A NOTE ON DETERMINANT FUNCTORS AND SPECTRAL SEQUENCES

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Abstract. The aim of this note is to prove certain compatibilities of determinant functors with spectral sequences and (co)homology thereby extending results of [3] and refining a description in [9]. It turns out that the determinant behaves as well as one would have expected in this regard, only that we were not able to find references for it in the literature. The results are crucial for descent calculations in the context of Iwasawa theory [12] or Equivariant Tamagawa Number Conjectures [4, 5, 6].

1. Introduction

As a generalisation or rather refinement of Euler-Poincare characteristics determinant functors have been first studied by Knudsen and Mumford [10] in the context of perfect complexes of $\mathcal{O}_X$-modules on a scheme X. The theory has been generalised by Deligne [7] and Knudsen [9] to determinants on exact categories with values in commutative Picard categories. Apart from applications in algebraic geometry determinant functors are a fundamental technical tool in Burns and Flach's formulation of Equivariant Tamagawa Number Conjectures [4, 5, 6], they have been used in Iwasawa theory by Kato, Perrin-Riou and Fontaine, Huber and Kings. In particular, for an associative ring R with unit, Fukaya and Kato present an adhoc construction of determinant functors on perfect complexes of R-modules in [8]. Witte used yet another approach in his thesis [15, 14], see also the nice survey article [11]. Determinant functors on triangulated categories have been constructed by Breuning [2], while the present note is strongly based on [3], see also [1].

Our motivation for this note stems from descent calculations in Iwasawa theory involving determinant functors. While from the theoretical point of view it is quite elegant to work with complexes, e.g. representing Galois cohomology like $R\Gamma(G,\mathbb{T})$ where G is the absolute Galois cohomology say of a global or local field and $\mathbb{T}$ denotes a big Galois representation, i.e., a module over some Iwasawa algebra with an additional compatible action by G, it often turns out in praxis that explicit descent calculations involve cohomology groups $\mathrm{H}^i(G,\mathbb{T})$ instead. Thus it seems crucial to the author to clarify how to compare the descent, i.e., the application of some derived or hyper Tor-functor to the complex and cohomology groups, respectively. From the technical point of view it concerns the question how the corresponding Tor-spectral sequence (3.34) behaves with respect to determinants and to the functor H which associates to a complex its (co)homology, see (3.35) subsection 3.1. In order to obtain an adequate statement we

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first recall and develop a general description for determinants in spectral sequences in section 2.

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2. Determinants and spectral sequences

2.1. **Spectral sequences.** Let $C = (C^n, \partial^n)_{n\in\mathbb{Z}}$ be a filtered cochain complex with values in some abelian category $\mathcal{E}$, i.e., an object in the category of cochain complexes together with a decreasing filtration

$$F: \quad \ldots \supseteq F^i(C) \supseteq F^{i+1}(C) \supseteq \ldots$$

within this category. We shall assume henceforth that the filtration is bounded, i.e., $F^t(C) = C$ and $F^s(C) = 0$ for some $s, t \in \mathbb{Z}$, as well as that the complex C itself is bounded - then we shall say for simplicity that C is a *bounded filtered complex*. We will now recall some basic facts and notation concerning the associated convergent (weakly convergent, biregular) cohomological spectral sequence

$$E_r^{\cdot,\cdot} = (E_r^{pq})_{pq} \Rightarrow \mathrm{H}^{p+q}(C), \qquad\qquad r \geq 0$$

with a specific viewpoint from the perspective of determinants. In particular it is often useful to consider a spectral sequence as a complex with respect to its total degree

$$E_r^{\cdot} = (E_r^n, \partial^n), \;\; E_r^n := \bigoplus_p E_r^{p,n-p}, \;\; \partial_r^n := \bigoplus \partial_r^{p,n-p}.$$

Recall that we have the canonical identification

$$E_0^{pq} = F^pC^{p+q}/F^{p+1}C^{p+q}, \;\; E_0^n = \bigoplus_p F^pC^n/F^{p+1}C^n$$

and that the differentials ∂_r^{pq} are induced from ∂^n of C. In the sequel we shall need a couple of different filtrations. To this end we define

$$\begin{aligned} A_r^{pq} : &= \ker(F^pC^{p+q} \to C^{p+q+1}/F^{p+r}C^{p+q+1}) = F^pC^{p+q} \cap \partial^{-1}(F^{p+r}C^{p+q+1}),\\ B_r^{pq} : &= \partial A_{r-1}^{p-r+1,q+r-2} \end{aligned}$$

and identify

$$E_r^{pq} = (A_r^{pq} + F^{p+1}C^{p+q})/(B_r^{pq} + F^{p+1}C^{p+q}) \cong A_r^{pq}/(B_r^{pq} + A_{r-1}^{p+1}). \tag{2.1}$$

The differentials of C induce differentials on A_r and E_r of degree r such that the following diagram is commutative

$$\begin{tikzcd} C^{p+q} \arrow[r, "\partial"] & C^{p+q+1} \\ A_r^{pq} \arrow[u, hook] \arrow[r, "\partial"] \arrow[d] & A_r^{p+r,q-r+1} \arrow[u, hook] \arrow[d] \\ E_r^{pq} \arrow[r, "\partial"] & E_r^{p+r,q-r+1}. \end{tikzcd}$$

Consider also the cycle complex $Z = Z(C)$ and boundary complex $B = B(C)$ with their induces filtration from C as well as the cohomology complex $H = \mathrm{H}(C)$ (all with trivial differentials) with induced filtration via the first of the two canonical complexes

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H \longrightarrow 0 \tag{2.2}$$

and

$$0 \longrightarrow Z \longrightarrow C \longrightarrow B[1] \longrightarrow 0\ . \tag{2.3}$$

Here, if C is a (cochain) complex, then $C[1]$ denotes the shifted one, i.e., $C[1]^i = C^{i+1}$ with differential $\partial_{C[1]} = -\partial_C$. In other words

$$F^p\mathrm{H}^n(C) = F^pZ + B^n/B^n \cong F^pZ^n/F^pB^n$$

and

$$\mathrm{gr}_pH^n = F^pH^n/F^{p+1}H^n \cong F^pZ^n + B^n/F^{p+1}Z^n + B^n \cong E_\infty^{pq},\ \ p+q=n. \tag{2.4}$$

