On Spectral Sequences for Iwasawa Adjoints à la Jannsen for Families

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Abstract.

Jannsen established several spectral sequences for (global and local) Iwasawa modules over (not necessarily commutative) Iwasawa algebras (mainly of p-adic Lie groups) over $\mathbb{Z}_p$, which are very useful for determining certain properties of such modules in arithmetic applications. Slight generalizations of said results are also obtained by Nekovář (for abelian groups and more general coefficient rings), by Venjakob (for products of not necessarily abelian groups, but with $\mathbb{Z}_p$-coefficients), and by Lim-Sharifi. Unfortunately, some of Jannsen's spectral sequences for families of representations as coefficients for (local) Iwasawa cohomology are still missing. We explain and follow the philosophy that all these spectral sequences are consequences or analogues of local cohomology and duality à la Grothendieck (and Tate for duality groups).

Contents

§1. Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with uniformizing element π and finite residue field. Consider the ring of formal power series

$R = \mathcal{O}[[X_1, \ldots, X_t]]$ in t variables, which is a complete regular local ring of dimension $d = t + 1$ with maximal ideal $\mathfrak{m}$. While our results hold under more general assumptions, this is the case we are most interested in.

First, there is Matlis duality: Denote with $\mathcal{E}$ an injective hull of $R/\mathfrak{m}$ as an R-module. Then $T = \mathrm{Hom}_R(-, \mathcal{E})$ induces a contravariant involutive equivalence between Noetherian and Artinian R-modules akin to Pontryagin duality.

Second, there is local duality: If $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$ denotes the right derivation of $M \longmapsto \varinjlim_k \mathrm{Hom}_R(R/\mathfrak{m}^k, M)$ in the derived category of R-modules, then

$$\mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \cong [-d] \circ T \circ \mathbf{R}\mathrm{Hom}_R(-, R)$$

on finitely generated R-modules.

Third, there is Koszul duality: The complex $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$ can be computed by means of Koszul complexes $K^\bullet$ which are self-dual: $K^\bullet = \mathrm{Hom}_R(K^\bullet, R)[d]$.

Finally, there is Tate duality: Let G be a pro-p duality group of dimension s. Then for finite p-torsion G-modules A we have

$$H^i(G, \mathrm{Hom}_{\mathbb{Z}}(A, I)) \cong H^{s-i}(G, A)^*$$

for a dualizing module I. Here $-^*$ denotes the abstract dual

$$(-)^* = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}).$$

Consider $\Lambda_R(G) = \varprojlim_U R[G/U]$ where U runs through the open normal subgroups of G for a topological group G. It is well known that $\Lambda_R(\mathbb{Z}_p^s) \cong R[[Y_1, \ldots, Y_s]]$ and $R \cong \Lambda_{\mathcal{O}}(\mathbb{Z}_p^r)$. The maximal ideal of $\Lambda_R(\mathbb{Z}_p^s)$ is then generated by the regular sequence

$$(\pi, X_1, \ldots, X_t, Y_1, \ldots, Y_s)$$

and no matter how we split up this regular sequence into two, they will remain regular. The Koszul complex then gives rise to a number of interesting spectral sequences and these should (at least morally) recover the spectral sequences

$$\mathrm{Tor}_n^{\mathbb{Z}_p}(D_m(M^\vee), \mathbb{Q}_p/\mathbb{Z}_p) \Longrightarrow \mathrm{Ext}^{n+m}_{\Lambda_{\mathbb{Z}_p}(G)}(M, \Lambda_{\mathbb{Z}_p}(G))^\vee \tag{1}$$

and

$$\varinjlim_k D_n(\mathrm{Tor}_m^{\mathbb{Z}_p}(\mathbb{Z}_p/p^k, M)^\vee) \Longrightarrow \mathrm{Ext}^{n+m}_{\Lambda_{\mathbb{Z}_p}(G)}(M, \Lambda_{\mathbb{Z}_p}(G))^\vee, \tag{2}$$

which show up in Jannsen's proof of [Jan89, 2.1 and 2.2]. The functors D_n stem from Tate's spectral sequence and are a corner stone in the theory of duality groups. In contrast to the abstract dual $-^*$ above, $-^\vee$ denotes the Pontryagin dual $\operatorname{Hom}_{\mathrm{cts}}(-, \mathbb{R}/\mathbb{Z})$.

This article is structured as follows: After briefly fixing a few conventions, we lay the ring theoretic groundwork in section 3, including a discussion of local cohomology. Afterwards, we compare Matlis with Pontryagin duality in section 4 and observe a relation between Tate cohomology and local cohomology in section 5. We will then show in section 6 that aforementioned spectral sequences (and many more) are consequences of the four duality principles laid out above. This also allows us to generalize Jannsen's spectral sequences to more general coefficients. For example, generalisations of eq. (1) and eq. (2) are subjects of proposition 6.6 and of proposition 6.10 respectively. While another spectral sequence for Iwasawa adjoints has already been generalized to more general coefficients (cf. theorem 8.1), the generalizations of the aforementioned spectral sequences are missing in the literature. We can even generalize an explicit calculation of Iwasawa adjoints (cf. [Jan89, corollary 2.6], [NSW08, (5.4.14)]) in theorem 6.15.

Furthermore, we generalize Venjakob's result on local duality for Iwasawa algebras ([Ven02, theorem 5.6]) to more general coefficients (cf. theorem 7.2). As an application we determine the torsion submodule of local Iwasawa cohomology generalizing a result of Perrin-Riou in the case $R = \mathbb{Z}_p$ in theorem 8.2.

§2. Conventions

A *ring* will always be unitary and associative, but not necessarily commutative. If not explicitly stated otherwise, "module" means left-module, "Noetherien" means left-Noetherien etc.

We will furthermore use the language of derived categories. If $\mathbf{A}$ is an abelian category, we denote with $\mathbf{D}(\mathbf{A})$ the derived category of unbounded complexes, with $\mathbf{D}^+(\mathbf{A})$ the derived category of complexes bounded below, with $\mathbf{D}^-(\mathbf{A})$ the derived category of complexes bounded above and with $\mathbf{D}^b(\mathbf{A})$ the derived category of bounded complexes.

As we simultaneously have to deal with left- and right-exact functors, both covariant and contravariant, recovering spectral sequences from isomorphisms in the derived category is a bit of a hassle regarding the indices. Suppose that $\mathbf{A}$ has enough injectives and projectives and that M is a (suitably bounded) complex of objects of $\mathbf{A}$. Then for a covariant functor $F\colon \mathbf{A} \longrightarrow \mathbf{A}$ we set $\mathbf{R}F(M) = F(Q)$ and $\mathbf{L}F(M) = F(P)$ with Q a complex of injective objects, quasi-isomorphic to M and P a complex of projectives, quasi-isomorphic to M. If F is contravariant, we set $\mathbf{L}F(M) = F(Q)$ and $\mathbf{R}F(M) = F(P)$. For indices, this implies the following: Assume that M is concentrated in degree zero. Then for F covariant, $\mathbf{R}F(M)$ has non-vanishing cohomology at most in non-negative degrees and $\mathbf{L}F(M)$ at most in non-positive degrees. For F contravariant, it's exactly the other way around. We set $\mathbf{L}^q F(M) = H^q(\mathbf{L}F(M))$ and $\mathbf{R}^q F(M) = H^q(\mathbf{R}F(M))$. Note that with these conventions

- $\mathbf{R}^p(-)^G(M) = H^p(G, A)$
- $\mathbf{L}^q(- \otimes N)(M) = \mathrm{Tor}_{-q}(M, N)$
- $\mathbf{R}^p \operatorname{Hom}(-, N)(M) = \operatorname{Ext}^p(M, N)$
- $\mathbf{L}^q(\varinjlim_U (-^U)^*)(M) = \varinjlim_U H^{-q}(U, M)^*$

If $F\colon \mathbf{A} \longrightarrow \mathbf{A}$ is exact, then F maps quasi-isomorphic complexes to quasi-isomorphic complexes. Its derivation $\mathbf{R}F$ (or $\mathbf{L}F$) is then given by simply applying F and we will make no distinction between F and $\mathbf{R}F\colon \mathbf{D}(\mathbf{A}) \longrightarrow \mathbf{D}(\mathbf{A})$ in this case.

For every integer $d \in \mathbb{Z}$ we have a *shift operator* $[d]$, so that for complexes C and $n \in \mathbb{Z}$ the following holds:

$$([d](C))^n = C^{n+d}.$$

We will at times write $C[d]$ instead of $[d](C)$. Note that although we occasionally cite [Wei94], we deviate from Weibel's conventions in this regard: Our $[d]$ is Weibel's $[-d]$. We furthermore set $\operatorname{Hom}(C^\bullet, D^\bullet)$ to be the complex with entries $\operatorname{Hom}(C^\bullet, D^\bullet)^i = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}(C^k, D^{k+i})$. Sign conventions won't matter in this paper.

If R is Noetherian, the category of finitely generated R-modules $\mathbf{f.g.}$-R-$\mathbf{Mod}$ is abelian. Its inclusion into the category of all R-modules induces equivalences $\mathbf{D}^*(\mathbf{f.g.}\text{-}R\text{-}\mathbf{Mod}) \cong \mathbf{D}^*_c(R\text{-}\mathbf{Mod})$ for $* \in \{+, b\}$ where subscript "c" means complexes with finitely generated cohomology. (The letter "c" actually stands for "coherent", which for Noetherian rings amounts to finitely generated.)

§3. A few facts on R-modules

3.1. Noncommutative rings

Let R be a ring. The intersection of all maximal left ideals coincides with the intersection of all maximal right ideals and is called the *Jacobson radical* of R and is hence a two-sided ideal, denoted by $J(R)$. For every $r \in J(R)$ the element $1 - r$ then has both a left and a right inverse and the following form of Nakayama's lemma holds, cf. e. g. [Lam91, (4.22)].

Lemma 3.1 (Azumaya-Krull-Nakayama)**.** *Let $M \in R$-**Mod** be a finitely generated R-module. If $J(R)M = M$, then $M = 0$.*

Recall that a ring is called local if it has a unique maximal left-ideal. This unique left ideal is then also the ring's unique maximal right-ideal and the group of two-sided units is the complement of this maximal ideal.

The following is well known and gives rise to the notion of "finitely presented" (or "compact") objects in arbitrary categories.

Lemma 3.2. *Let R be a ring and M an R-module. Then*

$$\mathrm{Hom}_R(M, -)$$

commutes with all direct limits if and only if M is finitely presented.

If R is Noetherien, this isomorphism extends to higher Ext-groups.

