Inaugural-Dissertation

zur Erlangung der Doktorwürde der
Naturwissenschaftlich-Mathematischen Gesamtfakultät der

Ruprecht-Karls-Universität Heidelberg

vorgelegt von

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geb. am 29. 06. 1989 in Rastatt

Tag der mündlichen Prüfung: ____________________

Thema:

A p-adic L-function with canonical motivic periods for families of modular forms

9. August 2017

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Abstract

We prove a version of the conjecture of Fukaya and Kato concerning the existence of p-adic L-functions for motives in the case of certain Hida families of modular forms and for commutative towers of fields, the novelty being the exact accordance with the conjectural interpolation formula. To do so, we first calculate the expressions in their conjectural formula as explicitly as possible and compare the result to the interpolation formula of the p-adic L-function constructed by Kitagawa. Here we use Eichler-Shimura isomorphisms to relate the periods appearing in Fukaya's and Kato's formula to the error terms appearing in Kitagawa's formula, which are defined in terms of modular symbols. Under a technical hypothesis on the Hida family we show that the conjectural interpolation formula differs from Kitagawa's one only by a unit in the Iwasawa algebra, so we find a p-adic L-function having the interpolation behaviour predicted by Fukaya and Kato (up to a non-constant sign).

Zusammenfassung

Wir beweisen eine Version der Vermutung von Fukaya und Kato über die Existenz p-adischer L-Funktionen für Motive im Falle gewisser Hida-Familien von Modulformen und für kommutative Körpertürme, wobei die wesentliche Neuerung in der exakten Übereinstimmung mit der vermuteten Interpolationsformel liegt. Dazu berechnen wir zunächst die Terme aus dieser vermuteten Formel so explizit wie möglich und vergleichen das Ergebnis mit der Interpolationsformel der von Kitagawa konstruierten p-adischen L-Funktion. Dieser Vergleich beruht auf Eichler-Shimura-Isomorphismen, die es uns erlauben, die bei Fukaya und Kato auftretenden Perioden in Beziehung zu den Fehlertermen aus Kitagawas Formel zu setzen, welche mittels modularer Symbole definiert sind. Unter einer technischen Annahme an die Hida-Familie zeigen wir, dass sich die vermutete und die von Kitagawa gefundene tatsächliche Interpolationsformel nur um eine Einheit in der Iwasawa-Algebra unterscheiden und finden so eine p-adische L-Funktion, die (bis auf ein nichtkonstantes Vorzeichen) das von Fukaya und Kato vorhergesagte Verhalten zeigt.

Contents

Preface

Introduction

A major theme in modern number theory and arithmetic geometry is the connection between special values of L-functions and purely algebraic invariants, and this surprising yet undeniable liaison is one of the most fascinating phenomena in the subject. Prominent examples of results or conjectures in this spirit include the analytic class number formula, the conjecture of Birch and Swinnerton-Dyer (BSD) or Kummer's criterion for irregular primes. A very promising approach for studying (some aspects of) such links is Iwasawa Theory, where the so-called Main Conjectures relate L-functions to groups of an arithmetic origin. One of the central actors in this story is a p-adic L-function (where p can be any prime), which lives somehow in-between the complex analytic and the algebraic world. These p-adic L-functions are thus of utmost interest since they allow to build a bridge between these seemingly distant worlds and to formulate precise statements connecting them.

The rough picture is as follows. The p-adic L-function is a function on a p-adic domain taking p-adic values, such that at certain special evaluation points its values are closely related to special values of complex L-functions, in a way to be described below – the slogan is that "p-adic L-functions interpolate complex L-values p-adically". As such, the p-adic L-function is an element of a certain ring, the Iwasawa algebra. On the other hand, the groups of arithmetic origin mentioned above (more precisely, their p-parts) are in a canonical way modules over the Iwasawa algebra, and the theory of such modules attaches to each a characteristic ideal in the Iwasawa algebra. The Main Conjecture then asserts that this characteristic ideal can be generated by the p-adic L-function. In this way, the p-adic L-function is related to both the complex analytic and the arithmetic-algebraic side of the picture.

These conjectures can be formulated for a very general class of objects of interest (more precisely, motives), and some special cases have been proven. However, in general not only the Main Conjecture is open, already the existence of the p-adic L-function is not clear.

Since the discovery of the first p-adic L-function by Kubota and Leopoldt in the 1960s (interpolating special values of Dirichlet L-functions), many other p-adic L-functions have been found, interpolating the complex L-functions of a variety of objects such as arithmetic Hecke characters, modular forms (thus also elliptic curves over $\mathbb{Q}$, by modularity), and some more general automorphic representations. However, there is no universal method to do this, rather one needs new ideas for each object. As a result the p-adic L-functions often have some ambiguity in that the interpolation formulas describing their relation to complex L-values contain expressions of a non-canonical nature. These expressions are rather artefacts of the construction and do not have much of a conceptual meaning.

From a modern point of view, Iwasawa Theory appears as a consequence of the Equivariant Tamagawa Number Conjecture (ETNC). This collection of conjectures, due to Burns and Flach building on work of Deligne, Beilison, Bloch, Kato, Perrin-Riou, Fontaine and others, is by design a common vast generalisations of the analytic class number formula and the BSD conjecture (which are statements not directly related to Iwasawa Theory). It describes in a satisfactory way the meaning of values of L-functions attached to motives at integer evaluation points in terms of cohomological invariants of the motive. Later it became clear that the ETNC is in fact strong enough to imply also Iwasawa Theory. More precisely, Fukaya and Kato show in [FK06] that the ETNC implies the existence of a p-adic L-function for a certain class of motives and moreover that the Main Conjecture holds. So the ETNC is very powerful indeed, but in return only very few cases are known.

Coming back to p-adic L-functions, these ideas provide us with a precise interpolation formula that these functions should conjecturally satisfy. There were also earlier efforts to conjecture how a general interpolation formula should look like, due to Coates and Perrin-Riou [CP89; Coa89; Coa88], but the formula obtained by Fukaya and Kato has the attractive feature of being a consequence of the ETNC, which gives it some deeper explanation. This result raises the question whether the interpolation formulas produced by particular known constructions of p-adic L-functions are in accordance with the conjectural one.

Let us give a flavour of Fukaya's and Kato's formula. Arguably, the most interesting expressions in their interpolation formulas are complex and p-adic periods defined via comparison isomorphisms between different cohomologies of the motive in consideration (coming from complex de Rham theory and p-adic Hodge theory). Roughly speaking, the conjecture predicts an interpolation behaviour of the form

$$\frac{\text{value of } p\text{-adic L-function}}{p\text{-adic period}} = (\text{some “easy” correction factors}) \cdot \frac{\text{value of complex L-function}}{\text{complex period}}$$

at certain evaluation points (see conjecture 1.3.41 for the precise formula). The above should be an equality of elements of $\overline{\mathbb{Q}}$, which means that the periods describe the transcendental parts of the L-values (as the correction factors are always algebraic). This underlines the significance of the periods. Unfortunately their rather abstract definition makes them difficult to compute. It is therefore a delicate task to check whether a particular p-adic L-function has exactly the conceptual interpolation behaviour predicted by Fukaya and Kato.

Our results for families of modular forms

Our purpose is to provide some evidence for Fukaya's and Kato's interpolation formula. We focus on elliptic modular forms and families of such, which are in some sense the easiest non-trivial example (in that they are “non-abelian”, i. e. of rank greater than one, but still rather accessible). For a newform f, by constructions of Deligne and Scholl [Del69; Sch90] we have a motive $\mathcal{M}(f)$ whose complex L-function is the L-function of f and for which Fukaya's and Kato's theory predicts a p-adic L-function. The methods of Fukaya and Kato can also be applied to (suitably defined) families of motives and yield then a conjectural p-adic L-function for the whole family interpolating the p-adic L-functions of the individual motives, as Barth showed in his thesis [Bar11]. Families of modular forms provide a natural example to which this framework applies.

For Hida families of modular forms there is a construction of a p-adic L-function due to Kitagawa [Kit94] whose interpolation formula at a first glance looks similar to the conjectural one. However, instead of actual periods, his formula contains expressions we want to call “error terms”, as they depend on non-canonical choices and a priori have no conceptual meaning. It is therefore natural to ask whether Kitagawa's function matches with Fukaya's and Kato's conjecture.

To illustrate the conjectures in this particular case, we aim to compute the expressions in its interpolation formula, most notably the periods, and to express these in terms of

Kitagawa's error terms. We find the following results.

Theorem: *(a) The complex period is essentially equal to Kitagawa's complex error term.*

(b) Impose a technical condition on the Hida family. Then the p-adic period differs from Kitagawa's p-adic error term essentially just by a unit which is global for the whole family (i. e. comes from a unit U in the whole Iwasawa algebra).

Here, "essentially" means: up to a Gauß sum and a power of $2\pi\mathrm{i}$ in the complex case, and up to a Gauß sum and an ε-factor in the p-adic case. These differences are expected and desirable. The technical condition is satisfied if the image of the Galois representation attached to the Hida family contains SL_2. See theorems IV.4.1, IV.4.9 and IV.4.10 for the precise statements and condition IV.4.4 for the precise condition. From a technical point of view, these are the main results of this work.

We continue and compute the other expressions in the interpolation formula, which turn out to be in good analogy to the expressions in Kitagawa's formula. The idea is then to alter Kitagawa's p-adic L-function by the unit U from the above theorem to obtain a p-adic L-function which matches nicely with the formula by Fukaya and Kato. At this point it turns out that unfortunately their conjectures seem to be slightly wrong. In fact, the final interpolation formula we obtain *differs* from the conjectural one by a non-constant sign that cannot be interpolated.[1] Thus it seems that Fukaya's and Kato's conjectures should be modified slightly in order to remedy this. While there are some suggestions, it lies beyond the scope of this work to study systematically how this could be resolved in the general setting of Fukaya and Kato.

To conclude, the main result we obtain in the end is the following:

Theorem: *Continue to impose the aforementioned technical condition. Then there exists a p-adic* L*-function for the Hida family whose interpolation behaviour is as predicted by Fukaya and Kato, up to the problematic sign mentioned above.*

See theorem IV.5.10 for the precise statement.

We now give a short overview of the content of this work and briefly explain our method of proof; see below for a more detailed account.

To compute the complex period, we give a precise description of the de Rham and Betti realisations of the motive $\mathcal{M}(f)$ and the complex comparison isomorphism. We find that the de Rham realisation is related to the space of cusp forms and that the Betti realisation is an Eichler-Shimura type cohomology group. The comparison isomorphism between them is essentially given by the (classical) Eichler-Shimura isomorphism, which has a rather explicit description. This allows us to compute the complex period.

In the p-adic case, Faltings [Fal87; Fal88] constructed an analogue of the Eichler-Shimura isomorphism between the de Rham and the p-adic realisation which is again essentially the comparison isomorphism. To study how this behaves in families, we use as the most important ingredient to our work the rather recent result that Faltings' p-adic Eichler-Shimura isomorphisms can be interpolated in families. This was conjectured by Ohta [Oht95] and proved by Kings, Loeffler and Zerbes [KLZ17] (building on work of Kato [Kato4]) for Hida families and Andreatta, Iovita and Stevens [AIS15] for overconvergent families. With this result at hand, we can define the constant U and express the p-adic period in terms of U and Kitagawa's p-adic error term.

[1] This problem was discovered independently by Y. Zaehringer. We describe it in section IV.3.

Chapter I contains some loosely connected preliminaries, most importantly an overview of the theory of motives and the work of Fukaya and Kato. In chapter II we provide a detailed description of modular curves and the motive $\mathcal{M}(f)$ attached to a modular form. Chapter III is the technical heart of this work: we introduce modular symbols and error terms and explain our most important ingredient, the p-adic Eichler-Shimura isomorphism in families. In the final chapter IV we put our previous work together, compute the periods and find the p-adic L-function we want.

Some remarks about our result should be given. The idea of using p-adic Eichler-Shimura isomorphisms to prove such a result was already mentioned by Ohta in [Oht95]. Ohta himself could not follow this strategy – while he constructed a p-adic Eichler-Shimura map in families, he was unable to prove that it indeed interpolates Faltings' maps. The author wants to remark that he was not aware of Ohta's article when the idea occurred to him.

A result similar to ours appears in Ochiai's unpublished work [Och05]. However, there are several differences. First, Ochiai uses a different definition of the p-adic period (although it is related to the one we use). Second, he does not work with the formula by Fukaya and Kato, but with yet another conjectural interpolation formula formulated by himself (which is not deduced from the ETNC and has a different shape in general). Third, his constant U lies a priori in a much larger ring than ours. Finally, the proof is only sketched there and an important point is omitted. It seems that Ochiai does not use p-adic Eichler-Shimura isomorphisms, so in any case our proof is different.

An advantage of our technique is that it is likely to work in more general situations. Here we indicate some possible generalisations, which we explain in more detail in the very last section IV.6.

The same methods should also apply to overconvergent families. For these, Bellaïche [Bel11] has constructed a p-adic L-function which has properties very similar to Kitagawa's, Zaehringer [Zae17] extended the work of Fukaya and Kato to cover the non-ordinary case and Andreatta, Iovita and Stevens [AIS15] provided the interpolation of the p-adic Eichler-Shimura isomorphism in families. Due to technical limitations, we only sketch the proof in this case, but nonetheless it should become clear how it will work. The author hopes to write down a complete proof in a future work. This in some sense completes the study of (commutative) p-adic L-functions for elliptic modular forms over $\mathbb{Q}$. As further generalisations it should be possible to extend these ideas to more general automorphic representations over larger fields, such as Hilbert modular forms. In some cases the necessary preliminary results are already known.

Outline of our methods and proofs

We give an overview of our construction, which will hopefully serve as a guideline to this work. For the sake of readability we will occasionally omit some details such as twists or unimportant factors here, at the cost of being slightly imprecise; also we assume that $p \neq 2$.

The main task is to "compute" the complex and p-adic motivic periods, as defined by Deligne, Fukaya and Kato, of the motive attached to a modular form. In the known constructions of p-adic L-functions for modular forms, there are expressions which we call "error terms" in the interpolation formulas, playing the role of the periods, and our aim is to compare the motivic periods to these error terms. Hence "compute" in this context should be understood as expressing them as explicitly as possible in terms of the error terms.

The motivic periods are defined essentially as determinants of comparison isomorphisms.

If M is a motive, we consider its Betti, de Rham and p-adic realisations, which we denote by M_{B}, M_{dR} and M_p, respectively. Between these realisations we have the comparison isomorphisms

$$\begin{aligned} \mathrm{cp}_\infty &: M_{\mathrm{B}} \otimes \mathbb{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbb{C}, \\ \mathrm{cp}_{\text{ét}} &: M_{\mathrm{B}} \otimes \mathbb{Q}_p \xrightarrow{\sim} M_p, \\ \mathrm{cp}_{\mathrm{dR}} &: M_p \otimes \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathrm{B}_{\mathrm{dR}}, \end{aligned}$$

where B_{dR} is Fontaine's field of p-adic periods. On the Betti side, we look at the subspace M_{B}^+ fixed by complex conjugation and on the de Rham side we look at the quotient $M_{\mathrm{dR}}/\mathrm{fil}^0 M_{\mathrm{dR}}$ and fix bases of these vector spaces. The complex period is then defined as the determinant of cp_∞, while the p-adic period is defined as the determinant of $\mathrm{cp}_{\mathrm{dR}} \circ \mathrm{cp}_{\text{ét}}$ (up to some factor which we ignore here), both maps viewed as going from M_{B}^+ to $M_{\mathrm{dR}}/\mathrm{fil}^0 M_{\mathrm{dR}}$ with appropriately extended coefficients, and both determinants calculated with respect to the same fixed bases. We explain this in sections I.3.1, I.3.3 and I.3.5.

We want to compute these periods for the motive $\mathcal{M}(f)$ attached to a newform f of weight $k \geq 2$ and level $N \geq 4$.[2] This motive is a submotive of a motive called ${}^N_k\mathcal{W}$; more precisely, on the realisations of ${}^N_k\mathcal{W}$ we have an action of the Hecke algebra, and the realisations of $\mathcal{M}(f)$ are the subspaces cut out by the Hecke eigenvalues of f. It turns out that it is more convenient to work with ${}^N_k\mathcal{W}$ and later specialise to these subspaces. We hence need to study the realisations and comparison isomorphisms of ${}^N_k\mathcal{W}$. For simplicity, we assume in this introduction that f has Fourier coefficients in $\mathbb{Z}$.

Let $X(N)$, $Y(N)$, $X_1(N)$, $Y_1(N)$ be the modular curves (over $\mathbb{Q}$) classifying elliptic curves with level N structure (see section II.1.1). Over each of these we have a universal elliptic curve, and we denote the maps from the latter to the modular curves by f. We then get for the Betti realisation (see theorem II.5.6)

$${}^N_k\mathcal{W}_{\mathrm{B}} = \mathrm{H}^1_{\mathrm{p}}(Y(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}}).$$

Using monodromy, we can write this more concretely as

$${}^N_k\mathcal{W}_{\mathrm{B}} = \mathrm{H}^1_{\mathrm{p}}(\Gamma(N), \mathrm{Sym}^{k-2} \mathbb{Q}^2).$$

For the de Rham realisation, we find an exact sequence (see propositions II.5.9 and II.5.10)

$$0 \longrightarrow \mathrm{S}_k(X(N), \mathbb{Q}) \longrightarrow {}^N_k\mathcal{W}_{\mathrm{dR}} \longrightarrow \mathrm{S}_k(X(N), \mathbb{Q})^\vee \longrightarrow 0$$

coming from the Hodge filtration; here the subspace of cusp forms $\mathrm{S}_k(X(N), \mathbb{Q})$ is precisely $\mathrm{fil}^0\, {}^N_k\mathcal{W}_{\mathrm{dR}}$. For the p-adic realisation we find

$${}^N_k\mathcal{W}_p = \mathrm{H}^1_{\mathrm{p,ét}}(Y(N) \underset{\mathbb{Q}}{\times} \overline{\mathbb{Q}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}}_p).$$

A central tool for our calculations is a canonical perfect pairing on the motive ${}^N_k\mathcal{W}$

$$\langle \cdot, \cdot \rangle : {}^N_k\mathcal{W} \times {}^N_k\mathcal{W} \longrightarrow \mathbb{Q},$$

[2] More precisely, we want to compute them for the *critical twists* of $\mathcal{M}(f)$, which gives us some extra factors which we ignore here. Also, the same methods should work for level less than 4, see remark II.1.6.

which we study in section II.8. The Hecke operators are self-adjoint with respect to this pairing. It induces a pairing on each realisation, and the various comparison isomorphisms respect this pairing. Moreover, it induces a perfect pairing

$$\mathcal{M}(f) \times \mathcal{M}(f) \longrightarrow \mathbb{Q}$$

and on the Betti side a perfect pairing

$$\mathcal{M}(f)^{\pm}_{\mathrm{B}} \times \mathcal{M}(f)^{\mp}_{\mathrm{B}} \longrightarrow \mathbb{Q},$$

while the restriction

$$\mathcal{M}(f)^{\pm}_{\mathrm{B}} \times \mathcal{M}(f)^{\pm}_{\mathrm{B}} \longrightarrow \mathbb{Q}$$

vanishes identically. On the de Rham side it induces the canonical pairing

$$\mathrm{S}_k(X(N), \mathbb{Q})^{\vee} \times \mathrm{S}_k(X(N), \mathbb{Q}) \longrightarrow \mathbb{Q}.$$

Let us first take a closer look at the complex period. We have the comparison isomorphism

$$\mathrm{cp}_{\infty} \colon\ {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C} \xrightarrow{\ \sim\ } {}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C}.$$

On the other hand, a classical result is the Eichler-Shimura isomorphism (theorem II.6.3)

$$\mathrm{ES} \colon \mathrm{H}^1_{\mathrm{p}}(\Gamma(N), \mathrm{Sym}^{k-2}\,\mathbb{C}^2) \xrightarrow{\ \sim\ } \mathrm{S}_k(X(N), \mathbb{C}) \oplus \overline{\mathrm{S}_k(X(N), \mathbb{C})}.$$

A crucial step in our calculation of the complex period is the observation that these are compatible in the following sense (see theorem II.6.7): we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{S}_k(X(N), \mathbb{C}) \arrow[dl, "\mathrm{Hodge}"'] \arrow[dr, "\mathrm{ES}"] & \\
{}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C} \arrow[rr, "\mathrm{cp}_{\infty}", "\sim"'] & & {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C}
\end{tikzcd}$$

where the left map comes from the Hodge filtration of ${}^{N}_{k}\mathcal{W}_{\mathrm{dR}}$, the right map is the Eichler-Shimura map and the bottom map is the comparison isomorphism. Since the Eichler-Shimura map admits a concrete description in terms of cocycles, this makes the comparison isomorphism a lot more explicit.

At this point we need to look at the definition of the complex error term. For this we need modular symbols, which are certain cohomology classes on modular curves (see section III.2 for their definition; they are closely related to the above group cohomology groups). We denote them by $\mathrm{MS}_k(N, R)$, where R is some coefficient ring. They carry an action of complex conjugation and a Hecke action, and the eigenspaces $\mathrm{MS}_k(N, \mathbb{Q})^{\pm}[f]$, where the complex conjugation acts as ± 1 and the Hecke algebra acts via the eigenvalues of f, are one-dimensional. Over $\mathbb{C}$, we have a canonical element $\xi_f \in \mathrm{MS}_k(N, \mathbb{C})$ attached to f, defined in terms of an explicit cocycle, which we can decompose as $\xi_f = \xi_f^+ + \xi_f^-$ with $\xi_f^{\pm} \in \mathrm{MS}_k(N, \mathbb{C})^{\pm}[f]$. If we now fix bases $\eta_f^{\pm} \in \mathrm{MS}_k(N, \mathbb{Q})^{\pm}[f]$ then, since modular symbols behave well with respect to base change, we find elements $\mathcal{E}_{\infty}(f, \eta_f^{\pm}) \in \mathbb{C}^{\times}$ such that $\xi_f^{\pm} = \mathcal{E}_{\infty}(f, \eta_f^{\pm})\eta_f^{\pm}$. These are the complex error terms.

The important observations to compare these to the complex motivic periods are: first there is a canonical map (see (III.2.1))

$$\mathrm{MS}_k(N,\mathbb{Q}) \longrightarrow {}^N_k\mathcal{W}_{\mathrm{B}} \tag{0.1}$$

which is Hecke equivariant and respects complex conjugation, and second, over $\mathbb{C}$ this map sends $\xi_f \in \mathrm{MS}_k(N,\mathbb{C})$ to $\mathrm{ES}(f) \in {}^N_k\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C}$, where ES is the Eichler-Shimura map (see lemma III.2.6).

In section IV.4.1 we choose bases of the Betti and de Rham side. By the above descriptions, the quotient $\mathcal{M}(f)_{\mathrm{dR}}/\mathrm{fil}^0\,\mathcal{M}(f)_{\mathrm{dR}}$ is the one-dimensional subspace of $\mathrm{S}_k(X(N),\mathbb{Q})^\vee$ generated by a linear form dual to f (i.e. sending f to 1 and all other vectors of a basis of $\mathrm{S}_k(X(N),K)$ containing f to 0). We fix such a linear form and call it δ. It thus satisfies $\langle\delta,f\rangle_{\mathrm{dR}} = 1$. On the Betti side, we can use the images of $\eta_f^\pm$ in ${}^N_k\mathcal{W}_{\mathrm{B}}^\pm$ under the above map (0.1) as a basis of ${}^N_k\mathcal{W}_{\mathrm{B}}^\pm$. Without loss of generality we assume that $\langle\eta_f^+,\eta_f^-\rangle_{\mathrm{B}} = 1$.

In section IV.4.2 we then compute the complex periods with respect to these bases. From the definition of $\mathcal{E}_\infty(f,\eta_f^\pm)$ and the fact that the pairing vanishes on $\mathcal{M}(f)^\pm \times \mathcal{M}(f)^\pm$, we first see that

$$\mathcal{E}_\infty(f,\eta_f^\pm) = \left\langle \eta_f^\mp, \xi_f \right\rangle_{\mathrm{B}}.$$

Transferring this to the de Rham side, writing $\rho^\mp$ for the image of $\eta_f^\mp$ under the comparison isomorphism and using that ξ_f becomes f on the de Rham side, we obtain

$$\left\langle \rho^\mp, f \right\rangle_{\mathrm{dR}} = \mathcal{E}_\infty(f,\eta_f^\pm).$$

This implies $\rho^\mp = \mathcal{E}_\infty(f,\eta_f^\pm)\delta$ and we see that the complex period equals the error term.

We now turn to the p-adic side. If we consider the comparison isomorphism for ${}^N_k\mathcal{W}$ over B_{HT} and look at its degree 0 part, we obtain (see theorem II.6.9)

$${}^N_k\mathcal{W}_p \otimes \mathbb{C}_p \xrightarrow{\ \sim\ } \mathrm{S}_k(X(N),\mathbb{C}_p) \;\oplus\; \mathrm{H}^1(X(N),\omega_{X(N)}^{2-k}) \otimes \mathbb{C}_p. \tag{0.2}$$

This was proved by Faltings and can be seen as a p-adic analogue of the Eichler-Shimura isomorphism. Note that from (0.1) and the comparison isomorphism $\mathrm{cp}_{\text{ét}}$ we have a canonical map

$$\mathrm{MS}_k(N,\mathbb{Z}_p) \longrightarrow {}^N_k\mathcal{W}_p. \tag{0.3}$$

In his construction of the two-variable p-adic L-function for a Hida family F, Kitagawa interpolates the modular symbols $\mathrm{MS}_k(Np^r,\mathbb{Z}_p)$ for varying $k \geq 2$ and $r \geq 1$. He obtains a large module $\mathbb{MS}^{\mathrm{ord}}(Np^\infty,\Lambda)$ of Λ-adic ordinary modular symbols (here $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ with $\Gamma = 1 + p\mathbb{Z}_p$ is the Iwasawa algebra) such that the reduction of $\mathbb{MS}^{\mathrm{ord}}(Np^\infty,\Lambda)$ modulo special prime ideals in Λ is the module of modular symbols of fixed level and weight (see sections III.4.1 and III.4.3). More precisely, if $P_{k,\varepsilon}$ is the kernel of the morphism $\Lambda \longrightarrow \mathbb{Z}_p$ induced by $\varepsilon\kappa^k$, where ε is a character of Γ of order p^r and $\kappa\colon \Gamma \longrightarrow \mathbb{Z}_p^\times$ is the canonical inclusion, then

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty,\Lambda)\Big/P_{k,\varepsilon} \cong \mathrm{MS}_k(Np^r,\mathbb{Z}_p)[\varepsilon]$$

(see theorem III.4.10 (a)).

Due to his technique, Kitagawa's two-variable p-adic L-function contains a p-adic error term in its interpolation formula which is defined as follows. On $\mathbb{MS}^{\mathrm{ord}}(Np^\infty,\Lambda)$ we have again a Hecke action and a complex conjugation, and $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda)^\pm[F]$ is free of rank 1 over Λ. If we fix a basis $\Xi^\pm$, then the image of $\Xi^\pm$ in $\mathrm{MS}_k(Np^r, \mathbb{Z}_p)$ may be written as

$$\Xi^\pm \bmod P_{k,\varepsilon} = \mathcal{E}_p(\Xi^\pm, \eta^\pm_{k,\varepsilon})\eta^\pm_{k,\varepsilon}$$

with some $\mathcal{E}_p(\Xi^\pm, \eta^\pm_{k,\varepsilon}) \in \mathbb{Z}_p$ if $\eta^\pm_{k,\varepsilon} \in \mathrm{MS}_k(Np^r, \mathbb{Z}_p)^\pm[f_{k,\varepsilon}]$ is a basis as before. We want to compare this error term to the p-adic period of $\mathcal{M}(f_{k,\varepsilon})$.

Now our most important ingredient, namely the fact that the p-adic Eichler-Shimura isomorphism can be interpolated in families, comes into play. This means that there is a canonical Hecke equivariant map

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda)$$

(the right side being the module of Λ-adic cusp forms) such that its reduction modulo $P_{k,\varepsilon}$ coincides with the map

$$\mathrm{MS}_k(N, \mathbb{Z}_p) \longrightarrow {}^N_k\mathcal{W}_p \otimes \mathbb{C}_p \longrightarrow\!\!\!\!\!\rightarrow \mathrm{S}_k(X(N), \mathbb{C}_p)$$

obtained from (0.3) and the projection to the first factor in (0.2). This was proved by Ohta, Kings, Loeffler and Zerbes (theorem III.5.11). Using this and the fact that $\mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda)[F]$ is free of rank 1 and generated by F, we define a constant $U \in \Lambda$ as the unique U such that Ξ^- is mapped to UF. Write $U_{k,\varepsilon} \in \mathbb{Z}_p$ for the reduction of U modulo $P_{k,\varepsilon}$.

In section IV.4.3 we compute the p-adic period. Reducing the equation defining U, we see that

$$\Xi^-_{k,\varepsilon} = \mathcal{E}_p(\Xi^-, \eta^-_{k,\varepsilon})\eta^-_{k,\varepsilon} \longmapsto U_{k,\varepsilon} f_{k,\varepsilon}$$

under the p-adic comparison isomorphism. Hence if we again write $\rho^+_{k,\varepsilon}$ for the image of $\eta^+_{k,\varepsilon}$ under the comparison isomorphism, we can perform a similar computation as in the complex case:

$$\left\langle \eta^+_{k,\varepsilon}, \frac{\mathcal{E}_p(\Xi^-, \eta^-_{k,\varepsilon})}{U_{k,\varepsilon}} \eta^-_{k,\varepsilon} \right\rangle_p = \frac{\mathcal{E}_p(\Xi^-, \eta^-_{k,\varepsilon})}{U_{k,\varepsilon}}$$

and therefore

$$\left\langle \rho^+_{k,\varepsilon}, f_{k,\varepsilon} \right\rangle_{\mathrm{dR}} = \frac{\mathcal{E}_p(\Xi^-, \eta^-_{k,\varepsilon})}{U_{k,\varepsilon}}.$$

It turns out that this works also with reversed signs, i. e. we also have

$$\left\langle \rho^-_{k,\varepsilon}, f_{k,\varepsilon} \right\rangle_{\mathrm{dR}} = \frac{\mathcal{E}_p(\Xi^+, \eta^+_{k,\varepsilon})}{U_{k,\varepsilon}}.$$

This shows $\rho^\mp_{k,\mathrm{e}} = \frac{\mathcal{E}_p(\Xi^\pm, \eta^\pm_{k,\varepsilon})}{U_{k,\varepsilon}}\delta$ and we conclude that the p-adic period differs from the error term only by the constant $U_{k,\varepsilon}$.

The constant U is global for the Hida family and we can moreover show that U is even a unit. This means that if we multiply Kitagawa's two-variable p-adic L-function by U^{-1}, the error term in his interpolation formula is transformed into the p-adic period, proving our main theorem.

Acknowledgements

First and foremost, I thank my advisor Otmar Venjakob for giving me the opportunity to work on these questions and for his constant support and interest in my work. He provided a pleasant atmosphere in which I enjoyed learning and doing mathematics, as well as the whole Mathematical Institute at the Faculty for Mathematics and Scientific Computing at Heidelberg University. At this point I also want to acknowledge financial support of the Deutsche Forschungsgemeinschaft (DFG) and the Mathematics Center Heidelberg (Match).

I express my gratitude to David Loeffler for answering numerous questions. His precise explanations helped me a lot in understanding many subtle points especially in the theory of modular forms.

Many other people have helped me during this work by discussing problems with me, answering my questions, proofreading or in other ways. For this support I want to thank Johannes Anschütz, Gebhard Böckle, Matthias Flach, Olivier Fouquet, Katharina Hübner, Fabian Januszewski, Jaclyn Lang, Marius Leonhardt, Tadashi Ochiai, Andreas Riedel, Birgit Schmoetten-Jonas, Oliver Thomas, Malte Witte, Jonathan Zachhuber and Yasin Zaehringer.

Finally I thank my family and especially Eva Maria Preinfalk for their support and confidence.

Heidelberg, 9th August, 2017 Michael Fütterer

Notations and conventions

General conventions

We denote by $\mathbb{N}$ the natural numbers beginning with 1 and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol p without further explanation denotes a prime number. Unless we explicitly state the contrary, a prime called p will assumed to be odd, for technical reasons.

All rings are supposed to have a unit and homomorphisms of rings are always unitary.

Numbers and Galois groups

We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and $\mathbb{C}$ of $\mathbb{R}$ and write $\mathbb{C}_p$ for the p-adic completion of $\overline{\mathbb{Q}}_p$. We further use the period rings B_{HT}, B_{dR}, B_{st} and $\mathrm{B}_{\mathrm{cris}}$ from p-adic Hodge theory. The Frobenius map on $\mathrm{B}_{\mathrm{cris}}$ will be denoted by φ_{cris}, and we use the same symbol for the Frobenius map on similar objects (such as B_{st}, $\mathrm{D}_{\mathrm{cris}}$, D_{st}).

We fix throughout the work a square root $\mathrm{i} \in \mathbb{C}$ of -1. By a *pair of embeddings of* $\overline{\mathbb{Q}}$, we mean a pair (ι_∞, ι_p) of embeddings $\iota_\infty \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \subseteq \mathbb{C}_p$. We provisionally fix such a pair of embeddings. This fixes a choice of a compatible system of p-power roots of unity $\xi = (\xi_n)_{n \geq 0}$ with $\xi_n \in \mathbb{Q}_p(\mu_{p^\infty})$ by saying that the pair of embeddings should identify ξ with the system $(\mathrm{e}^{2\pi \mathrm{i} p^{-n}})_{n \geq 0}$ of p-power roots of unity in $\mathbb{C}$. Our choice of (ι_∞, ι_p) is only provisional, we may change it at some point in this work. When we do so we thus also have to change ξ.[1]

Our fixed choice of a compatible system of p-power roots of unity $\xi = (\xi_n)_{n \geq 0}$ determines a uniformiser of $\mathrm{B}_{\mathrm{dR}}^+$, see [FO08, §4.2.2, §5.2.3]. We denote it by t_{dR}.

A *number field* is a finite extension of $\mathbb{Q}$. We do not view number fields as subfields of $\overline{\mathbb{Q}}$ a priori, so if we want to do so we have to choose an embedding $K \hookrightarrow \overline{\mathbb{Q}}$. If K is a number field and w is a place of K, then we denote by K_w the completion of K at w. If v is a place of $\mathbb{Q}$, we put $K_v := \prod_{w|v} K_w$, where the product runs over all places w of K lying above v. In particular, we then have $K_p = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$. If L is a finite extension of $\mathbb{Q}_p$, we denote by L^{nr} the maximal unramified extension of L and by $\hat{L}^{\mathrm{nr}}$ its p-adic completion.

We put $\mathrm{G}_{\mathrm{cyc}} := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ and write $\kappa_{\mathrm{cyc}} \colon \mathrm{G}_{\mathrm{cyc}} \xrightarrow{\sim} \mathbb{Z}_p^\times$ for the cyclotomic character.

For any field k we denote its absolute Galois group by G_k. We denote the nontrivial element of $\mathrm{G}_{\mathbb{R}}$ by Frob_∞. For a module M with an action of $\mathrm{G}_{\mathbb{R}}$, we denote by $M^\pm$ the submodule where Frob_∞ acts by multiplication with ± 1, respectively.

Note that our choice of a pair of embeddings of $\overline{\mathbb{Q}}$ induces various other choices. First, via restriction, it induces embeddings of local into global Galois groups, namely embeddings $\mathrm{G}_{\mathbb{R}} \hookrightarrow \mathrm{G}_{\mathbb{Q}}$ and $\mathrm{G}_{\mathbb{Q}_p} \hookrightarrow \mathrm{G}_{\mathbb{Q}}$. This then fixes a well-defined inertia subgroup at p of $\mathrm{G}_{\mathbb{Q}}$, which we denote by I_p. We denote by Frob_p a *geometric* Frobenius element in $\mathrm{G}_{\mathbb{Q}}$ at p. It is only well-defined up to conjugation and multiplication by elements of I_p, but this

[1] One says that the choice of i is like choosing an orientation of $\mathbb{C}$, while the choice of ξ is like choosing an orientation of $\mathbb{C}_p$. We thus require that $\mathbb{C}$ and $\mathbb{C}_p$ are oriented compatibly, in the above sense; this notion of compatibility depends on the pair of embeddings (ι_∞, ι_p).

will not be important in the where situations we use it. Further the inertia group fixes an embedding $\mathbb{Q}_p^{\mathrm{nr}} \hookrightarrow \overline{\mathbb{Q}}_p$ by identifying $\mathbb{Q}_p^{\mathrm{nr}}$ with the subfield of $\overline{\mathbb{Q}}_p$ fixed by I_p, and thus also $\hat{\mathbb{Q}}_p^{\mathrm{nr}} \hookrightarrow \mathbb{C}_p$.

If we have a number field K and fix further an embedding $K \hookrightarrow \overline{\mathbb{Q}}$, then this fixes embeddings $K \hookrightarrow \mathbb{C}$ and $K \hookrightarrow \overline{\mathbb{Q}}_p$, since we already fixed $\iota_\infty \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and a place $\mathfrak{p} \mid p$ of K as the kernel of

$$\mathcal{O}_K \hookrightarrow \mathcal{O}_{\mathbb{Q}_p} \twoheadrightarrow \mathcal{O}_{\mathbb{Q}_p} \big/ (p) \,.$$

Moreover, it fixes embeddings $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \hookrightarrow \mathrm{G}_{\mathbb{Q}}$ and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_{\mathfrak{p}}) \hookrightarrow \mathrm{G}_{\mathbb{Q}_p}$. The latter also fixes an embedding $K_{\mathfrak{p}} \hookrightarrow \mathrm{B}_{\mathrm{dR}}$ by identifying $K_{\mathfrak{p}}$ with $\mathrm{B}_{\mathrm{dR}}^{\mathrm{G}_{K_{\mathfrak{p}}}}$, and analogously with B_{HT}. Further, as above we get embeddings $K_{\mathfrak{p}}^{\mathrm{nr}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\hat{K}_{\mathfrak{p}}^{\mathrm{nr}} \hookrightarrow \mathbb{C}_p$.

We normalise the reciprocity map from class field theory such that it maps prime elements to *arithmetic* Frobenii. This is particularly important when we view Dirichlet characters as Galois characters.

Categories

We use the following categories:

$\mathcal{S}ets$	category of sets
$\mathcal{T}op$	category of nice[2] topological spaces
R-$\mathcal{M}od$, $\mathcal{M}od$-R	category of left resp. right modules over a ring R
$\mathcal{S}ch_{/S}$	category of schemes over a fixed base scheme S; if $S = \operatorname{Spec} R$ is affine we write $\mathcal{S}ch_{/R}$
$\mathcal{S}ch = \mathcal{S}ch_{/\mathbb{Z}}$	category of schemes
$\mathcal{R}ep_R(G)$	linear representations of a group G on finitely generated projective R-modules; continuous representations if R and G come with a topology

We denote limits (projective limits) by "$\varprojlim$" and colimits (direct limits, inductive limits) by "$\varinjlim$".

Schemes and group schemes

If X and T are S-schemes, we sometimes write $X_{/T} \coloneqq X \times_S T$ for the base change of X to T.

If S is a scheme and G is a group, we sometimes write $\underline{G}_{/S}$ for the constant group scheme. We write μ_n for the group scheme over $\mathbb{Z}$ of roots of unity of order dividing n.

[2] General topological spaces do not play an important role in this work. Therefore, we assume for simplicity that any topological space we consider is locally contractible, locally path-connected and semi-locally simply connected. Note that this implies that sheaf cohomology with constant coefficients agrees with singular cohomology, see [Voi02, Thm. 4.47].

Filtrations, gradings and Hodge-Tate weights

Let $i \in \mathbb{Z}$. If M is some decreasingly filtered module over some ring, we write $\mathrm{fil}^i M$ for the i-th filtration step, and we denote the graded pieces by $\mathrm{gr}^i M \coloneqq \mathrm{fil}^i M/\mathrm{fil}^{i+1} M$ and write

$$\mathrm{gr}(M) = \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i M$$

for the associated graded module. If M is some graded module, we write $\mathrm{gr}^i M$ for the submodule of degree i, or sometimes just M_i.

If M is a filtered or graded module, we define its Hodge-Tate weights to be those $i \in \mathbb{Z}$ such that $\mathrm{gr}^i M \neq 0$. If M is a vector space over a field K, then we define the multiplicity of the Hodge-Tate weight as $\dim_K \mathrm{gr}^i M$ for such an i.

The above convention implies in particular that the cyclotomic character has Hodge-Tate weight -1 (see fact 1.3.6).

If K is a field and V is a finite-dimensional decreasingly filtered K-vector space, then we define its *Hodge invariant* $\mathrm{t_H}(V) \in \mathbb{Z}$ as the sum of the Hodge-Tate weights with multiplicities

$$\mathrm{t_H}(V) \coloneqq \sum_{i\in\mathbb{Z}} i \dim_K \mathrm{gr}^i V.$$

In particular, if $\dim_K V = 1$, then $\mathrm{t_H}(V) = \max\{i \in \mathbb{Z} : \mathrm{fil}^i V = V\}$. By [FO08, §6.4.2], if $\dim_K V = n$, then

$$\mathrm{t_H}(V) = \mathrm{t_H}\Big(\bigwedge^n V\Big).$$

If $W = \bigoplus_{i\in\mathbb{Z}} W_i$ is a finite-dimensional graded K-vector space, we view it as a filtered K-vector space in the tautological way and write

$$\mathrm{t_H}(W) \coloneqq \sum_{i\in\mathbb{Z}} i \dim_K W_i.$$

In particular, if $\dim_K W = 1$, then $\mathrm{t_H}(W)$ is the unique $i \in \mathbb{Z}$ such that $W_i \neq 0$.

Homological algebra

Whenever we write "complex" without further specification, we will mean a *cochain* complex in some abelian category $\mathcal{A}$. If $C^\bullet$ is some complex in $\mathcal{A}$, we denote by $C[i]^\bullet$ the complex with $C[i]^n = C^{n+i}$, for $i \in \mathbb{Z}$, with the same differentials as the original one.[3] We denote the bounded below derived category of $\mathcal{A}$ by $\mathrm{D}^+(\mathcal{A})$. The image of some object $A \in \mathcal{A}$ in $\mathrm{D}^+(\mathcal{A})$ will be denoted by $A[0]$. More generally, the class of the complex having 0 everywhere except in degree n and having A in degree n will be denoted by $A[-n]$. We denote the class of some complex $C^\bullet$ in $\mathrm{D}^+(\mathcal{A})$ just by the same symbol. If we want to emphasise the grading of a complex, we write something like

$$\cdots \longrightarrow E \longrightarrow \underset{\substack{\uparrow \\ 0}}{F} \longrightarrow G \longrightarrow \cdots$$

to indicate that F is in degree 0.

We denote hypercohomology resp. hyper-derived functors with $\mathbf{H}^*$ resp. $\mathbf{R}^*$ as opposed to H^* resp. R^*.

[3] Note that there are several different conventions about how to define the differentials on $C[i]$; for example [GM03] defines it at $(-1)^i$ times the original one.

Cosets

To fix the notion of cosets, let G be a group and U a subgroup. If we let U act on G from the left by left multiplication $(u, g) \longmapsto ug$, then the orbit of $g \in G$ is Ug and we call this a left coset. If we let U act on G from the right by right multiplication $(g, u) \longmapsto gu$, then the orbit of a $g \in G$ is gU and we call this a right coset. We denote the set of left cosets by $U\backslash G$ and the set of right cosets by G/U. In parts of the literature, the notions of left and right cosets are interchanged.

Matrices

Let R be a ring and $n \in \mathbb{N}$. We denote the ring of $n \times n$ quadratic matrices with coefficients in R by $\mathrm{M}_n(R)$ and its unit group by $\mathrm{GL}_n(R)$. If R is an ordered ring (e. g. a subring of $\mathbb{R}$), then $\mathrm{M}_n^+(R)$ denotes the submonoid of the multiplicative monoid of $\mathrm{M}_2(R)$ consisting of matrices $A \in \mathrm{M}_2(R)$ with $\det A > 0$ and $\mathrm{GL}_n^+(R) := \mathrm{GL}_n(R) \cap \mathrm{M}_n^+(R)$.

If in a matrix some entry is left empty, this stands for a 0, while a "$*$" stands for an arbitrary element of R.

For $n = 2$, we define the *main involution* ι of $\mathrm{GL}_2(R)$ by

$$\alpha^\iota := (\det \alpha)\alpha^{-1}, \quad \alpha \in \mathrm{GL}_2(R).$$

Hence if $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Congruence subgroups and submonoids

We define some standard congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and congruence submonoids of $\mathrm{M}_2(\mathbb{Z})$. Let $N, M \in \mathbb{N}$.

We define

$$ə := \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

and write $\mathrm{G}_ə$ for the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ consisting of ə and the identity. Note that this matrix is often denoted by ε, but since ε may be used for various other things in this work, we denote it by ə and use this symbol exclusively for this matrix.

Define groups

$$\begin{aligned}
\Gamma(N) &:= \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\right\} \\
\Gamma_1(N) &:= \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N\right\} \\
\Gamma_0(N) &:= \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\right\} \\
\Gamma_{1,0}(N, M) &:= \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N,\ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod M\right\} \\
\Gamma_1^0(N, M) &:= \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N,\ \gamma \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod M\right\}
\end{aligned}$$

and monoids

$$\begin{aligned}
\mathrm{M}_2^{(N)} &:= \{\alpha \in \mathrm{M}_2(\mathbb{Z}) : \det\alpha \neq 0,\ (\det\alpha, N) = 1\}\\
\Delta_1(N) &:= \left\{\alpha \in \mathrm{M}_2^+(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod N\right\}\\
\Delta_0(N) &:= \left\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2^+(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N,\ (a, N) = 1\right\}.
\end{aligned}$$

We note that in the notation of [Shi71, §3.3], if we choose $\mathfrak{h} = (\mathbb{Z}/N)^\times$ and $t = 1$ we obtain $\Gamma' = \Gamma_0(N)$, $\Delta' = \Delta_0(N)$ and $\Delta'_N = \Delta_0(N) \cap \mathrm{M}_2^{(N)}$, whereas if we choose $\mathfrak{h} = \{1\}$ and $t = 1$ we obtain $\Gamma' = \Gamma_1(N)$, $\Delta' = \Delta_1(N)$ and $\Delta'_N = \Delta_1(N) \cap \mathrm{M}_2^{(N)}$ there. Further the Δ_N there is $\mathrm{M}_2^{(N)} \cap \mathrm{M}_2^+(\mathbb{Z})$.

All the groups defined above are normalised by ә. If Δ is any submonoid of $\mathrm{M}_2(\mathbb{Z})$, we write $\Delta^ә$ for the submonoid of $\mathrm{M}_2(\mathbb{Z})$ generated by Δ and ә. For example, we have

$$\Delta_0(N)^ә = \left\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N,\ (a, N) = 1,\ \det\alpha \neq 0\right\}.$$

This monoid is denoted $S_0(N)$ in [PS11, §2.1], [PS13, §2.1], [Ste94, (0.6)].

Gauß sums

Let $\chi\colon (\mathbb{Z}/N)^\times \longrightarrow K^\times$ be a Dirichlet character with values in some number field K. If we fix an embedding $\iota_K\colon K \hookrightarrow \overline{\mathbb{Q}}$, we can define its Gauß sum as

$$\mathrm{G}(\chi, \iota_K) := \sum_{a \in (\mathbb{Z}/N)^\times} \iota_K(\chi(a))\mathrm{e}^{2\pi\mathrm{i}a/N} \in \mathbb{C},$$

using our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. If L is the completion of K at the place induced by ι_K and our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, then ι_K induces $\iota_L\colon L \hookrightarrow \overline{\mathbb{Q}}_p$. If $N = p^m$ is a prime power, then $\mathrm{G}(\chi, \iota_K)$ corresponds via our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ to

$$\mathrm{G}(\chi, \iota_L) := \sum_{a \in (\mathbb{Z}/N)^\times} \iota_L(\chi(a))\xi_m^a \in \overline{\mathbb{Q}}_p,$$

where ξ_m is from our fixed compatible system of p-power roots of unity. In this way, we always view Gauß sums as elements of $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. We will often drop ι_K or ι_L from the notation, but one should keep in mind that Gauß sums depend on these embeddings.

Miscellaneous

If D is an integer coprime to a prime p, we put $\mathbb{Z}_{p,D} := \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/Dp^n$.

We denote by $\mathfrak{h} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ the complex upper half plane and by $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}) = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$ the completed upper half plane with the usual topology, see e. g. [DS05, §2.4].

If X is any topological space and A is an abelian group, then $\underline{A}$ denotes the constant sheaf defined by A. If $\mathcal{F}$ is some sheaf X, then we denote by $\mathrm{H}^i_{\mathrm{p}}(X, \mathcal{F})$ the i-th parabolic cohomology, which is defined as the image of cohomology with compact support $\mathrm{H}^i_c(X, \mathcal{F})$ in $\mathrm{H}^i(X, \mathcal{F})$, for any $i > 0$.

If V is a vector space over some field K, we denote by $V^\vee$ its K-linear dual. The field K will usually be clear from the context.

If V or ρ is some representation of some group, we write V^* resp. ρ^* for the *contragredient* (i. e. dual) representation.

We denote the symmetric group on n letters by $\mathfrak{S}_n$.

We denote the conductor of a Dirichlet character χ by $\operatorname{cond} \chi$.

For a prime p and $N \in \mathbb{N}$ we denote by $\operatorname{ord}_p N$ the p-adic valuation of N, i. e. the maximal $r \in \mathbb{N}_0$ such that $p^r \mid N$.

Chapter I.

Preliminaries

1. Abstract Hecke theory

"Hecke operators" act on a variety of objects. Unfortunately there are quite some slightly different conventions to define them in the literature, and comparing these can be quite messy in explicit situations. We therefore first standardise the notion of Hecke operators using the abstract Hecke algebra for a Hecke pair of a monoid and a subgroup, which was introduced by Shimura in [Shi59], see also [Shi71, §3.1]. We repeat this construction following closely the original sources; most of what we do here is also covered in [Miy89, §2.7].

Having defined the abstract Hecke algebra, we can give many groups a canonical module structure over it just by abstract nonsense. We develop this abstract theory to some extent and show Hecke equivariance statements for maps between these groups, whose proofs are mostly trivial in our abstract setting, so there is no need to do lengthy calculations. To connect our theory to more classical situations, we prove that the Hecke actions we define abstractly indeed become the actions defined in some texts in an ad-hoc way if one specialises to concrete situations. This provides a clean way to compare different definitions of Hecke operators.

In the applications we mentioned, the resulting statements are mostly clear or well-known to the experts, but nonetheless it seemed reasonable to give proofs for them, and the abstract setting we develop seemed to be the most elegant way to do this.

Traditionally, Hecke operators act from the right (i. e. we get right modules over the abstract Hecke algebra). We follow this convention, although in our applications the Hecke algebra will be commutative anyway. However, for group or monoid actions needed to get our machinery off the ground, there is no common convention in the literature: some authors use left actions and some use right actions. We chose to incorporate both conventions into one category, assuming that we have a fixed involution on the surrounding group, so that we get functors from things with left or right actions to right modules over the Hecke algebra. This makes the machinery somewhat unwieldy occasionally, but we think that in the applications we will benefit from that viewpoint.

1.1. Monoids with involution, actions and representations

Definition 1.1: A *monoid with involution* $(\Sigma, \star)$ consists of a monoid Σ with cancellation property (so that it lies in a group) and a map $\star\colon \Sigma \longrightarrow \Sigma$ (denoted $\alpha \longmapsto \alpha^\star$) which is an involution, that is, it fulfils $\alpha^{\star\star} = \alpha$ and $(\alpha\beta)^\star = \beta^\star\alpha^\star$ for all $\alpha, \beta \in \Sigma$ (in particular, it is bijective).

Example 1.2: Of course, any group with the inversion map as involution provides an example. The most important example to have in mind will be $\Sigma = \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ or $\Sigma = \mathrm{GL}_2^+(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ with the main involution ι (see page xx).

Definition 1.3: Let $(\Sigma, \star)$ be a monoid with involution, $\mathcal{C}$ be a category and $C \in \mathcal{C}$ an object.

(a) A *left action of* $(\Sigma, \star)$ *on* C is a homomorphism of monoids $\varphi\colon \Sigma \longrightarrow \operatorname{End}(C)$.

(b) A *right action of* $(\Sigma, \star)$ *on* C is a map $\varphi\colon \Sigma \longrightarrow \operatorname{End}(C)$ such that the composition $\Sigma \xrightarrow{\star} \Sigma \xrightarrow{\varphi} \operatorname{End}(C)$ is a homomorphism of monoids.

(c) Let X be the set of left and right actions of $(\Sigma, \star)$ on C. We define an equivalence relation $\sim$ on X by

$$\varphi \sim \psi :\Longleftrightarrow \varphi = \psi \text{ or } \varphi = \psi \circ \star.$$

An *action of* $(\Sigma, \star)$ *on* C is an equivalence class from the set X above. Obviously each equivalence class has two elements: one left action and one right action. We call these the left resp. right representative.

(d) Denote by $\mathcal{C}_{(\Sigma,\star)}$ the following category: objects are pairs (C, A) where $C \in \mathcal{C}$ is an object and A is an action on C. A morphism $f\colon (C, A) \longrightarrow (D, B)$ is a morphism $f\colon C \longrightarrow D$ in $\mathcal{C}$ such that the following equivalent conditions hold:

(i) For the actions A and B, choose representatives φ and ψ such that they are both a left action resp. both a right action. Then for any $\alpha \in \Sigma$, the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varphi(\alpha)"'] & D \arrow[d, "\psi(\alpha)"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

commutes.

(ii) For the actions A and B, choose representatives φ and ψ such that φ is a left action and ψ is a right action, or such that φ is a right action and ψ is a left action. Then for any $\alpha \in \Sigma$, the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varphi(\alpha)"'] & D \arrow[d, "\psi(\alpha^\star)"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

commutes.

In fact, the category $\mathcal{C}_{(\Sigma,\star)}$ is obviously equivalent to either just the category of objects $C \in \mathcal{C}$ with left actions or with right actions of Σ. We freely use this equivalence without further comments, thus regarding objects of $\mathcal{C}$ with either just a left or a right action as elements in $\mathcal{C}_{(\Sigma,\star)}$.

One could give a similar definition if we have a subsemigroup Δ of Σ which may not be stable under $\star$; an action should then be a right action of Δ or a left action of $\Delta^\star$ with the equivalence relation defined in the same way. For simplicity, we restrict to the case of a $\star$-stable semigroup, but in remark 1.25 (a) we will once consider this more general situation.

For a commutative ring R, we will work a lot with the category $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$, whose objects we call *R-linear representations of* $(\Sigma,\star)$. Such a representation is then an R-module M with an R-linear action of Σ either from the left or from the right. We will typically denote left actions by

$$\Sigma \times M \longrightarrow M, \quad (\alpha, m) \longmapsto \alpha \bullet m$$

and right actions by

$$M \times \Sigma \longrightarrow M, \quad (m, \alpha) \longmapsto m[\alpha]$$

(in the applications, α will often be a matrix, and then we write $m\left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right]$ instead of $m[\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)]$). It is clear that $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$ is an abelian category: it is equivalent to both the category of left or right $R[\Sigma]$-modules.

Sometimes the ring R will be a topological ring, and the modules should be topological modules. Everything we do can be extended to this situation. How to do this will be clear, and we will ignore the topological case, working just with abstract rings, so to not over-complicate matters.

Remark 1.4: The category $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$ admits an internal hom. To explain this, take two $M, N \in R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$. Choose the left representative for the action on M and the right representative for the action on N and put for $\phi \in \mathrm{Hom}_R(M,N)$ and $\alpha \in \Sigma$

$$\phi[\alpha](m) := \phi(\alpha \bullet m)[\alpha], \quad \text{for } m \in M.$$

Then $\phi[\alpha] \in \mathrm{Hom}_R(M,N)$ and this defines a right action of Δ on $\mathrm{Hom}_R(M,N)$. Via this action, we view $\mathrm{Hom}_R(M,N)$ as an element of $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$.

Of course, we could have chosen other representatives for the actions, and it is clear how the formula defining the action on $\mathrm{Hom}_R(M,N)$ should look like for the other representatives. We chose these representatives because this often occurs in the literature discussing the main application we have in mind (see section III.1).

In particular, this defines a notion of duals in $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$ if we take $N = R$ (with the trivial action of $(\Sigma,\star)$).

1.2. Hecke spaces and Hecke sheaves

Throughout the whole section, fix a monoid with involution $(\Sigma,\star)$ and a commutative ring R.

Definition 1.5: Let $X \in \mathcal{Top}_{(\Sigma,\star)}$, so X is a topological space[1] with an action of $(\Sigma,\star)$. View X as a ringed space by defining the structure sheaf to be the constant sheaf $\underline{R}$. Due to technical reasons assume further that Σ acts by automorphisms on X. Then we call X a Hecke space.

We define the notion of a Hecke sheaf which morally can be seen as a sheaf of R-modules on X with an action of $(\Sigma,\star)$. This generalises the notion of "G-sheaves", as they are called in [Gro57, chap. 5], or "sheaves with group action", as they are called in [Fu11, §9.1]: if we specialise to the case of the trivial Hecke pair $(G,1)$, we get this notion back. We follow these texts closely and refer to them for details.

If we choose a representative for the action of $(\Sigma,\star)$ on X, then for each $\alpha \in \Sigma$ we get a map $X \longrightarrow X$ given by the action of α. To make things clearer, we often denote this map by $\mathrm{L}\alpha$ resp. $\alpha\mathrm{R}$ for the left resp. right representative instead of just α. Note that by definition of the equivalence relation on actions, we have $\mathrm{L}\alpha^\star = \alpha\mathrm{R}$ for all $\alpha \in \Sigma$, and moreover $\mathrm{L}\alpha \circ \mathrm{L}\beta = \mathrm{L}\alpha\beta$, $\alpha\mathrm{R} \circ \beta\mathrm{R} = \beta\alpha\mathrm{R}$.

[1] Recall that we work only with "nice" topological spaces, see page xviii.

Definition 1.6: Let $\mathcal{F}$ be a sheaf of R-modules on X. Choose a representative for the action of $(\Sigma, \star)$ on X.

(a) For the left representative, define a *left Hecke sheaf structure on* $\mathcal{F}$ to be a collection $(\varphi_\alpha)_\alpha$ of isomorphisms $\varphi_\alpha : \mathcal{F} \xrightarrow{\sim} \mathrm{L}\alpha_* \mathcal{F}$ for every $\alpha \in \Sigma$, such that $\varphi_1 = \mathrm{id}_{\mathcal{F}}$ and for all $\alpha, \beta \in \Sigma$ the left diagram below commutes, or equivalently (by adjointness), a collection of isomorphisms $\varphi^\alpha : \mathrm{L}\alpha^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ such that $\varphi^1 = \mathrm{id}_{\mathcal{F}}$ and for all $\alpha, \beta \in \Sigma$ the right diagram below commutes:

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi_\alpha"] \arrow[d, "\varphi_{\alpha\beta}"'] & \mathrm{L}\alpha_* \mathcal{F} \arrow[d, "\mathrm{L}\alpha_*(\varphi_\beta)"] \\
(\mathrm{L}\alpha\beta)_* \mathcal{F} \arrow[r, "\sim"] & \mathrm{L}\alpha_*(\mathrm{L}\beta_* \mathcal{F})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{L}\alpha^* \mathcal{F} \arrow[r, "\varphi^\alpha"] & \mathcal{F} \\
\mathrm{L}\alpha^*(\mathrm{L}\beta^* \mathcal{F}) \arrow[r, "\sim"] \arrow[u, "\mathrm{L}\alpha^*(\varphi^\beta)"] & (\mathrm{L}\alpha\beta)^* \mathcal{F}. \arrow[u, "\varphi^{\beta\alpha}"']
\end{tikzcd}
$$

(b) For the right representative, define a *right Hecke sheaf structure on* $\mathcal{F}$ to be a collection of isomorphisms $\varphi_\alpha : \mathcal{F} \xrightarrow{\sim} \alpha\mathrm{R}_* \mathcal{F}$ for every $\alpha \in \Sigma$, such that $\varphi_1 = \mathrm{id}_{\mathcal{F}}$ and for all $\alpha, \beta \in \Sigma$ the left diagram below commutes, or equivalently, a collection of isomorphisms $\varphi^\alpha : \alpha\mathrm{R}^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ such that $\varphi^1 = \mathrm{id}_{\mathcal{F}}$ and for all $\alpha, \beta \in \Sigma$ the right diagram below commutes:

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi_\alpha"] \arrow[d, "\varphi_{\beta\alpha}"'] & \alpha\mathrm{R}_* \mathcal{F} \arrow[d, "\alpha\mathrm{R}_*(\varphi_\beta)"] \\
(\beta\alpha\mathrm{R})_* \mathcal{F} \arrow[r, "\sim"] & \alpha\mathrm{R}_*(\beta\mathrm{R}_* \mathcal{F})
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha\mathrm{R}^* \mathcal{F} \arrow[r, "\varphi^\alpha"] & \mathcal{F} \\
\alpha\mathrm{R}^*(\beta\mathrm{R}^* \mathcal{F}) \arrow[r, "\sim"] \arrow[u, "\alpha\mathrm{R}^*(\varphi^\beta)"] & (\beta\alpha\mathrm{R})^* \mathcal{F}. \arrow[u, "\varphi^{\alpha\beta}"']
\end{tikzcd}
$$

Similarly as we did in definition 1.3 (c), we want to define an equivalence relation on Hecke sheaf structures to unify them in a single category. Take a sheaf of R-modules $\mathcal{F}$ on X. Choose the left representative for the action on X and take left Hecke sheaf structure $(\varphi_\alpha)_\alpha$ on $\mathcal{F}$. Putting $\alpha^\star$ for $\alpha \in \Sigma$ and $\beta^\star$ for $\beta \in \Sigma$ in the left diagram in definition 1.6 (a) and using $\alpha^\star \beta^\star = (\beta\alpha)^\star$, we arrive at a commutative diagram

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi_{\alpha^\star}"] \arrow[d, "\varphi_{(\beta\alpha)^\star}"'] & \mathrm{L}\alpha^\star_* \mathcal{F} \arrow[d, "\mathrm{L}\alpha^\star_*(\varphi_{\beta^\star})"] \\
\mathrm{L}(\beta\alpha)^\star_* \mathcal{F} \arrow[r, "\sim"] & \mathrm{L}\alpha^\star_*(\mathrm{L}\beta^\star_* \mathcal{F}),
\end{tikzcd}
$$

for all $\alpha, \beta \in \Sigma$. Because we have $\mathrm{L}\alpha^\star = \alpha\mathrm{R}$, this diagram says that the collection of the $\varphi_{\alpha^\star}$ for all $\alpha \in \Sigma$ is a right Hecke sheaf structure on $\mathcal{F}$ for the right representative for the action on X. Similarly, if we start with a right Hecke sheaf $\mathcal{G}$, we obtain a left Hecke sheaf structure on it.

Definition 1.7: Define an equivalence relation on the set of left and right Hecke sheaf structures on a sheaf $\mathcal{F}$ by saying that two Hecke sheaf structures are equivalent if and only if they are either the same or one can be transformed into the other by the process we just described. Define a *Hecke sheaf on* X to be a sheaf of R-modules together with an equivalence class for this relation.

It is then clear that for each Hecke sheaf there is a unique representative which is a left Hecke sheaf and a unique one which is a right Hecke sheaf. We call them left and right representative, respectively.

Definition 1.8: Let $\mathcal{Sh}_R^{(\Sigma,\star)}(X)$ be the category whose objects are Hecke sheaves on X and in which a morphism $f\colon \mathcal{F} \longrightarrow \mathcal{G}$ between two Hecke sheaves is a morphism of sheaves $\mathcal{F} \longrightarrow \mathcal{G}$ such that for all $\alpha \in \Sigma$ the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \alpha_*\mathcal{F} \arrow[r] & \alpha_*\mathcal{G} \end{tikzcd} \quad \text{or equivalently} \quad \begin{tikzcd} \alpha^*\mathcal{F} \arrow[r] \arrow[d] & \alpha^*\mathcal{G} \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{G} \end{tikzcd}$$

commutes, if we choose the right (or left) representative for the action on X for both sheaves and the α in the diagram refers to $\alpha\mathrm{R}$ resp. $\mathrm{L}\alpha$. If we choose mixed representatives, one has to draw diagrams with "$\star$" on one of the sides; we don't write this down.

Proposition 1.9 (Grothendieck)**:** *The category $\mathcal{Sh}_R^{(\Sigma,\star)}(X)$ is an abelian category with enough injectives.*

Proof: If we specialise to the right representative and forget the left one, the category of Hecke sheaves is equivalent to the category of Σ-sheaves. The analogous statement for G-sheaves for a group G is proved in [Gro57, Prop. 5.1.1 and Thm. 1.10.1]. Investigating the proof there, one sees that it does not use anything special about groups and still works with monoids. $\square$

Now choose the left representative, say, for the action on X, let $p \in X$ be a point, $\alpha \in \Sigma$ and $\mathcal{F}$ a Hecke sheaf on X. By applying the "stalk at p" functor to the morphism $\mathrm{L}\alpha^*\mathcal{F} \longrightarrow \mathcal{F}$ and using the identification of stalks $(\mathrm{L}\alpha^*\mathcal{F})_p \cong \mathcal{F}_{\mathrm{L}\alpha(p)}$, we see that we get an induced map

$$\mathcal{F}_{\mathrm{L}\alpha(p)} \longrightarrow \mathcal{F}_p \tag{1.1}$$

for each $\alpha \in \Sigma$. Similarly, if we choose the right representative, we get a map $\mathcal{F}_{\alpha\mathrm{R}(p)} \longrightarrow \mathcal{F}_p$ for each $\alpha \in \Sigma$.

Construction 1.10: We want to have a notion of constant Hecke sheaves. Therefore, let M be an R-linear representation of $(\Sigma, \star)$.

Choose the left representative for the action of $(\Sigma, \star)$ on both X and M. Then for any open $U \subseteq X$ and any $\alpha \in \Sigma$, $\mathrm{L}\alpha$ induces a map between the set of connected components

$$\pi_0(\mathrm{L}\alpha^{-1}(U)) \longrightarrow \pi_0(U).$$

Define a map

$$\begin{aligned} \underline{M}(U) = \mathrm{Maps}(\pi_0(U), M) &\xrightarrow{(1)} \mathrm{Maps}(\pi_0(\mathrm{L}\alpha^{-1}(U)), M) \\ &\xrightarrow{(2)} \mathrm{Maps}(\pi_0(\mathrm{L}\alpha^{-1}(U)), M) = \underline{M}(\mathrm{L}\alpha^{-1}(U)) = \mathrm{L}\alpha_*\underline{M}(U) \end{aligned}$$

where (1) is induced by the above map on connected components and (2) is given by pointwise application of $\alpha^\star$ on M (in both cases, "left" and "right"!). Then one can check that the collection of morphisms of sheaves $\varphi_\alpha : \underline{M} \longrightarrow \mathrm{L}\alpha_* \underline{M}$ defined that way satisfies the necessary relations for a left Hecke sheaf structure on $\underline{M}$. One also easily checks that if we choose the right representative for the action, then the analogous construction defines a right Hecke sheaf structure which is equivalent to the previous left Hecke sheaf structure in the sense of definition 1.7. Therefore, we get a well-defined Hecke sheaf on X, which we denote by $\underline{M}$ and call the *constant Hecke sheaf.*

Going into the explicit proof of the fact that $\mathrm{L}\alpha_*$ and $\mathrm{L}\alpha^*$ are adjoint functors, one sees that the morphism $\mathrm{L}\alpha^* \underline{A} \longrightarrow \underline{A}$ corresponding to φ_α is explicitly given by

$$\begin{aligned}\mathrm{L}\alpha^* \underline{A}(U) = \underline{A}(\mathrm{L}\alpha(U)) = \mathrm{Maps}(\pi_0(\mathrm{L}\alpha(U)), A)& \\ \xrightarrow{(1)} \mathrm{Maps}(\pi_0(U), A) \xrightarrow{(2)} \mathrm{Maps}(\pi_0(U), A) = \underline{A}(U)&\end{aligned}$$

where again (1) is induced by the map on connected components and (2) is given by pointwise application of $\alpha^\star$ on A, if we chose the left representative, and analogously for the right representative.

Using this, we can see that for any point $p \in X$ and $\alpha \in \Sigma$, if we identify stalks with A in the usual way, the map (1.1) is just given by multiplication with $\alpha^\star$, for either representative.

Construction 1.11: Now we consider the situation where X and Y are Hecke spaces and $f : X \longrightarrow Y$ is a morphism in $\mathcal{T}op_{(\Sigma,\star)}$. We look at pushforwards and pullbacks of Hecke sheaves.

(a) Choose again the left representative, say, for the action of $(\Sigma, \star)$ on both X and Y. If $\mathcal{F}$ is a Hecke sheaf on X, then we have $\mathrm{L}\alpha_* f_* \mathcal{F} = f_* \mathrm{L}\alpha_* \mathcal{F}$ for each $\alpha \in \Sigma$ (where the first $\mathrm{L}\alpha$ is the action on Y, while the second is the action on X), so applying the functor f_* to the morphism φ_α gives a morphism $f_*\mathcal{F} \longrightarrow \mathrm{L}\alpha_* f_* \mathcal{F}$, and one easily checks that this makes $f_*\mathcal{F}$ a Hecke sheaf on X.

(b) Similarly, if $\mathcal{G}$ is a Hecke sheaf on Y, then applying the functor f^* to the morphism φ^α (the φ^α now being part of the Hecke sheaf structure of $\mathcal{G}$) and using a similar argument as before gives a morphism $\mathrm{L}\alpha^* f^* \mathcal{G} \longrightarrow \mathcal{G}$, and one checks that this makes $f^*\mathcal{G}$ a Hecke sheaf on X. This does not depend on the choice of the representative for the action.

(c) Since Hecke sheaves form an abelian category, we automatically get Hecke sheaf structures on all the higher direct images $\mathrm{R}^q f_* \mathcal{F}$. More explicitly: The family of functors $(\mathrm{R}^q f_*(\cdot))_{q \geq 0}$ is a δ-functor from the category of sheaves of abelian groups on X to the category of sheaves of abelian groups on Y, and as a derived functor it is universal. Since α_* (again α is an abbreviation for either $\mathrm{L}\alpha$ or $\alpha\mathrm{R}$) is exact (recall that we required that Σ acts by *automorphisms* on X!), $(\alpha_* \mathrm{R}^q f_*(\cdot))_{q \geq 0}$ is also a δ-functor. The morphism $f_*\mathcal{F} \longrightarrow \alpha_* f_* \mathcal{F}$ from before is a morphism between the degree 0 parts of these δ-functors and hence, by universality, induces a morphism of δ-functors
$$(\mathrm{R}^q f_*(\cdot))_{q \geq 0} \longrightarrow (\alpha_* \mathrm{R}^q f_*(\cdot))_{q \geq 0}.$$
For each fixed degree $q \geq 0$, the degree q part of this morphism gives the Hecke sheaf structure on $\mathrm{R}^q f_* \mathcal{F}$.

1.3. (Co)homology of Hecke spaces

We first look at singular homology of a Hecke space $X \in \mathcal{Top}_{(\Sigma,\star)}$. A singular simplex is a continuous map from a standard simplex to X, and composing such a map with $\mathrm{L}\alpha$ or $\alpha\mathrm{R}$ for $\alpha \in \Sigma$ produces a new singular simplex. In this way we view the R-modules of singular chains (i. e. the free R-modules over singular simplices) $C_n(X,R)$ as an element in $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$. It is clear that this is compatible with restriction to boundaries of simplices, so the action of Σ on singular chains defined that way induces an action of $(\Sigma,\star)$ on the homology groups $\mathrm{H}_i(X,R)$. Thus, the homology of Hecke spaces has values in $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$. It is also clear that if we have a Hecke subspace $A \subseteq X$, then its chains are stable under $(\Sigma,\star)$, so the long exact singular homology sequence of a pair of Hecke spaces is a sequence in $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$. Finally, under the usual identification of $\mathrm{H}_0(X,R)$ with the free R-module over the path-connected components of X, the action of $(\Sigma,\star)$ on $\mathrm{H}_0(X,R)$ is the one induced by the canonical action on path-connected components.

For $M \in R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$ we can define the singular cochains $C^n(X,M) = \mathrm{Hom}_R(C_n(X,R),M)$ by taking the internal hom in $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$. This makes the singular cohomology groups $\mathrm{H}^i(X,M)$ also elements of $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$, and the long exact singular cohomology sequence is a sequence in $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$.

Now we look at cohomology of Hecke sheaves. Let $\mathcal{F}$ be a Hecke sheaf on X, and choose the left representative, say. For each $\alpha \in \Sigma$, we have a map

$$\Gamma(X,\mathcal{F}) = \mathcal{F}(X) \xrightarrow{\varphi_\alpha} \mathrm{L}\alpha_*\mathcal{F}(X) = \mathcal{F}(\mathrm{L}\alpha^{-1}(X)) = \mathcal{F}(X) = \Gamma(X,\mathcal{F}) \tag{1.2}$$

and it is easily verified that this defines a left action of $(\Sigma,\star)$ on $\Gamma(X,\mathcal{F})$. Analogously, starting with the right representative, we get a right action of $(\Sigma,\star)$ equivalent to the previous left action. So we have seen:

Proposition 1.12: *Taking global sections defines a functor*

$$\mathcal{Sh}_R^{(\Sigma,\star)}(X) \longrightarrow R\text{-}\mathcal{Mod}_{(\Sigma,\star)}.$$

In particular, all cohomology groups of Hecke sheaves are elements of $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$.

There is also another way to see this (again using the left representative): functoriality of cohomology for $\mathrm{L}\alpha\colon X \longrightarrow X$ (for $\alpha \in \Sigma$) gives us a morphism $\mathrm{H}^q(X,\mathcal{F}) \longrightarrow \mathrm{H}^q(X,\mathrm{L}\alpha^*\mathcal{F})$, and φ^α induces an isomorphism $\mathrm{H}^q(X,\mathrm{L}\alpha^*\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^q(X,\mathcal{F})$. The composition of these gives an endomorphism of $\mathrm{H}^q(X,\mathcal{F})$, and in this way we get an action of Σ on $\mathrm{H}^q(X,\mathcal{F})$. To see that this describes the same action as before, it suffices to check this in degree 0, because sheaf cohomology, being a derived functor, is a universal δ-functor. Explicitly, because Σ acts by automorphisms, we have $\mathrm{L}\alpha(X) = \mathrm{L}\alpha^{-1}(X) = X$, so $\mathrm{L}\alpha_*\mathcal{F}(X) = \mathcal{F}(\mathrm{L}\alpha^{-1}(X)) = \mathcal{F}(X)$ and $\mathrm{L}\alpha^*\mathcal{F}(X) = \mathcal{F}(\mathrm{L}\alpha(X)) = \mathcal{F}(X)$. We then have to check that the map (1.2) coincides with the map

$$\mathcal{F}(X) = \mathcal{F}(\mathrm{L}\alpha(X)) = \mathrm{L}\alpha^*\mathcal{F}(X) \xrightarrow{\varphi^\alpha} \mathcal{F}(X).$$

This can be seen by going into the explicit proof of the adjointness of $\mathrm{L}\alpha_*$ and $\mathrm{L}\alpha^*$. We do not go further into detail.

The identification of sheaf cohomology with singular cohomology $\mathrm{H}^i(X,\underline{M}) \cong \mathrm{H}^i(X,M)$ is an isomorphism in $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$. Checking this in degree $i = 0$ is an easy calculation using the definition of the action of $(\Sigma,\star)$ on both sides, and by universality it extends to any $i \geq 0$.