As suggested by Knudsen in [9, §3] the derived filtrations $F_r^i(C) = DF_r^i = DF_r^i(C)$, $r \geq 0$, associated to $F(C)$ are probably the most appropriate ones in the context of determinants and spectral sequences. They are given as follows

$$\xrightarrow{\partial} F_r^i(C^n) := A_r^{p,q} \xrightarrow{\partial} F_r^i(C^{n+1}) = A_r^{p+r,q-r+1} \xrightarrow{\partial}$$

where the indices transform always as

$$p = i + nr, \quad q = n(1-r) - i, \quad n = p+q.$$

Note that $F_0(C) = F(C)$. We write $gr_r^i(C) := gr_{F_r}^i(C)$ for the associated graded complexes

$$\xrightarrow{\partial} gr_r^i(C^n) := F_r^i(C^n)/F_r^{i+1}(C^n) \xrightarrow{\partial} gr_r^i(C^{n+1}) = F_r^i(C^{n+1})/F_r^{i+1}(C^{n+1}) \xrightarrow{\partial}$$

and set

$$gr_r(C) := \bigoplus_i gr_r^i(C).$$

Then according to [9, prop. 3.5] it is easily checked that there is a canonical quasi-isomorphism

$$q_r : gr_r(C) \to E_r^{\cdot}$$

of complexes which is induced by the natural projections

$$A_r^{p,n-p}/A_r^{p+1,n-p-1} \twoheadrightarrow A_r^{p,n-p}/(B_r^{p,n-p} + A_{r-1}^{p+1,n-p-1})$$

of the summands of the individual objects. In particular, there is a canonical isomorphism

$$\mathrm{H}(q_r) : \mathrm{H}(gr_r(C)) \cong (E_{r+1}^{\cdot}, 0)$$

where both complexes are endowed with trivial differentials.

For later purposes we calculate the kernel of q_r or rather its ith component

$$K_r^i := \ker\left(gr_r^i(C) \to E_r^{i+\cdot r, \cdot(1-r)-i}\right)$$

such that $\ker(q_r) = \bigoplus_i K_r^i$. Then K_r^i is acyclic and looks like

(2.5)

$$(B_r^{p-r} + A_{r-1}^{p-r+1})/A_r^{p-r+1} \xrightarrow{\partial} (B_r^p + A_{r-1}^{p+1})/A_r^{p+1} \xrightarrow{\partial} (B_r^{p+r} + A_{r-1}^{p+r+1})/A_r^{p+r+1}$$

where the middle term is put in degree n and where we omit now and sometimes later the 2nd superscripts (= n minus the first one) for better readability. We obtain canonical isomorphisms

$$A_{r-1}^{p+1}/A_r^{p+1} = K_r^{i,n}/Z(K_r^{i,n}) \xrightarrow[\cong]{\partial} B(K_r^{i,n+1}) = (B_r^{p+r} + A_r^{p+r+1})/A_r^{p+r+1}\ , \tag{2.6}$$

and - on the level of complexes -

$$K_r^i/Z(K_r^i) \xrightarrow[\cong]{\partial} B(K_r^i)[1]. \tag{2.7}$$

2.2. **Determinants.** Recall that a (commutative) Picard category is a groupoid $\mathcal{P}$ with a product $\otimes : \mathcal{P}\times\mathcal{P} \to \mathcal{P}$ which satisfies compatible associativity, commutatitivity and unit constraints, and for which every object in $\mathcal{P}$ is invertible, see [9, Appendix A] for more details. In this note we will in general not explicitly mention the associativity and commutativity isomorphisms; by the coherence theorem for AC tensor categories this will not cause any confusion. We fix a unit object $\mathbf{1} = \mathbf{1}_{\mathcal{P}}$ which is unique up to unique isomorphism. Now we choose a determinant functor [9, 2] $\mathbf{d} : \mathcal{E}_{iso} \to \mathcal{P}$ on $\mathcal{E}$ with values in some (commutative) Picard category and extend it [9, thm. 2.3] to the category of bounded cochain complexes $\mathcal{C}^b(\mathcal{E})_{qis}$ of objects in $\mathcal{E}$ where we write 'qis' for the quasi-isomorphisms in $\mathcal{C}^b(\mathcal{E})$. Roughly speaking we thus have by definition

$$\mathbf{d}(C) = \prod_i \mathbf{d}(C^i)^{(-1)^i}$$

whence we obtain a canonical map

$$e_0 : \mathbf{d}(C) \overset{\mathbf{d}(F^\bullet)}{\cong} \prod_p \mathbf{d}(F^pC/F^{p+1}C) = \mathbf{d}(E_0^{\cdot}, \partial_0)$$

induced from the filtration $F^\bullet$ on C. Following [3, prop. 3.1] there is a canonical isomorphism

$$\eta_C : \mathbf{d}(C) \to \mathbf{d}(\mathrm{H}(C))$$

for each $C \in \mathcal{C}^b(\mathcal{E})$ which is induced by the sequences (2.2), (2.3) as well as by the canonical identification $\mu_B : \mathbf{d}(B)\mathbf{d}(B[1]) \cong \mathbf{1}$ as described in [3, lem. 2.3]. Thus we obtain canonical isomorphisms

$$\eta_r :\ \mathbf{d}(E_r^{\cdot}, \partial) \xrightarrow{\eta_{E_r}} \mathbf{d}(\mathrm{H}(E_r^{\cdot})) \cong \mathbf{d}(E_{r+1}^{\cdot}, 0) \cong \mathbf{d}(E_{r+1}^{\cdot}, \partial_r)$$

for all $r \geq 0$.