Proposition 3.3. *Let R be a Noetherian ring, M a finitely generated R-module, and $(N_i)_i$ a direct system of R-modules. Then*

$$\mathrm{Ext}^q_R(M, \varinjlim_i N_i) \cong \varinjlim_i \mathrm{Ext}^q_R(M, N_i).$$

Proof. As R is Noetherian, there exists a resolution of M by finitely generated projective R-Modules. As furthermore $\varinjlim$ commutes with homology, lemma 3.2 yields the result. ∎

Remark 3.4. Recall the following subtleties: Let R, S, T be rings, N a S-R-bimodule and P a S-T-bimodule. Then $\mathrm{Hom}_S(N, P)$ has the natural structure of an R-T-bimodule via $(rf)(n) = f(nr)$ and $(ft)(n) = f(n)t$.

Furthermore let M be a R-left-module. Then canonically

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P)) = \mathrm{Hom}_S(N \otimes_R M, P)$$

as T-right-modules.

Lemma 3.5. *If P is an R-R-bimodule that is flat as an R-right-module and Q an injective R-left-module, then $\mathrm{Hom}_R(P, Q)$ is again an injective R-left-module.*

Proof. $\mathrm{Hom}_R(-, \mathrm{Hom}_R(P, Q)) = \mathrm{Hom}_R(-, Q) \circ (P \otimes_R -)$ is a composition of exact functors and hence exact. ∎

Lemma 3.6. *Let N be an R-R-bimodule and M an R-left-module. Then*

$$\mathrm{Tor}^R_q(N, M) = 0$$

if and only if

$$\mathrm{Ext}^q_R(M, \mathrm{Hom}_R(N, Q)) = 0$$

for all injective R-left-modules Q.

Proof. The isomorphism of functors

$$\mathrm{Hom}_R(-, Q) \circ (N \otimes_R -) \cong \mathrm{Hom}_R(-, \mathrm{Hom}_R(N, Q))$$

yields an isomorphism

$$\mathrm{Hom}_R(-, Q) \circ (N \otimes^{\mathbf{L}}_R -) \cong \mathbf{R}\mathrm{Hom}_R(-, \mathrm{Hom}_R(N, Q))$$

in the derived category, which in turn yields

$$\mathrm{Hom}_R(\mathrm{Tor}^R_q(N, M), Q) \cong \mathrm{Ext}^q_R(M, \mathrm{Hom}_R(N, Q))$$

for all q. As the category of R-left-modules has sufficiently many injectives, this shows the proposition. ∎

Definition 3.7. Let R be a ring. A sequence $(r_1, \dots, r_d)$ of central elements in R is called *regular*, if for each i the residue class of r_{i+1} in $R/(r_1, \dots, r_i)$ is not a zero-divisor.

Definition 3.8. For a regular sequence $\underline{r} = (r_1, \dots, r_d)$ we denote by $\underline{r}^{(k)}$ the sequence $(r_1^k, \dots, r_d^k)$, which is again regular (cf. e. g. [Mat86, theorem 16.1]). If I is an ideal generated by a regular sequence $\underline{r}$, we will by abuse of notation refer to the ideal generated by $\underline{r}^{(k)}$ as $I^{(k)}$. Note that $I^{(k)}$ actually depends on the chosen regular sequence, which is either going to be clear from the context or arbitrary as long as chosen consistently.

Proposition 3.9. *Let R be a ring and $(r_1, \dots, r_k, s_1, \dots, s_l)$ such a regular sequence that the sequence $(s_1, \dots, s_l)$ is itself regular. Let $I = (r_i)_i$ and $J = (s_i)_i$ be the ideals generated by the first and second part of the regular sequence. Then for all $q \geq 1$*

$$\mathrm{Tor}^R_q(R/I, R/J) = 0$$

and

$$\varprojlim_n \mathrm{Tor}^R_q(R/I^n, R/J^n) = 0.$$

Proof. Let us first show that $\mathrm{Tor}^R_1(R/I, R/J) = 0$ and then reduce to this case by induction on l. Consider the exact sequence of R-modules

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

and apply $\operatorname{Tor}^R_\bullet(R/I, -)$. As R is flat,

$$\operatorname{Tor}^R_1(R/I, R/J) = \ker\Big(R/I \otimes_R J \longrightarrow R/I\Big) = \frac{I \cap J}{IJ}.$$

We argue by induction on l that this is zero: For $l = 1$, $x \in I \cap J$ implies $x = \lambda s_1 = s_1\lambda \in I$, so $\lambda = 0$ in R/I, so $\lambda \in I$ and hence $x \in IJ$. Denote with J' the ideal generated by $s_1, \dots, s_{l-1}$. By induction $I \cap J' = IJ'$. Let $x \in I \cap J = I \cap (J' + s_l R)$, so $x = a + s_l b$ with $a \in J'$ and $b \in R$. Clearly $s_l b = 0$ in $R/(I + J')$, so $b \in I + J'$ by regularity of the sequence and hence

$$I \cap J = I \cap (J' + s_l R) = I \cap (J' + s_l(I + J')) = I \cap (J' + s_l I) = I \cap J' + I \cap s_l I$$

as $s_l I \subseteq I$. Now $I \cap J' + I \cap s_l I = IJ' + s_l I = I(J' + s_l R) = IJ$, and this was to be shown.

We now argue by induction on l that $\operatorname{Tor}^R_q(R/I, R/J) = 0$ for all $q > 0$. For $l = 1$ we have the free resolution

$$0 \longrightarrow R \xrightarrow{s_1} R \longrightarrow R/J \longrightarrow 0$$

hence $\operatorname{Tor}^R_q(R/I, R/J) = 0$ for $q > 1$ and for $q = 1$ by what we saw above. Let J' be again the ideal generated by $s_1, \dots, s_{l-1}$. By induction we can assume that all $\operatorname{Tor}^R_q(R/I, R/J')$ vanish. Consider the sequence

$$0 \longrightarrow R/J' \xrightarrow{s_l} R/J' \longrightarrow R/J \longrightarrow 0,$$

which is exact as the subsequence $(s_1, \dots, s_l)$ is regular. Applying $\operatorname{Tor}^R_\bullet(R/I, -)$ shows that $\operatorname{Tor}^R_q(R/I, R/J) = 0$ for $q > 1$ by induction hypothesis — and by what we saw above also for $q = 1$.

Let $m' = \max\{k, l\}$. Clearly $I^{m'm} \subseteq I^{(m)} \subseteq I^m$ and the same is true for J, hence the natural map

$$\operatorname{Tor}^R_q(R/I^{m'm}, R/J^{m'm}) \longrightarrow \operatorname{Tor}^R_q(R/I^m, R/J^m)$$

factors through $\operatorname{Tor}^R_q(R/I^{(m)}, R/J^{(m)})$, which is zero. ■

3.2. The Koszul Complex

We recall a couple of well-known facts about the Koszul complex (cf. e. g. [Wei94, section 4.5]).

Definition 3.10. Denote by

$$K_\bullet(x) = 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

the chain complex (i. e., the degree decreases to the right) concentrated in degrees one and zero for a ring R and a central element $x \in Z(R)$. For central elements $x_1, \dots, x_d$ the complex

$$K_\bullet(x_1, \dots, x_d) = K_\bullet(x_1) \otimes \dots \otimes K_\bullet(x_d)$$

is called the *Koszul complex attached to* $x_1, \dots, x_d$. We will also consider the cochain complex $K^\bullet(x_1, \dots, x_d)$ with entries $K^p(x_1, \dots, x_d) = K_{-p}(x_1, \dots, x_d)$.

Remark 3.11. While this definition is certainly elegant, a more down to earth description is given as follows: $K_p(x_1, \dots, x_d)$ is the free R-module generated by the symbols $e_{i_1} \wedge \dots \wedge e_{i_p}$ with $i_1 < \dots < i_p$ with differential

$$d(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_p}.$$

This description emphasizes the importance of using central elements $(x_i)_i$.

Remark 3.12. The importance of the Koszul complex for our purposes stems from the following fact: If $x_1, \dots, x_d$ is a regular sequence, then $K_\bullet(x_1, \dots, x_d)$ is a free resolution of $R/(x_1, \dots, x_d)$, cf. e. g. [Wei94, corollary 4.5.5].

Proposition 3.13. *The complex* $K_\bullet = K_\bullet(x_1, \dots, x_d)$ *is isomorphic to the complex*

$$0 \longrightarrow \operatorname{Hom}_R(K_0, R) \longrightarrow \dots \longrightarrow \operatorname{Hom}_R(K_d, R) \longrightarrow 0,$$

where $\operatorname{Hom}_R(K_0, R)$ *is in degree* d *and* $\operatorname{Hom}_R(K_d, R)$ *in degree zero. Analogously*

$$K^\bullet \cong \operatorname{Hom}_R(K^\bullet, R)[d].$$

Proof. We have to describe isomorphisms $K_p \cong \operatorname{Hom}_R(K_{d-p}, R)$ such that all diagrams

$$\begin{tikzcd} K_p \arrow[r, "d"] \arrow[d, "\cong"] & K_{p-1} \arrow[d, "\cong"] \\ \operatorname{Hom}_R(K_{d-p}, R) \arrow[r, "d^*"] & \operatorname{Hom}_R(K_{d-p+1}, R) \end{tikzcd}$$

commute. Consider the map

$$e_{i_1} \wedge \dots \wedge e_{i_p} \longmapsto \big(e_{j_1} \wedge \dots \wedge e_{j_{d-p}} \longmapsto \operatorname{sgn}(\sigma)\big)$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma(1) = i_1, \dots, \sigma(p) = i_p, \sigma(p+1) = j_1, \dots, \sigma(d) = j_{d-p}$, i. e., in the exterior algebra $\bigwedge^d R^d$ we have

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-p}} = \operatorname{sgn}(\sigma) e_1 \wedge \cdots \wedge e_d.$$

(Note that $\operatorname{sgn}(\sigma) = 0$ if σ is not bijective.) It is then easy to verify that the diagram above does indeed commute: Identifying R with $\bigwedge^d R^d$, all but at most one summand vanishes in the ensuing calculation and the difference in sign is precisely the difference in the permutations. ∎

Proposition 3.14. *Let R be a ring and $x_1, \dots, x_d$ a regular sequence of central elements in R. Then in the bounded derived category of R-modules*

$$[d] \circ \mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_n), -) \cong R/(x_1, \dots, x_n) \otimes_R^{\mathbf{L}} -.$$

Proof. Denote with $K_\bullet$ the Koszul (chain) complex $K_\bullet(x_1, \dots, x_d)$ (concentrated in degrees $d, d-1, \dots, 0$) and with $K^\bullet$ the Koszul (cochain) complex (concentrated in degrees $-d, -d+1, \dots, 0$).