Lemma 1.13: *Let $X, Y \in \mathcal{T}op_{(\Sigma,\star)}$ be Hecke spaces, $f : X \longrightarrow Y$ a morphism, and let $\mathcal{F}$ be a Hecke sheaf on X. For the actions of Σ on X and Y as well as for the Hecke sheaf structure on $\mathcal{F}$, choose either the left or right representative for all of them. For each point $p \in Y$, let $X_p = f^{-1}(p)$ be the fibre at p, and for each $q \geq 0$ identify the stalk $(\mathrm{R}^q f_* \mathcal{F})_p$ with the cohomology group $\mathrm{H}^q(X_p, \mathcal{F}|_{X_p})$ in the usual way.*

Then for each $\alpha \in \Sigma$, the map on stalks

$$(\mathrm{R}^q f_* \mathcal{F})_{\mathrm{L}\alpha(p)} \longrightarrow (\mathrm{R}^q f_* \mathcal{F})_p \quad \textit{resp.} \quad (\mathrm{R}^q f_* \mathcal{F})_{\alpha\mathrm{R}(p)} \longrightarrow (\mathrm{R}^q f_* \mathcal{F})_p$$

as in (1.1) is the map

$$\mathrm{H}^q(X_{\mathrm{L}\alpha(p)}, \mathcal{F}|_{X_{\mathrm{L}\alpha(p)}}) \longrightarrow \mathrm{H}^q(X_p, \mathcal{F}|_{X_p}) \quad \textit{resp.} \quad \mathrm{H}^q(X_{\alpha\mathrm{R}(p)}, \mathcal{F}|_{X_{\alpha\mathrm{R}(p)}}) \longrightarrow \mathrm{H}^q(X_p, \mathcal{F}|_{X_p})$$

induced by functoriality of cohomology for the map $X_p \longrightarrow X_{\mathrm{L}\alpha(p)}$ (resp. $X_p \longrightarrow X_{\alpha\mathrm{R}(p)}$) given by the action of α on X and the map $\mathrm{L}\alpha^* \mathcal{F}|_{X_{\mathrm{L}\alpha(p)}} \longrightarrow \mathcal{F}|_{X_p}$ (resp. $\alpha\mathrm{R}^* \mathcal{F}|_{X_{\alpha\mathrm{R}(p)}} \longrightarrow \mathcal{F}|_{X_p}$) coming from the Hecke sheaf structure.

Sketch of proof: We work with the left representative.

Let $\mathcal{P}$ be the presheaf on Y mapping an open U to $\mathrm{H}^q(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$. Its sheafification is $\mathrm{R}^q f_* \mathcal{F}$ [Har77, Prop. III.8.1]. We *define* a morphism of presheaves $\mathcal{P} \longrightarrow \mathrm{L}\alpha_* \mathcal{P}$ such that for an open U the map $\mathrm{H}^q(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}) \longrightarrow \mathrm{H}^q(\mathrm{L}\alpha^{-1}(f^{-1}(U)), \mathcal{F}|_{\mathrm{L}\alpha^{-1}(f^{-1}(U))})$ is induced by functoriality of cohomology from the map $f^{-1}(U) \longrightarrow \mathrm{L}\alpha^{-1}(f^{-1}(U))$ given by the action of α and the map $\mathrm{L}\alpha^* \mathcal{F}|_{\mathrm{L}\alpha^{-1}(f^{-1}(U))} \longrightarrow \mathcal{F}|_{f^{-1}(U)}$ coming from the Hecke sheaf structure. After sheafification, this induces a morphism $\mathrm{R}^q f_* \mathcal{F} \longrightarrow \mathrm{L}\alpha_* \mathrm{R}^q f_* \mathcal{F}$.

To finish the proof, by construction 1.11 (c) it suffices to check that this construction, done for any i and any $\mathcal{F}$, defines a morphism of δ-functors

$$(\mathrm{R}^q f_*(\cdot))_{q \geq 0} \longrightarrow (\mathrm{L}\alpha_* \mathrm{R}^q f_*(\cdot))_{q \geq 0}$$

and that in degree 0 it gives back the original morphism $f_* \mathcal{F} \longrightarrow \mathrm{L}\alpha_* f_* \mathcal{F}$, because then the claim follows from the universality of $(\mathrm{R}^q f_*(\cdot))_{q \geq 0}$. This is a straightforward calculation which we omit here. $\square$

Corollary 1.14: *Let X, Y, f, p, X_p as in lemma 1.13. Let further A be an R-module which we consider as a trivial representation of $(\Sigma, \star)$. Identify the stalk $(\mathrm{R}^1 f_* \underline{A})_p$ with the dual space of the singular homology group $\mathrm{H}_1(X_p, A)$. Then for each $\alpha \in \Sigma$, the map on stalks*

$$(\mathrm{R}^1 f_* \underline{A})_{\mathrm{L}\alpha(p)} \longrightarrow (\mathrm{R}^1 f_* \underline{A})_p \quad \textit{resp.} \quad (\mathrm{R}^1 f_* \underline{A})_{\alpha\mathrm{R}(p)} \longrightarrow (\mathrm{R}^1 f_* \underline{A})_p$$

as in (1.1) is identified with the dual of the map on singular homology

$$\mathrm{H}_1(X_p, A) \longrightarrow \mathrm{H}_1(X_{\mathrm{L}\alpha(p)}, A) \quad \textit{resp.} \quad \mathrm{H}_1(X_p, A) \longrightarrow \mathrm{H}_1(X_{\alpha\mathrm{R}(p)}, A)$$

induced by the map $X_p \longrightarrow X_{\mathrm{L}\alpha(p)}$ (resp. $X_p \longrightarrow X_{\alpha\mathrm{R}(p)}$) given by the action of α on X.

1.4. Hecke pairs and the abstract Hecke algebra

Throughout the whole section, fix a monoid with involution $(\Sigma, \star)$.

Definition 1.15: A *Hecke pair*[2] (Δ, Γ) for $(\Sigma, \star)$ consists of a submonoid $\Delta \subseteq \Sigma$ and a subgroup $\Gamma \subseteq \Delta$ subject to the following conditions:

(a) If G is the smallest group containing Σ, then Δ is contained in the *commensurator* $\tilde{\Gamma}$ of Γ in G, which is the subgroup

$$\tilde{\Gamma} = \{g \in G : g\Gamma g^{-1} \text{ and } \Gamma \text{ are commensurable}\},$$

where two subgroups of a group are said to be *commensurable* if their intersection has finite index in both of them;

(b) The restriction of $\star$ to Γ is the inversion map $\gamma \longmapsto \gamma^{-1}$ (so in particular $\Gamma^\star = \Gamma$).

We call the Hecke pair (Δ, Γ) *central* if $\alpha^\star\alpha = \alpha\alpha^\star$ for all $\alpha \in \Delta$ and $\alpha^\star\alpha$ commutes with all $\gamma \in \Gamma$.

Example 1.16: (a) Any group G carries an involution given by inversion. Then $(G, 1)$ is a Hecke pair for the monoid with involution $(G, (\cdot)^{-1})$ which we call the *trivial Hecke pair*.[3] It is obviously central.

(b) In the most important type of example, we will consider one of the groups $G = \mathrm{GL}_2^+(\mathbb{Q})$ or $G = \mathrm{GL}_2(\mathbb{Q})$ with the main involution ι given by $\alpha^\iota = \det(\alpha)\alpha^{-1}$ and the submonoid $\Sigma = G \cap \mathrm{M}_2(\mathbb{Z})$. Then G is the smallest group containing Σ. If Δ is a any submonoid of Σ and $\Gamma \subseteq \Delta$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then (Δ, Γ) is a Hecke pair: If $\alpha \in G$, then $\alpha\Gamma\alpha^{-1} \cap \mathrm{SL}_2(\mathbb{Z})$ is again a congruence subgroup (see [DS05, Lem. 5.1.1], where $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, but the case of $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ obviously follows from this), and any two congruence subgroups are commensurable [DS05, Ex. 5.1.2], so $\tilde{\Gamma} = G$ in this case, and obviously the restriction of ι to $\mathrm{SL}_2(\mathbb{Z})$ is the inversion map. This Hecke pair is always central.

(c) One can find similar examples related to automorphic forms on other algebraic groups.

We now introduce the abstract Hecke algebra.

Definition 1.17: For a Hecke pair (Δ, Γ) and a commutative ring R, we define the abstract Hecke algebra to be the R-algebra generated over R by the double cosets $\Gamma\alpha\Gamma$ for all $\alpha \in \Delta$, and the product of two double cosets $\Gamma\alpha\Gamma$ and $\Gamma\beta\Gamma$ defined as follows: decompose the double cosets as a disjoint union of left cosets

$$\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i, \quad \Gamma\beta\Gamma = \bigsqcup_j \Gamma\beta_j$$

with $\alpha_i, \beta_j \in \Delta$ and define their product by

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum m_\xi \, \Gamma\xi\Gamma, \tag{1.3}$$

[2] There are various definitions of this notion in the literature. For our purposes, this will be the adequate one.

[3] Of course, (G, G) is also a Hecke pair which we could call the trivial one. But this one is even "too trivial": its abstract Hecke algebra we will define later will be trivial and the whole theory will loose its content then.

where the sum runs over all double cosets $\Gamma\xi\Gamma \subseteq \Gamma\alpha\Gamma\beta\Gamma$ and

$$m_\xi = \#\{(i,j) : \Gamma\alpha_i\beta_j = \Gamma\xi\}.$$

We denote this R-algebra by $\mathcal{H}_R(\Delta,\Gamma)$ and call it the *abstract Hecke algebra.*

Proposition 1.18 (Shimura)**:** *The above disjoint unions are finite, and $\mathcal{H}_R(\Delta,\Gamma)$ is a well-defined R-algebra.*

Proof: [Shi71, §3.1] $\square$

The abstract Hecke algebra is a generalisation of the concept of a group or monoid ring: if $(\Delta,\Gamma) = (G,1)$ is the trivial Hecke pair as in example 1.16 (a), then $\mathcal{H}_R(G,1)$ is just $R[G]$.

The following observation will sometimes be useful.

Remark 1.19: Fix $\alpha \in \Delta$ and decompose again

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^{e} \Gamma\alpha_i, \quad \alpha_i \in \Delta.$$

For each $\gamma \in \Gamma$ and each $i \in \{1,\dots,e\}$, the element $\alpha_i\gamma$ lies in a unique left coset in the above decomposition, i. e. there are unique $j \in \{1,\dots,e\}$ and $\gamma' \in \Gamma$ such that $\alpha_i\gamma = \gamma'\alpha_j$. Then for fixed γ, the map $i \longmapsto j$ is a permutation of the set $\{1,\dots,e\}$ which we denote by σ_γ, and for fixed i, we have a map $\rho_i\colon \Gamma \longrightarrow \Gamma$ given by $\gamma \longmapsto \gamma'$. So we have

$$\alpha_i\gamma = \rho_i(\gamma)\alpha_{\sigma_\gamma(i)}.$$

We now study relations between Hecke pairs and Hecke algebras.

Definition 1.20: Let (Δ,Γ) and (Δ',Γ') be Hecke pairs. We write $(\Delta,\Gamma) \prec (\Delta',\Gamma')$ if the following conditions hold:

(a) $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$,

(b) $\Gamma'\alpha\Gamma' = \Gamma'\alpha\Gamma$ for all $\alpha \in \Delta$,

(c) $\Gamma'\alpha \cap \Delta = \Gamma\alpha$ for all $\alpha \in \Delta$.

If in addition the condition

(d) $\Delta' = \Gamma'\Delta$

holds, then we write $(\Delta,\Gamma) \precapprox (\Delta',\Gamma')$.

A basic example for Hecke pairs (Δ,Γ) and (Δ',Γ') with $(\Delta,\Gamma) \prec (\Delta',\Gamma')$ is the case that $\Gamma = \Gamma'$ and $\Delta \subseteq \Delta'$. We will see more examples in section 1.8.

Proposition 1.21: *Let (Δ,Γ) and (Δ',Γ') be Hecke pairs. If $(\Delta,\Gamma) \prec (\Delta',\Gamma')$, then the map*

$$\mathcal{H}_R(\Delta,\Gamma) \longrightarrow \mathcal{H}_R(\Delta',\Gamma'), \quad \Gamma\alpha\Gamma \longmapsto \Gamma'\alpha\Gamma'$$

is a well-defined injective homomorphism of R-algebras. If $(\Delta,\Gamma) \precapprox (\Delta',\Gamma')$, then the above map is an isomorphism. In this case any double coset $\Gamma'\alpha\Gamma'$ can be represented by an $\alpha \in \Delta$ and

$$\mathcal{H}_R(\Delta',\Gamma') \longrightarrow \mathcal{H}_R(\Delta,\Gamma), \quad \Gamma'\alpha\Gamma' \longmapsto \Gamma\alpha\Gamma \quad (\alpha \in \Delta)$$

is a well-defined inverse.

Proof: See [Miy89, Thm. 2.7.6 (1)]. The statement is formulated there slightly less general, but this is what is really proved there. $\square$

1.5. Group cohomology of representations of Hecke pairs

Continue to fix a commutative ring R, a monoid with involution $(\Sigma, \star)$ and a Hecke pair (Δ, Γ). We will now work with the category $R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$ of representations of $(\Sigma, \star)$. For a representation $M \in R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$, we choose a representative for the action and can then form the R-submodule of Γ-invariants M^Γ. Since we required $\Gamma^\star = \Gamma$ in definition 1.15, this is a well-defined subset not depending on the choice of the representative for the action. We wish to make M^Γ a right $\mathcal{H}_R(\Delta, \Gamma)$-module.

Definition 1.22: For $\alpha \in \Delta$, we define an endomorphism $[\Gamma\alpha\Gamma]$ of M^Γ which we call a *double coset operator.* Decompose again

$$\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i, \quad \alpha_i \in \Delta$$

and choose a representative for the action of (Δ, Γ) on M.

(a) If the chosen representative is a left action, then define for $m \in M^\Gamma$

$$m[\Gamma\alpha\Gamma] = \sum_i \alpha_i^\star \bullet m.$$

(b) If the chosen representative is a right action, then define for $m \in M^\Gamma$

$$m[\Gamma\alpha\Gamma] = \sum_i m[\alpha_i].$$

It is obvious that this definition does not depend on the choice of the representative for the action (if it is at all well-defined). Thus, it suffices to prove the well-definedness in one of the two cases.

Lemma 1.23 (Shimura)**:** *The above is well-defined and makes M^Γ a right $\mathcal{H}_R(\Delta, \Gamma)$-module. If M' is another representation of $(\Sigma, \star)$ and $f : M \longrightarrow M'$ is a morphism, then the restricted morphism $M^\Gamma \longrightarrow (M')^\Gamma$ is $\mathcal{H}_R(\Delta, \Gamma)$-linear.*

Proof: [Miy89, Lem. 2.7.2, Lem. 2.7.4] □

Of course, in the case of a trivial Hecke pair $(G, 1)$, this defines just the canonical $R[G]$-module structure.

Example 1.24: Let $(\Sigma, \star)$ and (Δ, Γ) be as in example 1.16 (b) (with $G = \mathrm{GL}_2^+(\mathbb{Q})$), let $k \in \mathbb{N}_0$ and denote by M the $\mathbb{C}$-vector space of meromorphic functions on the upper half plane. Let Σ act on M from the right by saying that for $f \in M$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma$, the function $f[\alpha]_k$ on the upper half plane is given by[4]

$$f[\alpha]_k(\tau) = (\det \alpha)^{k-1}(c\tau + d)^{-k} f(\alpha\tau).$$

Then M^Γ is just the space $\mathcal{A}_k(\Gamma)$ of classical meromorphic modular forms for the group Γ. The action of a double coset $[\Gamma\alpha\Gamma]$ for $\alpha \in \Delta$ as defined in definition 1.22 in this case is

[4] Note that many texts, such as [Shi71], use a slightly different normalisation for this action in that the factor $(\det \alpha)^{k/2}$ instead of $\det \alpha^{k-1}$ is used. Our normalisation is used e. g. in [DS05; HidLFE]. Both normalisations give the same spaces of modular forms.

precisely the usual one.[5] So the resulting right $\mathcal{H}_{\mathbb{C}}(\Delta, \Gamma)$-module structure on $\mathcal{A}_k(\Gamma)$ is the classical one defined e. g. in [Shi71, §3.4].

Remark 1.25: (a) As is visible from the definition, to make the Γ-invariants a right $\mathcal{H}_R(\Delta, \Gamma)$-module we do not need an action of the full semigroup Σ. It would suffice to have either a right action of Δ or a left action of $\Delta^\star$. The $\mathcal{H}_R(\Delta, \Gamma)$-linearity statement from lemma 1.23 then just needs equivariance for this action.

Much of what we do could also be developed in this more general setting. However, some statements do need that the action comes from an action of the whole Σ (or at least of the subsemigroup of Σ generated by Δ and $\Delta^\star$). For reasons of clarity we therefore decided to fully develop the theory only in this case. Nevertheless, at one minor point, we will need the more general situation; see part (b) below.

(b) Let $\varepsilon\colon \Delta^\star \longrightarrow R^\times$ be a character. Then we can twist the $\mathcal{H}_R(\Delta, \Gamma)$-module structure by ε as follows. Decompose again

$$\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i, \qquad \alpha_i \in \Delta$$

and choose a representative for the action of (Δ, Γ) on M.

(i) If the chosen representative is a left action, then define for $m \in M^\Gamma$

$$m[\Gamma\alpha\Gamma] = \sum_i \varepsilon(\alpha_i^\star) \cdot \alpha_i^\star \bullet m.$$

(ii) If the chosen representative is a right action, then define for $m \in M^\Gamma$

$$m[\Gamma\alpha\Gamma] = \sum_i \varepsilon(\alpha_i^\star) \cdot m[\alpha_i].$$

Then one easily checks that this defines a right $\mathcal{H}_R(\Delta, \Gamma)$-module structure on M^Γ. We denote this new $\mathcal{H}_R(\Delta, \Gamma)$-module by $M^\Gamma(\varepsilon)$.

Of course, if we view M as a left Δ- resp. $\Delta^\star$-module (according to if we use the right or left representative), then we can twist the action of Δ resp. $\Delta^\star$ on M itself by $\varepsilon \circ \star$ resp. ε. If we then apply the previous construction to the twisted module $M(\varepsilon)$, we end up with the same $\mathcal{H}_R(\Delta, \Gamma)$-module $M^\Gamma(\varepsilon)$.

Thus if N is another R-linear representation of $(\Sigma, \star)$ and f is an R-linear map $M \longrightarrow N$ or $N \longrightarrow M$, then f will induce an $\mathcal{H}_R(\Delta, \Gamma)$-linear map $M^\Gamma(\varepsilon) \longrightarrow N^\Gamma$ or $N^\Gamma \longrightarrow M^\Gamma(\varepsilon)$ if and only if it is equivariant for the action of Δ resp. $\Delta^\star$, using the ε-twisted action on M.

This construction will occur only at one minor point in this work (namely in section III.4.2), and there we do not use any results from our abstract Hecke theory other then just the definition. Therefore we do not study these twisted modules further.

[5] In [DS05], the action of a double coset is defined by the same formula as in definition 1.22, whereas in [Shi71] there is an extra factor $(\det \alpha)^{k/2-1}$ in front of the sum. This cancels out the different normalisations in the definition of the Δ-action.

We have shown that we can consider taking Γ-invariants as a functor

$$R\text{-}\mathcal{Mod}_{(\Sigma,\star)} \longrightarrow \mathcal{Mod}\text{-}\mathcal{H}_R(\Delta,\Gamma). \tag{1.4}$$

It is obviously left exact and its derived functors are the usual group cohomology groups, so we automatically get a right $\mathcal{H}_R(\Delta,\Gamma)$-module structure on the cohomology groups $\mathrm{H}^q(\Gamma, M)$ for all $q \geq 0$ and all M. In particular, all maps in a long exact cohomology sequence attached to a short exact sequence of R-linear representations of $(\Sigma,\star)$ are $\mathcal{H}_R(\Delta,\Gamma)$-linear.

The $\mathcal{H}_R(\Delta,\Gamma)$-module structure on cohomology can be made very explicit. To state this, we use the representation of cohomology groups by homogeneous standard cochains.

Lemma 1.26 (Rhie/Whaples)**:** *Let $\tilde{c}\colon \Gamma^{q+1} \longrightarrow M$ be a homogeneous cocycle representing a cohomology class $c \in \mathrm{H}^q(\Gamma, M)$, and for $\alpha \in \Delta$ decompose*

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^{e} \Gamma\alpha_i.$$

Then the cohomology class $c[\Gamma\alpha\Gamma]$ is represented by the cocycle $\tilde{c}[\Gamma\alpha\Gamma]$ given by

$$\tilde{c}[\Gamma\alpha\Gamma](\gamma_0,\dots,\gamma_q) = \begin{cases} \displaystyle\sum_{i=1}^{e} \alpha_i^\star \bullet \tilde{c}(\rho_i(\gamma_0),\dots,\rho_i(\gamma_q)) & \textit{for the left representative,} \\[2ex] \displaystyle\sum_{i=1}^{e} \tilde{c}(\rho_i(\gamma_0),\dots,\rho_i(\gamma_q))[\alpha_i] & \textit{for the right representative.} \end{cases}$$

Proof: Of course, it suffices to prove this in one of the two cases. We prove it in the case of a left action.

The $\mathcal{H}_R(\Delta,\Gamma)$-module structure we have has the property that in degree 0 it is the one defined in definition 1.22 and all maps in a long exact cohomology sequence attached to a short exact sequence of representations of $(\Sigma,\star)$ are $\mathcal{H}_R(\Delta,\Gamma)$-linear. The latter means that multiplication by any $T \in \mathcal{H}_R(\Delta,\Gamma)$ is an endomorphism of the δ-functor $\mathrm{H}^*(\Gamma,-)$. Hence the $\mathcal{H}_R(\Delta,\Gamma)$-module structure is unique with the above property since $\mathrm{H}^*(\Gamma,-)$ is a universal δ-functor.

Thus to prove that the $\mathcal{H}_R(\Delta,\Gamma)$-action has the form claimed in the lemma, it suffices to show that if we *define* a right $\mathcal{H}_R(\Delta,\Gamma)$-action on each $\mathrm{H}^q(\Gamma, M)$ for all i and all M by these formulas, then we get back the original action in degree 0 and all maps in a long exact cohomology sequence attached to a short exact sequence of representations of $(\Sigma,\star)$ are $\mathcal{H}_R(\Delta,\Gamma)$-linear.

First, for $q = 0$, the class c corresponds to an $m \in M^\Gamma$ via the relation $m = \tilde{c}(1)$, and one sees immediately that $\rho_i(1) = 1$ for all i, so $c[\Gamma\alpha\Gamma]$ (as defined by the above formula) corresponds to

$$\tilde{c}[\Gamma\alpha\Gamma](1) = \sum_i \alpha_i^\star \tilde{c}(\rho_i(1)) = \sum_i \alpha_i^\star \bullet m,$$

which shows that we get back the original action in degree 0.

The second property is proved in [RW70] in the case of a left action. More precisely, in §II.1 they define a right action of double cosets on a different kind of cochain complex resolving M by some formula and prove that for this definition, all maps in a long exact cohomology sequence are $\mathcal{H}_R(\Delta, \Gamma)$-linear [RW70, Prop. 2.2, 2.3]. Then in [RW70, §II.2, Prop. 2.4] they prove that in the representation using homogeneous standard cochains, their formula becomes what we stated above. They work in the special case where Σ is a group G and the involution $\star$ is the inversion map $\delta \longmapsto \delta^{-1}$ on G, but investigating their calculations shows that they really use this only for elements of Γ, so their calculations remain valid in our setting because of the requirement in definition 1.15 (b). □

Since in many situations inhomogeneous cochains are preferred to homogeneous ones, it is useful to have an explicit description of the action of $\mathcal{H}_R(\Delta, \Gamma)$ also on inhomogeneous cochains. For simplicity, we state this only in degree 1, which will suffice for our purposes.

Corollary 1.27: *Let $\tilde{c}\colon \Gamma \longrightarrow M$ be an inhomogeneous 1-cocyle representing a cohomology class $c \in \mathrm{H}^1(\Gamma, M)$, and for $\alpha \in \Delta$ decompose*

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^{e} \Gamma\alpha_i.$$

Then the cohomology class $c[\Gamma\alpha\Gamma]$ is represented by the cocycle $\tilde{c}[\Gamma\alpha\Gamma]$ given by

$$\tilde{c}[\Gamma\alpha\Gamma](\gamma) = \begin{cases} \sum_{i=1}^{e} \alpha_i^\star \bullet \tilde{c}(\rho_i(\gamma)) & \text{for the left representative,} \\ \sum_{i=1}^{e} \tilde{c}(\rho_i(\gamma))[\alpha_i] & \text{for the right representative.} \end{cases}$$

Proof: The explicit isomorphism between the homogeneous and inhomogeneous standard resolutions is given in degree 1 by

$$\begin{aligned} \mathrm{Hom}_\Gamma(\mathbb{Z}[\Gamma^2], M) &\xleftrightarrow{\ \sim\ } \mathrm{Maps}(\Gamma, M) \\ c &\longmapsto \big[\gamma \longmapsto c(1, \gamma)\big] \\ \big[(\gamma_1, \gamma_2) \longmapsto f(\gamma_1^{-1}\gamma_2)\big] &\longleftarrow\!\!\!\mid f. \end{aligned}$$

Using this and lemma 1.26, one obtains the claim by an easy calculation. □

Example 1.28: In the situation when $(\Sigma, \star)$ and (Δ, Γ) are again as in example 1.16 (b) and M is an R-linear representation of $(\Sigma, \star)$, some texts define an action of $\mathcal{H}_R(\Delta, \Gamma)$ on $\mathrm{H}^q(\Gamma, M)$ (or just $\mathrm{H}^1(\Gamma, M)$) in an ad-hoc manner using the formulas from lemma 1.26 or corollary 1.27: see [Shi71, §8.3, (8.3.2)], [Hid86a, §4, p. 563], [HidLFE, §6.3]. So the actions defined there are the same as the one we defined abstractly.

Remark 1.29: In some situations one wants to consider parabolic group cohomology. We will need this only in degree 1. It can then be defined very explicitly: fix a subset $P \subseteq \Gamma$ and define

$$Z_P^1(\Gamma, M) = \{c \in Z^1(\Gamma, M) : \forall \pi \in P\colon c(\pi) \in (\pi - 1)M\},$$

where $Z^1(\Gamma, M)$ denotes inhomogeneous 1-cocycles on Γ with values in M. If we denote by $B^1(\Gamma, M)$ the corresponding coboundaries, then $B^1(\Gamma, M) \subseteq Z^1_P(\Gamma, M)$, and one defines then the first parabolic cohomology group with respect to P as

$$\mathrm{H}^1_P(\Gamma, M) = Z^1_P(\Gamma, M) \Big/ B^1(\Gamma, M) .$$

Of course one will need some conditions on P to guarantee that the subgroup $\mathrm{H}^1_P(\Gamma, M) \subseteq \mathrm{H}^1(\Gamma, M)$ is stable under the action of $\mathcal{H}_R(\Delta, \Gamma)$.

Instead of writing down such conditions in full generality, we just state that in our main application these will be satisfied. This main application is the case where Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and P is the subset of all parabolic elements. In this case the parabolic cohomology subgroup is indeed $\mathcal{H}_R(\Delta, \Gamma)$-stable. This is shown in [Shi71, §8.3].

There is yet another way to describe the action of double cosets on group cohomology. We need a preliminary lemma.

Lemma 1.30: *Let $\alpha \in \Delta$ and put $\Phi_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$, which is a subgroup of finite index of Γ; let e be this index. Let*

$$\Gamma = \bigsqcup_{i=0}^{e} \Phi_\alpha c_i, \quad c_i \in \Gamma$$

be a decomposition of Γ into left cosets and put $\alpha_i \coloneqq \alpha c_i$ for $i = 1, \dots, e$. Then

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^{e} \Gamma\alpha_i$$

is a disjoint decomposition of $\Gamma\alpha\Gamma$ into left cosets.

Proof: First, assume $\Gamma\alpha c_i = \Gamma\alpha c_j$ for $i, j \in \{1, \dots, e\}$. Then we can write $c_i = \alpha^{-1}\gamma\alpha c_j$ with some $\gamma \in \Gamma$, and since $c_i, c_j \in \Gamma$, we must have $\alpha^{-1}\gamma\alpha \in \Gamma$, so $\alpha^{-1}\gamma\alpha \in \Phi_\alpha$ and thus $i = j$. This proves that the union in the statement is indeed disjoint.

Since $\alpha_i = \alpha c_i \in \alpha\Gamma$ for all $i = 1, \dots, e$, we obviously have $\bigcup_i \Gamma\alpha_i \subseteq \Gamma\alpha\Gamma$.

Without loss of generality, assume that c_1 is the representative for the trivial left coset (i. e. $c_1 \in \Phi_\alpha$). We then have $\alpha c_1^{-1}\alpha^{-1} \in \Gamma$, so $\alpha = \alpha c_1^{-1}\alpha^{-1}\alpha c_1 \in \Gamma\alpha c_1 = \Gamma\alpha_1$, so we have $\Gamma\alpha \subseteq \bigcup_i \Gamma\alpha_i$.

Finally, for each $\gamma \in \Gamma$ and each $i \in \{1, \dots, e\}$ there is a $\sigma \in \Phi_\alpha$ and a $j \in \{1, \dots, e\}$ such that $c_i\gamma = \sigma c_j$. We have $\alpha\sigma\alpha^{-1} \in \Gamma$. Using this, we calculate for any $\gamma, \gamma' \in \Gamma$

$$\gamma'\alpha c_i\gamma = \gamma'\alpha\sigma c_j = \gamma'\alpha\sigma\alpha^{-1}\alpha c_j \in \Gamma\alpha c_j,$$

so $\Gamma\alpha c_i\Gamma \subseteq \Gamma\alpha c_j$ and hence $\Gamma\alpha\Gamma \subseteq \bigcup_i \Gamma\alpha c_i\Gamma \subseteq \bigcup_j \Gamma\alpha c_j = \bigcup_i \Gamma\alpha_i$. $\square$

Lemma 1.31: *Let $\alpha \in \Delta$ and put $\Phi_\alpha = \alpha^{-1}\Gamma\alpha \cap \Gamma$, $\Phi^\alpha = \alpha\Phi_\alpha\alpha^{-1}$. Then Φ_α, Φ^α are subgroups of Γ and we have the restriction and corestriction maps*

$$\mathrm{res}_{\Gamma|\Phi^\alpha} : \mathrm{H}^q(\Gamma, -) \longrightarrow \mathrm{H}^q(\Phi^\alpha, -), \quad \mathrm{cor}_{\Gamma|\Phi_\alpha} : \mathrm{H}^q(\Phi_\alpha, -) \longrightarrow \mathrm{H}^q(\Gamma, -) \quad (q \geq 0)$$

for cohomology of (left or right) Γ-modules.[6]

[6] The corestriction map is also called "transfer". For the definition of these maps, see e. g. [NSW13, §I.5 (3), (4)] or [Lan96, §II.1 (b), (e)]. There, only the case of left Γ-modules is treated. For the restriction map, this definition still works in the case of right Γ-modules. However, for the corestriction map, there is one subtlety, namely, one must sum over representatives of *left* cosets instead of right ones, otherwise the map is not well-defined. Note also that unfortunately the text [Lan96] calls left cosets what we call right cosets and vice versa.

(a) *Choose the right representative for the action of* $(\Sigma, \star)$ *on* M. *Define a map*

$$\alpha^{\complement} \colon \mathrm{H}^q(\Phi^\alpha, M) \longrightarrow \mathrm{H}^q(\Phi_\alpha, M)$$

to be induced by the map $\Phi_\alpha \longrightarrow \Phi^\alpha$, $\sigma \longmapsto \alpha\sigma\alpha^{-1}$ *and the map* $M \longrightarrow M$ *given by the action of* α.

(b) *Choose the left representative for the action of* $(\Sigma, \star)$ *on* M, *and assume in addition that* (Δ, Γ) *is central. Define a map*

$$\alpha^{\complement} \colon \mathrm{H}^q(\Phi^\alpha, M) \longrightarrow \mathrm{H}^q(\Phi_\alpha, M)$$

to be induced by the map $\Phi_\alpha \longrightarrow \Phi^\alpha$, $\sigma \longmapsto \alpha\sigma\alpha^{-1}$ *and the map* $M \longrightarrow M$ *given by the action of* $\alpha^\star$.

Then in both cases everything is well-defined and we have an equality

$$[\Gamma\alpha\Gamma] = \mathrm{cor}_{\Gamma|\Phi_\alpha} \circ \alpha^{\complement} \circ \mathrm{res}_{\Gamma|\Phi^\alpha}$$

of endomorphisms of $\mathrm{H}^q(\Gamma, M)$ *for any* $q \geq 0$.

Proof: In the case of the right representative, it is a straightforward calculation to check that the maps $\Phi_\alpha \longrightarrow \Phi^\alpha$ and $M \longrightarrow M$ are compatible in the sense of [NSW13, §I.5] and induce a well-defined map $\alpha^{\complement}$ on cohomology. Seeing this in the case of the left representative essentially amounts to the same; note that since (Δ, Γ) is central, we have $\alpha^\star\alpha\sigma\alpha^{-1} = \sigma\alpha^\star$ for all $\sigma \in \Phi_\alpha$, which one needs to make the calculation work in this case.

A statement similar to what we want to show is proved in [Lan96, Prop. II.1.14]. There, a dimension shifting argument is applied to reduce the proof to degree $q = 0$. An analogous dimension shifting argument also works in our case. To see this, one needs that all three maps $\mathrm{cor}_{\Gamma|\Phi_\alpha}$, $\alpha^{\complement}$ and $\mathrm{res}_{\Gamma|\Phi^\alpha}$ are functorial in the respective modules and commute with boundary homomorphisms. For this, see [NSW13, Prop. 1.5.2], whose proof works for all three maps; note that our $\alpha^{\complement}$ is essentially the same as the map called conjugation there (which is the reason why we use the notation "$(\cdot)^{\complement}$"). We omit the details of this dimension shifting.

So we are left to check the statement in degree 0. We first work with the right representative. The corestriction map is in this case induced by

$$M^{\Phi_\alpha} \longrightarrow M^\Gamma, \quad m \longmapsto \sum_{i=1}^{e} m[c_i],$$

where $c_1, \ldots, c_e$ are a complete set of representatives for the left cosets $\Gamma\backslash\Phi_\alpha$. The composite map

$$M^\Gamma \lhook\joinrel\longrightarrow M^{\Phi^\alpha} \xrightarrow{\ \alpha\ } M^{\Phi_\alpha} \longrightarrow M^\Gamma$$

then sends an $m \in M^\Gamma$ to $\sum_i m[\alpha c_i]$. Thus by definition of the endomorphism $[\Gamma\alpha\Gamma]$ in definition 1.22 (b), the claim follows from lemma 1.30.

To check the statement in degree 0 for the left representative, note that if $c_1, \ldots, c_e$ are representatives for the left cosets $\Gamma\backslash\Phi_\alpha$, then $c_1^{-1}, \ldots, c_e^{-1}$ are representatives for the right cosets Φ_α/Γ. Hence the corestriction map is in this case induced by

$$M^{\Phi_\alpha} \longrightarrow M^\Gamma, \quad m \longmapsto \sum_{i=1}^{e} c_i^{-1} \bullet m$$

and the composite map

$$M^\Gamma \hookrightarrow M^{\Phi^\alpha} \xrightarrow{\alpha^\star} M^{\Phi_\alpha} \longrightarrow M^\Gamma$$

sends an $m \in M^\Gamma$ to $\sum_i (c_i^{-1}\alpha^\star) \bullet m$. Since $c_i^{-1}\alpha^\star = (\alpha c_i)^\star$, again by definition of the endomorphism $[\Gamma\alpha\Gamma]$ in definition 1.22 (a) the claim follows from lemma 1.30.

Finally, since $[\Gamma\alpha\Gamma]$ does not depend on the choice of the representative α, we have shown in particular that also $\mathrm{cor}_{\Gamma|\Phi_\alpha} \circ \alpha^{\complement} \circ \mathrm{res}_{\Gamma|\Phi^\alpha}$ is independent of the choice of α and the representative for the action. ☐

We end this section by coming back to the relations between Hecke pairs we discussed at the end of the previous section. If (Δ, Γ) and (Δ', Γ') are Hecke pairs such that $(\Delta, \Gamma) \prec (\Delta', \Gamma')$, then forgetting the module structure along the map $\mathcal{H}_R(\Delta, \Gamma) \longrightarrow \mathcal{H}_R(\Delta', \Gamma')$ from proposition 1.21 induces a functor

$$\mathcal{M}od\text{-}\mathcal{H}_R(\Delta', \Gamma') \longrightarrow \mathcal{M}od\text{-}\mathcal{H}_R(\Delta, \Gamma).$$

Proposition 1.32: *If $(\Delta, \Gamma) \prec (\Delta', \Gamma')$ then the diagram of functors*

$$\begin{tikzcd} & R\text{-}\mathcal{M}od_{(\Sigma,\star)} \arrow[dl, "(-)^{\Gamma'}"'] \arrow[dr, "(-)^{\Gamma}"] & \\ \mathcal{M}od\text{-}\mathcal{H}_R(\Delta', \Gamma') \arrow[rr] & & \mathcal{M}od\text{-}\mathcal{H}_R(\Delta, \Gamma) \end{tikzcd}$$

commutes.

Proof: See [Miy89, Thm. 2.7.6 (2)]. The statement is proved there in slightly different situation, but the proof still works in our setting. ☐

1.6. Atkin-Lehner elements and adjoint Hecke algebras

In this short section, we define an abstract prototype of what is known classically as the Atkin-Lehner involution. Special elements of Σ which we call Atkin-Lehner elements give rise to extra endomorphisms of the Γ-invariants we studied that interact nicely with double coset operators. In our applications, a particular choice of such an element will define the Atkin-Lehner involution (note that in general the endomorphisms defined that way need not be involutions).

Fix a commutative ring R, a monoid with involution $(\Sigma, \star)$ and a Hecke pair (Δ, Γ). Further let M be an R-linear representation of $(\Sigma, \star)$.

We first start with the following obvious observation.

Remark 1.33: If $\sigma \in \Sigma$ is an element normalising Γ (note that we use here that Σ lies in some group), then $m \longmapsto m[\sigma]$ gives a well-defined R-linear endomorphism of M^Γ.

Definition 1.34: (a) We call an element $w \in \Sigma$ an *Atkin-Lehner element for* (Δ, Γ) if $w^{-1}\Gamma w = \Gamma$ and $w^{-1}\Delta w = \Delta^\star$.

(b) We call $\mathcal{H}_R(\Delta^\star, \Gamma)$ the *adjoint abstract Hecke algebra* for the pair (Δ, Γ) or the *Hecke algebra adjoint to* $\mathcal{H}_R(\Delta, \Gamma)$.

Lemma 1.35: *Fix an Atkin-Lehner element $w \in \Sigma$.*

(a) Let $\alpha \in \Delta$. If we have a decomposition

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^{e} \Gamma\alpha_i \quad (\alpha_i \in \Delta),$$

then

$$\Gamma w^{-1}\alpha w\Gamma = \bigsqcup_{i=1}^{e} \Gamma w^{-1}\alpha_i w.$$

(b) The map

$$\mathcal{H}_R(\Delta,\Gamma) \longrightarrow \mathcal{H}_R(\Delta^\star,\Gamma), \quad \Gamma\alpha\Gamma \longmapsto \Gamma w^{-1}\alpha w\Gamma$$

is a well-defined isomorphism of R-algebras. We denote it $T \longmapsto T^w$.

Proof: (a) First, if i, j are such that $\Gamma w^{-1}\alpha_i w \cap \Gamma w^{-1}\alpha_j w \neq \varnothing$, then there exists a $\gamma \in \Gamma$ with $w^{-1}\alpha_i w = \gamma w^{-1}\alpha_j w = w^{-1}w\gamma w^{-1}\alpha_j w$, so $\alpha_i = w\gamma w^{-1}\alpha_j$. Since $w\gamma w^{-1} \in \Gamma$, it follows $i = j$, so the union is indeed disjoint.

Of course $\bigsqcup_{i=1}^{e} \Gamma w^{-1}\alpha_i w \subseteq \Gamma w^{-1}\alpha w\Gamma$. Take an element $\gamma w^{-1}\alpha w\delta$ in the right hand side with $\gamma, \delta \in \Gamma$. Then $w\gamma w^{-1}\alpha w\delta w^{-1} \in \Gamma\alpha\Gamma$, so write $w\gamma w^{-1}\alpha w\delta w^{-1} = \varepsilon\alpha_i$ for some i with $\varepsilon \in \Gamma$. Then $\gamma w^{-1}\alpha w\delta = w^{-1}\varepsilon w w^{-1}\alpha_i w \in \Gamma w^{-1}\alpha_i w$.

(b) To see that the map is well-defined and bijective is easy and we omit this. We have to check that it is compatible with the multiplication of double cosets as defined in (1.3). To see this, it suffices to check

$$\Gamma\xi\Gamma \subseteq \Gamma\alpha\Gamma\beta\Gamma \iff \Gamma w^{-1}\xi w\Gamma \subseteq \Gamma w^{-1}\alpha w\Gamma w^{-1}\beta w\Gamma$$

and

$$\Gamma\alpha_i\beta_i = \Gamma\xi \iff \Gamma w^{-1}\alpha_i w w^{-1}\beta_i w = \Gamma w^{-1}\xi w.$$

The second statement is easy to see: just conjugate the left side of the equivalence with w and use $w\Gamma w^{-1} = \Gamma$. So let us prove the first statement.

Assume the left side of the equivalence and take $\gamma w^{-1}\xi w\delta \in \Gamma w^{-1}\xi w\Gamma$. Take $\gamma', \delta' \in \Gamma$ with $w\gamma = \gamma' w$ and $\delta w^{-1} = w^{-1}\delta$. Write $\gamma'\xi\delta' = \gamma''\alpha w\varepsilon w^{-1}\beta\delta''$ with $\gamma'', \delta'', \varepsilon \in \Gamma$, using the left side of the equivalence, and choose $\gamma''', \delta''' \in \Gamma$ with $w^{-1}\gamma'' = \gamma''' w^{-1}$, $\delta'' w = w\delta'''$. Then $\gamma w^{-1}\xi w\delta = \gamma''' w^{-1}\alpha w\varepsilon w^{-1}\beta w\delta''' \in \Gamma w^{-1}\alpha w\Gamma w^{-1}\beta w\Gamma$.

For the other implication, assume the right side of the equivalence and take $\gamma\xi\delta \in \Gamma\xi\Gamma$. Write $w^{-1}\gamma = \gamma' w^{-1}$ and $\delta w = w\delta'$ with $\gamma', \delta' \in \Gamma$. Using the right side of the equivalence, write $\gamma' w^{-1}\xi w\delta' = \gamma'' w^{-1}\alpha w\varepsilon w^{-1}\beta w\delta''$ with $\gamma'', \delta'', \varepsilon \in \Gamma$. Then $\gamma\xi\delta = w\gamma'' w^{-1}\alpha w\varepsilon w^{-1}\beta w\delta'' w^{-1} \in \Gamma\alpha\Gamma\beta\Gamma$. ☐

Let $w \in \Sigma$ an Atkin-Lehner element. The we have the endomorphism $m \longmapsto m[w]$ of M^Γ. By universality of group cohomology, this extends to endomorphisms of $\mathrm{H}^i(\Gamma, M)$ for all $i \geq 0$, which we denote by the same symbol. On the other hand, M^Γ has a right $\mathcal{H}_R(\Delta,\Gamma)$- as well as a right $\mathcal{H}_R(\Delta^\star,\Gamma)$-module structure.

Lemma 1.36: *We have*

$$[w]T^w = T[w]$$

as endomorphisms of $\mathrm{H}^i(\Gamma, M)$ *for* $T \in \mathcal{H}_R(\Delta, \Gamma)$ *and all* $i \geq 0$.[7]

Proof: It suffices to check this for $i = 0$, and this follows easily from the definition of the $\mathcal{H}_R(\Delta, \Gamma)$-module structure, using lemma 1.35 (a). □

1.7. Coverings and monodromy

Throughout the section, fix a commutative ring R, a monoid with involution $(\Sigma, \star)$, a Hecke pair (Δ, Γ) and a Hecke space $X \in \mathcal{Top}_{(\Sigma,\star)}$.

Because $\star$ is the inversion map on Γ, the left and right quotients $\Gamma \backslash X$ and X/Γ are the same space, so there is a well-defined quotient independently of the choice of the representative for the action on X. We denote this quotient by $\frac{X}{\Gamma}$ (when we want to explicitly use the left or right quotient, we still use the earlier notation). We write $\pi\colon X \longrightarrow \frac{X}{\Gamma}$ for the canonical projection. Note that since Γ may not be normal in Σ, the quotient space $\frac{X}{\Gamma}$ does not necessarily have an induced action of $(\Sigma, \star)$, so it is not an element of $\mathcal{Top}_{(\Sigma,\star)}$.

Definition 1.37: If $\mathcal{F}$ is a Hecke sheaf on X, then $\pi_*\mathcal{F}$ is a sheaf on $\frac{X}{\Gamma}$. Moreover, for each open $U \subseteq \frac{X}{\Gamma}$ and each $\gamma \in \Gamma$, using the left representative for all actions gives us a map

$$\begin{aligned}\pi_*\mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U)) \xrightarrow{\varphi_\gamma} \mathrm{L}\gamma_*\mathcal{F}(\pi^{-1}(U)) \\ = \mathcal{F}(\mathrm{L}\gamma^{-1}(\pi^{-1}(U))) = \mathcal{F}(\pi^{-1}(U)) = \pi_*\mathcal{F}(U),\end{aligned} \tag{1.5}$$

and it is easily verified that this defines an action of Γ on $\pi_*\mathcal{F}(U)$ independently of the choice of the representative. For $U \subseteq \frac{X}{\Gamma}$ open, we write $\pi_*^\Gamma\mathcal{F}(U)$ for the Γ-invariant sections. This defines a sheaf of R-modules $\pi_*^\Gamma\mathcal{F}$ on $\frac{X}{\Gamma}$, and this construction is clearly functorial in $\mathcal{F}$.

In the special case that $\mathcal{F}$ is the constant Hecke sheaf $\underline{M}$ for some representation M of $(\Sigma, \star)$, as in construction 1.10, the sheaf $\pi_*^\Gamma\underline{M}$ has an explicit description.

Lemma 1.38: *The sheaf* $\pi_*^\Gamma\underline{M}$ *is canonically isomorphic to the sheaf of continuous sections of the projection*

$$\frac{X \times M}{\Gamma} \longrightarrow \frac{X}{\Gamma},$$

where Γ *acts diagonally on* $X \times M$ *and* M *is endowed with the discrete topology.*

Sketch of proof: For $(x, a) \in X \times M$, we write $[x, a]$ for the class in the quotient $\frac{(X \times M)}{\Gamma}$, and analogously $[x]$ for the class of some $x \in X$. We work with the left representative of all actions.

Let $U \subseteq \Gamma\backslash X$ be open. By definition, the action of an $\alpha \in \Sigma$ on an $s \in \pi_*\underline{M}(U)$ is given by

$$(\alpha \cdot s)(C) = \alpha^\star \bullet s(\mathrm{L}\alpha(C))$$

for C being a connected component of $\pi^{-1}(U)$ (the "$\bullet$" denotes the action of Σ on M). So s is Γ-invariant if and only if $\gamma \bullet s(C) = s(\mathrm{L}\gamma(C))$, and such s are in one-to-one correspondence

[7] The notation here is for elements acting from the right, since we used this notation for the endomorphism $[w]$ and our modules are right $\mathcal{H}_R(\Delta, \Gamma)$-modules. If we view the endomorphisms as self-maps that can be composed, then the relation reads $T^w \circ [w] = [w] \circ T$.

with continuous maps $s\colon \pi^{-1}(U) \longrightarrow M$ satisfying $s(\mathrm{L}\gamma(x)) = \gamma \bullet s(x)$ for any $\gamma \in \Gamma$ and $x \in \pi^{-1}(U)$. For such a continuous map s, we define a section $\Gamma\backslash X \longrightarrow \Gamma\backslash(X \times M)$ by $[x] \longmapsto [x, s(x)]$.

On the other hand, take a section $t\colon \Gamma\backslash X \longrightarrow \Gamma\backslash(X \times M)$. We have to define a continuous map $s\colon \pi^{-1}(U) \longrightarrow M$ satisfying $s(\mathrm{L}\gamma(x)) = \gamma \bullet s(x)$. So take $x \in \pi^{-1}(U)$ and let $[y, b] := t([x])$, i. e. (y, b) is a representative for the image of $[x]$ under t. That t is a section means that there exists a $\gamma \in \Gamma$ such that $y = \mathrm{L}\gamma(x)$. We then define $s(x) = \gamma^{-1}b$.

One can now check that all this is well defined and gives mutually inverse bijections. We omit these calculations. □

Using the lemma (or just the definition), it is easy to see that the stalk of $\pi_*^\Gamma \underline{M}$ at any point in $\frac{X}{\Gamma}$ is M (and not M^Γ!).

From now on, we impose the following condition on the action of Γ on X:

Condition 1.39: Each $x \in X$ has a neighbourhood U such that $\gamma U \cap U \neq \varnothing$ implies $\gamma = 1$.

This condition implies in particular that Γ acts freely on X, i. e. without fixed points. It is the same as condition (∗) in [Hat02, §1.3, p. 72], or condition (D) in [Gro57, §5.3] together with the additional requirement that Γ acts freely. It is obviously independent of the choice of the representative for the action. The condition is satisfied if X is a Hausdorff space and Γ acts freely and properly discontinuously [Hat02, §1.3, Ex. 23].

Proposition 1.40: *In the special case $\Sigma = \Delta = \Gamma$, the functors π^* and π_*^Γ are quasi-inverse equivalences between the categories of Hecke sheaves on X (which are then just Γ-sheaves) and sheaves of R-modules on $\frac{X}{\Gamma}$.*

Proof: This is proved in [Gro57, p. 198/199] using condition 1.39. □

Construction 1.41: Let $\mathcal{F}$ be a Hecke sheaf on X. Although $\pi_*^\Gamma \mathcal{F}$ is not a Hecke sheaf (not even $\frac{X}{\Gamma} \in \mathcal{T}op_{(\Sigma,\star)}$), we have a canonical $\mathcal{H}_R(\Delta, \Gamma)$-module structure on its cohomology, as we now explain. The global sections of $\pi_* \mathcal{F}$ on $\frac{X}{\Gamma}$ are by definition the global sections of $\mathcal{F}$ on X, which comprise an R-linear representation of $(\Sigma, \star)$, as we saw in proposition 1.12. Hence all $\mathrm{H}^q(X, \mathcal{F})$ are in $R\text{-}\mathcal{M}od_{(\Sigma,\star)}$, and therefore $\mathrm{H}^p(\Gamma, \mathrm{H}^q(X, \mathcal{F}))$ carries a natural right $\mathcal{H}_R(\Delta, \Gamma)$-module structure for $p, q \geq 0$. In particular, the global sections of $\pi_*^\Gamma \mathcal{F}$ on $\frac{X}{\Gamma}$, which are $\mathrm{H}^0(\Gamma, \mathrm{H}^0(X, \mathcal{F}))$, are a right $\mathcal{H}_R(\Delta, \Gamma)$-module. By universality of the δ-functor $\mathrm{H}^q(\frac{X}{\Gamma}, -)$ this extends to a right $\mathcal{H}_R(\Delta, \Gamma)$-module structure on $\mathrm{H}^q(\frac{X}{\Gamma}, \pi_*^\Gamma \mathcal{F})$ for all $q \geq 0$.

It is easy to see from the definitions that the above construction also works for cohomology with compact support since $\mathrm{H}^0_c(\frac{X}{\Gamma}, \pi_*^\Gamma \mathcal{F})$ can be identified with the subgroup of $\mathrm{H}^0(\Gamma, \mathrm{H}^0(X, \mathcal{F}))$ consisting of Γ-invariant global sections in $\mathrm{H}^0(X, \mathcal{F})$ which give compactly supported sections on $\frac{X}{\Gamma}$. It is clear that this subgroup is stable under the action of $\mathcal{H}_R(\Delta, \Gamma)$.

Proposition 1.42: *For each Hecke sheaf $\mathcal{F}$ on X there is a convergent spectral sequence in the category of right $\mathcal{H}_R(\Delta, \Gamma)$-modules*

$$E_2^{pq} = \mathrm{H}^p(\Gamma, \mathrm{H}^q(X, \mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(\tfrac{X}{\Gamma}, \pi_*^\Gamma \mathcal{F}).$$

Proof: Let $\mathcal{A} = \mathcal{Sh}_R^{(\Sigma,\star)}(X)$, $\mathcal{B} = R\text{-}\mathcal{Mod}_{(\Sigma,\star)}$ and $C = \mathcal{Mod}\text{-}\mathcal{H}_R(\Delta,\Gamma)$, all of which are abelian categories. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be the global sections functor and $G\colon \mathcal{B} \longrightarrow C$ be the functor of Γ-invariants. We have seen that they are well-defined in this way, and of course they are left exact.

Let further $\mathcal{D}$ be the category of sheaves of R-modules on $\frac{X}{\Gamma}$. Then π_*^Γ is a functor $\mathcal{A} \longrightarrow \mathcal{D}$, and it is exact since it is the composition of the forgetful functor from $\mathcal{A}$ to the category of Γ-sheaves on X (which is exact) and the functor π_*^Γ from Γ-sheaves on X to $\mathcal{D}$, which is exact because it is an equivalence of categories by proposition 1.40. Therefore if $U\colon \mathcal{D} \longrightarrow C$ is the global sections functor, then $(\mathrm{R}^*U) \circ \pi_*^\Gamma$ is a δ-functor on $\mathcal{A}$ which coincides in degree 0 with $\mathrm{R}^*(G \circ F)$. Since the latter δ-functor is universal, the two δ-functors agree, which proves

$$\mathrm{R}^*(G \circ F) = \mathrm{H}^*(\tfrac{X}{\Gamma}, \pi_*^\Gamma(-)).$$

Using this, the claimed spectral sequence is just the Grothendieck spectral sequence for the composition of functors $G \circ F$. □

Corollary 1.43: *If $\mathcal{F}$ is a Hecke sheaf on X such that $\mathrm{H}^i(X,\mathcal{F})$ vanishes for $i > 0$, then there are canonical $\mathcal{H}_R(\Delta,\Gamma)$-linear isomorphisms*

$$\mathrm{H}^q(\Gamma, \mathrm{H}^0(X,\mathcal{F})) \xrightarrow{\ \sim\ } \mathrm{H}^q(\tfrac{X}{\Gamma}, \pi_*^\Gamma \mathcal{F})$$

for all $q \geq 0$.

Proof: The spectral sequence from proposition 1.42 is then a spectral sequence with only two rows (in fact, only one), so by [NSW13, Lem. 2.1.3 (i)] there is an exact sequence of right $\mathcal{H}_R(\Delta,\Gamma)$-modules

$$\ldots \longrightarrow \mathrm{H}^{q-1}(\tfrac{X}{\Gamma}, \pi_*^\Gamma \mathcal{F}) \longrightarrow \mathrm{H}^{q-2}(\Gamma, \mathrm{H}^1(X,\mathcal{F})) \longrightarrow$$
$$\longrightarrow \mathrm{H}^q(\Gamma, \mathrm{H}^0(X,\mathcal{F})) \longrightarrow \mathrm{H}^q(\tfrac{X}{\Gamma}, \pi_*^\Gamma \mathcal{F}) \longrightarrow \ldots$$

and the second term vanishes by assumption. □

From now on we further assume that X is connected, path connected and simply connected. Then the condition on $\mathrm{H}^i(X,\mathcal{F})$ in corollary 1.43 holds for any $\mathcal{F}$. Moreover, Γ is then the fundamental group of $\frac{X}{\Gamma}$ and X is the universal cover of $\frac{X}{\Gamma}$.

Consider again the special case that $\mathcal{F}$ is the constant Hecke sheaf $\underline{M}$ on X for some representation M of $(\Sigma,\star)$. Similarly as in the group cohomology situation, the action of $\mathcal{H}_R(\Delta,\Gamma)$ on the cohomology $\mathrm{H}^q(\frac{X}{\Gamma}, \pi_*^\Gamma \underline{M})$ has a rather explicit description if we assume in addition the following:

Condition 1.44: Assume that $\alpha\alpha^\star$ acts trivially on X for all $\alpha \in \Sigma$ and that (Δ,Γ) is central.[8]

Assume from now on that condition 1.44 is satisfied.

The description uses the trace map for coverings, which can be explained in a general setting, so let $\pi\colon E \longrightarrow F$ be a finite covering of degree e of topological spaces. We describe a natural transformation $\pi_*\pi^* \longrightarrow \mathrm{id}$ of endofunctors of the category of sheaves on F, so let $\mathcal{F}$ be a sheaf on F and $U \subseteq F$ an open set small enough that $\pi^{-1}(U)$ is a disjoint unit of e open sets of E homeomorphic to U via π. Then $\pi_*\pi^*\mathcal{F}(U) \cong \bigoplus \mathcal{F}(U)$, where the

[8] Strictly speaking, the centrality is only required in the description given below using the left representative. Since it will be satisfied anyway for all our examples, we assume it for simplicity.

sum runs over these e sets. We define the morphism $\pi_*\pi^*\mathcal{F}(U) \longrightarrow \mathcal{F}$ to be the sum map. This defines our natural transformation. Further, in this situation there are canonical δ-functorial isomorphisms $\mathrm{H}^*(E,-) \cong \mathrm{H}^*(F,\pi_*(-))$, which can be proved similarly as [Con09, Lem. 2.3.1.1]. Therefore we get induced δ-functorial maps

$$\mathrm{tr}_{E|F}\colon \mathrm{H}^*(E,\pi^*(-)) \xrightarrow{\sim} \mathrm{H}^*(F,\pi_*\pi^*(-)) \longrightarrow \mathrm{H}^*(F,-)$$

called trace maps.

We now consider the following special covering: let X, Γ and so on be as before and let $\Phi \subseteq \Gamma$ be a subgroup of finite index. Then we have a finite covering of topological spaces $\frac{X}{\Phi} \longrightarrow \frac{X}{\Gamma}$ and we denote the maps between the various spaces as indicated in the diagram

$$\begin{tikzcd} X \arrow[r, "\varpi"] \arrow[rr, bend right, "\pi"'] & \frac{X}{\Phi} \arrow[r, "s"] & \frac{X}{\Gamma}. \end{tikzcd}$$

We will need the following technical statement.

Lemma 1.45: *There is a canonical isomorphism $s^*\pi^\Gamma_*\underline{M} \xrightarrow{\sim} \varpi^\Phi_*\underline{M}$ of sheaves on $\frac{X}{\Phi}$.*

Proof: We prove only the case of the left representative, it is clear that the same proof also works for the right representative.

The sheaf $s_*\varpi^\Phi_*\underline{M}$ on $\Gamma\backslash X$ resp. X/Γ has as sections the Φ-invariant sections of $\pi_*\underline{M}$, so there is a natural morphism from $\pi^\Gamma_*\underline{M}$ into it. This defines a morphism $s^*\pi^\Gamma_*\underline{M} \longrightarrow \varpi^\Phi_*\underline{M}$ by adjointness. We want to prove that it is an isomorphism. To do so, we look at the stalks. Since the stalks of a presheaf and the stalks of its associated sheaf coincide, we can work with the presheaf s^*_{pre} whose associated sheaf is s^*. We know that all stalks are canonically isomorphic to M.

We use the proof of the adjointness of s_* and s^*_{pre}, see [Stacks, Tag 008N]. The morphism of presheaves comes from the composition

$$s^*_{\mathrm{pre}}\pi^\Gamma_*\underline{M} \longrightarrow s^*_{\mathrm{pre}}s_*\varpi^\Phi_*\underline{M} \longrightarrow \varpi^\Phi_*\underline{M}.$$

If we choose an open set $U \subseteq \Phi\backslash X$ small enough that $s\colon U \longrightarrow s(U)$ is a homeomorphism, then for this U, the sections of the left sheaf over U are $\underline{M}(\pi^{-1}(s(U)))^\Gamma$, while the sections of both the middle and the right sheaf over U are $\underline{M}((\varpi)^{-1}(U))^\Phi$, and both can be canonically identified with M. Since any neighbourhood of any point in $\Phi\backslash X$ contains such a U, we see that the canonical map between these sections induces the identity on M. $\square$

Remark 1.46: In this situation, composing the trace map we explained before with the inverse of the isomorphism from lemma 1.45 gives a morphism

$$\mathrm{tr}_{\Gamma|\Phi}\colon \mathrm{H}^q(\tfrac{X}{\Phi},\varpi^\Phi_*\underline{M}) \xrightarrow{\sim} \mathrm{H}^q(\tfrac{X}{\Phi},s^*\pi^\Gamma_*\underline{M}) \longrightarrow \mathrm{H}^q(\tfrac{X}{\Gamma},\pi^\Gamma_*\underline{M})$$

which we also call trace map. If we use the isomorphisms from corollary 1.43, we get a map

$$\mathrm{H}^q(\Phi,M) \longrightarrow \mathrm{H}^q(\Gamma,M),$$

and it is easy to see that this map is just the corestriction map (it suffices to check this for $q = 0$).

Now we look at the following special situation. Let $\alpha \in \Delta$ and put $\Phi_\alpha := \alpha^{-1}\Gamma\alpha \cap \Gamma$, $\Phi^\alpha := \alpha\Phi_\alpha\alpha^{-1}$.[9] We then have the following configurations of spaces and maps (for the left and right representative)

$$\begin{tikzcd} X \arrow[r, "\mathrm{L}\alpha"] \arrow[d, "\varpi_\alpha"] \arrow[dd, bend right=60, "\pi"'] & X \arrow[d, "\varpi^\alpha"'] \arrow[dd, bend left=60, "\pi"] \\ \Phi_\alpha\backslash X \arrow[r, "c_\alpha", "\sim"'] \arrow[d, "s_\alpha"] & \Phi^\alpha\backslash X \arrow[d, "s^\alpha"'] \\ \Gamma\backslash X & \Gamma\backslash X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha^\star\mathrm{R}"] \arrow[d, "\varpi_\alpha"] \arrow[dd, bend right=60, "\pi"'] & X \arrow[d, "\varpi^\alpha"'] \arrow[dd, bend left=60, "\pi"] \\ X/\Phi_\alpha \arrow[r, "c_\alpha", "\sim"'] \arrow[d, "s_\alpha"] & X/\Phi^\alpha \arrow[d, "s^\alpha"'] \\ X/\Gamma & X/\Gamma \end{tikzcd}$$

where all vertical arrows are canonical projections which we denote as indicated in the diagrams, and where the middle horizontal arrows are induced by $\mathrm{L}\alpha$ resp. $\alpha^\star\mathrm{R}$ and both denoted by c_α (we use the same symbols in the left and right situation to keep the notation "simple"). These diagrams commute, but keep in mind that they live in the category $\mathcal{T}op$, not $\mathcal{T}op_{(\Sigma,\star)}$!

By lemma 1.45 we have a canonical isomorphism $(s^\alpha)^*\pi_*^\Gamma\underline{M} \xrightarrow{\sim} (\varpi^\alpha)_*^{\Phi^\alpha}\underline{M}$ of sheaves on $\Phi^\alpha\backslash X$ resp. X/Φ^α, and analogously with Φ_α instead of Φ^α.

We need to introduce some further maps. First, the map s_α is a finite covering map, so we have the trace map as described above.

Next, the map $X \times M \longrightarrow X \times M$ given by $(x,m) \longmapsto (\mathrm{L}\alpha^\star(x), \alpha^\star \bullet m)$ for the left resp. $(x,m) \longmapsto (\alpha\mathrm{R}(x), m[\alpha])$ for the right representative induces a well-defined map $\Phi^\alpha\backslash(X \times M) \longrightarrow \Phi_\alpha\backslash(X \times M)$ resp. $(X \times M)/\Phi^\alpha \longrightarrow (X \times M)/\Phi_\alpha$ which fits into a commutative diagram

$$\begin{tikzcd} \Phi_\alpha\backslash(X \times M) \arrow[d] & \Phi^\alpha\backslash(X \times M) \arrow[l] \arrow[d] \\ \Phi_\alpha\backslash X \arrow[r, "\sim", "c_\alpha"'] & \Phi^\alpha\backslash X \end{tikzcd} \qquad \begin{tikzcd} (X \times M)/\Phi_\alpha \arrow[d] & (X \times M)/\Phi^\alpha \arrow[l] \arrow[d] \\ X/\Phi_\alpha \arrow[r, "\sim", "c_\alpha"'] & X/\Phi^\alpha. \end{tikzcd}$$

To check this, one has to do some calculations using the additional assumptions from above. Therefore, using the description in lemma 1.38, it induces a map of sheaves on $\Phi_\alpha\backslash X$ resp. X/Φ_α

$$\alpha^\dagger : c_\alpha^*(\varpi^\alpha)_*^{\Phi^\alpha}\underline{M} \longrightarrow (\varpi_\alpha)_*^{\Phi_\alpha}\underline{M}.$$

Finally, composing the isomorphism from lemma 1.45 with the map induced by s^α gives a morphism

$$\mathrm{res}_{\Gamma|\Phi^\alpha} : \mathrm{H}^q\big(\Gamma\backslash X, \pi_*^\Gamma\underline{M}\big) \xrightarrow{(s^\alpha)^*} \mathrm{H}^q\big(\Phi^\alpha\backslash X, (s^\alpha)^*\pi_*^\Gamma\underline{M}\big) \xrightarrow{\sim} \mathrm{H}^q\big(\Phi^\alpha\backslash X, (\varpi^\alpha)_*^{\Phi^\alpha}\underline{M}\big)$$

and analogously also for the right quotient spaces, which we call restriction map.

Proposition 1.47: *We have an equality*

$$[\Gamma\alpha\Gamma] = \mathrm{tr}_{\Gamma|\Phi_\alpha} \circ \alpha^\dagger \circ c_\alpha^* \circ \mathrm{res}_{\Gamma|\Phi^\alpha}$$

of endomorphisms of $\mathrm{H}^q(\frac{X}{\Gamma}, \pi_*^\Gamma\underline{M})$ *for all* $q \geq 0$, *for either choice of representative for the action. The same also works in cohomology with compact support.*

[9] Note that due to the requirement in the definition of a Hecke pair that Δ be contained in the commensurator of Γ (see definition 1.15 (a), we have indeed that both Φ_α and Φ^α have finite index in Γ.

Proof: Since all of the maps $\mathrm{tr}_{\Gamma|\Phi_\alpha}$, $\alpha^\dagger$, c_α^*, $\mathrm{res}_{\Gamma|\Phi^\alpha}$ are δ-functorial, it suffices to check this in degree $q = 0$, by definition of the $\mathcal{H}_R(\Delta, \Gamma)$-module structure. By construction, it is then clear that this will work for cohomology with compact support if it works for usual cohomology.

By the corresponding description for group cohomology in lemma 1.31, it suffices to check the commutativity of the three diagrams

$$\begin{tikzcd}
\mathrm{H}^0(\Gamma, M) \arrow[r, "\mathrm{res}_{\Gamma|\Phi^\alpha}"] \arrow[d] & \mathrm{H}^0(\Phi^\alpha, M) \arrow[d] \\
\mathrm{H}^0(\Gamma\backslash X, \pi_*^\Gamma \underline{M}) \arrow[r, "\mathrm{res}_{\Gamma|\Phi^\alpha}"] & \mathrm{H}^0(\Phi^\alpha\backslash X, (\varpi^\alpha)_*^{\Phi^\alpha} \underline{M})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{H}^0(\Phi^\alpha, M) \arrow[r, "\alpha^{\mathsf{C}}"] \arrow[d] & \mathrm{H}^0(\Phi_\alpha, M) \arrow[d] \\
\mathrm{H}^0(\Phi^\alpha\backslash X, (\varpi^\alpha)_*^{\Phi^\alpha} \underline{M}) \arrow[r, "\alpha^\dagger \circ c_\alpha^*"] & \mathrm{H}^0(\Phi_\alpha\backslash X, (\varpi_\alpha)_*^{\Phi_\alpha} \underline{M})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{H}^0(\Phi_\alpha, M) \arrow[r, "\mathrm{cor}_{\Gamma|\Phi_\alpha}"] \arrow[d] & \mathrm{H}^0(\Gamma, M) \arrow[d] \\
\mathrm{H}^0(\Phi_\alpha\backslash X, (\varpi_\alpha)_*^{\Phi_\alpha} \underline{M}) \arrow[r, "\mathrm{tr}_{\Gamma|\Phi_\alpha}"] & \mathrm{H}^0(\Gamma\backslash X, \pi_*^\Gamma \underline{M})
\end{tikzcd}$$

and the corresponding diagrams for the right representative, where the vertical maps are the canonical identifications described before.

We omit the detailed calculations necessary to prove the commutativity of these diagrams (for the third one, we mentioned it already in remark 1.46). One has to write out carefully the definitions of all sheaves and maps above and check the commutativity. $\square$

Remark 1.48: If $X = \mathfrak{h}$ is the upper half plane and Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we considered parabolic group cohomology in remark 1.29. On the other hand, in sheaf cohomology we have parabolic cohomology defined as the image of cohomology with compact support in usual cohomology. In this case, if we assume that condition 1.39 holds, the two notions of parabolic cohomology are identified by the isomorphism from corollary 1.43. This is proved in [Hid81, Prop. 1.1].[10]

1.8. Standard Hecke algebras for GL_2

In this section, we connect the theory we developed so far to more classical situations, giving important examples for our theory.

We first define abstract standard Hecke algebras. For this we use the (semi-)groups of matrices whose definitions were given on page xx.

Definition 1.49: For $N \in \mathbb{N}$ and a commutative ring R, define the *abstract standard Hecke algebras of level N* to be

$$\mathcal{H}_+(N)_R := \mathcal{H}_R(\Delta_0(N), \Gamma_0(N)), \quad \mathcal{H}(N)_R := \mathcal{H}_R(\Delta_0(N)^\flat, \Gamma_0(N)).$$

[10] In [Hid81, Prop. 1.1] there are two conditions on Γ which are denoted there $(1.2_{\mathrm{a,b}})$. These conditions are implied by condition 1.39.

Let $M \in \mathbb{N}$. We define the *abstract standard Hecke algebras of level N away from M* to be

$$\mathcal{H}_+^{(M)}(N)_R := \mathcal{H}_R(\Delta_0(N) \cap \mathrm{M}_2^{(M)}, \Gamma_0(N)), \quad \mathcal{H}^{(M)}(N)_R := \mathcal{H}_R(\Delta_0(N)^{ə} \cap \mathrm{M}_2^{(M)}, \Gamma_0(N)).$$

We define the *abstract standard Hecke algebras of level N away from the level*, also called *restricted abstract standard Hecke algebras of level N* to be

$$\mathcal{H}_+'(N)_R := \mathcal{H}_+^{(N)}(N)_R, \quad \mathcal{H}'(N)_R := \mathcal{H}^{(N)}(N)_R.$$

We will often consider the case $R = \mathbb{Z}$. In this case, or if the ring R is clear from the context, we omit the subscript "R" from the notation.

Obviously $(\Delta_0(N), \Gamma_0(N)) \prec (\Delta_0(N)^{ə}, \Gamma_0(N))$, so the map $\mathcal{H}_+(N) \longrightarrow \mathcal{H}(N)$ sending a double coset to itself is an injective ring homomorphism via which we view $\mathcal{H}_+(N)$ as a subring of $\mathcal{H}(N)$. Similarly, $\mathcal{H}_+^{(M)}(N)$ is a subring of $\mathcal{H}^{(M)}(N)$, $\mathcal{H}_+^{(M)}(N)$ is a subring of $\mathcal{H}_+(N)$ and $\mathcal{H}^{(M)}(N)$ is a subring of $\mathcal{H}(N)$.

We define some important elements of $\mathcal{H}(N)$. First, put for each prime p

$$T_p = \Gamma_0(N) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_0(N) \in \mathcal{H}_+(N).$$

More generally, we may define T_n for each $n \in \mathbb{N}$ to be the sum over all double cosets $\Gamma_0(N)\alpha\Gamma_0(N)$ where α runs over the elements of $\Delta_0(N)$ of determinant n, but we will not use these very often.

Then for each prime $\ell \nmid N$ define

$$S_\ell = \Gamma_0(N) \begin{pmatrix} \ell & \\ & \ell \end{pmatrix} \Gamma_0(N) \in \mathcal{H}_+'(N).$$

Finally we define the "Hecke operator at ∞"

$$\mathcal{E} = \Gamma_0(N)ə\Gamma_0(N) \in \mathcal{H}'(N). \tag{1.6}$$

From the definition of the multiplication, it is obvious that $\mathcal{E}^2 = 1$ in $\mathcal{H}'(N)$.

Lemma 1.50: *For each $\alpha \in \Delta_0(N)$, we have*

$$\Gamma_0(N)\alpha\Gamma_0(N) \cdot \mathcal{E} = \mathcal{E} \cdot \Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\alpha ə\Gamma_0(N) = \Gamma_0(N)ə\alpha\Gamma_0(N).$$

In particular, each $T \in \mathcal{H}_+(N)$ commutes with $\mathcal{E}$.

Proof: In this proof, write $\Gamma = \Gamma_0(N)$ for abbreviation. We can assume that α is a diagonal matrix by [Miy89, Lem. 4.5.2], and for such α we have $\alpha ə = ə\alpha$. This proves the last equality. We prove $\mathcal{E} \cdot \Gamma\alpha\Gamma = \Gamma ə\alpha\Gamma$, the equality $\Gamma\alpha\Gamma \cdot \mathcal{E} = \Gamma\alpha ə\Gamma$ is proved similarly. Since $ə = ə^{-1}$ and $ə$ normalises Γ, we have $\Gamma ə\Gamma\alpha\Gamma = \Gamma ə\Gamma ə ə\alpha\Gamma = \Gamma ə\alpha\Gamma$, so for a double coset $\Gamma\xi\Gamma$, we have that $\Gamma\xi\Gamma \subseteq \Gamma ə\Gamma\alpha\Gamma$ if and only if $\Gamma\xi\Gamma = \Gamma ə\alpha\Gamma$, and further $\Gamma ə\Gamma = \Gamma ə$. So the sum (1.3) used to define the multiplication in $\mathcal{H}$ has only one summand. To finish the proof, note that for a disjoint decomposition $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$, we have $\Gamma ə\alpha \subseteq \Gamma ə\alpha\Gamma = ə\Gamma\alpha\Gamma = ə(\sqcup_i \Gamma\alpha_i) = \sqcup_i ə\Gamma\alpha_i = \sqcup_i \Gamma ə\alpha_i$, so there has to be a unique i such that $\Gamma ə\alpha = \Gamma ə\alpha_i$, hence the coefficient m_ξ in the sum (1.3) is 1. □

Proposition 1.51 (Shimura): *(a) $\mathcal{H}_+(N)$ is commutative and generated as a ring by the T_p for all primes p and S_ℓ for all primes $\ell \nmid N$, and all these elements are algebraically independent.*

(b) The Hecke algebra $\mathcal{H}_+^{(M)}(N)$ away from $M \in \mathbb{N}$ is the subring of $\mathcal{H}_+(N)$ generated by the T_p for all primes $p \nmid M$ and the S_ℓ for all primes $\ell \nmid MN$.

(c) We have isomorphisms

$$\mathcal{H}_+^{(M)}(N)[X] \big/ (X^2) \xrightarrow{\sim} \mathcal{H}^{(M)}(N), \quad \mathcal{H}_+(N)[X] \big/ (X^2) \xrightarrow{\sim} \mathcal{H}(N)$$

via $X \longmapsto \mathcal{E}$.

(d) We have $T_m T_n = T_{mn}$ for coprime integers $m, n \in \mathbb{N}$ and a recursive relation

$$T_{\ell^r} = T_\ell T_{\ell^{r-1}} - pS_\ell T_{\ell^{r-2}}$$

for $r > 1$ and all primes $\ell \nmid N$.

Proof: Statement (a) is proved in [Shi71, Thm. 3.34 (1)]. The elements denoted $T'(p)$ or $T'(1, p)$ there are just our T_p and the $T'(\ell, \ell)$ there are our S_ℓ, as can be easily seen from their definition.

To prove statement (b), we introduce an auxiliary map

$$\det\colon \mathcal{H}(N) \longrightarrow \mathrm{Div}^+(\mathbb{Z}),$$

where by $\mathrm{Div}^+(\mathbb{Z})$ we mean the monoid of ideals in $\mathbb{Z}$, by mapping

$$\sum_{k=1}^{n} \Gamma\alpha_k\Gamma \longmapsto (\det\alpha_1, \ldots, \det\alpha_n)$$

(and $0 \longmapsto \mathbb{Z} = (1)$). It is then easy to check that this map is well-defined and has the properties

$$\det(A + B) = A + B, \quad \det(AB) = \det(A)\det(B) \quad \text{for all } A, B \in \mathcal{H}(N).$$

Clearly all T_ℓ and S_ℓ for all primes $\ell \nmid M$ lie in $\mathcal{H}_+^{(M)}(N)$. If $\mathcal{H}_+^{(M)}(N)$ contained a monomial F in which a T_p with $p \mid M$ occurred, then since $\det(T_p) = p$, the above properties imply $\det(F) + (M) \subseteq (p)$. But since $(\det\alpha, M) = 1$ for all $\alpha \in \Delta_0(N) \cap \mathrm{M}_2^{(M)}$, we have $\det(T) + (M) = \mathbb{Z}$ for all $T \in \mathcal{H}_+^{(M)}(N)$ again by the above properties, so $\mathcal{H}_+^{(M)}(N)$ cannot contain such a monomial.