On the other hand, associated with the derived filtration F_r^iC of C we have canonical maps (for each $r \geq 0$) which identity the determinant of C with the product over the determinants of all subquotients $gr_r^i(C)$ of F_rC. They induce isomorphisms

$$e_r : \mathbf{d}(C) \xrightarrow{\cong} \textstyle\prod_i \mathbf{d}(\mathrm{gr}_r^i(C)) \xrightarrow{\cong} \mathbf{d}(gr_r(C)) \xrightarrow{\mathbf{d}(q_r)} \mathbf{d}((E_r^{\cdot}, \partial_r))$$

and a diagram

$$\begin{tikzcd} \mathbf{d}(C) \arrow[r, "\cong"] \arrow[dr, "e_r"'] & \mathbf{d}(gr_r(C)) \arrow[r, "\eta_{gr_r(C)}"] \arrow[d, "\mathbf{d}(q_r)"'] & \mathbf{d}(\mathrm{H}(gr_r(C))) \arrow[d, "\mathbf{d}(\mathrm{H}(q_r))"] \\ & \mathbf{d}(E_r^\cdot) \arrow[r, "\eta_r"] & \mathbf{d}(E_{r+1}^\cdot) \end{tikzcd} \tag{2.8}$$

which is commutative by [3, prop. 3.1].

Lemma 2.1. *For all $r \geq 0$ we have*

$$e_r = \eta_{r-1} \circ \ldots \circ \eta_0 \circ e_0$$

upon identifying $\mathbf{d}((E_r, \partial_r)) = \mathbf{d}((E_r, 0))$.

We postpone the proof of the Lemma and consider first the case $r = \infty$: We set $e_\infty := e_r$ for r big enough such that $\partial_r = 0$, whence $E_\infty^\cdot = E_r^\cdot$.

Although the derived filtration will not stabilize for big r in the literal sense, it stabilizes if we consider it up to reindexing and up to omitting or inserting trivial filtration steps. The resulting filtration class is denoted by F_∞, for which a representing filtration can be described as follows. First consider *generalized good truncations*

$$\tau^j_{\leq n} : \ \cdots \xrightarrow{\partial} C^{n-2} \xrightarrow{\partial} \partial^{-1}(F^j(C^n)) \xrightarrow{\partial} F^j Z^n \xrightarrow{\partial} 0 \xrightarrow{\partial} \cdots$$

for $t \leq j \leq s$ and $m_0 := \min\{n | C^n \neq 0\} \leq n \leq n_0 := \max\{n | C^n \neq 0\}$ with associated graded complexes

$$gr^{(j,n)} : \ \cdots \xrightarrow{\partial} 0 \xrightarrow{\partial} \partial^{-1}(F^j(C^n))/\partial^{-1}(F^{j+1}(C^n)) \xrightarrow{\partial} F^j Z^n / F^{j+1} Z^n \xrightarrow{\partial} 0 \ .$$

Then it is easy to check that F_∞ can be represented by

$$F_\infty : C = \tau^t_{\leq n_0} \supseteq \cdots \supseteq \tau^j_{\leq n_0} \supseteq \cdots \supseteq \tau^s_{\leq n_0} = \tau^t_{\leq n_0 - 1} \supseteq \cdots \supseteq \tau^j_{\leq n} \supseteq \cdots \supseteq \tau^s_{\leq m_0} = 0.$$

Refining the latter by the filtration

$$0 \subseteq B \subseteq Z \subseteq C, \tag{2.9}$$

or by the slightly finer one

$$0 \subseteq B \subseteq \ldots \subseteq B + F^p Z \subseteq \ldots \subseteq Z \subseteq C, \tag{2.10}$$

we obtain the filtration

$$G : \ldots \supseteq \tau^j_{\leq n} \supseteq Z(\tau^j_{\leq n}) \supseteq B(\tau^j_{\leq n}) \supseteq \tau^{j+1}_{\leq n} \supseteq \ldots$$

with the property that

$$Z(\tau^j_{\leq n})/\tau^{j+1}_{\leq n} = Z(gr^{(j,n)}) \tag{2.11}$$

and

$$B(\tau^j_{\leq n})/\tau^{j+1}_{\leq n} = B(gr^{(j,n)}) \tag{2.12}$$

as can be easily checked. Then the associated graded complexes $gr^G(C)$ are given as follows

$$\tau^j_{\leq n}/Z(\tau^j_{\leq n}) = \partial^{-1}(F^j(C^n))/\partial^{-1}(F^{j+1}(C^n))[-(n-1)], \tag{2.13}$$

$$Z(\tau^j_{\leq n})/B(\tau^j_{\leq n}) = F^jZ^n/(F^jB^n+F^{j+1}Z^n)[-n] \cong E^{i,n-i}_\infty[-n] \cong gr^j\mathrm{H}^n(C)[-n] \tag{2.14}$$

and

$$B(\tau^j_{\leq n})/\tau^{j+1}_{\leq n} = (F^jB^n + F^{j+1}Z^n)/F^{j+1}Z^n[-n] \cong F^jB^n/F^{j+1}B^n[-n], \tag{2.15}$$

where for an object $M \in \mathcal{E}$ we write $M[-n]$ for the obvious complex concentrated in degree n (this convention is compatible with shifting in the following sense $(M[0])[n] = M[n]$). On the other hand the refinement $\mathcal{R}$ of the filtration (2.10) by $F_\infty = (\tau^j_{\leq n})$ looks like

$$\begin{gathered} 0 \subseteq B \cap \tau^{s-1}_{m_0} \subseteq \ldots \subseteq B \cap \tau^j_n \subseteq \ldots B \subseteq \ldots \subseteq B + F^{p+1}Z \subseteq \\ B + F^{p+1}Z + (B+F^pZ) \cap \tau^p_{\leq m_0} \subseteq B + F^{p+1}Z + (B+F^pZ) \cap \tau^p_{\leq m_0+1} \subseteq \ldots \\ B + F^{p+1}Z + (B+F^pZ) \cap \tau^p_{\leq n} \subseteq B + F^{p+1}Z + (B+F^pZ) \cap \tau^p_{\leq n} \subseteq \ldots \subseteq B + F^pZ \subseteq \ldots \\ Z \subseteq Z + \tau^{s-1}_{\leq m_0+1} \subseteq \ldots \subseteq Z + \tau^j_{\leq n} \subseteq \ldots \subseteq C \end{gathered}$$

with associated graded complexes $gr^?_{\mathcal{R}}(C)$

$$F^jB^n/F^{j+1}B^n[-n], \ldots, gr^p\mathrm{H}^n(C)[-n], \ldots, \partial^{-1}(F^jC^n)/\partial^{-1}(F^{j+1}C^n)[-(n-1)]. \tag{2.16}$$

Now we are ready to prove the following

Proposition 2.2. *For each bounded filtered complex C there is a canonical commutative diagram*

$$\begin{tikzcd} \mathbf{d}(C) \arrow[r, "e_\infty"] \arrow[d, "\eta_C"'] & \mathbf{d}(E^\cdot_\infty) \arrow[d, "\mathbf{d}(\psi)"] \\ \mathbf{d}(\mathrm{H}(C)) \arrow[r, "\mathbf{d}(F^\bullet H)"] & \prod_p \mathbf{d}(\mathrm{gr}^p\mathrm{H}(C)) \end{tikzcd} \tag{2.17}$$

where $\psi : E^\cdot_\infty \cong \bigoplus_p \mathrm{gr}^p\mathrm{H}(C)$ is the canonical isomorphism being part of the convergent spectral sequence associated to C and induced by (2.4).