As $x_1, \dots, x_d$ form a regular sequence, $K^\bullet$ is a free resolution of $R/(x_1, \dots, x_n)$ and hence allows us to calculate the derived functors as follows.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_n), M)[d] &= \mathrm{Hom}_R(K^\bullet, M)[d] \\
&\cong \mathrm{Hom}_R(K^\bullet[-d], R) \otimes_R M \\
&\cong K^\bullet \otimes_R M \\
&= R/(x_1, \dots, x_d) \otimes_R^{\mathbf{L}} M,
\end{aligned}$$

with the crucial isomorphisms being due to the fact that $K^\bullet$ is a complex of free modules and proposition 3.13. It is clear that these isomorphisms are functorial in M. ∎

Corollary 3.15. *Let R be a commutative ring, $x_1, \dots, x_d$ a regular sequence in R and $T = \mathrm{Hom}_R(-, Q)$ with Q injective. Then*

$$\mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_d), -) \circ T = T \circ [d] \circ \mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_d), -)$$

on the derived category of R-modules.

Proof. The functor T is exact and

$$\mathrm{Hom}_R(R/(x_1, \dots, x_d), -) \circ T = T \circ (R/(x_1, \dots, x_d) \otimes_R -)$$

by adjointness. Hence also

$$\mathbf{R}\mathrm{Hom}_R(R/(x_1,\dots,x_d),-)\circ T = T\circ(R/(x_1,\dots,x_d)\otimes_R^{\mathbf{L}} -).$$

By proposition 3.14, this is just $T\circ[d]\circ\mathbf{R}\mathrm{Hom}_R(R/(x_1,\dots,x_d),-)$. ■

Corollary 3.16. *Let R be a commutative ring and $x_1,\dots,x_d$ a regular sequence in R. Let further M be an R-module. Then*

$$\mathrm{Ext}_R^{d-p}(R/(x_1,\dots,x_d),M) = \mathrm{Tor}_p^R(R/(x_1,\dots,x_d),M).$$

Proof. This is just proposition 3.14, taking extra care of the indices:

$$\begin{aligned}\mathrm{Ext}_R^{d-p}(R/(x_1,\dots,x_d),M) &= \mathbf{R}^{d-p}\,\mathrm{Hom}_R(R/(x_1,\dots,x_d),M)\\ &= H^{-p}\,\mathbf{R}\mathrm{Hom}_R(R/(x_1,\dots,x_d)[-d],M)\\ &= H^{-p}(R/(x_1,\dots,x_d)\otimes_R^{\mathbf{L}} M)\\ &= \mathrm{Tor}_p^R(R/(x_1,\dots,x_d),M).\end{aligned}$$

■

3.3. Local Cohomology

Definition 3.17. Let R be a ring and $\underline{J} = (J_n)_{n\in\mathbb{N}}$ a decreasing sequence of two-sided ideals. (The classical example is to take a two-sided ideal J and set $\underline{J} = (J^n)_n$.) For an R-left-module M set

$$\Gamma_{\underline{J}}(M) = \{m\in M \mid J_n m = 0 \text{ for some } n\}.$$

It is clear that $\Gamma_{\underline{J}}$ is a left-exact functor with values in R-**Mod**. Denote its right-derived functor in the derived category $\mathbf{D}^+(R\text{-}\mathbf{Mod})$ by $\mathbf{R}\Gamma_{\underline{J}}$.

Remark 3.18. $\Gamma_{\underline{J}} = \varinjlim_n \mathrm{Hom}_R(R/J_n,-)$, so

$$\mathbf{R}\Gamma_{\underline{J}} = \varinjlim_n \mathbf{R}\mathrm{Hom}_R(R/J_n,-)$$

and

$$\mathbf{R}^q\Gamma_{\underline{J}} = \varinjlim_n \mathrm{Ext}_R^q(R/J_n,-).$$

Remark 3.19. Let I be an ideal generated by a regular sequence in a ring R. Then by cofinality of the systems

$$\Gamma_{(I^n)_n} = \Gamma_{(I^{(n)})_n}.$$

Lemma 3.20. *Let* $\mathbf{A}, \mathbf{B}$ *be abelian categories, with additive functors* $L\colon \mathbf{A} \longrightarrow \mathbf{B}$ *left adjoint to* $R\colon \mathbf{B} \longrightarrow \mathbf{A}$. *If* L *is exact,* R *preserves injective objects.*

Proof. This is well known, cf. e. g. [Wei94, proposition 2.3.10]. ∎

Remark 3.21. Let $\varphi\colon R \longrightarrow S$ be a homomorphism between unitary rings. Let $\underline{J}$ be decreasing sequence of two-sided ideals in R and denote with $\underline{J}S$ the induced sequence of two-sided ideals in S. If $\varphi(R)$ lies in the centre of S, then $\Gamma_{\underline{J}} \circ \operatorname{res}_\varphi = \operatorname{res}_\varphi \circ \Gamma_{\underline{J}S}$. If furthermore injective S-modules are also injective as R-modules, e. g., if S is a flat R-module via lemma 3.20, then $\mathbf{R}\Gamma_{\underline{J}} \circ \operatorname{res}_\varphi = \operatorname{res}_\varphi \circ \mathbf{R}\Gamma_{\underline{J}S}$. Local cohomology is thus independent of the base ring for flat extensions and we will omit $\operatorname{res}_\varphi$ and the distinction between $\underline{J}S$ and $\underline{J}$ in the future. Note especially that if R is complete, then $R[[G]] = R[G]^\wedge$ is a flat R-module.

Proposition 3.22. *Let* R *be a Noetherian local ring with maximal ideal* $\mathfrak{m}$ *and finite residue field. Let* M *be a finitely generated* R*-module. Then* $\Gamma_{\underline{\mathfrak{m}}}(M)$ *is the maximal finite submodule of* M.

Proof. Denote with T the maximal finite submodule of M (which exists as M is Noetherian). By Nakayama there exists a $k \in \mathbb{N}$ with $\mathfrak{m}^k T = 0$ and hence $T \subseteq \Gamma_{\underline{\mathfrak{m}}}(M)$. Conversely $R/\mathfrak{m}^k$ is a finite ring for each k, hence Rm is a finite module for each $m \in \Gamma_{\underline{\mathfrak{m}}}(M)$ and is thus contained in T. ∎

Proposition 3.23. *If* R *is a Noetherian ring and* $\underline{J}$ *a decreasing sequence of ideals, then* $\mathbf{R}\Gamma_{\underline{J}}$ *and* $\mathbf{R}^q\Gamma_{\underline{J}}$ *commute with direct limits.*

Proof. This is just proposition 3.3, as for Noetherian rings, direct limits of injective modules are again injective. ∎

Definition 3.24. For $\underline{I}$ and $\underline{J}$ decreasing sequences of two-sided ideals of a ring R set $(\underline{I} + \underline{J})_n = I_n + J_n$.

Remark 3.25. If I and J are two-sided ideals of a ring R, then generally $\underline{I} + \underline{J} \neq \underline{I + J}$. But as these two families are cofinal, $\Gamma_{\underline{I}+\underline{J}} = \Gamma_{\underline{I+J}}$.

Remark 3.26. Clearly $\Gamma_{\underline{I}+\underline{J}} = \Gamma_{\underline{I}} \circ \Gamma_{\underline{J}}$, but regrettably

$$\mathbf{R}\Gamma_{\underline{I}+\underline{J}} = \mathbf{R}\Gamma_{\underline{I}}\mathbf{R}\Gamma_{\underline{J}}$$

is in general false if the families $\underline{I}$ and $\underline{J}$ are not sufficiently independent from one another: For $R = \mathbb{Z}$, $\underline{I} = \underline{J} = (n_i\mathbb{Z})_i$ any descending sequence of non-trivial ideals and $M = \mathbb{Q}/\mathbb{Z}$, the five-term-sequence in cohomology would start as follows:

$$0 \longrightarrow \varinjlim_{i,j} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \operatorname{Hom}(\mathbb{Z}/n_j, \mathbb{Q}/\mathbb{Z})) \longrightarrow \varinjlim_i \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \mathbb{Q}/\mathbb{Z}) \\ \downarrow \\ \cdots$$

But clearly $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \mathbb{Q}/\mathbb{Z}) = 0$ and

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \operatorname{Hom}(\mathbb{Z}/n_j, \mathbb{Q}/\mathbb{Z})) = \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \mathbb{Z}/n_j) = \mathbb{Z}/(n_i, n_j),$$

hence

$$\varinjlim_{i,j} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \operatorname{Hom}(\mathbb{Z}/n_j, \mathbb{Q}/\mathbb{Z})) = \varinjlim_i \mathbb{Z}/n_i,$$

so the sequence above cannot possibly be exact.

This argument of course generalizes: Were $\mathbf{R}\Gamma_{\underline{I}+\underline{J}} = \mathbf{R}\Gamma_{\underline{I}}\mathbf{R}\Gamma_{\underline{J}}$, then [CE56, chapter XV, theorem 5.12] implied that

$$\varinjlim_i \operatorname{Ext}^p_R(R/I_i, \varinjlim_j \operatorname{Hom}_R(R/J_j, Q)) = 0$$

for all $p > 0$ and Q injective, i. e., if the isomorphism in the derived category holds, then because $\mathbf{R}\Gamma_{\underline{J}}$ mapped injective objects to $\Gamma_{\underline{I}}$-acyclics. Using lemma 3.6, a sufficient criterion for that to happen is that the transition maps eventually factor through $\operatorname{Ext}^p_R(R/\widetilde{I}_i, \operatorname{Hom}_R(R/\widetilde{J}_j, Q))$ for some $\widetilde{I}_i, \widetilde{J}_j$ with $\operatorname{Tor}^R_p(R/\widetilde{I}_i, R/\widetilde{J}_j) = 0$ for all $p > 0$ and this criterion appears to be close to optimal. The following proposition is a simple application of this principle.

Proposition 3.27. *Let R be a commutative ring and*

$$(r_1, \ldots, r_k, s_1, \ldots, s_l)$$

such a regular sequence, that $(s_1, \ldots, s_l)$ is itself again regular. Then for the ideals $I = (r_i)_i$ and $J = (s_i)_i$ we have

$$\mathbf{R}\Gamma_{\underline{I}+\underline{J}} = \mathbf{R}\Gamma_{\underline{I}}\mathbf{R}\Gamma_{\underline{J}} = \mathbf{R}\Gamma_{\underline{J}}\mathbf{R}\Gamma_{\underline{I}}.$$

Proof. The transition maps in the system

$$\varinjlim_i \operatorname{Ext}^p_R(R/I^i, \varinjlim_j \operatorname{Hom}_R(R/J^j, Q)) = \varinjlim_{i,j} \operatorname{Ext}^p_R(R/I^i, \operatorname{Hom}_R(R/J^j, Q))$$

eventually factor through $\operatorname{Ext}^p_R(R/I^{(n)}, \operatorname{Hom}_R(R/J^{(n)}, Q))$. But this vanishes by lemma 3.6 and proposition 3.9 for $p \geq 1$. As $\operatorname{Tor}^R_\bullet(-,-)$ is symmetrical for commutative rings, the same argument also applies for $\mathbf{R}\Gamma_{\underline{J}}\mathbf{R}\Gamma_{\underline{I}}$. ∎

§4. (Avoiding) Matlis Duality

First recall Pontryagin duality.