For an $\alpha \in \Delta_0(N)^\partial$ with $\det\alpha < 0$, we can write $\Gamma_0(N)\alpha\Gamma_0(N) = \mathcal{E} \cdot \Gamma_0(N)\partial\alpha\Gamma_0(N)$ by lemma 1.50, so $\mathcal{E}$ together with the T_p and S_ℓ generate $\mathcal{H}(N)$, and again by lemma 1.50 there are no relations other than $\mathcal{E}^2 = 1$ (and commutativity). This proves statement (c).

Statement (d) is proved in [Shi71, Thm. 3.34 (3), Thm. 3.24 (4), Thm. 3.35]. □

Lemma 1.52: *Let $N \in \mathbb{N}$ and Γ be a congruence subgroup with $\Gamma_1(N) \subseteq \Gamma \subseteq \Gamma_0(N)$. Put $\Delta = \Delta_1(N)\Gamma \subseteq \Delta_0(N)$. Then $(\Delta, \Gamma) \precapprox (\Delta_0(N), \Gamma_0(N))$. In particular, $(\Delta_1(N), \Gamma_1(N)) \precapprox (\Delta_0(N), \Gamma_0(N))$.*

Proof: See the proof of [Miy89, Thm. 4.5.18] and the comment after it. □

From now on we will identify the Hecke algebras which are isomorphic by the above lemma.

Remark 1.53: By the above lemma we know that the abstract standard Hecke algebra is isomorphic to $\mathcal{H}_{\mathbb{Z}}(\Delta_1(N)^\text{ə}, \Gamma_1(N))$. This is very important since we will often look at $\Gamma_1(N)$-invariants of modules. We denote the images of the standard elements T_p, S_ℓ and $\mathcal{E}$ in $\mathcal{H}_{\mathbb{Z}}(\Delta_1(N)^\text{ə}, \Gamma_1(N))$ by the same symbols. It is worth thinking about how they may be represented as double coset operators in $\mathcal{H}_{\mathbb{Z}}(\Delta_1(N)^\text{ə}, \Gamma_1(N))$. For T_p and $\mathcal{E}$ this is clear: the matrices $\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right)$ and ə used to represent them still lie in $\Delta_1(N)^\text{ə}$. But the matrix $\left(\begin{smallmatrix} \ell & \\ & \ell \end{smallmatrix}\right)$ used for S_ℓ does not. Instead, for a prime $\ell \nmid N$ choose a $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ with $\sigma_\ell \equiv \left(\begin{smallmatrix} \ell^{-1} & \\ & \ell \end{smallmatrix}\right) \bmod N$ (where we view $\ell \in (\mathbb{Z}/N)^\times$). Then we have $S_\ell = \Gamma_0(N)\ell\sigma_\ell\Gamma_0(N)$ in $\mathcal{H}_{\mathbb{Z}}(\Delta_0(N)^\text{ə}, \Gamma_0(N))$ since $\sigma_\ell \in \Gamma_0(N)$, but the matrix $\ell\sigma_\ell$ now lies in $\Delta_1(N)$, so in $\mathcal{H}_{\mathbb{Z}}(\Delta_1(N)^\text{ə}, \Gamma_1(N))$ we have also

$$S_\ell = \Gamma_1(N)\ell\sigma_\ell\Gamma_1(N).$$

The double coset $\Gamma_1(N)\ell\sigma_\ell\Gamma_1(N)$ is independent of the choice of σ_ℓ because another choice will differ by an element in $\Gamma(N) \subseteq \Gamma_1(N)$.

In some later proofs it will be useful to know an explicit decomposition of the double coset of $\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right)$, so we list this result.

Lemma 1.54: *For $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$ and any prime p, there is a decomposition*

$$\Gamma \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma = \begin{cases} \displaystyle\bigsqcup_{j=0}^{p-1} \Gamma \begin{pmatrix} 1 & j \\ & p \end{pmatrix} & \text{if } p \mid N, \\ \displaystyle\bigsqcup_{j=0}^{p-1} \Gamma \begin{pmatrix} 1 & j \\ & p \end{pmatrix} \sqcup \Gamma \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & \\ & 1 \end{pmatrix} & \text{if } p \nmid N, \text{ for } m, n \in \mathbb{Z} \text{ such that } mp - Nn = 1. \end{cases}$$

Of course the factor $\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)$ can be omitted if $\Gamma = \Gamma_0(N)$ since it lies in $\Gamma_0(N)$.

Proof: [DS05, (5.2)], [Miy89, Lem. 4.5.6] □

Lemma 1.55: *Let $M \in \mathbb{N}$ and p be a prime with $p \nmid M$.*

(a) For all $r \geq s \geq 1$ we have

$$(\Delta_1(Mp^r), \Gamma_1(Mp^r)) \precapprox (\Delta_1(Mp^s), \Gamma_1(Mp^s)).$$

(b) We have

$$(\Delta_1(Mp) \cap \mathrm{M}_2^{(p)}, \Gamma_1(Mp)) \precapprox (\Delta_1(M) \cap \mathrm{M}_2^{(p)}, \Gamma_1(M)).$$

Proof: In both cases the condition in definition 1.20 (a) is obvious, and condition (c) is also easy to see.

(a) The proof of this part is a modification of the proof of [Shi71, Prop. 3.32], which we follow very closely. We start with some preliminary matrix calculations.

Let $\alpha \in \Delta_1(Mp^s)$ and write $\det\alpha = mq$ with $(q, Mp) = 1$ and m having only prime factors that occur in Mp. For a prime ℓ, write $E_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$. Define

$$X(\alpha) := \{\beta \in \Delta_1(Mp^s) : \det\beta = \det\alpha,\ \forall\ell \mid q : E_\ell\beta E_\ell = E_\ell\alpha E_\ell\}.$$

Let $\beta = \left(\begin{smallmatrix} a & * \\ * & * \end{smallmatrix}\right) \in X(\alpha)$. Then since $\beta \in \Delta_1(Mp^s)$ we have $(a, mMp^r) = 1$, so there is an $e \in \mathbb{Z}$ with $ae \equiv 1 \bmod mMp^r$ and $e \equiv 1 \bmod Mp^s$. Choose $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma \equiv \left(\begin{smallmatrix} e & 0 \\ 0 & a \end{smallmatrix}\right) \bmod mMp^r$, then $\gamma \in \Gamma(Mp^s)$ and $\gamma\beta \equiv \left(\begin{smallmatrix} 1 & b \\ fMp^s & * \end{smallmatrix}\right) \bmod mMp^r$ with $b, f \in \mathbb{Z}$. Put $\delta = \left(\begin{smallmatrix} 1 & 0 \\ -fMp^s & 1 \end{smallmatrix}\right)$, then $\delta \in \Gamma_1(Mp^s)$ and $\delta\gamma\beta \equiv \left(\begin{smallmatrix} 1 & b \\ 0 & mq \end{smallmatrix}\right) \bmod mMp^r$, where the lower right entry here comes from the fact $\det\beta = mq$. Put now $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)$, $\varepsilon = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ and $\xi = \delta\gamma\beta\varepsilon^{-1}\eta^{-1}$. Then $\det\xi = q$ and $\xi \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) \bmod Mp^r$.

Now put $\xi' = \alpha\eta^{-1}$. Then $\det\xi' = q$ and $E_\ell\xi'E_\ell = E_\ell\alpha E_\ell$ for all $\ell \mid q$, so ξ' can be chosen as the element ξ_1 in the proof of [Shi71, Prop. 3.32]. The argument there then shows that there exist $\sigma \in \Gamma(Mp^s)$ and $\theta \in \Gamma(Mp^r)$ such that $\alpha = \sigma\xi'\theta^{-1}$. Our σ is $\varphi\omega$ for the φ and ω defined there, and in the construction of θ there, it is easy to see that we can in fact choose $\theta \in \Gamma(Mp^r)$ and not just $\theta \in \Gamma(Mp^s)$.

Let $\alpha \in \Delta_1(Mp^r) \subseteq \Delta_1(Mp^s)$. We first prove the condition in definition 1.20 (b), i. e. $\Gamma_1(Mp^s)\alpha\Gamma_1(Mp^s) = \Gamma_1(Mp^s)\alpha\Gamma_1(Mp^r)$. The inclusion "$\supseteq$" is clear. Moreover, we obviously have $\Gamma_1(Mp^s)\alpha\Gamma_1(Mp^s) \subseteq X(\alpha)$. To prove the other inclusion we show $X(\alpha) \subseteq \Gamma_1(Mp^s)\alpha\Gamma_1(Mp^r)$, so take $\beta \in X(\alpha)$. We use our previous calculations to see

$$\beta = \gamma^{-1}\delta^{-1}\xi\eta\varepsilon \in \Gamma_1(Mp^s)\xi\eta\Gamma_1(Mp^r) = \Gamma_1(Mp^s)\sigma\xi\eta\theta^{-1}\Gamma_1(Mp^r) = \Gamma_1(Mp^s)\alpha\Gamma_1(Mp^r).$$

We now prove the condition in definition 1.20 (d), i. e. $\Delta_1(Mp^s) = \Gamma_1(Mp^s)\Delta_1(Mp^r)$. The inclusion "$\supseteq$" is again clear. For the other inclusion, we use again the previous calculations to see that we can write any $\alpha \in \Delta_1(Mp^s)$ as

$$\alpha = \sigma\xi\eta\theta^{-1} \in \Gamma_1(Mp^s)\Delta_1(Mp^r).$$

(b) We first verify the condition in definition 1.20 (b). Let $\gamma_1, \gamma_2 \in \Gamma_1(M)$ and $\alpha \in \Delta_1(M) \cap \mathrm{M}_2^{(p)}$ be given, and write $\gamma_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then we must have $c \not\equiv 0$ or $d \not\equiv 0 \pmod{p}$, and also $c \not\equiv 0$ or $a \not\equiv 0 \pmod{p}$. Distinguishing these cases, we can find integers $x, y, z, w \in \mathbb{Z}$ such that

$$\begin{cases} \text{if } c \not\equiv 0: & \begin{array}{l} x \equiv 0 \\ y \equiv c^{-1} \\ w \equiv a \end{array} \\[2em] \text{if } d \not\equiv 0: & \begin{array}{l} \begin{cases} \text{if } a \not\equiv 0: & \begin{array}{l} x \equiv a^{-1} \\ y \equiv 0 \end{array} \\[1em] \text{if } c \not\equiv 0: & \begin{array}{l} x \equiv 0 \\ y \equiv c^{-1} \end{array} \end{cases} \\ w \equiv d^{-1}(1 + bc) \end{array} \end{cases}$$

and in any case $z \equiv -c$, where every "$\equiv$" is to be understood modulo p. Put $\delta := \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$. Then a direct calculation shows that $\delta\gamma_2 \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $\det\delta \equiv 1 \pmod p$. Using the Chinese Remainder Theorem we can moreover demand that $\delta \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo $C := M\det\alpha$, because by assumption $p \nmid C$. Using the surjectivity of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/C)$, we may take $\delta \in \mathrm{SL}_2(\mathbb{Z})$ without loss of generality. Then we have in $\mathrm{GL}_2(\mathbb{Q})$

$$\gamma_1\alpha\gamma_2 = \gamma_1\alpha(\delta^{-1}\alpha^{-1}\alpha\delta)\gamma_2 = (\gamma_1\alpha\delta^{-1}\alpha^{-1})\alpha(\delta\gamma_2)$$

and by construction $\delta\gamma_2 \in \Gamma_1(Mp)$. It remains to check that $\gamma_1\alpha\delta^{-1}\alpha^{-1} \in \Gamma_1(M)$, for which it suffices to see that the matrix $\alpha\delta^{-1}\alpha^{-1}$ has integral entries, because the congruence property is clear by construction. Let ℓ be a prime. If $\ell \mid \det\alpha$, then $\delta \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod \ell$ by construction, so $\alpha\delta^{-1}\alpha^{-1} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod \ell$. If $\ell \nmid \det\alpha$, then α is invertible modulo ℓ, so $\alpha\delta^{-1}\alpha^{-1} \in \mathrm{GL}_2(\mathbb{F}_\ell)$. So the denominators of the entries of $\alpha\delta^{-1}\alpha^{-1}$ are not divisible by any prime ℓ, hence are integral.

Now we check the condition in definition 1.20 (d). Let $\alpha \in \Delta_1(M) \cap \mathrm{M}_2^{(p)}$. Then since $p \nmid \det\alpha$ we have $\alpha^{-1}\left(\begin{smallmatrix} 1 & \\ & \det\alpha \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F}_p)$, and we can find a preimage $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ whose reduction modulo p is this matrix. By the Chinese Remainder Theorem we may take $\gamma \in \Gamma_1(M)$. Put $\beta := \gamma^{-1}\alpha$. Then by construction $\beta \in \Delta_1(Mp) \cap \mathrm{M}_2^{(p)}$, and $\alpha = \gamma\beta \in \Gamma_1(M)(\Delta_1(Mp) \cap \mathrm{M}_2^{(p)})$. □

Lemma 1.56: *Let* (Γ, Δ) *and* (Γ', Δ') *such that* Γ *and* Γ' *are normalised by* ∂.

(a) If $(\Gamma, \Delta) \prec (\Gamma', \Delta')$*, then* $(\Gamma, \Delta^{\partial}) \prec (\Gamma', \Delta'^{\partial})$.

(b) If $(\Gamma, \Delta) \precapprox (\Gamma', \Delta')$*, then* $(\Gamma, \Delta^{\partial}) \precapprox (\Gamma', \Delta'^{\partial})$.

Proof: We can assume that $\Delta, \Delta' \subseteq \mathrm{M}_2^+(\mathbb{Z})$, otherwise we have $\Delta = \Delta^{\partial}$ and $\Delta' = \Delta'^{\partial}$ and the claim is trivial.

Assume $(\Gamma, \Delta) \prec (\Gamma', \Delta')$. We have to look at the conditions in definition 1.20. Clearly, if $\Delta \subseteq \Delta'$ then $\Delta^{\partial} \subseteq \Delta'^{\partial}$, so condition (a) holds. We can write any $\alpha \in \Delta^{\partial} \setminus \Delta$ as $(\alpha\partial)\partial$ with $\alpha\partial \in \Delta$. We have for any such α

$$\Gamma'\alpha\Gamma' = \Gamma'\alpha\partial\partial\Gamma' = \Gamma'\alpha\partial\Gamma'\partial = \Gamma'\alpha\partial\Gamma\partial = \Gamma'\alpha\partial\partial\Gamma = \Gamma'\alpha\Gamma,$$

so condition (b) also holds. Now take a $\gamma' \in \Gamma'$ and $\alpha \in \Delta^{\partial} \setminus \Delta$. If $\gamma'\alpha \in \Gamma'\alpha \cap \Delta^{\partial}$, then $\det\gamma'\alpha\partial > 0$, so $\gamma'\alpha\partial \in \Gamma'\alpha\partial \cap \Delta = \Gamma\alpha\partial$ and thus $\gamma'\alpha = \gamma'\alpha\partial\partial \in \Gamma\alpha\partial\partial = \Gamma\alpha$, hence condition (c) holds. Now assume $(\Gamma, \Delta) \precapprox (\Gamma', \Delta')$, so $\Delta' = \Gamma'\Delta$. Then $\Delta'^{\partial} = \Gamma'\Delta^{\partial}$ and (d) is also satisfied. □

Corollary 1.57: *For any* $M \in \mathbb{N}$*, any prime* p *with* $p \nmid M$*, any* $r \geq 1$ *and any congruence subgroup* Γ *with* $\Gamma_1(Mp^r) \subseteq \Gamma \subseteq \Gamma_0(Mp^r)$ *and* $\Delta = \Delta_1(Mp^r)\Gamma$*, we have canonically*

$$\mathcal{H}_{\mathbb{Z}}(\Delta, \Gamma) \cong \mathcal{H}_+(Mp), \quad \mathcal{H}_{\mathbb{Z}}(\Delta^{\partial}, \Gamma) \cong \mathcal{H}(Mp).$$

If $\Delta' \supseteq \Delta$ *is a larger submonoid, then* $\mathcal{H}_+(Mp)$ *is canonically contained in* $\mathcal{H}_{\mathbb{Z}}(\Delta', \Gamma)$ *and* $\mathcal{H}(Mp)$ *is canonically contained in* $\mathcal{H}_{\mathbb{Z}}(\Delta'^{\partial}, \Gamma)$*. All this is compatible with the module structures over the various Hecke algebras defined in the preceding sections.*

Remark 1.58: Let $N \in \mathbb{N}$, let $M \mid N$ be a proper divisor and put $q := N/M$. Further let M be an R-linear representation of $(\Delta_1(N), \Gamma_1(N))$ for some ring R. Then $\Gamma_1(N) \subseteq \Gamma_1(M)$ is a subgroup, so we have an inclusion of R-modules $M^{\Gamma_1(M)} \subseteq M^{\Gamma_1(N)}$. By lemma 1.55 and proposition 1.21 we have an isomorphism of R-algebras $\mathcal{H}_+^{(q)}(N)_R \xrightarrow{\sim} \mathcal{H}_+^{(q)}(M)_R$ and by proposition 1.32 the inclusion $M^{\Gamma_1(M)} \subseteq M^{\Gamma_1(N)}$ is compatible with this isomorphism. More concretely, this inclusion respects the actions of the Hecke operators T_ℓ and S_ℓ for all primes $\ell \nmid q$. By lemma 1.55 (a), if M and N have the same prime divisors, then the inclusion is compatible with all Hecke operators. If not, then we can see from lemma 1.54 that the T_p for $p \mid q$ will in general act differently.

By lemma 1.56 the same holds if we replace $\Delta_1(N)$ by $\Delta_1(N)^\mathrm{a}$, i. e. $\mathcal{H}_+^{(q)}(N)_R$ by $\mathcal{H}^{(q)}(N)_R$.

Definition 1.59: Fix the Hecke pair $(\Delta_1(N)^\mathrm{a}, \Gamma_1(N))$ and an R-linear representation M of (Σ, ι). Since $\Gamma_1(N)$ is normal in $\Gamma_0(N)$, any element $\sigma \in \Gamma_0(N)$ induces an endomorphism $m \longmapsto m[\sigma]$ of $M^{\Gamma_1(N)}$ as in remark 1.33 called a *diamond operator*. Since there is an isomorphism

$$\Gamma_0(N) \big/ \Gamma_1(N) \xrightarrow{\sim} \left(\mathbb{Z} \big/ N\right)^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d$$

this gives an action of $(\mathbb{Z}/N)^\times$ on $M^{\Gamma_1(N)}$ and we denote it by $m \longmapsto m\langle d\rangle$ for $d \in (\mathbb{Z}/N)^\times$ (sometimes also by $m \longmapsto \langle d\rangle m$ if there is no need to distinguish between left and right actions, which we can do since $(\mathbb{Z}/N)^\times$ is abelian).

Obviously the matrix σ_ℓ from remark 1.53 normalises $\Gamma_1(N)$, and the automorphism of $M^{\Gamma_1(N)}$ it defines is just the diamond operator $\langle \ell \rangle$.

On the other hand, the diagonal matrix $\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)$ for some $\ell \in \mathbb{Z} \setminus \{0\}$ lies in the centre of Σ, so it induces endomorphism $m \longmapsto m\left[\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right]$ of $M^{\Gamma_1(N)}$ as in remark 1.33. It is then obvious that we have

$$m[S_\ell] = m\langle \ell \rangle \begin{bmatrix} \ell & 0 \\ 0 & \ell \end{bmatrix} \quad \text{for } m \in M^{\Gamma_1(N)} \tag{1.7}$$

for any prime $\ell \nmid N$. In particular, if the endomorphism defined by $\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)$ is invertible, then the diamond operators commute with all elements of the Hecke algebra $\mathcal{H}(N)$.

We end this section by defining the Atkin-Lehner endomorphism.

Definition 1.60: Let $N \in \mathbb{N}$ and put

$$w_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}.$$

It is easy to verify that w_N is an Atkin-Lehner element in the sense of definition 1.34 for $\Gamma = \Gamma_0(N)$ and $\Delta = \Delta_0(N)$ or for $\Gamma = \Gamma_1(N)$ and $\Delta = \Delta_1(N)$. Further we note that for any $A \in \mathrm{M}_2(\mathbb{Q})$ we have

$$w_N^{-1} A w_N = w_N A w_N^{-1}. \tag{1.8}$$

Definition 1.61: We call the endomorphism $[w_N]$ attached to w_N the *Atkin-Lehner endomorphism of level N* (see section 1.6).

The Atkin-Lehner endomorphism is often called Atkin-Lehner involution since in some applications the submonoid $\mathbb{Z}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \subseteq \Sigma$ acts trivially on the module under consideration, in which case $[w_N]$ is in fact an involution since w_N^2 lies in this submonoid. Since for a general module it may not be an involution, we call it Atkin-Lehner endomorphism.

Definition 1.62: (a) The *adjoint abstract standard Hecke algebra of level N* is the Hecke algebra adjoint to $\mathcal{H}(N)$, so $\mathcal{H}(N)^\iota = \mathcal{H}_\mathbb{Z}((\Delta_0(N)^\flat)^\iota, \Gamma_0(N))$. Define $\mathcal{H}_+(N)^\iota$ etc. analogously.

(b) Denote the images of T_p, S_ℓ and $\mathcal{E}$ under the isomorphism $\mathcal{H}(N) \xrightarrow{\sim} \mathcal{H}(N)^\iota$ from lemma 1.35 (b) by T_p^ι, S_ℓ^ι and $\mathcal{E}^\iota$, respectively.

Of course, $\mathcal{H}(N)^\iota$ is isomorphic to $\mathcal{H}_\mathbb{Z}((\Delta_1(N)^\flat)^\iota, \Gamma_1(N))$ and other Hecke algebras, analogous to corollary 1.57. Hereafter we again identify all these adjoint Hecke algebras.

It is easy to see that

$$T_p^\iota = \Gamma_0(N) \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_0(N)$$

and $\mathcal{E}^\iota = -\mathcal{E}$. For the diamond operators, it follows from the definition that

$$[w]\langle d\rangle = \langle d^{-1}\rangle[w] \tag{1.9}$$

for $d \in (\mathbb{Z}/N)^\times$. From this it is easy to see that $S_\ell^\iota = \left[\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right] \langle \ell\rangle^{-1}$. So from lemma 1.36 together with (1.8), we get the following:

Corollary 1.63: *If $M \in R\text{-}\mathcal{Mod}_{(\Sigma,\iota)}$, then we have the following relations*[11]

$$T_p[w] = [w]T_p^\iota, \quad T_p^\iota[w] = [w]T_p, \quad [w]\mathcal{E} = (-\mathcal{E})[w]$$

of endomorphisms of M^Γ.

1.9. Eigenalgebras

We define the notion of a Hecke eigenalgebra. This is a special case of the general definition of an eigenalgebra given in [Bel10, §1.2].

Let R be a commutative ring, $(\Sigma, \star)$ a monoid with involution and (Δ, Γ) a Hecke pair.

Definition 1.64: For a right $\mathcal{H}_R(\Delta,\Gamma)$-module M, denote by $\mathbf{T}_R^{(\Delta,\Gamma)}(M)$ the image of the canonical R-algebra morphism

$$\mathcal{H}_R(\Delta,\Gamma) \longrightarrow \mathrm{End}_R(M).$$

We call it the *Hecke eigenalgebra* of M.

The notation for Hecke algebras is not standard. Some texts use $\mathcal{H}$ or $\hbar$, some use $\mathbf{T}$ or $\mathbb{T}$. We follow the convention that we use $\mathcal{H}$ for abstract Hecke algebras, whereas we use $\mathbf{T}$ and similar symbols for Hecke eigenalgebras. When some of the parameters R, (Δ,Γ) or M are clear, we may drop them from the notation. For important choices of M, we will introduce a special notation.

In [Bel10, §1.3–5], basic ring-theoretic properties of eigenalgebras are proved, for example the following:

[11] The parentheses in the last relation are important! This is because $-I$, if I is the identity matrix, does not necessarily act as -1, it can also act trivially.

Lemma 1.65: *Let R be noetherian and let S be a commutative flat noetherian R-algebra. Then there is a canonical isomorphism*

$$\mathbf{T}_R^{(\Delta,\Gamma)}(M) \underset{R}{\otimes} S \xrightarrow{\sim} \mathbf{T}_S^{(\Delta,\Gamma)}(M \underset{R}{\otimes} S).$$

Proof: [Bel10, Prop. 1.4.1] □

If these eigenalgebras are commutative, their geometry is related to systems of eigenvalues, as we now explain.

Definition 1.66: A *system of Hecke eigenvalues* is an R-algebra morphism $\lambda\colon \mathcal{H}_R(\Delta,\Gamma) \longrightarrow R$. If M is a right $\mathcal{H}_R(\Delta,\Gamma)$-module and λ is a system of Hecke eigenvalues, then an $m \in M$ is called an *eigenvector* for λ if $m[T] = \lambda(T)m$ for all $T \in \mathcal{H}_R(\Delta,\Gamma)$. These eigenvectors comprise an R-submodule of M which we denote by $M[\lambda]$. If $m \in M[\lambda]$ we will also write $M[m]$ for $M[\lambda]$. We say that a system of eigenvalues λ *appears in* M if $M[\lambda] \neq 0$.

Proposition 1.67: *Let $R = K$ be a field, let M be a right $\mathcal{H}_K(\Delta,\Gamma)$-module and assume that $\mathbf{T}_K^{(\Delta,\Gamma)}(M)$ is commutative. Then for each extension field K' of K there is a canonical bijection between* $\operatorname{Spec}(\mathbf{T}_K^{(\Delta,\Gamma)}(M))(K) = \operatorname{Hom}_K(\mathbf{T}_K^{(\Delta,\Gamma)}(M), K')$ *and systems of Hecke eigenvalues that appear in $M \otimes_K K'$.*

Proof: [Bel10, Cor. 1.5.10] □

We now consider again the special case of the abstract standard Hecke algebra. The following observation is elementary but crucial.

Lemma 1.68: *Let $\Sigma = \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2^+(\mathbb{Q})$, $\Gamma = \Gamma_1(N)$, $\Delta = \Delta_1(N)$ and let M be an R-linear representation of (Σ,ι). Assume that there exists $n \in \mathbb{N}_0$ such that the element $\left(\begin{smallmatrix} \ell & \\ & \ell \end{smallmatrix}\right)$ acts on M as multiplication by ℓ^n for all primes $\ell \nmid N$. Then the following R-algebras are equal:*

(i) The eigenalgebra $\mathbf{T}_{(\mathrm{i})} := \mathbf{T}_R^{(\Delta,\Gamma)}(M)$,

(ii) the subalgebra $\mathbf{T}_{(\mathrm{ii})}$ of $\operatorname{End}_R(M)$ generated by T_p for all primes p and S_ℓ for all primes $\ell \nmid N$,

(iii) the subalgebra $\mathbf{T}_{(\mathrm{iii})}$ of $\operatorname{End}_R(M)$ generated by T_p for all primes p and $\langle\ell\rangle$ for all primes $\ell \nmid N$,

(iv) the subalgebra $\mathbf{T}_{(\mathrm{iv})}$ of $\operatorname{End}_R(M)$ generated by T_n for $n \in \mathbb{N}$.

Analogous statements hold for eigenalgebras away from the level. In particular, in this situation, any element of M^Γ which is an eigenvector of all T_n for all $n \in \mathbb{N}$ with $(n,N) = 1$ is automatically an eigenvector of all diamond operators.

Proof: We follow the proof of [DI95, Prop. 3.5.1]. By (1.7) and our assumption we have $S_\ell = \ell^{n+1}\langle\ell\rangle$, so we have inclusions $\mathbf{T}_{(\mathrm{ii})} \subseteq \mathbf{T}_{(\mathrm{iii})}$. From the relations in proposition 1.51 (d) we see that we also have an inclusion $\mathbf{T}_{(\mathrm{iv})} \subseteq \mathbf{T}_{(\mathrm{ii})}$. Further the equality $\mathbf{T}_{(\mathrm{i})} = \mathbf{T}_{(\mathrm{ii})}$ is clear by proposition 1.51 (a). It thus remains to show $\langle\ell\rangle \in \mathbf{T}_{(\mathrm{iv})}$ for all $\ell \nmid N$.

We use the relation $\ell^{n+2}\langle\ell\rangle = \ell S_\ell = T_\ell^2 - T_{\ell^2}$, which is a special case of proposition 1.51 (d). By Dirichlet's theorem on primes in arithmetic progressions we can find another prime

$q \neq \ell$ such that $\ell \equiv q \pmod N$ (i. e. $\langle \ell \rangle = \langle q \rangle$) and further integers $a, b \in \mathbb{Z}$ such that $a\ell^{n+2} + bq^{n+2} = 1$. Then we have

$$\langle \ell \rangle = (a\ell^{n+2} + bq^{n+2})\langle \ell \rangle = a(T_\ell^2 - T_{\ell^2}) + b(T_q^2 - T_{q^2}) \in \mathbf{T}_{(\mathrm{iv})}. \qquad \square$$

Definition 1.69: For a right $\mathcal{H}(N)_R$-module M we write

$$M^\pm := \{m \in M : m[\mathcal{E}] = \pm m\},$$

which is a right $\mathcal{H}_+(N)_R$-module. Note that if $\lambda\colon \mathcal{H}_+(N)_R \longrightarrow R$ is a system of Hecke eigenvalues, $M^\pm[\lambda]$ is a well-defined R-module.

2. Miscellaneous

2.1. Determinants

In this section, we very briefly recall the formalism of determinant functors as introduced in [FK06, §1.2]. There, determinant functors for modules over (non-commutative) rings are studied, but we will need this theory mostly just over fields, which simplifies matters considerably. We first introduce the general setting, but later we will specialise to the case of a field.

In this section, modules over rings should always be left modules.

Definition 2.1: Let Λ be a ring.

(a) Define a category $\mathcal{Det}_\Lambda$ as follows. Objects are pairs (P, Q) of finitely generated projective Λ-modules. The set $\mathrm{Hom}_{\mathcal{Det}_\Lambda}((P,Q),(P',Q'))$ for two such pairs (P, Q) and (P', Q') is empty unless $[P] - [Q] = [P'] - [Q']$ in $\mathrm{K}_0(\Lambda)$. In this case, take a finitely generated projective Λ-module R such that $P \oplus Q' \oplus R \cong P' \oplus Q \oplus R$ and define

$$\mathrm{Hom}_{\mathcal{Det}_\Lambda}((P,Q),(P',Q')) := \frac{(\mathrm{K}_1(\Lambda) \times \mathrm{Isom}(P \oplus Q' \oplus R, P' \oplus Q \oplus R))}{\mathrm{Aut}(P' \oplus Q \oplus R)},$$

where a $g \in \mathrm{Aut}(P' \oplus Q \oplus R)$ acts on an $(x, y) \in \mathrm{K}_1(\Lambda) \times \mathrm{Isom}(P \oplus Q' \oplus R, P' \oplus Q \oplus R)$ by $g(x, y) = (x\bar{g}, g^{-1}y)$ (with $\bar{g}$ being the image of g in $\mathrm{K}_1(\Lambda)$). This does not depend on the choice of R.

(b) For a finitely generated projective Λ-module P, define

$$\mathrm{Det}_\Lambda(P) := (P, 0) \in \mathcal{Det}_\Lambda.$$

An isomorphism $\varphi\colon P \longrightarrow Q$ of finitely generated projective Λ-modules induces a morphism $\mathrm{Det}_\Lambda(\varphi)\colon \mathrm{Det}_\Lambda(P) \longrightarrow \mathrm{Det}_\Lambda(Q)$ whose class is represented by $(1, \varphi)$, and this defines a functor Det_Λ from the category of finitely generated projective Λ-modules with isomorphisms to $\mathcal{Det}_\Lambda$.

(c) Let $(P, Q), (P', Q') \in \mathcal{Det}_\Lambda$. Define $(P, Q) \cdot (P', Q') := (P \oplus P', Q \oplus Q') \in \mathcal{Det}_\Lambda$. We identify the objects $(P, Q) \cdot (P', Q')$ and $(P', Q') \cdot (P, Q)$ of $\mathcal{Det}_\Lambda$ using the obvious canonical isomorphism. If we have isomorphisms $\varphi\colon P \longrightarrow Q$ and $\psi\colon P' \longrightarrow Q'$, then this defines a morphism

$$\mathrm{Det}_\Lambda(\varphi) \cdot \mathrm{Det}_\Lambda(\psi)\colon \mathrm{Det}_\Lambda(P) \cdot \mathrm{Det}_\Lambda(P') \longrightarrow \mathrm{Det}_\Lambda(Q) \cdot \mathrm{Det}_\Lambda(Q')$$

whose class is represented by $(1, \varphi \oplus \psi)$.

Let P be a finitely generated projective Λ-module. Then one has

$$\mathrm{End}_{\mathcal{D}et_\Lambda}(\mathrm{Det}_\Lambda(P)) = {(\mathrm{K}_1(\Lambda) \times \mathrm{Isom}(P,P))}\Big/{\mathrm{Aut}(P)},$$

which can be canonically identified with $\mathrm{K}_1(\Lambda)$.

Definition 2.2: For a finitely generated projective Λ-module P and a $\varphi \in \mathrm{Aut}_\Lambda(P)$, let

$$\det\nolimits_\Lambda^\times(\varphi) \in \mathrm{K}_1(\Lambda)$$

be the image of $\mathrm{Det}_\Lambda(\varphi) \in \mathrm{End}_{\mathcal{D}et_\Lambda}(\mathrm{Det}_\Lambda(P))$ in $\mathrm{K}_1(\Lambda)$. It is just the canonical image of φ in $\mathrm{K}_1(\Lambda)$.

For the rest of the section, we fix a field K and specialise to the case $\Lambda = K$. Any "Det" from now on should mean "Det_K", any "$\det^\times$" should mean "$\det_K^\times$" and any vector space should be a K-vector space. If V is some finite-dimensional vector space and $\varphi \in \mathrm{End}(V)$, we sometimes write $\det(\varphi, V)$ for the determinant of φ if we want to make clear on which space φ acts when this may not be clear from the context. If we moreover want to make clear that V is a K-vector space, then we write $\det_K(\varphi)$.

Definition 2.3: Let V, W be finite-dimensional vector spaces of equal dimension. Choose bases γ of V and δ of W.

(a) We write $\mathrm{is}_{\gamma,\delta}\colon V \longrightarrow W$ for the isomorphism between V and W that identifies the bases γ and δ. Note that $\mathrm{is}_{\gamma,\delta}^{-1} = \mathrm{is}_{\delta,\gamma}$.

(b) Let $\varphi\colon V \longrightarrow W$ be a morphism. Then we say that

$$\det_{\gamma,\delta}(\varphi) := \det(\mathrm{is}_{\delta,\gamma} \circ \varphi, V) = \det(\varphi \circ \mathrm{is}_{\delta,\gamma}, W)$$

is the determinant of φ with respect to γ and δ.

Definition 2.4: Let $n \geq 2$ and $V_1, \dots, V_n$ be vector spaces of equal finite dimension together with isomorphisms

$$V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} V_n \xrightarrow{\varphi_n} V_1.$$

Applying the determinant functor to these and multiplying, we get a morphism

$$\mathrm{Det}(\varphi_1) \cdots \mathrm{Det}(\varphi_n)\colon\ \mathrm{Det}(V_1) \cdots \mathrm{Det}(V_n) \longrightarrow \mathrm{Det}(V_2) \cdots \mathrm{Det}(V_n)\, \mathrm{Det}(V_1)$$

which, after identifying source an target, can be seen as an element in $\mathrm{K}_1(K) = K^\times$ as in definition 2.2. We denote this element by

$$\det\nolimits^\times(\varphi_1, \dots, \varphi_n) \in K^\times.$$

Lemma 2.5: *Assume we are in the situation of definition 2.4. Then*

$$\det\nolimits^\times(\varphi_1, \dots, \varphi_n) = (-1)^{n+1} \det(\varphi_n \circ \cdots \circ \varphi_1, V_1).$$

In particular, choosing bases γ of V_1 and δ of V_n and putting $\varphi_n = \mathrm{is}_{\delta,\gamma}$, one has

$$\det\nolimits^\times(\varphi_1, \dots, \varphi_{n-1}, \mathrm{is}_{\delta,\gamma}) = (-1)^{n+1} \det_{\gamma,\delta}(\varphi_{n-1} \circ \cdots \circ \varphi_1).$$

Proof: Spelling out explicitly what happens here, one sees that the morphism

$$\begin{aligned}\mathrm{Det}(\varphi_1)\cdots\mathrm{Det}(\varphi_n) &\in \mathrm{Hom}(\mathrm{Det}(V_1)\cdots\mathrm{Det}(V_n), \mathrm{Det}(V_2)\cdots\mathrm{Det}(V_n)\,\mathrm{Det}(V_1))\\ &= (K^\times \times \mathrm{Isom}(V_1\oplus\cdots\oplus V_n, V_2\oplus\cdots\oplus V_n\oplus V_1))\Big/\mathrm{Aut}(V_2\oplus\cdots V_n\oplus V_1)\end{aligned}$$

is represented by $(1, \varphi_1\oplus\cdots\oplus\varphi_n)$. If we choose bases of all V_i to get matrices A_i describing the φ_i, the isomorphism $\varphi_1\oplus\cdots\oplus\varphi_n$ is represented by

$$\begin{pmatrix} 0 & \cdots & 0 & A_n \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ & & A_{n-1} & 0 \end{pmatrix}$$

with respect to the combined basis and the resulting identification $V_1\oplus\cdots\oplus V_n \cong V_2\oplus\cdots\oplus V_n\oplus V_1$, while $\varphi_n\circ\cdots\circ\varphi_1$ is represented by $A_n\cdots A_1$ with respect to the basis chosen on V_1. By Laplace expansion, the claim follows. $\square$

Definition 2.6: Let V, W be finite-dimensional K-vector spaces,

$$f \in \mathrm{Hom}(\mathrm{Det}(V), \mathrm{Det}(W)) = (K^\times \times \mathrm{Isom}(V, W))\Big/\mathrm{Aut}(W)$$

a morphism whose class is represented by $(a, \tilde{f})$, and $c \in K^\times$. Then

$$c\cdot f \in \mathrm{Hom}(\mathrm{Det}(V), \mathrm{Det}(W))$$

is defined to be the morphism represented by $(ca, \tilde{f})$. See also [FK06, p. 43].

Lemma 2.7: *Let V, W be finite-dimensional K-vector spaces, let $\varphi\colon V \longrightarrow W$, $\psi\colon W\longrightarrow V$ be isomorphisms, and let $a \in K^\times$. Then*

$$\det{}^\times(a\cdot\varphi, \psi) = \det{}^\times(\varphi, a\cdot\psi) = a\det{}^\times(\varphi,\psi).$$

Proof: This follows directly from the definition of how to view an endomorphism of some determinant object as an element in $K^\times$. $\square$

Lemma 2.8: *Let V, W be K-vector spaces of equal finite dimension and let B be a K-algebra. Any "$\otimes$" below means "$\otimes_K$". Then there is a natural bijection*

$$(\mathrm{Isom}(V\otimes B, W\otimes B)\times B^\times)\Big/\mathrm{Aut}(W\otimes B) \cong \left(\left((\mathrm{Isom}(V,W)\times K^\times)\Big/\mathrm{Aut}(W)\right)\times B^\times\right)\Big/K^\times.$$

Here, $K^\times$ acts on $((\mathrm{Isom}(V,W)\times K^\times)/\mathrm{Aut}(W))\times B^\times$ by

$$l\cdot([\psi, a], b) = ([\psi, l^{-1}a], lb) \quad \textit{for } l\in K^\times, \psi\in\mathrm{Isom}(V,W), a\in K^\times, b\in B^\times.$$

Sketch of proof: We first define a map from the left hand side to the right hand side. Let $[\psi, c]$ be some class in the left hand side. Choose some isomorphism $\Phi \in \mathrm{Isom}(V, W)$. Then there exists an $s \in \mathrm{Aut}(W \otimes B)$ such that $s \circ \psi = \Phi$. We then map

$$[\psi, c] \longmapsto \left[[s \circ \psi, 1], c \det(s)^{-1}\right] .$$

In the other direction, let $[[\varphi, a], b]$ be in the right hand side. Then we map

$$[\varphi \otimes 1, ab] \longleftarrow\!\shortmid [[\varphi, a], b] .$$

It is now a long and tedious, but essentially trivial calculation to check that

- the action of $K^\times$ on $((\mathrm{Isom}(V, W) \times K^\times)/\mathrm{Aut}(W)) \times B^\times$ is well-defined,
- both maps above are well-defined, in particular the first map depends neither on the choice of Φ nor on the choice of class representatives,
- and the two maps are inverse to each other.

Since nothing interesting happens in this calculation, we omit it here. □

2.2. Actions of semidirect products

In this section we give a basic construction concerning actions of semidirect products of groups. This is mainly to fix notation and for future reference. In this subsection, every action occurring is a left action. Of course, similar statements are true also for right actions.

Let G be a group and Σ a monoid. For a homomorphism $\varphi\colon G \longrightarrow \mathrm{End}(\Sigma)$, we define the semidirect product $\Sigma \rtimes G$ to be the set $\Sigma \times G$ with multiplication $(m_1, g_1)(m_2, g_2) = (m_1\varphi_{g_1}(m_2), g_1g_2)$ (where we wrote φ_g for $\varphi(g)$, for $g \in G$). It is a monoid, and if Σ is a group it is the usual semidirect product of groups.

Lemma 2.9: *Let X be a topological space on which both G and Σ act continuously from the left in such a way that*

$$g(m(g^{-1}x)) = \varphi_g(m)x \tag{2.1}$$

for every $g \in G$, $m \in \Sigma$ and $x \in X$.

(a) Putting

$$(m, g)x = m(gx)$$

gives a well-defined left action of $\Sigma \rtimes G$ on X.

(b) Assume that Σ is a group. Writing $[x]$ for the orbit of $x \in X$ under Σ, we get a well-defined action of G on the quotient space $\Sigma\backslash X$ by defining its left representative by

$$g[x] = [gx].$$

Proof: This is an easy calculation using the relation (2.1). □

Example 2.10: An important application of this is the following. Here G is the group $\mathrm{G}_\text{ə}$. Let $\Sigma = \mathrm{GL}_2^+(\mathbb{R})$ act from the left on the complex upper half plane $\mathfrak{h}$ by fractional linear transformations, that is, by

$$\gamma z := \frac{az+b}{cz+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),\ z \in \mathfrak{h}.$$

(a) Observe that we can write $\mathrm{GL}_2(\mathbb{R})$ as the semidirect product

$$\mathrm{GL}_2(\mathbb{R}) = \mathrm{GL}_2^+(\mathbb{R}) \rtimes \mathrm{G}_\text{ə}$$

with $\mathrm{G}_\text{ə}$ acting by conjugation on $\mathrm{GL}_2^+(\mathbb{R})$. If we let $\text{ə} \in \mathrm{G}_\text{ə}$ act on the upper half plane $\mathfrak{h}$ as $\tau \longmapsto -\overline{\tau}$, where the bar means complex conjugation, it is an easy calculation to verify the relation (2.1). So by lemma 2.9 (a) we get an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathfrak{h}$.

(b) If we replace $\mathrm{GL}_2^+(\mathbb{R})$ by $\mathrm{GL}_2^+(\mathbb{Q})$ in (a), then we can extend the action of $\mathrm{GL}_2^+(\mathbb{Q})$ by linear fractional transformations to the set $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ using exactly the same formula as above (and the evident calculation rules for handling ∞, for example dividing by 0 should result in ∞ and so on), and similarly also for $\mathrm{G}_\text{ə}$, so we get an action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$ and thus on $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$.

(c) We can of course replace $\mathrm{GL}_2^+(\mathbb{Q})$ by any submonoid $\Delta \subseteq \mathrm{GL}_2^+(\mathbb{Q})$ which is normalised by ə. Then we have $\Delta^\text{ə} = \Delta \rtimes \mathrm{G}_\text{ə}$ and get an action of $\Delta^\text{ə}$ on $\mathfrak{h}$, $\mathbb{P}^1(\mathbb{Q})$ and $\mathfrak{h}^*$.

(d) Further replacing $\mathrm{GL}_2^+(\mathbb{Q})$ by $\mathrm{SL}_2(\mathbb{Z})$ in the above and writing

$$\mathrm{GL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \rtimes \mathrm{G}_\text{ə},$$

we similarly get actions of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathfrak{h}$, $\mathbb{P}^1(\mathbb{Q})$ and $\mathfrak{h}^*$. Furthermore, if $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a subgroup which is normalised by ə, we can form $\Gamma \rtimes \mathrm{G}_\text{ə}$. By lemma 2.9 (b) with $\Sigma = \Gamma$, we then get an action of $\mathrm{G}_\text{ə}$ on the quotients $\Gamma\backslash\mathfrak{h}$ and $\Gamma\backslash\mathfrak{h}^*$.

The motivation behind defining the action of ə on the upper half plane as $\tau \longmapsto -\overline{\tau}$ is the following. Since $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathfrak{h}$ with the stabiliser of $\mathrm{i} \in \mathfrak{h}$ being SO_2, there is an isomorphism of smooth manifolds

$$\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \xrightarrow{\sim} \mathfrak{h}, \quad \gamma\,\mathrm{SO}_2 \longmapsto \gamma\mathrm{i},$$

and one can check that the inclusion $\mathrm{SL}_2(\mathbb{R}) \hookrightarrow \mathrm{GL}_2(\mathbb{R})$ induces an isomorphism

$$\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R}) / \mathrm{SO}_2\,\mathbb{R}^\times$$

with inverse mapping the class of a $\gamma \in \mathrm{GL}_2(\mathbb{R})$ to the class of $(\sqrt{\det\gamma})^{-1}\gamma$ if $\det\gamma > 0$ and to the class of $(\sqrt{-\det\gamma})^{-1}\text{ə}\gamma$ if $\det\gamma < 0$. Now on $\mathrm{GL}_2(\mathbb{R})/\mathrm{SO}_2\,\mathbb{R}^\times$ we have a natural action of $\mathrm{GL}_2(\mathbb{R})$ by left multiplication, and an easy calculation shows that under the bijection

$$\mathrm{GL}_2(\mathbb{R}) / \mathrm{SO}_2\,\mathbb{R}^\times \cong \mathfrak{h}$$

this action becomes exactly the action defined in example 2.10 (a).[12]

There is another example that will be important later.

[12] In some texts, the matrix $-\text{ə}$ instead of ə is used. The same easy calculation shows that this gives the same actions of $\mathrm{GL}_2(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{Q})$ resp. $\mathrm{GL}_2(\mathbb{Z})$ on all sets considered above.

Example 2.11: Let $\Delta \subseteq \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2^+(\mathbb{Q})$ be any submonoid which is normalised by $\eth$. Define an action of Δ on $\mathbb{C} \times \mathfrak{h}$ by

$$\gamma(z,\tau) = (\det(\gamma)(c\tau + d)^{-1}z, \gamma\tau) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta,\ z \in \mathbb{C},\ \tau \in \mathfrak{h} \tag{2.2}$$

and let $\eth$ act on $\mathbb{C} \times \mathfrak{h}$ as $\eth(z,\tau) := (\overline{z}, \eth\tau) = (\overline{z}, -\overline{\tau})$. Then it is again an easy calculation to check the relation (2.1), so that we get a well-defined action of $\Delta^{\eth} = \Delta \rtimes \mathrm{G}_{\eth}$ on $\mathbb{C} \times \mathfrak{h}$. On the second factor $\mathfrak{h}$, this is just the action we defined in example 2.10 (c), so that the projection $\mathbb{C} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ is equivariant.

2.3. Some homological algebra

Here we collect some facts from homological algebra that we will need later on.

For simplicity of notation, we will denote every differential in every degree in every complex simply by d. For a morphism $f : C^\bullet \longrightarrow D^\bullet$ of complexes, we define the *mapping cone* following the convention in [GM03] to be the complex $\mathrm{cone}(f)^\bullet = C[1]^\bullet \oplus D^\bullet$ with differential given by the matrix[13]

$$\begin{pmatrix} -d & \\ f & d \end{pmatrix}.$$

Let $\mathcal{A}$ be an abelian category and

$$0 \longrightarrow R^\bullet \xrightarrow{f} S^\bullet \xrightarrow{g} T^\bullet \longrightarrow 0$$

an exact sequence of complexes in $\mathcal{A}$.

Lemma 2.12: *The map*

$$(f, 0) : R[1]^\bullet \longrightarrow \mathrm{cone}(g) = S[1]^\bullet \oplus T^\bullet$$

is a quasi-isomorphism.

Proof: We prove this by showing that the kernel and cokernel have vanishing cohomology. For the kernel this is obvious: it is even zero itself, since f is a monomorphism. The cokernel is isomorphic to $T[1]^\bullet \oplus T^\bullet$ with the differential given by the matrix

$$\begin{pmatrix} -d & \\ \mathrm{id} & d \end{pmatrix},$$

as one easily checks. We prove that this complex has zero cohomology by showing that its identity is nullhomotopic, using an argument inspired from [GM03, top of p. 158]. A contracting homotopy for it is given by

$$s^n : T^{n+1} \oplus T^n \longrightarrow T^n \oplus T^{n-1}, \quad (t^{n+1}, t^n) \longmapsto (t^n, 0),$$

as can be seen by a simple calculation. □

[13] Note that in [GM03, §III.3.2], the differential of the shifted complex $C[i]^\bullet$ is defined to be $(-1)^i$ times the original one, which is why their matrix does not contain the sign. Note also that we use a different sign convention as [Wei94, §1.5].

Lemma 2.13: *Let*

$$(0, \mathrm{id})\colon T^\bullet \longrightarrow \mathrm{cone}(g) = S[1]^\bullet \oplus T^\bullet$$

be the inclusion into the second factor. Identifying $\mathrm{H}^n(\mathrm{cone}(g))$ *with* $\mathrm{H}^n(R[1]^\bullet) = \mathrm{H}^{n+1}(R^\bullet)$ *using the quasi-isomorphism from lemma 2.12, the map*

$$\mathrm{H}^n(T^\bullet) \longrightarrow \mathrm{H}^{n+1}(R^\bullet)$$

induced by the above inclusion is the boundary map from the long exact cohomology sequence.

Proof: Note the similarity to [Wei94, Ex. 1.5.6].

We draw the snake lemma diagram used to construct the connecting homomorphism, together with an additional column for the mapping cone:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & 0 \arrow[d] \\
 & & & \mathrm{H}^n(T^\bullet) \arrow[rr, "(1)"] \arrow[d] & & \mathrm{H}^n(\mathrm{cone}(g)) \arrow[d] \\
 & \frac{R^n}{dR^{n-1}} \arrow[r, "f"] \arrow[d] & \frac{S^n}{dS^{n-1}} \arrow[r, "g"] \arrow[d] & \frac{T^n}{dT^{n-1}} \arrow[r] \arrow[d] & 0 & \frac{S^{n+1} \oplus T^n}{d(S^n \oplus T^{n-1})} \arrow[d] \\
0 \arrow[r] & Z^{n+1}(R^\bullet) \arrow[r, "f"] \arrow[dd] & Z^{n+1}(S^\bullet) \arrow[r, "g"] & Z^{n+1}(T^\bullet) & & Z^{n+1}(\mathrm{cone}(g)) \\
 & & 0 \arrow[d] & & & \\
 & \mathrm{H}^{n+1}(R^\bullet) \arrow[r, equal] \arrow[d] & \mathrm{H}^{n+1}(R^\bullet) \arrow[d] \arrow[uuuurrr, bend right, "\sim", "(2)"'] & & & \\
 & 0 & \frac{R^{n+1}}{dR^n} \arrow[d] & & & \\
 & & Z^{n+2}(R^\bullet). & & &
\end{tikzcd}$$

Here, (1) is induced by the inclusion into the second factor, as in the statement, and (2) comes from the quasi-isomorphism from lemma 2.12.

We argue by a diagram chase, i.e. using elements of the objects in the diagram. Following the proof of the snake lemma, start with a cohomology class in $\mathrm{H}^n(T^\bullet)$ and take a $t \in T^n$ representing it. Let $s \in S^n$ be a preimage, i.e. $g(s) = t$, map it to $Z^{n+1}(S^\bullet)$ with the differential, and then take a preimage $r \in R^{n+1}$, i.e. $f(r) = d(s)$. This r then represents the image in $\mathrm{H}^{n+1}(R^\bullet)$ of the class from $\mathrm{H}^n(T^\bullet)$ we started with under the boundary map.

The image under the map (1) of the class in $\mathrm{H}^n(T^\bullet)$ represented by t in $\mathrm{H}^n(\mathrm{cone}(g))$ is represented by $(0, t)$, whereas the image under the map (2) of the class in $\mathrm{H}^{n+1}(R^\bullet)$ represented by r in $\mathrm{H}^n(\mathrm{cone}(g))$ is represented by $(f(r), 0)$. The difference between these two elements of $S^{n+1} \oplus T^n$ is $(-d(s), g(s))$ and is thus the image of $(s, 0) \in S^n \oplus T^{n-1}$ under the differential of the mapping cone. Hence the $(0, t)$ and $(f(r), 0)$ represent the same cohomology class in $\mathrm{H}^n(\mathrm{cone}(g))$. Since (2) is an isomorphism, the claim follows. $\square$

Now consider the following situation: let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in an abelian category $\mathcal{A}$ and let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor. This gives us a connecting homomorphism

$$\partial\colon \mathrm{R}^0\mathcal{F}(C) \longrightarrow \mathrm{R}^1\mathcal{F}(A).$$

On the other hand, in the derived category $\mathrm{D}^+(\mathcal{A})$

$$A[1] = (A \longrightarrow \underset{\substack{\uparrow\\ 0}}{0}) \quad \text{and} \quad (B \longrightarrow \underset{\substack{\uparrow\\ 0}}{C})$$

define isomorphic objects, and we have an obvious morphism in $\mathrm{D}^+(\mathcal{A})$

$$C[0] \longrightarrow (B \longrightarrow \underset{\substack{\uparrow\\ 0}}{C}).$$

Applying the total derived functor $\mathrm{R}\mathcal{F}$ to this and then taking cohomology in degree zero, this gives a morphism

$$\mathrm{R}^0\mathcal{F}(C) = \mathrm{H}^0(\mathrm{R}\mathcal{F}(C[0])) \longrightarrow \mathrm{H}^0(\mathrm{R}\mathcal{F}(B \longrightarrow \underset{\substack{\uparrow\\ 0}}{C}) \cong \mathrm{H}^0(\mathrm{R}\mathcal{F}(A[1])) = \mathrm{R}^1\mathcal{F}(A). \tag{2.3}$$

Lemma 2.14: *In the situation described above, the morphism (2.3) coincides with the boundary homomorphism*

$$\partial\colon \mathrm{R}^0\mathcal{F}(C) \longrightarrow \mathrm{R}^1\mathcal{F}(A).$$

Proof: Choose injective resolutions $B \longrightarrow I^\bullet$ and $C \longrightarrow J^\bullet$ of B and C, respectively, and write $f\colon I^\bullet \longrightarrow J^\bullet$ for the map between them induced by the map $B \longrightarrow C$. By lemma 2.12, $\operatorname{cone}(f)$ is then an injective resolution of $A[1]$ and hence also of the complex

$$B \longrightarrow \underset{\substack{\uparrow\\ 0}}{C}.$$

The map

$$\mathrm{H}^0(\mathrm{R}\mathcal{F}(C[0])) \longrightarrow \mathrm{H}^0(\mathrm{R}\mathcal{F}(B \longrightarrow \underset{\substack{\uparrow\\ 0}}{C}))$$

is induced by the inclusion into the second factor

$$J^\bullet \longrightarrow \operatorname{cone}(f).$$

Therefore, the claim follows from lemma 2.13. $\square$

2.4. Some p-adic Hodge theory

We fix a finite extension L of $\mathbb{Q}_p$ and an extension F of $\mathbb{Q}_p$ which is either finite or the maximal unramified extension of a finite extension of $\mathbb{Q}_p$. Let F_0 denote the maximal subfield of F which is unramified over $\mathbb{Q}_p$ and $\hat{F}$ resp. $\hat{F}_0$ the completions of F and F_0, respectively. Note that then $\hat{F}$ is a p-adic field in the sense of [BC09, Def. 1.3.1] and we have $\mathrm{G}_F = \mathrm{G}_{\hat{F}}$ by [BC09, Ex. 1.4.4 (2)], and likewise for $\hat{F}_0$.

Central to the study of representations of G_F on L-vector spaces via p-adic Hodge theory is the formalism of B-admissible representations, where B is some regular period ring. This formalism is developed in an abstract setting e. g. in [BC09, §1.5] or [FO08, §2.1.2]. Unfortunately, the setting there only considers the case $L = \mathbb{Q}_p$ (but see [BC09, Exerc. 6.4.3, 8.4.3]), but it is clear that the statements there hold analogously in this more general situation.

We use the period rings B_{HT}, B_{dR}, B_{st} and $\mathrm{B}_{\mathrm{cris}}$, see [FO08, §5.1, §5.2.2, §6.1.4, §6.1.1]. The formalism of admissible representations provides us with functors from $\mathcal{Rep}_L(\mathrm{G}_F)$ to categories of "linear algebra objects". Let F_0 be the maximal subfield of F which is unramified over $\mathbb{Q}_p$. We summarise the relevant statements:

Theorem 2.15: *Let ? be "HT", "dR", "st" or "cris" and for $V \in \mathcal{Rep}_L(\mathrm{G}_F)$ let*

$$\mathrm{D}_{?,F}(V) := (\mathrm{B}_? \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{G}_F},$$

where G_F acts diagonally on the tensor product. Then $\mathrm{D}_{?,F}$ defines a functor as follows:

(1) $\mathrm{D}_{\mathrm{HT},F}$ goes from $\mathcal{Rep}_L(\mathrm{G}_F)$ to the category of graded $\hat{F} \otimes_{\mathbb{Q}_p} L$-modules of finite rank.

(2) $\mathrm{D}_{\mathrm{dR},F}$ goes from $\mathcal{Rep}_L(\mathrm{G}_F)$ to the category of filtered $\hat{F} \otimes_{\mathbb{Q}_p} L$-modules of finite rank.

(3) $\mathrm{D}_{\mathrm{st},F}$ goes from $\mathcal{Rep}_L(\mathrm{G}_F)$ to the category of $\hat{F}_0 \otimes_{\mathbb{Q}_p} L$-modules of finite rank with an L-linear and F_0-semilinear (with respect to the arithmetic Frobenius on F_0) automorphism φ_{cris}, a nilpotent endomorphism N such that $N\varphi_{\mathrm{cris}} = p\varphi_{\mathrm{cris}}N$ and a filtration on $F \otimes_{F_0} \mathrm{D}_{\mathrm{st},F}(-)$.

(4) $\mathrm{D}_{\mathrm{cris},F}$ goes from $\mathcal{Rep}_L(\mathrm{G}_F)$ to the category of $\hat{F}_0 \otimes_{\mathbb{Q}_p} L$-modules of finite rank with an automorphism φ_{cris} and a filtration as in the previous case.

Here each filtration is decreasing, separated and exhaustive.

Let

$$\alpha_V \colon \mathrm{B}_? \underset{\hat{F}_?}{\otimes} \mathrm{D}_{?,F}(V) \longrightarrow \mathrm{B}_? \underset{\hat{F}_?}{\otimes} (\mathrm{B}_? \underset{\mathbb{Q}_p}{\otimes} V) = (\mathrm{B}_? \underset{\hat{F}_?}{\otimes} \mathrm{B}_?) \underset{\mathbb{Q}_p}{\otimes} V \longrightarrow \mathrm{B}_? \underset{\mathbb{Q}_p}{\otimes} V$$

be the canonical map, where $\hat{F}_?$ is $\hat{F}$ if $? \in \{\mathrm{HT}, \mathrm{dR}\}$ and $\hat{F}_0$ if $? \in \{\mathrm{st}, \mathrm{cris}\}$. Then α_V is injective and we denote by $\mathcal{Rep}_L^?(\mathrm{G}_F)$ the subcategory of such V for which α_V is an isomorphism, called Hodge-Tate-, de Rham-, semistable and crystalline representations, respectively. Then the following hold:

(a) $\mathcal{Rep}_L^?(\mathrm{G}_F)$ is closed under subrepresentations, duals, quotients and tensor products.

(b) Restricted to $\mathcal{Rep}_L^?(\mathrm{G}_F)$, the functor $\mathrm{D}_{?,F}$ is an exact and faithful tensor functor. To be precise, there are canonical isomorphisms

$$\mathrm{D}_{?,F}(V_1) \underset{\hat{F}_? \otimes_{\mathbb{Q}_p} L}{\otimes} \mathrm{D}_{?,F}(V_2) \cong \mathrm{D}_{?,F}(V_1 \underset{L}{\otimes} V_2),$$

$$\mathrm{D}_{?,F}(\mathrm{Hom}_L(V,L)) \cong \mathrm{Hom}_{\hat{F}_? \otimes_{\mathbb{Q}_p} L}(\mathrm{D}_{?,F}(V), \hat{F}_? \underset{\mathbb{Q}_p}{\otimes} L).$$

Proof: All this is well-known and in the special case $L = \mathbb{Q}_p$ this is proven in [BC09, Thm. 5.2.1]. One can check that the same proof still works in this more general setting (using that $\mathrm{B}_?^{\mathrm{G}_F} = \hat{F}_?$). See also [FO08, Thm. 2.13]. □

Remark 2.16: The statement about tensor products in theorem 2.15 (b) can be refined. If $V_1, V_2 \in \mathcal{Rep}_L(\mathrm{G}_F)$ are any representations (not necessarily in $\mathcal{Rep}_L^?(\mathrm{G}_F)$), then we always have an *injective* map

$$\mathrm{D}_{?,F}(V_1) \underset{\hat{F}_? \otimes_{\mathbb{Q}_p} L}{\otimes} \mathrm{D}_{?,F}(V_2) \hookrightarrow \mathrm{D}_{?,F}(V_1 \underset{L}{\otimes} V_2),$$

and the map is an isomorphism if and only if both V_1 and V_2 are in $\mathcal{Rep}_L^?(\mathrm{G}_F)$. This is shown during the proof of the statement in the above references.

We will often consider only the case where $F = \mathbb{Q}_p$, in which we write $\mathrm{D}_?$ instead of $\mathrm{D}_{?,\mathbb{Q}_p}$. The $\mathrm{D}_?$ are then L-vector spaces for any "?" and the automorphism φ_{cris} of D_{st} and $\mathrm{D}_{\mathrm{cris}}$ is L-linear.

Proposition 2.17: *Let ? be any of the properties "Hodge-Tate", "de Rham", "semistable" or "crystalline" and $V \in \mathcal{Rep}_L(\mathrm{G}_F)$. Let $F_? = F$ if $? \in \mathrm{HT}, \mathrm{dR}$ and $F_? = F_0$ if $? \in \{\mathrm{st}, \mathrm{cris}\}$. Then there is a canonical isomorphism*

$$\hat{F}_?^{\mathrm{nr}} \underset{F_?}{\otimes} \mathrm{D}_{?,F}(V) \xrightarrow{\sim} \mathrm{D}_{?,F^{\mathrm{nr}}}(V).$$

In particular, V has the property ? if and only if the restriction of V to $\mathrm{G}_{F^{\mathrm{nr}}}$ has the property P.

Proof: [BC09, Prop. 6.3.8, Prop. 9.3.1] □

Lemma 2.18: *Let $V \in \mathcal{Rep}_L(\mathrm{G}_F)$ be unramified. Then V is crystalline and we have a canonical isomorphism*

$$\mathrm{D}_{\mathrm{cris},F}(V) \cong (\hat{F}^{\mathrm{nr}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{Gal}(F^{\mathrm{nr}}/F)}.$$

The action of φ_{cris} on the left hand side corresponds on the right hand side to the action of an arithmetic Frobenius Frob_p^{-1} on the first tensor factor.

Proof: Let $n = \dim_{\mathbb{Q}_p} V$. That V is crystalline is clear from proposition 2.17. Let $I = \mathrm{G}_{F^{\mathrm{nr}}}$. We have a canonical injection

$$\hat{F}^{\mathrm{nr}} \underset{\mathbb{Q}_p}{\otimes} V = \mathrm{B}_{\mathrm{cris}}^I \underset{\mathbb{Q}_p}{\otimes} V^I \hookrightarrow (\mathrm{B}_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} V)^I = \mathrm{D}_{\mathrm{cris},F^{\mathrm{nr}}}(V).$$

Again by proposition 2.17 and the definition of "crystalline" the right hand side is an n-dimensional $\hat{F}^{\mathrm{nr}}$-vector space, as is the left hand side, so the above map is in fact an isomorphism. We conclude

$$\mathrm{D}_{\mathrm{cris},F}(V) = (\mathrm{B}_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{G}_F} = ((\mathrm{B}_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} V)^I)^{\mathrm{Gal}(F^{\mathrm{nr}}/F)} \cong (\hat{F}^{\mathrm{nr}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{Gal}(F^{\mathrm{nr}}/F)}.$$

The last statement is clear from the definition of φ_{cris} on $\mathrm{B}_{\mathrm{cris}}$. □

Lemma 2.19: *(a) Let $\psi\colon \mathrm{G}_{\mathbb{Q}_p} \longrightarrow L^\times$ be a continuous character. Then the following are equivalent:*

(i) ψ is de Rham.

(ii) ψ is Hodge-Tate.

(iii) ψ is a product of a finitely ramified character and a crystalline character.

(b) Let $\psi\colon \mathrm{G}_{\mathbb{Q}_p} \longrightarrow L^\times$ be a continuous character. Then the following are equivalent:

(i) ψ is crystalline.

(ii) ψ is semistable.

(iii) ψ is a Tate twist of an unramified character.

Proof: [BC09, Ex. 6.3.9, Cor. 9.3.2] □

Lemma 2.20: *Let $\psi\colon \mathrm{G}_{\mathbb{Q}_p} \longrightarrow L^\times$ be a crystalline character and write $\psi = \psi_{\mathrm{nr}}\kappa_{\mathrm{cyc}}^n$ with $n \in \mathbb{Z}$ and ψ_{nr} unramified. Then φ_{cris} acts on $\mathrm{D}_{\mathrm{cris}}(\psi)$ as multiplication by $\psi_{\mathrm{nr}}(\mathrm{Frob}_p)p^{-n}$.*

Proof: Since $\mathrm{D}_{\mathrm{cris}}$ commutes with tensor products, it suffices to show independently that φ_{cris} acts on $\mathrm{D}_{\mathrm{cris}}(\kappa_{\mathrm{cyc}})$ as multiplication by p^{-1} and on $\mathrm{D}_{\mathrm{cris}}(\psi_{\mathrm{nr}})$ by multiplication with $\psi_{\mathrm{nr}}(\mathrm{Frob}_p)$.

We first consider $\mathrm{D}_{\mathrm{cris}}(\kappa_{\mathrm{cyc}}) = (\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1))^{\mathrm{G}_{\mathbb{Q}_p}}$. Let ξ be any nonzero element of $\mathbb{Q}_p(1)$ and consider $\mathrm{t}_{\mathrm{dR}} \in \mathrm{B}_{\mathrm{cris}}^\times$. Then $\sigma \in \mathrm{G}_{\mathbb{Q}_p}$ acts on both ξ and t_{dR} as multiplication by $\kappa_{\mathrm{cyc}}(\sigma)$ (see [BC09, top of p. 62] for t_{dR}). Hence $\xi \otimes \mathrm{t}_{\mathrm{dR}}^{-1}$ lies in $\mathrm{D}_{\mathrm{cris}}(\kappa_{\mathrm{cyc}})$, and since the latter is a 1-dimensional $\mathbb{Q}_p$-vector space, it is a basis. Now φ_{cris} acts on t_{dR} as multiplication by p by [BC09, top of p. 133] and it acts trivially on ξ, so the claim follows.

Now let V be a one-dimensional L-vector space on which $\mathrm{G}_{\mathbb{Q}_p}$ acts via ψ_{nr}. By lemma 2.18, we have then

$$\mathrm{D}_{\mathrm{cris}}(V) = (\hat{\mathbb{Q}}_p^{\mathrm{nr}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p)}$$

and φ_{cris} acts as the arithmetic Frobenius Frob_p^{-1} on the first tensor factor $\hat{\mathbb{Q}}_p^{\mathrm{nr}}$. Put $u := \psi_{\mathrm{nr}}(\mathrm{Frob}_p)$ and take an element

$$x = \sum_i b_i \otimes v_i \in (\hat{\mathbb{Q}}_p^{\mathrm{nr}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p)}.$$

Then ux is also $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p)$-invariant, so

$$ux = \mathrm{Frob}_p^{-1}(ux) = u\,\mathrm{Frob}_p^{-1}(x) = u\Big(\sum_i \varphi_{\mathrm{cris}}(b_i) \otimes \mathrm{Frob}_p^{-1}(v_i)\Big) = \sum_i \varphi_{\mathrm{cris}}(b_i) \otimes v_i = \varphi_{\mathrm{cris}}(x).$$

□

We now recall the important functor $\mathrm{D}_{\mathrm{pst}}$.

Definition 2.21: For $V \in \mathcal{Rep}_L(\mathrm{G}_{\mathbb{Q}_p})$, put

$$\mathrm{D}_{\mathrm{pst}}(V) := \varinjlim_{F \supseteq \mathbb{Q}_p} (\mathrm{B}_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{G}_F} = \varinjlim_{F \supseteq \mathbb{Q}_p} \mathrm{D}_{\mathrm{st},F}(V)$$

where F ranges over all finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$.

It is a $\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} L$-module endowed with an L-linear and $\mathbb{Q}_p^{\mathrm{nr}}$-semilinear (with respect to the arithmetic Frobenius) automorphism φ_{cris}. We call V *potentially semistable* if $\mathrm{D}_{\mathrm{pst}}(V)$ is free of rank $\dim_L V$ over $\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} L$.

On $\mathrm{D}_{\mathrm{pst}}(V)$ we have a (diagonal) action of $\mathrm{G}_{\mathbb{Q}_p}$ which is L-linear and $\mathbb{Q}_p^{\mathrm{nr}}$-semilinear (explicitly: $\sigma(ax) = \sigma(a)\sigma(x)$ for $a \in \mathbb{Q}_p^{\mathrm{nr}}$, $x \in \mathrm{D}_{\mathrm{pst}}(V)$ and $\sigma \in \mathrm{G}_{\mathbb{Q}_p}$). The functor $\mathrm{D}_{\mathrm{pst}}$ commutes with tensor products, which follows from the fact that D_{st} does so.

One of the most important results in p-adic Hodge theory is Berger's p-adic monodromy theorem:

Theorem 2.22 (Berger): *Any de Rham representation is potentially semistable.*

Proof: [Ber02, Thm. 0.7] □

Remark 2.23: The above result can be used to describe $\mathrm{D}_{\mathrm{pst}}$ more explicitly. Let V be a de Rham representation and fix a finite Galois extension $F/\mathbb{Q}_p$ that $V|_{\mathrm{G}_F}$ is semistable. Then for each finite extension F'/F the canonical map

$$F'_0 \underset{F_0}{\otimes} \mathrm{D}_{\mathrm{st},F}(V) = F'_0 \underset{F_0}{\otimes} (\mathrm{B}_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{G}_F} \hookrightarrow F'_0 \underset{F_0}{\otimes} \mathrm{B}_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} V \longrightarrow \mathrm{B}_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} V$$

is injective since it is a restriction of the map α_V, and moreover its image is $\mathrm{G}_{F'}$-invariant, so it induces an isomorphism $F'_0 \otimes_{F_0} \mathrm{D}_{\mathrm{st},F}(V) \xrightarrow{\sim} \mathrm{D}_{\mathrm{st},F'}(V)$. Consequently, we get $\mathrm{D}_{\mathrm{pst}}(V) = \mathbb{Q}_p^{\mathrm{nr}} \otimes_{F_0} \mathrm{D}_{\mathrm{st},F}(V)$. Note that the action of $\mathrm{G}_{\mathbb{Q}_p}$ on $\mathrm{D}_{\mathrm{pst}}(V)$ corresponds to the diagonal action on the right hand side, where $\mathrm{G}_{\mathbb{Q}_p}$ acts as usual via is quotient $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p)$ on $\mathbb{Q}_p^{\mathrm{nr}}$ and via its quotient $\mathrm{Gal}(F/\mathbb{Q}_p)$ on $\mathrm{D}_{\mathrm{st},F}(V)$. Further φ_{cris} also acts diagonally, not only on $\mathrm{D}_{\mathrm{st},F}(V)$! This is because $\mathbb{Q}_p^{\mathrm{nr}}$ has to be considered as a subring of B_{st} here, and thus φ_{cris} acts as an arithmetic Frobenius Frob_p^{-1} on this factor.

2.5. Galois representations and families

We fix a profinite group G (in the applications, it will mostly be either $\mathrm{G}_{\mathbb{Q}}$ or $\mathrm{G}_{\mathbb{Q}_p}$).

Definition 2.24: Let R be a commutative ring. A *representation* of G with coefficients in R is a finitely generated projective R-module M together with a (continuous[14]) homomorphism

$$\rho\colon G \longrightarrow \mathrm{Aut}_R(M).$$

The category of such representations will be denoted by $\mathcal{Rep}_R(G)$. We say that a representation has rank $n \in \mathbb{N}_0$ if the module M has constant rank n.

Definition 2.25: Let M be a representation as above and assume that M has finite length as an $R[G]$-module. Then the *semisimplification* of M is defined as the semisimplification as an $R[G]$-module, i. e. as the direct sum of the composition factors in a composition series of M as an $R[G]$-module. By the Jordan-Hölder theorem this is well-defined up to isomorphism. We will mostly use this only when R is a field, in which case M has automatically finite length.

Theorem 2.26 (Brauer/Nesbitt): *Let M and N be k-linear representations of G, where k is a perfect field. Then the semisimplifications of M and N are isomorphic if and only if we have an equality of characteristic polynomials*

$$\det(1 - gT, M) = \det(1 - gT, N)$$

in $k[T]$ for all $g \in G$.

[14] In the applications, M will usually have some natural topology.

Proof: The "only if" statement means that the above characteristic polynomials depend only on the semisimplification of a representation. This is clear because if we have a composition series

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

of M and we write ρ_i for the homomorphism describing the action on the quotients M_i/M_{i-1}, then the action on M may be described by a matrix of the form

$$\begin{pmatrix} \rho_1 & * & \dots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & \rho_n \end{pmatrix},$$

hence the characteristic polynomials coincide with the corresponding ones where only the diagonal entries are present.

For the other direction see [CR62, Thm. 30.16]. The statement is formulated there for a finite group G, and the proof contains an argument incorporating the finitely many isomorphism classes of irreducible representations of G. But inspecting the proof and its ingredients one sees that the argument still goes through if we replace these finitely many isomorphism classes by the isomorphism classes of irreducible representations occurring as subquotients of M or N, which are of course still finitely many since M and N are finite-dimensional over k. □

In general, any representation $G \longrightarrow \mathrm{Aut}_R(M)$ can be viewed as a whole family of representations: each ring homomorphism $R \longrightarrow S$ to some other ring S induces a representation $G \longrightarrow \mathrm{Aut}_S(M \otimes_R S)$, and we get a family of representations parametrised by such ring homomorphisms. Of particular interest are the ring homomorphisms obtained by reducing R modulo some prime ideal, so we get for instance a family of representations parametrised by $\mathrm{Spec}\, R$. Therefore, in some situations we call a representation also a *family of representations* and speak of a representation in the strict sense only if R is a field; we hope this does not lead to confusions.

Of course, one could define this notion also more geometrically: we can see M as a locally free sheaf on $\mathrm{Spec}\, R$, and we can then define not only affine families of Galois representations, but also such parametrised by general schemes. One can also work in other geometric contexts, for example one can use affinoid algebras instead of just commutative rings and then consider families of representations parametrised by rigid analytic spaces. The rigid analytic setting seems to be a very natural context for questions related to p-adic L-functions, but in some situations the algebraic language is also important, and we study some relations between the two notions. For simplicity, we restrict to the affine resp. affinoid case.