Proof. Recall first that η_C is induced by the filtration (2.9) above using the exact sequences (2.3) and (2.2). Recall that

$$\begin{aligned} &\mathbf{d}(gr_{\mathcal{R}}(C)) = \\ &\prod_{n,j}\mathbf{d}(\partial^{-1}(F^jC^n)/\partial^{-1}(F^{j+1}C^n)[-n-1]) \prod_{p,n}\mathbf{d}(gr^p\mathrm{H}^n(C)[-n]) \prod_{n,j}\mathbf{d}(F^jB^n/F^{j+1}B^n[-n]) \\ &\cong \prod_j \mathbf{d}(\partial^{-1}(F^jC)/\partial^{-1}(F^{j+1}C)) \prod_p \mathbf{d}(gr^p\mathrm{H}(C)) \prod_j \mathbf{d}(F^jB/F^{j+1}B) \end{aligned}$$

and set

$$R := \mathbf{d}(gr_\infty/Z(gr_\infty))\mathbf{d}(\mathrm{H}(gr_\infty))\mathbf{d}(B(gr_\infty)) \cong \mathbf{d}(B(gr_\infty)[1])\mathbf{d}(\mathrm{H}(gr_\infty))\mathbf{d}(B(gr_\infty)).$$

By the refinement principle [9, prop. 1.9] applied to the filtrations F_∞ and (2.10) we obtain the following diagram

$$\begin{tikzcd}
\mathbf{d}(C) \arrow[r, "\mathbf{d}(F_\infty)"] \arrow[dddd, "can"'] & \mathbf{d}(gr_\infty(C)) \arrow[r, "\eta"] \arrow[d] & \mathbf{d}(\mathrm{H}(gr_\infty(C))) \arrow[r] & \mathbf{d}(E_\infty) \arrow[ddddddd, "\mathbf{d}(\psi)"] \\
& \mathbf{d}(gr_G(C)) \arrow[r] \arrow[d] & R \arrow[u, "\mu_{B(gr_\infty)}"'] & \\
& \mathbf{d}(gr_{\mathcal{R}}(C)) & & \\
& & & \\
\mathbf{d}(C/Z)\mathbf{d}(Z/B)\mathbf{d}(B) \arrow[r] \arrow[d] & \mathbf{d}(C/Z)\mathbf{d}(\mathrm{gr}^{\cdot}\mathrm{H}(C))\mathbf{d}(B) \arrow[uu] \arrow[d] & (?) & \\
\mathbf{d}(B[1])\mathbf{d}(H)\mathbf{d}(B) \arrow[r] \arrow[d, "\mu_B"'] & \mathbf{d}(B[1])\mathbf{d}(\mathrm{gr}^{\cdot}\mathrm{H}(C))\mathbf{d}(B) \arrow[d, "\mu_B"'] & & \\
\mathbf{d}(H) \arrow[r, "can"] & \mathbf{d}(\mathrm{gr}^{\cdot} H) \arrow[rr, equal] & & \mathbf{d}(\mathrm{gr}^{\cdot} H)
\end{tikzcd}$$

in which all interior rectangles clearly commute except possibly (?). Here we write $\mathrm{gr}^{\cdot} H$ for the complex $\bigoplus_p \mathrm{gr}^p H$ (with trivial differentials). Note that the left vertical map is just η_C while the upper horizontal map is e_∞ by diagram (2.8), because η_∞ is the identity. After canceling $\mathrm{gr}_p \mathrm{H}(C)$ and the part corresponding to the middle part of $\mathcal{R}$ between B and Z, respectively, and taking into account the identification

$$\partial^{-1}(F^j(C^n))/\partial^{-1}(F^{j+1}(C^n)) \cong F^j B^n / F^{j+1} B^n$$

the commutativity of (?) boils down to the commutativity of the following diagram (which is upside down in comparison with the previous diagram!)

$$\begin{tikzcd}
\mathbf{d}(B[1])\mathbf{d}(B) \arrow[r, "\mu_B"] \arrow[d] & \mathbf{1} \arrow[d, equal] \\
\prod_{i,n} \mathbf{d}(F^i B^n / F^{i+1} B^n[-n+1]) \prod_{i,n} \mathbf{d}(F^i B^n / F^{i+1} B^n[-n]) \arrow[r, "\prod \mu_{F^i B^n / F^{i+1} B^n}"] \arrow[d] & \mathbf{1} \arrow[d, equal] \\
\mathbf{d}(B(gr_\infty)[1])\mathbf{d}(B(gr_\infty)) \arrow[r, "\mu_{B(gr_\infty)}"] & \mathbf{1}
\end{tikzcd}$$

which follows immediately from [3, lem. 2.3]. Here the upper square uses (part of) $\mathcal{R}$, while the lower one uses the identifications (2.12) and (2.15), i.e.,

$$B(gr_\infty) \cong \bigoplus_{j,n} F^i B^n / F^{i+1} B^n[-n].$$

$\square$

Remark 2.3. By similar, but simpler considerations regarding the filtration (2.9) and the (usual) good truncation filtration $\tau_{\leq n} (= \tau^t_{\leq n})$ one sees that η_C can also be described using the *canonical* filtration

$$\ldots \supseteq \tau_{\leq n} \supseteq Z(\tau_{\leq n}) \supseteq B(\tau_{\leq n}) \supseteq \tau_{\leq n-1} \supseteq \ldots$$

with

$$Z(\tau_{\leq n}):\ \cdots \xrightarrow{\partial} C^{n-2} \xrightarrow{\partial} Z^{n-1} \xrightarrow{\partial} Z^n \xrightarrow{\partial} 0 \longrightarrow \cdots$$

and

$$B(\tau_{\leq n}):\ \cdots \xrightarrow{\partial} C^{n-2} \xrightarrow{\partial} Z^{n-1} \xrightarrow{\partial} B^n \xrightarrow{\partial} 0 \longrightarrow \cdots$$

together with the identifications $C^{n-1}/Z^{n-1} \cong B^n$ induced by ∂.

