact convex set is the convex hull of its 0-dimensional faces.

Roadmap: [Ch. 2.6] Definition of face, k-face and extreme points with the formula just above Example 2.6.1. All results from Theorem 2.6.2 to Theorem 2.6.16 with proof but skip Corollary 2.6.9. Explain the example illustrated in Figure 2.10.

6 Konvexe Funktionen einer Variable (Seminar)

Speakers: Jonas Stähle, Jingyi Zhang

In this talk, we make the acquaintance of convex functions $f : \mathbb{R}^n \to \mathbb{R}$. They satisfy

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y), \qquad \forall\, x, y \in \mathbb{R}^n,\ \lambda, \mu \geq 0,\ \lambda + \mu = 1. \tag{6.1}$$

As we will discover in Talk 9, these are exactly the functions with convex epigraph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq y\}$. In this talk, you will give geometric properties of convex functions on the real line, namely when $n = 1$. You will further discuss analytic properties of these functions. Are they continuous or even differentiable? What can be said in that case, about their derivatives? You will see that a central player in these questions is the support of f at a point x which you can interpret as the support hyperplane of the epigraph (see Talk 4) at the point $(x, f(x))$ under cover.

Roadmap: Present the definitions and theorems (with proof) of Chapter 5.1.

7 Die Jensen-Ungleichung und ihre Verwandten (Seminar)

Speakers: Leona Gerlinger, Barbara Riehl

You will see how many classical inequalities such as the one between the geometric and arithmetic mean can be proved applying (6.1) (or better a swift generalization of it called Jensen inequality) to a given convex function. In this way you get, for instance, Hölder and Minkowski inequality that perhaps you know from analysis. More in general, you will consider the sequence of weighted means of order t and show that they are increasing in t. Finally, you will introduce log-convex functions of which we see an important example in the next talk.

Roadmap: Present the definitions and theorems (with proof) of Chapter 5.2. Right after the proof of Hölder and Minkowski, give immediately also the proof of the integral version of these inequalities. For Hölder you can see Theorem 5.3.1 in the next section. Define log-convex functions and prove that the sum and the product of log-convex functions is log-convex [Ch. 5.3].

8 Die Gamma-Funktion und der Satz von Artin (Seminar)

Speakers: Felix Rugel, Luis Elvis Schneck

Euler's gamma function Γ is a famous log-convex function generalizing the factorial of a natural number. It plays a prominent role in number theory and complex analysis. You will show some important properties of the Gamma function and prove a theorem of Artin (called also Bohr-Mollerup Theorem in the literature) saying that Γ is the only function satisfying such properties. In passing, you will establish the Gauß formula for Γ. Artin's theorem enables one to prove some interesting identities about Γ, one of which is the Gaussian integral $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$. This integral plays a central role in the Stirling's formula about the asymptotic expansion of the factorial.

Roadmap: Present all the results with proofs from the definition of the gamma function on page 208 to the proof of Stirling's formula on page 215. Mention that Γ gives a formula for the volume ω_n of the n-dimensional unit ball: $\omega_n = \pi^{n/2}/\Gamma(n/2+1)$ [Bar]. This will be used in Talk 14. Only if you have time, you can give the proof following [Bar, Lemma 3.4].

9 Konvexe Funktionen mehrerer Variablen I: Epigraph und Stetigkeit (Seminar)

Speakers: Amelie Haberzettl, Leonie Rick

We venture in the realm of convex functions of several variables. You will establish the promised relationship between convex functions and their epigraph and use it to show that the supremum of a family of convex functions is convex. A central result that you will prove is the existence of a support of a convex function at a point using what we saw in Talk 4. As an intermezzo you will discuss the special class of positively homogeneous convex functions and use them to reprove Minkowski's inequality. In the last part, you will show that convex functions are locally Lipschitz and, hence, continuous.

Roadmap: Present the definition and theorems in Chapter 5.4. Leave the proof of Theorem 5.4.3 and of Theorem 5.4.2 as exercises. When giving the definition of support at a point, observe explicitly that the set of supports is a closed convex subset of the space of affine transformations from $\mathbb{R}^n$ to $\mathbb{R}$. Present just Theorem 5.5.1 and its proof. In the statement, change the last sentence to "Then f is locally Lipschitz and, in particular, continuous on X". Observe indeed that the last inequality of the proof shows that f is Lipschitz in the ball $B[x_0; r]$.

10 Konvexe Funktionen mehrerer Variablen II: Differenzierbarkeit (Seminar)

Speakers: Kevin Reiber, Luise Schneider

You generalize the result about differentiability that we saw in Talk 6 to the case of convex functions in several variables. The two milestone results that you will meet are that a function with unique support at a point is differentiable at that point and that a twice differentiable function is convex if and only if its Hessian is semipositive definite.

Roadmap: Present all the definitions and theorems (with proofs) of Chapter 5.5 except Theorem 5.5.1, which has already been covered in the previous talk.

11 Eine Brücke zwischen konvexen Mengen und Funktionen: die Stützfunktion (Seminar)

Speakers: Christian Reibold, Ahmad Seeno

You will discover a nice connection between compact convex sets $C \subset \mathbb{R}^n$ and positively homogeneous convex functions $h : \mathbb{R}^n \to \mathbb{R}$. Namely, you will associate to C its support function h_C: $h_C(u)$ tells you how "tall" C is in direction $u \in \mathbb{R}^n$. You will see that $C \mapsto h_C$ is a bijection and explore its algebraic properties. In the last part of the talk, you will switch topic and prove a result needed in the next talk about matrix inequalities: Important examples of convex sets C are given by the set of solutions of systems of linear equations and inequalities such as $C = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ and your task will be to characterize the extreme points of C.

Roadmap: Present the first part of Chapter 5.6 until Theorem 5.6.5 including proofs. Present Chapter 4.4 up to Theorem 4.4.2 with the included proof.

12 Matrix-Ungleichungen (Seminar)

Speakers: Emma Behringer, Max Zoller

You will use the theory of convex functions to prove two remarkable inequalities about the determinant of a square matrix of dimension n. Minkowski inequality says that the positively homogeneous function $A \mapsto -\sqrt[n]{\det A}$ is convex on the space of positive definite matrices. Hadamard inequality tries to answer the question: How large can the determinant of a matrix whose entries have absolute value smaller than 1 be? Hadamard inequality yields the upper bound $n^{n/2}$. It is remarkable that for many values of n, it is not known if a matrix with the given property exists whose determinant is exactly $n^{n/2}$.

Roadmap: Present Chapter 5.8 until Theorem 5.8.6 with proofs. In the statement of Theorem 5.8.4 add that equality holds if and only if $v_1, \dots, v_n$ are eigenvectors. Present Minkowski inequality before Hadamard inequality. Add to the statement of the first Hadamard inequality that equality holds if and only if the columns of A are pairwise orthogonal. The second inequality is an equality if and only if A is diagonal. Observe that Hadamard's inequality implies the bound $|\det A| \leq n^{n/2}$ where A belongs to the set S of matrices for a matrix whose entries have absolute value smaller than one. Using page 42 in [AZ], state the Hadamard's determinant problem and observe that the maximum of the determinant on S is achieved by a matrix whose entries have values in the set $\{-1, +1\}$. If time permits and you are interested, you can define Hadamard's matrices and present some of their fascinating properties using the discussion starting from equation (6) on page 43 of [AZ].

References

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Further Readings

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