Theorem 4.1 (Pontryagin duality, e. g. [NSW08, (1.1.11)]). *The functor* $\Pi = \mathrm{Hom}_{\mathrm{cts}}(-, \mathbb{R}/\mathbb{Z})$ *induces a contravariant auto-equivalence on the category of locally compact Hausdorff abelian groups and interchanges compact with discrete groups. The isomorphism* $A \longrightarrow \Pi(\Pi(A))$ *is given by* $a \longmapsto (\varphi \longmapsto \varphi(a))$.

If G *is pro-*p*, then* $\Pi(G) = \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{Q}_p/\mathbb{Z}_p)$. *If* D *is a discrete torsion group or a topologically finitely generated profinite group, then* $\Pi(D) = \mathrm{Hom}_{\mathbb{Z}}(D, \mathbb{Q}/\mathbb{Z})$.

We will write $-^\vee$ *for* Π *if it is notationally more convenient.*

Matlis duality is commonly stated as follows:

Theorem 4.2 (Matlis duality, [BH93, theorem 3.2.13]). *Let* R *be a complete Noetherian commutative local ring with maximal ideal* $\mathfrak{m}$ *and* $\mathcal{E}$ *a fixed injective hull of the* R*-module* $R/\mathfrak{m}$. *Then* $\mathrm{Hom}_R(-, \mathcal{E})$ *induces an equivalence between the finitely generated modules and the Artinian modules with inverse* $\mathrm{Hom}_R(-, \mathcal{E})$.

Example 4.3. If R is a discrete valuation ring, then $Q(R)/R$ is an injective hull of its residue field.

Matlis duality – using an *abstract* dualizing module instead of a topological one – behaves very nicely in relation to local cohomology. In applications however the Matlis module $\mathcal{E}$ is cumbersome and in general not particularly easy to construct.

Example 4.4. Consider the rings $R = \mathbb{Z}_p$, $S_1 = \mathbb{Z}_p[\pi]$ and $S_2 = \mathbb{Z}_p[[T]]$ with π a uniformizer of $\mathbb{Q}_p(\sqrt{p})$. Clearly the homomorphisms $R \longrightarrow S_i$ are local and flat and their respective residue fields agree. But while $\mathcal{E}_R = \mathbb{Q}_p/\mathbb{Z}_p, \mathcal{E}_{S_1} \cong \mathbb{Q}_p/\mathbb{Z}_p^{\oplus 2}$ as an abelian group. Furthermore, $\mathcal{E}_{S_2} \cong \bigoplus_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p$ as an abelian group by proposition 4.8.

The best we can hope for in general is the following.

Proposition 4.5 ([Stacks, Tag 08Z5]). *Let* $R \longrightarrow S$ *be a flat and local homomorphism between Noetherian local rings with respective maximal ideals* $\mathfrak{m}$ *and* $\mathfrak{M}$. *Assume that* $R/\mathfrak{m}^n \cong S/\mathfrak{M}^n$ *for all* n. *Then an injective hull of* $S/\mathfrak{M}$ *as an* S*-module is also an injective hull of* $R/\mathfrak{m}$ *as an* R*-module.*

Starting with pro-p local rings, Matlis modules are however intimately connected with Pontryagin duality.

Lemma 4.6. *Let R be a pro-p local ring with maximal ideal $\mathfrak{m}$. Then there exists an isomorphism of R-modules $R/\mathfrak{m} \cong \operatorname{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}, \mathbb{Q}_p/\mathbb{Z}_p)$.*

Proof. As $R/\mathfrak{m}$ is finite and hence a commutative field, both objects are isomorphic as abelian groups. As vector spaces of the same finite dimension over $R/\mathfrak{m}$ they are hence isomorphic as $R/\mathfrak{m}$-modules and thus as R-modules. ∎

Lemma 4.7. *Let R be a pro-p local ring with maximal ideal $\mathfrak{m}$ and M a finitely presented or a discrete R-module. Then $\Pi(M) = \operatorname{Hom}_R(M, \Pi(R))$.*

Proof. Let first $M = \varinjlim_i M_i$ be an arbitrary direct limit of finitely presented R-modules. Then by lemma 3.2

$$\begin{aligned}\operatorname{Hom}_R(\varinjlim_i M_i, \Pi(R)) &= \varprojlim_i \operatorname{Hom}_R(M_i, \varinjlim_k \Pi(R/\mathfrak{m}^k)) \\ &= \varprojlim_i \varinjlim_k \operatorname{Hom}_R(M_i, \operatorname{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}^k, \mathbb{Q}_p/\mathbb{Z}_p)) \\ &= \varprojlim_i \varinjlim_k \operatorname{Hom}_{\mathbb{Z}_p}(M_i/\mathfrak{m}^k, \mathbb{Q}_p/\mathbb{Z}_p) \\ &= \varprojlim_i \Pi(M_i).\end{aligned}$$

If M itself is finitely presented, this shows the proposition. If M is discrete, we can take the M_i to be discrete and finitely presented (i. e., finite). The projective limit of their duals exists in the category of compact R-modules and it follows that $\varprojlim_i \Pi(M_i) = \Pi(M)$. ∎

Proposition 4.8. *Let R be a Noetherian pro-p local ring with maximal ideal $\mathfrak{m}$. Then $\Pi(R) = \operatorname{Hom}_{\mathrm{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$ is an injective hull of $R/\mathfrak{m}$ as an R-module.*

Proof. $\Pi(R)$ is injective as an abstract R-module: By Baer's criterion it suffices to show that $\operatorname{Hom}_R(R, \Pi(R)) \longrightarrow \operatorname{Hom}_R(I, \Pi(R))$ is surjective for every (left-)ideal I of R. By lemma 4.7, this is equivalent to the surjectivity of $\Pi(R) \longrightarrow \Pi(I)$, which is clear.

In lieu of lemma 4.6 it hence suffices to show that

$$\operatorname{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}, \mathbb{Q}_p/\mathbb{Z}_p) \subseteq \operatorname{Hom}_{\mathrm{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$$

is an essential extension, so take

$$H \leq \operatorname{Hom}_{\mathrm{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$$

an R-submodule and $0 \neq f \in H$. Then by continuity, f descends to

$$f \colon R/\mathfrak{m}^{k+1} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

with k minimal. It follows that there exists an element $r \in \mathfrak{m}^k$ with $f(r) \neq 0$. $rf \colon R \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ is consequentially also not zero, lies in H but now descends to

$$rf \colon R/\mathfrak{m} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

i. e., $H \cap \mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}, \mathbb{Q}_p/\mathbb{Z}_p) \neq 0$. ∎

Corollary 4.9. *Let R be a commutative pro-p Noetherian commutative local ring. Then if M is finitely generated or Artinian, Matlis and Pontryagin duality agree.*

Proof. Immediate from lemma 4.7 and proposition 4.8. ∎

Proposition 4.10. *Let R satisfy Matlis duality via*

$$T = \mathrm{Hom}_R(-, \mathcal{E}).$$

Let $\underline{I}$ be a decreasing family of ideals generated by regular sequences of length d. Then

$$\mathbf{R}\Gamma_{\underline{I}} = \varinjlim_n T \circ [d] \circ \mathbf{R}\mathrm{Hom}(R/I_n, -) \circ T$$

on $\mathbf{D}_c^+(R\text{-}\mathbf{Mod})$.

Proof. By corollary 3.15 it follows that

$$\mathbf{R}\Gamma_{\underline{I}} = \varinjlim_n \mathbf{R}\mathrm{Hom}(R/I_n, -) \circ T \circ T = \varinjlim_n T \circ [d] \circ \mathbf{R}\mathrm{Hom}(R/I_n, -) \circ T.$$

∎

§5. Tate Duality and Local Cohomology

Remark 5.1. Working in the derived category makes a number of subtleties more explicit than working only with cohomology groups. Assume that R is a complete local commutative Noetherian ring with finite residue field of characteristic p and G an analytic pro-p group. Then every $\Lambda = R[[G]]$-module has a natural topology via the filtration of augmentation ideals of Λ. It is furthermore obvious to consider the following two categories:

- $\mathcal{C}(\Lambda)$, the category of compact Λ-modules (with continuous G-action),

- $\mathcal{D}(\Lambda)$, the category of discrete Λ-modules (with continuous G-action).

Pontryagin duality then induces equivalences between $\mathcal{C}(\Lambda)$ and $\mathcal{D}(\Lambda^\circ)$, where $-^\circ$ denotes the opposite ring. It is furthermore well-known that both categories are abelian, that $\mathcal{C}(\Lambda)$ has exact projective limits and enough projectives and analogously that $\mathcal{D}(\Lambda)$ has exact direct limits and enough injectives, cf. e.g. [RZ00, chapter 5]. It is important to note that the notion of continuous Λ-homomorphisms and abstract ones often coincides: If M is finitely generated with the quotient topology and N is either compact or discrete, every Λ-homomorphism $M \longrightarrow N$ is continuous, cf. [Lim12, lemma 3.1.4].

In what follows we want to compare Tate cohomology, i.e. $\mathbf{L}D$ as defined below, with other cohomology theories such as local cohomology. Now Tate cohomology is defined on the category of *discrete* G-modules and we hence have a contravariant functor

$$\mathbf{L}D\colon \mathbf{D}^+(\mathcal{D}(\Lambda)) \longrightarrow \mathbf{D}^-(\Lambda^\circ\text{-}\mathbf{Mod})$$

Local cohomology on the other hand is defined on $\mathbf{D}^+(\Lambda\text{-}\mathbf{Mod})$ or any subcategory that contains sufficiently many acyclic (e.g. injective) modules. This is not necessarily satisfied for $\mathbf{D}^-(\mathcal{C}(\Lambda))$. A statement such as

$$\mathbf{L}D \circ \Pi = [d] \circ \mathbf{R}\Gamma_{\underline{I}}$$

without further context hence does not make much sense: The implication would be that this would be an isomorphism of functors defined on $\mathbf{D}^b(\mathcal{C}(\Lambda))$, but $\mathbf{R}\Gamma_{\underline{I}}$ doesn't exist on $\mathbf{D}^b(\mathcal{C}(\Lambda))$.