Now we come back to the

Proof (of Lemma 2.1). Denote by $\mathcal{R}_0$ the refinement of the filtration $F_r(C)$ such that its graded pieces correspond to

$$gr_r^i/Z(gr_r^i):\ A_r^{p-r}/A_{r+1}^{p-r} \longrightarrow A_r^p/A_{r+1}^{p+1} \longrightarrow A_r^{p+r}/A_{r+1}^{p+r},$$

$$H(gr_r^i):\ E_{r+1}^{p-r} \xrightarrow{0} E_{r+1}^p \xrightarrow{0} E_{r+1}^{p+r},$$

and

$B(gr_r^i):$

$$(B_{r+1}^{p-r} + A_r^{p-r+1})/A_r^{p-r+1} \longrightarrow (B_{r+1}^p + A_r^{p+1})/A_r^{p+1} \longrightarrow (B_{r+1}^{p+r} + A_r^{p+r+1})/A_r^{p+r+1}.$$

We write $\mathcal{R}_1$ for the refinement of $F_{r+1}(C)$ by $\mathcal{R}_0$. Then, by the refinement principle applied to $F_{r+1}(C)$ and $\mathcal{R}_0$ the left upper square in the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{d}(C) \arrow[r] \arrow[d] & \mathbf{d}(gr_r(C)) \arrow[r] & \mathbf{d}(gr_{\mathcal{R}_0}(C)) \arrow[r, "a"] \arrow[d] & \mathbf{d}(\mathrm{H}(gr_r(C))) \arrow[d] \\
\mathbf{d}(gr_{r+1}(C)) \arrow[rr] \arrow[d, equal] & & \mathbf{d}(gr_{\mathcal{R}_1}(C)) \arrow[r, "b"] & \mathbf{d}((E^{\cdot}_{r+1}, 0)) \arrow[d, equal] \\
\mathbf{d}(gr_{r+1}(C)) \arrow[rrr, "\mathbf{d}(q_r)"] & & & \mathbf{d}((E^{\cdot}_{r+1}, \partial_{r+1}))
\end{tikzcd}$$

Note that the composite from the left upper corner to the right lower corner via the left lower corner is just e_{r+1}. The upper horizontal line here is defined such that it coincides with the upper line of the diagram (2.8), in particular, by the definition of η_{gr_r} the map a is induced by the identifications

$$A_r^p/A_{r+1}^{p+1} \xrightarrow{\partial} (B_{r+1}^p + A_r^{p+1})/A_r^{p+1}\ \text{, i.e.,}\quad gr_r^i/Z(gr_r^i) \xrightarrow{\partial} B(gr_r^i)[1] \tag{2.18}$$

plus $\mu_{B(gr_r^i)}$ as above. The map b makes by definition the lower rectangle commutative, i.e., it is induced by the identifications of the canonical map $\mathbf{d}(\ker(q_r)) \to \mathbf{d}(\mathrm{H}(\ker(q_r))) = \mathbf{1}$ as the involved complex is acyclic. Thus b corresponds to the identifications given by (2.6) or (2.7), which are the same as in (2.18) for a whence the whole diagram is commutative and combined with diagram (2.8) the lemma is proved. Indeed, upon identifying $\mathbf{d}((E_r, \partial_r)) = \mathbf{d}((E_r, 0))$ the composite from the left upper corner to the right lower corner via the right upper corner is nothing else than $\eta_r \circ e_r$. $\square$

Lemma 2.4. *Let* $\Delta: \ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be a strictly exact sequence of bounded filtered complexes,i.e.,* $0 \longrightarrow F^pA \longrightarrow F^pB \longrightarrow F^pC \longrightarrow 0$ *is exact for any* p*. Then the sequence of complexes*

$$E_0(\Delta): \ 0 \longrightarrow E_0^{\cdot}(A) \longrightarrow E_0^{\cdot}(C) \longrightarrow E_0^{\cdot}(C) \longrightarrow 0$$

is also exact and gives rise to the bottom line in the following commutative diagrams

$$\begin{tikzcd} \mathbf{d}(B) \arrow[r, "\mathbf{d}(\Delta)"] \arrow[d, "e_i(B)"'] & \mathbf{d}(A)\mathbf{d}(C) \arrow[d, "e_i(A)e_i(C)"] \\ \mathbf{d}(E_i(B)) \arrow[r, "\mathbf{d}(E_i(\Delta))"] & \mathbf{d}(E_i(A))\mathbf{d}(E_i(C)) \end{tikzcd}$$

for $i = 0, 1$ *with* $\mathbf{d}(E_1(\Delta)) = \mathbf{d}(\mathrm{H}(E_0(\Delta)))$.

Proof. The exactness for the E_0-terms follows immediately from the strict exactness while the commutative diagram for $i = 0$ is another easy consequence of the refinement principle [9, prop. 1.9] applied to the filtrations

$$0 \subseteq A \subseteq B$$

and

$$F^\bullet B,$$

the details of which we leave to the reader. For $i = 1$ simply apply the functoriality [3, thm. 3.3] of η with respect to short exact sequences (see also diagram (3.24)). $\square$

Our treatment of determinants of spectral sequences should be compared to that in [9, §3] where apart from the derived also the spectral filtration of C is used. As in (loc. cit.) our description also generalizes immediately to exact categories if the admissibility (loc. cit., def. 1.6) and compatibility (loc. cit., def. 1.8) of all involved filtrations is granted. Also note that analogous results hold for homological spectral sequences, of course.

