22.04.2025

Seminar: Locally analytic vector bundles on the Fargues-Fontaine-curve

AG Venjakob

Summer semester 25

Organisation:	We meet every Thursday at 11:15 in SR 8 (Mathematikon). Online participation is possible - please contact us for the Zoom data.
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Overview

The aim is to study [Po1], in which a Sen theory for equivariant vector bundles on the Fargues-Fontaine curve is developed. Let $K/\mathbb{Q}_p$ be a finite extension and consider the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of finite-dimensional $\mathbb{Q}_p$ representations of G_K. On the one hand, "the analytic dogma" - there is a fully faithful embedding

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \hookrightarrow \Phi\Gamma(\mathcal{R})$$

in the category $\Phi\Gamma(\mathcal{R})$ of (φ, Γ)-modules over the Robba ring $\mathcal{R}$. Here $\Gamma = Gal(K_\infty/K)$ denotes the Galois group of the p-cyclotomic extension K_∞ of K.

On the other hand, - "the geometric dogma" - there is a fully faithful embedding

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \hookrightarrow VB_\Gamma(\mathcal{X})$$

into the category $VB_\Gamma(\mathcal{X})$ of Γ-equivariant vecor bundles on the Fargues-Fontaine curve $\mathcal{X}$. In fact, Fargues and Fontaine show that there is an equivalence

$$\Phi\Gamma(\mathcal{R}) \cong VB_\Gamma(\mathcal{X}) \tag{1}$$

compatible with each of the aforementioned embeddings of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. This links the analytic and geometric perspectives.

Theorem A, the main result of the article - states that taking the locally analytic vectors within an equivariant vector bundle on $\mathcal{X}$ gives rise to an equivalence of categories

$$VB_\Gamma(\mathcal{X}) \cong VB^{l.a.}(\mathcal{X})$$

of $VB_\Gamma(\mathcal{X})$ with the category $VB^{l.a.}(\mathcal{X})$ of locally analytic vector bundles on $\mathcal{X}$, which recovers Sen-theory and should be compared via (1) to the decompletion of (φ, Γ)-modules in the style of the famous theorem of Cherbonnier and Colmez stating that any (φ, Γ)-module is overconvergent.

This framework leads - again via (1) - to a geometric interpretation (Theorem B) of a result of Berger, which we get to know during the seminar. From a technical point of view, Theorem C is of independent interest in investigating the vanishing of higher derivatives of the functor "taking locally analytic vectors".

The Talks

The first 3 talks have the task to give a survey on vector bundles on the Fargues-Fontaine curve with its relation to (φ, Γ)-modules. Give proofs only where explicitly required or if they are instructive and time permits. The speakers of these mini-series should coordinate among each other.

Talk 1: The adic Fargue-Fontaine curve – Max Witzelsperger (24.4.)

Explain [F, §1-2] (Assume F algebraically closed, if necessary) and cover [Po1, 3A,3B], compare also [Durham, §1-5] for more details or [FF, Preface §3] respectively [SW, §13.5]. Give a proof of [FF, Prop. 10.1.1].

Talk 2: (Equivariant) vector bundles – Otmar Venjakob (8.5.)

Explain [F, 3.1,3.2] (Assume F algebraically closed, if necessary), parts of [Beijing, §1, 6] or [FF, Preface §4-5] (e.g. Thm. 4.7, Rem. 4.9,) and cover [Po1, 3C], compare also [Durham, 6.1-6.3] for more details.

Talk 3: Comparison to (φ, Γ)-modules – Marvin Schneider (15.5.)

Cover [Po1, 7A] and compare with [FF, Preface §4-5] (e.g. Thm. 5.9, Rem. 5.10). Explain Example 3.4 of [Po1] in detail. Prove Proposition 7.2 in [Po1] using the discussion before [SW, Def. 13.4.3] and sketch the proof of [Po1, Theorem 7.3], see also [KL, Thm. 9.5.8].

Talk 4: Locally analytic vectors – Anna Blanco-Cabanillas (22.5.)

Start by defining (higher) locally analytic vectors as in [Po1, §2] and explain the spectral sequences [Po1, Proposition 2.5.], without giving a detailed proof. Then move on and cover [Po1, §4a+b]. The focus should be on providing the proof of the flatness result [Po1, Proposition 4.2].

Talk 5: Locally analytic vector bundles – Otmar Venjakob (12.6.)

Cover [Po1, §4c] and explain the proof of [Po1, Proposition 4.8.] in detail.

Talk 6: Acyclicity of locally analytic vectors – Marvin Schneider (26.6.)

Provide a survey of the CST method and strategy of proof in [Po1, §5]. One could also report on the stronger result from [Po2].

Talk 7: Descent to locally analytic vectors – Max Witzelsperger (3.7.)

The goal of this talk is to provide a proof of the first main result [Po1, Theorem 6.1] in as much detail as possible. This is done in [Po1, §6].

Talk 8: Locally analytic vector bundles and p-adic differential equations – Ruth Wild (10.7.)

The first goal of this talk is to extend the results of talk 3 using the results of the previous talk. This is done in [Po1, §7b], especially [Po1, Theorem 7.5]. The second part of the talk should focus on covering [Po1, §8a-8d]. For some motivation for the material, it might be helpful to consult [Ber08].

Talk 9: The solution functor and Theorem B – Marlon Kocher (17.7.)

Goal of this talk is to cover [Po1, §8e] and provide a proof of [Po1, Theorem 8.12]. The speaker should also look at [Ber08] and try to provide some motivation for the main result.

References

[Po1] Porat, Gal, *Locally analytic vector bundles on the Fargues-Fontaine curve*, Algebra & Number Theory Vol. 18, `http://dx.doi.org/10.2140/ant.2024.18.899`

[Po2] Porat, Gal. *Locally analytic vectors and decompletion in mixed characteristic.* arXiv preprint arXiv:2407.19791 (2024).`https://doi.org/10.48550/arXiv.2407.19791`

[SW] Scholze, Peter; Weinstein, Jared. *Berkeley lectures on p-adic geometry.* Annals of Mathematics Studies, 207. Princeton University Press, Princeton, NJ, 2020.

[FF] Fargues, Laurent; Fontaine, Jean-Marc. *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* With a preface by Pierre Colmez. Astérisque No. 406 (2018)

[F] Fargues L.: *Quelques resultats et conjectures concernant la courbes.* `http://webusers.imj-prg.fr/~laurent.fargues/AuDela.pdf`

[Beijing] Fargues, Laurent; Fontaine, Jean-Marc. *Vector bundles and p-adic Galois representations.* Fifth International Congress of Chinese Mathematicians. Part 1, 2, 77–113, AMS/IP Stud. Adv. Math., 51, pt. 1, 2, Amer. Math. Soc., Providence, RI, 2012.

[Durham] Fargues, Laurent; Fontaine, Jean-Marc. *Vector bundles on curves and p-adic Hodge theory.* Automorphic forms and Galois representations. Vol. 2, 17–104, London Math. Soc. Lecture Note Ser., 415, Cambridge Univ. Press, Cambridge, 2014

[KL] Kedlaya, Kiran S.; Liu, Ruochuan. *Relative p-adic Hodge theory: foundations.* Astérisque No. 371 (2015)

[Ber08] Berger, L. Equations differentielles p-adiques et (phi,N)-modules filtres. *HAL* (2008)