Definition 5.2. Let G be a profinite group and A a discrete G-module. Denote with D the functor

$$D\colon A \longmapsto \varinjlim_U (A^U)^*$$

where $N^* = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$, the limit runs over the open normal subgroups of G with the dual of the corestriction being the transition maps (cf. [NSW08, II.5 and III.4]). D is right exact and contravariant and $D(A)$ has a continuous action of G *from the right.* Denote its left derivation in the derived category of discrete G-modules by

$$\mathbf{L}D\colon \mathbf{D}^+(\mathcal{D}(\widehat{\mathbb{Z}}[[G]])) \longrightarrow \mathbf{D}^-(\widehat{\mathbb{Z}}[[G]]^\circ\text{-}\mathbf{Mod})$$

(where $-^\circ$ denotes the opposite ring), so

$$D_i(A) = \mathbf{L}^{-i}D(A) = \varinjlim_U H^i(U, A)^*.$$

If G is a profinite group, R a profinite ring and A a discrete $R[[G]]$-module, then $D(A)$ is again an $R[[G]]$-module, so

$$\mathbf{L}D\colon \mathbf{D}^+(\mathcal{D}(R[[G]])) \longrightarrow \mathbf{D}^-(R[[G]]^\circ\text{-}\mathbf{Mod}),$$

where $\mathcal{D}(R[[G]])$ denotes the category of discrete $R[[G]]$-modules. Furthermore, we can of course also look at the functor

$$\mathbf{L}D\colon \mathbf{D}^+(R[[G]]\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^-(R[[G]]^\circ\text{-}\mathbf{Mod}).$$

Naturally, these functors don't necessarily coincide.

Proposition 5.3. *Let R be such a profinite ring, that the structure morphism $\widehat{\mathbb{Z}} \longrightarrow R$ gives it the structure of a finitely presented flat $\widehat{\mathbb{Z}}$-module. Let G be a profinite group such that $R[[G]]$ is a Noetherian local ring with finite residue field. (This is the case if G is a p-adic analytic group and R is the valuation ring of a finite extension over $\mathbb{Z}_p$.)*

Then an injective discrete $R[[G]]$-module is an injective discrete G-module.

Proof. By lemma 3.20 it suffices to show that $?\colon \mathcal{D}(R[[G]]) \longrightarrow \mathcal{D}(\widehat{\mathbb{Z}}[[G]])$ has an exact left adjoint. It is clear that

$$M \longmapsto R[[G]] \otimes_{\widehat{\mathbb{Z}}[[G]]} M$$

is an algebraic exact left adjoint, so it remains to show that $R \otimes_{\widehat{\mathbb{Z}}} M = R[[G]] \otimes_{\widehat{\mathbb{Z}}[[G]]} M$ is a discrete $R[[G]]$-module. Now M is the direct limit of finite modules, hence so is $R \otimes_{\widehat{\mathbb{Z}}} M$. But for a finite $R[[G]]$-module N this is clear as then $\mathfrak{m}^k M_i = 0$ for some k with $\mathfrak{m}$ the maximal ideal of $R[[G]]$ by Nakayama. ∎

Corollary 5.4. *The following diagram commutes if R is a finitely presented flat $\widehat{\mathbb{Z}}$-module with $R[[G]]$ Noetherian and local with finite residue field:*

$$\begin{array}{ccc}
\mathbf{D}^+(\mathcal{D}(R[[G]])) & \xrightarrow{\;?\;} & \mathbf{D}^+(\mathcal{D}(\widehat{\mathbb{Z}}[[G]])) \\
\Big\downarrow {\scriptstyle \mathbf{L}D} & & \Big\downarrow {\scriptstyle \mathbf{L}D} \\
\mathbf{D}^-(R[[G]]^\circ\text{-}\mathbf{Mod}) & \xrightarrow{\;?\;} & \mathbf{D}^-(\widehat{\mathbb{Z}}[[G]]^\circ\text{-}\mathbf{Mod})
\end{array}$$

Proof. Clearly the forgetful functors and D all commute on the level of categories of modules. The result then follows from proposition 5.3. ∎

Proposition 5.5 ([Lim12, corollary 3.1.6, proposition 3.1.8]). *Let M, N be Λ-modules.*

(1) *If M is Artinian, then $\Lambda \times M \longrightarrow M$ is continuous if we give M the discrete topology.*
(2) *If N is Noetherian, then N is compact if we give it the topology induced by Λ.*
(3) *The functor*

$$\mathbf{f.g.}\text{-}\Lambda\text{-}\mathbf{Mod} \longrightarrow \mathcal{C}(\Lambda)$$

maps projective objects to projectives.

Proposition 5.6. *Let $\mathcal{O}$ be a pro-p discrete valuation ring, $R = \mathcal{O}[[X_1, \dots, X_t]]$ with maximal ideal $\mathfrak{m}$, $G = \mathbb{Z}_p^d$ and $\Lambda = \varprojlim_i R[G/G^{p^i}]$ with generalized augmentation ideals $I_i = \ker \Lambda \longrightarrow R[G/G^{p^i}]$. Then the following holds in $\mathbf{D}_c^b(\Lambda\text{-}\mathbf{Mod})$:*

$$\mathbf{L}D \circ T \cong [d] \circ \mathbf{R}\Gamma_{\underline{I}},$$

Especially the following diagram commutes:

$$\begin{tikzcd}
\mathbf{D}_c^b(\Lambda\text{-}\mathbf{Mod}) \arrow[r, "\cong"] \arrow[d, hook] & \mathbf{D}^b(\mathbf{f.g.}\text{-}\Lambda\text{-}\mathbf{Mod}) \arrow[r] & \mathbf{D}^b(\mathcal{C}(\Lambda)) \arrow[r, "T"] & \mathbf{D}^b(\mathcal{D}(\Lambda^\circ)) \arrow[d, "\mathbf{L}D"] \\
\mathbf{D}^b(\Lambda\text{-}\mathbf{Mod}) \arrow[rrr, "{[d]\circ\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}}"] & & & \mathbf{D}^b(\Lambda\text{-}\mathbf{Mod})
\end{tikzcd}$$

Proof. Λ is a regular local ring with maximal ideal generated by $(\pi, X_1, \dots, X_t, \gamma_1 - 1, \dots, \gamma_d - 1)$ for any set of topological generators $(\gamma_i)_i$ of G and uniformizer π of $\mathcal{O}$. One immediately verifies that the sequences $\gamma_1^{p^i} - 1, \dots, \gamma_d^{p^i} - 1$ are again regular and generate the ideals I_i.

By proposition 4.10,

$$[d] \circ \mathbf{R}\Gamma_{\underline{I}} = \varinjlim_n T \circ \mathbf{R}\mathrm{Hom}_\Lambda(\Lambda/I_n, -) \circ T.$$

Take a bounded complex M of finitely generated R-modules that is quasi-isomorphic to a bounded complex P of finitely generated projective modules. The resulting complex $T(P)$ is then not only a bounded complex of injective discrete modules by Pontryagin duality and proposition 5.5, but also a bounded complex of injective abstract Λ-modules by lemma 3.5. In all relevant derived categories $T(M) \cong T(P)$ holds. As $\mathrm{Hom}_\Lambda(\Lambda/I_n, -) = (-)^{G^{p^n}}$ by construction, we can compute $[d] \circ \mathbf{R}\Gamma_{\underline{I}}(M)$

as follows (keeping corollary 4.9 in mind):

$$\begin{aligned}
[d] \circ \mathbf{R}\Gamma_{\underline{I}}(M) &= [d] \circ \mathbf{R}\Gamma_{\underline{I}}(P) \\
&= \varinjlim_n T \circ \mathbf{R}\mathrm{Hom}_\Lambda(\Lambda/I_n, -) \circ T(P) \\
&= \varinjlim_n T(\mathrm{Hom}_\Lambda(\Lambda/I_n, T(P))) \\
&= \varinjlim_n T(T(P)^{G^{p^n}}) \\
&= \varinjlim_n \Pi(T(P)^{G^{p^n}}) \\
&= D(T(P)) = \mathbf{L}D \circ T(M).
\end{aligned}$$

■

Lemma 5.7. *Let R be a commutative Noetherian ring with unit and $R \longrightarrow S$ a flat ring extension with R contained in the centre of S and S again (left-)Noetherian.*
Then

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^-(R\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^+(R\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^+(R\text{-}\mathbf{Mod})$$

extends to

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^-(R\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^+(S\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^+(S\text{-}\mathbf{Mod}),$$

which in turn restricts to

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^b_c(R\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^b_c(S\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^b_c(S\text{-}\mathbf{Mod}),$$

Proof. First note that if M is an R-left-module and N an S-left-module, then $\mathrm{Hom}_R(M, N)$ carries the structure of an S-left-module via $(sf)(m) = sf(m)$. Then $\mathrm{Hom}_R(R, S) \cong S$ as S-left-modules and the following diagram commutes:

$$\begin{array}{ccc}
S\text{-}\mathbf{Mod} & \xrightarrow{\mathrm{Hom}_R(M,-)} & S\text{-}\mathbf{Mod} \\
\downarrow{\scriptstyle ?} & & \downarrow{\scriptstyle ?} \\
R\text{-}\mathbf{Mod} & \xrightarrow{\mathrm{Hom}_R(M,-)} & R\text{-}\mathbf{Mod}
\end{array}$$

As S is a flat R-module, ? preserves injectives by lemma 3.20 and we can compute $\mathbf{R}\mathrm{Hom}_R(M, -)$ in either category.

If M is a finitely generated R-module and N a finitely generated S-module, then $\mathrm{Hom}_R(M, N)$ is again a finitely generated S-module, as S is left-Noetherian. If M is a bounded complex of finitely generated

R-modules, then it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. The result then follows at once. ■

Remark 5.8. Note however that $\mathbf{R}\mathrm{Hom}_R$ does *not* extend to a functor

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^-(S\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^+(R\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^+(S\text{-}\mathbf{Mod}).$$

Even in those cases where we can give $\mathrm{Hom}_R(M, A)$ the structure of an S-module (e. g. when S has a Hopf structure with antipode $s \longmapsto \overline{s}$ via $(sf)(m) = f(\overline{s}m)$), projective S-modules in general are not projective. This is specially true for $R[[G]]$, which is a flat, but generally not a projective R-module.