3. (Co)homology and spectral sequences

Let $F : \mathcal{E} \to \mathcal{D}$ be a right exact, additive functor between two abelian categories with finite homological dimension assuming that $\mathcal{E}$ has sufficiently many projectives. In the following we will not distinguish between chain and cochain complexes as they can be identified using the usual renumeration. We write $D^b(\mathcal{E})$ for the triangulated derived category of bounded complexes in $\mathcal{E}$. Assume that $\mathbf{d}_{\mathcal{E}} : D^b(\mathcal{E})_{qis} \to \mathcal{P}_{\mathcal{E}}$ and $\mathbf{d}_{\mathcal{D}} : D^b(\mathcal{D})_{qis} \to \mathcal{P}_{\mathcal{D}}$ are F-compatible determinant functors (for determinant functors of triangulated categories we refer the reader to [2]), i.e., such that there exists a commutative diagram of functors (modulo a natural isomorphism q)

$$\begin{tikzcd} D^b(\mathcal{E})_{qis} \arrow[r, "\mathbf{d}_{\mathcal{E}}"] \arrow[d, "\mathbf{L}F"'] & \mathcal{P}_{\mathcal{E}} \arrow[d, "\psi^{\mathcal{E}}_{\mathcal{D}}"] \\ D^b(\mathcal{D})_{qis} \arrow[r, "\mathbf{d}_{\mathcal{D}}"] & \mathcal{P}_{\mathcal{D}} \end{tikzcd}$$

in which $\psi^{\mathcal{E}}_{\mathcal{D}}$ is a AC-tensor functor in the language of [9, Appendix A] and such that we have a morphism of determinants

$$(\psi^{\mathcal{E}}_{\mathcal{D}} \circ \mathbf{d}_{\mathcal{E}}) \to (\mathbf{d}_{\mathcal{D}} \circ \mathbf{L}F)$$

in the sense of (loc. cit., def. 1.11) or [2, def. 3.2].

Here the derived functor $\mathbf{L}F$ has to be calculated by a finite resolution of F-acyclic objects, if necessary. For simplicity we assume henceforth that $\mathcal{E}$ has finite projective dimension.

Now let

$$\Delta: \ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{3.19}$$

be an exact sequence in $\mathcal{E}$ (or more generally in $\mathcal{C}^b(\mathcal{E})$ using hyper-derived functors $\mathbf{L}F$ below). Applying $\mathbf{L}F$ gives a distinguished triangle

$$\mathbf{L}F(\Delta): \ \mathbf{L}F(A) \longrightarrow \mathbf{L}F(B) \longrightarrow \mathbf{L}F(C) \longrightarrow \mathbf{L}F(A)[1] \tag{3.20}$$

which in turn induces the long exact LF-sequence

$$\begin{gathered} L_iF(A) \longrightarrow L_iF(B) \longrightarrow L_iF(C) \xrightarrow{\delta} \\ \cdots \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0, \end{gathered} \tag{3.21}$$

in which $L_iF(A) = L_iF(B) = L_iF(C) = 0$ for large i by assumption. Then the additivity relation

$$\mathbf{d}_{\mathcal{E}}(\Delta) : \mathbf{d}_{\mathcal{E}}(B) \cong \mathbf{d}_{\mathcal{E}}(A)\mathbf{d}_{\mathcal{E}}(C) \tag{3.22}$$

induces via $\psi^{\mathcal{E}}_{\mathcal{D}}$ the additivity relation

$$\mathbf{d}_D(\mathbf{L}F(\Delta)) : \mathbf{d}_{\mathcal{D}}(\mathbf{L}F(B)) \cong \mathbf{d}_{\mathcal{D}}(\mathbf{L}F(A))\mathbf{d}_{\mathcal{D}}(\mathbf{L}F(C)) \tag{3.23}$$

corresponding to (3.20) by the F-compatibility.

Lemma 3.1. *There is a commutative diagram*

$$\begin{tikzcd} \mathbf{d}_{\mathcal{D}}(\mathbf{L}F(B)) \arrow[r, "\mathbf{d}_D(\mathbf{L}F(\Delta))"] \arrow[d, "\eta_{\mathbf{L}F(B)}"'] & \mathbf{d}_{\mathcal{D}}(\mathbf{L}F(A))\mathbf{d}_{\mathcal{D}}(\mathbf{L}F(C)) \arrow[d, "\eta_{\mathbf{L}F(A)}\eta_{\mathbf{L}F(C)}"] \\ \prod_i \mathbf{d}_{\mathcal{D}}(L_iF(B))^{(-1)^i} \arrow[r, "\mathbf{d}_D(\mathrm{H}(\mathbf{L}F(\Delta)))"] & \prod_i \mathbf{d}_{\mathcal{D}}(L_iF(A))^{(-1)^i}\prod_i \mathbf{d}_{\mathcal{D}}(L_iF(C))^{(-1)^i} \end{tikzcd}$$

where the bottom line is induced by the long exact sequence (3.21).

Proof. [3, thm. 3.3]. □

On the other hand the same reference applied to (3.19) (for complexes) leads to a commutative diagram

$$\begin{tikzcd} \mathbf{d}_{\mathcal{E}}(B) \arrow[r, "\mathbf{d}_{\mathcal{E}}(\Delta)"] \arrow[d, "\eta_B"'] & \mathbf{d}_{\mathcal{E}}(A)\mathbf{d}_{\mathcal{E}}(C) \arrow[d, "\eta_A\eta_C"] \\ \mathbf{d}_{\mathcal{E}}(\mathrm{H}B) \arrow[r, "\mathbf{d}_{\mathcal{E}}(\mathrm{H}(\Delta))"] & \mathbf{d}_{\mathcal{E}}(\mathrm{H}A)\mathbf{d}_{\mathcal{E}}(\mathrm{H}C) \end{tikzcd} \tag{3.24}$$

where again the bottom line is induced by the long exact homology sequence attached to (3.19). Now we may apply $\mathbf{L}F$ to each homology complex like $\mathrm{H}A$ which boils down to form $L_iF(H_jA)$ for all i, j. Recall that for a bounded complex A there is a convergent (homological) spectral sequence

$$E^2_{pq} = L_pF(\mathrm{H}_q(A)) \Rightarrow \mathbb{L}F_{p+q}(A) \tag{3.25}$$

More precisely, consider a Cartan-Eilenberg resolution of A, i.e., double complexes D_A, D_Z, D_B and D_H consisting of projective objects together with

- short exact sequences of double complexes

$$0 \longrightarrow D_B \longrightarrow D_Z \longrightarrow D_H \longrightarrow 0 \tag{3.26}$$

$$0 \longrightarrow D_Z \longrightarrow D_A \xrightarrow{\partial} D_B[1]_h \longrightarrow 0 \tag{3.27}$$

  where $[1]_h$ means shift in the horizontal direction, i.e., in the direction parallel to B.
- augmentation maps $D_A \to A$, $D_Z \to Z$ etcetera such that $\mathrm{tot}(D_A) \to A$ etcetera are quasi-isomorphisms. Moreover, in the vertical direction one has

$$\mathrm{H}^v_i(D_A) = \begin{cases} A, & i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

  and similarly for B, Z and $H = \mathrm{H}(A)$.