We will be interested in two particular types of rings and ideals. For both we need to fix a finite extension L of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. The two situations we consider are as follows:

- Let $\mathcal{K}$ be a finite extension of $\mathrm{Quot}(\mathcal{O}[\![T]\!])$ and $\mathcal{I}$ the integral closure of $\mathcal{O}[\![T]\!]$ in $\mathcal{K}$. We consider the case $R = \mathcal{I}$. Then $\mathcal{I}$ is 2-dimensional, it is finite and flat over $\mathcal{O}[\![T]\!]$ by [BGR84, §4.2, Thm. 1] and [Bou89, chap. III, §3.4 Cor.], and it is moreover noetherian. Further $\mathcal{I}$ is a local ring, as can be seen using [Bou89, chap. V, §2.1, Prop. 1, Thm. 1]. Let $\mathfrak{m}_{\mathcal{I}}$ be is its maximal ideal. We endow $\mathcal{I}$ with the $\mathfrak{m}_{\mathcal{I}}$-adic topology. Then it is complete and Hausdorff by [Bou89, chap. III, §2.12 Cor. 1, §2.2 Prop. 6], and also compact since it is a finitely generated $\mathcal{O}[\![T]\!]$-module. So in particular, $\mathcal{I}$ is a profinite ring by [RZ00, Prop. 5.2.1]. We further endow any finitely generated $\mathcal{I}$-module $\mathcal{T}$ also with the $\mathfrak{m}_{\mathcal{I}}$-adic topology. This defines a topology on $\mathrm{Aut}_{\mathcal{I}}(\mathcal{T})$ such that we have an isomorphism of profinite groups $\mathrm{Aut}_{\mathcal{I}}(\mathcal{T}) \xrightarrow{\sim} \varprojlim_{n\in\mathbb{N}} \mathrm{Aut}_{\mathcal{I}}(\mathcal{T}/\mathfrak{m}_{\mathcal{I}}^n\mathcal{T})$.

- Let A be an affinoid algebra over L. We consider the case $R = A$.

In the following we denote the Tate algebra over L in n indeterminates by $L\langle X_1, \ldots, X_n\rangle$.

Definition 2.27: In the first case, we call a representation with coefficients in $\mathcal{I}$ also an *algebraic (p-adic) family of Galois representations.* In the second case we call a representation with coefficients in A also an *analytic (p-adic) family of Galois representations.* We will mostly omit the word "p-adic" here.

We call each $\mathcal{O}$-algebra morphism $\phi\colon \mathcal{I} \longrightarrow \overline{\mathbb{Q}}_p$ a *specialisation of* $\mathcal{I}$ and each L-algebra morphism $\phi\colon A \longrightarrow \overline{\mathbb{Q}}_p$ a *specialisation of* A.

Algebraic families of Galois representations are studied in [Bar11, §2.2]. There, specialisations are defined as maps $L[\![T]\!] \longrightarrow \overline{\mathbb{Q}}_p$. But since $L[\![T]\!]$ is a discrete valuation ring, it has a unique prime ideal, so there is only one such map. This must have been overlooked in this work since otherwise the theory is not interesting. A p-adic family of Galois representations is defined there as a representation into $\mathrm{GL}_n(L[\![T]\!])$. Later in [Bar11, Cor. 2.11] it is proved that any such representation contains a free $\mathrm{G}_{\mathbb{Q}}$-stable $\mathcal{O}[\![T]\!]$-lattice. This proof is also not correct since it uses that there is a canonical map from $L[\![T]\!]$ into the quotient field of $\mathcal{O}[\![T]\!]$, which is not true. Anyways, later in [Bar11], Barth always chooses such a lattice and works with specialisations $\mathcal{O}[\![T]\!] \longrightarrow \overline{\mathbb{Q}}_p$, so the later results are not affected by this problem. This situation is covered by our definition by choosing $\mathcal{I} = \mathcal{O}[\![T]\!]$,[15] so this seems to be a reasonable generalisation.

The algebraic families of Galois representations we defined should more precisely be called "one-parameter families of Galois representations", since we work over integral extensions of the one-variable power series ring $\mathcal{O}[\![T]\!]$. In this work we just call them "families" because we will not consider multi-parameter families. Working with multiple variables instead gives an analogous notion of multi-parameter families. Such an approach is followed for example in [Hid96].

Lemma 2.28: *Let $\phi\colon \mathcal{I} \longrightarrow \overline{\mathbb{Q}}_p$ be a specialisation, let R be its image and F be the quotient field of R. Then R is an integral ring extension of $\mathcal{O}$ and F is a finite extension of L.*

Proof: We first prove this in the special case $\mathcal{I} = \mathcal{O}[\![T]\!]$. The kernel of ϕ is a prime ideal of height 1, so by [NSW13, Lem. 5.3.7] it is of the form (f) where $f \in \mathcal{O}[T]$ is an irreducible Weierstraß polynomial. Let $d = \deg f$. By the Division Lemma [NSW13, Lem. 5.3.1], $\mathcal{O}[\![T]\!]/(f) \cong R$ is a free $\mathcal{O}$-module of rank d. Then F is canonically isomorphic to $L \otimes_{\mathcal{O}} \mathcal{O}[\![T]\!]/(f)$, which is an L-vector space of dimension d, so F/L is a finite extension.

Now let $\mathcal{I}$ be as in the general case, let P be the kernel of ϕ and let (f) be the kernel of $\phi|_{\mathcal{O}[\![T]\!]}$, as above. Since $\mathcal{I}$ is a finitely generated $\mathcal{O}[\![T]\!]$-module, by base change $\mathcal{I}/(f)$ is a finitely generated $\mathcal{O}[\![T]\!]/(f)$-module and thus a finitely generated $\mathcal{O}$-module. Then so is $\mathcal{I}/P \cong R$ since it is a quotient of $\mathcal{I}/(f)$. The field F can be identified with $L \otimes_{\mathcal{O}} R$, so it is a finite-dimensional L-vector space. □

[15] In [Bar11] there is the additional assumption that $\mathcal{T}$ be free, but since $\mathcal{O}[\![T]\!]$ is a local ring and we required $\mathcal{T}$ to be projective, this is automatically fulfilled.

Definition 2.29: (a) The *field of coefficients* of an algebraic specialisation ϕ is the subfield L_ϕ of $\overline{\mathbb{Q}}_p$ generated by the image of ϕ, which is a finite extension of L by lemma 2.28. We write $\mathcal{O}_\phi$ for its ring of integers. Note that we can view ϕ also as a map $\phi\colon \mathcal{I} \longrightarrow \mathcal{O}_\phi$ since the image of ϕ is a subring of $\mathcal{O}_\phi$, again by lemma 2.28. In this situation we put further $K_\phi = L_\phi \cap \overline{\mathbb{Q}}$, which is a finite extension of K, and write $\mathfrak{P}_\phi$ for the place of K_ϕ such that the completion at this place is L_ϕ.

(b) The *field of coefficients* of an analytic specialisation $\phi\colon A \longrightarrow \overline{\mathbb{Q}}_p$ is its image, which we denote L_ϕ. It is clear that L_ϕ is a subfield of $\overline{\mathbb{Q}}_p$, and by the rigid analytic Nullstellensatz [BGR84, §6.1.2, Cor. 3] it is a finite extension of L. As in the algebraic case we write $\mathcal{O}_\phi$ for its ring of integers, $K_\phi = L_\phi \cap \overline{\mathbb{Q}}$ and $\mathfrak{P}_\phi$ for the place of K_ϕ such that the completion at this place is L_ϕ.

Definition 2.30: Let $(\mathcal{T}, \rho)$ be an (algebraic or analytic) family of Galois representations of rank n and ϕ a specialisation. In the algebraic case, set $\mathcal{T}_\phi := \mathcal{T} \otimes_{\mathcal{I},\phi} \mathcal{O}_\phi$, which is a free $\mathcal{O}_\phi$-module of rank n. In the analytic case, set $\mathcal{T}_\phi := \mathcal{T} \otimes_{A,\phi} L_\phi$, which is an n-dimensional L_ϕ-vector space. We write

$$\rho_\phi\colon G \longrightarrow \operatorname{Aut}(\mathcal{T}_\phi)$$

for the representation induced from ρ and call it the *specialisation of ρ at ϕ*.

Proposition 2.31: *Let $\mathcal{I}$ be as above and set $A := \mathcal{I} \otimes_{\mathcal{O}[\![T]\!]} L\langle T\rangle$. Then A is an affinoid L-algebra. Every specialisation of $\mathcal{I}$ induces a specialisation of A, which we denote by the same symbol. Every algebraic family $\mathcal{T}^{\mathrm{alg}}$ of Galois representations over $\mathcal{I}$ naturally induces an analytic family of Galois representations $\mathcal{T}^{\mathrm{rig}}$ over A such that for each specialisation ϕ, $\mathcal{T}^{\mathrm{alg}}_\phi$ is a Galois-stable lattice in $\mathcal{T}^{\mathrm{rig}}_\phi$.*

Proof: Since $\mathcal{I}$ is finite over $\mathcal{O}[\![T]\!]$, by base change A is finite over $L\langle T\rangle$, hence A is affinoid by [BGR84, §6.1.1, Prop. 6]. So tensoring an algebraic family with $L\langle T\rangle$ yields an analytic family. If $\phi\colon \mathcal{I} \longrightarrow \overline{\mathbb{Q}}_p$ is an algebraic specialisation, then $\alpha := \phi(T) \in \overline{\mathbb{Q}}_p$ has absolute value ≤ 1, so we can define a morphism $L\langle T\rangle \longrightarrow L_\phi$ by $T \longmapsto \alpha$ (see [BGR84, §6.1.1, Prop. 4]). This induces an analytic specialisation which we denote again by ϕ. The rest is then clear. □

Definition 2.32: (a) Let $\mathcal{V}$ be a finite-dimensional $\mathcal{K}$-vector space. An $\mathcal{I}$*-lattice* in $\mathcal{V}$ is a finitely generated $\mathcal{I}$-submodule which generates $\mathcal{V}$ as a $\mathcal{K}$-vector space. See also [Bou89, chap. VII, §4.1, Prop. 1 & Cor.].

(b) Let $\rho\colon G \longrightarrow \operatorname{Aut}_{\mathcal{K}}(\mathcal{V})$ be a representation of G on a finite-dimensional $\mathcal{K}$-vector space $\mathcal{V}$. Then ρ is called *continuous* if $\mathcal{V}$ contains a G-stable $\mathcal{I}$-lattice $\mathcal{T}$ such that the induced map

$$\rho\colon G \longrightarrow \operatorname{Aut}_{\mathcal{I}}(\mathcal{T})$$

is continuous with respect to the profinite topology. See also [Hid86a, p. 557] and [Bar11, §2.2].

Lemma 2.33: *Let $\rho\colon G \longrightarrow \operatorname{Aut}_{\mathcal{K}}(\mathcal{V})$ be a continuous representation of G on a finite-dimensional $\mathcal{K}$-vector space $\mathcal{V}$. Then one can find a free G-stable $\mathcal{I}$-sublattice $\mathcal{T}$ of $\mathcal{V}$.*

Proof: This can be proved with exactly the same argument as [Bar11, Cor. 2.11]. □

2.6. (φ, Γ)-modules and families

We shortly review (φ, Γ)-modules over the Robba ring and families of such, following Berger, Colmez and Bellovin [Ber11; Col04; Col05; BC08; Bel15]. Throughout the section we fix a finite extension L of $\mathbb{Q}_p$.

Definition 2.34: The *Robba ring* $\mathrm{B}^{\dagger}_{\mathrm{rig},L}$ over L is the ring of Laurent series $\sum_{n\in\mathbb{Z}} a_n X^n$ with coefficients $a_n \in L$ for which there exists a real number $0 \leq r < 1$ such that the series converges on the annulus $\{x \in L : r \leq |x| < 1\}$.

Since L will usually be clear from the context, we denote the Robba ring mostly just by $\mathrm{B}^{\dagger}_{\mathrm{rig}}$. In other texts it is denoted by $\mathcal{R}$ or $\mathcal{R}_L$.

The Robba ring carries a Frobenius endomorphism φ_{rig} and an action of $\mathrm{G}_{\mathrm{cyc}}$. In the literature, the group which we call $\mathrm{G}_{\mathrm{cyc}}$ is usually called Γ in this context, so we follow this convention and let $\Gamma := \mathrm{G}_{\mathrm{cyc}}$ for this section.

Definition 2.35: Let A be an affinoid L-algebra. A (φ, Γ)-*module* over A is a free $A\hat{\otimes}_L \mathrm{B}^{\dagger}_{\mathrm{rig}}$-module M of finite rank with an endomorphism φ and an action of Γ such that the following properties hold:

- φ is A-linear and φ_{rig}-semilinear,
- if B is a basis of M, then also $\varphi(B)$ is a basis,
- the action of Γ is A-linear and semilinear with respect to the Γ-action on $\mathrm{B}^{\dagger}_{\mathrm{rig}}$,
- if B is some basis of M and $\gamma \in \Gamma$ then also $\gamma(B)$ is a basis of M,
- the action of Γ commutes with φ.

We denote the category of (φ, Γ)-modules over A by $\mathrm{B}^{\dagger}_{\mathrm{rig}}\text{-}\mathcal{M}od_A^{(\varphi,\Gamma)}$. It carries natural notions of direct sums, duals and tensor products.

If $A = L$,[16], the (φ, Γ)-module is called *étale* if with respect to some basis of M the endomorphism φ is described by a matrix whose entries are Laurent series in $\mathrm{B}^{\dagger}_{\mathrm{rig}}$ all of whose coefficients have absolute value ≤ 1.

In the case $A = L$, we call a (φ, Γ)-module over L just a (φ, Γ)-module. Just as in the case of Galois representations, a (φ, Γ)-module over A can be seen as a whole family of (φ, Γ)-modules parametrised by $\operatorname{Sp} A$, the rigid space associated to A.

Theorem 2.36 (Fontaine, Cherbonnier/Colmez, Kedlaya, Liu, Berger, Bellovin)**:** *There is a fully faithful functor*

$$\mathrm{D}^{\dagger}_{\mathrm{rig}} \colon \mathcal{R}ep_A(\mathrm{G}_{\mathbb{Q}_p}) \longrightarrow \mathrm{B}^{\dagger}_{\mathrm{rig}}\text{-}\mathcal{M}od_A^{(\varphi,\Gamma)}.$$

This functor commutes with direct sums and tensor products and is compatible with base change in A. If A = L, then its essential image is the subcategory of étale (φ, Γ)*-modules.*

Proof: For the general case see [Bel15, Cor. 2.2.11]; note that since we work with an affinoid algebra, the associated space is automatically quasi-compact and quasi-separated. For the case $A = L$ see [Col05, Prop. 2.7]. □

[16] This notion can also be defined for general A, but since we don't need this we omit it.

The compatibility with base change means in particular that the functor $\mathrm{D}^{\dagger}_{\mathrm{rig}}$ behaves well with families: it does not matter whether we first apply it to a family of Galois representations and then specialise the resulting (φ, Γ)-module or whether we first specialise a family of Galois representations and then apply $\mathrm{D}^{\dagger}_{\mathrm{rig}}$.

We now restrict to the case $A = L$. It is possible to extend the definitions of D_{HT}, D_{dR}, D_{st}, $\mathrm{D}_{\mathrm{pst}}$ and $\mathrm{D}_{\mathrm{cris}}$ to(φ, Γ)-modules, generalising the definitions for Galois representations. We do not repeat the definitions here, but just summarise this fact in the next theorem. In fact it is even possible to this for general A (see [Bel15, §4]), but we will not use this.

Theorem 2.37 (Berger, Colmez, Bellovin)**:** *There are functors* $\mathrm{D}_?$, *with* ? *being one of "*HT*", "*dR*", "*st*", "*pst*" or "*cris*", going from the category* $\mathrm{B}^{\dagger}_{\mathrm{rig}}$ *-$\mathcal{M}od_A^{(\varphi,\Gamma)}$ to the category of L-vector spaces*[17] *with additional structure,*[18] *such that if V is an L-linear representation of* $\mathrm{G}_{\mathbb{Q}_p}$, *then*

$$\mathrm{D}_?(\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)) = \mathrm{D}_?(V)$$

compatibly with these additional structures.

Proof: See [Bel15, §4.2] and the references there. □

Definition 2.38: We say that a (φ, Γ)-module M is Hodge-Tate, de Rham, semistable resp. crystalline if $\dim_L \mathrm{D}_?(M)$ equals the rank of M, for the corresponding "?".

We finally cite the following important definition from [Ber11, Def. 2.3.1].

Definition 2.39: We say that a (φ, Γ)-module M is *trianguline* if for some finite extension F of L the base change $\mathrm{B}^{\dagger}_{\mathrm{rig},F} \otimes_{\mathrm{B}^{\dagger}_{\mathrm{rig},L}} M$ is a successive extension of (φ, Γ)-modules of rank 1. We say that an L-linear representation of $\mathrm{G}_{\mathbb{Q}_p}$ is *trianguline* if $\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)$ is trianguline.

2.7. Galois cohomology and Selmer groups

In this section we briefly recall some definitions and statements related to certain Galois cohomology groups. All cohomology groups in this section are continuous cochain cohomology groups, which are often denoted "$\mathrm{H}^*_{\mathrm{cts}}$" or similarly, but we omit this and write just "H^*". As is common, we denote the cohomology of the absolute Galois group of a field k by $\mathrm{H}^*(k, -)$ instead of $\mathrm{H}^*(\mathrm{G}_k, -)$. We denote the complex of continuous cochains whose cohomology is $\mathrm{H}^*(k, V)$ (for a representation V of G_k) by $\mathrm{R}\Gamma(k, V)$, and we denote its image in the derived category by the same symbol.

In this whole section we fix a prime p and a finite extension $L/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Before we treat actual Galois cohomology, we record the following facts about continuous cochain cohomology of profinite groups.

Theorem 2.40: *Let G be a profinite group having the following property:*

> *For each finite discrete p-primary G-module A and each $i \geq 0$ the groups $\mathrm{H}^i(G, A)$ are finite.* (2.4)

Let V be a finite-dimensional L-vector space with a continuous G-action and fix a G-stable $\mathcal{O}$-lattice $T \subseteq V$. Then the following hold for all $i \geq 0$:

[17] In the case "pst", the functor goes to $\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} L$-modules.

[18] The additional structures are the same as in the case of Galois representations, e. g. filtrations, Frobenius endomorphisms and so on.

(a) *We have* $\mathrm{H}^i(G,T) \cong \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i(G,T/p^nT)$.

(b) *We have* $\mathrm{H}^i(G,V) \cong \mathrm{H}^i(G,T) \otimes_{\mathcal{O}} L$.

(c) *The* $\mathcal{O}$*-modules* $\mathrm{H}^i(G,T)$ *are finitely generated. The* L*-vector spaces* $\mathrm{H}^i(G,V)$ *are finite-dimensional.*

(d) *The canonical homomorphism* $\mathrm{H}^i(G,T) \otimes_{\mathcal{O}} L/\mathcal{O} \longrightarrow \mathrm{H}^i(G,V/T)$ *has finite kernel and cokernel.*

Proof: [NSW13, Cor. 2.7.6, Cor. 2.7.9, Cor. 2.7.10] □

Proposition 2.41: *The property (2.4) holds for the following groups:*

- *The local absolute Galois group* $\mathrm{G}_{\mathbb{Q}_\ell}$ *for any prime* ℓ*;*
- *the Galois group* $\mathrm{G}_{\mathbb{Q},S}$ *of the maximal extension of* $\mathbb{Q}$ *unramified outside a finite set of places* S *of* $\mathbb{Q}$ *containing* p *and the archimedean place.*

Proof: [NSW13, Thm. 7.1.8 (iii), Thm. 8.3.20 (i)] □

Now we turn to Galois representations. Let V be an L-linear representation of $\mathrm{G}_{\mathbb{Q}}$ which is unramified outside a finite set of places of $\mathbb{Q}$ and de Rham at all primes. We first look at the local behaviour. By proposition 2.41, we may then use the results from theorem 2.40.

Definition 2.42: For each place v of $\mathbb{Q}$ we define a complex $\mathrm{R\Gamma_f}(\mathbb{Q}_v,V)$ as

$$\mathrm{R\Gamma_f}(\mathbb{Q}_v,V) := \begin{cases} V^{\mathrm{I}_v} \xrightarrow{1-\mathrm{Frob}_v} V^{\mathrm{I}_v}, & v \neq p,\infty, \\ \mathrm{D_{cris}}(V) \xrightarrow{(1-\varphi_{\mathrm{cris}},\,1)} \mathrm{D_{cris}}(V) \oplus \mathrm{D_{dR}}(V)/\mathrm{fil}^0\,\mathrm{D_{dR}}(V), & v = p, \\ \mathrm{R\Gamma}(\mathbb{R},V), & v = \infty, \end{cases}$$

where the entries are in degree 0 and 1, respectively. We denote its cohomology groups by $\mathrm{H}^*_{\mathrm{f}}(\mathbb{Q}_p,V)$.

Lemma 2.43: *Let* v *be a place of* $\mathbb{Q}$.

(a) $\mathrm{H}^0_{\mathrm{f}}(\mathbb{Q}_v,V) = \mathrm{H}^0(\mathbb{Q}_v,V)$.

(b) $$\dim_L \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v,V) = \begin{cases} \dim_L \mathrm{H}^0(\mathbb{Q}_v,V), & v \neq p,\infty, \\ \dim_L \mathrm{H}^0(\mathbb{Q}_p,V) + \dim_L \mathrm{D_{dR}}(V)/\mathrm{fil}^0\,\mathrm{D_{dR}}(V), & v = p, \\ 0, & v = \infty. \end{cases}$$

From now on, let $v = \ell$ *be a prime number.*

(c) *If* $\ell \neq p$*, then* $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell,V) \cong \mathrm{H}^1(\mathbb{F}_\ell,V^{\mathrm{I}_\ell}) \cong \ker\left(\mathrm{H}^1(\mathbb{Q}_\ell,V) \xrightarrow{\mathrm{res}} \mathrm{H}^1(\mathrm{I}_\ell,V)\right)$.

(d) *If* $\ell = p$*, then* $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p,V) \cong \ker\left(\mathrm{H}^1(\mathbb{Q}_p,V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p,\mathrm{B_{cris}} \otimes_{\mathbb{Q}_p} V)\right)$.

(e) *Under the perfect cup product pairing*

$$\mathrm{H}^1(\mathbb{Q}_\ell, V) \times \mathrm{H}^1(\mathbb{Q}_\ell, V^*(1)) \longrightarrow \mathrm{H}^2(\mathbb{Q}_\ell, L(1)) = L$$

the subspaces $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)$ *and* $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V^*(1))$ *are orthogonal complements of each other.*

Proof: The statements for $v = \infty$ are trivial, so we assume $v = \ell$ is a prime number.

We first treat the case $\ell \neq p$. Then (a) is clear from the definition and (b) follows from the exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathrm{f}}(\mathbb{Q}_\ell, V) \longrightarrow V^{\mathrm{I}_\ell} \xrightarrow{1-\mathrm{Frob}_\ell} V^{\mathrm{I}_\ell} \longrightarrow \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V) \longrightarrow 0.$$

To prove (c) we use the long exact sequence attached to the short exact sequence of $\mathrm{G}_{\mathbb{F}_\ell}$-representations

$$0 \longrightarrow (1-\mathrm{Frob}_\ell)V^{\mathrm{I}_\ell} \longrightarrow V^{\mathrm{I}_\ell} \longrightarrow \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V) \longrightarrow 0,$$

which looks like

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0(\mathbb{F}_\ell, (1-\mathrm{Frob}_\ell)V^{\mathrm{I}_\ell}) \longrightarrow \mathrm{H}^0(\mathbb{F}_\ell, V^{\mathrm{I}_\ell}) \longrightarrow \mathrm{H}^0(\mathbb{F}_\ell, \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)) \\ &\longrightarrow \mathrm{H}^1(\mathbb{F}_\ell, (1-\mathrm{Frob}_\ell)V^{\mathrm{I}_\ell}) \longrightarrow \mathrm{H}^1(\mathbb{F}_\ell, V^{\mathrm{I}_\ell}) \xrightarrow{(*)} \mathrm{H}^1(\mathbb{F}_\ell, \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)) \\ &\longrightarrow \mathrm{H}^2(\mathbb{F}_\ell, (1-\mathrm{Frob}_\ell)V^{\mathrm{I}_\ell}) \longrightarrow \cdots. \end{aligned}$$

One knows that for any $\mathrm{G}_{\mathbb{F}_\ell}$-representation V' the dimensions of $\mathrm{H}^i(\mathbb{F}_\ell, V')$ for $i = 0, 1$ are equal and are 0 for $i \geq 2$: this follows from [NSW13, Prop. 1.7.7] and theorem 2.40 (a), using the exact sequence

$$0 \longrightarrow (V')^G \longrightarrow V' \xrightarrow{1-\mathrm{Frob}_\ell} V' \longrightarrow (V')_G \longrightarrow 0,$$

where Frob_ℓ is here a topological generator of $\mathrm{G}_{\mathbb{F}_\ell}$. Hence the arrow labelled $(*)$ in the above long exact sequence is surjective, and counting dimensions we see that it is an isomorphism. Since $\mathrm{G}_{\mathbb{F}_\ell} \cong \widehat{\mathbb{Z}}$ is the free profinite group of rank 1 and acts trivially on $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)$, we get $\mathrm{H}^1(\mathbb{F}_\ell, \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)) \cong \mathrm{Hom}(\mathrm{G}_{\mathbb{F}_\ell}, \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)) \cong \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)$, completing the proof of the first isomorphism. The second one now follows from the inflation-restriction exact sequence [NSW13, Prop. 1.6.7] (using again theorem 2.40 (a)).

In the case $\ell = p$, the statements (a), (b) and (d) follow from [BK90, Cor. 3.8.4].

Finally, (e) is shown in [BK90, Prop. 3.8] (for any ℓ). □

Definition 2.44: We define the following variant:

$$\mathrm{H}^1_{\mathrm{g}}(\mathbb{Q}_p, V) \coloneqq \ker\left(\mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathrm{B}_{\mathrm{dR}} \underset{\mathbb{Q}_p}{\otimes} V)\right).$$

Lemma 2.45: *We have*

$$\dim_L \mathrm{H}^1_{\mathrm{g}}(\mathbb{Q}_p, V) = \dim_L \left(\mathrm{D}_{\mathrm{dR}}(V) \big/ \mathrm{fil}^0\, \mathrm{D}_{\mathrm{dR}}(V) \right) + \dim_L \mathrm{H}^0(\mathbb{Q}_p, V) + \dim_L \mathrm{D}_{\mathrm{cris}}(V(1))^{\varphi_{\mathrm{cris}}=1}.$$

In particular, $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p, V) = \mathrm{H}^1_{\mathrm{g}}(\mathbb{Q}_p, V)$ *if* $\mathrm{D}_{\mathrm{cris}}(V^*(1))^{\varphi_{\mathrm{cris}}=1} = 0$.

Proof: The dimension formula is proved in [BK90, Cor. 3.8.4] and the last claim follows from lemma 2.43 (b). ☐

Lemma 2.46: *Let ℓ be a prime number.*

(a) The group $\mathrm{H}^1(\mathbb{Q}_\ell, V)$ parametrises equivalence classes of extensions of $\mathrm{G}_{\mathbb{Q}_\ell}$-representations

$$0 \longrightarrow V \longrightarrow E \longrightarrow L \longrightarrow 0. \tag{2.5}$$

(b) Let $\ell \neq p$. The class of an extension as above lies in $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_\ell, V)$ if and only if the sequence (2.5) remains exact after taking I_ℓ-invariants.

(c) Let $\ell = p$. The class of an extension as above lies in $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p, V)$ if and only if the sequence (2.5) remains exact after applying $\mathrm{D}_{\mathrm{cris}}$.

(d) Let $\ell = p$. The class of an extension as above lies in $\mathrm{H}^1_{\mathrm{g}}(\mathbb{Q}_p, V)$ if and only if the sequence (2.5) remains exact after applying D_{dR}.

Proof: Statement (a) is well-known and follows e. g. from [Wei94, §3.4]. The cohomology class belonging to an extension as in (2.5) is the image of $1 \in L = \mathrm{H}^0(\mathbb{Q}_\ell, L)$ under the boundary map $\mathrm{H}^0(\mathbb{Q}_\ell, L) \longrightarrow \mathrm{H}^1(\mathbb{Q}_\ell, V)$ in the attached long exact sequence. Using this description, (b) follows easily from lemma 2.43 (c). The proof of (c) resp. (d) works exactly the same, using lemma 2.43 (d) resp. definition 2.44. ☐

We now turn to the global Galois cohomology. Since V is unramified outside a finite set of places S, we work throughout with the cohomology groups $\mathrm{H}^*(\mathrm{G}_{\mathbb{Q},S}, V)$, for which we can use theorem 2.40 by proposition 2.41.

Definition 2.47: We define complexes for each place v of $\mathbb{Q}$

$$\mathrm{R}\Gamma_{/\mathrm{f}}(\mathbb{Q}_v, V) := \operatorname{cone}\Big(\mathrm{R}\Gamma_{\mathrm{f}}(\mathbb{Q}_v, V) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_v, V)\Big)$$

and for a finite set S of places of $\mathbb{Q}$ containing p, the places where V ramifies and the archimedean prime, we put

$$\mathrm{R}\Gamma_{\mathrm{f}}(\mathbb{Q}, V) := \operatorname{cone}\left(\mathrm{R}\Gamma(\mathrm{G}_{\mathbb{Q},S}, V) \longrightarrow \bigoplus_{v \in S} \mathrm{R}\Gamma_{/\mathrm{f}}(\mathbb{Q}_v, V)\right)[-1].$$

We denote the cohomology of these complexes by $\mathrm{H}^*_{/\mathrm{f}}(\mathbb{Q}_\ell, V)$ and $\mathrm{H}^*_{\mathrm{f}}(\mathbb{Q}, V)$, respectively. The group $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V)$ is called the (Bloch-Kato-)Selmer group of V (but we shall not use this terminology).

See [BF96, §1.2.1–2] for a more precise explanation of the above definitions and for an explanation why the definition of $\mathrm{R}\Gamma_f(\mathbb{Q}, V)$ is independent of the choice of S; see also [Ven07, §7].

The concrete meaning of these definitions is described by the following lemma.

Lemma 2.48: *For $i = 0,1$ there is a canonical isomorphism*

$$\mathrm{H}^i_{\mathrm{f}}(\mathbb{Q}, V) \xrightarrow{\sim} \ker\left(\mathrm{H}^i(\mathrm{G}_{\mathbb{Q},S}, V) \longrightarrow \bigoplus_{v\in S} \frac{\mathrm{H}^i(\mathbb{Q}_v, V)}{\mathrm{H}^i_{\mathrm{f}}(\mathbb{Q}_v, V)}\right),$$

i. e. a cohomology class lies globally in $\mathrm{H}^i_{\mathrm{f}}$ if and only if it does so everywhere locally. In particular, $\mathrm{H}^0_{\mathrm{f}}(\mathbb{Q}, V) = \mathrm{H}^0(\mathbb{Q}, V)$.

Proof: Let v be a place. By definition of $\mathrm{R}\Gamma_{/\mathrm{f}}(\mathbb{Q}_\ell, V)$ we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0_{\mathrm{f}}(\mathbb{Q}_v, V) \longrightarrow \mathrm{H}^0(\mathbb{Q}_v, V) \longrightarrow \mathrm{H}^0_{/\mathrm{f}}(\mathbb{Q}_v, V) \\ \longrightarrow \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_v, V) \longrightarrow \mathrm{H}^1_{/\mathrm{f}}(\mathbb{Q}_v, V) \longrightarrow 0 \end{aligned}$$

and since the map $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_v, V)$ is injective, we obtain

$$\mathrm{H}^i_{/\mathrm{f}}(\mathbb{Q}_v, V) = \frac{\mathrm{H}^i(\mathbb{Q}_v, V)}{\mathrm{H}^i_{\mathrm{f}}(\mathbb{Q}_v, V)} \quad \text{for } i = 0,1.$$

In particular, $\mathrm{H}^0_{/\mathrm{f}}(\mathbb{Q}_v, V) = 0$ by lemma 2.43 (a). Then by definition of $\mathrm{R}\Gamma_{\mathrm{f}}(\mathbb{Q}, V)$ we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0_{\mathrm{f}}(\mathbb{Q}, V) \longrightarrow \mathrm{H}^0(\mathrm{G}_{\mathbb{Q},S}, V) \longrightarrow \bigoplus_{v\in S} \mathrm{H}^0_{/\mathrm{f}}(\mathbb{Q}_v, V) \\ \longrightarrow \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V) \longrightarrow \mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, V) \longrightarrow \bigoplus_{v\in S} \mathrm{H}^1_{/\mathrm{f}}(\mathbb{Q}_v, V) \longrightarrow \cdots \end{aligned}$$

and the claim follows. □

From now on we fix a $\mathrm{G}_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice T inside V.

Definition 2.49: (a) Let $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, T)$ be the preimage of $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V)$ under the canonical map $\mathrm{H}^1(\mathbb{Q}, T) \longrightarrow \mathrm{H}^1(\mathbb{Q}, V)$.

(b) Let $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V/T)$ be the image of $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V)$ under the canonical map $\mathrm{H}^1(\mathbb{Q}, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}, V/T)$.

Analogously we define $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, T)$ and $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V/T)$ for places v.

Proposition 2.50: *If $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V/T)$ is finite, then $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V) = 0$.*

Proof: From the definitions and lemma 2.48 we easily obtain canonical isomorphisms

$$\begin{aligned} \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, T) &\xrightarrow{\sim} \ker\left(\mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, T) \longrightarrow \bigoplus_{v\in S0} \frac{\mathrm{H}^1(\mathbb{Q}_v, T)}{\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, T)}\right), \\ \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V/T) &\xrightarrow{\sim} \ker\left(\mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, V/T) \longrightarrow \bigoplus_{v\in S} \frac{\mathrm{H}^1(\mathbb{Q}_v, V/T)}{\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V/T)}\right). \end{aligned} \qquad (*_1)$$

By theorem 2.40 (d), the canonical map $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, T) \otimes_{\mathcal{O}} L/\mathcal{O} \longrightarrow \mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, V/T)$ has finite kernel and cokernel. One easily checks, using the definition, that it maps $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, T) \otimes L/\mathcal{O}$ into $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V/T)$ for each place v, so that by $(*_1)$ we get a well-defined map

$$\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, T) \underset{\mathcal{O}}{\otimes} L/\mathcal{O} \longrightarrow \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V/T) \qquad (*_2)$$

which of course still has finite kernel and cokernel.

Similarly, again using the definitions and theorem 2.40 (b), we see that the canonical map $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, T) \otimes_{\mathcal{O}} L \longrightarrow \mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, V)$ is an isomorphism which induces isomorphisms $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, T) \otimes_{\mathcal{O}} L \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V)$ for all places v. Then again by $(*_1)$ we obtain an isomorphism

$$\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, T) \underset{\mathcal{O}}{\otimes} L \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V). \tag{$*_3$}$$

It follows from $(*_2)$ that $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, T) \otimes L/\mathcal{O}$ is finite. Since $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q},S}, T)$ is finitely generated by theorem 2.40 (c), $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, T)$ must therefore be a torsion group. Using the isomorphism $(*_3)$, the claim follows. ☐

Lemma 2.51: *We have the dimension formula*

$$\begin{aligned}\dim_L \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V) = \dim_L \mathrm{H}^0(\mathbb{Q}, V) - \dim_L \mathrm{H}^0(\mathbb{Q}, V^*(1)) + \dim_L \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V^*(1)) \\ + \dim_L \left(\mathrm{D}_{\mathrm{dR}}(V|_{\mathrm{G}_{\mathbb{Q}_p}}) \Big/ \mathrm{fil}^0 \mathrm{D}_{\mathrm{dR}}(V|_{\mathrm{G}_{\mathbb{Q}_p}}) \right) - \dim_L \mathrm{H}^0(\mathbb{R}, V).\end{aligned}$$

Proof: We use a result from [NSW13, §VIII.7]. There finite Galois modules are studied, but using again theorem 2.40 (a) we can use the results there also in our situation. By lemma 2.43 (c), (e), the groups $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V)$ for all places v define a "collection of local conditions" with dual $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_v, V^*(1))$ in the sense of [NSW13, Def. 8.7.8], whose corresponding global groups are $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V)$ resp. $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V^*(1))$ by lemma 2.48. Using lemma 2.43 (b), the statement then follows from [NSW13, Thm. 8.7.9]. ☐

3. Motives, periods and related conjectures

Motives play a central role in this work and in the theory behind the Equivariant Tamagawa Number Conjecture and p-adic L-functions. Many texts on these subjects do not rigorously define what they mean by a motive, and there are good reasons for this – this work will be no exception to this custom. Nevertheless we want to at least explain some of the theoretical background, which will hopefully illustrate two points: first, that there is a concept underlying the ad-hoc formalism used in the literature, and second, why it is okay for us to use a pragmatic approach to motives. For a rigorous treatment of motives we refer to [And04], and for a nice introduction to the main ideas see [Mil13].

The wish that stimulated the theory of motives is that one wants to have a "universal cohomology theory" for smooth projective varieties[19] over a field k through which any "good" cohomology theory factors. More precisely, there is the notion of a *Weil cohomology theory* [And04, §3.3], which is a functor from the category $\mathcal{V}ar(k)$ of smooth projective varieties over k to some abelian category, and one wants a universal abelian category $\mathcal{M}ot(k)$ of so-called "motives" with a fully faithful functor $h\colon \mathcal{V}ar(k) \longrightarrow \mathcal{M}ot(k)$ such that any Weil cohomology theory factors uniquely through h. The objects of $\mathcal{M}ot(k)$ should be thought of as "pieces in the cohomology of varieties" – $\mathcal{M}ot(k)$ clearly must contain more objects than $\mathcal{V}ar(k)$, which is not abelian, for example kernels and cokernels of morphisms, and thus allows to decompose varieties into smaller pieces although they may be irreducible as geometric objects (see example 3.16 (d) for an illustration of this phenomenon). Until today it

[19] Here, "variety" means an integral, separated scheme of finite type over a field. In particular, each variety is connected.

is not known whether such a category exists, although candidates can be constructed using some abstract nonsense tricks. To carry this out one needs the following notion.

Definition 3.1: Let X, Y be varieties over k. A *correspondence* from X to Y is a formal finite sum of closed subvarieties of $X \times_k Y$ of codimension $\dim X$.[20]

Example 3.2: If $f\colon Y \longrightarrow X$ is a morphism of varieties over k, then its graph, which is the image of the morphism $f \times \mathrm{id}_Y \colon Y \times_k Y \longrightarrow X \times_k Y$, is a closed subvariety of codimension $\dim X$ (which can easily be seen on affine pieces), hence defines a correspondence from X to Y.

To construct candidates for the category of motives, one starts with the category $\mathcal{V}ar(k)$ and chooses as morphisms the free abelian group of correspondences modulo a certain equivalence relation and tensored with $\mathbb{Q}$ (or some other field of coefficients). Then one extends this category by formally adjoining kernels and images of idempotent endomorphisms and the Tate motive (which we describe in fact 3.6 below). We do not give any details about this construction (see [And04, §4.1] for this), but one should keep in mind that correspondences induce morphisms between motives.

The construction of this category depends on the choice of an appropriate equivalence relation on correspondences, and there are various natural choices for this equivalence relation. The finest reasonable equivalence relation one can use here is called *rational equivalence*, while the coarsest is called *numerical equivalence* (in a sense that can be made precise, see [And04, Déf. 3.1.1.1, Lem. 3.2.2.1, Ex. 3.2.7.2]). The resulting categories of motives are denoted $\mathcal{M}ot_{\mathrm{rat}}(k)$ and $\mathcal{M}ot_{\mathrm{num}}(k)$ and called *Chow motives* and *Grothendieck motives*, respectively. Both have advantages and disadvantages. While any Weil cohomology theory factors uniquely through $\mathcal{M}ot_{\mathrm{rat}}(k)$, essentially by construction (see [And04, §4.2.4–5]), the analogous statement for $\mathcal{M}ot_{\mathrm{num}}(k)$ requires an unproven conjecture of Grothendieck and is thus not known. On the other hand, $\mathcal{M}ot_{\mathrm{num}}(k)$ is an abelian category as desired and is even semisimple [And04, Thm. 4.5.1.1], while $\mathcal{M}ot_{\mathrm{rat}}(k)$ is not abelian in general. Further $\mathcal{M}ot_{\mathrm{num}}(k)$ has the advantage that it allows more decompositions than $\mathcal{M}ot_{\mathrm{rat}}(k)$ does: Every Chow motive is a Grothendieck motive, but the converse may be false, i. e. some "pieces" in the cohomology of varieties only exist as a Grothendieck motive, but maybe not as a Chow motive.

The category $\mathcal{M}ot_{\mathrm{rat}}(k)$ thus has *realisation functors* (see [And04, §4.2.5]), i. e. functors to abelian categories induced by Weil cohomology theories. For what we want to do with motives, we only need these realisations, but unfortunately the main example we are interested in is a Grothendieck motive.

We now specialise to the case $k = \mathbb{Q}$. Then we have the following Weil cohomology theories (see [And04, §3.4.1]): Betti cohomology (i. e. singular cohomology of the complex manifold obtained by taking complex points of a variety), algebraic de Rham cohomology and ℓ-adic étale cohomology for every prime ℓ. Between these we have comparison isomorphisms after tensoring with appropriate period rings (see [And04, §3.4.2]). We thus obtain a functor from $\mathcal{M}ot_{\mathrm{rat}}(\mathbb{Q})$ to the (abelian!) category $\mathcal{P}re\mathcal{M}ot(\mathbb{Q})$ whose objects are tuples consisting of a vector space for each above-mentioned Weil cohomology theory, together with certain additional structures and comparison isomorphisms between them. Such objects are also called *premotivic structures* (see [DFG04, §1.1.1] or [Ven07, §2]), and we explain

[20] More precisely, this is a correspondence of degree 0, but we will not use correspondences of higher degree, so we omit this.

them in section 3.1. Of course that functor cannot be fully faithful since $\mathcal{Mot}_{\mathrm{rat}}(\mathbb{Q})$ is not abelian. Nonetheless yet another way to define a category of motives over $\mathbb{Q}$ would be as the smallest abelian subcategory of $\mathcal{PreMot}(\mathbb{Q})$ containing the essential image of this functor. Conjecturally there should be an analogous functor $\mathcal{Mot}_{\mathrm{num}}(\mathbb{Q}) \longrightarrow \mathcal{PreMot}(\mathbb{Q})$, and if it were fully faithful then yet another possibility to define a motive would be as an object in the essential image of this latter functor.

The pragmatic solution to this dilemma that most texts implicitly choose is to work just with the category $\mathcal{PreMot}(\mathbb{Q})$ of premotivic structures, keeping in mind that the realisations one works with should somehow come from geometry. We follow this practice, keeping in mind the background explained above.

3.1. Realisations and comparison isomorphisms

From now on we fix a number field K which we use as coefficients of the motives we consider and recall that we have fixed a pair of embeddings (ι_∞, ι_p) of $\overline{\mathbb{Q}}$ for each prime p. We imagine motives just as being given by their realisations, and we list those that will be important in this work and the Weil cohomology theories that induce them. There are other realisations, but we shall not need them. Each motive has a *rank*, which is a non-negative integer $n \in \mathbb{N}_0$, and the following realisations:

- The *Betti realisation*: an n-dimensional K-vector space M_{B} with a K-linear action of $\mathrm{G}_{\mathbb{R}}$ (coming from singular cohomology).[21]
- The *de Rham realisation*: an n-dimensional K-vector space M_{dR} with a decreasing and exhaustive filtration $\mathrm{fil}^i M_{\mathrm{dR}}$ ($i \in \mathbb{Z}$) called the *Hodge filtration* (coming from algebraic de Rham cohomology).
- For each finite place $\mathfrak{p}$ of K the $\mathfrak{p}$*-adic (étale) realisation*: an n-dimensional $K_{\mathfrak{p}}$-vector space $M_{\mathfrak{p}}$ with a continuous $K_{\mathfrak{p}}$-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (coming from p-adic étale cohomology).

Then there are the following variants which we use occasionally.

- The *Hodge realisation*: an n-dimensional graded K-vector space M_{H}, which is the graded vector space associated to M_{dR}.
- For each rational prime p the p*-adic (étale) realisation*: this is a free K_p-module M_p of rank n which is the product of $M_{\mathfrak{p}}$ for all places $\mathfrak{p} \mid p$ of K.

Definition 3.3: For a motive M, we define its *tangent space* by

$$\mathrm{t}_M \coloneqq {}^{M_{\mathrm{dR}}}\big/_{\mathrm{fil}^0 M_{\mathrm{dR}}} .$$

Next we list the comparison isomorphisms that relate these realisation.

[21] This space has as a further feature a Hodge structure, but we will never use it in this work. It is related to the Hodge filtration on the de Rham realisation, and everything we will need from the Hodge structure can also be obtained from this filtration.

(1) The complex comparison isomorphism

$$\mathrm{cp}_\infty\colon M_{\mathrm{B}} \underset{\mathbb{Q}}{\otimes} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{dR}} \underset{\mathbb{Q}}{\otimes} \mathbb{C},$$

which is an isomorphism of $K \otimes_{\mathbb{Q}} \mathbb{C}$-modules with a linear action of $\mathrm{G}_{\mathbb{R}}$. Here $\mathrm{G}_{\mathbb{R}}$ acts diagonally on the left side and through the factor $\mathbb{C}$ on the right side.

(2) The p-adic (étale) comparison isomorphism of K_p-modules

$$\mathrm{cp}_{\text{ét}}\colon M_{\mathrm{B}} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_p = M_{\mathrm{B}} \underset{K}{\otimes} K_p \xrightarrow{\sim} M_p.$$

It is compatible with the action of $\mathrm{G}_{\mathbb{R}}$, which on M_p is induced by the fixed embedding $\iota_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and is trivial on $\mathbb{Q}_p$.

(3) The comparison isomorphism from p-adic Hodge theory

$$\mathrm{cp}_{\mathrm{dR}}\colon M_p \underset{\mathbb{Q}_p}{\otimes} \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}} \underset{\mathbb{Q}}{\otimes} \mathrm{B}_{\mathrm{dR}}$$

which is an isomorphism of filtered $(\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K_p)$-modules. It respects the action of $\mathrm{G}_{\mathbb{Q}_p}$, where the $\mathrm{G}_{\mathbb{Q}_p}$-action on M_p comes from the fixed embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\mathrm{G}_{\mathbb{Q}_p}$ acts on the left side diagonally and on the right side through the factor B_{dR}.

There are the following variants of these, for which we have to fix an embedding $K \hookrightarrow \overline{\mathbb{Q}}$. This fixes a place $\mathfrak{p}$ of K and embeddings $K_{\mathfrak{p}} \hookrightarrow \mathrm{B}_{\mathrm{dR}}$ and $K_{\mathfrak{p}} \hookrightarrow \mathrm{B}_{\mathrm{HT}}$. By abuse of notation, we use the same names for the comparison isomorphisms again.

(1′) An isomorphism of $\mathbb{C}$-vector spaces with $\mathrm{G}_{\mathbb{R}}$-action

$$\mathrm{cp}_\infty\colon M_{\mathrm{B}} \underset{K}{\otimes} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{dR}} \underset{K}{\otimes} \mathbb{C}.$$

(2′) An isomorphism of $K_{\mathfrak{p}}$-vector spaces

$$\mathrm{cp}_{\text{ét}}\colon M_{\mathrm{B}} \underset{K}{\otimes} K_{\mathfrak{p}} \xrightarrow{\sim} M_{\mathfrak{p}}$$

respecting the action of $\mathrm{G}_{\mathbb{R}}$.

(3′) An isomorphism of filtered B_{dR}-vector spaces

$$\mathrm{cp}_{\mathrm{dR}}\colon M_{\mathfrak{p}} \underset{K_{\mathfrak{p}}}{\otimes} \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}} \underset{K}{\otimes} \mathrm{B}_{\mathrm{dR}}$$

respecting the action of $\mathrm{G}_{\mathbb{Q}_p}$.

There are further variants.

(3″) After taking $\mathrm{G}_{\mathbb{Q}_p}$-invariants, $\mathrm{cp}_{\mathrm{dR}}$ gives an isomorphism of filtered K_p-modules

$$\mathrm{cp}_{\mathrm{dR}}\colon\ \mathrm{D}_{\mathrm{dR}}(M_p) \xrightarrow{\sim} M_{\mathrm{dR}} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_p.$$

Since this isomorphism respects the filtrations, it induces further an isomorphism of K_p-modules

$$\mathrm{cp}_{\mathrm{dR}}\colon\ \mathrm{D}_{\mathrm{dR}}(M_p) \big/ \mathrm{fil}^0\, \mathrm{D}_{\mathrm{dR}}(M_p) \xrightarrow{\sim} \mathrm{t}_M \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_p.$$

(3‴) If we tensor the above isomorphisms over K_p with $K_\mathfrak{p}$, we obtain isomorphisms of $K_\mathfrak{p}$-vector spaces

$$\mathrm{cp}_{\mathrm{dR}}\colon\ \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}) \xrightarrow{\ \sim\ } M_{\mathrm{dR}} \underset{K}{\otimes} K_\mathfrak{p}$$

(which respects filtrations) and

$$\mathrm{cp}_{\mathrm{dR}}\colon\ \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}) \Big/ \mathrm{fil}^0\, \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}) \xrightarrow{\ \sim\ } \mathrm{t}_M \underset{K}{\otimes} K_\mathfrak{p}.$$

(4) Finally there is the Hodge-Tate version

$$\mathrm{cp}_{\mathrm{HT}}\colon M_p \underset{\mathbb{Q}_p}{\otimes} \mathrm{B}_{\mathrm{HT}} \xrightarrow{\ \sim\ } M_{\mathrm{H}} \underset{\mathbb{Q}}{\otimes} \mathrm{B}_{\mathrm{HT}},$$

which is an isomorphism of graded $(K_p \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{HT}})$-modules, and its $\mathfrak{p}$-adic variant

$$\mathrm{cp}_{\mathrm{HT}}\colon M_\mathfrak{p} \underset{K_\mathfrak{p}}{\otimes} \mathrm{B}_{\mathrm{HT}} \xrightarrow{\ \sim\ } M_{\mathrm{H}} \underset{K}{\otimes} \mathrm{B}_{\mathrm{HT}},$$

which is an isomorphism of graded B_{HT}-modules. Both respect the actions of $\mathrm{G}_{\mathbb{Q}_p}$.

Definition 3.4: A collection of realisations as listed above together with comparison isomorphisms between them is called a *premotivic structure*. The category of these objects, with morphisms being maps between the realisations respecting all additional structures and all comparison isomorphisms, makes up the abelian category of premotivic structures over $\mathbb{Q}$ with coefficients in K, denoted by $\mathcal{PreMot}(\mathbb{Q})_K$.

When we write "motive", we will from now on mostly mean just such a premotivic structure.

Remark 3.5: The category of motives has some additional features. First, it admits a tensor product. We do not explain here where it comes from. The only thing that will be important for us is that it commutes with realisations, i. e. the realisations of the tensor product of motives are just the tensor products of the realisations. Second, there is a notion of duals, which on realisations is given by taking the dual space. Finally there is a notion of extension of scalars: if $K \subseteq F \subseteq \overline{\mathbb{Q}}$ are number fields, then there is a functor from the category of motives over $\mathbb{Q}$ with coefficients in K to the category of motives over $\mathbb{Q}$ with coefficients in F, which we denote by $M \longmapsto M \otimes_K F$. On realisations it is given by tensoring each vector space with F over K resp. with $F_\mathfrak{p}$ or $F_\mathfrak{P}$ over $K_\mathfrak{p}$ for the $\mathfrak{p}$-adic realisation, where $\mathfrak{P} \mid \mathfrak{p}$ is the place of F lying over $\mathfrak{p}$ fixed by $F \hookrightarrow \overline{\mathbb{Q}}$.

3.2. Examples: the Tate motive and Artin motives

In this section we give two basic examples of motives that will be important later. We do not introduce them rigorously but just list their realisations and comparison isomorphisms. References for this are [Ven07, Ex. 2.1] and [DFG04, §1.1.3].

Fact 3.6: *The Tate motive $\mathbb{Q}(1)$ is a motive of rank 1 over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. It has the following realisations:*

- *$\mathbb{Q}(1)_{\mathrm{B}}$ is a one-dimensional $\mathbb{Q}$-vector space with a canonical[22] basis $b_{\mathrm{B}}^{\mathbb{Q}(1)}$. Complex conjugation acts as -1 on $\mathbb{Q}(1)_{\mathrm{B}}$.*

[22] The basis depends on the choice of a root i of -1 in $\mathbb{C}$. Once we have fixed this choice (as we have throughout this work), the basis is canonical.

- $\mathbb{Q}(1)_{\mathrm{dR}}$ *is a one-dimensional* $\mathbb{Q}$*-vector space with a canonical basis* $b_{\mathrm{dR}}^{\mathbb{Q}(1)}$*. The Hodge filtration on* $\mathbb{Q}(1)_{\mathrm{dR}}$ *is given by* $\operatorname{fil}^i \mathbb{Q}(1)_{\mathrm{dR}} = \mathbb{Q}(1)_{\mathrm{dR}}$ *for* $i \leq -1$ *and* $\operatorname{fil}^i \mathbb{Q}(1)_{\mathrm{dR}} = 0$ *for* $i \geq 0$.
- $\mathbb{Q}(1)_p$ *is a one-dimensional* $\mathbb{Q}_p$*-vector space with a canonical*[23] *basis* $b_p^{\mathbb{Q}(1)}$*. The action of* $\mathrm{G}_{\mathbb{Q}}$ *on* $\mathbb{Q}(1)_p$ *is given by the cyclotomic character.*

The comparison isomorphisms are as follows:

- $\mathrm{cp}_\infty \colon \mathbb{Q}(1)_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathbb{Q}(1)_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ *sends* $b_{\mathrm{B}}^{\mathbb{Q}(1)} \otimes 1$ *to* $b_{\mathrm{dR}}^{\mathbb{Q}(1)} \otimes 2\pi\mathrm{i}$.
- $\mathrm{cp}_{\text{ét}} \colon \mathbb{Q}(1)_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{Q}(1)_p$ *sends* $b_{\mathrm{B}}^{\mathbb{Q}(1)} \otimes 1$ *to* $b_p^{\mathbb{Q}(1)} \otimes 1$.
- $\mathrm{cp}_{\mathrm{dR}} \colon \mathbb{Q}(1)_p \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} \mathbb{Q}(1)_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathrm{B}_{\mathrm{dR}}$ *sends* $b_p^{\mathbb{Q}(1)} \otimes 1$ *to* $b_{\mathrm{dR}}^{\mathbb{Q}(1)} \otimes \mathrm{t}_{\mathrm{dR}}$.

Note that by the above fact, the cyclotomic character has Hodge-Tate weight -1.

Next we describe Artin motives, which one should think of as "pieces" of $\mathrm{H}^0(\operatorname{Spec} F)$, where F is a number field. By an *Artin representation*, we mean a finite-dimensional vector space V over a number field K with a homomorphism $\rho\colon \mathrm{G}_{\mathbb{Q}} \longrightarrow \operatorname{Aut}_K(V)$ with finite image.

Fact 3.7: *The Artin motive* $\mathcal{M}(\rho)$ *attached to* ρ *is a motive of rank* $\dim_K V$ *over* $\mathbb{Q}$ *with coefficients in* K*. It has the following realisations:*

- $\mathcal{M}(\rho)_{\mathrm{B}} = V$*. The action of* Frob_∞ *comes via* ρ *from the embedding* $\mathrm{G}_{\mathbb{R}} \hookrightarrow \mathrm{G}_{\mathbb{Q}}$.
- $\mathcal{M}(\rho)_{\mathrm{dR}} = (V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^{\mathrm{G}_{\mathbb{Q}}}$*, where* $\mathrm{G}_{\mathbb{Q}}$ *acts diagonally on both factors. The Hodge filtration on* $\mathcal{M}(\rho)_{\mathrm{dR}}$ *is given by* $\operatorname{fil}^i \mathcal{M}(\rho)_{\mathrm{dR}} = \mathcal{M}(\rho)_{\mathrm{dR}}$ *for* $i \leq 0$ *and* $\operatorname{fil}^i \mathcal{M}(\rho)_{\mathrm{dR}} = 0$ *for* $i \geq 1$.
- $\mathcal{M}(\rho)_{\mathrm{p}} = V \otimes_K K_{\mathrm{p}}$ *with* $\mathrm{G}_{\mathbb{Q}}$ *acting just on* V *via* ρ.
- *Both comparison isomorphisms* cp_∞ *and* $\mathrm{cp}_{\mathrm{dR}}$ *are induced by the inclusion*

  $$(V \underset{\mathbb{Q}}{\otimes} \overline{\mathbb{Q}})^{\mathrm{G}_{\mathbb{Q}}} \hookrightarrow V \underset{\mathbb{Q}}{\otimes} \overline{\mathbb{Q}},$$

  while the comparison isomorphism $\mathrm{cp}_{\text{ét}}$ *is the identity.*

We describe the comparison isomorphisms more explicitly in the following special case. Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Then we can view χ as an Artin representation as above with $K = \mathbb{Q}(\mu_f)$. Choose an embedding $K \hookrightarrow \overline{\mathbb{Q}}$ and let $\zeta \in K^\times$ be the primitive root of unity which is sent to $\mathrm{e}^{2\pi\mathrm{i}/f}$ by the induced embedding $K \hookrightarrow \mathbb{C}$. We may view ζ then also as an element in $\overline{\mathbb{Q}}_p$ and thus as an element in B_{dR}. Let $\mathrm{G}(\chi) = \sum_a \chi(a)\zeta^a$ be the Gauß sum of χ with respect to our chosen embeddings, where a runs over $(\mathbb{Z}/f)^\times$. The following is easily derived from the above fact.

Fact 3.8: *The Dirichlet motive attached to* χ *is a motive* $\mathcal{M}(\chi)$ *of rank* 1 *over* $\mathbb{Q}$ *with coefficients in* K*. It has the following realisations:*

- $\mathcal{M}(\chi)_{\mathrm{B}}$ *is a one-dimensional* K*-vector space with a canonical basis* b_{B}^χ*. Complex conjugation acts as multiplication by* $\chi(-1)$.

[23] The basis depends on the choice of a compatible system $(\xi_n)_{n\geq 1}$ of p-power roots of unity in $\overline{\mathbb{Q}}_p$. Once we have fixed this choice, the basis is canonical.

- *$\mathcal{M}(\chi)_{\mathrm{dR}}$ is a one-dimensional K-vector space with a canonical basis b^{χ}_{dR}. The Hodge filtration on $\mathcal{M}(\chi)_{\mathrm{dR}}$ is given by $\mathrm{fil}^i\,\mathcal{M}(\chi)_{\mathrm{dR}} = \mathcal{M}(\chi)_{\mathrm{dR}}$ for $i \leq 0$ and $\mathrm{fil}^i\,\mathcal{M}(\chi)_{\mathrm{dR}} = 0$ for $i \geq 1$.*

- *$\mathcal{M}(\chi)_{\mathrm{p}}$ is a one-dimensional K-vector space with a canonical basis b^{χ}_{p} and with $\mathrm{G}_{\mathbb{Q}}$ acting via χ.*

The comparison isomorphisms are as follows:

- $\mathrm{cp}_{\infty}\colon \mathcal{M}(\chi)_{\mathrm{B}} \otimes_K \mathbb{C} \xrightarrow{\sim} \mathcal{M}(\chi)_{\mathrm{dR}} \otimes_K \mathbb{C}$ *sends* $b^{\chi}_{\mathrm{B}} \otimes 1$ *to* $b^{\chi}_{\mathrm{dR}} \otimes \mathrm{G}(\chi)^{-1}$.

- $\mathrm{cp}_{\text{ét}}\colon \mathcal{M}(\chi)_{\mathrm{B}} \otimes_K \mathbb{Q}_p \xrightarrow{\sim} \mathcal{M}(\chi)_{\mathrm{p}}$ *sends* $b^{\chi}_{\mathrm{B}} \otimes$ *to* $b^{\chi}_{\mathrm{p}} \otimes 1$.

- $\mathrm{cp}_{\mathrm{dR}}\colon \mathcal{M}(\chi)_{\mathrm{p}} \otimes_{K_{\mathrm{p}}} \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} \mathcal{M}(\chi)_{\mathrm{dR}} \otimes_K \mathrm{B}_{\mathrm{dR}}$ *sends* $b^{\chi}_{p} \otimes 1$ *to* $b^{\chi}_{\mathrm{dR}} \otimes \mathrm{G}(\chi)^{-1}$.

Definition 3.9: We define the following using remark 3.5.

(a) For $n \in \mathbb{N}$, we define $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$. We let $\mathbb{Q}(-n)$ be the dual of $\mathbb{Q}(n)$ for $n \in \mathbb{N}$. If K is a number field, then we define $K(n) := \mathbb{Q}(n) \otimes_{\mathbb{Q}} K$.

The canonical bases $b^{\mathbb{Q}(1)}_{?}$ for $? \in \{\mathrm{B}, \mathrm{dR}, p\}$ of the realisations of $\mathbb{Q}(1)$ induce canonical bases of the realisations of $\mathbb{Q}(n)$ for each $n \in \mathbb{Z}$. We denote these bases by $b^{\mathbb{Q}(n)}_{?}$, respectively. For $K(n)$ we denote them by $b^{K(n)}_{?}$.

(b) For a motive M with coefficients in a number field K, we define $M(n) := M \otimes_K K(n)$ and $M(\rho) := M \otimes \mathcal{M}(\rho)$ for $n \in \mathbb{Z}$ resp. ρ an Artin representation.[24]

Lemma 3.10: *(a) Let R be a ring with $2 \in R^{\times}$ and let M and N be $R[\mathrm{G}_{\mathbb{R}}]$-modules. Then*

$$(M \underset{R}{\otimes} N)^{+} = \left(M^{+} \underset{R}{\otimes} N^{+}\right) \oplus \left(M^{-} \underset{R}{\otimes} N^{-}\right).$$

(b) Let M be a motive over a number field K, let $\rho\colon G \longrightarrow \mathrm{GL}_r(K')$ be a representation with coefficients in a finite extension K' of K and let $n \in \mathbb{N}$. Then

$$M(\rho)(n)^{+}_{\mathrm{B}} = M^{s}_{\mathrm{B}} \otimes \mathcal{M}(\rho)^{+}_{\mathrm{B}} \otimes K(n)_{\mathrm{B}} \oplus M^{-s}_{\mathrm{B}} \otimes \mathcal{M}(\rho)^{-}_{\mathrm{B}} \otimes K(n)_{\mathrm{B}}$$

with $s = (-1)^n$. In particular, if $\rho = \chi$ is a Dirichlet character, then

$$M(\chi)(n)^{+}_{\mathrm{B}} = M^{s}_{\mathrm{B}} \otimes \mathcal{M}(\chi) \otimes K(n)_{\mathrm{B}}$$

with $s = \chi(-1)(-1)^n$.

Proof: Part (a) is an easy calculation. For (b), note that $M(n)^{+}_{\mathrm{B}} = M^{s}_{\mathrm{B}} \otimes_K K(n)$ for $s = (-1)^n$. Using this, the claim follows immediately from (a). □

[24] More precisely: Let F be the field of coefficients of ρ and let $F' = KF$ be the composite field. Then we mean $M(\rho) = (M \otimes_K F') \otimes_{F'} (\mathcal{M}(\rho) \otimes_F F')$. We will often ignore this detail.

3.3. L-functions, criticality and complex periods

Fix a number field K and a motive M over $\mathbb{Q}$ with coefficients in K.

Most of the content of this section is from [Del79], to where we refer for more details. Recall that we have fixed a pair of embeddings of $\overline{\mathbb{Q}}$ for each prime p. Choose further an embedding $K \hookrightarrow \overline{\mathbb{Q}}$. Everything we do in this section can also be done without such a choice, but as we will have fixed an embedding most of the time, we omit explaining this.

We first introduce motivic L-functions.

Definition 3.11: Let ℓ, p be primes, let $\mathfrak{p} \mid p$ be a place of K and let $L = K_{\mathfrak{p}}$. Further let $\rho\colon \mathrm{G}_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}_L(V)$ be a representation of $\mathrm{G}_{\mathbb{Q}_p}$ on a finite-dimensional L-vector space V. Define a polynomial $\mathrm{P}_\ell(V,T) \in L[T]$ by

$$\mathrm{P}_\ell(V,T) := \begin{cases} \det_L(1-\rho(\mathrm{Frob}_\ell)T, V^{I_\ell}), & \ell \neq p, \\ \det_L(1-\varphi_{\mathrm{cris}}T, \mathrm{D}_{\mathrm{cris}}(V)), & \ell = p. \end{cases}$$

Conjecture 3.12: *The polynomial* $\mathrm{P}_\ell(M_{\mathfrak{p}}, T)$ *lies in fact in* $K[T]$ *and is independent of the choice of* p *and* $\mathfrak{p}$.

We assume that conjecture 3.12 is true and write $\mathrm{P}_\ell(M,T) := \mathrm{P}_\ell(M_{\mathfrak{p}},T)$.

Definition 3.13: Define the complex L-*function of* M by the Euler product

$$\mathrm{L}(M,s) := \prod_{\ell \text{ prime}} \mathrm{P}_\ell(M, \ell^{-s})^{-1}, \quad \operatorname{Re} s \gg 0.$$

Here we use the embedding $K \hookrightarrow \mathbb{C}$.

Remark 3.14: By theorem 2.26 the L-function of a motive depends only on the Galois representations up to semisimplification. More precisely: if M and M' are motives with coefficients in K such that for all finite places $\mathfrak{p}$ of K the representations $M_{\mathfrak{p}}$ and $M'_{\mathfrak{p}}$ of $\mathrm{G}_{\mathbb{Q}}$ have isomorphic semisimplifications, then $\mathrm{L}(M,s) = \mathrm{L}(M',s)$ for all $s \in \mathbb{C}$ (where the L-functions are defined).

Conjecture 3.15: *The* L*-function* $\mathrm{L}(M,s)$ *has a meromorphic continuation to all of* $\mathbb{C}$ *and satisfies a functional equation*

$$\mathrm{L}_\infty(M,s)\mathrm{L}(M,s) = \varepsilon(M,s)\mathrm{L}_\infty(M^*(1),-s)\mathrm{L}(M^*(1),-s), \quad s \in \mathbb{C},$$

where the L_∞ *are certain "Euler factors at* ∞*" and* ε *is an* ε*-factor, which will be explained in section 3.4.*

We assume also that this conjecture is true. We will not need the functional equation in this work; in particular we will not need the precise formulas for the Euler factors at ∞. These are built from the Γ-function and we refer to [Ser70, §3.2, (25)] for their definition.

Example 3.16: (a) The L-function of the Tate motive is the shifted Riemann zeta function:

$$\mathrm{L}(\mathbb{Q}(1),s) = \zeta(s+1) \quad (s \in \mathbb{C} \smallsetminus \{0\}).$$

In general, it is easy to check that for any motive M we have $\mathrm{L}(M(n),s) = \mathrm{L}(M,s+n)$ for $s \in \mathbb{C}$.

(b) The L-function of the Artin motive is the Artin L-function for the representation ρ, if we define the latter using *geometric* Frobenii:

$$L(\mathcal{M}(\rho), s) = L(\rho, s) \quad (s \in \mathbb{C}).$$

(c) The L-function of the Dirichlet motive $L(\mathcal{M}(\chi^*), s)$ is the classical Dirichlet L-function $L(\chi, s)$:

$$L(\mathcal{M}(\chi^*), s) = L(\chi, s) \quad (s \in \mathbb{C}).$$

This perhaps confusing formula comes from our normalisation of class field theory: the isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q}) \cong (\mathbb{Z}/f)^\times$ identifies each prime $\ell \in (\mathbb{Z}/f)^\times$ with an *arithmetic* Frobenius, whereas in the definition of the Euler factors of the motivic L-functions the *geometric* Frobenius is used; see also [DFG04, §1.1.3].

(d) For any number field F we have a corresponding motive which comes from the variety $\operatorname{Spec} F$ over $\mathbb{Q}$. Its L-function is the Dedekind zeta function ζ_F attached to F. For details see again [DFG04, §1.1.3].

Assume that F is Galois over $\mathbb{Q}$. Using the Artin formalism (i. e. the inductivity of Artin L-functions, see [Del73, Prop. 3.8]), we know that ζ_F is the Artin L-function of the regular representation of $\mathrm{Gal}(F/\mathbb{Q})$. This representation can be decomposed into irreducible ones. More precisely, if $\rho_1, \dots, \rho_k$ are all irreducible representations of $\mathrm{Gal}(F/\mathbb{Q})$ on $\mathbb{C}$-vector spaces, then the regular representation is isomorphic to $\rho_1^{\dim \rho_1} \oplus \cdots \oplus \rho_k^{\dim \rho_k}$. The L-function can then be written as a product of Artin L-functions

$$\zeta_K(s) = \prod_{i=1}^{k} L(\rho_i^{\dim \rho_i}, s).$$

This product representation is the manifestation on the L-functions side of the fact that the motive $\operatorname{Spec} F$ decomposes into irreducible pieces. This illustrates the usefulness of the concept of motives: it really provides a finer look at varieties, since such a decomposition does not exist at the level of varieties.

A similar phenomenon also happens with modular forms: using modular curves one can construct for each $N \geq 4$ and $k \geq 2$ a so-called *Kuga-Sato variety* $\mathrm{KS}(N, k)$, which is a projective smooth variety over $\mathbb{Q}$. The motive it defines decomposes into pieces belonging to the newforms of level N and weight k. We will study this in detail in section II.5.

Definition 3.17: We define the *period map*

$$\mathrm{cp}_\infty^+ \colon M_{\mathrm{B}}^+ \underset{K}{\otimes} \mathbb{C} \longrightarrow \mathrm{t}_M \underset{K}{\otimes} \mathbb{C}$$

to be the composition

$$M_{\mathrm{B}}^+ \underset{K}{\otimes} \mathbb{C} \hookrightarrow M_{\mathrm{B}} \underset{K}{\otimes} \mathbb{C} \xrightarrow{\mathrm{cp}_\infty} M_{\mathrm{dR}} \underset{K}{\otimes} \mathbb{C} \twoheadrightarrow \mathrm{t}_M \underset{K}{\otimes} \mathbb{C}.$$

If cp_∞^+ is an isomorphism, M is called *critical*.

See [Coa91, Lem. 3] for a proof why this definition of criticality is equivalent to the one given in [Del79].

Definition 3.18: Choose K-bases γ of M_{B}^{+} and δ of t_M. Then define

$$\Omega_\infty^{\gamma,\delta}(M) := \det_{\gamma,\delta}(\mathrm{cp}_\infty^{+}) \in \mathbb{C}.$$

This is the *complex period* of the motive M. Obviously, if M is critical, then $\Omega_\infty^{\gamma,\delta}(M) \in \mathbb{C}^\times$.

Deligne's conjecture is then the following:

Conjecture 3.19 (Deligne)**:** *Assume the conjectures 3.12 and 3.15. If M is critical, then*

$$\frac{\mathrm{L}(M,0)}{\Omega_\infty^{\gamma,\delta}(M)} \in K.$$

Obviously the truth of this conjecture does not depend on the choice of γ and δ.

3.4. ε-factors and ε-isomorphisms

We briefly recall ε-factors and ε-isomorphisms. The former appear in the conjectural functional equation for L-functions of a motive M with coefficients in a number field K, see conjecture 3.15. The expression $\varepsilon(M,s)$ appearing there is a product of local expressions $\varepsilon_v(M,s)$ for each place v of $\mathbb{Q}$ which roughly arise as follows. In the rank 1 case, the L-functions of motives are covered by those of Hecke characters (by Class Field Theory and lemma 2.19 (a)), and for these the functional equation is known to hold. Tate's thesis shows that in this situation there is a canonical way to define the expressions $\varepsilon_v(M,s)$ to make the functional equation hold. If the rank of M is now larger than 1, one needs a result due to Langlands and Deligne showing that there is a unique way to extend them to higher-dimensional objects such that some reasonable properties (like inductivity) are fulfilled and one still has Tate's original ε-factors in the one-dimensional situation (see [Del73, Thm. 4.1]).

The higher-dimensional objects alluded to above are representations of Weil groups (together some other data). We recall the definition of the Weil group of $\mathbb{Q}_p$ for a prime p, which is the only one we need. If $v\colon\ \mathrm{G}_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p) \xrightarrow{\sim} \widehat{\mathbb{Z}}$ is the canonical map, where the last isomorphism sends an *arithmetic* Frobenius to 1, then the *Weil group* is defined as $\mathrm{W}_{\mathbb{Q}_p} := v^{-1}(\mathbb{Z})$. There is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{I}_p \arrow[r] \arrow[d, equal] & \mathrm{W}_{\mathbb{Q}_p} \arrow[r, "v"] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathrm{I}_p \arrow[r] & \mathrm{G}_{\mathbb{Q}_p} \arrow[r, "v"] & \widehat{\mathbb{Z}} \arrow[r] & 0.
\end{tikzcd}$$

The Weil group is considered as a topological group not with the subspace topology, but by requiring the inertia group I_p, which should still carry its profinite topology, to be open. We choose an element $\tau \in \mathrm{W}_{\mathbb{Q}_p}$ such that $v(\tau) = 1$ and fix it for the following.

Let E be a field of characteristic 0 containing all p-power roots of unity and let W be a finite-dimensional E-vector space with a continuous action of the Weil group $\mathrm{W}_{\mathbb{Q}_p}$ (where W is endowed with the discrete topology). Further let $\psi\colon \mathbb{Q}_p \longrightarrow E^\times$ be a continuous character with open kernel and μ a Haar measure on $\mathbb{Q}_p$. The result of Deligne attaches to such a set of data a scalar

$$\varepsilon(W,\psi,\mu) \in E^\times$$

called the *ε-factor* attached to (W, ψ, μ). We will always fix the Haar measure to be the one which gives $\mathbb{Z}_p$ the measure 1 and omit it from the notation.

Definition 3.20: For a finite-dimensional E-vector space W with a continuous action of $\mathrm{W}_{\mathbb{Q}_p}$, let

$$\varepsilon(W, \psi) := \varepsilon(W, \psi, \mu)$$

be the ε-factor as defined in [Del73] with ψ and μ as above. If $\rho\colon \mathrm{W}_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}_E(W)$ is the homomorphism describing the action, then we also write $\varepsilon(\rho, \psi)$ instead. If the character ψ is clear from the context we may omit it.

To define the local ε-factors of the motive M, one needs the following more general concept.

Definition 3.21: (a) A *Weil-Deligne representation* of $\mathrm{W}_{\mathbb{Q}_p}$ on an E-vector space W is a pair (ρ, N) consisting of a continuous homomorphism $\rho\colon \mathrm{W}_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}_E(W)$ and $N \in \mathrm{End}_E(W)$ such that

$$\rho(\tau^{-1})N\rho(\tau) = p^{-1}N.$$

(b) For a Weil-Deligne representation (ρ, N) and a character ψ as before we define its ε-factor as

$$\varepsilon_{\mathrm{WD}}(W, \psi) := \varepsilon(\rho, \psi) \det(-\tau^{-1}, W^{\mathrm{I}_p}/(\ker N)^{\mathrm{I}_p}),$$

where $\varepsilon(\rho, \psi)$ is the ε-factor from definition 3.20.

One can also define a Weil group at infinity $\mathrm{W}_{\mathbb{R}}$ and corresponding Weil-Deligne representations, but we will not need this. If now v is a place of $\mathbb{Q}$, then one can produce a Weil-Deligne representation of the corresponding Weil group out of M, and the local ε-factor $\varepsilon_v(M, s)$ is then defined to be the one from definition 3.21 (b) for this representation.[25]

We will not need the archimedean ε-factors, for which we refer to [Del73, §5.B], and thus let $v = p$ be a prime. Then one can get a Weil-Deligne representation of $\mathrm{W}_{\mathbb{Q}_p}$ out of M in various ways. Fix a finite place λ of K. If $p \nmid \lambda$ there is a classical way to produce a Weil-Deligne representation of $\mathrm{W}_{\mathbb{Q}_p}$ out of the $\mathrm{G}_{\mathbb{Q}_p}$-representation M_λ, due to Grothendieck. While this is important in the theory of ε-factors in general, we will not use this definition; see [Tat79, §4.1–2, esp. §4.1.6, §4.2.4] for details. Instead, using p-adic Hodge theory one can also produce a Weil-Deligne representation out of M_λ if $p \mid \lambda$, which we describe below. Similarly as for the Euler factor in the L-function in conjecture 3.12, it is conjectured that the ε-factor from definition 3.21 (b) does not depend on the choice of λ.

To describe the Weil-Deligne representation coming from the λ-adic realisation ($\lambda \mid p$) we do not need that the representation comes from a motive, and in fact we do not need a representation at all – the definition works for (φ, Γ)-modules, for which it was introduced by Nakamura [Nak15, §3.3] generalising [FK06, §3.3.4]. Since we will need this later only for true representations, we omit discussing the more general case of (φ, Γ)-modules, but one should keep in mind that the theory works essentially the same also in this situation.