An essential ingredient in the proof of this section's main theorem is Grothendieck local duality. It is commonly stated as follows:

Theorem 5.9 (Local duality, [Har66, theorems V.6.2, V.9.1])**.** *Let R be a commutative regular local ring of dimension d with maximal ideal $\mathfrak{m}$, and $\mathcal{E}$ a fixed injective hull of the R-module $R/\mathfrak{m}$. Denote with $R[d]$ the complex concentrated in degree $-d$ with entry R. Then*

$$\mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \cong T \circ \mathbf{R}\mathrm{Hom}_R(-, R[d]) = [-d] \circ T \circ \mathbf{R}\mathrm{Hom}_R(-, R)$$

on $\mathbf{D}^b_c(R\text{-}\mathbf{Mod})$.

The regularity assumption on R can be weakened if one is willing to deal with a dualizing complex that is not concentrated in just one degree (loc. cit.). Relaxing commutativity however is more subtle and will be the focus of section 7.

Theorem 5.10. *Let $\mathcal{O}$ be a pro-p discrete valuation ring, $R = \mathcal{O}[[X_1, \dots, X_t]]$ with maximal ideal $\mathfrak{m}$, $G = \mathbb{Z}_p^s$ and $\Lambda = R[[G]]$. Then*

$$T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda) \cong [t+1] \circ \mathbf{R}\Gamma_{\mathfrak{m}} \circ \mathbf{L}D \circ T$$

on $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$. The right hand side can furthermore be expressed as

$$\mathbf{R}\Gamma_{\mathfrak{m}} \circ \mathbf{L}D \circ T \cong \varinjlim_k \mathbf{L}D \circ T \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -).$$

Proof. Λ is again a regular local ring, now of dimension $t + s + 1$. Denote its maximal ideal by $\mathfrak{M}$. By theorem 5.9

$$\mathbf{R}\Gamma_{\underline{\mathfrak{M}}} \cong T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda[s+t+1]) = [-s-t-1] \circ T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda).$$

Now $\mathfrak{M} = \mathfrak{m} + (\gamma_1 - 1, \dots, \gamma_s - 1)$ and if $x_1, \dots, x_{t+1}$ is a regular sequence in R, then $x_1, \dots, x_{t+1}, \gamma_1 - 1, \dots, \gamma_s - 1$ is a regular sequence in Λ.

Furthermore, the sequence $\gamma_1 - 1, \dots, \gamma_s - 1$ is of course itself regular in Λ. Let it generate the ideal I. Then we can apply proposition 3.27, i. e.,

$$\mathbf{R}\Gamma_{\mathfrak{M}} \cong \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ \mathbf{R}\Gamma_{\underline{I}}.$$

By proposition 5.6, we have $\mathbf{R}\Gamma_{\underline{I}} = [-s] \circ \mathbf{L}D \circ T$, which shows the first isomorphism.

Consider furthermore the functor $\varinjlim_k \mathbf{L}D \circ T \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -)$. By lemma 5.7 we can compute it on $\mathbf{D}^b_c(\Lambda\textbf{-Mod})$ as

$$\begin{aligned}
\varinjlim_k \mathbf{L}D \circ T \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -) &\cong \varinjlim_k [s] \circ \mathbf{R}\Gamma_{\underline{I}} \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -) \\
&\cong [s] \circ \mathbf{R}\Gamma_{\underline{I}} \circ \varinjlim_k \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -) \\
&= [s] \circ \mathbf{R}\Gamma_{\underline{I}} \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \\
&\cong [s] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ \mathbf{R}\Gamma_{\underline{I}} \\
&\cong [s] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ [-s] \circ \mathbf{L}D \circ T \\
&= \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ \mathbf{L}D \circ T,
\end{aligned}$$

as by proposition 3.23, local cohomology commutes with direct limits, $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$ and $\mathbf{R}\Gamma_{\underline{I}}$ commute by proposition 3.27, and by two applications of proposition 5.6. ∎

Remark 5.11. Proposition 5.6 and theorem 5.10 should together be compared with the duality diagram [Nek06, (0.4.4)].

Remark 5.12. If we express theorem 5.10 in terms of a spectral sequence, it looks like this:

$$\varinjlim_k \mathbf{L}^p D(T(\mathrm{Ext}^q_R(R/\mathfrak{m}^k, M))) \Longrightarrow T(\mathrm{Ext}^{t+1-p-q}_\Lambda(M, \Lambda)).$$

Writing $\mathrm{E}^\bullet_\Lambda$ for $\mathrm{Ext}^\bullet_\Lambda(-, \Lambda)$, flipping the sign of p and shifting $q \longmapsto t+1-q$ then yields

$$\varinjlim_k D_p(\mathrm{Ext}^{t+1-q}_R(R/\mathfrak{m}^k, M)^\vee) \Longrightarrow \mathrm{E}^{p+q}_\Lambda(M)^\vee$$

and the following exact five term sequence:

$$\begin{tikzcd}
\mathrm{E}^2_\Lambda(M)^\vee \arrow[r] & \varinjlim_k D_2(\mathrm{Ext}^{t+1}_R(R/\mathfrak{m}^k, M)^\vee) \arrow[r] & \varinjlim_k D(\mathrm{Ext}^{t}_R(R/\mathfrak{m}^k, M)^\vee) \arrow[dll, out=0, in=180] \\
\mathrm{E}^1_\Lambda(M)^\vee \arrow[r] & \varinjlim_k D_1(\mathrm{Ext}^{t+1}_R(R/\mathfrak{m}^k, M)^\vee) \arrow[r] & 0
\end{tikzcd}$$

§6. Iwasawa Adjoints

In this section let R be a pro-p commutative local ring with maximal ideal $\mathfrak{m}$ and residue field of characteristic p. Let G be a compact p-adic Lie group and $\Lambda = \Lambda(G) = \varprojlim_U R[[G/U]]$, where U ranges over the open normal subgroups of G. As is customary, we set again $\mathrm{E}^\bullet_\Lambda(M) = \mathrm{Ext}^\bullet_\Lambda(M, \Lambda)$.

Remark 6.1. Note that if M is a left Λ-module, it also has an operation of Λ from the right given by $mg = g^{-1}m$. This of course does not give M the structure of a Λ-bimodule, as the actions are not compatible. We can however still give $\mathrm{Hom}_\Lambda(M, \Lambda)$ the structure of a left Λ-module by $(g.\varphi)(m) = \varphi(m)g^{-1}$.

The following lemma is based on an observation in the proof of [Jan89, theorem 2.1].

Lemma 6.2. $\mathrm{E}^0_\Lambda(M) = \varprojlim_U \mathrm{Hom}_R(M_U, R)$ *for finitely generated* Λ*-modules* M*, where the transition map for a pair of open normal subgroups* $U \leq V$ *are given by the dual of the trace map*

$$M_V \longrightarrow M_U, m \longmapsto \sum_{g \in V/U} gm.$$

Proof. Note first that as $\mathrm{Hom}_\Lambda(M, -)$ commutes with projective limits,

$$\mathrm{Hom}_\Lambda(M, \Lambda) = \varprojlim_U \mathrm{Hom}_\Lambda(M, R[G/U]) = \varprojlim_U \mathrm{Hom}_{R[G/U]}(M_U, R[G/U]).$$

For U an open normal subgroups of G, consider the trace map

$$\begin{aligned} \mathrm{Hom}_R(M_U, R) &\longrightarrow \mathrm{Hom}_{R[G/U]}(M_U, R[G/U]) \\ \varphi &\longmapsto \left(m \longmapsto \sum_{g \in G/U} \varphi(g^{-1}m) \cdot g \right), \end{aligned}$$

which is clearly an isomorphism of R-modules and induces the required isomorphism to the projective system mentioned in the proposition. ■

Proposition 6.3. *On* $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$ *we have*

$$\Pi \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda) \cong \varinjlim_U \Pi \circ \mathbf{R}\mathrm{Hom}_R(-, R) \circ \mathbf{L}(-)_U.$$

Proof. Immediate by lemma 6.2, as $(-)_U$ clearly maps finitely generated free Λ-modules to finitely generated free R-modules. ■

Remark 6.4. The spectral sequence attached to proposition 6.3 looks like this:

$$\varinjlim_U \mathrm{Ext}^p_R(H_q(U,M),R)^\vee \Longrightarrow \mathrm{E}^{p+q}_\Lambda(M)^\vee$$

Its five term exact sequence is given by

$$\mathrm{E}^2_\Lambda(M)^\vee \longrightarrow \varinjlim_U \mathrm{Ext}^2_R(M_U,R)^\vee \longrightarrow \varinjlim_U \mathrm{Hom}_R(H_1(U,M),R)^\vee \longrightarrow \mathrm{E}^1_\Lambda(M)^\vee \longrightarrow \varinjlim_U \mathrm{Ext}^1_R(M_U,R)^\vee \longrightarrow 0$$

The following lemma is also based on an observation in the proof of [Jan89, theorem 2.1].

Lemma 6.5. $\mathrm{Hom}_\Lambda(M,\Lambda)^\vee \cong \varinjlim_U R^\vee \otimes_R M_U$ *for finitely generated* Λ*-modules* M*.*

Proof.

$$\begin{aligned}\mathrm{Hom}_\Lambda(M,\Lambda)^\vee &\cong \varinjlim_U \Pi(\mathrm{Hom}_R(M_U,R))\\ &\cong \varinjlim_U \Pi\,\mathrm{Hom}_R(\Pi(R),\Pi(M_U))\\ &\cong \Pi\circ\Pi(\varinjlim_U M_U \otimes_R \Pi(R))\\ &\cong \varinjlim M_U \otimes_R R^\vee.\end{aligned}$$

■

Proposition 6.6. $\Pi \circ \mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda) \cong \left(R^\vee \otimes^{\mathbf{L}}_R -\right) \circ \mathbf{L}D \circ \Pi$ *on* $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$*.*

Proof. Using lemmas 6.2 and 6.5, usual Pontryagin duality, and the fact that tensor products commute with direct limits, we get:

$$\begin{aligned}\mathrm{Hom}_\Lambda(M,\Lambda)^\vee &\cong \varinjlim_U R^\vee \otimes_R M_U\\ &\cong R^\vee \otimes_R \varinjlim \Pi(\Pi(M)^U)\\ &= R^\vee \otimes_R D(\Pi(M)).\end{aligned}$$

It hence remains to show that $(D\circ\Pi)$ maps projective objects to $R^\vee \otimes_R -$-acyclics and it actually suffices to check this for the module Λ. But $D(\Pi(\Lambda)) = \varinjlim_U R[G/U]$ is clearly $R^\vee \otimes_R -$-acyclic. ■

Remark 6.7. The spectral sequence attached to proposition 6.6 looks like this:

$$\operatorname{Tor}_p^R(R^\vee, D_q(M^\vee)) \Longrightarrow \mathrm{E}_\Lambda^{p+q}(M)^\vee,$$

which yields the following five term exact sequence:

$$\begin{tikzcd}
\mathrm{E}_\Lambda^2(M)^\vee \arrow[r] & \operatorname{Tor}_2^R(R^\vee, D(M^\vee)) \arrow[r] & R^\vee \otimes_R D_1(M^\vee) \arrow[dll, out=0, in=180] \\
\mathrm{E}_\Lambda^1(M)^\vee \arrow[r] & \operatorname{Tor}_1^R(R^\vee, D(M^\vee)) \arrow[r] & 0
\end{tikzcd}$$

This also proves that $\mathrm{E}_\Lambda^p(M) = 0$ if $p > \dim G + \dim R$. If $\dim R = 1$, the spectral sequence degenerates and we can compute $\mathrm{E}_\Lambda^p(\operatorname{tor}_R M)$ and $\mathrm{E}_\Lambda^p(M/\operatorname{tor}_R M)$ akin to [NSW08, (5.4.13)]. The spectral sequence for $R = \mathbb{Z}_p$ first appeared in [Jan89, theorem 2.1].