We set

$$\mathcal{A} = \mathrm{tot}F(D_A) = \mathbf{L}F(A) \qquad \mathcal{Z} = \mathrm{tot}F(D_Z) \tag{3.28}$$

$$\mathcal{B} = \mathrm{tot}F(D_B) \qquad \mathcal{H} = \mathrm{tot}F(D_H) = \mathbf{L}F(H) = \bigoplus_j \mathbf{L}F(H_j[-j]) \tag{3.29}$$

and recall that

$$\mathrm{H}_i(\mathcal{A}) = \mathbb{L}_iF(A) \qquad \mathrm{H}_i(\mathcal{H}) = \mathbb{L}_iF(H) = \bigoplus_j L_{i-j}F(H_i))$$

as the differentials of H and hence the horizontal differentials ∂^h of D_H are all trivial. Note also that (3.25) is just the spectral sequences associated to the double complex $F(D_A)$ using the filtration by rows [13, def. 5.6.2]. With (3.26) also

$$0 \longrightarrow \mathrm{tot}D_B \longrightarrow \mathrm{tot}D_{\mathcal{Z}} \longrightarrow \mathrm{tot}D_H \longrightarrow 0$$

is exact and consists of projectives, whence

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{Z} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{3.30}$$

and similarly

$$0 \longrightarrow \mathcal{Z} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}[1] \longrightarrow 0 \tag{3.31}$$

is also exact. The latter two sequences give rise to an isomorphism

$$\vartheta : \mathbf{d}_{\mathcal{D}}(\mathcal{A}) \cong \mathbf{d}_{\mathcal{D}}(\mathcal{H}). \tag{3.32}$$

By the functoriality 3.1 of η we obtain the canonical commutative diagram

$$\begin{tikzcd} \mathbf{d}_{\mathcal{D}}(\mathcal{A}) \arrow[r, "\vartheta"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathbf{d}_{\mathcal{D}}(\mathcal{H}) \arrow[d, "\eta_{\mathcal{H}}"] \\ \mathbf{d}_{\mathcal{D}}(\mathrm{H}\mathcal{A}) \arrow[r, "\mathrm{H}(\vartheta)"] & \mathbf{d}_{\mathcal{D}}(\mathrm{H}\mathcal{H}). \end{tikzcd} \tag{3.33}$$

Now we will use the homological versions of the results of section 2.1.

Theorem 3.2. *The following diagram commutes*

$$\begin{tikzcd} \mathbf{d}_{\mathcal{D}}(\mathbf{L}F(A)) \arrow[r, "\vartheta"] \arrow[d, "e^{\infty}\approx\eta"'] & \mathbf{d}_{\mathcal{D}}(\mathcal{H}) = \prod_i \mathbf{d}_{\mathcal{D}}(\mathbf{L}F(\mathrm{H}_i(A)))^{(-1)^i} \arrow[d, "e^2(\mathcal{H})=e^{\infty}(\mathcal{H})\approx\eta_{\mathcal{H}}\approx"', "\prod\eta"] \\ \mathbf{d}_{\mathcal{D}}(E^{\infty}(\mathcal{A}))\approx\prod_j \mathbf{d}_{\mathcal{D}}(\mathbb{L}_jF(A))^{(-1)^i} & \mathbf{d}_{\mathcal{D}}(E^2(\mathcal{A}))\approx\prod_{i,j}\mathbf{d}_{\mathcal{D}}(L_jF(\mathrm{H}_i(A)))^{(-1)^i} \arrow[l, "e^{2\infty}"] \end{tikzcd}$$

where the top line is induced by (3.32) *and* (3.29)*, while the lower line* $e^{2\infty} := \eta^{r-1}\circ \ldots \circ \eta^2$ *is induced from the spectral sequence* (3.25) *using the notation from section 2.1 and assuming* $E^r = E^{\infty}$*. Moreover the diagram is compatible with* (3.33) *and the diagrams arising from Proposition 2.2.*

Proof. First note that $(E^1_{\cdot}(\mathcal{H}),\partial) \cong (E^1_{\cdot}(\mathcal{A}),\partial)$ as complexes (with $E^1_{pq} = D_{H\,qp}$) and $E^2_{pq}(\mathcal{H}) \cong E^2_{pq}(\mathcal{A}) \cong LF_p(H_q)$ on objects (but in general not as complexes!). Applying Lemma 2.4 to the exact sequences (3.30),(3.31) (taking into account that F applied to the sequences (3.26) again gives exact sequences which grant the hypothesis of the lemma) leads to the commutative diagram

$$\begin{tikzcd} \mathbf{d}(\mathcal{A}) \arrow[r, "\vartheta"] \arrow[d, "e^1"'] & \mathbf{d}(\mathcal{H}) \arrow[d, "e^1"] \\ \mathbf{d}(E^1(\mathcal{A})) \arrow[r, "E^1(\vartheta)", "\approx"'] \arrow[d, "\eta"'] & \mathbf{d}(E^1(\mathcal{H})) \arrow[d, "\eta"'] \\ \mathbf{d}(E^2(\mathcal{A})) \arrow[r, "\mathrm{H}(E^1(\vartheta))", "\approx"'] \arrow[d, "e^{2\infty}"'] & \mathbf{d}(E^2(\mathcal{H})) \arrow[d, equal] \\ \mathbf{d}(E^{\infty}(\mathcal{A})) \arrow[d, "\mathbf{d}(F^{\bullet}H)^{inverse}\circ\mathbf{d}(\psi)"', "\approx"] & \mathbf{d}(E^{\infty}(\mathcal{H})) \arrow[d, "\approx"', "\mathbf{d}(F^{\bullet}H)^{inverse}\circ\mathbf{d}(\psi)"] \\ \mathbf{d}_{\mathcal{D}}(\mathrm{H}\mathcal{A}) \arrow[r, "\mathrm{H}(\vartheta)"] & \mathbf{d}_{\mathcal{D}}(\mathrm{H}\mathcal{H}) \end{tikzcd}$$

where according to the above calculation the isomorphisms $E^1(\vartheta)$ and hence $\mathrm{H}(E^1(\vartheta))$ may be considered as a natural identification ($\approx$)(only this identification defines the latter isomorphism). Note that the outer diagram is just diagram (3.33). Taking all these identifications into account the result follows. □