Construction 3.22: Let L be a finite extension of $\mathbb{Q}_p$, fix an embedding $\iota_L\colon L \hookrightarrow \overline{\mathbb{Q}}_p$ and let V be a de Rham representation of $\mathrm{G}_{\mathbb{Q}_p}$ with coefficients in L. Let further $E = \overline{\mathbb{Q}}_p$. We construct a Weil-Deligne representation with coefficients in $\overline{\mathbb{Q}}_p$ out of $\mathrm{D}_{\mathrm{pst}}(V)$ (which is

[25] More precisely, this is $\varepsilon_v(M, 0)$, see [Tat79, §4.1.6] for the formula for general $s \in \mathbb{C}$. Also one needs an embedding of E into $\mathbb{C}$ to consider $\varepsilon_v(M, s)$ as complex numbers.

a $\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} L$-module). Via the embedding $\mathbb{Q}_p^{\mathrm{nr}} \hookrightarrow \overline{\mathbb{Q}}_p$ and our fixed ι_L, we get a map $\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} L \longrightarrow \overline{\mathbb{Q}}_p$. If we tensor $\mathrm{D}_{\mathrm{pst}}(V)$ with $\overline{\mathbb{Q}}_p$ along this map, we get a $\overline{\mathbb{Q}}_p$-vector space W_{ι_L}, carrying a semilinear action of $\mathrm{G}_{\mathbb{Q}_p}$ coming from the one on $\mathrm{D}_{\mathrm{pst}}(V)$. We need a $\overline{\mathbb{Q}}_p$-linear action of $\mathrm{W}_{\mathbb{Q}_p} \subseteq \mathrm{G}_{\mathbb{Q}_p}$ on W_{ι_L}. To obtain it, we use the endomorphism φ_{cris} of W_{ι_L} coming from φ_{cris} on B_{st}, which is also semilinear. If we define an action of $\mathrm{W}_{\mathbb{Q}_p}$ on W_{ι_L}, denoted by "$\bullet$" to distinguish it from the original action of $\mathrm{G}_{\mathbb{Q}_p}$, by

$$\sigma \bullet x := \sigma(x)\varphi_{\mathrm{cris}}^{-v(\sigma)}(x) \quad (\sigma \in \mathrm{W}_{\mathbb{Q}_p}),$$

then one can then check that this action is linear. From remark 2.23 it follows easily that a finite index subgroup of the inertia group acts trivially, hence by the definition of the topology on $\mathrm{W}_{\mathbb{Q}_p}$ the action on W_{ι_L} is continuous. On the other hand, the endomorphism N of $\mathrm{D}_{\mathrm{pst}}(V)$ induces an endomorphism N of W_{ι_L} satisfying the relation in definition 3.21 (a). So W_{ι_L} is a Weil-Deligne representation of $\mathrm{W}_{\mathbb{Q}_p}$.

Being now in the situation $E = \overline{\mathbb{Q}}_p$, we can define a character $\psi\colon \mathbb{Q}_p \longrightarrow \overline{\mathbb{Q}}_p^\times$ as the unique one with kernel $\mathbb{Z}_p$ satisfying $\psi(p^{-n}) = \xi_n$ for all $n \geq 0$, where $\xi = (\xi_n)_n$ is our fixed system of p-power roots of unity. From now on we only use this character to define ε-factors.

Definition 3.23: For a de Rham representation V, define

$$\varepsilon(V, \iota_L) := \varepsilon_{\mathrm{WD}}(W_{\iota_L}) \in \overline{\mathbb{Q}}_p^\times,$$

where $\varepsilon_{\mathrm{WD}}(-)$ on the right hand side is from definition 3.21 (b).

We may sometimes drop ι_L from the notation but one should keep in mind that ε depends on this choice.

Lemma 3.24: *We have*

$$\varepsilon(V, \iota_L) = \varepsilon(W_{\iota_L})\iota_L(\det(-\varphi_{\mathrm{cris}}, \mathrm{D}_{\mathrm{st}}(V)/\mathrm{D}_{\mathrm{cris}}(V)))$$

where $\varepsilon(W_{\iota_L})$ is as in definition 3.20.

Proof: Since $\mathrm{B}_{\mathrm{cris}}$ is the subring of B_{st} which is the kernel of N, and consequently $\mathrm{D}_{\mathrm{cris}}(-) = \mathrm{D}_{\mathrm{st}}(-)^{N=0}$, it suffices to check that $\mathrm{D}_{\mathrm{pst}}(V)^{\mathrm{I}_p} = \mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{st}}(V)$.

By theorem 2.22 we can choose a finite Galois extension $F/\mathbb{Q}_p$ such that $V|_{\mathrm{G}_F}$ is semistable, and we then have $\mathrm{D}_{\mathrm{pst}}(V)^{\mathrm{I}_p} = \mathbb{Q}_p^{\mathrm{nr}} \otimes_{F_0} \mathrm{D}_{\mathrm{st},F}(V)^{\mathrm{I}_p}$ by remark 2.23. There is an obvious injection

$$F_0 \underset{\mathbb{Q}_p}{\otimes} \mathrm{D}_{\mathrm{st}}(V) \hookrightarrow \mathrm{D}_{\mathrm{st},F}(V)^{\mathrm{I}_p}. \tag{$*$}$$

Consider the sequence of maps

$$\hat{\mathbb{Q}}_p^{\mathrm{nr}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{D}_{\mathrm{st}}(V) \hookrightarrow \hat{\mathbb{Q}}_p^{\mathrm{nr}} \underset{F_0}{\otimes} \mathrm{D}_{\mathrm{st},F}(V)^{\mathrm{I}_p} = \big(\hat{\mathbb{Q}}_p^{\mathrm{nr}} \underset{F_0}{\otimes} (\mathrm{B}_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{G}_F}\big)^{\mathrm{I}_p}$$
$$\xhookrightarrow{(+)} (\mathrm{B}_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} V)^{\mathrm{I}_p} \xrightarrow{\sim} \hat{\mathbb{Q}}_p^{\mathrm{nr}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{D}_{\mathrm{st}}(V).$$

Here the map labelled (+) is injective since it is a restriction of the map α_V from theorem 2.15 and the last isomorphism comes from proposition 2.17. It follows that all maps in the above composition are isomorphisms and hence the F_0-vector spaces in $(*)$ have the same dimension, completing the proof. ☐

In [Nak15], the ε-factor of a de Rham (φ, Γ)-module is defined using the formula from lemma 3.24. This is different from the definition given by Fukaya and Kato, which does not contain the second factor, but it should be the correct one since it is multiplicative in short exact sequences, while the one by Fukaya and Kato is not, as Nakamura explains in [Nak15, Rem. 3.6], and is moreover consistent with the one "away from p" from [Tat79]. Note however that if V is one-dimensional, then the two definitions coincide since one-dimensional representations are semistable if and only if they are crystalline (see lemma 2.19 (b)).

We will later apply this only to one-dimensional de Rham Galois representations. By lemma 2.19 we have an explicit description of de Rham characters. In this case, using some well-known formal properties of ε-factors and of the functor $\mathrm{D}_{\mathrm{pst}}$, the ε-factors can be expressed quite explicitly, as the next proposition shows.

Proposition 3.25: *Let $\chi\colon (\mathbb{Z}/p^m)^\times \longrightarrow \mathcal{O}^\times$ be a primitive Dirichlet character viewed as a finitely ramified character of $\mathrm{G}_{\mathbb{Q}_p}$, let ψ be an unramified character of $\mathrm{G}_{\mathbb{Q}_p}$ and $n \in \mathbb{Z}$. Let $u = \iota_L(\psi(\mathrm{Frob}_p))$. Then*

$$\varepsilon(\chi \otimes \psi \otimes \kappa_{\mathrm{cyc}}^n, \iota_L) = u^m p^{-nm} \mathrm{G}(\chi^*, \iota_L).$$

Proof: [LVZ15, Prop. 2.3.3] □

From the ε-factors for de Rham representations one can build ε-de Rham-isomorphisms, which play a central role in Fukaya's and Kato's local ε-conjecture in [FK06, §3]. This conjecture concerns the question whether the Equivariant Tamagawa Number Conjecture (ETNC) for the motive $M^*(1)$ can be deduced from the ETNC for a motive M. It is shown that this is the case if certain "equivariant ε-isomorphisms" exist for each prime, and the content of the conjecture is this existence. We are only interested in the prime p, where the equivariant ε-isomorphism is characterised by the property that it interpolates the aforementioned ε-de Rham-isomorphisms. For us only the latter ones will be important since they also appear in the definition of p-adic periods, we will not need equivariant ε-isomorphisms. For a detailed explanation of these and the conjecture see [Ven07, §5].

To explain ε-de Rham-isomorphisms, let again V be a de Rham representation of $\mathrm{G}_{\mathbb{Q}_p}$ of dimension n with coefficients in L. We have the canonical map α_V from theorem 2.15, which is an isomorphism of $\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-modules

$$\alpha_V\colon \mathrm{B}_{\mathrm{dR}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{D}_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathrm{B}_{\mathrm{dR}} \underset{\mathbb{Q}_p}{\otimes} V$$

to which we apply the determinant functor to obtain

$$\mathrm{B}_{\mathrm{dR}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{Det}_L(\mathrm{D}_{\mathrm{dR}}(V)) \longrightarrow \mathrm{B}_{\mathrm{dR}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{Det}_L(V). \tag{3.1}$$

The definition of the p-adic period involves the so-called ε-isomorphism for V, a modification of the above isomorphism which we now describe.

For each choice of an embedding $\iota_L\colon L \hookrightarrow \overline{\mathbb{Q}}_p$, we have $\varepsilon(V, \iota_L)$ as defined in definition 3.23. Since

$$\overline{\mathbb{Q}}_p \underset{\mathbb{Q}_p}{\otimes} L = \prod_{\iota_L} \overline{\mathbb{Q}}_p,$$

where ι_L ranges over all embeddings $L \hookrightarrow \overline{\mathbb{Q}}_p$, the collection of the $\varepsilon(V, \iota_L)$ for all ι_L defines an element

$$\varepsilon_L(V) \in (\overline{\mathbb{Q}}_p \underset{\mathbb{Q}_p}{\otimes} L)^\times.$$

Recall further the Hodge invariant

$$\mathrm{t_H}(V) = \sum_{r \in \mathbb{Z}} r \dim_L(\mathrm{gr}^r \mathrm{D_{dR}}(V)).$$

Definition 3.26: We define

$$\varepsilon_{\mathrm{dR}}(V) \in \mathrm{Hom}(\mathrm{B_{dR}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{Det}_L(\mathrm{D_{dR}}(V)), \mathrm{B_{dR}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{Det}_L(V))$$

to be $\mathrm{t_{dR}^{-t_H(V)}}\varepsilon_L(V)$ times the isomorphism (3.1), using the multiplication from definition 2.6.

This element of course depends on ξ, but since we have fixed ξ, we do not include it into the notation, in contrast to [FK06].

Proposition 3.27 (Fukaya/Kato)**:** *We have in fact*

$$\varepsilon_{\mathrm{dR}}(V) \in \mathrm{Hom}(\tilde{L} \underset{L}{\otimes} \mathrm{Det}_L(\mathrm{D_{dR}}(V)), \tilde{L} \underset{L}{\otimes} \mathrm{Det}_L(V)).$$

Proof: [FK06, Prop. 3.3.5] □

Remark 3.28: At first sight the definition of $\varepsilon_{\mathrm{dR}}(V)$ in [FK06] looks a bit different from definition 3.26. They put $I = \mathrm{Isom}(\mathrm{Det}_L(\mathrm{D_{dR}}(V)), \mathrm{Det}_L(V))$ and regard the isomorphism (3.1) as an element of $(I \times (\mathrm{B_{dR}} \otimes_{\mathbb{Q}_p} L)^\times)/L^\times$. Then they define a scalar multiplication on this set in a way analogous to definition 2.6. They do not say how the action of $L^\times$ on $I \times (\mathrm{B_{dR}} \otimes_{\mathbb{Q}_p} L)^\times$ looks like, but if we assume it is as in lemma 2.8 with the K there being L, the V there being $\mathrm{D_{dR}}(V)$, the W there being V and the B there being $\mathrm{B_{dR}} \otimes_{\mathbb{Q}_p} L$, then we get a bijection

$$(I \times (\mathrm{B_{dR}} \otimes_{\mathbb{Q}_p} L)^\times) \Big/ L^\times \cong \mathrm{Hom}(\mathrm{B_{dR}} \underset{L}{\otimes} \mathrm{Det}_L(\mathrm{D_{dR}}(V)), \mathrm{B_{dR}} \underset{L}{\otimes} \mathrm{Det}_L(V)).$$

This bijection is compatible with our definition of the scalar multiplication, so our definition is in fact the same as the one in [FK06]. To obtain proposition 3.27 above, on has to use lemma 2.8 again, with the K, V and W there as before but now with the B there being $\tilde{L}$. Since $\varepsilon_{\mathrm{dR}}(V)$ is afterwards used in the form given here, we prefer this to viewing it as an element in $(I \times \tilde{L}^\times)/L^\times$.

3.5. p-adic periods of motives after Fukaya and Kato

3.5.1. The Dabrowski-Panchishkin condition

Fix a number field K, a critical motive with coefficients in K, an embedding $K \hookrightarrow \mathbb{Q}$ (or equivalently a place $\mathfrak{p} \mid p$ of K) and put $L := K_{\mathfrak{p}}$.

Condition 3.29: (a) We say that the motive M satisfies the *strong Dabrowski-Panchishkin condition* at $\mathfrak{p}$ if there is a subspace $M_\mathfrak{p}^{\mathrm{DP}} \subseteq M_\mathfrak{p}$ stable under the action of $\mathrm{G}_{\mathbb{Q}_p}$ such that the inclusion $M_\mathfrak{p}^{\mathrm{DP}} \subseteq M_\mathfrak{p}$ induces an isomorphism

$$\mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}^{\mathrm{DP}}) \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}) \Big/ \operatorname{fil}^0 \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p})$$

of L-vector spaces. See [FK06, §4.2.3, (C2)].

(b) We say that the motive M satisfies the *weak Dabrowski-Panchishkin condition* at $\mathfrak{p}$ if the (φ, Γ)-module $D := \mathrm{D}_{\mathrm{rig}}^{\dagger}(M_\mathfrak{p})$ contains a sub-(φ, Γ)-module D^{DP} such that the inclusion $D^{\mathrm{DP}} \subseteq D$ induces an isomorphism

$$\mathrm{D}_{\mathrm{dR}}(D^{\mathrm{DP}}) \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}) \Big/ \operatorname{fil}^0 \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p})$$

of L-vector spaces.

If M satisfies the strong Dabrowski-Panchishkin condition, then the subspace $M_\mathfrak{p}^{\mathrm{DP}}$ is unique: it is the largest $\mathrm{G}_{\mathbb{Q}_p}$-stable subspace of $M_\mathfrak{p}$ whose $\operatorname{fil}^0 \mathrm{D}_{\mathrm{dR}}$ vanishes. Analogously if M satisfies the weak Dabrowski-Panchishkin condition, then D^{DP} is unique.

From theorems 2.36 and 2.37 it is clear that the strong Dabrowski-Panchishkin condition implies the weak Dabrowski-Panchishkin condition. Since the (φ, Γ)-module D^{DP} in condition 3.29 (b) does not need to be étale, the weak Dabrowski-Panchishkin condition is really weaker than the strong one.

Lemma 3.30: *Assume that M is critical and let $\rho\colon G \longrightarrow \mathrm{GL}_r(K')$ be a representation with coefficients in a finite extension K' of K and $n \in \mathbb{N}$ such that $M(\rho)(n)$ is still critical.*

(a) If M satisfies the strong Dabrowski-Panchishkin condition, then $M(\rho)(n)$ still satisfies the strong Dabrowski-Panchishkin condition and

$$M(\rho)(n)_\mathfrak{p}^{\mathrm{DP}} = M_\mathfrak{p}^{\mathrm{DP}} \otimes \mathcal{M}(\rho) \otimes K(n)_\mathfrak{p}.$$

(b) If M satisfies the weak Dabrowski-Panchishkin condition, then $M(\rho)(n)$ still satisfies the weak Dabrowski-Panchishkin condition and

$$\mathrm{D}_{\mathrm{rig}}^{\dagger}(M(\rho)(n)_\mathfrak{p})^{\mathrm{DP}} = \mathrm{D}_{\mathrm{rig}}^{\dagger}(M_\mathfrak{p})^{\mathrm{DP}} \otimes \mathrm{D}_{\mathrm{rig}}^{\dagger}(\mathcal{M}(\rho)) \otimes \mathrm{D}_{\mathrm{rig}}^{\dagger}(K(n)_\mathfrak{p}).$$

Proof: We give the proof only in the case of the strong Dabrowski-Panchishkin condition, the weak case works similarly.

We first show this for $n = 0$. For simplicity of notation, we assume without loss of generality that ρ has coefficients in L. Since tensoring a motive with $\mathcal{M}(\rho)$ does not change the Hodge filtration, we have $\operatorname{fil}^0 \mathrm{D}_{\mathrm{dR}}(M(\rho)_\mathfrak{p}) = \operatorname{fil}^0 \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}) \otimes_L \mathcal{M}(\rho)_\mathfrak{p}$. This shows the statement for $n = 0$.

Then we assume that ρ is trivial, which suffices to complete the proof. If M and $M(n)$ are both critical, then by dimension counting one easily sees that $\operatorname{fil}^0 M_{\mathrm{dR}} = \operatorname{fil}^n M_{\mathrm{dR}}$, using the definition of criticality. This implies $\operatorname{fil}^0 M(n)_{\mathrm{dR}} = \operatorname{fil}^0 M_{\mathrm{dR}} \otimes_K K(n)_{\mathrm{dR}}$ and thus $\operatorname{fil}^0 \mathrm{D}_{\mathrm{dR}}(M(n)_\mathfrak{p}) = \operatorname{fil}^0 \mathrm{D}_{\mathrm{dR}}(M_\mathfrak{p}) \otimes_L K(n)_\mathfrak{p}$. This completes the proof. □

3.5.2. The p-adic period

We first introduce the following notation following [FK06, §3.1.1].

Definition 3.31: (a) A ring Λ is called *p-adic*[26] if it contains a two-sided ideal I such that for each $n \in \mathbb{N}$ the quotient Λ/I^n is finite of p-power order and

$$\Lambda \xrightarrow{\sim} \varprojlim_n \Lambda\big/ I^n .$$

For such a ring, we denote by J_Λ its Jacobson radical. We denote by *p-$\mathcal{A}dic\mathcal{R}ings$* the category of p-adic rings, where a morphism $\varphi\colon \Lambda_1 \longrightarrow \Lambda_2$ should be a ring homomorphism such that $\varphi(J_{\Lambda_1}) \subseteq J_{\Lambda_2}$.

(b) For a p-adic ring Λ, we define

$$\tilde{\Lambda} := \varprojlim_n \left(\mathrm{W}(\overline{\mathbb{F}}_p) \underset{\mathbb{Z}_p}{\otimes} \Lambda\big/ J_\Lambda^n \right).$$

(c) If L is a ring such that $L = \Lambda[\frac{1}{p}]$ for a p-adic ring Λ, then we write further

$$\tilde{L} := \tilde{\Lambda}[\tfrac{1}{p}].$$

It is easy to check that definition 3.31 (b) defines a functor

$$\widetilde{(\cdot)}\colon p\text{-}\mathcal{A}dic\mathcal{R}ings \longrightarrow \mathrm{W}(\overline{\mathbb{F}}_p)\text{-}\mathcal{A}lg.$$

Further note that if L is a finite extension of $\mathbb{Q}_p$, we have $\tilde{L} = \hat{\mathbb{Q}}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} L$.

Now let K, L and M be as before and assume that M satisfies the strong Dabrowski-Panchishkin condition. We denote the isomorphism given by the condition by dp.

The definition of the p-adic period relies on a fixed choice of an isomorphism[27]

$$\beta\colon \tilde{L} \underset{L}{\otimes} M_{\mathrm{p}}^{+} \xrightarrow{\sim} \tilde{L} \underset{L}{\otimes} M_{\mathrm{p}}^{\mathrm{DP}},$$

so assume that such an isomorphism β exists and fix it for the following.

The definition of the p-adic period further uses the ε-isomorphism $\varepsilon_{\mathrm{dR}} = \varepsilon_{\mathrm{dR}}(M_{\mathrm{p}}^{\mathrm{DP}})$ attached to[28] $M_{\mathrm{p}}^{\mathrm{DP}}$ as defined in definition 3.26 and the comparison isomorphisms for the motive M. Finally we choose K-bases γ of M_{B}^{+} and δ of t_M, which gives us

$$\mathrm{is}_{\delta,\gamma}: \mathrm{t}_M \xrightarrow{\sim} M_{\mathrm{B}}^{+}.$$

[26] This is the condition called $(*)$ in [FK06, §1.4.1]. In other texts, e. g. [Bar11, Def. 1.11], a ring with this property is simply called *adic*, but since the notion depends on p we use the term *p-adic* instead.

[27] Fukaya and Kato choose β in [FK06, §4.1.3] as an isomorphism of determinant objects $\tilde{L}\otimes_L \mathrm{Det}_L(M_{\mathrm{p}}^{+}) \xrightarrow{\sim} \tilde{L}\otimes_L \mathrm{Det}_L(M_{\mathrm{p}}^{\mathrm{DP}})$. Of course our choice of $\beta\colon \tilde{L}\otimes_L M_{\mathrm{p}}^{+} \xrightarrow{\sim} \tilde{L}\otimes_L M_{\mathrm{p}}^{\mathrm{DP}}$ induces such an isomorphism of determinant objects and the converse is *not* true, so we are requiring here more than Fukaya and Kato. In our application we will always choose β directly as an isomorphism between the vector spaces, so for simplicity we decided to explain the theory only in this case. There is not much generality lost conceptually, but some arguments become a little simpler this way.

[28] Note that $M_p \otimes_{K_p} K_{\mathrm{p}} = M_{\mathrm{p}}$. This is why the p-adic representation used in [FK06, §4.1.10] is in fact (isomorphic to) M_{p}.

Definition 3.32: The isomorphisms listed above induce isomorphisms of determinant objects

$$\begin{aligned}
\varepsilon_{\mathrm{dR}}^{-1}\colon \tilde{L} \underset{L}{\otimes} \mathrm{Det}_L(M_{\mathrm{p}}^{\mathrm{DP}}) &\longrightarrow \tilde{L} \underset{L}{\otimes} \mathrm{Det}_L(\mathrm{D}_{\mathrm{dR}}(M_{\mathrm{p}}^{\mathrm{DP}})) \\
\mathrm{dp}\colon \mathrm{Det}_L(\mathrm{D}_{\mathrm{dR}}(M_p^{\mathrm{DP}})) &\longrightarrow \mathrm{Det}_L\left(\mathrm{D}_{\mathrm{dR}}(M_p)\middle/\mathrm{fil}^0\,\mathrm{D}_{\mathrm{dR}}(M_p)\right) \\
\mathrm{cp}_{\mathrm{dR}}\colon \mathrm{Det}_L\left(\mathrm{D}_{\mathrm{dR}}(M_{\mathrm{p}})\middle/\mathrm{fil}^0\,\mathrm{D}_{\mathrm{dR}}(M_{\mathrm{p}})\right) &\longrightarrow L \underset{K}{\otimes} \mathrm{Det}_K(\mathrm{t}_M) \\
\mathrm{is}_{\delta,\gamma}\colon \mathrm{Det}_K(\mathrm{t}_M) &\longrightarrow \mathrm{Det}_K(M_{\mathrm{B}}^+) \\
\mathrm{cp}_{\text{ét}}\colon L \underset{K}{\otimes} \mathrm{Det}_K(M_{\mathrm{B}}^+) &\longrightarrow \mathrm{Det}_L(M_{\mathrm{p}}^+) \\
\beta\colon \tilde{L} \underset{L}{\otimes} \mathrm{Det}_L(M_{\mathrm{p}}^+) &\longrightarrow \tilde{L} \underset{L}{\otimes} \mathrm{Det}_L(M_{\mathrm{p}}^{\mathrm{DP}}).
\end{aligned}$$

Tensoring these with $\tilde{L}$, multiplying all these isomorphisms together and applying $\det^\times$, this gives us an element $\tilde{\Omega} \in \tilde{L}^\times$ (see definition 2.4). Using the canonical map $\tilde{L} \longrightarrow \hat{L}^{\mathrm{nr}}$ induced by the embedding $L \hookrightarrow \overline{\mathbb{Q}}_p$, we obtain an element $\Omega_{\mathrm{p}}^{\gamma,\delta,\beta}(M) \in (\hat{L}^{\mathrm{nr}})^\times$ which is the *p-adic period attached to M*.

This definition (which is the original definition from [FK06]) is rather inexplicit and not so useful for calculating periods. First, the use of determinant functors is not very concrete. Second, during the process of the definition our fixed embedding $L \hookrightarrow \overline{\mathbb{Q}}_p$ is not used until the very end.

To change the second point, note that if we tensor β with $\hat{L}^{\mathrm{nr}}$ along the map $\tilde{L} \longrightarrow \hat{L}^{\mathrm{nr}}$ induced by the embedding $L \hookrightarrow \overline{\mathbb{Q}}_p$, we get an isomorphism

$$\beta\colon \hat{L}^{\mathrm{nr}} \underset{L}{\otimes} M_{\mathrm{p}}^+ \xrightarrow{\sim} \hat{L}^{\mathrm{nr}} \underset{L}{\otimes} M_{\mathrm{p}}^{\mathrm{DP}}.$$

Also $\varepsilon_{\mathrm{dR}}$ can be defined as an isomorphism over $\hat{L}^{\mathrm{nr}}$ instead of $\tilde{L}$. To explain this, we abbreviate $V = M_{\mathrm{p}}^{\mathrm{DP}}$ and come back to the isomorphism of $\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-modules

$$\alpha_V\colon \mathrm{B}_{\mathrm{dR}} \underset{\mathbb{Q}_p}{\otimes} \mathrm{D}_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathrm{B}_{\mathrm{dR}} \underset{\mathbb{Q}_p}{\otimes} V$$

and tensor it over $\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$ with B_{dR} along the map $\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L \longrightarrow \mathrm{B}_{\mathrm{dR}}$ induced by $L \hookrightarrow \overline{\mathbb{Q}}_p$. This gives us an isomorphism of B_{dR}-vector spaces

$$\alpha_V\colon \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathrm{D}_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} V.$$

We now modify it using just $\varepsilon(V, \iota_L)$ for our fixed embedding ι_L instead of $\varepsilon_L(V)$ to obtain $\varepsilon_{\mathrm{dR}}$. Using these definitions over $\hat{L}^{\mathrm{nr}}$ instead of $\tilde{L}$ directly gives us element in $(\hat{L}^{\mathrm{nr}})^\times$ after applying $\det^\times$, and it is clear that this element equals $\Omega_{\mathrm{p}}^{\gamma,\delta,\beta}(M)$.

In this way we can get rid of the second complication mentioned above. The first is then accomplished by the following proposition, which describes the period in a rather accessible way.

Proposition 3.33: *Denote by ϕ the composition*

$$\mathrm{B_{dR}} \underset{K}{\otimes} M_{\mathrm{B}}^{+} \xrightarrow{\mathrm{cp_{\acute{e}t}}} \mathrm{B_{dR}} \underset{L}{\otimes} M_{\mathrm{p}}^{+} \xrightarrow{\beta} \mathrm{B_{dR}} \underset{L}{\otimes} M_{\mathrm{p}}^{\mathrm{DP}} \xrightarrow{\alpha^{-1}_{M_{\mathrm{p}}^{\mathrm{DP}}}}$$

$$\xrightarrow{\alpha^{-1}_{M_{\mathrm{p}}^{\mathrm{DP}}}} \mathrm{B_{dR}} \underset{L}{\otimes} \mathrm{D_{dR}}(M_{\mathrm{p}}^{\mathrm{DP}}) \xrightarrow{\mathrm{dp}} \mathrm{B_{dR}} \underset{L}{\otimes} \left(\mathrm{D_{dR}}(M_{\mathrm{p}}) \middle/ \mathrm{fil}^0\, \mathrm{D_{dR}}(M_{\mathrm{p}})\right) \xrightarrow{\mathrm{cp_{dR}}} \mathrm{B_{dR}} \underset{K}{\otimes} \mathrm{t}_M .$$

Then

$$\Omega_{\mathrm{p}}^{\gamma,\delta,\beta}(M) = -\frac{\mathrm{t}_{\mathrm{dR}}^{\mathrm{t_H}(M_{\mathrm{p}}^{\mathrm{DP}})}}{\varepsilon(M_{\mathrm{p}}^{\mathrm{DP}}, \iota_L)} \det_{\gamma,\delta}(\phi) \in \mathrm{B}_{\mathrm{dR}}^{\times} .$$

In particular, the above expression lies in fact in $(\hat{L}^{\mathrm{nr}})^{\times}$.

Proof: If we denote, by abuse of notation, the scalar extensions of the morphisms in definition 3.32 to $\hat{L}^{\mathrm{nr}}$ by still the same symbol, then we have with the notation of definition 2.4 (with $n = 6$ there)

$$\Omega_{\mathrm{p}}^{\gamma,\delta,\beta}(M) = \det{}^{\times}(\varepsilon_{\mathrm{dR}}^{-1}, \mathrm{dp}, \mathrm{cp_{dR}}, \mathrm{is}_{\delta,\gamma}, \mathrm{cp_{\acute{e}t}}, \beta).$$

The statement then follows directly from lemmas 2.5 and 2.7, using the easy observation that

$$\det{}^{\times}(\varphi_i, \ldots \varphi_n, \varphi_1, \ldots, \varphi_{i-1}) = \det{}^{\times}(\varphi_1, \ldots, \varphi_n)$$

for any sequence $\varphi_1, \ldots, \varphi_n$ of isomorphisms between vector spaces and each $i \in \{1, \ldots, n-1\}$. ☐

Remark 3.34: The above proposition shows in particular that

$$\det_{\gamma,\delta}(\phi) \in \mathrm{t}_{\mathrm{dR}}^{\mathbb{Z}} \mathbb{C}_p \subseteq \mathrm{B_{dR}} .$$

Therefore, to compute the p-adic period we do not have to tensor up to $\mathrm{B_{dR}}$, it can already be computed over $\mathrm{B_{HT}}$. More precisely: let V and W be K-vector spaces with bases γ and δ, respectively, and let W be filtered. Let $\phi\colon \mathrm{B_{dR}} \otimes_K V \xrightarrow{\sim} \mathrm{B_{dR}} \otimes_K W$ be an isomorphism of filtered vector spaces such that $\det_{\gamma,\delta}(\phi) = \alpha \mathrm{t}_{\mathrm{dR}}^{h}$ with $\alpha \in \mathbb{C}_p$ and $h \in \mathbb{Z}$. We apply the functor gr to ϕ to obtain an isomorphism of graded vector spaces $\phi'\colon \mathrm{B_{HT}} \otimes_K V \xrightarrow{\sim} \mathrm{B_{HT}} \otimes_K \mathrm{gr}(W)$. If we use $\mathrm{t_{dR}}$ to identify $\mathrm{B_{HT}}$ with $\mathbb{C}_p[\mathrm{t_{dR}}, \mathrm{t}_{\mathrm{dR}}^{-1}]$, then by construction we know that still $\det_{\gamma,\delta}(\phi') = \alpha \mathrm{t}_{\mathrm{dR}}^{h}$.

Let us now assume that W is pure of weight $h \in \mathbb{Z}$ (i. e. $\mathrm{fil}^h W = W$, $\mathrm{fil}^{h+1} W = 0$), which is for example the case if W is one-dimensional. It then even suffices to tensor with $\mathbb{C}_p$. More precisely, we have then

$$\mathrm{B_{HT}} \underset{K}{\otimes} \mathrm{gr}(W) = \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_p(q-h) \underset{K}{\otimes} W$$

and since $1 \otimes \gamma \in \mathrm{B_{HT}} \otimes_K V$ lies in the weight 0 part, its image under ϕ' also lies in the weight 0 part, i. e. $\alpha t^h \otimes \delta \in \mathbb{C}_p(-h) \otimes_K W$. Therefore if we define $\phi'' := \mathrm{gr}^0(\phi)$, which is an isomorphism

$$\phi''\colon \mathbb{C}_p \underset{K}{\otimes} V \xrightarrow{\sim} \mathbb{C}_p(-h) \underset{K}{\otimes} W,$$

then we know that $\det_{\gamma,\delta}(\phi) = \det_{\gamma,\delta}(\phi'')$.

As mentioned before, the definition of the ε-constant of a de Rham representation in definition 3.23 also works for any de Rham (φ,Γ)-module (not only Galois representations), and the definition of ε-isomorphisms can also be extended to (φ,Γ)-modules, see [Nak15, §3.3]. In fact, given an isomorphism

$$\beta\colon \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} M_{\mathfrak{p}}^{+} \xrightarrow{\ \sim\ } \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathrm{D}_{\mathrm{dR}}(\mathrm{D}^{\dagger}_{\mathrm{rig}}(M_{\mathfrak{p}})^{\mathrm{DP}})$$

one can extend the definition of the p-adic period to motives satisfying only the weak Dabrowski-Panchishkin condition. If ϕ denotes the composition

$$\mathrm{B}_{\mathrm{dR}} \underset{K}{\otimes} M_{\mathrm{B}}^{+} \xrightarrow{\mathrm{cp}_{\text{ét}}} \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} M_{\mathfrak{p}}^{+} \xrightarrow{\beta}$$
$$\mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathrm{D}_{\mathrm{dR}}(\mathrm{D}^{\dagger}_{\mathrm{rig}}(M_{\mathfrak{p}})^{\mathrm{DP}}) \xrightarrow{\mathrm{dp}} \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \left(\mathrm{D}_{\mathrm{dR}}(M_{\mathfrak{p}}) \big/ \mathrm{fil}^0\, \mathrm{D}_{\mathrm{dR}}(M_{\mathfrak{p}})\right) \xrightarrow{\mathrm{cp}_{\mathrm{dR}}} \mathrm{B}_{\mathrm{dR}} \underset{K}{\otimes} \mathrm{t}_M.$$

then we have similarly as before

$$\Omega_{\mathfrak{p}}^{\gamma,\delta,\beta}(M) = -\frac{\mathrm{t}_{\mathrm{dR}}^{\mathrm{t_H}(\mathrm{D}_{\mathrm{dR}}(\mathrm{D}^{\dagger}_{\mathrm{rig}}(M_{\mathfrak{p}})^{\mathrm{DP}}))}}{\varepsilon(\mathrm{D}^{\dagger}_{\mathrm{rig}}(M_{\mathfrak{p}})^{\mathrm{DP}}, \iota_L)} \det_{\gamma,\delta}(\phi) \in \mathrm{B}_{\mathrm{dR}}^{\times}.$$

We will not use this in this work, it will only be mentioned in the final section IV.6.

3.6. Families of motives

We now define families of motives following and generalising [Bar11, §2.2]. However, in one aspect we deal with a special case, namely, we assume that the number field F there is just $\mathbb{Q}$. This simplifies the notation while not much is lost conceptually.

Let K be a number field, $\mathfrak{p}$ a finite place of K, $L = K_{\mathfrak{p}}$ the completion and $\mathcal{O} = \mathcal{O}_L$ be the ring of integers. We use the notations and results from section 2.5, so in particular we write $\mathcal{I}$ for the integral closure of $\mathcal{O}[\![T]\!]$ in a finite extension $\mathcal{K}$ of $\mathrm{Quot}(\mathcal{O}[\![T]\!])$ and A for an affinoid algebra over L.

Definition 3.35: A *p-adic (analytic resp. algebraic) family of motives* $(\Sigma, \mathcal{T}, \rho, (M(\phi))_{\phi\in\Sigma})$ consists of a p-adic (analytic resp. algebraic) family of Galois representations $\rho\colon \mathrm{G}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(\mathcal{T})$, a set Σ of (analytic resp. algebraic) specialisations and a motive $M(\phi)$ over K_ϕ for each $\phi \in \Sigma$ such that for each $\phi \in \Sigma$ the specialisation $\mathcal{T}_\phi$ is isomorphic to a $\mathrm{G}_{\mathbb{Q}}$-stable $\mathcal{O}_\phi$-lattice in the $\mathfrak{P}_\phi$-adic realisation $M(\phi)_{\mathfrak{P}_\phi}$ of $M(\phi)$ in the algebraic case, resp. $\mathcal{T}_\phi$ is isomorphic to the $\mathfrak{P}_\phi$-adic realisation $M(\phi)_{\mathfrak{P}_\phi}$ of $M(\phi)$ in the analytic case.

One often assumes that the set Σ is Zariski dense in $\mathrm{Spec}\,\mathcal{I}$ resp. dense in $\mathrm{Sp}\,A$, which is a reasonable assumption with regard on p-adic L-functions (see below).

Condition 3.36: Let $(\Sigma, \mathcal{T}, \rho, (M(\phi))_{\phi\in\Sigma})$ be an analytic family of motives.

(a) We say that the family satisfies the *strong Dabrowski-Panchishkin condition* at $\mathfrak{p}$ if every motive $M(\phi)$ satisfies the strong Dabrowski-Panchishkin condition at $\mathfrak{P}_\phi$ and moreover there is a free A-submodule $\mathcal{T}^{\mathrm{DP}}$ of $\mathcal{T}$ which is a direct summand and stable under $\mathrm{G}_{\mathbb{Q}_p}$ such that its image in $\mathcal{T}_\phi$, which we denote $\mathcal{T}_\phi^{\mathrm{DP}}$, is the subspace $M(\phi)_{\mathfrak{P}_\phi}^{\mathrm{DP}}$ from the strong Dabrowski-Panchishkin condition for the motive $M(\phi)$.

(b) We say that the family satisfies the *weak Dabrowski-Panchishkin condition* at $\mathfrak{p}$ if every motive $M(\phi)$ satisfies the weak Dabrowski-Panchishkin condition $\mathfrak{P}_\phi$ and moreover the (φ, Γ)-module $D := \mathrm{D}^\dagger_{\mathrm{rig}}(\mathcal{T})$ contains a sub-(φ, Γ)-module $\mathcal{T}^{\mathrm{DP}}$ such that the base change $\mathcal{T}^{\mathrm{DP}} \otimes_{A \hat{\otimes} \mathrm{B}^\dagger_{\mathrm{rig}}, \phi} (L_\phi \hat{\otimes} \mathrm{B}^\dagger_{\mathrm{rig}})$ is the sub-(φ, Γ)-module $\mathrm{D}^\dagger_{\mathrm{rig}}(M(\phi)_{\mathfrak{P}_\phi})^{\mathrm{DP}}$ from the weak Dabrowski-Panchishkin condition for the motive $M(\phi)$.

Now let $(\Sigma, \mathcal{T}, \rho, (M(\phi))_{\phi\in\Sigma})$ be an algebraic family of motives.

(c) We say that the algebraic family satisfies the strong resp. weak Dabrowski-Panchishkin condition at $\mathfrak{p}$ if the associated analytic family (see proposition 2.31) satisfies the strong resp. weak Dabrowski-Panchishkin condition.

The strong Dabrowski-Panchishkin condition for algebraic families is then the condition used in [Bar11, Cond. 2.12].

By theorems 2.36 and 2.37, it is again clear that the strong Dabrowski-Panchishkin condition implies the weak Dabrowski-Panchishkin condition for families of motives.

3.7. Conjectural p-adic L-functions

As mentioned in the introduction (page viii), the existence of p-adic L-functions interpolating complex L-values of motives can be deduced from the Equivariant Tamagawa Number Conjecture (ETNC). Since we do not need the ETNC in this work, we here just state the resulting existence statements for p-adic L-functions as conjectures.

Let K be a number field inside $\overline{\mathbb{Q}}$, $\mathfrak{p}$ the thereby fixed place of K, let $L = K_{\mathfrak{p}}$ be the completion and $\mathcal{O} = \mathcal{O}_L$ the ring of integers.

3.7.1. The isomorphism β

The p-adic L-functions constructed from the ETNC depend on a choice of a certain isomorphism β. We first collect some technical statements about this which we will need later.

We begin with the situation for a single motive. For this we fix the following data:

- An almost everywhere unramified Galois extension $F_\infty/\mathbb{Q}$ with Galois group G such that G has a topologically finitely generated pro-p open normal subgroup. The latter is the condition called $(**)$ in [FK06, §1.4.2]. We assume that outside p the extension $F_\infty/\mathbb{Q}$ is at most finitely ramified.[29] We further assume that F_∞ contains $\mathbb{Q}(\mu_{p^\infty})$, so that $\mathrm{G}_{\mathrm{cyc}}$ is a quotient of G. Write $\Lambda = \mathcal{O}[\![G]\!]$.

- A critical motive M over a number field K that satisfies the strong Dabrowski-Panchishkin condition at $\mathfrak{p}$. Let t be an $\mathcal{O}$-stable lattice in $M_{\mathfrak{p}}$ and put $t^{\mathrm{DP}} := t \cap M_{\mathfrak{p}}^{\mathrm{DP}}$.

- Define $T := \Lambda \otimes_{\mathcal{O}} t$ and $T^{\mathrm{DP}} := \Lambda \otimes_{\mathcal{O}} t^{\mathrm{DP}}$, see [FK06, §4.2.7]. Let $g \in \mathrm{G}_{\mathbb{Q}}$ act on T by $x \otimes y \longmapsto xg^{-1} \otimes gy$ and analogously on T^{DP}. Then fix an isomorphism of $\tilde{\Lambda}$-modules $\beta\colon \tilde{\Lambda} \otimes_\Lambda T^+ \xrightarrow{\sim} \tilde{\Lambda} \otimes_\Lambda T^{\mathrm{DP}}$. Such an isomorphism exists by [FK06, Lem. 4.2.8].[30]

[29] This assumption is for simplicity, it is not necessary for the theory. It implies that the set called Υ defined in [FK06, §4.2.13] is empty, which simplifies our discussion a little.

[30] Fukaya and Kato choose β as an isomorphism $\tilde{\Lambda} \otimes_\Lambda \mathrm{Det}_\Lambda(T^+) \xrightarrow{\sim} \tilde{\Lambda} \otimes_\Lambda \mathrm{Det}_\Lambda(T^{\mathrm{DP}})$. This is a little bit more general, but our choice suffices for our purposes and simplifies some arguments a little.

Lemma 3.37: *Let $\rho\colon G \longrightarrow \mathrm{GL}_r(K')$ be a representation with coefficients in a finite extension K' of K and $n \in \mathbb{N}$ such that $M(\rho)(n)$ is still critical. Then the fixed β induces canonically an isomorphism*

$$\beta(\rho, n)\colon \tilde{L} \underset{L}{\otimes} M(\rho)(n)^+_{\mathfrak{p}} \longrightarrow \tilde{L} \underset{L}{\otimes} M(\rho)(n)^{\mathrm{DP}}_{\mathfrak{p}}.$$

Proof: We first note that if Λ is a p-adic ring and $n \in \mathbb{N}$, then $\mathrm{M}_n(\Lambda)$ is also a p-adic ring and $\widetilde{\mathrm{M}_n(\Lambda)} = \mathrm{M}_n(\tilde{\Lambda})$. This is easy to see from the definitions. Hence by functoriality, ρ induces $\tilde{\rho}\colon \tilde{\Lambda} \longrightarrow \mathrm{M}_n(\tilde{L})$. In the same way the cyclotomic character induces $\tilde{\kappa}_{\mathrm{cyc}}\colon \tilde{\Lambda} \longrightarrow \mathrm{W}(\overline{\mathbb{F}}_p)$. Let

$$\psi = \tilde{\rho} \otimes \tilde{\kappa}_{\mathrm{cyc}}\colon \tilde{\Lambda} \longrightarrow \mathrm{M}_n(\tilde{L}) \underset{\mathrm{W}(\overline{\mathbb{F}}_p)}{\otimes} \mathrm{W}(\overline{\mathbb{F}}_p) = \mathrm{M}_n(\tilde{L}).$$

We regard $\tilde{L}^n$ as a $\tilde{\Lambda}$-module via ψ. Then we have

$$T \underset{\Lambda}{\otimes} \tilde{\Lambda} \underset{\tilde{\Lambda},\psi}{\otimes} \tilde{L}^n = t \underset{\mathcal{O}}{\otimes} \Lambda \underset{\Lambda}{\otimes} \tilde{\Lambda} \underset{\tilde{\Lambda},\psi}{\otimes} \tilde{L}^n = t \underset{\mathcal{O},\psi}{\otimes} \tilde{L}^n \cong M_{\mathfrak{p}} \underset{L,\psi}{\otimes} \tilde{L}^n = M(n)(\rho)_{\mathfrak{p}} \underset{L}{\otimes} \tilde{L}.$$

In the same way we see $T^{\mathrm{DP}} \otimes_\Lambda \tilde{\Lambda} \otimes_{\tilde{\Lambda},\psi} \tilde{L}^n \cong M(n)(\rho)^{\mathrm{DP}}_{\mathfrak{p}} \otimes_L \tilde{L}$. Here we used lemma 3.30.

Now we discuss T^+. Note that $\tilde{\Lambda}$ also carries a $\mathrm{G}_{\mathbb{R}}$-action. Let V be any $\tilde{\Lambda}$-module with an action of $\mathrm{G}_{\mathbb{R}}$. One can easily check the $V^\pm \otimes_{\tilde{\Lambda}} \tilde{\Lambda}^\mp = 0$. From this and lemma 3.10 (a), one can easily derive that $V \otimes_{\tilde{\Lambda}} \tilde{\Lambda}^\pm = V^\pm$. Similarly one sees that $\Lambda^\pm \otimes_\Lambda \tilde{\Lambda} = \tilde{\Lambda}^\pm$. Therefore

$$T^+ \underset{\Lambda}{\otimes} \tilde{\Lambda} \underset{\tilde{\Lambda}}{\otimes} V = (t^+ \underset{\mathcal{O}}{\otimes} \Lambda^+ \oplus t^- \underset{\mathcal{O}}{\otimes} \Lambda^-) \underset{\Lambda}{\otimes} \tilde{\Lambda} \underset{\tilde{\Lambda}}{\otimes} V = t^+ \underset{\mathcal{O}}{\otimes} V^+ \oplus t^- \underset{\mathcal{O}}{\otimes} V^- = (t \underset{\mathcal{O}}{\otimes} V)^+.$$

Putting $V = \tilde{L}^n$ here gives $T^+ \otimes_\Lambda \tilde{\Lambda} \otimes_{\tilde{\Lambda}} \tilde{L}^n = M(n)(\rho)^+_{\mathfrak{p}} \otimes_L \tilde{L}$.

Altogether, we see that tensoring β with $\tilde{L}^n$ over $\tilde{\Lambda}$ along ψ induces $\beta(\rho, n)$ as desired.□

We now turn to the setting for algebraic families. For this we fix the following data:

- A ring $\mathcal{I}$ as in section 3.6 and a set of specialisations Σ. We use the notations introduced there.
- A Galois extension $F_\infty/\mathbb{Q}$ with Galois group G just as before. Put[31] $\mathbb{\Lambda} = \mathcal{I}[\![G]\!]$ and $\Lambda_\phi = \mathcal{O}_\phi[\![G]\!]$ for each $\phi \in \Sigma$. Note that any ϕ induces a map $\mathbb{\Lambda} \longrightarrow \Lambda_\phi$ which we also denote by ϕ.
- A family of critical motives $(M(\phi))_{\phi\in\Sigma}$ given by $\rho\colon\ \mathrm{G}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{I}}(\mathcal{T})$ that satisfies the strong Dabrowski-Panchishkin condition and such that all the motives $M(\phi)$ are critical.
- Define[32] $\mathbb{T} \coloneqq \mathbb{\Lambda} \otimes_{\mathcal{I}} \mathcal{T}$ and $\mathbb{T}^{\mathrm{DP}} \coloneqq \mathbb{\Lambda} \otimes_{\mathcal{I}} \mathcal{T}^{\mathrm{DP}}$. Let $g \in \mathrm{G}_{\mathbb{Q}}$ act on $\mathbb{T}$ by $x \otimes y \longmapsto xg^{-1} \otimes gy$ and analogously on $\mathbb{T}^{\mathrm{DP}}$. Then fix an isomorphism of $\mathbb{\Lambda}$-modules $\beta\colon \mathbb{T}^+ \xrightarrow{\ \sim\ } \mathbb{T}^{\mathrm{DP}}$. Such an isomorphism exists by [Bar11, Lem. 2.14].

[31] The ring $\mathbb{\Lambda}$ should play the role of the ring Λ in [Bar11, §2.2], which is defined there as $\mathcal{O}[\![G]\!][\![T]\!]$. Using the notion of completed tensor products of profinite rings [RZ00, §5.5], we see that $\mathcal{O}[\![G]\!][\![T]\!] = \mathcal{O}[\![G]\!]\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[\![T]\!] = \mathcal{O}[\![T]\!][\![G]\!]$, so if we specialise to the case $\mathcal{I} = \mathcal{O}[\![T]\!]$ our definition coincides with the one in [Bar11]. We thus could have defined $\mathbb{\Lambda} = \mathcal{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[\![G]\!]$.

[32] In [Bar11, p. 22] the tensor product in the definition of $\mathbb{T}$ is formed over $\mathcal{O}$ instead of $\mathcal{I}$ (which is $\mathcal{O}[\![T]\!]$ there). We think that this is a typo since otherwise the statement in lemma 3.38 below is not true.

Lemma 3.38: *Let $\phi \in \Sigma$. We have canonically $\mathbb{T} \otimes_{\mathbb{\Lambda},\phi} \Lambda_\phi = \mathcal{T}_\phi \otimes_{\mathcal{O}_\phi} \Lambda_\phi$, $\mathbb{T}^{\mathrm{DP}} \otimes_{\mathbb{\Lambda},\phi} \Lambda_\phi = \mathcal{T}_\phi^{\mathrm{DP}} \otimes_{\mathcal{O}_\phi} \Lambda_\phi$ and $\mathbb{T}^{+} \otimes_{\mathbb{\Lambda},\phi} \Lambda_\phi = \mathcal{T}_\phi^{+} \otimes_{\mathcal{O}_\phi} \Lambda_\phi$. In particular, β induces an isomorphism*

$$\beta_\phi\colon \mathcal{T}_\phi^{+} \underset{\mathcal{O}_\phi}{\otimes} \Lambda_\phi \xrightarrow{\ \sim\ } \mathcal{T}_\phi^{\mathrm{DP}} \underset{\mathcal{O}_\phi}{\otimes} \Lambda_\phi.$$

Proof: The first two statements are obvious and the third one can be proved by similar arguments as lemma 3.37. See also [Bar11, Lem. 2.14]. □

3.7.2. The conjectures

First note that if M is a critical motive and we choose bases γ of M_{B}^{+} and δ of t_M, then the ratio

$$\frac{\Omega_{\mathrm{p}}^{\gamma,\delta,\beta}(M)}{\Omega_{\infty}^{\gamma,\delta}(M)}$$

is well-defined independently of the choices of γ and δ. More precisely: the ratio above does not make sense since the two objects lie in different rings, but if we change γ to γ' and δ to δ', then $\Omega_{\mathrm{p}}^{\gamma,\delta,\beta}(M)\Omega_{\mathrm{p}}^{\gamma',\delta',\beta}(M)^{-1} = \Omega_{\infty}^{\gamma,\delta}(M)\Omega_{\infty}^{\gamma',\delta'}(M)^{-1}$. This follows directly from the definitions. We therefore omit γ and δ in the following when we write down such an expression and always mean that implicitly some γ and δ are chosen.

Definition 3.39: Let M be a critical motive over K that satisfies the strong Dabrowski-Panchishkin condition. Define the *local correction factor at p* by

$$\mathrm{LF}_p(M) := \left.\frac{\mathrm{P}_p(M_{\mathrm{p}},T)}{\mathrm{P}_p(M_{\mathrm{p}}^{\mathrm{DP}},T)}\right|_{T=1} \cdot \mathrm{P}_p((M_{\mathrm{p}}^{\mathrm{DP}})^*(1),1) \in L,$$

where P_p is defined in definition 3.11.

A p-adic L-function in the general setting should be an element of some localised K-group that can be evaluated at certain representations of G and produces values in $\hat{L}^{\mathrm{nr}}$. We do not recall the details about this but refer to [FK06, §4.1.2–5].

There is a technical restriction on the evaluation points at which we can hope to describe the value of the p-adic L-function. We thus introduce the following notion.

Definition 3.40: Fix a critical motive M satisfying the strong Dabrowski-Panchishkin condition, a finite extension L' of L, an Artin representation $\rho\colon G \longrightarrow \mathrm{GL}_r(L')$ and $n \in \mathbb{N}$. We say that (ρ, n) is an *appropriate pair* for M if

(1) $M(\rho^*)(n)$ is still critical,

(2) $\mathrm{LF}_p(M(\rho^*)(n)) \neq 0$,

(3) $\mathrm{H}_{\mathrm{f}}^i(\mathbb{Q}, V) = \mathrm{H}_{\mathrm{f}}^i(\mathbb{Q}, V^*(1)) = 0$ for $V = M(\rho^*)(n)_{\mathrm{p}}$ and $i = 0, 1$.

These conditions are formulated in [FK06, Prop. 4.2.21]. There is a further condition related to a set called Υ there, but as we remarked in footnote 29 this set is empty in our setting, so this further condition is vacuous.

Conjecture 3.41 (Fukaya/Kato)**:** *Fix a critical motive M satisfying the strong Dabrowski-Panchishkin condition, an extension $F_\infty/\mathbb{Q}$ and an isomorphism β as in section 3.7.1. Attached to this data, there is a p-adic* L*-function such that for each appropriate pair (ρ, n) for M the value of the p-adic* L*-function at $\rho\kappa_{\mathrm{cyc}}^{-n}$ is*

$$\prod_{j\geq 1}(j-1)!^{\dim_K \mathrm{gr}^{n-j} M_{\mathrm{dR}}}\, \mathrm{LF}_p(M(\rho^*)(n)) \frac{\Omega_{\mathrm{p}}^{\beta(\rho^*,n)}(M(\rho^*)(n))}{\Omega_\infty(M(\rho^*)(n))} \mathrm{L}(M(\rho^*), n).$$

Here $\beta(\rho^*, n)$ is from lemma 3.37.

We now turn to families of motives. In this setting the p-adic L-function should also be an element of a localised K-group that can be evaluated (in particular) at pairs (ϕ, ψ) where $\phi\colon \mathcal{I} \longrightarrow \mathbb{Q}_p$ is a morphism and ψ is a representation of G, and again produces values in $\hat{L}^{\mathrm{nr}}$, see [Bar11, §4.2–3].

Conjecture 3.42 (Fukaya/Kato, Barth)**:** *Fix $\mathcal{I}$, Σ, $F_\infty/\mathbb{Q}$, an algebraic family of critical motives $(M(\phi))_{\phi\in\Sigma}$ satisfying the strong Dabrowski-Panchishkin condition and an isomorphism β as in section 3.7.1. Attached to this data, there is a p-adic* L*-function such that for each $\phi \in \Sigma$ and each appropriate pair (ρ, n) for $M(\phi)$, the value of the p-adic* L*-function at $(\phi, \rho\kappa_{\mathrm{cyc}}^{-n})$ is*

$$\prod_{j\geq 1}(j-1)!^{\dim_K \mathrm{gr}^{n-j} M(\phi)_{\mathrm{dR}}}\, \mathrm{LF}_p(M(\phi)(\rho^*)(n)) \frac{\Omega_{\mathrm{p}}^{\beta_\phi(\rho^*,n)}(M(\phi)(\rho^*)(n))}{\Omega_\infty(M(\phi)(\rho^*)(n))} \mathrm{L}(M(\phi)(\rho^*), n).$$

Here β_ϕ is from lemma 3.38 and $\beta_\phi(\rho^*, n)$ is from lemma 3.37.

We finally remark that in both situations there is an Iwasawa Main Conjecture which says that the p-adic L-function is a characteristic element (suitably defined) for some Selmer complex constructed using certain Iwasawa modules attached to the motive resp. the family. Since this work is focused on the analytic side of Iwasawa Theory, we do not repeat these Main Conjectures here.

As mentioned before, the above interpolation formulas are in fact consequences of the Equivariant Tamagawa Number Conjecture (ETNC), see [FK06, Thm. 4.2.22] resp. [Bar11, Thm. 4.31][33] and also [Ven07]. Moreover the ETNC also implies the above-mentioned Main Conjecture.

As a further generalisations of these results and conjectures, it seems plausible to expect the existence of such a p-adic L-function also for analytic families and also in the case where the families satisfy only the weak Dabrowski-Panchishkin condition. Note that each expression in the interpolation formula (in particular the p-adic period) is still well-defined in this setting, so we can hope for the same interpolation formula. Also it is natural to ask whether the existence of the p-adic L-function can still be deduced from the ETNC by methods similar to the ones of Fukaya, Kato and Barth. We do not pursue these questions further in this work. The recent thesis [Zae17] shows that the work of Fukaya and Kato can be generalised to the case of a single motive satisfying the weak Dabrowski-Panchishkin condition.

[33] In [Bar11], this is only proved for families for which $\mathcal{I} = \mathcal{O}[\![T]\!]$. In view of this result conjecture 3.42 seems to be a reasonable generalisations.

Chapter II.

Modular curves and motives for modular forms

We introduce and study modular curves, modular forms and motives attached to them. This chapter contains no new results, but it collects a large amount of properties of these objects. Almost everything in this chapter is well-known to the experts, but some statements or their proofs are rarely to be found in the literature.

For the whole chapter we fix an integer $N \geq 1$ (later we will assume $N \geq 4$) which will be the level. In this chapter we will only look at one fixed level (apart from section 7), while in the next chapter the level can vary.

1. Modular curves

1.1. Arithmetic theory of modular curves

In this section, we introduce modular curves as moduli spaces for elliptic curves (or generalised elliptic curves) with level structure, in an arithmetic-geometric setting. Although this theory is well-known, the results we need are scattered around the literature, so we collect some facts and references here.

Over rings or schemes on which N is invertible, it is not too difficult to see that the functors classifying elliptic curves with an appropriately defined level structure are representable by affine schemes (which after complexification are quotients of the upper half plane $\mathfrak{h}$). There are then two points on which one wants to improve: first, to get rid of the requirement that N be invertible. Second, to "compactify" the moduli schemes to proper ones (which after complexification should give quotients of the extended upper half plane $\mathfrak{h}^*$) by classifying more general objects.

There are three major texts (among others) about moduli of elliptic curves with level structures. First Deligne and Rapoport [DR73] introduces generalised elliptic curves by allowing certain singularities and then considers moduli of these generalised elliptic curves with level structures. This gives compactifications of the moduli schemes and moduli interpretations of the cusps, as intended. The disadvantage of their approach is that a level structure as they define it can only exist if the level N is invertible on the base scheme, so that the moduli schemes live over $\mathbb{Z}[1/N]$. On the other hand, Katz and Mazur [KM85] uses so-called Drinfeld level structures which do not need N to be invertible. This gives moduli schemes of elliptic curves over $\mathbb{Z}$, which are then compactified using a normalisation construction. Unfortunately, this does not give a moduli interpretation of the cusps.

The two approaches are unified by Conrad in [Con07], which provides schemes over $\mathbb{Z}$ with a modular interpretation also of the cusps. We follow this text and cite the main results, supplying them with results from other articles where it seems necessary.

An *elliptic curve* is a proper smooth irreducible curve $E \longrightarrow S$ over some scheme S whose geometric fibres are connected curves of genus one, together with a fixed section $S \longrightarrow E$.

A *generalised elliptic curve* is a stable curve of genus 1, i. e. a proper, flat, finitely presented morphism of relative dimension ≥ 1 whose smooth geometric fibres are connected curves of genus one and whose non-smooth geometric fibres are Néron polygons, together with some additional structure. For the precise definition see [DR73, Déf. II.1.12].

For a generalised elliptic curve $E \longrightarrow S$ we write $E[N]$ for its N-torsion, i. e. the kernel of the multiplication by N.

Proposition 1.1: *$E[N]$ is a finite flat group scheme over S, locally free of rank N^2. It is étale over S if and only if N is invertible on S. In this case it is étale locally isomorphic to the constant group scheme $\underline{(\mathbb{Z}/N)^2}_{/S}$.*

Proof: [DR73, §II.1.18–20], [KM85, Thm. 2.3.1, Cor. 2.3.2] □

The following definition uses (relative) effective Cartier divisors, see [KM85, (1.1.1–2), Lem. 1.2.2] for the necessary background on these. Further it uses the Weil pairing on the N-torsion of a generalised elliptic curve $E \longrightarrow S$, which is a pairing $e_N : E[N] \times E[N] \longrightarrow \mu_N$ of group schemes over S; see [KM85, §2.8.5] and [DR73, IV.3.21].

Definition 1.2: Let $E \longrightarrow S$ be a generalised elliptic curve.

(a) A *naive* $\Gamma(N)$*-structure* on E is a homomorphism of group schemes over S

$$\varphi\colon \underline{\left(\mathbb{Z}\big/N\right)^2}_{/S} \longrightarrow E[N]$$

such that there is an equality of effective Cartier divisors on E

$$E[N] = \sum_{(a,b)\in(\mathbb{Z}/N)^{\times 2}} [\varphi(a,b)]$$

and the above Cartier divisor meets each irreducible component in each geometric fibre.

(b) A *naive* $\Gamma_1(N)$*-structure* on E is a homomorphism of group schemes over S

$$\varphi\colon \underline{\mathbb{Z}\big/N}_{/S} \longrightarrow E[N]$$

such that the effective Cartier divisor

$$\sum_{a\in(\mathbb{Z}/N)^{\times}} [\varphi(a)]$$

is a subgroup scheme of E and meets each irreducible component in each geometric fibre.

(c) An *arithmetic* $\Gamma(N)$*-structure* on E is an isomorphism of group schemes over S

$$\phi\colon \mu_N \times \underline{\mathbb{Z}\big/N}_{/S} \xrightarrow{\sim} E[N]$$

of determinant 1. The latter condition means that if we view φ as a pair of embeddings $\varphi_1 : \mu_N \hookrightarrow E[N]$, $\varphi_2 : \mathbb{Z}/N \hookrightarrow E[N]$, then $e_N(\varphi_1(\zeta), \varphi_2(n)) = \zeta^n$ under the Weil pairing, for all ζ and n.

(d) An *arithmetic* $\Gamma_1(N)$*-structure* on E is a closed immersion of group schemes over S

$$\phi\colon \mu_N \hookrightarrow E[N].$$

See also [Con07, Def. 2.4.1–2], [Kat76, §2.0].

Note that any S-homomorphism from a finite constant group scheme over S to any separated S-scheme is automatically a closed immersion, because if $f: X \longrightarrow S$ is any morphism of schemes with X separated, then each section $s\colon S \longrightarrow X$ is a closed immersion by [EGA1, Cor. 5.4.6]. In particular, naive $\Gamma(N)$- or $\Gamma_1(N)$-level structures are closed immersions.

We now define 8 moduli functors for elliptic curves.

Definition 1.3: For $*$ being nothing or "1" and ? being "naive" or "arith" let

$$X_*(N)^?\colon \mathcal{Sch} \longrightarrow \mathcal{Sets}$$

be the functor associating to a scheme S the set of isomorphism classes of pairs (E, φ), where E is a generalised elliptic curve over S and φ is a naive resp. arithmetic $\Gamma_*(N)$-structure. Let $Y_*(N)^? \subseteq X_*(N)^?$ be the subfunctor associating to S the set of isomorphism classes of pairs (E, φ) with E a (usual, non-generalised) elliptic curve and φ as before.

Remark 1.4: (a) If a generalised elliptic curve is a usual elliptic curve, then the requirement about the irreducible components in the geometric fibres is automatically fulfilled since elliptic curves are irreducible, so this gives back the definition in [KM85, (3.1–2)].

(b) If S is a $\mathbb{Z}[1/N]$-scheme and $E \longrightarrow S$ is a generalised elliptic curve, then a full level N structure on E is just an isomorphism of group schemes over S

$$\underline{\left(\mathbb{Z}/N\right)^2}_{/S} \xrightarrow{\sim} E[N],$$

while point of exact order N on E is just a monomorphism of group schemes over S

$$\underline{\mathbb{Z}/N}_{/S} \hookrightarrow E[N]$$

whose image meets each irreducible component in each geometric fibre, which is the definition given in [DR73, §2.3, §4.7]. This is shown in [KM85, Lem. 1.4.4, Lem. 1.5.3].

(c) Over $\mathbb{Z}[1/N]$ there is a canonical isomorphism of functors

$$Y(N)^{\mathrm{naive}} \xrightarrow{\sim} \mu_N^\times \times Y(N)^{\mathrm{arith}}$$

where $\mu_N^\times$ is the group scheme of primitive roots of unity, see [Kat76, (2.0.8)] or [DR73, chap. V, (4.4.1)].

(d) Over $\mathbb{Z}[\mu_N]$ there is then a canonical isomorphism of functors

$$v_N\colon Y_1(N)^{\mathrm{naive}} \xrightarrow{\sim} Y_1(N)^{\mathrm{arith}}$$

since the group schemes $\underline{\mathbb{Z}/N}$ and μ_N are canonically isomorphic over this ring. Later we will often be working over a ring containing the N-th roots of unity (see remark 5.5), so we can then identify these modular curves.

Theorem 1.5 (Shimura, Igusa, Katz/Mazur, Deligne/Rapoport, Conrad)**:** *If $N \geq 4$,[1] then all 8 moduli functors from definition 1.3 are representable over $\mathbb{Z}$ by a scheme which we denote by the same symbol as the corresponding functor. They have the following geometric properties:*

(a) $Y(N)^{\mathrm{naive}}$ and $Y_1(N)^{\mathrm{naive}}$ are finite flat regular curves over $\mathbb{Z}$. Over $\mathbb{Z}[1/N]$ these curves are affine finite étale.

(b) $X(N)^{\mathrm{naive}}$ and $X_1(N)^{\mathrm{naive}}$ are proper flat regular curves over $\mathbb{Z}$. Over $\mathbb{Z}[1/N]$ these curves are projective and smooth.

(c) $Y(N)^{\mathrm{arith}}$ and $Y_1(N)^{\mathrm{arith}}$ are smooth affine curves over $\mathbb{Z}$.

(d) $X(N)^{\mathrm{arith}}$ and $X_1(N)^{\mathrm{arith}}$ are smooth curves over $\mathbb{Z}$. Over $\mathbb{Z}[1/N]$ theses curve are proper.

Proof: We begin with the statements about the naive moduli problems. The claims about $Y(N)^{\mathrm{naive}}$ and $Y_1(N)^{\mathrm{naive}}$ follow from [KM85, Cor. 2.7.2–3, Thm. 5.1.1, Scholie 4.7.0, Cor. 4.7.1].

By [Con07, Thm. 3.1.7], the functors $X(N)^{\mathrm{naive}}$ and $X_1(N)^{\mathrm{naive}}$ are represented each by a Deligne-Mumford stack over $\mathbb{Z}$. We claim that this Deligne-Mumford stack is an algebraic space. This is equivalent to the fact that its geometric points have no non-trivial automorphisms, see [DR73, §VI.2.1].[2] For elliptic curves with level structure, this is proved in [KM85, Cor. 2.7.2–3], while for Néron polygons with a chosen point of exact order N, this is easy to see using the description of automorphisms of Néron polygons in [DR73, §II.1.9]. Recall that we assumed $N \geq 4$.

The properness over $\mathbb{Z}$ is proved in [Con07, Thm. 3.2.7]. That it is a curve, i. e. of pure relative dimension 1, is proved in [Con07, Thm. 3.3.1]. The regularity is proved in [Con07, Thm. 4.1.1].

That the algebraic spaces are in fact schemes follows from the general fact that a regular algebraic space over $\mathbb{Z}$ of relative dimension 1 is always a scheme. This is claimed in [DR73, p. 69, after Cor. IV.2.9]. See also the proof of [Con07, Thm. 4.2.1 (2)].

Now we turn to the arithmetic moduli problems. The statements about $Y(N)^{\mathrm{arith}}$ and $X(N)^{\mathrm{arith}}$ follow similarly as a above from [DR73, §V.4.4]. More precisely, the rigidity of the moduli problem follows from the isomorphism in remark 1.4 (c) and the fact that the naive $\Gamma(N)$-problem is rigid. By [DR73, Lem. V.4.5, V.4.7] the moduli problem is represented by a smooth Deligne-Mumford stack, which is hence regular, and it follows as above that is a scheme.

The geometric properties of the arithmetic $\Gamma(N)$-moduli problems are not stated in our references, but they follow from the corresponding ones for the arithmetic $\Gamma_1(N)$-moduli problems (see below) since the $\Gamma(N)$-functors are étale $(\mathbb{Z}/N)^{\times}$-torsors over the $\Gamma_1(N)$-functors and the properties are étale local and stable under base change.

For $Y_1(N)^{\mathrm{arith}}$, see [KM85, (4.9–10), p. 120] and [DI95, Var. 8.2.2]. For $X_1(N)^{\mathrm{arith}}$, see [DI95, Thm. 9.3.7] and [Gro90, Prop. 2.1]. □

Remark 1.6: For general N, the moduli functors are represented by Artin stacks instead of schemes. This is shown in the same references cited in the above proof, but we will not

[1] For the $\Gamma_1(N)$-structures, it suffices to assume $N \geq 3$. In what follows, we will often assume that $N \geq 4$ although in some situations it might suffice to assume $N \geq 3$; but see remark 1.6.

[2] Note that what Deligne and Rapoport call "champ algébrique" is what nowadays is called a Deligne-Mumford stack.

use this fact. Nevertheless, let us remark at this point that although we will later always assume $N \geq 4$, everything we are going to do should carry over to low level by replacing modular curves by modular stacks. This makes the theory technically more complicated, but conceptually the same techniques should still work. However, we will not check the details.

Definition 1.7: (a) The curves from theorem 1.5 are called modular curves. From now on, whenever we write $X_*(N)$ or $Y_*(N)$ (for $*$ being nothing or "1") without a superscript, we will always mean the *naive* versions of the modular curves. The curves classifying arithmetic $\Gamma(N)$-structures will play no role in this work and were mentioned only for completeness. If we are working over some ring containing the N-th roots of unity, then we identify the curves classifying naive and arithmetic $\Gamma_1(N)$-structures using the isomorphism from remark 1.4 (d). Thus in this setting $X_1(N)$ and $Y_1(N)$ denote also the arithmetic versions.

(b) We denote the universal elliptic resp. generalised elliptic curves over these moduli schemes by $E_*(N)^?$ and $\overline{E}_*(N)^?$, respectively, with $*$ and ? as before, and by abuse of notation we denote all the maps from any of these universal curves to the bases by f and every unit section by e.

(c) We denote the complements $X_*(N)^? \smallsetminus Y_*(N)^?$ by $C_*(N)^?$. They are closed subschemes equal to the locus over which the universal generalised elliptic curve $\overline{E}_*(N)^?$ is not smooth, and for ? being "naive" they define relative effective Cartier divisors [Con07, Thm. 4.1.1 (1)]. We call them the *cusps* or the *cuspidal divisors.*

Remark 1.8: Note that if $\mathcal{P}$ is any one of these moduli functors which is representable, M is the representing object and $f : E \longrightarrow M$ is the universal (maybe generalised) elliptic curve, then E represents the functor

$$\begin{aligned} \mathcal{S}ch &\longrightarrow \mathcal{S}ets, \\ T &\longmapsto \{(t,l) : t \in \mathcal{P}(T) = \mathrm{Hom}(T,M),\ l \in \mathrm{Hom}(T,E) \text{ s. th. } f \circ l = t\}. \end{aligned}$$

Although this is trivial, it means that giving a point on the universal elliptic curve is the same as giving a point in M, that is, an elliptic curve with level structure, and a point on this elliptic curve.

The group $\mathrm{GL}_2(\mathbb{Z}/N)$ acts on naive $\Gamma(N)$-structures on (generalised) elliptic curves by precomposing them with automorphisms of $\underline{\mathbb{Z}/N}^2$. We normalise this action by saying that $\mathrm{GL}_2(\mathbb{Z}/N)$ should act on $(\mathbb{Z}/N)^2$ by multiplication on the left *with the transpose matrix*, so this gives a right action on level N structures and thus by universality a right action on $E(N) \longrightarrow Y(N)$ and $\overline{E}(N) \longrightarrow X(N)$. This is described more explicitly e. g. in [Con09, §4.2.3].

Every full level N structure on a (generalised) elliptic curve yields a point of exact order N by precomposing it with the embedding

$$\mathbb{Z}\big/N \times \{0\} \hookrightarrow \left(\mathbb{Z}\big/N\right)^2. \tag{1.1}$$

This defines morphisms of functors

$$Y(N) \longrightarrow Y_1(N), \quad X(N) \longrightarrow X_1(N). \tag{1.2}$$

Remark 1.9: If $E \longrightarrow S$ is an elliptic curve and $\varphi\colon \underline{\mathbb{Z}/N}_{/S} \longrightarrow E[N]$ is a naive $\Gamma_1(N)$-structure, then φ is uniquely determined by $P := \varphi(1) \in E[N]$. We may use this to define morphisms between the moduli functors we introduced, which by the Yoneda lemma correspond to morphisms between modular curves. For example, any automorphism σ of $\mathbb{Z}/N$ gives rise to another torsion point $\sigma(P) := \varphi(\sigma(1))$, and we can define an automorphism of $Y_1(N)^{\text{naive}}$ by $(E, P) \longmapsto (E, \sigma(P))$ (see definition 3.1). We will often use this method to define similar morphisms between modular curves without further explanation. As a further example, we can write the morphism (1.2) as $(E, P, Q) \longmapsto (E, P)$.

Lemma 1.10: *The quotient of $E(N) \longrightarrow Y(N)$ by the subgroup*

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_2\left(\mathbb{Z}/N\right)$$

exists as a scheme and is canonically isomorphic to $E_1(N) \longrightarrow Y_1(N)$. Here we mean the naive quotient in the sense of [HidGMF, §1.8.1]. The analogous statement is true for the compactified versions.

Proof: This subgroup is obviously the one fixing pointwise the image of (1.1) (recall that we act by the transposed matrix!). Hence the quotient functor of $Y(N)$ by this subgroup is $Y_1(N)$, from which the claim follows. For details, see [HidGMF, §1.8.1]. See also [KM85, Thm. 7.4.2 (3)] or [DR73, Prop. IV.3.10 (iii)]. ☐

Lemma 1.11: *There is a cartesian diagram of schemes*

$$\begin{tikzcd} E(N) \arrow[r] \arrow[d] & E_1(N) \arrow[d] \\ Y(N) \arrow[r] & Y_1(N) \end{tikzcd}$$

in which the bottom arrow is the map (1.2). The same holds for the corresponding compactified versions.

Proof: We will prove the statement only for the non-compactified versions, the proof for the compactified versions works similarly.

We use the language of stacks as explained in [Beh+06]. The stacks $\underline{Y(N)}$ and $\underline{Y_1(N)}$ defined by $Y(N)$ and $Y_1(N)$ classify elliptic curves with the respective level structure. Let $\mathcal{M}_{1,1}$ be the moduli stack of elliptic curves (see [Beh+06, §1.5]) and let $\mathcal{E}$ be the stack of genus 1 curves with two sections (defined like $\mathcal{M}_{1,2}$ in [Beh+06, Ex. 1.1c] except that the two sections need not be disjoint). Via the morphism $\mathcal{E} \longrightarrow \mathcal{M}_{1,1}$ forgetting the second section, $\mathcal{E}$ can be seen as the universal elliptic curve over $\mathcal{M}_{1,1}$: this can be shown similarly as in [Beh+06, Ex. 2.25 (4)]. We have morphisms from both $\underline{Y(N)}$ and $\underline{Y_1(N)}$ to $\mathcal{M}_{1,1}$, sending an elliptic curve with level structure to just the underlying elliptic curve. Now, as stacks, the universal objects $\underline{E(N)}$ resp. $\underline{E_1(N)}$ are in fact isomorphic to the pullbacks of $\mathcal{E}$ along these respective morphisms, that is, the diagrams

$$\begin{tikzcd} \underline{E(N)} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \underline{Y(N)} \arrow[r] & \mathcal{M}_{1,1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \underline{E_1(N)} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \underline{Y_1(N)} \arrow[r] & \mathcal{M}_{1,1} \end{tikzcd}$$

are 2-cartesian. We will prove this for $\underline{E(N)}$, the other case being similar. The statement then follows from this.