Lemma 6.8. $R^\vee \cong \varinjlim_k R/\mathfrak{m}^{(k)}$ *if R is regular.*

Proof. R satisfies local duality by assumption, hence by corollaries 4.9 and 3.16 $R^\vee = T(R) \cong \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}(R) = \varinjlim_k \operatorname{Ext}_R^d(R/\mathfrak{m}^{(k)}, R) \cong \varinjlim_k R/\mathfrak{m}^{(k)}$. ∎

Lemma 6.9. $\operatorname{Hom}_\Lambda(M,\Lambda)^\vee \cong \varinjlim_{U,k}(M/\mathfrak{m}^{(k)})_U$ *for finitely generated Λ-modules M and regular R.*

Proof.

$$\begin{aligned}
\operatorname{Hom}_\Lambda(M,\Lambda)^\vee &\cong R^\vee \otimes_R \varinjlim_U M_U \\
&\cong \varinjlim_U \varinjlim_k R/\mathfrak{m}^{(k)} \otimes_R M_U \\
&\cong \varinjlim_U \varinjlim_k (M/\mathfrak{m}^{(k)} M)_U
\end{aligned}$$

using lemmas 6.5 and 6.8. ∎

Proposition 6.10. *If R is regular,* $\Pi \circ \mathbf{R}\operatorname{Hom}_\Lambda(-,\Lambda) \cong \varinjlim_k \mathbf{L}D \circ \Pi \circ \left(R/\mathfrak{m}^{(k)} \otimes_R^{\mathbf{L}} -\right) \cong \varinjlim_k \mathbf{L}D \circ \Pi \circ [d] \circ \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{m}^{(k)}, -)$ *on* $\mathbf{D}_c^b(\Lambda\text{-}\mathbf{Mod})$.

Proof. By lemma 6.9

$$\begin{aligned}\mathrm{Hom}_\Lambda(M,\Lambda)^\vee &\cong \varinjlim_U \varinjlim_k (M/\mathfrak{m}^{(k)}M)_U \\ &\cong \varinjlim_{U,k} \Pi(\Pi(M/\mathfrak{m}^{(k)}M)^U) \\ &= \varinjlim_k D(\Pi(R/\mathfrak{m}^{(k)} \otimes_R M)).\end{aligned}$$

By proposition 3.14 it suffices to show that $(\Lambda/\mathfrak{m}^{(k)})^\vee$ is D-acyclic. But

$$\mathbf{L}^{-i}D((\Lambda/\mathfrak{m}^{(k)})^\vee) = \varinjlim_U H^i(U, R/\mathfrak{m}^{(k)}[[G]]^\vee)^* = \varinjlim_U H_i(U, R/\mathfrak{m}^{(k)}[[G]]),$$

which is zero for $i > 0$ by Shapiro's Lemma.

The other isomorphism now follows from proposition 3.14. ■

Remark 6.11. Writing D_p for $\mathbf{L}^{-p}D$, the spectral sequences attached to proposition 6.10 look like this:

$$\varinjlim_k D_p(\mathrm{Tor}_q^R(R/\mathfrak{m}^{(k)}, M)^\vee) \Longrightarrow \mathrm{E}_\Lambda^{p+q}(M)^\vee$$

and

$$\varinjlim_k D_p(\mathrm{Ext}_R^{d-q}(R/\mathfrak{m}^{(k)}, M)^\vee) \Longrightarrow \mathrm{E}_\Lambda^{p+q}(M)^\vee$$

with exact five term sequences

$$\begin{aligned}&\mathrm{E}_\Lambda^2(M)^\vee \longrightarrow \varinjlim_k D_2((M/\mathfrak{m}^{(k)}M)^\vee) \longrightarrow \varinjlim_k D(\mathrm{Tor}_1^R(R/\mathfrak{m}^{(k)}, M)^\vee) \\ &\longrightarrow \mathrm{E}_\Lambda^1(M)^\vee \longrightarrow \varinjlim_k D_1((M/\mathfrak{m}^{(k)}M)^\vee) \longrightarrow 0\end{aligned}$$

and

$$\begin{aligned}&\mathrm{E}_\Lambda^2(M)^\vee \longrightarrow \varinjlim_k D_2(\mathrm{Ext}_R^d(R/\mathfrak{m}^{(k)}, M)^\vee) \longrightarrow \varinjlim_k D(\mathrm{Ext}_R^{d-1}(R/\mathfrak{m}^{(k)}, M)^\vee) \\ &\longrightarrow \mathrm{E}_\Lambda^1(M)^\vee \longrightarrow \varinjlim_k D_1(\mathrm{Ext}_R^d(R/\mathfrak{m}^{(k)}, M)^\vee) \longrightarrow 0\end{aligned}$$

respectively. For $R = \mathbb{Z}_p$, these appear in the proof of [Jan89, theorem 2.1].

Lemma 6.12 ([Lim12, proposition 3.1.7])**.** *Let M be a finitely generated Λ-module. Then $M \cong \varprojlim_k M/\mathfrak{M}^k M$ algebraically and topologically.*

Lemma 6.13. $\Pi \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^{(k)}, -)$ *maps bounded complexes of Λ-modules with finitely generated cohomology to bounded complexes whose cohomology modules are discrete p-torsion G-modules. If M is a complex in $\varinjlim_U \mathbf{D}^b_c(R[G/U]\text{-}\mathbf{Mod})$, then all cohomology groups of*

$$\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^{(k)}, M)^\vee$$

are furthermore finite.

Proof. The groups $\mathrm{Ext}^q_R(R/\mathfrak{m}^{(k)}, M)$ for M finitely generated over Λ are clearly p-torsion and finitely generated as Λ-modules, hence compact by lemma 6.12, and consequentially topologically profinite and pro-p. Their Pontryagin duals are thus discrete p-torsion G-modules.

If M is finitely generated over some $R[G/U]$, it is also finitely generated over R and $\mathrm{Ext}^q_R(R/\mathfrak{m}^{(k)}, M)$ finitely generated over $R/\mathfrak{m}^{(k)}$, hence finite. ∎

Proposition 6.14. *Assume that R is regular. Let G be a duality group (cf. [NSW08, (3.4.6)]) of dimension s at p. Then*

$$\begin{aligned} \mathrm{E}^m_\Lambda(M)^\vee &\cong \varinjlim_k \mathbf{L}^{-s} D\, \mathrm{Ext}^{d-(m-s)}_R(R/\mathfrak{m}^{(k)}, M)^\vee \\ &\cong \varinjlim_k \mathbf{L}^{-s} D\, \mathrm{Tor}^R_{m-s}(R/\mathfrak{m}^{(k)}, M)^\vee. \end{aligned}$$

for finitely generated $R[G/U]$-modules M. Especially $\Pi \circ \mathrm{E}^s_\Lambda$ is then right-exact.

Proof. As G is a duality group of dimension s at p, the complex $\mathbf{L}D(M')$ has trivial cohomology outside of degree $-s$ if M' is a finite discrete p-torsion G-module. Together with lemma 6.13 this implies that the spectral sequence attached to proposition 6.10 degenerates and gives

$$\mathrm{E}^m_\Lambda(M)^\vee \cong \varinjlim_k \mathbf{L}^{-s} D\, \mathrm{Ext}^{d-(m-s)}_R(R/\mathfrak{m}^{(k)}, M)^\vee.$$

The other isomorphism follows with exactly the same argument. Note furthermore that as $\dim R = d$, $\mathrm{Ext}^{d-(m-s)}_R(R/\mathfrak{m}^{(k)}, M) = 0$ if $m-s < 0$, hence $\mathrm{E}^m_\Lambda(M)^\vee = 0$ if $m < s$. ∎

Theorem 6.15. *Assume that R is regular and that G is a Poincaré group at p of dimension s with dualizing character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ (i. e., $\varinjlim_\nu D_s(\mathbb{Z}/p^\nu) \cong \mathbb{Q}_p/\mathbb{Z}_p(\chi)$, cf. [NSW08, (3.7.1)]), which gives rise to* *the "twisting functor"* $\boldsymbol{\chi}\colon M \longmapsto M(\chi)$. *Assume that* R *is a commutative complete Noetherian ring of global dimension* d *with maximal ideal* $\mathfrak{m}$. *Then*

$$T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda) = \boldsymbol{\chi} \circ [d+s] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$$

on $\varinjlim_U \mathbf{D}^b_c(R[G/U]\text{-}\mathbf{Mod})$.

Proof. Let $I = \mathbb{Q}_p/\mathbb{Z}_p(\chi) = \boldsymbol{\chi}(\mathbb{Q}_p/\mathbb{Z}_p)$ be the dualizing module of G. For any p-torsion G-module A we have by [NSW08, (3.7)] that $\mathbf{L}^{-s}D(A) = \varinjlim_U H^s(U,A)^* \cong \varinjlim_U \mathrm{Hom}_{\mathbb{Z}_p}(A,I)^U = \mathrm{Hom}_{\mathbb{Z}_p}(A,I)$, as I is also a dualizing module for every open subgroup of G.

Note that

$$H^0(\boldsymbol{\chi} \circ [d] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}}) = \boldsymbol{\chi} \circ \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}$$

and that $\boldsymbol{\chi} \circ \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}$ is hence right-exact. Note furthermore that

$$\begin{aligned}
H^0([-s] \circ T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda)) &= \mathrm{E}^s(-)^\vee \\
&\cong \varinjlim_k \mathbf{L}^{-s} D\, \mathrm{Ext}^d_R(R/\mathfrak{m}^{(k)}, -)^\vee \\
&\cong \varinjlim_k \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ext}^d_R(R/\mathfrak{m}^{(k)}, -)^\vee, I) \\
&\cong \boldsymbol{\chi} \circ \varinjlim_k \mathrm{Ext}^d_R(R/\mathfrak{m}^{(k)}, -) \\
&= \boldsymbol{\chi} \circ \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}
\end{aligned}$$

using proposition 6.14 and Pontryagin duality.