3.1. **Application to the Tor-Functor.** Let R, R' be two regular rings and Y a (R, R')-bimodule, which is finitely generated and projective as R'-module. Taking for $\mathcal{E}$ and $\mathcal{D}$ the category of finitely generated R- and R'-modules, respectively, for $\mathbf{d}_R : D^b(\mathcal{E})_{qis} \to \mathcal{P}_R$ and $\mathbf{d}_{R'} : D^b(\mathcal{D})_{qis} \to \mathcal{P}_{R'}$ the determinant functor of Fukaya and Kato [8, §1], setting $L(-) = Y \otimes_R -$ and letting $\psi^R_{R'} := \psi^{\mathcal{E}}_{\mathcal{D}} : \mathcal{P}_R \to \mathcal{P}_{R'}$ be the canonical functor induced by L we are in the situation of the previous subsection. In particular there is a convergent (homological) spectral sequence

$$E^2_{pq} = \mathrm{Tor}^R_p(Y, \mathrm{H}_q(A)) \Rightarrow \mathbb{T}or^R_{p+q}(Y, A) \tag{3.34}$$

and the canonical identification

$$\mathbf{d}_R(A) \cong \mathbf{d}_R(\mathrm{H}A) \cong \prod_i \mathbf{d}_R(\mathrm{H}^i(A))^{(-1)^i}$$

induces by (3.22), (3.23) and Theorem 3.2 the first line of the following commutative diagram

$$\begin{tikzcd}
\mathbf{d}_{R'}(Y \otimes^{\mathbb{L}} A) \arrow[r] \arrow[d] & \prod_i \mathbf{d}_{R'}(Y \otimes^{\mathbb{L}} \mathrm{H}_i(A))^{(-1)^i} \arrow[d] \\
\prod_j \mathbf{d}_{R'}(\mathbb{T}or^R_j(Y, A))^{(-1)^j} & \prod_{i,j} \mathbf{d}_{R'}(\mathrm{Tor}^R_j(Y, \mathrm{H}_i(A))^{(-1)^{i+j}}) \arrow[l]
\end{tikzcd} \tag{3.35}$$

where the bottom row is induced by the spectral sequence (3.34).

Remark 3.3. Strictly speaking Fukaya and Kato's determinant functor goes from $\mathcal{C}^b(\mathrm{Pmod}(R))_{qis}$ to $\mathcal{P}_R$ where $\mathrm{Pmod}(R)$ denotes the exact category of finitely generated projective (say left) R-modules. But due to our regularity assumption on R we may choose finite projective resolutions for each finitely generated R-module and consider the functor $\mathcal{E} \to \mathcal{C}^b(\mathrm{Pmod}(R))_{qis} \to \mathcal{P}_R$ which then extends also to give a determinant on $\mathcal{C}^b(\mathcal{E})_{qis}$. Note also that by [10, cor. 2 of thm. 2] in this situation we have canonical isomorphism for all distinguished triangles (instead of short exact sequences of complexes) whence we obtain the desired determinant functor from $D^b(\mathcal{E})_{qis}$ to $\mathcal{P}_R$. This should also be compared with [2, cor. 5.3] applied to the triangulated derived category $D^b(\mathcal{E})$ of bounded complexes in $\mathcal{E}$. In this way more natural examples arise where the results of this section apply.

3.2. **Compatibility of the Snake lemma and Spectral sequences.** Finally we just state a result concerning the comparison of spectral sequences versus long exact sequences: Consider a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{3.36}$$

of bounded cochain complexes concentrated in degree greater or equal to 0. Setting

$$\begin{aligned}
D^{0\bullet} &:= A^\bullet, & d_v^{0\bullet} &: = d_A^\bullet, \\
D^{1\bullet} &:= B^\bullet, & d_v^{1\bullet} &: = -d_B^\bullet, \\
D^{2\bullet} &:= C^\bullet, & d_v^{2\bullet} &: = d_C^\bullet, \\
D^{i\bullet} &:= 0 \text{ otherwise}, & &
\end{aligned}$$

we obtain a (cochain) double complex D with horizontal differentials

$$d_h^{0\bullet} = f^\bullet, \qquad d_h^{1\bullet} = g^\bullet \qquad \text{and 0 otherwise.}$$

Lemma 3.4. *In the above situation the total complex* $\mathrm{tot}(D)$ *is acyclic and the isomorphism* $e^{1\infty} := \eta_{r-1} \circ \ldots \circ \eta_1$

$$\prod_i \mathbf{d}(\mathrm{H}^i(A))^{(-1)^i} \prod_i \mathbf{d}(\mathrm{H}^i(B))^{(-1)^{i+1}} \prod_i \mathbf{d}(\mathrm{H}^i(C))^{(-1)^i} \cong \mathbf{d}(\mathrm{tot}(D)) \cong \mathbf{1} \tag{3.37}$$

arising from the spectral sequence

$$E_1^{pq} = \mathrm{H}_v^q(D_{p\bullet}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(D)) = 0$$

coincides with the isomorphism arising form the long exact cohomology sequence associated with (3.36)*.*

The easy proof is left to the reader. In particular, identifications via the Snake Lemma are compatible with spectral sequences.

Remark 3.5. If one shifts the double complex to different degrees, the differentials may change their signs, but these changes are compatible with usual sign conventions for shifting and for the relation between double complexes and cochain complexes over the category of cochain complexes [13, Sign trick 1.2.5].

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