Let T be a scheme. By definition of the fibre product of categories fibred in groupoids [Beh+06, §2.5], an object over T in the fibre product $\underline{Y(N)} \times_{\mathcal{M}_{1,1}} \mathcal{E}$ consists of

$$\text{a morphism } T \longrightarrow Y(N) \text{ (which corresponds to an elliptic curve } C' \longrightarrow T \text{ with level structure), a genus 1 curve } C \longrightarrow T \text{, two sections } \sigma, \tau\colon T \longrightarrow C \text{ and an isomorphism } \theta\colon C' \xrightarrow{\sim} C \text{ over } T \text{ compatible with the unit section of } C' \text{ and the section } \sigma \text{ of } C. \tag{1.3}$$

An object of $\underline{E(N)}$ over T consists by remark 1.8 of

$$\text{a morphism } T \longrightarrow Y(N) \text{ (which corresponds to an elliptic curve } C'' \longrightarrow T \text{ with level structure) and a section } \tilde{\tau}\colon T \longrightarrow C''. \tag{1.4}$$

We define maps between the sets of data of this type. Sending an element of type (1.4) to the same morphism $T \longrightarrow Y(N)$, the genus 1 curve $C'' \longrightarrow T$, the unit section of the elliptic curve $C'' \longrightarrow T$ and the section $\tilde{\tau}$, together with the identity isomorphism $C'' \xrightarrow{\sim} C''$ gives a map of the set of elements of type (1.4) to the set of elements of type (1.3). In the other direction, we send an element of type (1.3) to the same morphism $T \longrightarrow Y(N)$ and the section $\tilde{\tau} := \theta^{-1} \circ \tau\colon T \longrightarrow C'$.

It is not difficult to verify that the collection of these maps for every T defines a natural isomorphism between the fibre functors (functors of points) of the stacks $\underline{E(N)}$ and $\underline{Y(N)} \times_{\mathcal{M}_{1,1}} \mathcal{E}$, and hence an equivalence of categories between these stacks. □

1.2. Complex analytic theory of modular curves and GAGA

We now work in the category of complex analytic spaces. An *analytic elliptic curve* is a proper smooth map $f\colon E \longrightarrow S$ of analytic spaces of relative dimension 1 whose fibres are curves of genus 1, together with a specified section $e\colon S \longrightarrow E$. In the complex analytic setting, modular curves (as Riemann surfaces) are often introduced as quotients of the complex upper half plane $\mathfrak{h}$. We explicitly construct two (isomorphic) elliptic curves over $\mathfrak{h}$ and then look at quotients by congruence subgroups. In this way we get analytic elliptic curves over these quotients that have a universal property similar as in the arithmetic situation. We follow closely [Con09, chap. 1, esp. ex. 1.1.1.16].

In this section, for $z, w \in \mathbb{C}$ being two $\mathbb{R}$-linearly independent vectors in the complex plane, we write $[z, w]$ for the $\mathbb{Z}$-lattice generated by them.

Definition 1.12: (a) Define Λ as the image of

$$\mathbb{Z}^2 \times \mathfrak{h} \hookrightarrow \mathbb{C} \times \mathfrak{h}, \quad (m, n, \tau) \longmapsto (m\tau + n, \tau), \tag{1.5}$$

so we have

$$\Lambda = \bigcup_{\tau \in \mathfrak{h}} [\tau, 1] \times \{\tau\}, \tag{1.6}$$

and put $E_{\mathrm{Lat}} := (\mathbb{C} \times \mathfrak{h})/\Lambda$. Equivalently, we could define E_{Lat} as the quotient of $\mathbb{C} \times \mathfrak{h}$ by the left action of $\mathbb{Z}^2$ given by

$$(m, n)(z, \tau) = (z + m\tau + n, \tau) \qquad (m, n \in \mathbb{Z}). \tag{1.7}$$

The map $f_{\mathrm{Lat}}\colon E_{\mathrm{Lat}} \longrightarrow \mathfrak{h}$ is defined as the projection onto the second factor and with the obvious identity section this makes E_{Lat} an elliptic curve over $\mathfrak{h}$.

(b) Let $\wp_\tau$ for a $\tau \in \mathfrak{h}$ denote the Weierstraß function for the lattice $[1, \tau]$ and let $g_2 = 60G_4$ and $g_3 = 140G_6$ denote the usual Eisenstein series on $\mathfrak{h}$ occurring in the differential equation describing $\wp$ (see [Sil86, Rem. 3.5.1]). We put

$$E_{\mathrm{Wei}} := \{(\tau, [x : y : z]) \in \mathfrak{h} \times \mathbb{P}^2(\mathbb{C}) \mid y^2 z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3\}$$

and let

$$f_{\mathrm{Wei}} \colon E_{\mathrm{Wei}} \longrightarrow \mathfrak{h}$$

be the projection on the first factor. The identity section is again the obvious one.

Here, "Wei" stands for "Weierstraß" and "Lat" stands for "lattice".

The map

$$\mathbb{C} \times \mathfrak{h} \longrightarrow E_{\mathrm{Wei}}, \quad (z, \tau) \longmapsto (\tau, [\wp_\tau(z) : \wp'_\tau(z) : 1]),$$

which is obviously invariant under the action of $\mathbb{Z}^2$ on $\mathbb{C} \times \mathfrak{h}$, induces an isomorphism of elliptic curves $E_{\mathrm{Lat}} \xrightarrow{\sim} E_{\mathrm{Wei}}$ over $\mathfrak{h}$.

The upper half plane is in fact also the solution of a moduli problem for elliptic curves, which can be interpreted as a "level ∞ moduli problem" (it has no solution in the category of schemes, but does have one in the category of analytic spaces). Both E_{Wei} and E_{Lat} are then universal elliptic curves over $\mathfrak{h}$. We won't need this, for details see [Con09, Thm. 1.4.3.1].

Let $\Sigma := \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$. In example 1.2.11, we defined an action of Σ on $\mathbb{C} \times \mathfrak{h}$. On the other hand, we defined an action of $\mathbb{Z}^2$ on $\mathbb{C} \times \mathfrak{h}$ in (1.7). One can then check that we get a well-defined left action of Σ on the quotient E_{Lat}.[3]

Since f_{Lat} is equivariant for this action, we can take quotients by any subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ to get $f_{\mathrm{Lat}} \colon \Gamma\backslash E_{\mathrm{Lat}} \longrightarrow \Gamma\backslash\mathfrak{h}$ (which has the same name as before, by abuse of notation).

The compatibility between the analytic and the arithmetic theory is stated in the following proposition.

Theorem 1.13: *There is a commutative diagram of analytic spaces*

$$\begin{array}{ccc} E_1(N)^{\mathrm{an}} & \xrightarrow{\sim} & \Gamma_1(N)\backslash E_{\mathrm{Lat}} \\ \downarrow & & \downarrow \\ Y_1(N)^{\mathrm{an}} & \xrightarrow{\sim} & \Gamma_1(N)\backslash\mathfrak{h} \end{array}$$

in which the horizontal maps are isomorphisms.

Hence $\Gamma_1(N)\backslash\mathfrak{h}$ is the solution for the moduli problem on analytic spaces associating to an analytic space the set of isomorphism classes of elliptic curves with (naive) level $\Gamma_1(N)$-structure.[4] *The fibre over some $\tau \in \mathfrak{h}$ is $E_\tau := \mathbb{C}/[1, \tau]$ and the point of exact order N in this fibre is $\frac{1}{N}$.*

Proof: [Con09, Thm. 4.2.6.2; §2.1.3][5] □

---

[3] Note that we cannot use lemma 1.2.9 (b) here because the relation (1.2.1) does not hold; nevertheless this can be checked by a direct calculation. More precisely: For $(m, n) \in \mathbb{Z}^2$, $\gamma \in \Sigma$ and $(z, \tau) \in \mathbb{C} \times \mathfrak{h}$ one has to find $(u, v) \in \mathbb{Z}^2$ such that $\gamma((m, n)(z, \tau)) = (u, v)(\gamma(z, \tau))$. One checks that $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) := {}^{\intercal}\gamma^{\iota}\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right)$ does the job if $\det\gamma > 0$. For $\gamma = ə$ one chooses $(u, v) = (-m, n)$.

[4] The definition of a $\Gamma_1(N)$-structure on an analytic elliptic curve is totally analogous to the algebraic case, see definition 1.2 (b). The requirement about the geometric fibres is vacuous in this case since we are dealing with smooth elliptic curves, whose fibres are connected anyway.

[5] We remark here that in the text [Con09], *right* actions of $\mathrm{SL}_2(\mathbb{Z})$ on both $\mathfrak{h}$ and E_{Lat} are used. However the corresponding left actions obtained from these via the involution ι are exactly the left actions defined here; in particular the quotient spaces we get are the same. To see this, we refer to [Con09, Thm. 1.5.2.2, p. 73] for the action on $\mathfrak{h}$ and to [Con09, proof of Lem. 1.5.4.4, p. 84] for the action on E_{Lat}.

In the following, we identify $Y_1(N)^{\mathrm{an}}$ with $\Gamma_1(N)\backslash\mathfrak{h}$.

Remark 1.14: (a) Note that the above statement holds for both naive and arithmetic $\Gamma_1(N)$-structures since we are working over $\mathbb{C}$. Hence for analytic considerations it does not matter which version of the $\Gamma_1(N)$ modular curve is used.

(b) The analogue of the above theorem for $Y(N)^{\mathrm{an}}$ is not quite true: the analytic space $\Gamma(N)\backslash\mathfrak{h}$ does not classify *all* full level N structures in the analytic setting, but only those with a fixed Weil pairing, see [Con09, §2.1.2] or [DS05, §1.5]. Since the possible values of the Weil pairing are the primitive N-th roots of unity, the analytification of $Y(N)$ can be identified with the disjoint union of $\varphi(N)$ copies of $\Gamma(N)\backslash\mathfrak{h}$ (where φ is the Euler totient function). See for this also [DR73, Introduction, p. 15] or [Kat04, §1.8].

(c) However, for $(Y(N)^{\mathrm{arith}})^{\mathrm{an}}$ the analogous statement is true, which follows easily from remark 1.4 (d).

1.3. The action of complex conjugation

In this section we identify the groups $\mathrm{G}_{\mathbb{R}}$ and $\mathrm{G}_{\text{ə}}$ in the only possible way. Note that then $\mathrm{G}_{\mathbb{R}}$ acts on $\mathfrak{h}$ by the action we defined in example 1.2.10. Our purpose is, loosely speaking, to prove that it is reasonable to make this identification.

We begin in the analytic setting. Denote again by $[z, w]$ the $\mathbb{Z}$-lattice generated by two $\mathbb{R}$-linearly independent vectors $z, w \in \mathbb{C}$ in the complex plane. The spaces $\mathbb{C}$ and $\mathbb{P}^n(\mathbb{C})$ will be endowed with the canonical action by complex conjugation.

Lemma 1.15: *We get induced actions of* $\mathrm{G}_{\mathbb{R}}$ *on* E_{Wei} *and* E_{Lat} *and all maps in the diagram*

$$\begin{tikzcd}
E_{\mathrm{Lat}} \arrow[rr, "\sim"] \arrow[dr, "f_{\mathrm{Lat}}"'] & & E_{\mathrm{Wei}} \arrow[dl, "f_{\mathrm{Wei}}"] \\
& \mathfrak{h} &
\end{tikzcd}$$

are $\mathrm{G}_{\mathbb{R}}$*-equivariant.*

Proof: First we observe that for any $\tau \in \mathfrak{h}$, the lattice $[\text{ə}\tau, 1]$ is just the image of the lattice $[\tau, 1] \subseteq \mathbb{C}$ under complex conjugation on $\mathbb{C}$. This means that $\Lambda \subseteq \mathbb{C} \times \mathfrak{h}$ is invariant under the diagonal action of $\mathrm{G}_{\mathbb{R}}$ on $\mathbb{C} \times \mathfrak{h}$, hence we have a well-defined action on the quotient E_{Lat}, and moreover, the map f_{Lat} is $\mathrm{G}_{\mathbb{R}}$-equivariant.

We let $\mathrm{G}_{\mathbb{R}}$ act diagonally on $\mathfrak{h} \times \mathbb{P}^2(\mathbb{C})$. To see that this induces a well-defined action on E_{Wei}, it suffices to note that Eisenstein series have real Fourier coefficients, which implies that $\overline{G_k(\tau)} = G_k(-\overline{\tau}) = G_k(\text{ə}\tau)$ for every even $k \geq 2$ and $\tau \in \mathfrak{h}$. This shows at the same time that f_{Wei} is $\mathrm{G}_{\mathbb{R}}$-equivariant.

Finally, to see that the isomorphism $E_{\mathrm{Lat}} \xrightarrow{\sim} E_{\mathrm{Wei}}$ is $\mathrm{G}_{\mathbb{R}}$-equivariant, we observe

$$\overline{\wp_\tau(z)} = \wp_{-\overline{\tau}}(\overline{z}), \quad \overline{\wp'_\tau(z)} = \wp'_{-\overline{\tau}}(\overline{z}) \quad \text{for } \tau \in \mathfrak{h},\ z \in \mathbb{C}. \qquad \square$$

From example I.2.10 we have an action of $\mathrm{G}_{\mathbb{R}} = \mathrm{G}_{\text{ə}}$ on the quotient of $\mathfrak{h}$ by $\Gamma_1(N)$ since $\Gamma_1(N)$ is normalised by ə. For the diagonal action of $\mathrm{G}_{\mathbb{R}}$ on $\mathbb{C} \times \mathfrak{h}$ and the action of $\mathrm{SL}_2(\mathbb{Z})$ on the same space defined in (I.2.2), it is an easy calculation to verify relation (I.2.1), so we also get an action of $\mathrm{G}_{\mathbb{R}}$ on $\Gamma_1(N)\backslash E_{\mathrm{Lat}}$. On the other hand, we have a natural action of $\mathrm{G}_{\mathbb{R}}$ on the set of complex points of the schemes $Y_1(N)$ and $E_1(N)$ from theorem 1.5, hence also on the corresponding analytifications.

Proposition 1.16: *All maps in the diagram in theorem 1.13 are* $\mathrm{G}_{\mathbb{R}}$*-equivariant.*

Proof: By universality of E_{Lat} and E_{Wei} and lemma 1.15, we can also prove the claim with E_{Lat} replaced by E_{Wei}, and by surjectivity of the vertical maps we have to prove $\mathrm{G}_{\mathbb{R}}$-equivariance only for the isomorphism in the top row of the diagram.

So let $y\colon \operatorname{Spec}\mathbb{C} \longrightarrow E_1(N)$ a complex point of $E_1(N)$ and let $x\colon \operatorname{Spec}\mathbb{C} \longrightarrow Y_1(N)$ the point in $Y_1(N)(\mathbb{C})$ below it. By the universal property of $Y_1(N)$, the fibre of the universal elliptic curve $E_1(N) \longrightarrow Y_1(N)$ at x is an elliptic curve E_x over $\mathbb{C}$ that comes equipped with a point $P \in E_x[N]$ of exact order N, and y is a point in $E_x(\mathbb{C})$ (see remark 1.8). Under the isomorphism $Y_1(N)(\mathbb{C}) \cong \Gamma_1(N)\backslash\mathfrak{h}$, x can be lifted to some $\tau \in \mathfrak{h}$ and we see that the elliptic curve E_x can be embedded into projective space as the curve described by the Weierstraß equation $Y^2 = 4X^3 - g_2(\tau)X - g_3(\tau)$, with the point P having coordinates $[\wp_\tau(\frac{1}{N}) : \wp'_\tau(\frac{1}{N}) : 1]$.

The action of complex conjugation on x and y (denoted by a bar) is given by the diagram

$$\begin{tikzcd}
E_{\overline{x}} \arrow[r] \arrow[d] & E_x \arrow[r] \arrow[d] & E_1(N) \arrow[d] \\
\operatorname{Spec}\mathbb{C} \arrow[r] \arrow[urr, "\overline{y}"] \arrow[rr, bend right, "\overline{x}"'] & \operatorname{Spec}\mathbb{C} \arrow[r, "x"'] \arrow[ur, "y"'] & Y_1(N),
\end{tikzcd}$$

where the left map in the bottom row is induced by complex conjugation and both squares are cartesian. It is immediate to check that the elliptic curve $E_{\overline{x}}$ can be given by complex conjugating the coefficients in the Weierstraß equation for E_x and that the point $\overline{y}$ is obtained from y by complex conjugating its coordinates, if we view it as a point on $E_{\overline{x}}$ embedded into projective space. The same happens to the point P. Hence this action is compatible with the action of $\mathrm{G}_{\mathbb{R}}$ on E_{Wei}. □

2. Some sheaves on modular curves

2.1. The symmetric power local system

Fix $n \geq 0$. If f is the map from any of the universal elliptic curves described so far to the corresponding moduli space, then the local systems $\mathrm{R}^1 f_* \underline{\mathbb{Z}}$ and $\mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}}$ on this moduli space will play a fundamental role. We call the latter one the symmetric power local system.

For a discussion of symmetric powers and actions on them see appendix A.1. We will use the content of this section without further comments.

First, we study the local system $\mathrm{R}^1 f_* \underline{\mathbb{Z}}$ in the analytic setting, so let f be either f_{Wei} or f_{Lat} and look at the local system $\mathrm{R}^1 f_* \underline{\mathbb{Z}}$ on $\mathfrak{h}$. Since $\mathfrak{h}$ is contractible, this local system is in fact constant, so it is isomorphic to $\underline{\mathbb{Z}}^2$ (as one can see by looking at a stalk, using the topological proper base change theorem [Con09, Thm. 1.2.1.1]). However, there is no canonical isomorphism, we have to choose one. We choose the same one as in [Con09, (1.2.1.6), p. 34/35] and call it α. The same choice is also made (less explicitly) in [Kato4, §4.7].

To describe this explicitly, we focus on $f = f_{\mathrm{Lat}}$. For $\tau \in \mathfrak{h}$, write E_τ for the fibre of f_{Lat} over τ, so $E_\tau = \mathbb{C}/[1,\tau]$. The choice of a trivialisation $\alpha\colon \mathrm{R}^1 f_* \underline{\mathbb{Z}} \xrightarrow{\sim} \underline{\mathbb{Z}}^2$ is equivalent to a consistent choice of bases of the homology groups $H_1(E_\tau, \mathbb{Z}) \cong [1,\tau]$ for all τ, since its dual basis on cohomology defines such an isomorphism on each stalk, which suffices since the local system is constant. We choose once and for all the ordered basis $(\tau, 1)$ of this homology group.

Of course this choice is somewhat arbitrary, we could also choose another basis. Some consequences of this choice and how they compare with other situations in the literature are listed in appendix A.

From the action of $\Sigma = \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ on E_{Lat} and $\mathfrak{h}$, we get a Hecke sheaf structure on $\mathrm{R}^1 f_{\mathrm{Lat}*}\underline{\mathbb{Z}}$ (where we view $\mathbb{Z}$ as a trivial Σ-module, which gives us the constant Hecke sheaf $\underline{\mathbb{Z}}$ on E_{Lat}). On the other hand, the constant sheaf $\underline{\mathbb{Z}}^2$ on $\mathfrak{h}$ has a natural Hecke sheaf structure coming from the canonical left action of $\Sigma \subseteq \mathrm{M}_2(\mathbb{Z})$ on $\mathbb{Z}^2$ by left multiplication.

Lemma 2.1: *α is an isomorphism of Hecke sheaves on $\mathfrak{h}$.*

Proof: If we let Σ act on $\mathbb{Z}^2$ from the right by left multiplication with the transposed matrix, we can consider the attached constant Hecke sheaf $\underline{\mathbb{Z}}^2$ on $\mathfrak{h}$ coming from that action, and of course the claim is equivalent to the claim that the dual map $\alpha^\vee : \underline{\mathbb{Z}}^2 \xrightarrow{\sim} (\mathrm{R}^1 f_* \underline{\mathbb{Z}})^\vee$ is a morphism of Hecke sheaves for this Hecke sheaf structure on $\underline{\mathbb{Z}}^2$. This can be checked on stalks.

Let $\tau \in \mathfrak{h}$ and $\gamma \in \Sigma$ be given, and abbreviate $f = f_{\mathrm{Lat}}$. We use corollary 1.1.14 with $X = E_{\mathrm{Lat}}$, $Y = \mathfrak{h}$, $\mathcal{F} = \underline{\mathbb{Z}}$, $R = A = \mathbb{Z}$ and $p = \tau \in \mathfrak{h}$, which makes $X_p = E_\tau$. This tells us that the map on stalks $(\mathrm{R}^1 f_* \mathbb{Z})_{\gamma\tau} \longrightarrow (\mathrm{R}^1 f_* \mathbb{Z})_\tau$ is the dual of the map $\mathrm{H}_1(E_\tau, \mathbb{Z}) \longrightarrow \mathrm{H}_1(E_{\gamma\tau}, \mathbb{Z})$ induced by the map $\mathbb{C} \times \mathfrak{h} \longrightarrow \mathbb{C} \times \mathfrak{h}$ given by the action of γ. We thus have to show that the diagram

$$\begin{tikzcd} \mathrm{H}_1(E_\tau, \mathbb{Z}) \arrow[r, "\gamma"] & \mathrm{H}_1(E_{\gamma\tau}, \mathbb{Z}) \\ \mathbb{Z}^2 \arrow[u, "\alpha_\tau^\vee", "\sim"'] \arrow[r, "{}^{\mathsf{T}}\gamma'"] & \mathbb{Z}^2 \arrow[u, "\sim", "\alpha_{\gamma\tau}^\vee"'] \end{tikzcd}$$

commutes, where the top map is induced by the action of γ (on $\mathbb{C}$; we will denote it by "$\bullet$" below) and the bottom map is multiplication by ${}^{\mathsf{T}}\gamma'$ from the left (for the reason why this really has to be ${}^{\mathsf{T}}\gamma'$ and not ${}^{\mathsf{T}}\gamma$, see the last sentence in construction 1.1.10).

We first assume $\det\gamma > 0$ and write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. First note that ${}^{\mathsf{T}}\gamma' \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d \\ -b \end{smallmatrix}\right)$. Using this and the definition of $\alpha^\vee$, we calculate

$$\begin{aligned} \alpha_{\gamma\tau}^\vee({}^{\mathsf{T}}\gamma' \begin{pmatrix} 1 \\ 0 \end{pmatrix}) &= d \cdot \frac{a\tau + b}{c\tau + d} - b \cdot 1 \\ &= \frac{d(a\tau + b) - b(c\tau + d)}{c\tau + d} \\ &= (\det\gamma)(c\tau + d)^{-1}\tau, \end{aligned}$$

while $\gamma \bullet (\alpha_\tau^\vee \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = \gamma \bullet \tau = (\det\gamma)(c\tau + d)^{-1}\tau$ by definition of $\alpha^\vee$ and the definition of the action in (1.2.2). A similar calculation can be done with $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ instead of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. This shows the commutativity of the above diagram for $\det\gamma > 0$.

To complete the proof, it suffices to check the commutativity for $\gamma = \partial$. This works totally analogously to the above calculation. ☐

We now turn to $Y = Y_1(N)^{\mathrm{an}}$, so from now on we write f also for the map $E_1(N)^{\mathrm{an}} \longrightarrow Y_1(N)^{\mathrm{an}}$ or $\Gamma\backslash E_{\mathrm{Lat}} \longrightarrow \Gamma\backslash\mathfrak{h}$.

Lemma 2.2: *The sheaf* $\mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}}$ *on* $Y_1(N)^{\mathrm{an}} \cong \Gamma\backslash\mathfrak{h}$ *is isomorphic to* $\pi_*^{\Gamma} \underline{\mathrm{Sym}^n \mathbb{Z}^2}$ *as a* $\mathrm{G}_{\mathbb{R}}$*-sheaf, where* $\pi : \mathfrak{h} \longrightarrow \Gamma\backslash\mathfrak{h}$ *is the canonical projection.*

Proof: It is easy to see that the diagram of analytic spaces

$$\begin{array}{ccc} E_{\mathrm{Lat}} & \longrightarrow & \Gamma\backslash E_{\mathrm{Lat}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{h} & \xrightarrow{\pi} & \Gamma\backslash\mathfrak{h} \end{array}$$

is cartesian, where f also stands for the left map and π is the projection. By lemma 1.15, the maps are $\mathrm{G}_{\mathbb{R}}$-equivariant. By proposition I.1.40, the sheaf $\mathrm{R}^1 f_* \underline{\mathbb{Z}}$ on $\Gamma\backslash\mathfrak{h}$ is isomorphic to $\pi_*^{\Gamma} \pi^* \mathrm{R}^1 f_* \underline{\mathbb{Z}}$. Since the formation of the sheaf $\mathrm{R}^1 f_*(-)$ is compatible with base change, $\pi^* \mathrm{R}^1 f_* \underline{\mathbb{Z}}$ is the corresponding sheaf $\mathrm{R}^1 f_* \underline{\mathbb{Z}}$ on $\mathfrak{h}$. By lemma 2.1, the latter sheaf is isomorphic to the constant Hecke sheaf $\underline{\mathbb{Z}}^2$ on $\mathfrak{h}$, where $\mathbb{Z}^2$ carries the canonical action of $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ by left multiplication. ☐

Lemma 2.3: *The groups* $\mathrm{H}^1_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}})$ *are free of finite rank, for* ? *being "*c*" or "*p*". The group* $\mathrm{H}^1(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}})$ *is finitely generated.*

Proof: For ? = p and the final statement see [Con09, Lem. 2.3.2.4] and the comment before it. The case ? = c will be proved later in proposition III.2.2 (b). ☐

Note that the morphism $Y(N) \longrightarrow Y_1(N)$ from section 1.1 induces a map

$$\mathrm{H}^1_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}}) \longrightarrow \mathrm{H}^1_?(Y(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}}) \tag{2.1}$$

for ? being nothing, "c" or "p" by lemma 1.11 and the fact that the formation of the sheaf $\mathrm{Sym}^n_{\underline{\mathbb{Z}}} \mathrm{R}^1 f_* \underline{\mathbb{Z}}$ is compatible with base change.

We remark at this point that for any abelian group which is flat as a $\mathbb{Z}$-module, we have canonical isomorphisms

$$\mathrm{R}^1 f_* \underline{\mathbb{Z}} \otimes \underline{A} \xrightarrow{\sim} \mathrm{R}^1 f_* \underline{A}$$

and

$$\mathrm{H}^1_{\mathrm{p}}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}}) \otimes A \longrightarrow \mathrm{H}^1_{\mathrm{p}}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{A})$$

by [Con09, Lem. 1.7.7.2]. We will use this in the following without further comment.

2.2. Sheaves of differentials on modular curves

We begin with some algebraic considerations.

Definition 2.4: Let S be a scheme and $f\colon E \longrightarrow S$ be a generalised elliptic curve over it with unit section $e\colon S \longrightarrow E$. Define a sheaf on S by

$$\omega_{E/S} := e^*\Omega^1_{E/S}.$$

This sheaf has the following properties:

Lemma 2.5: *(a) The formation of the sheaf $\omega_{E/S}$ is compatible with base change along any $S' \longrightarrow S$.*

(b) The sheaves $\Omega^1_{E/S}$ and $\omega_{E/S}$ are line bundles.

(c) There is a canonical isomorphism $\omega_{E/S} \cong f_*\Omega^1_{E/S}$.

Proof: Statement (a) follows easily from the fact that the formation of $\Omega^1_{E/S}$ is compatible with base change. For (b) and (c) note that in our case $\Omega^1_{E/S}$ is the relative dualising sheaf for the morphism f; it is denoted by $\omega_{E/S}$ in [DR73]. With this observation the two claims follow from [DR73, Prop. I.1.6 (ii)]. □

This definition applies in particular when S is a modular curve and E is the universal (maybe generalised) elliptic curve.

Proposition 2.6: *Let Y be one of the modular curves $Y_*(N)^?$ we considered. Let X be the corresponding compactification, let E resp. $\overline{E}$ be the universal elliptic resp. generalised elliptic curve over Y resp. X, and let $C = C_*(N)^? = X \setminus Y$ be the cuspidal divisor. There is an isomorphism of line bundles on Y*

$$\omega^{\otimes 2}_{E/Y} \xrightarrow{\sim} \Omega^1_Y$$

called the Kodaira-Spencer map, which can be extended to X to an isomorphism

$$\omega^{\otimes 2}_{\overline{E}/X} \xrightarrow{\sim} \Omega^1_X(C).$$

Proof: This holds in fact on the stack $\overline{\mathcal{M}}_{1,1}$ of all (generalised) elliptic curves used also in the proof of lemma 1.11 by [DR73, §VI.4.5]. So by base change it holds on any modular curve. See also [KM85, (10.13.10), Thm. 10.13.11].

See [Con09, §1.5.3, Thm. 1.5.4.1, Thm. 1.5.7.1] for a proof in the analytic setting. □

Now we work in the analytic situation. For the rest of this section let Y denote either $Y(N)^{\mathrm{an}}$ or $Y_1(N)^{\mathrm{an}}$, let X be the corresponding compactification and $f\colon E \longrightarrow Y$ the universal elliptic curve over it. In the following, all sheaves occurring should be thought as living in the category of sheaves of just abelian groups.

Lemma 2.7: *The complex $\Omega^\bullet_{E/Y}$ is a resolution of $f^{-1}\mathcal{O}_Y$.*

Proof: The question is local on E. The map f is a proper submersion of relative dimension 1, so it is a family of complex manifolds in the sense of [Voi02, Def. 9.2] and the results from there apply. By [Voi02, Thm. 9.3] f is locally on E of the form

$$D \times Y \longrightarrow Y$$

where D is the unit circle in $\mathbb{C}$. So it suffices to prove the claim for f being the second projection $D \times Y \longrightarrow Y$, and we write p for the first projection. By the Poincaré Lemma, $\Omega_D^\bullet$ is a resolution of $\underline{\mathbb{C}}$ on D, so $p^{-1}\Omega_D^\bullet$ is a resolution of $\underline{\mathbb{C}}$ on $D \times Y$. We have

$$\Omega_{E/Y}^\bullet = f^{-1}\mathcal{O}_Y \underset{\underline{\mathbb{C}}}{\otimes} p^{-1}\Omega_D^\bullet$$

by [Har77, Prop. 8.10], so $\Omega_{E/Y}^\bullet$ is quasi-isomorphic to

$$f^{-1}\mathcal{O}_Y \underset{\underline{\mathbb{C}}}{\otimes} \underline{\mathbb{C}} = f^{-1}\mathcal{O}_Y. \qquad \square$$

Lemma 2.8: *There is a commutative diagram of sheaves on Y*

$$\begin{tikzcd}
\mathbf{R}^1 f_* \Omega_E^\bullet \arrow[r] \arrow[d, "\sim"'] & \mathbf{R}^1 f_* \Omega_{E/Y}^\bullet \arrow[d, "\sim"'] \\
\mathrm{R}^1 f_* \underline{\mathbb{C}} \arrow[r] & \mathrm{R}^1 f_* \underline{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_Y
\end{tikzcd}$$

in which the vertical maps are isomorphisms.

Proof: In the diagram of sheaves on E

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{C}} \arrow[r] \arrow[d] & \mathcal{O}_E \arrow[r] \arrow[d, equal] & \Omega_E^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_E \arrow[r] & \Omega_{E/Y}^1 \arrow[r] & 0
\end{tikzcd}$$

the rows are exact by lemma 2.7 (and the usual Poincaré lemma), and it is obvious that both squares commute. Hence we get a commutative square of morphisms of complexes of sheaves on E

$$\begin{tikzcd}
\Omega_E^\bullet \arrow[r] \arrow[d, "\mathrm{qis}"'] & \Omega_{E/Y}^\bullet \arrow[d, "\mathrm{qis}"'] \\
\underline{\mathbb{C}}[0] \arrow[r] & \underline{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} f^{-1}\mathcal{O}_Y[0]
\end{tikzcd}$$

in which the vertical maps are quasi-isomorphisms. By applying $\mathbf{R}^1 f_*$ to this square and using the projection formula, we get the desired diagram. $\square$

3. Hecke operators and related topics

In this section we explain how Hecke operators are defined algebraically. This is very important because it gives us Hecke actions on (most) cohomology groups attached to modular curves, compatible with additional structures, whereas the abstract Hecke theory from section 1.1 is limited to a more concrete setting. The two approaches will be compared in section 3.3.

3.1. Diamond automorphisms and Hecke correspondences

We first define certain automorphisms of modular curves giving rise to diamond operators. These are defined directly as automorphisms of modular curves. For details, see [Con09, §2.3.1, §4.2.7].

Definition 3.1: Let S be a scheme and let (E,P) be an S-valued point of $Y_1(N)$ or $X_1(N)$, given by a (maybe generalised) elliptic curve $E \longrightarrow S$ and a point $P \in E[N]$ of exact order N. Then for any $d \in (\mathbb{Z}/N)^\times$, dP is again a point of exact order N. The association $P \longmapsto dP$ defines an automorphism of the scheme $Y_1(N)$ or $X_1(N)$ which we denote by $\langle d\rangle$ and call a *diamond automorphism* (see remark 1.9).

This induces endomorphisms of cohomology groups of $X_1(N)$ or $Y_1(N)$. If $\mathcal{F}$ is any sheaf on $Y_1(N)$ or $X_1(N)$ and we are given a morphism $\langle d\rangle^*\mathcal{F} \longrightarrow \mathcal{F}$, then we get an induced endomorphism of $\mathrm{H}^i_?(Z,\mathcal{F})$, where Z stands for $X_1(N)$ or $Y_1(N)$ and ? is nothing, "c" or "p". We denote it again by $\langle d\rangle$.

Next, we define Hecke correspondences on modular curves and an abstract prototype of Hecke operators. For simplicity, we restrict to the case of the curve $Y_1(N)$ and the modules attached to it. Similar definitions can be made also for $Y(N)$ and the arithmetic versions, however we do not give the definitions here but refer to [Del69, (3.13)–(3.18)] for this.

Continue to assume $N \geq 4$. We need to study yet another moduli problem for elliptic curves. For this, let p be a prime. The $\Gamma_1(N,p)$ moduli problem classifies triples

$$(E \longrightarrow S, P, C)$$

with $E \longrightarrow S$ a generalised elliptic curve, P a (naive) $\Gamma_1(N)$-structure and C a locally free subgroup scheme of order of the smooth locus of E which is cyclic of order p, subject to some extra conditions. See [Con07, Def. 2.4.3] for the precise definition. It is representable over $\mathbb{Z}$ by a proper flat regular curve $X_1(N,p)$ which is projective and smooth over $\mathbb{Z}[1/N]$. This follows from [Con07, Thm. 3.1.7] by exactly the same argument used in the proof of theorem 1.5. If we restrict to usual (non-generalised) elliptic curve, we get a finite flat regular subscheme which is affine finite étale over $\mathbb{Z}[1/N]$, and which we denote by $Y_1(N,p)$. The universal elliptic curve over it will be denoted by $E_1(N,p) \longrightarrow Y_1(N,p)$.

We define two maps called *degeneracy maps*

$$\begin{aligned} \pi_1\colon\ & Y_1(N,p) \longrightarrow Y_1(N), \\ & (E,P,C) \longmapsto (E,P), \\ \pi_2\colon\ & Y_1(N,p) \longrightarrow Y_1(N), \\ & (E,P,C) \longmapsto (E/C, P \bmod C). \end{aligned} \tag{3.1}$$

By [Con07, Thm. 4.4.3] the morphisms π_1, π_2 extend uniquely to finite flat morphisms

$$\pi_1,\pi_2\colon X_1(N,p) \longrightarrow X_1(N).$$

We picture these morphisms in the diagram

$$\begin{tikzcd} & Y_1(N,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_1(N) & & Y_1(N) \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} & X_1(N,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1(N) & & X_1(N). \end{tikzcd} \tag{3.2}$$

Remark 3.2: Since $X_1(N)$ is proper over $\mathbb{Z}$ by theorem 1.5 and π_1 and π_2 are finite, hence also proper, it follows easily that the natural morphism $X_1(N,p) \longrightarrow X_1(N) \times_{\mathbb{Z}} X_1(N)$ arising from (3.2) is proper. Moreover its image has codimension $1 = \dim X_1(N)$, so this defines a correspondence (see definition I.3.1) on $X_1(N)$ called the p-th *Hecke correspondence.*

From the above maps we can build a diagram

$$\begin{tikzcd}
 & E_1(N,p) \arrow[rr, "\varphi"] \arrow[dl, "w_1"'] \arrow[dr, "u_1"] & & E_1(N,p)/C \arrow[dl, "u_2"'] \arrow[dr, "w_2"] & \\
E_1(N) \arrow[dr, "f"'] & & Y_1(N,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & E_1(N) \arrow[dl, "f"] \\
 & Y_1(N) & & Y_1(N) &
\end{tikzcd} \tag{3.3}$$

in which the squares are cartesian and C is the order p subgroup from the universal $\Gamma_1(N,p)$-structure on $E_1(N,p)$. Here φ is the canonical quotient map with kernel C, so it may be seen as the universal p-isogeny.

We now define an abstract prototype of Hecke operators. We are being a bit imprecise here; one could formulate this in full generality using Grothendieck's six functors formalism, but we want to apply this only for the following three types of sheaves: Zariski sheaves, étale sheaves, or sheaves on the analytification.

Suppose we are given a sheaf $\mathcal{F}$ on $E_1(N)$ (in this sense) and further a morphism of sheaves $\varphi^* w_2^* \mathcal{F} \longrightarrow w_1^* \mathcal{F}$ on $E_1(N,p)$ (in our application, there will always be a natural choice of such a morphism). Using the natural morphism $w_2^*\mathcal{F} \longrightarrow \varphi_*\varphi^* w_2^* \mathcal{F}$ (the unit for the adjunction $\varphi^* \dashv \varphi_*$), we get from this a morphism $w_2^*\mathcal{F} \longrightarrow \varphi_* w_1^* \mathcal{F}$. We want to apply $\mathrm{R}^i u_{2*}$ to this (for any $i \geq 0$). Since φ is an isogeny, it is a finite morphism, so φ_* is exact and we have $\mathrm{R}^i(u_2 \circ \varphi)_* = (\mathrm{R}^i u_{2*}) \circ \varphi_*$. Using $u_1 = u_2 \circ \varphi$, we thus get a morphism $\mathrm{R}^i u_{2*} w_2^* \mathcal{F} \longrightarrow \mathrm{R}^i u_{1*} w_1^* \mathcal{F}$. Since the two squares are cartesian, this is the same as a morphism

$$\varphi^* \colon \pi_2^* \mathrm{R}^i f_* \mathcal{F} \longrightarrow \pi_1^* \mathrm{R}^i f_* \mathcal{F}.$$

In the situations we described we have further a trace map[6]

$$\mathrm{tr}_{\pi_1} \colon \mathrm{H}^*(Y_1(N,p), \pi_1^*(\cdot)) \longrightarrow \mathrm{H}^*(Y_1(N), \cdot).$$

Here H^* means the appropriate cohomology for our situation, i. e. Zariski, étale or analytic sheaf cohomology. This allows us to make the following definition.

Definition 3.3: The *abstract p-th Hecke operator* is defined as

$$T_p \coloneqq \mathrm{tr}_{\pi_1} \circ \varphi^* \circ \pi_2^* \in \mathrm{End}(\mathrm{H}^q(Y_1(N), \mathrm{R}^i f_* \mathcal{F})),$$

for any $q \geq 0$, $i \geq 0$ and $\mathcal{F}$.

We will study this in more concrete situations in the next section and in section 4.3.

[6] For the analytic situation see remark I.1.46. For Zariski cohomology it comes from the construction explained in [Stacks, Tag 0BSY]. For étale cohomology see [SGA4.3, exp. XVII, §6.2].

3.2. Hecke operators in Betti and étale cohomology

We first apply the abstract definition of Hecke operators in the analytic situation and where the sheaf $\mathcal{F}$ on $E_1(N)^{\mathrm{an}}$ is the constant sheaf $\mathbb{Z}$ and $i = 1$. In this situation we have natural isomorphisms $w_2^*\underline{\mathbb{Z}} \cong \varphi^* w_1^* \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}$ since the maps w_1, w_2 and φ are surjective, so we have a natural choice for the morphism $w_2^*\mathcal{F} \longrightarrow \varphi^* w_1^* \mathcal{F}$. We can of course apply Sym^n for any $n \geq 0$ to all the morphisms of sheaves. We then get endomorphisms T_p of $\mathrm{H}^q_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}})$ for ? being nothing, "c" or "p". Further, it is easy to see that on $Y_1(N)^{\mathrm{an}}$ there is an isomorphism of sheaves $\langle d\rangle^* \mathrm{R}^1 f_* \underline{\mathbb{Z}} \cong \mathrm{R}^1 f_* \underline{\mathbb{Z}}$, so we get also diamond operators on $\mathrm{H}^q_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathbb{Z}})$.

Let now $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$. We can then look at étale cohomology of modular curves with coefficients in $\mathcal{O}/p^t$ (for integers $t \geq 0$) and sheaves constructed out of this, and by taking the limit we can take $\mathcal{O}$ as coefficients. Totally analogously as in the analytic situation described above, we can define Hecke and diamond operators also on these étale cohomology groups.

Proposition 3.4: *For ? being nothing, "c" or "p", there are canonical isomorphisms of $\mathcal{O}$-modules*

$$\mathrm{H}^1_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathcal{O}}) \cong \mathrm{H}^1_{\text{ét},?}(Y_1(N) \underset{\mathbb{Z}}{\times} \overline{\mathbb{Q}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathcal{O}}).$$

These isomorphisms respect the Hecke operators T_p and the diamond operators. The natural action of $\mathrm{G}_{\mathbb{Q}}$ on the right hand side commutes with these operators.

Proof: The comparison theorem of étale cohomology and its variant for compact support [SGA4.3, Exp. XI, Thm. 4.4; Exp. XVII, Cor. 5.3.5] show that the left hand side is isomorphic to $\mathrm{H}^1_{\text{ét},?}(Y_1(N) \times_{\mathbb{Q}} \mathbb{C}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathcal{O}})$. That this in turn is isomorphic to $\mathrm{H}^1_{\text{ét},?}(Y_1(N) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathrm{Sym}^n \mathrm{R}^1 f_* \underline{\mathcal{O}})$ follows from the smooth base change theorem in étale cohomology, see [SGA4½, Exp. V, Thm. 3.2 and Cor. 3.3].

That the isomorphisms are compatible with the Hecke operators is clear by construction. The final statement is also clear since the Hecke correspondences and all the maps involved in the definition of the T_p operators and the diamond operators are defined over $\mathbb{Q}$. ☐

Of course, one can perform similar constructions also with $Y(N)$ instead of $Y_1(N)$, but we omit this since we did not introduce Hecke correspondences in this context.

3.3. Comparison with the abstract Hecke theory

By lemma 2.2, we have an isomorphism

$$\mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Z}} \cong \pi_*^\Gamma \underline{\mathrm{Sym}^{k-2} \mathbb{Z}^2}$$

of $\mathrm{G}_{\mathbb{R}}$-sheaves on $Y_1(N)^{\mathrm{an}} \cong \Gamma\backslash\mathfrak{h}$, and $\underline{\mathrm{Sym}^{k-2} \mathbb{Z}^2}$ is a Hecke sheaf on $\mathfrak{h}$. Hence as explained in section 1.1.7, we have an action of the abstract standard Hecke algebra $\mathcal{H}(N)$ on its cohomology groups. The purpose of this section is to prove that this gives the same action as the definitions from the previous section.

In case of the diamond operators, it suffices to check that for $d \in (\mathbb{Z}/N)^\times$ the automorphism $\langle d\rangle$ of $Y_1(N)^{\mathrm{an}}$ corresponds under the identification $Y_1(N)^{\mathrm{an}} \cong \Gamma_1(N)\backslash\mathfrak{h}$ to the automorphism of $\Gamma_1(N)\backslash\mathfrak{h}$ induced by the action of a matrix $\sigma_d \in \mathrm{SL}_2(\mathbb{Z})$ with

$$\sigma_d \equiv \begin{pmatrix} d^{-1} & \\ & d \end{pmatrix} \bmod N$$

on $\mathfrak{h}$. This can be checked explicitly using the modular description of points on $\mathfrak{h}$, see e. g. [DS05, p. 175].

We now discuss the T_p operators. Put

$$\alpha := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix},$$

$\Phi_\alpha := \alpha^{-1}\Gamma_1(N)\alpha \cap \Gamma_1(N)$ and $\Phi^\alpha := \alpha\Phi_\alpha\alpha^{-1}$. It is easy to see that $\Phi_\alpha = \Gamma_1^0(N,p)$ and $\Phi^\alpha = \Gamma_{1,0}(N,p) = \Gamma_1(N) \cap \Gamma_0(Np)$. Further abbreviate $\Gamma = \Gamma_1(N)$. We then have a diagram

$$\begin{tikzcd} \Phi_\alpha\backslash\mathfrak{h} \arrow[r, "\alpha", "\sim"'] \arrow[d, "s_\alpha"'] & \Phi^\alpha\backslash\mathfrak{h} \arrow[d, "s^\alpha"] \\ \Gamma\backslash\mathfrak{h} & \Gamma\backslash\mathfrak{h} \end{tikzcd} \tag{3.4}$$

in which the vertical maps are the canonical projections and the map in the top row is induced by the action of α on $\mathfrak{h}$. Note that the latter map is an isomorphism with inverse given by the action of α^ι (because $\alpha\alpha^\iota$ is a scalar matrix, which acts trivially on $\mathfrak{h}$).

The analytic $\Gamma_1(N,p)$-moduli problem from section 3.1 can also be solved by a quotient of the upper half plane, as stated in the following theorem. Note the similarity to theorem 1.13.

Theorem 3.5: *There is a commutative diagram of analytic spaces*

$$\begin{tikzcd} E_1(N,p)^{\mathrm{an}} \arrow[r, "\sim"] \arrow[d] & \Phi_\alpha\backslash E_{\mathrm{Lat}} \arrow[d] \\ Y_1(N,p)^{\mathrm{an}} \arrow[r, "\sim"] & \Phi_\alpha\backslash\mathfrak{h} \end{tikzcd}$$

in which the horizontal maps are isomorphisms. The class of a point $\tau \in \mathfrak{h}$ corresponds to the triple $(E_\tau, 1/N, C_\tau)$ where E_τ is the complex elliptic curve $\mathbb{C}/[1,\tau]$ (with $[1,\tau]$ again denoting the $\mathbb{Z}$-sublattice of $\mathbb{C}$ generated by 1 and τ) and C_τ being the subgroup of E_τ generated by τ/p. In the diagram (3.4), the maps s_α and $s^\alpha \circ \alpha\colon \Phi_\alpha\backslash\mathfrak{h} \longrightarrow \Gamma\backslash\mathfrak{h}$ correspond to the maps π_1 and π_2 from (3.1).

Proof: That there is a bijection of sets is easy to see. This is stated e. g. in [DS05, Ex. 1.5.6], as well as the final statement about the projections. That the maps are analytic isomorphisms can be seen as in the proof of [Con09, Thm. 4.2.6.2]. $\square$

Denote the isomorphism $Y_1(N,p)^{\mathrm{an}} \xrightarrow{\sim} \Phi_\alpha\backslash\mathfrak{h}$ by ρ. This allows us to extend the diagram (3.4) to

$$\begin{tikzcd} E_1(N,p)^{\mathrm{an}} \arrow[r, "\sim"] \arrow[d, "u"'] & \Phi_\alpha\backslash E_{\mathrm{Lat}} \arrow[r, "\alpha"] \arrow[d] & \Phi^\alpha\backslash E_{\mathrm{Lat}} \arrow[d] & \\ Y_1(N,p)^{\mathrm{an}} \arrow[r, "\rho", "\sim"'] \arrow[d, "\pi_1"'] & \Phi_\alpha\backslash\mathfrak{h} \arrow[r, "\alpha", "\sim"'] \arrow[d, "s_\alpha"'] & \Phi^\alpha\backslash\mathfrak{h} \arrow[r, "\rho^{-1}\circ\alpha^\iota", "\sim"'] \arrow[d, "s^\alpha"] & Y_1(N,p)^{\mathrm{an}} \arrow[d, "\pi_2"] \\ Y_1(N)^{\mathrm{an}} \arrow[r, "\sim"] & \Gamma\backslash\mathfrak{h} & \Gamma\backslash\mathfrak{h} \arrow[r, "\sim"] & Y_1(N)^{\mathrm{an}}. \end{tikzcd}$$

The map $\alpha\colon \Phi_\alpha\backslash E_{\mathrm{Lat}} \longrightarrow \Phi^\alpha\backslash E_{\mathrm{Lat}}$ has degree p, and in the fibre over some $\tau \in \mathfrak{h}$, it is just the quotient map $E_\tau \longrightarrow E_{\tau/p}$, whose kernel is the subgroup C_τ generated by τ/p. Hence this map is a model of the universal p-isogeny. Further it is easy to see that the square

$$\begin{tikzcd}
\Phi^\alpha\backslash E_{\mathrm{Lat}} \arrow[r] \arrow[d] & \Gamma\backslash E_{\mathrm{Lat}} \arrow[dd] \\
\Phi^\alpha\backslash\mathfrak{h} \arrow[d, "\alpha'"', "\sim"] & \\
\Phi_\alpha\backslash\mathfrak{h} \arrow[r] & \Gamma\backslash\mathfrak{h}
\end{tikzcd}$$

is cartesian, so the left vertical map corresponds to v. This allows us to complete our previous diagram to

$$\begin{tikzcd}
E_1(N,p)^{\mathrm{an}} \arrow[r, "\sim"] \arrow[d, "u"'] \arrow[rrr, bend left, "\varphi"] & \Phi_\alpha\backslash E_{\mathrm{Lat}} \arrow[r, "\alpha"] \arrow[d] & \Phi^\alpha\backslash E_{\mathrm{Lat}} \arrow[r, "\sim"] \arrow[d] & (E_1(N,p)/C)^{\mathrm{an}} \arrow[d, "v"] \\
Y_1(N,p)^{\mathrm{an}} \arrow[r, "\rho", "\sim"'] \arrow[d, "\pi_1"'] & \Phi_\alpha\backslash\mathfrak{h} \arrow[r, "\alpha", "\sim"'] \arrow[d, "s_\alpha"'] & \Phi^\alpha\backslash\mathfrak{h} \arrow[r, "\rho^{-1}\circ\alpha'", "\sim"'] \arrow[d, "s^\alpha"] & Y_1(N,p)^{\mathrm{an}} \arrow[d, "\pi_2"] \\
Y_1(N)^{\mathrm{an}} \arrow[r, "\sim"] & \Gamma\backslash\mathfrak{h} & \Gamma\backslash\mathfrak{h} \arrow[r, "\sim"] & Y_1(N)^{\mathrm{an}}.
\end{tikzcd} \tag{3.5}$$

This diagram includes the (analytification of the) diagram (3.3).

Using this diagram, the definition of the T_p operator from definition 3.3 and proposition I.1.47, one can see that the action of T_p on $\mathrm{H}^i(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}})$ corresponds to the action of the element $T_p \in \mathcal{H}_+(N)$ on $\mathrm{H}^i(\frac{\mathfrak{h}}{\Gamma}, \underline{\pi_*^\Gamma \mathrm{Sym}^{k-2}\,\mathbb{Z}^2})$, for each $i \geq 0$. We omit the details necessary to verify this.

Henceforth we view $\mathrm{H}^i(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}})$ as modules over $\mathcal{H}_+(N)$. This gives a definition of operators T_n for any $n \in \mathbb{N}$.

3.4. The Atkin-Lehner involution

Over $\mathbb{Z}[\mu_N]$, the curve $Y_1(N)$ carries an important involution which enriches the theory of modular curves. We define it here and briefly state important properties. Similar definitions are made in [DR73, §VI.4.4], [KM85, §11.3] or [Con09, §2.3.6], but not precisely in our setting. Our exposition follows [FK12, §1.4.2].

Recall from remark 1.4 (d) that over $\mathbb{Z}[\mu_N]$ there is a canonical isomorphism of modular curves

$$v_N\colon Y_1(N)^{\mathrm{naive}}_{/\mathbb{Z}[\mu_N]} \xrightarrow{\sim} Y_1(N)^{\mathrm{arith}}_{/\mathbb{Z}[\mu_N]}.$$

There is another isomorphism between these modular curves which will be important.

We define

$$w_N\colon Y_1(N)^{\mathrm{arith}} \longrightarrow Y_1(N)^{\mathrm{naive}}$$

as the unique morphism which is given on points over $\mathbb{Z}[1/N]$ as follows (see remark 1.9). Let $(E,\alpha) \in Y_1(N)^{\mathrm{arith}}(S)$, so $E \longrightarrow S$ is an elliptic curve and $\alpha\colon \mu_N \hookrightarrow E[N]$ is a closed immersion. We then define

$$w_N(E, \alpha) \in Y_1(N)^{\mathrm{naive}}(S) := \left(E\big/\mathrm{im}(\alpha), \beta\right)$$

where β sends $1 \in \mathbb{Z}/N$ to the unique element $e \in E[N]$ such that $e_N(\alpha(\zeta), e) = \zeta$, where ζ is an N-th root of unity and e_N is the Weil pairing.

The resulting endomorphisms $w_N \circ v_N^{-1}$ of $Y_1(N)^{\mathrm{naive}}_{/\mathbb{Z}[\mu_N]}$ and $v_N^{-1} \circ w_N$ of $Y_1(N)^{\mathrm{arith}}_{/\mathbb{Z}[\mu_N]}$ are in fact involutions.

Definition 3.6: The endomorphisms $w_N \circ v_N^{-1}$ of $Y_1(N)^{\mathrm{naive}}_{/\mathbb{Z}[\mu_N]}$ and $v_N^{-1} \circ w_N$ of $Y_1(N)^{\mathrm{arith}}_{/\mathbb{Z}[\mu_N]}$ are called *Atkin-Lehner involutions.*

Following a common convention, we denote them again just by w_N because we will hardly ever use one of the isomorphisms v_N or w_N alone. We hope that this will not cause any confusion.

One can show that there is an equality of automorphisms

$$w_N \circ \langle d \rangle = \langle d \rangle^{-1} \circ w_N$$

for any $d \in (\mathbb{Z}/N)^{\times}$.

As in the case of diamond operators, the Atkin-Lehner involution induces endomorphisms of cohomology modules attached to $Y_1(N)$. For example we get an endomorphism w_N of $\mathrm{H}^i_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}})$ for any $k, i \geq 0$ and ? being nothing, "c" or "p", as we now explain. Look at the diagram

$$\begin{tikzcd}
E_1(N) \arrow[r, "\varphi"] \arrow[dr, "f"'] & E_1(N)/C \arrow[r, "\psi", "\sim"'] \arrow[d, "g"] & E_1(N) \arrow[d, "f"] \\
& Y_1(N) \arrow[r, "w_N"] & Y_1(N)
\end{tikzcd}$$

where C is the image of the universal $\Gamma_1(N)$-structure on $E_1(N)$ and the square is cartesian. This induces a morphism (see [Stacks, Tag 02N7])

$$w_N^* \mathrm{R}^1 f_* \underline{\mathbb{Z}} \longrightarrow \mathrm{R}^1 g_* \psi^* \underline{\mathbb{Z}} \longrightarrow \mathrm{R}^1 g_* \underline{\mathbb{Z}} \longrightarrow \mathrm{R}^1 f_* \underline{\mathbb{Z}}$$

of sheaves on $Y_1(N)^{\mathrm{an}}$ and thus an endomorphism of $\mathrm{H}^i_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}})$. It is important to note that this endomorphism is *not* an involution: we have $w_N^2 = (-N)^{k-2}$ as endomorphisms of $\mathrm{H}^1_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}})$ by [Con09, Lem. 2.3.6.1], so it is not even invertible. This may seem surprising, but the reason is that the top map $E_1(N) \longrightarrow E_1(N)$ in the diagram above is no longer an isomorphism, it has degree N.

Lemma 3.7: *Under the isomorphism from theorem 1.13, w_N becomes the involution of $\Gamma_1(N)\backslash\mathfrak{h}$ induced by the action of the matrix*

$$w_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}$$

on $\mathfrak{h}$.

Proof: [Con09, §2.3.6] ☐

The matrix w_N is the standard Atkin-Lehner element of level N from definition 1.1.60. By lemma 2.2 and corollary 1.1.43, there are isomorphisms

$$\mathrm{H}^1(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Z}}) \cong \mathrm{H}^1(\frac{\mathfrak{h}}{\Gamma_1(N)}, \pi_*^{\Gamma} \underline{\mathrm{Sym}^{k-2} \mathbb{Z}^2}) \cong \mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2} \mathbb{Z}^2). \tag{3.6}$$

On rightmost space the matrix w_N defines an endomorphism, as explained in section 1.1.6. Under the isomorphism (3.6), it corresponds to the one we defined before on the leftmost space.

Inspired from lemma 1.1.36, let us define the following:

Definition 3.8: Define endomorphisms of $\mathrm{H}^1_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}})$, for ? being nothing, "c" or "p", by putting

$$T_n^\iota := w_N^{-1} T_n w_N, \quad \langle d \rangle^\iota := w_N^{-1} \langle d \rangle w_N.$$

Define analogous endomorphisms in étale cohomology.

Note that since we used $\mathbb{Q}$ as coefficients, the Atkin-Lehner endomorphism w_N is in fact invertible on $\mathrm{H}^1_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}})$, so the above is well-defined. Moreover, it is clear that these endomorphisms correspond under the isomorphism (3.6) to the ones with the same name defined in definition 1.1.62 (b). As endomorphisms of $\mathrm{H}^1_?(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}})$ we have $\langle d \rangle^\iota = \langle d \rangle^{-1}$ and $T_n^\iota = \langle n \rangle^{-1} T_n$, see (1.1.9) and [Miy89, Thm. 4.5.4 (1)].

4. Modular forms

In this section, when we write "ring" we will always mean a commutative ring.

4.1. Definition(s) of modular forms

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and $k \geq 2$ an integer (we will not consider weights less than 2 in this work). The classical way to define modular forms (of weight k and level Γ) is to view them as holomorphic functions on the upper half plane $f: \mathfrak{h} \longrightarrow \mathbb{C}$ satisfying the transformation rule $f[\gamma]_k = f$ with respect to elements $\gamma \in \Gamma$ (where $[\cdot]_k$ was defined in example 1.1.24) and such that the limit

$$\lim_{y \to \infty} f[\gamma]_k(iy)$$

exists for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$; if the latter limit is always 0, then they are called cusp forms. We denote the $\mathbb{C}$-vector spaces consisting of classical modular resp. cusp forms by $\mathrm{M}_k(\Gamma)$ resp. $\mathrm{S}_k(\Gamma)$; we will however rarely use them. Here we shall define modular forms algebraically using the modular curves introduced before, which makes the arithmetic nature of modular forms more apparent. The resulting spaces will be denoted slightly differently to distinguish them from the classical modular forms.

From now on, we fix a level $N \geq 4$ and a weight $k \geq 2$ for the whole section. Let X be one of the modular curves $X_*(N)^?$, with $*$ being nothing or "1" and ? being "naive" or "arith", and let $\overline{E}$ be the universal generalised elliptic curve over it. Let Y be the corresponding open modular curve and E the universal elliptic curve over it. Let $C = X \setminus Y$ be the cuspidal divisor. Further let R be any ring, and write $X_{/R} := X \times_{\mathbb{Z}} R$ and for a sheaf $\mathcal{F}$ on X, we write $\mathcal{F}_{/R} := p^*\mathcal{F}$ for the pullback of $\mathcal{F}$ to $X_{/R}$ along the projection $p: X \times_{\mathbb{Z}} R \longrightarrow X$.

Definition 4.1 (Modular forms): (a) The R-module of *weight k modular forms on X* is defined as

$$\mathrm{M}_k(X,R) \coloneqq \mathrm{H}^0(X_{/R}, (\omega_{E/X}^{k-2} \underset{\mathcal{O}_X}{\otimes} \Omega_X^1(C))_{/R}).$$

(b) The R-module of *weight k cusp forms on X* is defined as

$$\mathrm{S}_k(X,R) \coloneqq \mathrm{H}^0(X_{/R}, (\omega_{E/X}^{k-2} \underset{\mathcal{O}_X}{\otimes} \Omega_X^1)_{/R}).$$

It is clear that $\mathrm{S}_k(X,R)$ is an R-submodule of $\mathrm{M}_k(X,R)$.

Proposition 4.2: *If R is a ring and S is a flat R-algebra, then we have natural isomorphisms*

$$\mathrm{M}_k(X,S) = \mathrm{M}_k(X,R) \underset{R}{\otimes} S, \quad \mathrm{S}_k(X,S) = \mathrm{S}_k(X,R) \underset{R}{\otimes} S.$$

Proof: This follows from [EGA3, Prop. 1.4.15] by choosing $X = X_{/R}$, $Y = \operatorname{Spec} R$ and $Y' = \operatorname{Spec} S$ there, together with the fact that base changes of surjective morphisms are surjective [Stacks, Tag 01S1], since the morphism $X \longrightarrow \operatorname{Spec} \mathbb{Z}$ is clearly surjective. □

Proposition 4.3: *There are canonical isomorphisms*

$$\mathrm{M}_k(X,R) \cong \mathrm{H}^0(X_{/R}, (\omega_{E/X}^{k})_{/R}), \quad \mathrm{S}_k(X,R) \cong \mathrm{H}^0(X_{/R}, (\omega_{E/X}^{k}(-C))_{/R}).$$

Proof: This follows from proposition 2.6. □

Remark 4.4: In view of proposition 4.3, we could also have defined modular forms and cusp forms by the cohomology groups appearing there on the right hand sides. This may seem more natural at a first glance, but in fact it isn't (at least for the applications we have in mind). First, in the way we initially defined the spaces of modular forms and cusp forms, they fit better with the motivic viewpoint, in that the space of cusp forms is naturally a subspace of the de Rham realisation of a motive called ${}^{N}_{k}\mathcal{W}$ to be introduced in section 5 below (see proposition 5.9 and the comments after it), and the definition of the Eichler-Shimura map is also more natural in this context (see section 6.1.1). Second, the definition of Hecke operators looks more natural in this description, see section 4.3 below.

Proposition 4.5: *For a ring R the R-modules $\mathrm{M}_k(X,R)$ and $\mathrm{S}_k(X,R)$ are free of finite rank.*

Proof: By proposition 4.2 it suffices to prove this for $R = \mathbb{Z}$. Of course it suffices to consider $\mathrm{M}_k(X,R)$. We distinguish the naive and arithmetic modular curves.

The naive modular curves are proper and flat over $\mathbb{Z}$. Since pushforwards of coherent sheaves under proper morphisms remain coherent, it is clear that $\mathrm{M}_k(X,\mathbb{Z})$ are finitely generated. So it remains to see that these groups are torsion free. To see this, cover X by affine schemes $\operatorname{Spec} A_1, \dots, \operatorname{Spec} A_r$. Then $\mathrm{M}_k(X,R)$ embeds into the direct sum

$$\mathrm{H}^0(\operatorname{Spec} A_1, \omega_{E/X}^k|_{\operatorname{Spec} A_1}) \oplus \cdots \oplus \mathrm{H}^0(\operatorname{Spec} A_r, \omega_{E/X}^k|_{\operatorname{Spec} A_r})$$

and it suffices to see that these summands are torsion free. Since X is flat, the A_i are flat $\mathbb{Z}$-modules. Since the sheave $\omega^k_{\overline{E}/X}$ is a line bundle, it is flat over A_r, hence over $\mathbb{Z}$, and the claim follows.

For the arithmetic modular curve the statement follows from the q-expansion principle below, see theorem 4.15 and remark 4.16. □

We now explain another way to define modular forms, which often goes by the name "Katz modular forms" or "geometric modular forms" in the literature, see for example [Kat73, §§1.1–2].

To begin with, let X be a scheme and $\mathcal{F}$ a sheaf on it. Then giving a global section $s \in \mathrm{H}^0(X, \mathcal{F})$ is equivalent to giving the following data: for each scheme $f: X_f \longrightarrow X$ over X a global section $s_f \in \mathrm{H}^0(X_f, f^*\mathcal{F})$ subject to the obvious compatibility relations: if we have morphisms

$$X_{f\circ g} \xrightarrow{g} X_f \xrightarrow{f} X$$

and $s_f \in \mathrm{H}^0(X_f, f^*\mathcal{F})$ and $s_{f\circ g} \in \mathrm{H}^0(X_{f\circ g}, g^* f^*\mathcal{F})$ are the attached global sections, then the natural map

$$\mathrm{H}^0(X_f, f^*\mathcal{F}) \longrightarrow \mathrm{H}^0(X_{f\circ g}, g^* f^*\mathcal{F})$$

should send s_f to $s_{f\circ g}$. This is true for trivial reasons: any global section $s \in \mathrm{H}^0(X, \mathcal{F})$ produces such a collection of compatible global sections on all X-schemes by pulling back, and any such collection gives a global section $s = s_{\mathrm{id}_X}$ on X by looking at $\mathrm{id}: X \longrightarrow X$, and these two processes are obviously inverse to each other.

If now X represents some set-valued functor $\mathcal{P}$ on $\mathcal{S}ch$,[7] then giving an X-scheme is equivalent to giving an element of $\mathcal{P}(X)$, so global sections in $\mathrm{H}^0(X, \mathcal{F})$ are equivalent to compatible global sections parametrised by $\mathcal{P}(X)$.

Let us now come back to the situation where X is a moduli scheme of elliptic curves with level structure as before. Then since formation of the sheaves $\omega_{\overline{E}/X}$ and Ω^1_X is compatible with base change, the above discussion easily shows that modular forms in $\mathrm{M}_k(X, R)$ are bijection with functions associating to each isomorphism class of generalised elliptic curves $E \longrightarrow S$ over an R-scheme S with appropriate level structure a global section of in $\mathrm{H}^0(S, \omega^k_{E/S})$, compatible with base change along morphisms of R-schemes in the above sense, and similarly for cusp forms.

Since any scheme is a sheaf for the Zariski topology,[8] it suffices to consider *affine* schemes S in the above description. If $S = \operatorname{Spec} A$ is affine, then since $\omega_{E/S}$ and Ω^1_S are line bundles, $\omega^{k-2} \otimes_{\mathcal{O}_S} \Omega^1_S$ corresponds to a free rank 1 A-module. This proves the following proposition.

Proposition 4.6 (Katz)**:** *The R-module of modular forms $\mathrm{M}_k(X, R)$ is canonically in bijection with "rules" f which assign to each triple $(E/A, \omega, \varphi)$ consisting of a generalised elliptic curve over an R-algebra A, a global section $\omega \in \mathrm{H}^0(S, \omega^{k-2}_{E/S} \otimes_{\mathcal{O}_S} \Omega^1_S)$ and a level structure φ (depending on what X is) an element $f(E/A, \omega, \varphi) \in R$ subject to the following conditions:*

(a) $f(E/A, \omega, \varphi)$ depends only on the R-isomorphism class of $(E/A, \omega, \varphi)$.

[7] Of course any scheme represents its own Hom functor. This is most interesting when $\mathcal{P}$ is a reasonably interesting moduli problem. Moreover, this can be used to define global sections on functors which are not representable.

[8] More precisely, we mean that the functor of points $\mathrm{Hom}_{\mathcal{S}ch}(-, S)$ of a scheme S is a sheaf on the big Zariski site, i. e. the category of all schemes with covering families being jointly surjective families of Zariski open immersions.

(b) f *is homogeneous of degree* k *in the sense* $f(E/A, \lambda\omega, \varphi) = \lambda^{-k} f(E/A, \omega, \varphi)$ *for* $\lambda \in R$.

(c) If $g\colon A \longrightarrow B$ *is a* R*-algebra morphism, then* $f(E \otimes_A B/B, \omega_B, \varphi_B) = g(f(E/A, \omega, \varphi))$ *(where* ω_B, φ_B *denote the pullbacks to* $E \otimes_A B$*).*

The above characterisation obviously does not need that the moduli functors for elliptic curves with level structures are representable. Thus it gives a way to define modular forms also for small level.

Remark 4.7: In some texts, modular forms are defined as rules obeying similar axioms as in proposition 4.6. Many texts do not use generalised elliptic curves as above, but just usual ones, and call the resulting objects meromorphic modular forms. Using the theory of q-expansions, one then defines holomorphic modular forms as the ones whose q-expansions are power series (see section 4.2). The element ω is often replaced by a nowhere vanishing translation invariant differential[9] ω on E. The reason for this is that for a usual elliptic curve $f: E \longrightarrow S$, for each global section $\omega_0 \in \mathrm{H}^0(S, \omega_{E/S})$ there exists a unique translation invariant differential $\omega \in \mathrm{H}^0(E, \Omega^1_{E/S})$ such that $e^*\omega = \omega_0$, see [BLR90, Prop. 4.2.1].

We end this section by briefly explaining why the spaces of modular forms introduced before give back the classical definition from the beginning of the section if we specialise to $R = \mathbb{C}$ and $X = X_1(N)$. For details see [Kato4, §3.8] or [Cono9, §1.5].

By GAGA it is clear that $\mathrm{S}_k(X_1(N), \mathbb{C})$ is isomorphic to the corresponding cohomology group on the analytification of $X_1(N)$, with the corresponding analytic sheaves of differentials (this is the definition given in [Cono9, Def. 1.5.7.3]). Using the cartesian diagram

$$\begin{array}{ccc} E_{\mathrm{Lat}} & \longrightarrow & E_1(N)^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathfrak{h} & \longrightarrow & Y_1(N)^{\mathrm{an}} \end{array}$$

we can pull back modular forms to $\mathfrak{h}$. More precisely, we obtain a map

$$\mathrm{S}_k(X_1(N), \mathbb{C}) \cong \mathrm{H}^0(X_1(N)^{\mathrm{an}}, ((\omega_{\overline{E}_1(N)/X_1(N)})^{\mathrm{an}})^{k-2} \underset{\mathcal{O}_{X_1(N)^{\mathrm{an}}}}{\otimes} \Omega^1_{X_1(N)^{\mathrm{an}}}) \longrightarrow$$
$$\mathrm{H}^0(Y_1(N)^{\mathrm{an}}, ((\omega_{E_1(N)/Y_1(N)})^{\mathrm{an}})^k) \longrightarrow \mathrm{H}^0(\mathfrak{h}, (\omega_{E_{\mathrm{Lat}}/\mathfrak{h}})^k),$$

with E_{Lat} as in section 1.2 and $\omega_{E_{\mathrm{Lat}}/\mathfrak{h}} := (f_{\mathrm{Lat}})_* \Omega^1_{E_{\mathrm{Lat}}/\mathfrak{h}}$. The latter sheaf is still a line bundle and thus globally trivial since $\mathfrak{h}$ is contractible (see [Cono9, §1.2.2, esp. Lem. 1.2.2.1]), and if we write (z, τ) for the standard coordinate on $\mathbb{C} \times \mathfrak{h}$ and recall that E_{Lat} is a quotient of $\mathbb{C} \times \mathfrak{h}$ we can write the pullback of an $f \in \mathrm{M}_k(X_1(N), \mathbb{C})$ to $\mathfrak{h}$ as

$$\tilde{f}(2\pi \mathrm{i} dz)^{\otimes k}$$

with a holomorphic function $\tilde{f}$ on $\mathfrak{h}$. The function $\tilde{f}$ satisfies the usual property

$$\tilde{f}(\gamma\tau) = (c\tau + d)^k f(\tau) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$$

[9] See [BLR90, §4.2] for the precise meaning of "translation invariant differential."

because the form $\tilde{f}(2\pi i \mathrm{d}z)^{\otimes k}$ has to be $\Gamma_1(N)$-invariant. This pullback establishes an isomorphism between $\mathrm{M}_k(X_1(N), \mathbb{C})$ and the space $\mathrm{M}_k(\Gamma_1(N))$ of classical modular forms for the group $\Gamma_1(N)$ as defined before.

Note that the above works for both $X_1(N)^{\text{naive}}$ and $X_1(N)^{\text{arith}}$ since they are isomorphic over $\mathbb{C}$. However if we replace $X_1(N)$ by $X(N)^{\text{naive}}$, then the space $\mathrm{M}_k(X(N)^{\text{naive}}, \mathbb{C})$ is *not* isomorphic to the space of classical modular forms for the group $\Gamma(N)$, but rather to a direct sum of $\varphi(N)$ copies of it, where φ is the Euler function. The reason for this is explained in remark 1.14 (b). If we use $X(N)^{\text{arith}}$ instead, by remark 1.14 (c) we do get the space of classical modular forms for the group $\Gamma(N)$.

From now on we will use the classical viewpoint as well and we will switch freely between the viewpoints without further comments, thus identifying f with $\tilde{f}$ as above if we are working over $\mathbb{C}$. With the link to the classical situation we have at once all theorems from the classical theory at our disposal. Some of these will be cited during the next sections, using that we now know how to translate between the viewpoints.

4.2. The Tate curve and q-expansions

The *Tate curve* E_{Tate} is a generalised elliptic curve over the ring $\mathbb{Z}[[q]]$ which has certain special properties and therefore plays an important role. It can be given explicitly as the curve in $\mathbb{P}^2_{\mathbb{Z}[q]}$ defined by a certain equation [DI95, Ex. 8.1.3], but there is a more conceptual construction due to Raynaud using formal schemes and algebraisation which is explained in [DR73, §VII.1] and [Con07, §2.5]. In this description the Tate curve can also be viewed as a quotient of $\mathrm{G_m}$ over $\mathbb{Z}[[q]]$.

We do not repeat the construction here, but we just list the properties we need. These are stated in [KM85, §8.8], [DR73, chap. VII, (1.16.1–4)] and [Con07, §2.5].

Fact 4.8: *The Tate curve E_{Tate} has the following properties:*

(a) Over $\mathbb{Z}((q))$ it is a smooth elliptic curve and its fibre over $q = 0$ is a Neron 1*-gon.*

(b) It carries a canonical translation invariant differential ω_{Tate}.

(c) For each N there a short exact sequence of group schemes over $\mathbb{Z}((q))$

$$0 \longrightarrow \mu_N \xrightarrow{a_N} E_{\text{Tate}}[N] \xrightarrow{b_N} \mathbb{Z}/N \longrightarrow 0$$

such that for any $\mathbb{Z}((q))$-algebra R, any $\zeta \in \mu_N(R)$ and any $x \in E_{\text{Tate}}[N](R)$ we have

$$e_N(a_N(\zeta), x) = \zeta^{b_N(x)}.$$

Corollary 4.9: *(a) The Tate curve has a canonical arithmetic $\Gamma_1(N)$-structure $\varphi^{\text{arith}}_{\text{Tate}}$.*

(b) Over $\mathbb{Z}[[q]] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_N]$ the Tate curve has a canonical naive $\Gamma_1(N)$-structure $\varphi^{\text{naive}}_{\text{Tate}}$.

Proof: Claim (a) follows from fact 4.8 (c). Claim (b) follows from this since μ_N and $\mathbb{Z}/N$ are canonically isomorphic over $\mathbb{Z}[\mu_N]$. ☐

Hence $(E_{\text{Tate}}, \omega_{\text{Tate}}, \varphi^{?}_{\text{Tate}})$ with ? being “arith” or “naive” is a triple as in proposition 4.6, and we can evaluate modular forms at this triple.

Definition 4.10: Let ? be either "arith" or "naive". Let R be any ring in the first case and a $\mathbb{Z}[\zeta_N]$-algebra in the second case.

Let $f \in \mathrm{M}_k(X_1(N)^?, R)$. We call $f(E_{\mathrm{Tate}}, \omega_{\mathrm{Tate}}, \varphi^?_{\mathrm{Tate}}) \in R[\![q]\!]$ the *q-expansion of f*. We call the resulting map

$$q\colon \mathrm{M}_k(X_1(N)^?, R) \longrightarrow R[\![q]\!], \quad f \longmapsto f(E_{\mathrm{Tate}}, \omega_{\mathrm{Tate}}, \varphi^?_{\mathrm{Tate}})$$

the *q-expansion map.* The coefficients of the power series are called the *Fourier coefficients* of f. We call f *normalised* if the coefficient of q^1 in its q-expansion is 1.

For the $\Gamma(N)$-moduli problems (both naive and arithmetic), one can of course also define q-expansion maps, however there are no canonical ones. For the $\Gamma_1(N)$-problems there are of course also more (non-canonical) q-expansion map, one for each choice of a level structure. We will not need these other q-expansion maps.

Remark 4.11: The following viewpoint on q-expansions is also important. Let again ? be either "arith" or "naive". Write as abbreviations $\mathbb{Z}^?$ to denote $\mathbb{Z}$ in the first case and $\mathbb{Z}[\zeta_N]$ in the second case, and $\mathbb{Z}[\![q]\!]^?$ to denote $\mathbb{Z}[\![q]\!]$ in the first case and $\mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_N]$ in the second case. Let R be a $\mathbb{Z}^?$-algebra.