By [Har66, proposition I.7.4] the left derivation of $\mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}$ is $[d] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$: The complex $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}(R)$ is concentrated in degree d and hence every module is a quotient of a module with this property, as local cohomology commutes with arbitrary direct limits. ∎

Note that even though this theorem suspiciously looks like local duality, the local cohomology on the right hand side is with respect to the maximal ideal of the coefficient ring, not the whole Iwasawa algebra. The local duality result is subject of the next section.

We end this section with a generalization of [Jan89, corollary 2.6], where $R = \mathbb{Z}_p$ was considered.

Corollary 6.16. *In the setup of theorem 6.15 assume that* M *is a finitely generated* $R[G/U]$*-module. Then the following hold:*

(1) *If* M *is free over* R*, then*

$$\mathrm{E}^q_\Lambda(M)^\vee \cong \begin{cases} M \otimes_R R^\vee(\chi) & \text{if } q = s \\ 0 & \text{else} \end{cases}$$

(2) *If M is R-torsion, then* $\mathrm{E}^q_\Lambda(M) = 0$ *for all* $q \leq s$.
(3) *If M is finite, then*

$$\mathrm{E}^q_\Lambda(M)^\vee \cong \begin{cases} M(\chi) & \text{if } q = d+s \\ 0 & \text{else} \end{cases}$$

Proof. By theorem 6.15, we have

$$\mathrm{E}^q_\Lambda(M)^\vee \cong \mathbf{R}^{d+s-q}\Gamma_{\underline{\mathfrak{m}}}(M)(\chi)$$

in any case.

In the first case, this is just $M \otimes_R R^\vee$ for $q = s$ and zero else. In the second case, local duality yields $\mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}(M) \cong \mathrm{Hom}_R(M, R)^\vee = 0$. In the third case, we note that M has an injective resolution by modules that are the direct limit of finite modules. Proposition 3.23 together with proposition 3.22 then imply the result. ∎

§7. Local Duality for Iwasawa Algebras

This section gives a streamlined proof of a local duality result for Iwasawa algebras as first published in [Ven02] and generalizes the result to more general coefficient rings. Let G be a pro-p Poincaré group of dimension s with dualizing character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ and R a commutative Noetherian pro-p regular local ring with maximal ideal $\mathfrak{m}$ of global dimension d. Set $\Lambda = R[[G]]$, which is of global dimension $r = d + s$.

Proposition 7.1. $\mathbf{R}\Gamma_{\underline{\mathfrak{M}}}(\Lambda) \cong \Lambda^\vee[-d-s]$ *and* $\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M}^l, \Lambda) \cong \mathbf{R}\Gamma^{d+s-i}_{\underline{\mathfrak{m}}}(\Lambda/\mathfrak{M}^l)(\chi)$.

Proof. By proposition 6.14,

$$\begin{aligned} \mathbf{R}^i\Gamma_{\underline{\mathfrak{M}}}(\Lambda) &= \varinjlim_l \mathrm{E}^i(\Lambda/\mathfrak{M}^l) \\ &\cong \varinjlim_l \left(\varinjlim_k \mathbf{L}^{-s} D(\mathrm{Ext}^{d-(i-s)}_R(R/\mathfrak{m}^{(k)}, \Lambda/\mathfrak{M}^l)^\vee) \right)^\vee. \end{aligned}$$

As in the proof of theorem 6.15, we can express

$$\mathbf{L}^{-s} D(\mathrm{Ext}^{d-(i-s)}_R(R/\mathfrak{m}^{(k)}, \Lambda/\mathfrak{M}^l)^\vee)$$

as

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ext}^{d-(i-s)}_R(R/\mathfrak{m}^{(k)}, \Lambda/\mathfrak{M}^l)^\vee, \mathbb{Q}_p/\mathbb{Z}_p(\chi)),$$

which we again see is isomorphic to

$$\mathrm{Ext}^{d-(i-s)}_R(R/\mathfrak{m}^{(k)}, \Lambda/\mathfrak{M}^l)(\chi).$$

In the direct limit over k this becomes $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}^{d-(i-s)}(\Lambda/\mathfrak{M}^l)(\chi)$.

$\Gamma_{\underline{\mathfrak{m}}}$ restricted to the subcategory of finite Λ- (or R-)modules is the identity. As $\Gamma_{\underline{\mathfrak{m}}}$ commutes with arbitrary direct limits, this is also true for the category of discrete Λ-modules. As the latter category contains sufficiently many injectives, $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}(N) = N$ if N is a complex of discrete Λ-modules.

Now $\Lambda/\mathfrak{M}^l$ is such a finite module, hence

$$\mathbf{R}\Gamma_{\underline{\mathfrak{M}}}(\Lambda) = [-d-s] \circ \varinjlim_{l} (\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}(\Lambda/\mathfrak{M}^l)(\chi))^\vee = \Lambda(\chi)^\vee[-r].$$

The proposition now follows at once if we observe that $\Lambda \cong \Lambda(\chi)$ as a Λ-module via $g \longmapsto \chi(g)g$. ∎

Theorem 7.2 (Local duality for Iwasawa algebras)**.**

$$\mathbf{R}\Gamma_{\underline{\mathfrak{M}}} \cong [-r] \circ \Pi \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda)$$

on $\mathbf{D}_c^b(\Lambda)$.

Proof. Because of proposition 7.1, this follows verbatim as in [Har67, theorem 6.3]: The functors $\mathbf{R}^r\Gamma_{\underline{\mathfrak{M}}}$ and $\mathrm{Hom}_\Lambda(-, \Lambda)^\vee$ are related by a pairing of Ext-groups, are both covariant and right-exact and agree on Λ, hence also agree on finitely generated modules. As the complex $\mathbf{R}\Gamma_{\underline{\mathfrak{M}}}(\Lambda)$ is concentrated in degree r, the same argument as in theorem 6.15 shows that the left derivation of $\mathbf{R}^r\Gamma_{\underline{\mathfrak{M}}}$ is just $[r] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{M}}}$ and the result follows. ∎

§8. Torsion in Iwasawa Cohomology

There are notions of both local and global Iwasawa cohomology. Our result about their torsion below holds in both cases and we will first deal with the local case.

In both subsections, R is a commutative Noetherian pro-p local ring of residue characteristic p.

8.1. Torsion in Local Iwasawa Cohomology

Let K be a finite extension of $\mathbb{Q}_p$ and $K_\infty|K$ a Galois extension with an analytic pro-p Galois group G without elements of finite order. Let T be a finitely generated $\Lambda = R[[G]]$-module and set $A = T \otimes_R R^\vee$. Due to Lim and Sharifi we have the following spectral sequence (stemming from an isomorphism of complexes in the derived category). Write $H^i_{\mathrm{Iw}}(K_\infty, T) = \varprojlim_{K'} H^i(G_{K'}, T)$ where the limit is taken with respect to the corestriction maps over all finite field extensions $K'|K$

contained in K_∞, and where we denote by G_L the absolute Galois group of a field L.

Theorem 8.1. *There is a convergent spectral sequence*

$$\mathrm{E}^i_\Lambda(H^j(G_{K_\infty}, A)^\vee) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K, T).$$

Proof. This is [LS13, theorem 4.2.2, remark 4.2.3], which generalizes a local version of the main result of [Jan14] to more general coefficients. ∎

Theorem 8.2. *If G is a pro-p Poincaré group of dimension $s \geq 2$ with dualizing character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ and if R is regular, then*

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) = 0.$$

If $s = 1$, then

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) \cong \operatorname{Hom}_R((T^*)_G, R)(\chi^{-1}),$$

where $T^* = \operatorname{Hom}_R(T, R)$.

Proof. The exact five-term sequence attached to the spectral sequence of theorem 8.1 starts like this:

$$\begin{array}{l} 0 \longrightarrow \mathrm{E}^1_\Lambda(H^0(G_{K_\infty}, A)^\vee) \longrightarrow H^1_{\mathrm{Iw}}(K_\infty, T) \longrightarrow \mathrm{E}^0_\Lambda(H^1(G_{K_\infty}, A)^\vee) \\ \hfill \downarrow \quad \\ \hfill \mathrm{E}^2_\Lambda(H^0(G_{K_\infty}, A)^\vee) \end{array}$$

Note that $\mathrm{E}^2_\Lambda(M)$ is pseudo-null and hence Λ-torsion for every finitely generated module M, which follows from the spectral sequence attached to the isomorphism $\mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda) \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda) \cong \mathrm{id}$ on $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$. Furthermore $\mathrm{E}^0_\Lambda(M)$ is always Λ-torsion free, as Λ is integral. It follows that $\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) \subseteq \mathrm{E}^1_\Lambda(H^0(G_{K_\infty}, A)^\vee)$. As the latter is Λ-torsion,

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) = \mathrm{E}^1_\Lambda(H^0(G_{K_\infty}, A)^\vee).$$

The result now follows immediately from corollary 6.16. ∎

8.2. Torsion in Global Iwasawa Cohomology

Let K be a finite extension of $\mathbb{Q}$ and S a finite set of places of K. Let K_S be the maximal extension of K which is unramified outside S and $K_\infty|K$ a Galois extension contained in K_S. Suppose that $G = G(K_\infty|K)$ is an analytic pro-p group without elements of finite order. Let T be a finitely generated $\Lambda = R[[G]]$-module and set $A = T \otimes_R$

$R^\vee$. Due to Lim and Sharifi we have the following spectral sequence (stemming from an isomorphism of complexes in the derived category). Write $H^i_{\mathrm{Iw}}(K_\infty, T) = \varprojlim_{K'} H^i(G(K_S|K'), T)$ where the limit is taken with respect to the corestriction maps over all finite field extensions $K'|K$ contained in K_∞.

Theorem 8.3. *There is a convergent spectral sequence*

$$\mathrm{E}^i_\Lambda(H^j(G(K_S|K_\infty), A)^\vee) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K_\infty, T).$$

Proof. This follows from [LS13, theorem 4.5.1], which generalizes the main result of [Jan14] to more general coefficients. ∎

From this we derive the following.

Theorem 8.4. *If G is a pro-p Poincaré group of dimension $s \geq 2$ with dualizing character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ and if R is regular, then*

$$\mathrm{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) = 0.$$

If $s = 1$, then

$$\mathrm{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) \cong \mathrm{Hom}_R((T^*)_G, R)(\chi^{-1}),$$

where $T^* = \mathrm{Hom}_R(T, R)$.

Proof. Replace "G_{K_∞}" with "$G(K_S|K_\infty)$" in the proof of theorem 8.2. ∎

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