The pair $(E_{\mathrm{Tate}}, \varphi^?_{\mathrm{Tate}})$ corresponds by the universal property of the modular curve $X_1(N)^?$ to a morphism $\operatorname{Spec} \mathbb{Z}[\![q]\!]^? \longrightarrow X_1(N)^?$. Pulling back a given $f \in \mathrm{M}_k(X_1(N)^?, R)$ along this morphism gives rise to a global section

$$f' \in \mathrm{H}^0(\operatorname{Spec}(R \underset{\mathbb{Z}^?}{\otimes} \mathbb{Z}[\![q]\!]^?), (e^*\Omega^1_{E_{\mathrm{Tate}} \times_{\mathbb{Z}[q]^?}(R \otimes_{\mathbb{Z}^?} \mathbb{Z}[\![q]\!]^?)/(R \otimes_{\mathbb{Z}^?} \mathbb{Z}[\![q]\!]^?)})^k_{/R}).$$

The latter is a free $R \otimes_{\mathbb{Z}^?} \mathbb{Z}[\![q]\!]^?$-module of rank 1 and ω^k_{Tate} is a canonical basis. Then $q(f)$ is the unique element in $R \otimes_{\mathbb{Z}^?} \mathbb{Z}[\![q]\!]^?$ such that $f' = q(f)\omega^k_{\mathrm{Tate}}$.

Lemma 4.12: *Let ? be either "arith" or "naive", and let $\mathbb{Z}[\![q]\!]^?$ be as in remark 4.11. Let $\infty_? \in X_1(N)^?$ be the image of the prime ideal (q) under the morphism $\operatorname{Spec} \mathbb{Z}[\![q]\!]^? \longrightarrow X_1(N)^?$ from remark 4.11. Then $\infty_?$ is a cusp of $X_1(N)^?$, i. e. $\infty_? \in C_1(N)^?$.*

Proof: We have to show that $\infty_?$ is not contained in $Y_1(N)^?$. For this it suffices to see that $\overline{E}_1(N)^?$ is not smooth over $\infty_?$. If it were smooth, then by base change E_{Tate} would be smooth over (q). But this is not the case by fact 4.8 (a). □

Definition 4.13: We call $\infty_?$ the *cusp at infinity* of $X_1(N)^?$.

Corollary 4.14: *Let ? and R be as before. If $f \in \mathrm{S}_k(X_1(N)^?, R)$ is a cusp form then $q(f) \in qR[\![q]\!]$.*

Proof: If f' is defined as in remark 4.11, then we have

$$f' \in \mathrm{H}^0(\operatorname{Spec}(R \underset{\mathbb{Z}^?}{\otimes} \mathbb{Z}[\![q]\!]^?), (e^*\Omega^1_{E_{\mathrm{Tate}} \times_{\mathbb{Z}[q]^?}(R \otimes_{\mathbb{Z}^?} \mathbb{Z}[\![q]\!]^?)/(R \otimes_{\mathbb{Z}^?} \mathbb{Z}[\![q]\!]^?)}(-C))^k_{/R})$$

because f is a cusp form. This means in particular that f' vanishes at $\infty_?$ by lemma 4.12, i. e. the pullback of f' to a global section on $\operatorname{Spec}(R \otimes_{\mathbb{Z}^?} \mathbb{Z}[\![q]\!]^?/(q))$ is zero. This means that $q(f)$ has to be divisible by q. □

Theorem 4.15 (Katz, q-expansion principle)**:** *Let ? and R be as before. Then the following hold:*

(a) *The q-expansion map $q\colon \mathrm{M}_k(X_1(N)^?, R) \longrightarrow R[\![q]\!]$ is injective.*

(b) *If $S \subseteq R$ is a subring resp. a $\mathbb{Z}[\zeta_N]$-subalgebra, then the diagram*

$$\begin{tikzcd} \mathrm{M}_k(X_1(N)^?, S) \arrow[r] \arrow[d, "q"'] & S[\![q]\!] \arrow[d, "q"] \\ \mathrm{M}_k(X_1(N)^?, R) \arrow[r] & R[\![q]\!] \end{tikzcd}$$

is cartesian. It other words, a modular form $f \in \mathrm{M}_k(X_1(N)^?, R)$ comes from a modular form in $\mathrm{M}_k(X_1(N)^?, S)$ if and only if its Fourier coefficients lie in S.

The analogous statement for cusp forms instead of modular forms is also true.

Proof: See [DI95, Thm. 12.3.4] for the case $X = X_1(N)^{\mathrm{arith}}$. The other case follow from this since $X_1(N)^{\mathrm{naive}}$ and $X_1(N)^{\mathrm{arith}}$ are isomorphic over $\mathbb{Z}[\zeta_N]$. ☐

Remark 4.16: For the modular curve $X(N)^{\mathrm{arith}}$, we also have a q-expansion principle (over $\mathbb{Z}$) similar to the one in theorem 4.15; see [Hid86b, p. 237]. We will not need this.

In the case $R = \mathbb{C}$ our notion of q-expansion as introduced above coincides with the classical q-expansion of modular forms defined via the theory of Fourier series. Using the description in remark 4.11, this follows from [Con09, §1.6.6, Def. 1.6.6.1, Thm. 1.6.6.2] (note that the analytification of our Tate curve E_{Tate} is denoted "$\underline{\mathrm{Tate}}_1$" there, see [Con09, Cor. 1.6.2.6 and its proof], and that the integer h there is 1 in our case).

Corollary 4.17: *For a subring R of $\mathbb{C}$, let $\mathrm{M}_k(\Gamma_1(N), R)$ be the R-submodule of classical modular forms $\mathrm{M}_k(\Gamma_1(N))$ whose Fourier coefficients lie in R.*

Then $\mathrm{M}_k(X_1(N)^{\mathrm{arith}}, R)$ is canonically isomorphic to $\mathrm{M}_k(\Gamma_1(N), R)$. For an arbitrary commutative ring R we have

$$\mathrm{M}_k(X_1(N)^{\mathrm{arith}}, R) = \mathrm{M}_k(\Gamma_1(N), \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} R.$$

Analogous statements hold also for cusp forms instead of modular forms.

The q-expansion principle is the main reason why one considers arithmetic level structures. The analogous statement for corollary 4.17 over $X_1(N)^{\mathrm{naive}}$, namely that for *any* subring R of $\mathbb{C}$ (in particular $R = \mathbb{Z}$) the R-module $\mathrm{M}_k(X_1(N)^{\mathrm{naive}}, R)$ is canonically isomorphic to the R-submodule of classical modular forms $\mathrm{M}(\Gamma_1(N), \mathbb{C})$ with Fourier coefficients in R, is *false*, see [Con07, Rem. 4.4.2]. The problem is that the cusp ∞_{naive} which belongs to the Tate curve is not defined over $\mathbb{Z}$ in this case, but only over $\mathbb{Z}[\mu_N]$, while for $X_1(N)^{\mathrm{arith}}$ the cusp ∞_{arith} is defined over $\mathbb{Z}$; see also [DI95, Rem. 12.3.6].

4.3. Hecke and Atkin-Lehner operators on modular forms

We now study Hecke operators on modular forms. For simplicity, we restrict our attention to $X_1(N)$, since we did not introduce Hecke correspondences for $X(N)$. Everything we do in this section works similarly also for $X_1(N)^{\mathrm{arith}}$.

So we go back to the setting from the end of section 3.1 and work with Zariski sheaves there. We choose $\mathcal{F} = \Omega^1_{E_1(N)/Y_1(N)}$ as the sheaf $\mathcal{F}$ on $E_1(N)$ from section 3.1, and $i = 0$

there. Then by base change, we have $w_1^*\Omega^1_{E_1(N)/Y_1(N)} = \Omega^1_{E_1(N,p)/Y_1(N,p)}$ and $w_2^*\Omega^1_{E_1(N)/Y_1(N)} = \Omega^1_{(E_1(N,p)/C)/Y_1(N,p)}$, so

$$\pi_1^*\omega_{E_1(N)/Y_1(N)} = \omega_{E_1(N,p)/Y_1(N,p)}, \quad \pi_2^*\omega_{E_1(N)/Y_1(N)} = \omega_{(E_1(N,p)/C)/Y_1(N,p)}.$$

Further φ induces a natural map

$$\varphi^*\Omega^1_{(E_1(N,p)/C)/Y_1(N,p)} \longrightarrow \Omega^1_{E_1(N,p)/Y_1(N,p)}$$

as required in the definition of the abstract Hecke operators. So we now have the map

$$\varphi^* \colon \pi_2^*\omega_{E_1(N)/Y_1(N)} \longrightarrow \pi_1^*\omega_{E_1(N)/Y_1(N)}.$$

By [Con09, Thm. 4.4.3] it extends to a map

$$\varphi^* \colon \pi_2^*\omega_{\overline{E}_1(N)/X_1(N)} \longrightarrow \pi_1^*\omega_{\overline{E}_1(N)/X_1(N)}$$

of sheaves on $X_1(N,p)$. Hence we can apply the construction from section 3.1 to get endomorphisms of cohomology groups on $X_1(N)$.

However, there is one subtlety when we apply this to modular forms: we do not just take k-th tensor powers of the above construction, but we rather take $(k-2)$-th tensor powers and then tensor with $\Omega^1_{X_1(N)}$. In this way we get endomorphisms of $\mathrm{M}_k(X_1(N),\mathbb{Z})$ and $\mathrm{S}_k(X_1(N),\mathbb{Z})$, and by base change also for other coefficient rings. The reason to do this is explained in remark 4.4. For the definition of Hecke operators on $\mathrm{H}^0(X_1(N),\omega^k_{E_1(N)/X_1(N)})$ using k-th powers the isomorphism from proposition 4.3 is *not* Hecke equivariant, rather the T_p on $\mathrm{M}_k(X_1(N),\mathbb{Z})$ corresponds to pT_p on $\mathrm{H}^0(X_1(N),\omega^k_{E_1(N)/X_1(N)})$ (see [Con07, §4.5]).

Since for $d \in (\mathbb{Z}/N)^\times$ we have a canonical morphism $\Omega^1_{E_1(N)/Y_1(N)} \longrightarrow \langle d\rangle_*\Omega^1_{E_1(N)/Y_1(N)}$, we get also diamond operators on modular forms, i. e. we have an action of $(\mathbb{Z}/N)^\times$.

One can prove that this definition of Hecke operators gives back the classical definition on the complex analytic fibre. This is shown during the proof of [Con09, Thm. 2.3.3.1]. Therefore we can now view $\mathrm{M}_k(X_1(N),R)$ and $\mathrm{S}_k(X_1(N),R)$ for any commutative ring R as modules over the standard Hecke algebra $\mathcal{H}_+(N)_R$.

Definition 4.18: Let $\Phi \subseteq (\mathbb{Z}/N)^\times$ be a subgroup and $\varepsilon \colon \Phi \longrightarrow R^\times$ be a character.

(a) If $f \in \mathrm{S}_k(X_1(N),R)$ is such that the action of Φ on f is given by the character ε, then we call ε the *nebentype* of f with respect to Φ.

(b) We denote by $\mathrm{S}_k(X_1(N),\Phi,\varepsilon,R)$ the submodule of $\mathrm{S}_k(X_1(N),R)$ of forms which have nebentype ε with respect to Φ. If $\Phi = (\mathbb{Z}/N)^\times$ we write just $\mathrm{S}_k(X_1(N),\varepsilon,R)$.

If the order of Φ is invertible in R, we have a decomposition

$$\mathrm{S}_k(X_1(N),R) = \bigoplus_{\varepsilon} \mathrm{S}_k(X_1(N),\Phi,\varepsilon,R)$$

where ε runs over all characters of Φ.

In most situations we will consider only the case $\Phi = (\mathbb{Z}/N)^\times$.

Definition 4.19: Let $f \in \mathrm{M}_k(X_1(N), R)$. Then f is called an *eigenform* if it is an eigenvector of all T_p for p prime and of all diamond operators $\langle d\rangle$ with $d \in (\mathbb{Z}/N)^\times$. By lemma 1.1.68 this is equivalent to f being an eigenvector of all T_n for $n \in \mathbb{N}$. It is called an *eigenform away from the level* if it is an eigenvector of all T_p for all primes $p \nmid N$ and of all diamond operators $\langle d\rangle$ with $d \in (\mathbb{Z}/N)^\times$, or again equivalently an eigenvector of all T_n for $(n, N) = 1$.

We now look at the Atkin-Lehner involution. Similarly as explained in section 3.4, we get an Atkin-Lehner endomorphism w_N of spaces of modular forms like $\mathrm{S}_k(X_1(N), R)$ for a $\mathbb{Z}[\mu_N]$-algebra R (which is not an involution, of course). Note that since we now work over $\mathbb{Z}[\mu_N]$, the naive and arithmetic modular curves are isomorphic.

From definition 3.8 we see that if f is an eigenform, then so is $w_N f$. However it need not be normalised even if f was normalised.[10]

Definition 4.20: Let R be a $\mathbb{Z}[\mu_N]$-algebra. For a normalised eigenform $f \in \mathrm{S}_k(X_1(N), R)$ we define the *dual eigenform* f^* to be the unique normalised eigenform which is a scalar multiple of $w_N f$.

Proposition 4.21 (Li): *Let K be a Galois number field containing the N-th roots of unity, and fix an embedding $K \subseteq \mathbb{C}$. Let $f \in \mathrm{S}_k(X_1(N), K)$ be a normalised eigenform and write*

$$q(f) = \sum_{n=1}^{\infty} a_n q^n, \quad a_n \in \mathbb{C}.$$

Then

$$q(f^*) = \sum_{n=1}^{\infty} \overline{a}_n q^n,$$

where $\overline{a}_n$ is the complex conjugate of a_n.

Proof: Define a cusp form $\tilde{f} \in \mathrm{S}_k(X_1(N), K)$ by its q-expansion

$$q(\tilde{f}) = \sum_{n=1}^{\infty} \overline{a}_n q^n.$$

This definition is also made at [Li75, top of p. 296]. The notation there is different: our $\tilde{f}$ is denoted $f|K$ there and our $w_N f$ is denoted $F|H_N$ there. Then at [Li75, bottom of p. 296] it is shown that $\tilde{f}$ is again an eigenform and that $w_N f$ is a nonzero multiple of $\tilde{f}$. Since $\tilde{f}$ is obviously normalised, we must have $\tilde{f} = f^*$. □

Definition 4.22: Let K be a number field and fix an embedding $K \subseteq \mathbb{C}$. Inspired from the above proposition, we define for any $f \in \mathrm{S}_k(X_1(N), K)$ the *dual cusp form* to be the $f^* \in \mathrm{S}_k(X_1(N), K)$ such that

$$q(f^*) = \sum_{n=1}^{\infty} \overline{a}_n q^n.$$

Similarly we define f^* for $f \in \mathrm{S}_k(X(N), K)$.

[10] Note that we defined the property "normalised" in terms of q-expansions, so we really need to work over a $\mathbb{Z}[\mu_N]$-algebra here.

4.4. Hecke eigenalgebras and their duality to modular forms

Let R be a ring. For simplicity we assume in this section that R is noetherian and flat over $\mathbb{Z}$, although some statements below are still true for general R (see the references for this). In our applications R will always satisfy these requirements.

Definition 4.23: Let $\Phi \subseteq (\mathbb{Z}/N)^\times$ be a subgroup and $\varepsilon\colon \Phi \longrightarrow R^\times$ be a character, and write X for either $X_1(N)^{\mathrm{arith}}$ or $X_1(N)^{\mathrm{naive}}$. We introduce the following notations for Hecke eigenalgebras, using the notation from section I.1.9:

- $\mathbf{T}_k(N,\Phi,\varepsilon,R) \coloneqq \mathbf{T}_R^{(\Delta_1(N),\Gamma_1(N))}(\mathrm{M}_k(X,\Phi,\varepsilon,R))$
- $\mathbf{T}_k^\iota(N,\Phi,\varepsilon,R) \coloneqq \mathbf{T}_R^{(\Delta_1(N)^\iota,\Gamma_1(N))}(\mathrm{M}_k(X,\Phi,\varepsilon,R))$

In the case $\Phi = (\mathbb{Z}/N)^\times$ we again omit Φ from the notation, and in the case $\Phi = \{1\}$ we omit both Φ and ε from the notation (i. e. $\mathbf{T}_k(N,R) \coloneqq \mathbf{T}_R^{(\Delta_1(N),\Gamma_1(N))}(\mathrm{M}_k(X,R))$ and so on). Analogously we can define eigenalgebras with modular forms replaced by cusp forms, and we denote these in the same way with "$\mathbf{T}$" replaced by "$\mathbf{t}$".

Note that our notation is not really well-defined yet, because on the right hand sides we could choose X to be either $X_1(N)^{\mathrm{arith}}$ or $X_1(N)^{\mathrm{naive}}$. But we will see below in lemma 4.25 that the Hecke eigenalgebras for both versions of the modular curve are in fact canonically isomorphic, so we do not distinguish them notationally.

Since the R-modules of modular resp. cusp forms are finitely generated by proposition 4.5, their endomorphism rings are also finitely generated as R-modules, and hence so are all the Hecke eigenalgebras defined above.

All these are quotients of the abstract standard Hecke algebra of level N, $\mathcal{H}_+(N)_R$, resp. the adjoint abstract standard Hecke algebra of level N, $\mathcal{H}_+(N)_R^\iota$. We denote the images of $T_n, S_\ell \in \mathcal{H}_+(N)$ and $T_n^\iota \in \mathcal{H}_+(N)^\iota$ in these eigenalgebras still by the same symbols. By lemma I.1.68 they contain the diamond operators $\langle d\rangle$ for $d \in (\mathbb{Z}/N)^\times$, and they are generated by these and the T_p for all primes p and also by all T_n for all $n \in \mathbb{N}$.

Lemma 4.24: *If S is a noetherian flat commutative R-algebra, then there is a canonical isomorphism*

$$\mathbf{T}_k(N,R) \underset{R}{\otimes} S \xrightarrow{\sim} \mathbf{T}_k(N,S),$$

and similarly for all the other Hecke eigenalgebras defined above.

Proof: This follows immediately from proposition 4.2 and lemma I.1.65. □

Lemma 4.25: *The Hecke eigenalgebras do not depend on whether we choose the naive or arithmetic version of $X_1(N)$ in their definition. If we define*

$$\tilde{\mathbf{T}}_k(N,R) \coloneqq \mathbf{T}_R^{(\Delta_1(N),\Gamma_1(N))}(\mathrm{H}^1_{\mathrm{c}}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_* \underline{R})),$$
$$\tilde{\mathbf{t}}_k(N,R) \coloneqq \mathbf{T}_R^{(\Delta_1(N),\Gamma_1(N))}(\mathrm{H}^1_{\mathrm{p}}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_* \underline{R})),$$

then there are canonical isomorphisms

$$\tilde{\mathbf{t}}_k(N,R) \cong \mathbf{t}_k(N,R), \quad \tilde{\mathbf{T}}_k(N,R) \cong \mathbf{T}_k(N,R).$$

Similar statements hold also for the other Hecke eigenalgebras introduced in definition 4.23.

We will prove this lemma later in section 6.1.4. Hereafter we identify the Hecke algebras from above which are isomorphic.

We now introduce an important pairing. To begin with, we define for each $n \in \mathbb{N}$ a linear form $a_n \colon \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, R) \longrightarrow R$ as the composition

$$\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, R) \xrightarrow{q} R[\![q]\!] \longrightarrow R,$$

where the right arrow maps a power series to the coefficient at q^n.

Lemma 4.26: *Let $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, R)$ and p be a prime.*

(a) We have $a_1(T_n f) = a_n(f)$.

(b) If f is an eigenvector of T_n with eigenvalue $a \in R$, then $a \cdot a_1(f) = a_n(f)$.

Proof: (a) By proposition 4.2 and R-linearity of the involved maps, it suffices to consider the case $R = \mathbb{Z}$. In this case the claim follows from the classical formulas for the action of Hecke operators on Fourier coefficients [Shi71, (3.5.12)].

(b) Using part (a) we get $a \cdot a_1(f) = a_1(T_n f) = a_n(f)$. ☐

Corollary 4.27: *(a) If f is a normalised eigenform, $\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, R)[f]$ is free of rank 1 over R, generated by f.*

(b) Normalised eigenforms in $\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, R)$ are in bijection with R-valued points in $\operatorname{Spec} \mathbf{t}_k(N, R)$. The bijection maps $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, R)$ to the morphism $\mathbf{t}_k(N, R) \longrightarrow R$ sending a Hecke operator to the corresponding eigenvalue.

Proof: We have

$$a_n(g) = a_1(T_n g) = \langle g, T_n \rangle = \langle T_n g, T_1 \rangle = a_n(f)\langle g, T_1 \rangle = a_1(g) a_n(f)$$

for all $n \in \mathbb{N}$, so by the q-expansion principle (theorem 4.15) it follows $g = a_1(g) f$. This proves the first statement. It follows that normalised eigenforms are in bijection with systems of Hecke eigenvalues that appear in $\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, R)$ (see definition 1.1.66). By proposition 1.1.67, these are in turn in bijection with R-valued points in $\operatorname{Spec} \mathbf{t}_k(N, R)$, so we also have the second statement. ☐

For Φ and ε as before, we now define a pairing

$$\langle \cdot, \cdot \rangle \colon \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \Phi, \varepsilon, R) \times \mathbf{t}_k(N, \Phi, \varepsilon, R) \longrightarrow R, \quad (f, T) \longmapsto a_1(Tf). \tag{4.1}$$

This pairing is obviously bilinear and Hecke equivariant in the following sense: for $T_1, T_2 \in \mathbf{t}_k(N, \Phi, \varepsilon, R)$ and $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \Phi, \varepsilon, R)$ we have $\langle T_1 f, T_2 \rangle = \langle f, T_1 T_2 \rangle$. The following important fact is derived essentially from the q-expansion principle.

Theorem 4.28: *The pairing (4.1) is perfect.*

Proof: [HidMFG, Thm. 3.17] ☐

Thus $\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \Phi, \varepsilon, R)$ can be identified with $\mathrm{Hom}_R(\mathbf{t}_k(N, \Phi, \varepsilon, R), R)$, where "$\mathrm{Hom}_R$" means morphisms of R-modules, and the normalised eigenforms correspond under this identification to the morphisms of R-algebras.

Corollary 4.29: *The Fourier coefficients of any $f \in S_k(X_1(N)^{\mathrm{arith}}, R)$ lie in a finitely generated $\mathbb{Z}$-submodule of R, and if f is a normalised eigenform, then they lie in a subring which is finitely generated over $\mathbb{Z}$ and hence integral.*

Proof: This follows from theorem 4.28 since $\mathbf{t}_k(N, R)$ is a finitely generated R-module and $\mathrm{Hom}_R(\mathbf{t}_k(N, R), R) = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{t}_k(N, \mathbb{Z}), R)$. $\square$

Remark 4.30: If R is a domain of characteristic 0, then one can always view the Fourier coefficients of any $f \in S_k(X_1(N)^{\mathrm{arith}}, R)$ as lying in some subfield of $\mathbb{C}$ by choosing some embedding, so one can view such modular forms as classical ones.

4.5. Newforms

Definition 4.31: Let X be one of the modular curves $X_*(N)^?$ with $*$ being nothing or "1" and ? being "naive" or "arith" and R be a commutative ring. Let $f \in S_k(X, R)$ be nonzero. Then f is called a *newform* if it is a normalised eigenform and moreover the following property holds:

> There do not exist a proper divisor $M \mid N$, a finite ring extension S of R and a $g \in S_k(X, S)$ which is an eigenform of almost all Hecke operators with the same eigenvalues as f. (4.2)

Newforms are usually studied in the classical context of the theory of Atkin, Lehner and Li, and they are sometimes also called "primitive forms". This theory is exposed in [Miy89, §4.6], and we first explain why our definition of newforms agrees with the definition of primitive forms there if $R = \mathbb{C}$ and $X = X_1(N)$.

If $f \in S_k(X_1(N), \mathbb{C})$ is a primitive form in the sense of [Miy89, p. 164, bottom], then by definition and [Miy89, Thm. 4.6.13] it is a normalised eigenform. If property (4.2) was violated, i. e. there would exist M and g as there, then by [Miy89, Lem. 4.6.2] g considered as an element of $S_k(X_1(N), \mathbb{C})$ would still have the same eigenvalues as f at almost all Hecke operators, so by [Miy89, Thm. 4.6.12] it would be a scalar multiple of f, but since it comes from a lower level, it would follow from the definition of primitive forms that g must be 0, which is a contradiction. In the other direction, if $f \in S_k(X_1(N), \mathbb{C})$ is not primitive in the sense of [Miy89], then by [Miy89, Cor. 4.6.14] there exists a proper divisor $M \mid N$ and an eigenform $g \in S_k(X_1(M), \mathbb{C})$ which is an eigenform of almost all Hecke operators with the same eigenvalues as f, so such an f is not a newform in our sense.

Theorem 4.32: *Let K be a field of characteristic 0 and $f \in S_k(X_1(N)^{\mathrm{arith}}, K)$ be a normalised eigenform away from the level. Write $S_k(X_1(N)^{\mathrm{arith}}, K)[f]'$ for the f-eigenspace of the restricted Hecke algebra $\mathcal{H}'_+(N)_K$, i. e. the submodule where the Hecke operators T_n for $(n, N) = 1$ act by the same eigenvalues as on f. Then the following are equivalent:*

(i) f is a newform.

(ii) f is a normalised eigenform away from the level and has property (4.2).

(iii) f is normalised, an eigenvector of T_p for almost all primes p and has property (4.2).

(iv) For each finite extension K' of K, $S_k(X_1(N)^{\mathrm{arith}}, K')[f]'$ is 1-dimensional.

If these hold, then for each finite extension K' of K there exists a unique (normalised) eigenform in $S_k(X_1(N)^{\mathrm{arith}}, K')$ whose eigenvalues away from the level coincide with the ones of f.

Proof: First note that by remark 4.30 we can assume without loss of generality that K is a subfield of $\mathbb{C}$. This enables us to exploit results from the classical theory.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial. Further if $S_k(X_1(N)^{\mathrm{arith}}, K')[f]'$ is 1-dimensional, then all its elements are eigenforms because Hecke operators commute and hence the space is stable under all Hecke operators, so (iv) implies the final statement.

For the implication (iii) $\Rightarrow$ (iv) we use that in the special case $K = \mathbb{C}$ the implication follows from [Miy89, Thm. 4.6.12] or [DS05, Thm. 5.8.2]; note that the statement there assumes the stronger condition from (ii), but the proof works in fact with the weaker one from (iii). For general K, if (ii) holds for f, then it follows from (4.2) that it still holds for f considered as an element in $S_k(X_1(N)^{\mathrm{arith}}, \mathbb{C})$, so $S_k(X_1(N)^{\mathrm{arith}}, \mathbb{C})[f]'$ is 1-dimensional. But $S_k(X_1(N)^{\mathrm{arith}}, K)[f]' \otimes_K \mathbb{C} \subseteq S_k(X_1(N)^{\mathrm{arith}}, \mathbb{C})[f]'$ and $S_k(X_1(N)^{\mathrm{arith}}, K)[f]'$ is non-trivial since it contains f, so it must be 1-dimensional. This shows (iii) $\Rightarrow$ (iv).

It remains to show (iv) $\Rightarrow$ (i). Over $\mathbb{C}$, this follows from [Miy89, Thm. 4.6.14] and [DS05, Thm. 5.8.3] (see also [Bel10, Thm. 1.6.4]). In general, if (iv) holds, then $S_k(X_1(N)^{\mathrm{arith}}, \mathbb{C})[f]'$ is also one-dimensional, hence f is a newform over $\mathbb{C}$. But then f has to be already a newform over K, which concludes the proof. ☐

Theorem 4.33: *Let K be a field of characteristic 0 and $f \in S_k(X_1(N)^{\mathrm{arith}}, K)$ be normalised eigenform away from the level. Then there is a divisor $M \mid N$, a finite extension K' of K and a newform $f_0 \in S_k(X_1(M)^{\mathrm{arith}}, K')$ whose eigenvalues for T_p with $(N, p) = 1$ are the same as the ones for f.*

Proof: Again by remark 4.30 we can assume without loss of generality that K is a subfield of $\mathbb{C}$. Then the result follows from [Miy89, Cor. 4.6.14] ☐

5. The motives ${}^{N}_{k}\mathcal{W}$ and $\mathcal{M}(f)$

We now introduce a motive called ${}^{N}_{k}\mathcal{W}$ which is defined in [Sch90] and a submotive $\mathcal{M}(f)$ which is the motive attached to a newform f. In Scholl's article the explanation of the construction is rather brief, but there is a draft of an unpublished book [Sch97] whose chapter 7 supplies a lot more details and also some motivation; see also [Sch94, §6] for some motivation. The construction is also given in [Sch88, §v.1] and [Kat04, §11.1–3].

As in section 1.3, we can view the motive ${}^{N}_{k}\mathcal{W}$ as given by its realisations and comparison isomorphisms. In this manner, the motive is described in detail in [DFG04, §1.2], [DFG01, §2]. Though, in [Sch90] ${}^{N}_{k}\mathcal{W}$ is introduced as a Chow motive in the rigorous sense.

5.1. Construction of the motives

Fix $N \geq 4$, $k \geq 2$. We explain the construction of the motives ${}^{N}_{k}\mathcal{W}$ and $\mathcal{M}(f)$ following [Sch97, chap. 7]. To be consistent with great parts of the literature on this topic, we use the modular curve $X(N)$ below, but it should be clear the same motive $\mathcal{M}(f)$ can also be constructed using $X_1(N)$ instead; this approach is taken in [Sch88, §v.1].

In this section we consider the modular curves as curves over $\mathbb{Q}$, but often we omit writing down the base change from $\mathbb{Z}$ to $\mathbb{Q}$ explicitly. We define $\mathrm{preKS}(N, k)$ to be the $(k-2)$-fold fibre product of the universal generalised elliptic curve $\overline{E}(N)$ with itself over $X(N)$. It is a singular projective variety over $\mathbb{Q}$.

Theorem 5.1 (Deligne, Scholl)**:** *There is a canonical desingularisation of* $\mathrm{preKS}(N,k)$*, i. e. a smooth projective variety* $\mathrm{KS}(N,k)$ *together with a canonical birational projective morphism* $\mathrm{KS}(N,k) \longrightarrow \mathrm{preKS}(N,k)$ *which is an isomorphism on the regular locus.*

Proof: [Sch90, Thm. 3.1.0]; [Sch97, chap. 7, esp. §7.4.1] □

Definition 5.2: The desingularisation $\mathrm{KS}(N,k)$ from theorem 5.1 is called the *Kuga-Sato variety of level* N *and weight* k.

We have several groups acting here. By remark 1.8, a point in $\overline{E}(N)$ over $\mathbb{Q}$ is a quadruple $(E/S, P, Q, x)$, where E is a generalised elliptic curve over a $\mathbb{Q}$-scheme S, P and Q are a basis of $E[N]$ and $x \in E(S)$. If we put $(a,b)(E/S,P,Q,x) := (E/S,P,Q,x+aP+bQ)$ for $a,b \in \mathbb{Z}/N$, then this defines an action of $(\mathbb{Z}/N)^2$ on $\overline{E}(N)$ (over $\mathbb{Q}$). Further sending $(E/S,P,Q,x)$ to $(E/S,P,Q,-x)$ defines an action of $\{\pm 1\}$, and we get an action of the semidirect product $(\mathbb{Z}/N)^2 \rtimes \{\pm 1\}$ on $\overline{E}(N)$. On $\mathrm{preKS}(N,k)$ we thus have an action of $((\mathbb{Z}/N)^2 \rtimes \{\pm 1\})^{k-2}$ and moreover of the symmetric group $\mathfrak{S}_{k-2}$ permuting the factors. Altogether we now have an action of

$$G(N,k) := \left(\left(\mathbb{Z}\big/N \right)^2 \rtimes \{\pm 1\} \right)^{k-2} \rtimes \mathfrak{S}_{k-2}$$

on $\mathrm{preKS}(N,k)$. This action extends to an action on $\mathrm{KS}(N,k)$, see [Sch97, §7.4.1].

As indicated at the beginning of section I.3, a motive may be described by a projective smooth variety and an idempotent endomorphism (and maybe a Tate twist). Hence the following is well-defined.

Definition 5.3: (a) Let $\varepsilon\colon G(N,k) \longrightarrow \{\pm 1\}$ be the character that is trivial on each factor $\mathbb{Z}/N$, is the product map on $\{\pm 1\}^{k-2}$ and the sign on $\mathfrak{S}_{k-2}$.

(b) Let $\pi_\varepsilon \in \mathbb{Q}[\Gamma_k]$ be the projector onto the ε-eigenspace and define a motive ${}^N_k\mathcal{W} := (\mathrm{KS}(N,k), \pi_\varepsilon)$. ${}^N_k\mathcal{W}$ is sometimes called the *parabolic cohomology motive*, see [Sch97, Def. 7.4.1.1].

One can show (see [Sch90, §4]) that the Hecke correspondences from remark 3.2 extend to correspondences on $\mathrm{KS}(N,k)$ and thus induce endomorphisms of the motive ${}^N_k\mathcal{W}$. This defines actions of Hecke operators on ${}^N_k\mathcal{W}$, and thus also on its realisations. Further the action of $\mathrm{GL}_2(\mathbb{Z}/N)$ on $E(N) \longrightarrow Y(N)$ also carries over to endomorphisms of ${}^N_k\mathcal{W}$. This raises the question whether it is possible to decompose ${}^N_k\mathcal{W}$ into Hecke eigenspaces.

Now fix a newform $f \in S_k(X_1(N), K)$ with coefficients in a number field K. In [Sch90, §4.2.0], Scholl explains that if we view ${}^N_k\mathcal{W} \otimes_{\mathbb{Q}} K$ as a Grothendieck motive, then one can indeed look at these eigenspaces, so that the following definition makes sense.

Definition 5.4: The motive $\mathcal{M}(f)$ attached to f is defined as the (Grothendieck) submotive of ${}^N_k\mathcal{W} \otimes_{\mathbb{Q}} K$ on which the Hecke operator T_p acts precisely by the Hecke eigenvalue a_p for all primes $p \nmid N$ (where the a_p are the Hecke eigenvalues of f) and where the subgroup

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_2\left(\mathbb{Z}\big/N\right)$$

acts trivially.

The first condition above is plausible from theorem 4.32. In view of lemma 1.10, the latter condition can be explained by the fact that f is a modular form for $\Gamma_1(N)$. We remark that Scholl uses the subgroup $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ instead, but he does not explain where in his case the $\mathrm{GL}_2(\mathbb{Z}/N)$-action comes from. This subgroup, acting from the left on level N structures (i.e. without transposing) fixes the subgroup $\{0\} \times \mathbb{Z}/N \subseteq (\mathbb{Z}/N)^2$, so it would correspond to another morphism $Y(N) \longrightarrow Y_1(N)$, forgetting the first basis vector of the N-torsion instead of the second. Probably Scholl uses this morphism, however we will work with the subgroup as above, since it seems to be the more common convention (e.g. used also in [Del79, §7.4], [KM85, Thm. 7.4.2 (3)]).

For working with the motive ${}^{N}_{k}\mathcal{W}$, we will mostly need only the attached premotivic structure. In the following sections, we describe the relevant realisations and comparison isomorphisms of ${}^{N}_{k}\mathcal{W}$ (which is a Chow motive, so its realisations are well-defined). Ultimately, we will be interested in the motive $\mathcal{M}(f)$, but since it is only a Grothendieck motive, strictly speaking it does not have realisations right-away (see the beginning of section 1.3). But by its construction, we may take the subspaces of the realisations of ${}^{N}_{k}\mathcal{W}$ tensored with K where the Hecke correspondences act by the eigenvalues given by f and $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$ acts trivially as the realisations of $\mathcal{M}(f)$, so this particular Grothendieck motive does have realisations and we have an attached premotivic structure.

Remark 5.5: Let us remark at this point that whenever we later work with the motive $\mathcal{M}(f)$, it will not do any harm to assume that K is large enough to contain the N-th roots of unity. Under this assumption the modular curves $X_1(N)^{\mathrm{naive}}$ and $X_1(N)^{\mathrm{arith}}$ become canonically isomorphic. In our descriptions of the motives we always work with the naive version to be consistent with the literature, while in many other situations we work with the arithmetic version because it is better suited for q-expansions. Assuming that K is large enough will thus allow us to use results that initially require the arithmetic version also in motivic situations. We will not always write this down explicitly.

5.2. The Betti realisation

Theorem 5.6: *The Betti realisation of ${}^{N}_{k}\mathcal{W}$ is the parabolic cohomology group*

$$ {}^{N}_{k}\mathcal{W}_{\mathrm{B}} = \mathrm{H}^1_{\mathrm{p}}(Y(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}}). $$

The isomorphism comes from the Leray spectral sequence for the morphism $\mathrm{KS}(N,k) \longrightarrow X(N)$. *The action of* $G_{\mathbb{R}}$ *comes from its natural action on* $Y(N)^{\mathrm{an}}$ *and* $E(N)^{\mathrm{an}}$ *and the resulting* $G_{\mathbb{R}}$*-sheaf structure on* $\mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}}$.

Proof: [Sch90, Thm. 1.2.1]; [Sch97, Thm. 7.4.1.3] □

On the group $\mathrm{H}^1_{\mathrm{p}}(Y(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}})$, we have Hecke operators as explained in section 3.2. The motive ${}^{N}_{k}\mathcal{W}$ also carries Hecke operators induced by the Hecke correspondences on $\mathrm{KS}(N,k)$. These definitions are compatible by [Sch90, Prop. 4.1.1].

Remark 5.7: The Betti realisation can also be described more explicitly as follows. The Riemann surface $Y(N)^{\mathrm{an}}$ is a disjoint union of $\varphi(N)$ copies of $Y := \Gamma(N)\backslash\mathfrak{h}$ (see remark 1.14 (b)). By choosing any point x_0 on Y as a base point, we can identify $\Gamma(N)$ with the fundamental group $\pi_1(Y, x_0)$. The stalk of $\mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Q}}$ at x_0 is isomorphic to $\mathrm{Sym}^{k-2} \mathbb{Q}^2$ (after choosing a base), and by monodromy, it carries a $\Gamma(N)$-action. By lemma 2.1, this action is just the canonical one coming from left multiplication. We therefore have a canonical isomorphism (using corollary I.1.43)

$$ {}^N_k\mathcal{W}_{\mathrm{B}} \cong \bigoplus \mathrm{H}^1_{\mathrm{p}}(\Gamma(N), \mathrm{Sym}^{k-2}\,\mathbb{Q}^2), $$

where "$\oplus$" means that we have $\varphi(N)$ copies of the right group. Moreover, by lemma 2.1 this isomorphism respects the action of the abstract standard Hecke algebra $\mathcal{H}(N)$. This allows us to use corollary I.1.27, which gives us a rather explicit description of Hecke operators. Further it says in combination with lemma 1.15 and proposition 1.16 that complex conjugation on ${}^N_k\mathcal{W}$ is induced by the action of the matrix ∂.

5.3. The de Rham realisation

A description of the de Rham realisation of ${}^N_k\mathcal{W}$ is given in [DFG04, §1.2.4]. There is also a more extensive unpublished version [DFG01] of [DFG04] which contains more details. We briefly recall this construction here; for details, see the above-mentioned articles. We use the language of log schemes [Kat89].

We use again the notations from section 1.1. The cuspidal divisor $C(N) = X(N) \smallsetminus Y(N)$ defines a logarithmic structure on $X(N)$ and its preimage in $\overline{E}(N)$ defines a logarithmic structure on $\overline{E}(N)$. Then there is the notion of a sheaf of logarithmic relative differentials, see [Kat89, §1.7]. It is denoted ω in the reference [DFG01] and Ω in [Kat89], but to avoid any confusion with the usual differentials or the sheaf ω defined in definition 2.4, we will denote it by $\mho$.

We look at the sheaves of logarithmic relative differentials

$$ \mho^1_{\overline{E}(N)/X(N)} \quad \text{and} \quad \mho^1_{X(N)/\mathbb{Z}} =: \mho^1_{X(N)}. $$

The latter one is just the sheaf of differentials with logarithmic poles $\Omega^1_{X(N)}(\log C(N))$. There is the logarithmic de Rham complex, which has nonzero entries only in degree 0 and 1, since $X(N)$ is a curve, and hence looks like

$$ \mho^\bullet_{\overline{E}(N)/X(N)} = (\underset{\substack{\uparrow \\ 0}}{\mathcal{O}_{X(N)}} \xrightarrow{\mathrm{d}} \mho^1_{\overline{E}(N)/X(N)}). $$

We put

$$ \mathcal{E} = \mathbf{R}^1 f_* \mho^\bullet_{\overline{E}(N)/X(N)}, \qquad \mathcal{E}_k = \mathrm{Sym}^{k-2}_{\mathcal{O}_{X(N)}} \mathcal{E}, \qquad \mathcal{E}_{k,\mathrm{c}} = \mathcal{E}_k(-C(N)). $$

On $\mathcal{E}$, there is the logarithmic Gauß-Manin connection

$$ \nabla \colon \mathcal{E} \longrightarrow \mathcal{E} \underset{\mathcal{O}_{X(N)}}{\otimes} \mho^1_{X(N)}, $$

and this induces flat logarithmic connections on $\mathcal{E}_k$ and $\mathcal{E}_{k,\mathrm{c}}$ which we denote by ∇_k and $\nabla_{k,\mathrm{c}}$, respectively. We denote the complexes of sheaves on $X(N)$ defined by these connections by $\mathcal{E}^\bullet_k$ and $\mathcal{E}^\bullet_{k,\mathrm{c}}$, respectively.

Theorem 5.8: *The de Rham realisation of ${}^N_k\mathcal{W}$ is*

$$ {}^N_k\mathcal{W}_{\mathrm{dR}} = \mathrm{image}(\mathbf{H}^1(X(N), \mathcal{E}^\bullet_{k,\mathrm{c}}) \longrightarrow \mathbf{H}^1(X(N), \mathcal{E}^\bullet_k)) \otimes \mathbb{Q}. $$

The isomorphism comes from the Leray spectral sequence for the morphism $\mathrm{KS}(N,k) \longrightarrow X(N)$.

Proof: This is claimed without proof in [DFG01, §2.2, p. 15]. The analogous statement for Betti and étale cohomology is proved in [Sch97, Thm. 7.4.1.3], and the technique used there can be adapted to algebraic Rham cohomology. □

5.4. The Hodge filtration and the Hodge realisation

Proposition 5.9: *The Hodge filtration of the de Rham realisation of ${}^N_k\mathcal{W}$ is given by*

$$\mathrm{fil}^i\, {}^N_k\mathcal{W}_{\mathrm{dR}} = \begin{cases} {}^N_k\mathcal{W}_{\mathrm{dR}} & \text{for } i \leq 0, \\ \mathrm{S}_k(X(N), \mathbb{Q}) & \text{for } 1 \leq i \leq k-1, \\ 0 & \text{for } i \geq k. \end{cases}$$

Proof: This is [Kato4, (11.2.5)]. The middle term in the filtration is denoted $S_k(X(N))$ there; to see that it equals our $\mathrm{S}_k(X(N), \mathbb{Q})$ see its definition in [Kato4, §3.1] and note that Kato considers $X(N)$ as a curve over $\mathbb{Q}$. □

The de Rham realisation of course also carries a Hecke action, since the Hecke operators are endomorphisms of the motive ${}^N_k\mathcal{W}$. The embedding of the cusp forms is Hecke equivariant.

We will now describe explicitly where the embedding of the intermediate filtration step $\mathrm{S}_k(X(N), \mathbb{Q})$ comes from. For this, we continue to use the notations from the preceding sections.

Over $Y(N)$, $\omega_{Y(N)}$ is isomorphic to $f_*\Omega^1_{E(N)/Y(N)}$ and $\mathcal{E}$ is isomorphic to $\mathbf{R}^1 f_*\Omega^\bullet_{E(N)/Y(N)}$, and this gives a canonical map

$$\omega_{Y(N)} \longrightarrow \mathcal{E}|_{Y(N)}$$

just as in (6.1), which is just the Hodge filtration of $\mathcal{E}|_{Y(N)}$. It can be extended to the cusps to give a morphism

$$\omega^{k-2}_{X(N)} \underset{\mathcal{O}_{X(N)}}{\otimes} \Omega^1_{X(N)} \longrightarrow \mathcal{E}_{k,\mathrm{c}} \underset{\mathcal{O}_{X(N)}}{\otimes} \mho^1_{X(N)}. \tag{5.1}$$

The injection

$$\mathrm{S}_k(X(N), \mathbb{Z}) \hookrightarrow \mathbf{H}^1(X(N), \mathcal{E}^\bullet_{k,\mathrm{c}})$$

is then obtained by considering $\omega^{k-2}_{X(N)} \otimes_{\mathcal{O}_{X(N)}} \Omega^1_{X(N)}$ as a complex concentrated in degree 1, which via (5.1) gives us a morphism of complexes

$$(\omega^{k-2}_{X(N)} \underset{\mathcal{O}_{X(N)}}{\otimes} \Omega^1_{X(N)})[-1] \longrightarrow \mathcal{E}^\bullet_{k,\mathrm{c}}$$

to which we can apply $\mathbf{H}^1$. That it is an injection can be proved similarly as [Sch85, Thm. 2.7 (i)].

We also want to say how the cokernel of the above injection, i.e. $\mathrm{gr}^0\, {}^N_k\mathcal{W}_{\mathrm{dR}}$, looks like.

Proposition 5.10: *We have canonically*

$$\mathrm{gr}^0\, {}^N_k\mathcal{W}_{\mathrm{dR}} = \mathrm{H}^1(X(N)_{/\mathbb{Q}}, (\omega^{2-k}_{X(N)})_{/\mathbb{Q}}) \cong \mathrm{S}_k(X(N), \mathbb{Q})^\vee.$$

Here $(\cdot)^\vee$ denotes the dual $\mathbb{Q}$-vector space and the last isomorphism comes from Serre duality.

We will prove this proposition later in section 6.1.4. The statement about Serre duality is obvious since $(\Omega^1_{X(N)})_{/\mathbb{Q}}$ is a dualising sheaf on the proper smooth curve $X(N)_{/\mathbb{Q}}$.

Corollary 5.11: *The Hodge realisation of* ${}^{N}_{k}\mathcal{W}$ *is*

$$ {}^{N}_{k}\mathcal{W}_{\mathrm{H}} = \mathrm{H}^1(X(N)_{/\mathbb{Q}}, (\omega^{2-k}_{X(N)})_{/\mathbb{Q}}) \oplus \mathrm{S}_k(X(N), \mathbb{Q}) $$

with the first summand sitting in degree 0 *and the second summand sitting in degree* $k-1$.

5.5. The p-adic realisation

Fix a prime p, a number field K, a prime $\mathfrak{p} \mid p$ of K and a newform $f \in \mathrm{S}_k(X_1(N), K)$. Write $L = K_{\mathfrak{p}}$ and $\mathcal{O}$ for its ring of integers.

Theorem 5.12: *(a) The p-adic realisation of* ${}^{N}_{k}\mathcal{W}$ *is*

$$ {}^{N}_{k}\mathcal{W}_p = \mathrm{H}^1_{\mathrm{p,\acute{e}t}}(Y(N) \underset{\mathbb{Z}}{\times} \overline{\mathbb{Q}}, \mathrm{Sym}^{k-2}\, \mathrm{R}^1 f_* \underline{\mathbb{Q}_p}). $$

It has Hodge-Tate weights $k-1$ *and* 0*. The isomorphism comes from the Leray spectral sequence for the morphism* $\mathrm{KS}(N,k) \longrightarrow X(N)$.

(b) The $\mathfrak{p}$*-adic realisation of* $\mathcal{M}(f)$ *is Deligne's Galois representation attached to* f*. More precisely, it is a two-dimensional odd irreducible L-linear representation of* $\mathrm{G}_{\mathbb{Q}}$ *which is unramified outside* $Np\infty$ *and such that for each prime* ℓ *we have*

$$ \mathrm{P}_\ell(\mathcal{M}(f)_{\mathfrak{p}}, T) = 1 - a_\ell T + \psi(\ell)\ell^{k-1}T^2, $$

where ψ *is the Nebentype of* f *and* P_ℓ *is as in definition I.3.11 (and* $\psi(\ell) = 0$ *if* $\ell \mid N$*). Note that this includes both the case* $\ell = p$ *and* $\ell \mid N$*, so in particular* $\mathcal{M}(f)_{\mathfrak{p}}$ *is crystalline at p if and only if* $p \nmid N$*. In particular, conjecture I.3.12 holds for* $\mathcal{M}(f)$.

(c) The determinant of the representation $\mathcal{M}(f)_{\mathfrak{p}}$ *is the character* $\psi^* \kappa_{\mathrm{cyc}}^{1-k}$.

Proof: For (a) see [Sch90, §1.2.0–1] or [Sch97, Thm. 7.4.1.3]. For the case $\ell \nmid N$ (including $\ell = p$) in (b) see [Sch90, Thm. 1.2.4], for the case $\ell \mid N$ see [Sai97]. See also [Kat04, (14.10.3–4)]. Part (c) follows directly from part (b) using Chebotarev's density theorem. □

At this point we define the notions of slopes and ordinariness, which are p-adic properties of modular forms.

Definition 5.13: Let $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \mathcal{O})$ be an eigenform and let $a_p \in K$ be the eigenvalue of T_p on f. The *slope of* f *at* $\mathfrak{p}$ is the $\mathfrak{p}$-adic valuation of a_p. We call f *ordinary at* $\mathfrak{p}$ if it has slope 0 (i. e. $a_p \in \mathcal{O}^\times$), *of finite slope at* $\mathfrak{p}$ if its slope is finite (i. e. $a_p \neq 0$) and *of critical slope at* $\mathfrak{p}$ if its slope is $k-1$.

Since Hecke eigenvalues are algebraic integers, slopes of eigenforms are always non-negative. The name "critical slope" is justified by the following fact.

Proposition 5.14: *The slope of an eigenform* f *of weight* k *is at most* $k-1$.

Proof: [Pol14, §3.7] □

For ordinary newforms the associated Galois representation is ordinary in the sense of [Per94], which is the content of theorem 5.15 (a) below.

Theorem 5.15: *If* f *is ordinary at* $\mathfrak{p}$*, then the following statements hold.*

(a) *There is a* 1*-dimensional unramified* $\mathrm{G}_{\mathbb{Q}_p}$*-subrepresentation* $\mathcal{M}(f)^0_{\mathfrak{p}}$ *of* $\mathcal{M}(f)_{\mathfrak{p}}$.

(b) $\mathcal{M}(f)(1)$ *satisfies the strong Dabrowski-Panchishkin condition with*

$$\mathcal{M}(f)(1)^{\mathrm{DP}}_{\mathfrak{p}} = \mathcal{M}(f)^0_{\mathfrak{p}} \underset{L}{\otimes} K(1)_{\mathfrak{p}}.$$

(c) *The polynomial*

$$X^2 - a_p X + \psi(p)p^{k-1} \tag{5.2}$$

has a unique root $\alpha \in \mathcal{O}^\times$ *which is a* p*-adic unit. If* $p \mid N$*, then* $\alpha = a_p$.

(d) *With* α *as in (c) and* δ *being the unramified character of* $\mathrm{G}_{\mathbb{Q}_p}$ *describing the action on* $\mathcal{M}(f)^0_{\mathfrak{p}}$*, one has* $\delta(\mathrm{Frob}_{\mathfrak{p}}) = \alpha$.

Proof: (a) See [HidMFG, Thm. 3.26 (2)]. The representation there has a 1-dimensional unramified quotient, but since Hida uses arithmetic Frobenii instead of geometric ones, his representation is dual to ours, and we get a subrepresentation.

(b) Note that it suffices to see that the inclusion $\mathcal{M}(f)^0_{\mathfrak{p}} \hookrightarrow \mathcal{M}(f)_{\mathfrak{p}}$ induces an injection $\mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)^0_{\mathfrak{p}}) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)_{\mathfrak{p}})/\mathrm{fil}^1\,\mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)_{\mathfrak{p}})$. This is because by proposition 5.9 and the fact that $\mathrm{cp}_{\mathrm{dR}}$ respects Hodge filtrations, we know the filtration on $\mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)_{\mathfrak{p}})$, so the right side is of dimension 1 and the injection will be an isomorphism which after tensoring with $K(1)_{\mathfrak{p}}$ gives the isomorphism from the Dabrowski-Panchishkin condition.

Since the map $\mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)^0_{\mathfrak{p}}) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)_{\mathfrak{p}})$ is a map of filtered vector spaces, it suffices to see that $\mathrm{fil}^0\,\mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)^0_{\mathfrak{p}}) = \mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)^0_{\mathfrak{p}})$ and $\mathrm{fil}^1\,\mathrm{D}_{\mathrm{dR}}(\mathcal{M}(f)^0_{\mathfrak{p}}) = 0$, i. e. that $\mathcal{M}(f)^0_{\mathfrak{p}}$ has Hodge-Tate weight 0. This follows from [Per94, §2.3, Lemme].

(c) For $p \nmid N$ see [Kat04, Prop. 17.1]. For $p \mid N$ this is obvious.

(d) See [HidMFG, Thm. 3.26 (2)]. □

Definition 5.16: The polynomial (5.2) is called the p-th *Hecke polynomial* of f and if f is ordinary then the root α from theorem 5.15 (c) is called the *unit root.*

Theorem 5.17 (Kisin, Colmez)**:** *If* f *is of finite slope (not necessarily ordinary),* $\mathrm{D}^\dagger_{\mathrm{rig}}(\mathcal{M}(f)_{\mathfrak{p}})$ *contains a rank* 1 *sub-*(φ,Γ)*-module* $\mathrm{D}^\dagger_{\mathrm{rig}}(\mathcal{M}(f)_{\mathfrak{p}})^0$. $\mathcal{M}(f)(1)$ *satisfies the weak Dabrowski-Panchishkin condition with* $\mathrm{D}^\dagger_{\mathrm{rig}}(\mathcal{M}(f)(1)_{\mathfrak{p}})^{\mathrm{DP}} = \mathrm{D}^\dagger_{\mathrm{rig}}(\mathcal{M}(f)_{\mathfrak{p}})^0 \otimes_L K(1)_{\mathfrak{p}}$*. In particular,* $\mathcal{M}(f)_{\mathfrak{p}}$ *is trianguline.*

Proof: In the ordinary case the statements are clear by theorem 5.15, so assume that f is non-ordinary.

Colmez has given in [Col08] a complete classification of 2-dimensional trianguline representations and proved some further theorems about them. He also gives a complete classification of rank 1 (φ,Γ)-modules, whose isomorphism classes are in bijection with characters $\delta\colon \mathbb{Q}_p^\times \longrightarrow L^\times$ by [Col08, Prop. 3.1]. Denote the (φ,Γ)-module belonging to such a character δ by $\mathcal{R}(\delta)$. Two-dimensional trianguline representations are parametrised by certain triples $(\delta_1,\delta_2,\mathcal{L})$, where $\delta_1,\delta_2\colon \mathbb{Q}_p^\times \longrightarrow L^\times$ are again characters and $\mathcal{L}$ is an element of either $\mathbb{P}^0(L)$ or $\mathbb{P}^1(L)$ (which will not be important for us; see [Col08, §§0.2–4]).

We will use [Col08, Thm. 0.8 (ii)],[11] which says that if V is a 2-dimensional irreducible L-linear representation of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights $(0,q)$ such that there exists an $\alpha \in L^\times$ with $v_p(\alpha) > 0$ and

$$(\mathrm{B}^+_{\mathrm{cris}} \underset{L}{\otimes} V)^{G_{\mathbb{Q}_p}, \varphi_{\mathrm{cris}}=\alpha} \neq 0, \tag{$*$}$$

then V is trianguline. If further $(\delta_1, \delta_2, \mathcal{L})$ is the triple from the classification and $v_p(\alpha) \leq q$, then δ_1 is trivial on $\mathbb{Z}_p$ and $\mathcal{R}(\delta_1)$ is a sub-(φ,Γ)-module of $\mathrm{D}^\dagger_{\mathrm{rig}}(V)$.

To be able to use this result, we need to check that the representation $\mathcal{M}(f)_\mathfrak{p}$ satisfies $(*)$ above. The α there will be the p-th Hecke eigenvalue a_p. We know already that $\mathcal{M}(f)_\mathfrak{p}$ has the other properties needed for Colmez's result: its Hodge-Tate weights are $(0, k-1)$ by theorem 5.12 (a) and the inequality $0 < v_p(a_p) \leq k-1$ holds by proposition 5.14. The property $(*)$ is equivalent to a result of Kisin, as we now explain.

For the moment, let V be any L-linear $G_{\mathbb{Q}_p}$-representation. We begin with the canonical biduality isomorphism $V \xrightarrow{\sim} \mathrm{Hom}_L(\mathrm{Hom}_L(V,L),L)$. It is easy to check that there is a canonical isomorphism

$$\mathrm{Hom}_L(\mathrm{Hom}_L(V,L),L) \underset{\mathbb{Q}_p}{\otimes} \mathrm{B}^+_{\mathrm{cris}} \cong \mathrm{Hom}_L(\mathrm{Hom}_L(V,L), L \underset{\mathbb{Q}_p}{\otimes} \mathrm{B}^+_{\mathrm{cris}}),$$

so tensoring the biduality map with $\mathrm{B}^+_{\mathrm{cris}}$ gives an isomorphism of $\mathrm{B}^+_{\mathrm{cris}}$-modules

$$\mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} V \xrightarrow{\sim} \mathrm{Hom}_L(\mathrm{Hom}_L(V,L), \mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} L). \tag{$**$}$$

We endow both sides with left actions of $G_{\mathbb{Q}_p}$. On $\mathrm{B}^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$ we let it act diagonally, on $\mathrm{Hom}_L(V,L)$ by the usual contragredient action given by $(\sigma f)(v) \coloneqq f(\sigma^{-1}v)$ $(\sigma \in G_{\mathbb{Q}_p}, v \in V)$, on $\mathrm{B}^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L$ we let it act only on the first factor and on $\mathrm{Hom}_L(\mathrm{Hom}_L(V,L), \mathrm{B}^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L)$ as $(\sigma\Phi)(f) \coloneqq \sigma(\Phi(f \circ \sigma))$. Then one can check that the map $(**)$ is $G_{\mathbb{Q}_p}$-equivariant and that

$$\big(\mathrm{Hom}_L(\mathrm{Hom}_L(V,L), \mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} L)\big)^{G_{\mathbb{Q}_p}} = \mathrm{Hom}_{L[G_{\mathbb{Q}_p}]}(\mathrm{Hom}_L(V,L), \mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} L).$$

We now endow both sides with the Frobenius endomorphism induced from φ_{cris} (which on each side acts only on $\mathrm{B}^+_{\mathrm{cris}}$). Then it is easy to check that the Frobenius commutes with the $G_{\mathbb{Q}_p}$-action, the map $(**)$ respects the Frobenius action and that

$$\big(\mathrm{Hom}_L(\mathrm{Hom}_L(V,L), \mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} L)\big)^{\varphi_{\mathrm{cris}}=\alpha} = \mathrm{Hom}_L(\mathrm{Hom}_L(V,L), (\mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} L)^{\varphi_{\mathrm{cris}}=\alpha}).$$

Therefore $(**)$ induces an isomorphism

$$(\mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} V)^{G_{\mathbb{Q}_p}, \varphi_{\mathrm{cris}}=\alpha} \xrightarrow{\sim} \mathrm{Hom}_{L[G_{\mathbb{Q}_p}]}(\mathrm{Hom}_L(V,L), (\mathrm{B}^+_{\mathrm{cris}} \underset{\mathbb{Q}_p}{\otimes} L)^{\varphi_{\mathrm{cris}}=\alpha}),$$

and we see that $(*)$ is equivalent to the existence of a nonzero L-linear $G_{\mathbb{Q}_p}$-equivariant map $V^* \longrightarrow (\mathrm{B}^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L)^{\varphi_{\mathrm{cris}}=\alpha}$ (V^* being the contragredient representation).

The existence of such a map for $V = \mathcal{M}(f)_\mathfrak{p}$ and $\alpha = a_p$ is proved in [Kis03, Thm. 6.3]. Note that Kisin uses arithmetic Frobenii (see [Kis03, §1.4]), so his representation is indeed the contragredient one.

[11] Note that for Colmez the Hodge-Tate weight of the cyclotomic character is 1, not −1 as in this work; he writes this at the end of §0.1. Therefore the Hodge-Tate weights in his paper are the negatives of the ones in our citation here.

So by Colmez's result, $\mathcal{M}(f)_{\mathfrak{p}}$ is trianguline and has a rank 1 sub-(φ, Γ)-module $\mathcal{R}(\delta_1)$, with δ_1 being trivial on $\mathbb{Z}_p^\times$. To finish, we need to derive from this that $\mathcal{M}(f)(1)$ satisfies the weak Dabrowski-Panchishkin condition. For this we can use exactly the same arguments as in the proof of theorem 5.15 (b), which reduces us to showing that $\mathcal{R}(\delta_1)$ has Hodge-Tate weight 0. This follows from the fact that δ_1 is trivial on $\mathbb{Z}_p^\times$ by [Col08, §2.2]. □

6. Comparison isomorphisms and Eichler-Shimura isomorphisms

The classical Eichler-Shimura isomorphism [Shi71, §8.2] relates the space of cusp forms over $\mathbb{C}$ to a certain group cohomology. From a more abstract point of view, this group cohomology group is closely related to the Betti realisation of ${}^N_k\mathcal{W}$ (tensored with $\mathbb{C}$), and by proposition 5.9 the space of cusp forms is related to the de Rham realisation of ${}^N_k\mathcal{W}$ (tensored with $\mathbb{C}$). Between these realisations we also have the complex comparison isomorphism cp_∞, and it turns out that these two maps are essentially the same. This provides a powerful tool to deal with the comparison isomorphism since the Eichler-Shimura isomorphism has a rather explicit description.

There is also a p-adic variant of the Eichler-Shimura isomorphism, and it is essentially equal to the p-adic comparison isomorphism $\mathrm{cp}_{\mathrm{HT}}$.

In this section we make these well-known statements precise.

6.1. The complex situation

6.1.1. The complex Eichler-Shimura isomorphism

The Eichler-Shimura isomorphism relates the space of cusp forms to the parabolic cohomology group of the local system $\mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}}$ on a modular curve. We explain the construction of the Eichler-Shimura map following [Con09, §1.7.1]. There is also a variant of the Eichler-Shimura isomorphism for modular forms instead of cusp forms. In our explanation of how the Eichler-Shimura map is constructed, we omit this for simplicity, but we state the result in theorem 6.3. The construction is also explained for both cases in [Kat04, §4.10], where it is called "period map".

In this whole section we write Y for either $Y(N)^{\mathrm{an}}$ or $Y_1(N)^{\mathrm{an}}$, $f\colon E \longrightarrow Y$ for the universal elliptic curve over it and X for the corresponding compactification.

From lemma 2.8, we get an isomorphism

$$\mathbf{R}^1 f_*\Omega^\bullet_{E/Y} \cong \mathrm{R}^1 f_*\underline{\mathbb{C}} \underset{\underline{\mathbb{C}}}{\otimes} \mathcal{O}_Y \cong \mathrm{R}^1 f_*\underline{\mathbb{R}} \underset{\underline{\mathbb{R}}}{\otimes} \mathcal{O}_Y.$$

The Hodge filtration of $\mathbf{R}^1 f_*\Omega^\bullet_{E/Y}$ is given by the injection

$$\omega_Y \lhook\joinrel\longrightarrow \mathbf{R}^1 f_*\Omega^\bullet_{E/Y}, \tag{6.1}$$

which comes from applying $\mathbf{R}^1 f_*$ to the morphism of complexes

$$\Omega^1_{E/Y}[-1] \longrightarrow \Omega^\bullet_{E/Y}.$$

We take $(k-2)$-th powers and then tensor with Ω^1_Y over $\mathcal{O}_Y$. Using the above isomorphism, this gives us a map

$$\begin{aligned}\omega_Y^{k-2} \underset{\mathcal{O}_Y}{\otimes} \Omega^1_Y &\longrightarrow \operatorname{Sym}^{k-2}_{\mathcal{O}_Y}(\mathbf{R}^1 f_* \Omega^\bullet_{E/Y}) \underset{\mathcal{O}_Y}{\otimes} \Omega^1_Y \\ &\xrightarrow{\sim} \operatorname{Sym}^{k-2}_{\mathcal{O}_Y}(\mathrm{R}^1 f_* \underline{\mathbb{R}} \underset{\underline{\mathbb{R}}}{\otimes} \mathcal{O}_Y) \underset{\mathcal{O}_Y}{\otimes} \Omega^1_Y = \operatorname{Sym}^{k-2}_{\underline{\mathbb{R}}} \mathrm{R}^1 f_* \underline{\mathbb{R}} \underset{\underline{\mathbb{R}}}{\otimes} \Omega^1_Y. \end{aligned} \tag{6.2}$$

Let us write $\mathcal{D}^k_R$ for $\operatorname{Sym}^{k-2}_{\underline{R}} \mathrm{R}^1 f_* \underline{R}$ for $R = \mathbb{R}$ or $R = \mathbb{C}$ in the following.

On the other hand, we have an exact sequence

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \longrightarrow 0$$

which we tensor over $\underline{\mathbb{R}}$ with the (locally free, hence flat) sheaf $\mathcal{D}^k_{\mathbb{R}}$ to obtain

$$0 \longrightarrow \mathcal{D}^k_{\mathbb{C}} \longrightarrow \mathcal{D}^k_{\mathbb{R}} \underset{\underline{\mathbb{R}}}{\otimes} \mathcal{O}_Y \xrightarrow{d} \mathcal{D}^k_{\mathbb{R}} \underset{\underline{\mathbb{R}}}{\otimes} \Omega^1_Y \longrightarrow 0. \tag{6.3}$$

Lemma 6.1: *Consider the composition*

$$\begin{aligned}\mathrm{S}_k(X,\mathbb{C}) = \mathrm{H}^0(X, \omega^{k-2}_{E/X} \underset{\mathcal{O}_X}{\otimes} \Omega^1_X) &\longrightarrow \mathrm{H}^0(Y, \omega^{k-2}_Y \underset{\mathcal{O}_Y}{\otimes} \Omega^1_Y) \\ &\longrightarrow \mathrm{H}^0(Y, \mathcal{D}^k_{\mathbb{R}} \underset{\underline{\mathbb{R}}}{\otimes} \Omega^1_Y) \longrightarrow \mathrm{H}^1(Y, \mathcal{D}^k_{\mathbb{C}}) \end{aligned}\tag{6.4}$$

where the first map is the restriction map, the second map is induced by (6.2) and the third map is the boundary homomorphism in the long exact cohomology sequence attached to (6.3).

Then its image lies in the parabolic cohomology group $\mathrm{H}^1_{\mathrm{p}}(Y, \mathcal{D}^k_{\mathbb{C}})$.

Proof: [Con09, §1.7.4] □

Definition 6.2: The *Eichler-Shimura map* is defined to be the map (6.4)

$$\mathrm{ES}\colon \mathrm{S}_k(X,\mathbb{C}) \longrightarrow \mathrm{H}^1_{\mathrm{p}}(Y, \operatorname{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{C}}).$$

Denote by $\overline{\mathrm{S}_k(X,\mathbb{C})}$ the complex conjugate vector space of $\mathrm{S}_k(X,\mathbb{C})$, that is, the complex vector space consisting of the same underlying set, but with the scalar multiplication twisted by complex conjugation. We consider $\overline{\mathrm{S}_k(X,\mathbb{C})}$ as the space of antiholomorphic cusp forms and denote both ($\mathbb{R}$-linear, but not $\mathbb{C}$-linear!) maps $\mathrm{S}_k(X,\mathbb{C}) \longrightarrow \overline{\mathrm{S}_k(X,\mathbb{C})}$ and $\overline{\mathrm{S}_k(X,\mathbb{C})} \longrightarrow \mathrm{S}_k(X,\mathbb{C})$ which are the identity on the underlying set by $\overline{(\cdot)}\colon f \longmapsto \overline{f}$. The complex vector space $\mathrm{H}^1_{\mathrm{p}}(Y, \operatorname{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{C}}) = \mathrm{H}^1_{\mathrm{p}}(Y, \mathcal{D}^k_{\mathbb{R}} \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}})$ has a canonical real structure, that is, an antilinear involution σ, coming from the complex conjugation on the second tensor factor $\underline{\mathbb{C}}$. One easily checks that the map

$$\overline{\mathrm{ES}} \coloneqq \sigma \circ \mathrm{ES} \circ \overline{(\cdot)}\colon \overline{\mathrm{S}_k(X,\mathbb{C})} \longrightarrow \mathrm{H}^1_{\mathrm{p}}(Y, \operatorname{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{C}})$$

is $\mathbb{C}$-linear.

Theorem 6.3 (Eichler/Shimura): *The Eichler-Shimura map induces a Hecke equivariant isomorphism of complex vector spaces*

$$\mathrm{ES} \oplus \overline{\mathrm{ES}} \colon \mathrm{S}_k(X, \mathbb{C}) \oplus \overline{\mathrm{S}_k(X, \mathbb{C})} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{p}}(Y, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{C}}).$$

There is also the variant for modular forms

$$\mathrm{ES} \oplus \overline{\mathrm{ES}} \colon \mathrm{M}_k(X, \mathbb{C}) \oplus \overline{\mathrm{S}_k(X, \mathbb{C})} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{c}}(Y, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{C}}).$$

Proof: For $Y = Y_1(N)$, the first statement follows from [Con09, Thm. 1.7.1.1, Lem. 1.7.7.2 and Thm. 2.3.2.1] with $\Gamma = \Gamma_1(N)$. For $Y = Y(N)$, it follows from the same with $\Gamma = \Gamma(N)$, together with remark 1.14 (b) and the remark after definition 4.1.

For the second statement see [PS13, footnote 5, proof of Lem. 6.6]. □

There is a more classical description of the Eichler-Shimura isomorphism using group cohomology which has the advantage of being rather explicit. This description is used e. g. in the classical reference [Shi71, chap. 8], although in a slightly different formulation. We briefly describe it here, for simplicity only for the $\Gamma_1(N)$ situation. So for the moment we specialise to the case $X = X_1(N)^{\mathrm{an}}$, $Y = Y_1(N)^{\mathrm{an}}$ and put $\Gamma = \Gamma_1(N)$. Similarly as in remark 5.7 we get a Hecke-equivariant isomorphism

$$\mathrm{H}^1_{\mathrm{p}}(\Gamma, \mathrm{Sym}^{k-2} \mathbb{C}^2) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{p}}(Y, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{C}}). \tag{6.5}$$

We refer to the discussion of symmetric powers in appendix A.1 and we view the elements of $\mathrm{Sym}^{k-2} \mathbb{C}^2$ as homogeneous polynomials of degree $k-2$ in two variables, as in lemma A.1.2 (a).

Proposition 6.4: *(a) The Eichler-Shimura isomorphism followed by the inverse of (6.5) maps $f \in \mathrm{S}_k(X, \mathbb{C})$ to the cocycle*

$$\gamma \longmapsto \int_{\tau_0}^{\gamma\tau_0} \omega_f, \qquad \gamma \in \Gamma$$

with $\omega_f = (2\pi \mathrm{i})^{k-1}(zX + Y)^{k-2} f \,\mathrm{d}z$ (a $\mathrm{Sym}^{k-2}\mathbb{C}^2$-valued 1-form) and $\tau_0 \in \mathfrak{h}$ being a lift of $x_0 \in Y$. In particular, the above cocycle is independent of the choice of τ_0.

(b) The Eichler-Shimura isomorphism composed with the inverse map of (6.5) maps $\overline{g} \in \overline{\mathrm{S}_k(X, \mathbb{C})}$ to the cocycle

$$\gamma \longmapsto \int_{\tau_0}^{\gamma\tau_0} \omega_{\overline{g}}, \qquad \gamma \in \Gamma$$

with $\omega_{\overline{g}} = (-2\pi \mathrm{i})^{k-1}(\overline{z}X + Y)^{k-2}\overline{g}\,\mathrm{d}\overline{z}$ (a $\mathrm{Sym}^{k-2}\mathbb{C}^2$-valued 1-form) and $\tau_0 \in \mathfrak{h}$ being a lift of $x_0 \in Y$. Here, the $\overline{g}$ in the definition of $\omega_{\overline{g}}$ now literally means the complex conjugate function of the $\mathbb{C}$-valued function g.

Proof: The claim in (a) is [Con09, Lem. 1.7.5.1]. In the formula there, the upper integration bound is $\gamma^{-1}\tau_0$ instead of $\gamma\tau_0$, but in Conrad's text the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}$ is the *right* action dual to the usual one (see [Con09, Thm. 1.5.2.2]).

We now prove (b). By definition of the real structure on $\mathrm{H}^1_{\mathrm{p}}(Y, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{C}})$ and the corresponding real structure on $\mathrm{H}^1_{\mathrm{p}}(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2)$, the composite map

$$\overline{\mathrm{S}_k(X,\mathbb{C})} \longrightarrow \mathrm{H}^1_{\mathrm{p}}(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2)$$

of $\overline{\mathrm{ES}}$ and the identification (6.5) then sends a $\overline{g} \in \overline{\mathrm{S}_k(X,\mathbb{C})}$ to the cocycle

$$\gamma \longmapsto \overline{\int_{\tau_0}^{\gamma\tau_0} \omega_g}, \qquad \gamma \in \Gamma.$$

If we define $\omega_{\overline{g}} = (-2\pi \mathrm{i})^{k-1}(\overline{z}X + Y)^{k-2}\overline{g}\mathrm{d}\overline{z}$ then this cocycle is equal to the cocycle

$$u_{\overline{g}} \colon \gamma \longmapsto \int_{\tau_0}^{\gamma\tau_0} \omega_{\overline{g}}, \qquad \gamma \in \Gamma,$$

as one can see by an easy calculation using the definition of the complex curve integral. □

Now we examine how the complex conjugation on $\mathrm{H}^1_{\mathrm{p}}(Y, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}})$ behaves under the Eichler-Shimura isomorphism. This works again in the general setting, so X may now be $X(N)^{\mathrm{an}}$, Y may be $Y(N)^{\mathrm{an}}$ and so on. The action of $\mathrm{G}_{\mathbb{R}}$ on Y and the fact that $\mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}}$ is a $\mathrm{G}_{\mathbb{R}}$-sheaf gives an action of $\mathrm{G}_{\mathbb{R}}$ on $\mathrm{H}^1_{\mathrm{p}}(Y, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}})$ and hence also on $\mathrm{H}^1_{\mathrm{p}}(Y, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{C}}) = \mathrm{H}^1_{\mathrm{p}}(Y, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}}) \otimes \mathbb{C}$, where we let $\mathrm{G}_{\mathbb{R}}$ act *trivially* on the second tensor factor $\mathbb{C}$.

Definition 6.5: We define an action of $\mathrm{G}_{\mathbb{R}}$ on $\mathrm{S}_k(X,\mathbb{C}) \oplus \overline{\mathrm{S}_k(X,\mathbb{C})}$ by letting the nontrivial element Frob_∞ act as

$$f \oplus \overline{g} \longmapsto -(g^* \oplus \overline{f^*}),$$

where f^* is defined as in definition 4.22.

Lemma 6.6: *With the above definition, the Eichler-Shimura isomorphism is* $\mathrm{G}_{\mathbb{R}}$*-equivariant.*

Proof: To simplify notation, we prove this only in the case $X = X_1(N)$, so $\Gamma = \Gamma_1(N)$; the proof for $X = X(N)$ works similar.

Let

$$u_{\overline{g}} \colon \gamma \longmapsto \int_{\tau_0}^{\gamma\tau_0} \omega_{\overline{g}}, \qquad \gamma \in \Gamma,$$

with $\omega_{\overline{g}} = (-2\pi \mathrm{i})^{k-1}(\overline{z}X + Y)^{k-2}\overline{g}\mathrm{d}\overline{z}$ be the cocycle from proposition 6.4 (b). We have to show that complex conjugation on $\mathrm{H}^1_{\mathrm{p}}(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2)$ sends u_f to $-u_{\overline{f^*}}$.

As explained in remark 5.7, the action of complex conjugation is given by the matrix $\mathsf{ə}$, and by corollary I.1.27, this action maps a cocycle u to the cocycle

$$\gamma \longmapsto \mathsf{ə}^{\iota} \bullet u(\mathsf{ə}\gamma\mathsf{ə}).$$

In general, if h is any smooth $\mathrm{Sym}^{k-2}\,\mathbb{C}^2$-valued function on $\mathfrak{h}$ such that for any $\gamma \in \Gamma$ the integral

$$\int_{\tau_0}^{\gamma\tau_0} h(z)\mathrm{d}z, \quad \tau_0 \in \mathfrak{h}$$

does not depend on the choice of τ, then an easy calculation using the definition of the complex curve integral shows that for such an h and any $\gamma \in \Gamma$, $\tau_0 \in \mathfrak{h}$ we have

$$\int_{\tau_0}^{\partial\gamma\partial\tau_0} h(z)\mathrm{d}z = \int_{\tau_0}^{\gamma\tau_0} h(\partial z)\mathrm{d}\overline{z}.$$

For $h(z) = (2\pi\mathrm{i})^{k-1}(zX+Y)^{k-2}f(z)$ it is easy to check that $h(\partial z) = (2\pi\mathrm{i})^{k-1}(-\overline{z}X+Y)^{k-2}\overline{f^*(z)}$. To this we then have to apply the action of $\partial^\iota = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ on $\mathrm{Sym}^{k-2}\mathbb{C}^2$. Altogether we see that u_f is sent to the cocycle

$$\begin{aligned}\gamma \longmapsto & \int_\tau^{\gamma\tau} (2\pi\mathrm{i})^{k-1}(-\overline{z}X - Y)^{k-2}\overline{f^*(z)}\mathrm{d}\overline{z} \\ &= -\int_\tau^{\gamma\tau} (-2\pi\mathrm{i})^{k-1}(\overline{z}X + Y)^{k-2}\overline{f^*(z)}\mathrm{d}\overline{z} \\ &= -u_{\overline{f^*}},\end{aligned}$$

as desired. This completes the proof. □

6.1.2. Explicit description of the comparison isomorphism

A description of the complex comparison isomorphism of ${}^N_k\mathcal{W}$ is given in [DFG01, §2.2]. We recall this here and refer to this text for more details.

Write $j\colon Y(N) \longrightarrow X(N)$ for the inclusion. Via the analytification map $X(N)^{\mathrm{an}} \longrightarrow X(N) \times_{\mathbb{Z}} \mathbb{C}$, we can pull back all the involved sheaves and connections to the analytic setting. By abuse of notation, we write j also for the analytic inclusion. In [DFG01, §2.2] it is proved that this does not change the cohomology groups and the de Rham realisation (over $\mathbb{C}$, of course). So we can work in the analytic category and write for the rest of the section $X = X(N)^{\mathrm{an}}$, $Y = Y(N)^{\mathrm{an}}$ and $E = E(N)^{\mathrm{an}}$.

In this section, let us write again $\mathcal{D}^k_{\mathbb{C}} = \mathrm{Sym}^{k-2}_{\underline{\mathbb{C}}}\,\mathrm{R}^1 f_*\underline{\mathbb{C}}$, as we did in section 6.1.1. We further use some of the notation from section 5.3. Write $\mathcal{G}^\bullet_k$ for the restriction of $\mathcal{E}^\bullet_k$ or $\mathcal{E}^\bullet_{k,\mathrm{c}}$ to Y (both are the same), which is by definition just the complex

$$(\underset{\underset{0}{\uparrow}}{\mathcal{G}_k} \longrightarrow \mathcal{G}_k \underset{\mathcal{O}_Y}{\otimes} \Omega^1_Y), \quad \text{with } \mathcal{G}_k = \mathrm{Sym}^{k-2}_{\mathcal{O}_Y}\mathbf{R}^1 f_*\Omega^\bullet_{E/Y}.$$

Using lemma 2.8 and taking $(k-2)$-th symmetric powers, one can construct an exact sequence

$$0 \longrightarrow \mathcal{D}^k_{\mathbb{C}} \longrightarrow \mathcal{G}_k \longrightarrow \mathcal{G}_k \underset{\mathcal{O}_Y}{\otimes} \Omega^1_Y \longrightarrow 0, \tag{6.6}$$

where the right map is just the restriction of the connection ∇_k to Y. This means that the two right entries in this sequence are precisely the complex $\mathcal{G}^\bullet_k$, so this gives rise to a quasi-isomorphism of complexes

$$\mathcal{G}^\bullet_k \longrightarrow \mathcal{D}^k_{\mathbb{C}}[0]$$

which after applying $\mathbf{H}^1$ gives an isomorphism

$$\mathbf{H}^1(Y, \mathcal{G}^\bullet_k) \xrightarrow{\sim} \mathrm{H}^1(Y, \mathcal{D}^k_{\mathbb{C}}).$$

The exact sequence above can be extended to the cusps to

$$0 \longrightarrow j_!\mathcal{D}^k_{\mathbb{C}} \longrightarrow \mathcal{E}_{k,\mathrm{c}} \longrightarrow \mathcal{E}_{k,\mathrm{c}} \underset{\mathcal{O}_X}{\otimes} \mho^1_X \longrightarrow 0 \tag{6.7}$$

where the right map is now $\nabla_{k,c}$. This gives in the same way an isomorphism

$$\mathbf{H}^1(X, \mathcal{E}^\bullet_{k,c}) \xrightarrow{\sim} \mathrm{H}^1_c(Y, \mathcal{D}^k_{\mathbb{C}}).$$

Finally, the restriction morphism

$$\mathbf{H}^1(X, \mathcal{E}^\bullet_k) \longrightarrow \mathbf{H}^1(Y, \mathcal{G}^\bullet_k)$$

is an isomorphism by [DFG01, §2.4, p. 21].

We assemble the isomorphisms we have so far in a diagram

$$\begin{tikzcd}
\mathbf{H}^1(X, \mathcal{E}^\bullet_{k,c}) \arrow[r, "(1)"] \arrow[d, "\sim"'] & \mathbf{H}^1(X, \mathcal{E}^\bullet_k) \arrow[r, "\sim"] & \mathbf{H}^1(Y, \mathcal{G}^\bullet_k) \arrow[d, "\sim"'] \\
\mathrm{H}^1_c(Y, \mathcal{D}^k_{\mathbb{C}}) \arrow[rr, "(2)"] & & \mathrm{H}^1(Y, \mathcal{D}^k_{\mathbb{C}}).
\end{tikzcd} \tag{6.8}$$

The image of the map (1) is ${}^N_k\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C}$ and the image of the map (2) is ${}^N_k\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C}$. Therefore this provides us with the desired comparison isomorphism

$${}^N_k\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C} \xrightarrow{\sim} {}^N_k\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C}.$$

6.1.3. Relating Eichler-Shimura and the comparison isomorphism

In this section we prove the compatibility of the Eichler-Shimura isomorphism from theorem 6.3 with the canonical map from cusp forms to the de Rham realisation of ${}^N_k\mathcal{W}$ from the Hodge filtration (see proposition 5.9) and the comparison isomorphism from section 6.1.2. This is also stated without proof in [Kat04, §11.3].

Theorem 6.7: *The diagram*

$$\begin{tikzcd}
& \mathrm{S}_k(X(N), \mathbb{C}) \arrow[dl, "Hodge"'] \arrow[dr, "\mathrm{ES}"] & \\
{}^N_k\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C} \arrow[rr, "\sim"] & & {}^N_k\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C}
\end{tikzcd}$$

commutes. Here the left map comes from the Hodge filtration of ${}^N_k\mathcal{W}_{\mathrm{dR}}$ *as in proposition 5.9, the right map is the Eichler-Shimura map and the bottom map is the comparison isomorphism.*

Proof: In this proof, we abbreviate $X = X(N)^{\mathrm{an}}$, $Y = Y(N)^{\mathrm{an}}$ and $E = E(N)^{\mathrm{an}}$.

Recall the complex $\mathcal{E}^\bullet_{k,c}$ of sheaves on X from section 5.3 and its restriction $\mathcal{G}^\bullet_k$ to Y from section 6.1.2, as well as the sheaf $\mathcal{D}^k_{\mathbb{C}}$ on Y.

From lemma 2.8 we have an isomorphism

$$\mathcal{G}_k \cong \mathcal{D}^k_{\mathbb{C}} \underset{\underline{\mathbb{C}}}{\otimes} \mathcal{O}_Y \tag{6.9}$$

and this allows us to identify the exact sequence (6.3) used in the definition of the Eichler-Shimura map with the exact sequence (6.6) used to define the comparison isomorphism to obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}^k_{\mathbb{C}} \arrow[r] \arrow[d, equal] & \mathcal{D}^k_{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_Y \arrow[r] \arrow[d, "\sim"] & \mathcal{D}^k_{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} \Omega^1_Y \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathcal{D}^k_{\mathbb{C}} \arrow[r] & \mathcal{G}_k \arrow[r] & \mathcal{G}_k \otimes_{\mathcal{O}_Y} \Omega^1_Y \arrow[r] & 0.
\end{tikzcd} \tag{6.10}$$

Using this and the definition of the map from the Hodge filtration (labelled "Hodge" below), we extend the diagram (6.8), which is part (1) below, to

$$\begin{tikzcd}
\mathrm{H}^0(X, \omega^{k-2}_{\overline{E}/X} \otimes_{\mathcal{O}_X} \Omega^1_X) \arrow[r] \arrow[dd, "\text{Hodge}"] \arrow[rrddd, dashed, bend left=40] \arrow[rrddd, dashed, bend right=40] & \mathrm{H}^0(Y, \omega^{k-2}_Y \otimes_{\mathcal{O}_Y} \Omega^1_Y) \arrow[r] \arrow[rd, bend right] \arrow[d, phantom, "(2)"] \arrow[r, phantom, shift right=6ex, "(3)"] & \mathrm{H}^0(Y, \mathcal{D}^k_{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} \Omega^1_Y) \arrow[d, "\sim"', "(6.9)"] \arrow[ddd, bend left=60, "\partial"] \\
 & {} & \mathrm{H}^0(Y, \mathcal{G}_k \otimes_{\mathcal{O}_Y} \Omega^1_Y) \arrow[d] \arrow[dd, bend left=50, "\partial"] \arrow[u, phantom, shift left=10ex, "(4)"] \\
\mathbf{H}^1(X, \mathcal{E}^\bullet_{k,\mathrm{c}}) \arrow[r] \arrow[d, "\sim"'] & \mathbf{H}^1(X, \mathcal{E}^\bullet_k) \arrow[r, "\sim"] \arrow[d, phantom, "(1)"] & \mathbf{H}^1(Y, \mathcal{G}^\bullet_k) \arrow[d, "\sim"', "(5)"] \\
\mathrm{H}^1_\mathrm{c}(Y, \mathcal{D}^k_{\mathbb{C}}) \arrow[rr] & {} & \mathrm{H}^1(Y, \mathcal{D}^k_{\mathbb{C}}),
\end{tikzcd}$$

where the two maps labelled "∂" are boundary maps in the long exact cohomology sequences attached to the short exact sequences (6.3) and (6.6), respectively.

We want to prove that the outermost (dashed) arrows coincide, since the composition along the lower dashed arrow is the composition of the Hodge filtration map with the comparison isomorphism, while the composition along the upper dashed arrow is the Eichler-Shimura map. We prove this by showing that each of the partial diagrams (1)–(5) commutes.

We know already that (1) commutes. That (2) commutes is clear since both ways are basically the same map. Part (3) commutes just by definition of the map $\omega_Y^{k-2} \otimes_{\mathcal{O}_Y} \Omega^1_Y \longrightarrow \mathcal{D}^k_{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} \Omega^1_Y$ in (6.2). Part (4) commutes because (6.10) commutes.

To see that (5) commutes, we are hence left to prove that the boundary map

$$\mathrm{H}^0(Y, \mathcal{G}_k \underset{\mathcal{O}_Y}{\otimes} \Omega^1) \longrightarrow \mathrm{H}^1(Y, \mathcal{D}^k_{\mathbb{C}})$$

for the exact sequence (6.6) is equal to the composition of the map

$$\mathrm{H}^0(Y, \mathcal{G}_k \underset{\mathcal{O}_Y}{\otimes} \Omega^1) \longrightarrow \mathbf{H}^1(Y, \mathcal{G}^\bullet_k)$$

induced by the inclusion of complexes

$$(\mathcal{G}_k \underset{\mathcal{O}_Y}{\otimes} \Omega^1)[-1] \longrightarrow \mathcal{G}^\bullet_k$$

with the comparison isomorphism. By the description of the comparison isomorphism in section 6.1.2, this is precisely the content of lemma 1.2.14, applied to the exact sequence (6.6) for $\mathcal{F}$ being the global sections functor. $\square$

Corollary 6.8: *The diagram*

$$\begin{tikzcd}
\mathrm{S}_k(X_1(N),\mathbb{C}) \arrow[r, equal] & \mathrm{H}^0(X_1(N)^{\mathrm{an}}, \omega^{k-2}_{X_1(N)^{\mathrm{an}}} \otimes_{\mathcal{O}_{X_1(N)^{\mathrm{an}}}} \Omega^1_{X_1(N)^{\mathrm{an}}}) \arrow[r, "\mathrm{ES}"] \arrow[dd, "(4)"] & \mathrm{H}^1_{\mathrm{p}}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\mathrm{R}^1 f_*\underline{\mathbb{C}}) \arrow[d, "(1)"] \\
 & & {}^N_k\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C} \\
\mathrm{S}_k(X(N),\mathbb{C}) \arrow[r, equal] & \mathrm{H}^0(X(N)^{\mathrm{an}}, \omega^{k-2}_{X(N)^{\mathrm{an}}} \otimes_{\mathcal{O}_{X(N)^{\mathrm{an}}}} \Omega^1_{X(N)^{\mathrm{an}}}) \arrow[r, "(3)"] & {}^N_k\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C} \arrow[u, "(2)", "\sim"']
\end{tikzcd}$$

commutes. Here (1) and (4) come from the map (2.1), (2) is the comparison isomorphism and (3) is the map given by the Hodge filtration of ${}^N_k\mathcal{W}_{\mathrm{dR}}$.

Proof: In the proof, we omit "an" superscripts to simplify the notation.

By lemma 1.11 and the fact that the formation of all sheaves occurring in our situation is compatible with base change, each square in the diagram

$$\begin{tikzcd}
\mathrm{H}^0(X_1(N), \omega^{k-2}_{X_1(N)} \otimes_{\mathcal{O}_{X_1(N)}} \Omega^1_{X_1(N)}) \arrow[r] \arrow[d] & \mathrm{H}^0(Y_1(N), \omega^{k-2}_{Y_1(N)} \otimes_{\mathcal{O}_{Y_1(N)}} \Omega^1_{Y_1(N)}) \arrow[r] \arrow[d] & \cdots \\
\mathrm{H}^0(X(N), \omega^{k-2}_{X(N)} \otimes_{\mathcal{O}_{X(N)}} \Omega^1_{X(N)}) \arrow[r] & \mathrm{H}^0(Y(N), \omega^{k-2}_{Y(N)} \otimes_{\mathcal{O}_{Y(N)}} \Omega^1_{Y(N)}) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^0(Y_1(N), \mathcal{D}^k_{\mathbb{C}} \otimes_{\mathcal{O}_{Y_1(N)}} \Omega^1_{Y_1(N)}) \arrow[r] \arrow[d] & \mathrm{H}^1(Y_1(N), \mathcal{D}^k_{\mathbb{C}}) \arrow[d] \\
\cdots \arrow[r] & \mathrm{H}^0(Y(N), \mathcal{D}^k_{\mathbb{C}} \otimes_{\mathcal{O}_{Y(N)}} \Omega^1_{Y(N)}) \arrow[r] & \mathrm{H}^1(Y(N), \mathcal{D}^k_{\mathbb{C}})
\end{tikzcd}$$

commutes (where by abuse of notation we used the same symbol $\mathcal{D}^k_{\mathbb{C}} = \mathrm{Sym}^{k-2}\mathrm{R}^1 f_*\underline{\mathbb{C}}$ for the respective sheaf on the two curves). Thus the claim follows from theorem 6.7. □

6.1.4. Some complements

In this section we give some missing proofs from before which use the previous results.

Proof of lemma 4.25: We explain the proof for $\mathbf{t}$, for $\mathbf{T}$ it works exactly the same. We prove that the second statement ($\tilde{\mathbf{t}} \cong \mathbf{t}$) holds for both possibilities to define $\mathbf{t}$, once using the naive and once using the arithmetic modular curve. Since $\tilde{\mathbf{t}}$ is independent of this choice, this implies the first statement. Let X be either $X_1(N)^{\mathrm{arith}}$ or $X_1(N)^{\mathrm{naive}}$. The statements we use below will be valid for both versions.

Since the formation of the Hecke eigenalgebra $\mathbf{t}_k(N,R)$ is compatible with base change by lemma 4.24 and the same holds for $\tilde{\mathbf{t}}_k(N,R)$ with exactly the same proof, it suffices to prove this in the case $R = \mathbb{Z}$.

We abbreviate $\mathrm{H}^1_{\mathrm{p}} := \mathrm{H}^1_{\mathrm{p}}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\mathrm{R}^1 f_*\underline{\mathbb{Z}})$ and $\mathrm{S}_k := \mathrm{S}_k(X,\mathbb{Z})$ and further $\mathbf{T}(M) := \mathbf{T}^{(\Delta_1(N),\Gamma_1(N))}_{\mathbb{Z}}(M)$ for a $\mathcal{H}_+(N)$-module M, so that $\mathbf{T}(\mathrm{H}^1_{\mathrm{p}}) = \tilde{\mathbf{t}}_k(N,\mathbb{Z})$ and $\mathbf{T}(\mathrm{S}_k) = \mathbf{t}_k(N,\mathbb{Z})$.

Since $\mathrm{H}^1_{\mathrm{p}}$ is free of finite rank by lemma 2.3, it is easy to see that $\mathbf{T}(\mathrm{H}^1_{\mathrm{p}}) = \mathbf{T}(\mathrm{H}^1_{\mathrm{p}} \otimes_{\mathbb{Z}} \mathbb{C})$. The same argument shows that $\mathbf{T}(\mathrm{S}_k) = \mathbf{T}(\overline{\mathrm{S}_k(X,\mathbb{C})})$ (using propositions 4.2 and 4.5). By definition of the Hecke action on $\mathrm{S}_k(X,\mathbb{C}) \oplus \overline{\mathrm{S}_k(X,\mathbb{C})}$ we have further $\mathbf{T}(\mathrm{S}_k(X,\mathbb{C})) = \mathbf{T}(\mathrm{S}_k(X,\mathbb{C}) \oplus \overline{\mathrm{S}_k(X,\mathbb{C})})$. Finally by the Eichler-Shimura isomorphism we have $\mathbf{T}(\mathrm{S}_k(X,\mathbb{C}) \oplus \overline{\mathrm{S}_k(X,\mathbb{C})}) \cong \mathbf{T}(\mathrm{H}^1_{\mathrm{p}} \otimes_{\mathbb{Z}} \mathbb{C})$. This completes the proof. □

Proof of proposition 5.10: Write A for the cokernel of the map

$$\mathrm{S}_k(X(N),\mathbb{Q}) \hookrightarrow {}^N_k\mathcal{W}_{\mathrm{dR}}.$$

We first compute the dimension of A. From the Eichler-Shimura isomorphism from theorem 6.3 we can obtain a short exact sequence

$$0 \longrightarrow \mathrm{S}_k(X(N),\mathbb{C}) \longrightarrow {}^N_k\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C} \longrightarrow \overline{\mathrm{S}_k(X(N),\mathbb{C})} \longrightarrow 0.$$

Now using the comparison isomorphism for ${}^N_k\mathcal{W}$, we can extend this to a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{S}_k(X(N),\mathbb{C}) \arrow[r, "\mathrm{ES}"] \arrow[d, equal] & {}^N_k\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C} \arrow[r] & \overline{\mathrm{S}_k(X(N),\mathbb{C})} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{S}_k(X(N),\mathbb{C}) \arrow[r] & {}^N_k\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C} \arrow[u, "\sim"] & &
\end{tikzcd}$$

in which the square commutes by theorem 6.7. Hence we know

$$\dim_{\mathbb{Q}} A = \dim_{\mathbb{C}} \mathrm{S}_k(X(N),\mathbb{C}).$$

Now by the description of the motive ${}^N_k\mathcal{W}$ in [DFG01] (called $M_!$ there) and in particular its de Rham realisation with its Hodge filtration on p. 15/16 there, we know

$$A = \mathrm{image}(\mathrm{H}^1(X(N),\omega^{2-k}_{X(N)}(-C(N))) \longrightarrow \mathrm{H}^1(X(N),\omega^{2-k}_{X(N)})).$$

But since $\mathrm{H}^1(X(N),\omega^{2-k}_{X(N)})$ is isomorphic to $\mathrm{S}_k(X(N),\mathbb{Q})^\vee$ by Serre duality, as we already pointed out after proposition 5.10, the result follows. ☐

6.2. The p-adic situation

We look at the p-adic Hodge-Tate comparison isomorphism for ${}^N_k\mathcal{W}$

$$\mathrm{cp}_{\mathrm{HT}}\colon\ {}^N_k\mathcal{W}_p \underset{\mathbb{Q}_p}{\otimes} \mathrm{B}_{\mathrm{HT}} \xrightarrow{\ \sim\ } {}^N_k\mathcal{W}_{\mathrm{H}} \underset{\mathbb{Q}}{\otimes} \mathrm{B}_{\mathrm{HT}}\,.$$

It is an isomorphism of graded vector spaces. Let us take its degree 0 part. From theorem 5.12 and corollary 5.11 we have an explicit description of the vector spaces involved, and we obtain the following, which can be viewed as a p-adic analogue of the Eichler-Shimura isomorphism.

Theorem 6.9 (Faltings)**:** *There is a canonical $\mathrm{G}_{\mathbb{Q}_p}$-equivariant and Hecke equivariant isomorphism*

$$\mathrm{H}^1_{\mathrm{p,\acute{e}t}}(Y_?(N) \underset{\mathbb{Z}}{\times} \overline{\mathbb{Q}}_p, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}}_p) \underset{\mathbb{Z}_p}{\otimes} \mathbb{C}_p \xrightarrow{\ \sim\ }$$
$$\mathrm{S}_k(X_?(N),\mathbb{C}_p)(1-k) \ \oplus\ \mathrm{H}^1(X_?(N),\omega^{2-k}_{X(N)}) \underset{\mathcal{O}}{\otimes} \mathbb{C}_p.$$

Here, "?" is either nothing or "1" and $\mathrm{G}_{\mathbb{Q}_p}$ acts diagonally on the left side and through $\mathbb{C}_p$ on the right side.

In fact, this theorem was proved by Faltings in two ways. In [Fal88] he proved the existence of the general Hodge-Tate comparison isomorphism for smooth projective varieties (which later was interpreted in the context of motives), and in the earlier paper [Fal87] he proved the above special case, which is essentially Thm. 6 (iii) there.

We indicate how to obtain the above statement from [Fal87, Thm. 6 (iii)]. There an analogous statement for $X(N)^{\mathrm{arith}}$ is proved. The theorem is derived from abstract results proved earlier. An investigation of the proof shows that these abstract results can as well be applied to $X(N)^{\mathrm{naive}}$ or $X_1(N)^{\mathrm{naive}}$. Due to his method of proof, Faltings needs to consider these curves as curves over a ring $\mathcal{O}$ (there called V) which is an integer ring in some finite extension of $\mathbb{Q}_p$ large enough such that the curves have stable reduction over $\mathcal{O}$. All occurring sheaves are then pulled back to the curves over $\mathcal{O}$. But since all occurring sheaves are compatible with base change and we tensor with $\mathbb{C}_p$ in the end anyway, the statement can also be formulated with that ring $\mathcal{O}$ replaced by $\mathbb{Z}$. Finally, we remark that the k there is what we call $k-2$. Using this, we get from [Fal87, Thm. 6 (iii)] a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{p}}(Y_?(N) \underset{\mathbb{Z}}{\times} \overline{\mathbb{Q}}_p, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{Z}}_p)(k-1) \underset{\mathbb{Z}_p}{\otimes} \mathbb{C}_p \xrightarrow{\sim}$$
$$\mathrm{H}^1(X_?(N), \omega^{2-k}_{X(N)}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}_p(k-1) \ \oplus\ \mathrm{H}^0(X_?(N), \omega^{k-2}_{X(N)} \otimes \Omega^1_{X(N)}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}_p.$$

Twisting $1-k$ times gives the claim in the form above.

Although it seems to be well-known and plausible that the p-adic Eichler-Shimura isomorphism from [Fal87, Thm. 6 (iii)] coincides with the map from theorem 6.9 (which can be viewed as a p-adic analogue of the statement in theorem 6.7), the author does not know of an explicit proof in the literature and since he is not very familiar with the methods used by Faltings, he was not able to give a such proof by himself.

7. Level changing and refinements

Let $N \geq 4$ be an integer and p be a prime. The construction of p-adic L-functions for modular forms of level N (which we recall in appendix B) requires N to be divisible by p for technical reasons. If $p \nmid N$, then there is a process called "refinement" that associates to each newform of level f two eigenforms f_α and f_β of level Np. In fact, any eigenform f of any level can be refined, but if p divides the level then one refinement will be just f itself. If $p \nmid N$, then both refinements are different from f, and if f is ordinary, then exactly one of its refinements is ordinary. For a non-refined newform f (whose level is hence not divisible by p) one can define its p-adic L-function to be that of one of its refinements and then relate it to the L-function of f itself (using proposition IV.1.4 below). Unless f is ordinary, this is non-canonical since f has two refinements.

So the natural objects to which one can associate p-adic L-functions are refinements of newforms, which we introduce in section 7.1. The refinement of a newform of level not divisible by p is *not* a newform any more. On the other hand, the natural objects to which one can associate Galois representations and motives are newforms. Since our aim is to compare existing constructions of p-adic L-functions to conjectural ones phrased using the language of motives, we thus ultimately have to compare periods of motives for non-refined forms to error terms of the corresponding refinements.

For this purpose it seems most convenient to have a theory of refinements also on the motivic side. What we seek is a morphism of motives ${}^{N}_{k}\mathcal{W} \longrightarrow {}^{Np}_{k}\mathcal{W}$ such that the morphism it induces on the intermediate subspace of the Hodge filtration of the de Rham realisations, which are cusp forms by proposition 5.9, maps f to a refinement. Such a morphism of motives will then induce morphisms on all realisations that are compatible with all comparison isomorphisms. We describe such a motivic theory of refinements in section 7.2.

7.1. Refinements of classical modular forms

In this section we fix a prime p. Let K be a number field with ring of integers $\mathcal{O}$ and choose an embedding $K \hookrightarrow \overline{\mathbb{Q}}$. Via our fixed pair of embeddings of $\overline{\mathbb{Q}}$, this defines a prime $\mathfrak{p}$ of K lying over p; let $L := K_{\mathfrak{p}}$ be the completion. Further this allows us to use the classical viewpoint since it induces an embedding $K \hookrightarrow \mathbb{C}$. This sections follows closely [Bel10, §III.7.1].

From the classical viewpoint on modular forms as functions on the upper half plane it is clear that we have an inclusion of $\mathcal{O}$-modules $\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \mathcal{O}) \hookrightarrow \mathrm{S}_k(X_1(Np)^{\mathrm{arith}}, \mathcal{O})$. One has to be careful here because this inclusion does *not* respect the action of T_p (but it does respect all other Hecke operators), see remark I.1.58.

Let $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \psi, \mathcal{O})$ be an eigenform away from the level and denote the eigenvalue of T_ℓ by a_ℓ ($\ell \nmid N$). We look at the polynomial

$$X^2 - a_p X + \psi(p) p^{k-1} \tag{7.1}$$

and call its roots α and β; we assume for simplicity that they both lie in $\mathcal{O}$. Then define two functions on the upper half plane by

$$f_\alpha(\tau) = f(\tau) - \beta f(p\tau), \quad f_\beta(\tau) = f(\tau) - \alpha f(p\tau) \quad (\tau \in \mathfrak{h}).$$

They are again cusp forms with special properties. Instead of proving this directly, we formulate a more general lemma that also applies to other situations.

Lemma 7.1: *Fix $N \in \mathbb{Z}$ and a prime p. Let K be a field of characteristic $\neq p$, M a K-linear representation of (Σ, ι) (where $\Sigma = \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$). Assume there exists $l \in \mathbb{Z}$ such that the matrix $\begin{pmatrix} p & \\ & p \end{pmatrix}$ acts on M as multiplication by p^l. Fix further $a, \zeta \in K$, let α, β be the roots of the polynomial $X^2 - aX + \zeta p^{l+1}$ and assume they lie in K. Define an endomorphism*

$$\mathrm{Ref}_\alpha := \mathrm{id} - p^{-l-1}\beta \begin{bmatrix} p & \\ & 1 \end{bmatrix}$$

of M (where $[\cdot]$ denotes the action of Σ on M, viewed as a right action). Then the following hold:

(a) Ref_α *induces a morphism* $\mathrm{Ref}_\alpha \colon M^{\Gamma_1(N)} \longrightarrow M^{\Gamma_1(Np)}$.

(b) Identify the restricted abstract standard Hecke algebras $\mathcal{H}^{(p)}(N)_K$ and $\mathcal{H}^{(p)}(Np)_K$ using lemma I.1.55 (b) and proposition I.1.21. Then the morphism $\mathrm{Ref}_\alpha \colon M^{\Gamma_1(N)} \longrightarrow M^{\Gamma_1(Np)}$ *is $\mathcal{H}^{(p)}(N)_K$-linear.*

(c) Denote by $T_p^{(N)}$ resp. $T_p^{(Np)}$ the p-th Hecke operators in $\mathcal{H}(N)_K$ resp. $\mathcal{H}(Np)_K$. If $m \in M^{\Gamma_1(N)}$ satisfies $T_p^{(N)} m = am$ and $\langle p \rangle m = \zeta m$, then $T_p^{(Np)} \mathrm{Ref}_\alpha(m) = \alpha \mathrm{Ref}_\alpha(m)$.

Proof: For (a) it suffices to note that

$$\Gamma_1(Np) \subseteq \Gamma_1(N), \qquad \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_1(Np) \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1} \subseteq \Gamma_1(N).$$

We now prove (b). By lemma I.1.55 (b) and proposition I.1.32, the "id" part in the definition of Ref_α is $\mathcal{H}^{(p)}(N)_K$-linear, and we have to check that the action of $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ is also $\mathcal{H}^{(p)}(N)_K$-linear. By proposition I.1.51 (b), (c) we need to check that it respects the operators T_q for primes $q \neq p$, S_ℓ for primes $\ell \nmid Np$ and $\mathcal{E}$. For $\mathcal{E}$ this is clear, for S_ℓ this follows easily from remark I.1.53 and for T_q this can be proved as in [Miy89, Lem. 4.6.2].

To prove (c) we follow [Bel10, Lem. III.7.2]. We now have to be careful which space we are acting on. Also we now let Hecke operators act from the right because we use this convention also for the action of Σ (i. e. we write $m[T_p]$ instead of $T_p m$, and similarly for diamond operators). Using the description in terms of double coset operators and lemma I.1.54, we see that $[T_p^{(N)}] = [T_p^{(Np)}] + \langle p \rangle \left[\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right] [T_p^{(Np)}] = p^l$. From this and the polynomial (7.1) we get

$$\begin{aligned} \mathrm{Ref}_\alpha(m)[T_p^{(Np)}] &= \left(m - p^{-l} \beta m \begin{bmatrix} p & \\ & 1 \end{bmatrix} \right) [T_p^{(Np)}] \\ &= m[T_p^{(Np)}] - \beta m = am - m \langle p \rangle \begin{bmatrix} p & \\ & 1 \end{bmatrix} - \beta m \\ &= \alpha m - \zeta m \begin{bmatrix} p & \\ & 1 \end{bmatrix} = \alpha m - \alpha \beta p^{-l-1} m \begin{bmatrix} p & \\ & 1 \end{bmatrix} = \alpha \mathrm{Ref}_\alpha(f). \qquad \square \end{aligned}$$

Corollary 7.2: *The functions f_α and f_β define cusp forms in $\mathrm{S}_k(X_1(Np)^{\mathrm{arith}}, \psi, \mathcal{O})$, where ψ is now viewed as a character of $(\mathbb{Z}/Np)^\times$. They are eigenforms away from the level with the same eigenvalues as f and they are moreover eigenvectors of T_p with eigenvalues α and β, respectively. If f was a normalised eigenform, then so are f_α and f_β, with the same eigenvalues for all Hecke operators except T_p.*

Proof: This follows from lemma 7.1 with M as in example I.1.24, $l = k-2$, $a = a_p$, $\zeta = \psi(p)$. Note that $f\left[\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right](\tau) = p^{k-1} f(p\tau)$. $\square$

Definition 7.3: The forms f_α and f_β are called the *refinements* of f at p. They are also commonly referred to as the *p-stabilisations* of f.

Note that if $p \mid N$ then one of α and β equals a_p and the other one is 0, so one of f_α and f_β is just f itself, and we usually consider it as an element of $\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \psi, \mathcal{O})$ (as opposed to $\mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \psi, \mathcal{O})$).

Remark 7.4: Assume $p \nmid N$. From the polynomial (7.1) we see immediately that

$$v_\mathfrak{p}(\alpha) + v_\mathfrak{p}(\beta) = v_\mathfrak{p}(\alpha\beta) = v_\mathfrak{p}(\psi(p)p^{k-1}) = k-1$$

and

$$v_\mathfrak{p}(a_p) = v_\mathfrak{p}(\alpha + \beta) \geq \min(v_\mathfrak{p}(\alpha), v_\mathfrak{p}(\beta)),$$

with equality if $v_\mathfrak{p}(\alpha) \neq v_\mathfrak{p}(\beta)$. This implies in particular that if f is ordinary, then one of its refinements is ordinary and the other one is of critical slope. By theorem 5.15 (c), if f is ordinary then exactly one of α and β is a p-adic unit, and the refinement coming from this unit root is the ordinary one. This latter statement also holds if $p \mid N$: then one of the refinements is just f itself, while the other one is in the kernel of T_p, so there is still a unique ordinary refinement.

7.2. Change of level on modular curves and a motivic theory of refinements

Fix integers $M, N \geq 4$ with $N \mid M$. Using remark 1.9, we define two morphisms between modular curves

$$\begin{aligned} \sigma_{M,N}\colon X(M) \longrightarrow X(N), \quad (E,P,Q) &\longmapsto \left(E, \frac{M}{N}P, \frac{M}{N}Q\right), \\ \theta_{M,N}\colon X(M) \longrightarrow X(N), \quad (E,P,Q) &\longmapsto \left({}^{E}\!/_{NP}, P, Q\right), \end{aligned}$$

and similarly, denoted by the same symbols,

$$\begin{aligned} \sigma_{M,N}\colon X_1(M) \longrightarrow X_1(N), \quad (E,P) &\longmapsto \left(E, \frac{M}{N}P\right), \\ \theta_{M,N}\colon X_1(M) \longrightarrow X_1(N), \quad (E,P) &\longmapsto \left({}^{E}\!/_{NP}, P\right), \end{aligned}$$

and call them the *change of level morphisms.* Further using remark 1.8 we define morphisms

$$\begin{aligned} \Sigma_{M,N}\colon \overline{E}(M) \longrightarrow \overline{E}(N), \quad (E,P,Q,x) &\longmapsto \left(E, \frac{M}{N}P, \frac{M}{N}Q, x\right), \\ \Theta_{M,N}\colon \overline{E}(M) \longrightarrow \overline{E}(N), \quad (E,P,Q,x) &\longmapsto \left({}^{E}\!/_{NP}, P, Q, x\right), \\ \Sigma_{M,N}\colon \overline{E}_1(M) \longrightarrow \overline{E}_1(N), \quad (E,P,x) &\longmapsto \left(E, \frac{M}{N}P, x\right), \\ \Theta_{M,N}\colon \overline{E}_1(M) \longrightarrow \overline{E}_1(N), \quad (E,P,x) &\longmapsto \left({}^{E}\!/_{NP}, P, x\right) \end{aligned} \tag{7.2}$$

lying over the ones from before. It is easy to see that the diagrams

$$\begin{tikzcd} \overline{E}_?(M) \arrow[r] \arrow[d] & \overline{E}_?(N) \arrow[d] \\ X_?(M) \arrow[r] & X_?(N), \end{tikzcd} \tag{7.3}$$

where "?" is either 1 or nothing, the vertical maps are f and the horizontal ones are either $\Sigma_{M,N}$ and $\sigma_{M,N}$ or $\Theta_{M,N}$ and $\theta_{M,N}$, are cartesian in all cases. The maps $\sigma_{M,N}$ and $\Sigma_{M,N}$ are also introduced in [Sch97, §5.2.2], where they are called $\rho_{M,N}$ resp. $\sigma^{\text{naive}}_{M,N}$.

Proposition 7.5: *For each $k \geq 2$, the maps introduced above induce morphisms of motives*

$$\sigma_{M,N}, \theta_{M,N}\colon {}^{N}_{k}\mathcal{W} \longrightarrow {}^{M}_{k}\mathcal{W}.$$

We call them change of level morphisms, too.

Proof: It is clear that both $\Sigma_{M,N}$ and $\Theta_{M,N}$ induce morphisms $\text{preKS}(M,k) \longrightarrow \text{preKS}(N,k)$. By construction it is easy to see that these are equivariant for the actions of the groups $G(M,k)$ resp. $G(N,k)$ introduced in section 5.1 (via the natural morphism $G(M,k) \longrightarrow G(N,k)$).

Hence the graphs of these morphisms in $\mathrm{preKS}(N,k) \times_{\mathbb{Q}} \mathrm{preKS}(M,k)$ (see example 1.3.2) are invariant under $G(N,k) \times G(M,k)$. Since the actions extend to the desingularisations, the same thus holds for the closures of their preimages under the morphism $\mathrm{KS}(N,k) \times_{\mathbb{Q}} \mathrm{KS}(M,k) \longrightarrow \mathrm{preKS}(N,k) \times_{\mathbb{Q}} \mathrm{preKS}(M,k)$ from theorem 5.1, and they are still closed subvarieties of codimension $\dim \mathrm{KS}(N,k)$. Hence we get induced morphisms of motives. See also [Sch97, §7.4.2]. $\square$

The morphisms thus induce maps on all realisations of ${}^N_kW$. In particular, by looking at the intermediate step in the Hodge filtration on de Rham realisations and using proposition 5.9, we get maps

$$\sigma_{M,N}, \theta_{M,N} \colon \mathrm{S}_k(X(N), \mathbb{Q}) \longrightarrow \mathrm{S}_k(X(M), \mathbb{Q}). \tag{7.4}$$

Proposition 7.6: *After tensoring with $\mathbb{C}$, we have:*

(a) The map $\sigma_{M,N} \colon \mathrm{S}_k(X(N), \mathbb{C}) \longrightarrow \mathrm{S}_k(X(M), \mathbb{C})$ sends an f, viewed as function on the upper half plane,[12] to itself.

(b) The map $\theta_{M,N} \colon \mathrm{S}_k(X(N), \mathbb{C}) \longrightarrow \mathrm{S}_k(X(M), \mathbb{C})$ sends an f, viewed as function on the upper half plane, to the function $\tau \longmapsto \left(\frac{M}{N}\right)^k f(\frac{M}{N}\tau)$.

Proof: Let τ stand for either $\sigma_{M,N}$ or $\theta_{M,N}$. Because the diagrams (7.3) are cartesian, we have a canonical isomorphism $\tau^*\omega_{X(N)} \xrightarrow{\sim} \omega_{X(M)}$ by lemma 2.5 (a), so τ induces a morphism

$$\mathrm{H}^0(X(N), \omega_{X(N)}^{\otimes k}) \longrightarrow \mathrm{H}^0(X(M), \omega_{X(M)}^{\otimes k}). \tag{$*$}$$

Since the explicit descriptions of the realisations of ${}^N_kW$ come from the Leray spectral sequence for the morphism $\mathrm{KS}(N,k) \longrightarrow X(N)$ (see theorem 5.8), the morphism (7.4) is the restriction of the morphism $(*)$ (tensored with $\mathbb{Q}$) to $\mathrm{S}_k(X(N), \mathbb{Q})$. The morphism $(*)$ can be constructed analogously with $X(N)$ replaced by $X_1(N)$. For simplicity and to avoid the technical complications mentioned in footnote 12, we prove the claim for this morphism instead; it should be clear that the actual statement can be proved in the same way.

To prove this, we define maps

$$\begin{aligned}
\Sigma^{\mathrm{an}}_{M,N} \colon \mathbb{C} \times \mathfrak{h} &\longrightarrow \mathbb{C} \times \mathfrak{h}, \quad (z,\tau) \longmapsto (z,\tau)\\
\Theta^{\mathrm{an}}_{M,N} \colon \mathbb{C} \times \mathfrak{h} &\longrightarrow \mathbb{C} \times \mathfrak{h}, \quad (z,\tau) \longmapsto (\tfrac{M}{N}z, \tfrac{M}{N}\tau)\\
\sigma^{\mathrm{an}}_{M,N} \colon \mathfrak{h} &\longrightarrow \mathfrak{h}, \quad \tau \longmapsto \tau\\
\theta^{\mathrm{an}}_{M,N} \colon \mathfrak{h} &\longrightarrow \mathfrak{h}, \quad \tau \longmapsto \tfrac{M}{N}\tau
\end{aligned}$$

and show that they induce the corresponding maps named in the same way without "an" on the analytifications of the modular curves. To simplify the notation, we henceforth omit the subscripts "$_{M,N}$". First observe that obviously Σ^{an} lies over σ^{an} and Θ^{an} lies over θ^{an}. Next we check that Σ^{an} and Θ^{an} are compatible with the matrix action introduced in example 1.2.11. For Σ^{an} this is trivial. For Θ^{an} we note that it is just the action of the matrix $\left(\begin{smallmatrix} M/N & \\ & 1 \end{smallmatrix}\right)$ and that

$$\begin{pmatrix} M/N & \\ & 1 \end{pmatrix} \Gamma_1(M) \begin{pmatrix} M/N & \\ & 1 \end{pmatrix}^{-1} \subseteq \Gamma_1(N).$$

[12] Strictly speaking, since we use the modular curve $X(N)$ here, f is a $\varphi(N)$-tuple of functions on the upper half plane, where $\varphi(N)$ is the Euler totient function, see remark 1.14 (b). The description of the map given here has to be applied to each entry in the tuple.

Hence we get induced maps

$$\Sigma^{\mathrm{an}}, \Theta^{\mathrm{an}} \colon \ \Gamma_1(M)\Big\backslash(\mathbb{C} \times \mathfrak{h}) \longrightarrow \Gamma_1(N)\Big\backslash(\mathbb{C} \times \mathfrak{h})$$

and analogously with σ^{an} and θ^{an}. Finally we need to check that for the "relative lattice" $\Lambda \subseteq \mathbb{C} \times \mathfrak{h}$ introduced in definition 1.12 (a) we have $\Sigma^{\mathrm{an}}(\Lambda), \Theta^{\mathrm{an}}(\Lambda) \subseteq \Lambda$. But this is clear from the representation (1.6) and the definitions of Σ^{an} and Θ^{an}.

To summarise, using theorem 1.13 we now have commuting maps

$$\begin{array}{ccc} E_1(M)^{\mathrm{an}} & \xrightarrow{\Sigma^{\mathrm{an}}} & E_1(N)^{\mathrm{an}} \\ f\big\downarrow & & f\big\downarrow \\ Y_1(M)^{\mathrm{an}} & \xrightarrow{\sigma^{\mathrm{an}}} & Y_1(N)^{\mathrm{an}} \end{array} \qquad\qquad \begin{array}{ccc} E_1(M)^{\mathrm{an}} & \xrightarrow{\Theta^{\mathrm{an}}} & E_1(N)^{\mathrm{an}} \\ f\big\downarrow & & f\big\downarrow \\ Y_1(M)^{\mathrm{an}} & \xrightarrow{\theta^{\mathrm{an}}} & Y_1(N)^{\mathrm{an}}. \end{array}$$

In terms of the moduli description from theorem 1.13, by definition the maps are given by

$$\Sigma^{\mathrm{an}}\left(E_\tau, \tfrac{1}{M}, z\right) = \left(E_\tau, \tfrac{1}{N}, z\right), \quad \Theta^{\mathrm{an}}\left(E_\tau, \tfrac{1}{M}, z\right) = \left(E_{\frac{M}{N}\tau}, \tfrac{1}{N}, \tfrac{M}{N}z\right).$$

We need to compare this to the definitions in (7.2). For Σ it is clear that the definitions are compatible because $\frac{M}{N} \cdot \frac{1}{M} = \frac{1}{N}$. For Θ we note that $\frac{M}{N}[1, \tau] \subseteq [1, \frac{M}{N}\tau]$, so the multiplication-by-$\frac{M}{N}$ map $\mathbb{C} \longrightarrow \mathbb{C}$ induces a surjective homomorphism $E_\tau \longrightarrow\!\!\!\!\!\to E_{\frac{M}{N}\tau}$ with kernel $\frac{N}{M}$. Hence as a point in the moduli space $E_1(N)^{\mathrm{an}}$ we have

$$\left(E_{\frac{M}{N}\tau}, \frac{1}{N}, \frac{M}{N}z\right) = \left(E_\tau \Big/ \tfrac{N}{M}, \frac{1}{M}, z\right),$$

so the definitions of Θ are also compatible.

Now statement (a) is clear. For (b) we identify f with the differential form $\tilde{f}(2\pi i \mathrm{d}z)^{\otimes k}$ as in the description of how to view modular forms as functions on $\mathfrak{h}$ at the end of section 4.1. Then θ sends f to

$$(\theta^{\mathrm{an}})^*(\tilde{f}(2\pi i\mathrm{d}z)^{\otimes k}) = (\tilde{f} \circ \theta^{\mathrm{an}})(2\pi i \mathrm{d}(z \circ \Theta^{\mathrm{an}}))^{\otimes k} = \left(\frac{M}{N}\right)^k (\tilde{f} \circ \theta^{\mathrm{an}})(2\pi i\mathrm{d}z)^{\otimes k}. \qquad \square$$

Corollary 7.7: *Let $N \geq 4$, p be a prime, K be a number field, $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, K)$ be an eigenform away from the level and let α, β be the roots of the p-th Hecke polynomial (7.1). Assume that they lie in K and that K contains the Np-th roots of unity. Then there exist two canonical morphisms of motives*

$$\mathrm{Ref}_\alpha, \mathrm{Ref}_\beta \colon \ {}^N_k\mathcal{W} \underset{\mathbb{Q}}{\otimes} K \longrightarrow {}^{Np}_{\ k}\mathcal{W} \underset{\mathbb{Q}}{\otimes} K$$

such that the induced morphisms on the intermediate step in the Hodge filtration on de Rham realisations $\mathrm{S}_k(X(N), K) \longrightarrow \mathrm{S}_k(X(Np), K)$ map f to its two refinements f_α and f_β, respectively.

Proof: Just define the morphisms as $\sigma_{Np,N} + p^{-k}\gamma\theta_{Np,N}$ for $\gamma \in \{\alpha, \beta\}$ with σ and θ as in proposition 7.5. The claim then follows from the definition of the refinements and proposition 7.6. $\square$

8. Poincaré duality for ${}^{N}_{k}W$

8.1. The cup product pairing in singular cohomology of modular curves

There are important pairings in the cohomology of modular curves. Their construction is explained in detail in [DFG01, §2.4] or [Con09, §2.3.3], to where we refer for more details. We do not repeat the precise constructions but only say that using the cup product one can construct a pairing

$$\langle\cdot,\cdot\rangle : \mathrm{H}^1(Y_?(N), \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}}) \times \mathrm{H}^1_{\mathrm{c}}(Y_?(N), \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}}) \longrightarrow \mathbb{Z}$$

inducing perfect pairings

$$\langle\cdot,\cdot\rangle : \mathrm{H}^1(Y_?(N), \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}}) \times \mathrm{H}^1_{\mathrm{c}}(Y_?(N), \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}}) \longrightarrow \mathbb{Q}$$

and

$$\langle\cdot,\cdot\rangle : \mathrm{H}^1_{\mathrm{p}}(Y_?(N), \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}}) \times \mathrm{H}^1_{\mathrm{p}}(Y_?(N), \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}}) \longrightarrow \mathbb{Q}$$

for ? being nothing or “1”.

Via the Eichler-Shimura isomorphism, we can interpret the pairing $\langle\cdot,\cdot\rangle$ on the space $\mathrm{S}_k(X,\mathbb{C}) \oplus \overline{\mathrm{S}_k(X,\mathbb{C})}$. It is closely related to the classical Petersson scalar product on $\mathrm{S}_k(X,\mathbb{C})$, which we denote by $\langle\cdot,\cdot\rangle_{\mathrm{Pet}}$.

Proposition 8.1: *We have*

$$\left\langle \mathrm{ES}(f_1 \oplus \overline{g}_1), \mathrm{ES}(f_2 \oplus \overline{g}_2) \right\rangle = C \cdot \left(\langle f_1, g_2 \rangle_{\mathrm{Pet}} + (-1)^{k+1} \langle f_2, g_1 \rangle_{\mathrm{Pet}} \right)$$

with a nonzero constant $C \in \mathbb{C}^\times$ depending only on k.

Proof: [Con09, Thm. 2.3.3.1]. □

The exact value of the constant C is also given at the above reference, but it will not be important for us.

Proposition 8.2: *The adjoints of the Hecke operators T_p and $\langle d\rangle$ with respect to the pairing $\langle\cdot,\cdot\rangle$ on $\mathrm{H}^1_{\mathrm{p}}(Y_?(N), \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}})$ are T_p^ι and $\langle d\rangle^\iota$, respectively.*

Proof: [Con09, Thm. 2.3.7.3] □

8.2. Motivic interpretation, and some complements

There is a general theory of a Poincaré duality for motives which is explained e. g. in [Lev98]. In this book, Levine constructs a category $\mathcal{DM}(k)$ for k a field (he does it even in greater generality) which should be viewed as a candidate for the derived category of the category of mixed motives over k. The category of Chow motives over k admits a fully faithful embedding into $\mathcal{DM}(k)$ [Lev98, §VI.2.1.5], and the category $\mathcal{DM}(k)$ admits a duality [Lev98, §VI.1] which on realisations recovers the Poincaré duality required in the axioms of a Weil cohomology theory. We will not expand further on this theory, it is fully exposed in the given reference.

In view of this theoretical background, we can reinterpret the pairing from the previous sections. In fact, we should view the pairing not as a pairing with values in $\mathbb{Q}$, but we should identify the one-dimensional $\mathbb{Q}$-vector space where it takes its values with $\mathbb{Q}(1-k)_{\mathrm{B}}$, the Betti realisation of $\mathbb{Q}(1-k)$, so we get pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{B}}\colon\ {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \times {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \longrightarrow b_{\mathrm{B}}^{\mathbb{Q}(1-k)}\mathbb{Q} = \mathbb{Q}(1-k)_{\mathrm{B}}. \tag{8.1}$$

This is then just the incarnation on the Betti realisation (which is why we denote it $\langle\cdot,\cdot\rangle_{\mathrm{B}}$ from now on) of a perfect pairing of motives

$$\langle\cdot,\cdot\rangle\colon\ {}^{N}_{k}\mathcal{W} \times {}^{N}_{k}\mathcal{W} \longrightarrow \mathbb{Q}(1-k) \tag{8.2}$$

which comes from the above-mentioned Poincaré duality theory for motives. See [DFG01, §2.4–5] for background on this (note that our motive ${}^{N}_{k}\mathcal{W}$ is denoted $M_!$ or $M_{N,!}$ there). On the other realisations of ${}^{N}_{k}\mathcal{W}$ we therefore also have pairings, and these pairings are identified by the respective comparison isomorphisms.

For an explicit description of the pairings

$$\langle\cdot,\cdot\rangle_{\mathrm{dR}}\colon\ {}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \times {}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \longrightarrow \mathbb{Q}(1-k)_{\mathrm{dR}}, \tag{8.3}$$

$$\langle\cdot,\cdot\rangle_{p}\colon\ {}^{N}_{k}\mathcal{W}_{p} \times {}^{N}_{k}\mathcal{W}_{p} \longrightarrow \mathbb{Q}(1-k)_{p}, \tag{8.4}$$

in terms of the wedge product of differential forms resp. the cup product in étale cohomology, we refer to [DFG01, §2.4], [Del69, (3.20)] and [Con09, §5.2.11].

The compatibility of the pairings with the comparison isomorphisms means the following: if $?_1, ?_2 \in \{\mathrm{B}, \mathrm{dR}, p\}$ and B is the period ring using to compare the $?_1$- and $?_2$-realisations (i. e. $B \in \{\mathbb{C}, \mathbb{Q}_p, \mathrm{B}_{\mathrm{dR}}\}$), then the diagram

$$\begin{tikzcd}
({}^{N}_{k}\mathcal{W}_{?_1} \otimes B) \times ({}^{N}_{k}\mathcal{W}_{?_1} \otimes B) \arrow[r, "\langle\cdot,\cdot\rangle_{?_1}"] \arrow[d, "\sim"'] & \mathbb{Q}(1-k)_{?_1} \otimes B \arrow[d, "\sim"'] \\
({}^{N}_{k}\mathcal{W}_{?_2} \otimes B) \times ({}^{N}_{k}\mathcal{W}_{?_2} \otimes B) \arrow[r, "\langle\cdot,\cdot\rangle_{?_2}"] & \mathbb{Q}(1-k)_{?_2} \otimes B
\end{tikzcd}$$

commutes (where the vertical maps are the comparison isomorphisms). To make this really explicit, we state the following proposition. It follows from the above diagram and the explicit description of the comparison isomorphism of the Tate motive in fact 1.3.6.

Proposition 8.3: *(a) Let $x_{\mathrm{B}}, y_{\mathrm{B}} \in {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ and let $x_{\mathrm{dR}}, y_{\mathrm{dR}}$ be their images under the comparison isomorphism ${}^{N}_{k}\mathcal{W}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$. Let further $c \in \mathbb{C}$. Then*

$$\langle x_{\mathrm{B}}, y_{\mathrm{B}}\rangle_{\mathrm{B}} = c \cdot b_{\mathrm{B}}^{\mathbb{Q}(1-k)} \iff \langle x_{\mathrm{dR}}, y_{\mathrm{dR}}\rangle_{\mathrm{dR}} = (2\pi\mathrm{i})^{1-k} c \cdot b_{\mathrm{dR}}^{\mathbb{Q}(1-k)}.$$

(b) Let $x_{\mathrm{B}}, y_{\mathrm{B}} \in {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and let x_p, y_p be their images under the comparison isomorphism ${}^{N}_{k}\mathcal{W}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} {}^{N}_{k}\mathcal{W}_p$. Let further $c \in \mathbb{Q}_p$. Then

$$\langle x_{\mathrm{B}}, y_{\mathrm{B}}\rangle_{\mathrm{B}} = c \cdot b_{\mathrm{B}}^{\mathbb{Q}(1-k)} \iff \big\langle x_p, y_p\big\rangle_p = c \cdot b_p^{\mathbb{Q}(1-k)}.$$

(c) Let $x_p, y_p \in {}^{N}_{k}\mathcal{W}_p \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ and let $x_{\mathrm{dR}}, y_{\mathrm{dR}}$ be their images under the comparison isomorphism ${}^{N}_{k}\mathcal{W}_p \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} {}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathrm{B}_{\mathrm{dR}}$. Let further $c \in \mathrm{B}_{\mathrm{dR}}$. Then

$$\big\langle x_p, y_p\big\rangle_p = c \cdot b_p^{\mathbb{Q}(1-k)} \iff \langle x_{\mathrm{dR}}, y_{\mathrm{dR}}\rangle_{\mathrm{dR}} = \mathrm{t}_{\mathrm{dR}}^{1-k} c \cdot b_{\mathrm{dR}}^{\mathbb{Q}(1-k)}.$$

Proposition 8.4: *Via the exact sequence*

$$0 \longrightarrow \mathrm{S}_k(X(N),\mathbb{Q}) \longrightarrow {}^N_k\mathcal{W}_{\mathrm{dR}} \longrightarrow \mathrm{H}^1(X(N),\omega^{2-k}_{X(N)}) \longrightarrow 0$$

coming from the Hodge filtration (see propositions 5.9 and 5.10), the pairing (8.3) induces a perfect pairing

$$\mathrm{S}_k(X(N),\mathbb{Q}) \times \mathrm{H}^1(X(N),\omega^{2-k}_{X(N)}) \longrightarrow \mathbb{Q}(1-k)_{\mathrm{dR}}$$

and this pairing coincides with the Serre duality pairing under the canonical identification $\mathbb{Q}(1-k)_{\mathrm{dR}} \cong \mathbb{Q}$.

Proof: For well-definedness, we have to show that the pairing $\langle\cdot,\cdot\rangle_{\mathrm{dR}}$ from (8.3) restricted to $\mathrm{fil}^0\, {}^N_k\mathcal{W}_{\mathrm{dR}} = \mathrm{S}_k(X(N),\mathbb{Q})$ vanishes. By theorem 6.7, this is equivalent to the vanishing of the pairing on $\mathrm{S}_k(X(N),\mathbb{C})$ induced by the pairing $\langle\cdot,\cdot\rangle_{\mathrm{B}}$ via the Eichler-Shimura isomorphism. This vanishing property follows from proposition 8.1. That it coincides with the Serre duality pairing (as predicted by the general theory of Poincaré duality for motives) is stated in [DFG01, p. 21]. From this fact follows its perfectness. $\square$

Corollary 8.5: *The space* $\mathrm{gr}^0\, \mathcal{M}(f)_{\mathrm{dR}}$ *is canonically isomorphic to the dual space of the subspace of* $\mathrm{S}_k(X_1(N),K)$ *generated by* $w_N f$.

Proof: This follows easily from propositions 5.10, 8.2 and 8.4 and the fact that the comparison isomorphism respects the pairings. Note that although the Atkin-Lehner involution w_N may not be defined over K unless K contains all N-th roots of unity, the Fourier coefficients of $w_N f$ lie again in K, see [Li75, p. 296]. $\square$

By the adjointness relations for the Hecke operators (see proposition 8.2), the pairing $\langle\cdot,\cdot\rangle_{\mathrm{B}}$ induces a pairing

$$\mathcal{M}(f)_{\mathrm{B}} \times \mathcal{M}(f^*)_{\mathrm{B}} \longrightarrow K(1-k).$$

Since it may be handy to have the Hecke operators self-adjoint, it is common in the literature modify the pairing in the following way: Let

$$\langle\cdot,\cdot\rangle^{\iota}_{\mathrm{B}} := \langle\cdot, w_N\cdot\rangle_{\mathrm{B}}$$

where w_N is the Atkin-Lehner involution (we will use this only on the Betti realisation). This modified version is also called *twisted Poincaré duality pairing* by some authors. It is clear that the Hecke operators are self-adjoint with respect to it. We will not use the modified pairing too much, but some statements are easier to formulate using the modified version.

Lemma 8.6: *The pairing* $\langle\cdot,\cdot\rangle^{\iota}_{\mathrm{B}}$ *restricts to perfect pairings*

$$\mathcal{M}(f)^{\pm}_{\mathrm{B}} \times \mathcal{M}(f)^{\mp}_{\mathrm{B}} \longrightarrow K(1-k),$$

while its restriction to $\mathcal{M}(f)^{\pm}_{\mathrm{B}} \times \mathcal{M}(f)^{\pm}_{\mathrm{B}}$ *vanishes.*

Proof: The perfectness of the pairing can be tested after tensoring with $\mathbb{C}$. Put $b := \mathrm{ES}(f) \in \mathcal{M}(f)_{\mathrm{B}} \otimes_K \mathbb{C}$ and $b^{\pm} := b \pm \mathrm{Frob}_\infty(b) \in \mathcal{M}(f)^{\pm}_{\mathrm{B}} \otimes_K \mathbb{C}$, where Frob_∞ is complex conjugation. By lemma 6.6, we have $b^{\pm} = \mathrm{ES}(f \pm \mathrm{Frob}_\infty(f))$ with Frob_∞ acting as in definition 6.5, so explicitly

$$b^{\pm} = \mathrm{ES}(f \oplus (\mp\overline{f^*})).$$

In particular $b^\pm \neq 0$, and since $\mathcal{M}(f)^\pm_{\mathrm{B}}$ is one-dimensional, $b^\pm$ is a basis. Hence it suffices to show that $\langle b^\pm, w_N b^\pm\rangle_{\mathrm{B}} = 0$ and $\langle b^\pm, w_N b^\mp\rangle_{\mathrm{B}} \neq 0$. Explicitly, this is

$$\Big\langle \mathrm{ES}(f \oplus (\mp\overline{f^*})), \mathrm{ES}(w_N f \oplus w_N(\mp\overline{f^*}))\Big\rangle_{\mathrm{B}} = 0,$$
$$\Big\langle \mathrm{ES}(f \oplus (\mp\overline{f^*})), \mathrm{ES}(w_N f \oplus w_N(\pm\overline{f^*}))\Big\rangle_{\mathrm{B}} \neq 0.$$

We transfer this problem to $\mathrm{S}_k(X_1(N),\mathbb{C})\oplus\overline{\mathrm{S}_k(X_1(N),\mathbb{C})}$ via the Eichler-Shimura isomorphism. The pairing there that corresponds to $\langle\cdot,\cdot\rangle_{\mathrm{B}}$ is, up to a constant, basically the Petersson scalar product, as shown in proposition 8.1. Using the formula there and the relation $w_N(f^*) = (-1)^k(w_N f)^*$ from [Li75, bottom of p. 296], what we have to show becomes

$$\mp(-1)^k\langle f, (w_N f)^*\rangle_{\mathrm{Pet}} \mp (-1)^{k-1}\langle f^*, w_N f\rangle_{\mathrm{Pet}} = 0,$$
$$\pm(-1)^k\langle f, (w_N f)^*\rangle_{\mathrm{Pet}} \mp (-1)^{k-1}\langle f^*, w_N f\rangle_{\mathrm{Pet}} \neq 0.$$

By proposition 4.21, $(w_N f)^*$ is a nonzero multiple of f, and thus also $w_N f$ is a nonzero multiple of f^* by the same factor. Further the Petersson norm of any modular form and its image under w_N are equal; this is also shown at [Li75, middle of p. 296]. Hence the desired properties follow from the definiteness of the Petersson scalar product. □

Proposition 8.7: *Let $n \in \mathbb{Z}$ and ρ be an Artin representation. The motive $\mathcal{M}(f)(\rho)(n)$ is critical if and only if $1 \leq n \leq k-1$.*

Proof: From the descriptions of the Tate motive and the motive associated to an Artin representation in section 1.3.2, we observe that tensoring a motive with $\mathcal{M}(\rho)$ does not change its Hodge filtration, whereas tensoring with $\mathbb{Q}(n)$ moves it by n steps. From the Hodge filtration of ${}^{N}_{k}\mathcal{W}$ (see proposition 5.9) we therefore see by counting dimensions that $\mathcal{M}(f)(\rho)(n)$ can only be critical if $1 \leq n \leq k-1$. So fix such an n, and without loss of generality assume that ρ is trivial. We have $\mathcal{M}(f)(n)^+_{\mathrm{B}} = \mathcal{M}(f)^{(-1)^n}_{\mathrm{B}} \otimes_K K(n)_{\mathrm{B}}$.

Let $b^\pm \in \mathcal{M}(f)^\pm_{\mathrm{B}}\otimes_K\mathbb{C}$ be as in the proof of lemma 8.6 and let $c^\mp \in \mathcal{M}(f)^\mp_{\mathrm{B}}\otimes_K\mathbb{C}$ be elements such that $\langle b^\pm, c^\mp\rangle^\iota_{\mathrm{B}} \neq 0$. Then by theorem 6.7 and the fact that the comparison isomorphism respects the pairings, the image of $c^\mp \otimes_K (b^{\mathbb{Q}(1)}_{\mathrm{B}})^{\otimes n}$ for $\mp = (-1)^{n+1}$ in $\mathrm{t}_{\mathcal{M}(f)(n)} \otimes_K \mathbb{C} = (\mathrm{gr}^0\,\mathcal{M}(f)_{\mathrm{dR}} \otimes_K K(n)_{\mathrm{dR}}) \otimes_K \mathbb{C} \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathrm{S}_k(X(N),\mathbb{C}),\mathbb{C})$ is a linear form whose evaluation at $w_N f$ does not vanish. □

Chapter III.

Modular symbols and p-adic families

Modular symbols are certain cohomology classes on modular curves which are used in the first place as a technical device to construct p-adic L-functions for modular forms (see appendix B for this). We first introduce them in a general setting, using the abstract Hecke theory from section I.1. After that we introduce the classical modular symbols which were used to construct p-adic L-functions for a single modular form.

To handle families of modular forms, one needs to put the classical modular symbols together into families. For Hida families, this is achieved by Kitagawa's construction of $\mathcal{I}$-adic modular symbols. After some preliminary discussion of Hida families we introduce and study these.

The most important ingredient in this whole work is the $\mathcal{I}$-adic Eichler-Shimura isomorphism, which relates $\mathcal{I}$-adic modular symbols and $\mathcal{I}$-adic cusp forms. We cite this statement, proved by Ohta, Kings, Loeffler and Zerbes [Oht95; KLZ17], at the end of the chapter. Before that, we need to include a rather technical section on a different viewpoint on $\mathcal{I}$-adic modular symbols because parts of the literature use that viewpoint.

While in the previous chapter we considered modular forms and modular curves for a fixed level N, in this chapter the level will be varying (except in sections 1 and 2). More precisely, the role which was played by N in the previous chapter will now be played by Np^r, where N is a fixed integer prime to p and $r \geq 1$ is varying. The reader should not be confused by these different roles.

1. General theory of modular symbols

We first define modular symbols in a rather general setting. To do so, we use the abstract Hecke theory developed in section I.1.

We consider the group $\mathrm{GL}_2(\mathbb{Q})$ with the main involution ι and put $\Sigma := \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$. Fix a Hecke pair (Δ, Γ) for (Σ, ι) and a commutative ring R.

The group $\mathrm{GL}_2(\mathbb{Q})$ acts on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ as explained in example I.2.10 (b). Let $\mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ be the free abelian group over the set $\mathbb{P}^1(\mathbb{Q})$ and let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the subgroup consisting of elements of degree 0. The left action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$ induces a left action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ and $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$. So these groups are a $\mathbb{Z}$-linear representations of (Σ, ι) and $\mathbb{P}^1(\mathbb{Q})$ is a Hecke space.

Definition 1.1: Let M be an R-linear representation of (Σ, ι). Then

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M) = \mathrm{Hom}_R(R \underset{\mathbb{Z}}{\otimes} \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)$$

is again an R-linear representation of (Σ, ι), as explained in remark I.1.4. We define the right

$\mathcal{H}_R(\Delta,\Gamma)$-module of *modular symbols with coefficients in* M to be the Γ-invariants

$$\mathrm{MSymb}(\Gamma, M) := \mathrm{H}^0(\Gamma, \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)).$$

This is exactly the definition given in [PS13, §2.1], except that there a concrete choice for (Δ,Γ) is used.

Clearly MSymb defines a left exact functor from R-linear representations of (Σ,ι) to right $\mathcal{H}_R(\Delta,\Gamma)$-modules.

Proposition 1.2 (Ash/Stevens)**:** *Let M be an R-linear representation of (Σ,ι). Assume that $\mathfrak{h}$ and Γ satisfy condition I.1.39 and further that $\frac{\mathfrak{h}^*}{\Gamma}$ is compact and $\frac{\mathbb{P}^1(\mathbb{Q})}{\Gamma}$ is finite. Write $\pi\colon \mathfrak{h} \longrightarrow \frac{\mathfrak{h}}{\Gamma}$ for the canonical projection.*

(a) There is a canonical $\mathcal{H}_R(\Delta,\Gamma)$-linear isomorphism

$$\mathrm{MSymb}(\Gamma, M) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{c}}(\tfrac{\mathfrak{h}}{\Gamma}, \pi_*^\Gamma \underline{M})$$

which is functorial in M.

(b) There is a canonical commutative diagram of $\mathcal{H}_R(\Delta,\Gamma)$-modules

$$\begin{tikzcd} \mathrm{MSymb}(\Gamma, M) \arrow[r, two heads, "\partial"] \arrow[d, "\sim"'] & \mathrm{H}^1(\Gamma, M) \arrow[d, "\sim"] \\ \mathrm{H}^1_{\mathrm{c}}(\tfrac{\mathfrak{h}}{\Gamma}, \pi_*^\Gamma \underline{M}) \arrow[r, two heads] & \mathrm{H}^1(\tfrac{\mathfrak{h}^*}{\Gamma}, \pi_*^\Gamma \underline{M}). \end{tikzcd}$$

Here the right isomorphism is from corollary I.1.43, the upper horizontal map is a boundary map (see the proof for details) and the lower map is the tautological one. The horizontal maps are surjective.

This is again functorial in M.

(c) If M and $\mathrm{H}^1(\Gamma, M)$ are finitely generated R-modules, then also $\mathrm{MSymb}(\Gamma, M)$ is finitely generated over R.

Proof: We argue similarly as in the proofs of proposition I.1.42 and corollary I.1.43. Write π also for the canonical projection $\mathfrak{h}^* \longrightarrow \frac{\mathfrak{h}^*}{\Gamma}$ as well as for its restriction to $\mathbb{P}^1(\mathbb{Q})$. Let $\mathcal{A} := \mathcal{Sh}_R^{(\Sigma,\iota)}(\mathfrak{h}^*)$, $\mathcal{B} := R\text{-}\mathcal{Mod}_{(\Sigma,\iota)}$ and $\mathcal{C} := \mathcal{Mod}\text{-}\mathcal{H}_R(\Delta,\Gamma)$, all of which are abelian categories. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be the functor sending a sheaf $\mathcal{F}$ on $\mathfrak{h}^*$ the submodule of $\mathrm{H}^0(\mathfrak{h}^*, \mathcal{F})$ consisting of sections vanishing in $\mathbb{P}^1(\mathbb{Q})$. We denote its right derived functors by $\mathrm{H}^i(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), -)$, and it is easy to see that for constant coefficients it coincides with the usual singular relative cohomology group. Let $G\colon \mathcal{B} \longrightarrow \mathcal{C}$ be the functor of Γ-invariants.

If $\mathcal{F} \in \mathcal{A}$, then elements of $G(F(\mathcal{F}))$ are global sections of $\pi_*^\Gamma \mathcal{F}$ on $\frac{\mathfrak{h}^*}{\Gamma}$ that vanish in $\frac{\mathbb{P}^1(\mathbb{Q})}{\Gamma}$. Using elementary topology arguments, it is easy to see that such sections are in bijection with sections of $\pi_*^\Gamma \mathcal{F}$ on $\frac{\mathfrak{h}}{\Gamma}$ with compact support.[1] Arguing as in the proof of proposition I.1.42,

[1] Indeed, if a section on $\frac{\mathfrak{h}}{\Gamma}$ has compact support, it has to vanish on a small neighbourhood around each cusp. Conversely, if a section on $\frac{\mathfrak{h}^*}{\Gamma}$ vanishes at a cusp, then since the support of sections of sheaves is always closed [Stacks, Tag 01AU], it has to vanish on a small neighbourhood of the cusp, so its support is a compact subset of $\frac{\mathfrak{h}}{\Gamma}$.

we thus see

$$\mathrm{R}^*(G \circ F) = \mathrm{H}^*_{\mathrm{c}}(\tfrac{\mathfrak{h}}{\Gamma}, \pi^\Gamma_*(-)).$$

Using this, the Grothendieck spectral sequence for the composition of functors $G \circ F$ becomes

$$E_2^{pq} = \mathrm{H}^p(\Gamma, \mathrm{H}^q(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), \mathcal{F})) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{c}}(\tfrac{\mathfrak{h}}{\Gamma}, \pi^\Gamma_*\mathcal{F}).$$

We put $\mathcal{F} = \underline{M}$ there. Since $\mathfrak{h}^*$ and $\mathbb{P}^1(\mathbb{Q})$ are zero-dimensional, $\mathrm{H}^q(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M)$ and therefore E_2^{pq} vanishes for $q > 1$ and all p. We thus get an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(\Gamma, \mathrm{H}^0(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M)) &\longrightarrow \mathrm{H}^1_{\mathrm{c}}(\tfrac{\mathfrak{h}}{\Gamma}, \pi^\Gamma_*\underline{M}) \\ &\longrightarrow \mathrm{H}^0(\Gamma, \mathrm{H}^1(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M)) \longrightarrow \mathrm{H}^2(\Gamma, \mathrm{H}^0(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M)) \longrightarrow \cdots. \end{aligned}$$

From the commutative diagram in $\mathbb{Z}\text{-}\mathcal{M}od_{(\Sigma,\iota)}$ with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_1(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), \mathbb{Z}) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}_0(\mathbb{P}^1(\mathbb{Q}), \mathbb{Z}) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}_0(\mathfrak{h}^*, \mathbb{Z}) \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \arrow[r] & \mathrm{Div}(\mathbb{P}^1(\mathbb{Q})) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

where the upper row is the long exact singular homology sequence for the pair $(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}))$ of Hecke spaces, the lower right map is the degree and the vertical maps are canonical isomorphisms, we see that $\mathrm{H}_0(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M) = 0$. Thus the first and fourth term in the sequence above vanish, and we get an isomorphism $\mathrm{H}^1_{\mathrm{c}}(\tfrac{\mathfrak{h}}{\Gamma}, \pi^\Gamma_*\underline{M}) \xrightarrow{\sim} \mathrm{H}^0(\Gamma, \mathrm{H}^1(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M))$.

Our next arguments follow [Kit94, §3.2] closely. We apply the functor $\mathrm{Hom}_{\mathbb{Z}}(-, M) = \mathrm{Hom}_R(- \otimes_{\mathbb{Z}} R, M)$ to the above diagram to obtain a diagram in $R\text{-}\mathcal{M}od_{(\Sigma,\iota)}$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_R(\mathrm{Div}(\mathbb{P}^1(\mathbb{Q})), M) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_R(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M) \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & \mathrm{H}^0(\mathfrak{h}^*, M) \arrow[r] & \mathrm{H}^0(\mathbb{P}^1(\mathbb{Q}), M) \arrow[r] & \mathrm{H}^1(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M) \arrow[r] & 0 \end{tikzcd}$$

whose rows are still exact since all $\mathbb{Z}$-modules in the previous diagram were free and hence the sequences there split (non-canonically). This gives us an isomorphism as claimed in statement (a). By construction it is clear that it is functorial in M.

We now take group cohomology for Γ of this diagram, which gives us two long exact sequences in $\mathcal{M}od\text{-}\mathcal{H}_R(\Delta, \Gamma)$. We add a third exact sequence below, using statement (a) and corollary 1.1.43 for the spaces $\mathfrak{h}^*$ and $\mathbb{P}^1(\mathbb{Q})$. This gives us the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^0(\Gamma, M) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^0(\Gamma, \mathrm{Hom}_R(\mathrm{Div}(\mathbb{P}^1(\mathbb{Q})), M)) \arrow[r] \arrow[d, "\sim"] & {} \\ \cdots \arrow[r] & \mathrm{H}^0(\Gamma, \mathrm{H}^0(\mathfrak{h}^*, M)) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^0(\Gamma, \mathrm{H}^0(\mathbb{P}^1(\mathbb{Q}), M)) \arrow[r] \arrow[d, "\sim"] & {} \\ \cdots \arrow[r] & \mathrm{H}^0(\tfrac{\mathfrak{h}^*}{\Gamma}, \pi^\Gamma_*\underline{M}) \arrow[r] & \mathrm{H}^0(\tfrac{\mathbb{P}^1(\mathbb{Q})}{\Gamma}, \pi^\Gamma_*\underline{M}) \arrow[r] & {} \end{tikzcd}$$

$$\begin{array}{ccccc}
\longrightarrow \mathrm{H}^0(\Gamma, \mathrm{Hom}_R(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)) & \longrightarrow & \mathrm{H}^1(\Gamma, M) & \longrightarrow & \cdots \\
\downarrow \sim & & \downarrow \sim & & \\
\longrightarrow \mathrm{H}^0(\Gamma, \mathrm{H}^1(\mathfrak{h}^*, \mathbb{P}^1(\mathbb{Q}), M)) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathrm{H}^0(\mathfrak{h}^*, M)) & \longrightarrow & \cdots \\
\downarrow \sim & & \downarrow \sim & & \\
\longrightarrow \mathrm{H}^1_c(\frac{\mathfrak{h}}{\Gamma}, \pi_*^\Gamma \underline{M}) & \longrightarrow & \mathrm{H}^1(\frac{\mathfrak{h}^*}{\Gamma}, \pi_*^\Gamma \underline{M}) & \longrightarrow & \cdots .
\end{array}$$

The lower squares commute since all isomorphisms come from the spectral sequence. This proves the commutativity of the diagram in statement (b). Moreover the surjectivity of the horizontal maps follows from $\mathrm{H}^1(\frac{\mathbb{P}^1(\mathbb{Q})}{\Gamma}, \pi_*^\Gamma \underline{M}) = 0$ (which holds since $\frac{\mathbb{P}^1(\mathbb{Q})}{\Gamma}$ is a finite set of points) and the above diagram.

Finally we see from the above exact sequences that statement (c) also holds, because we required that $\frac{\mathbb{P}^1(\mathbb{Q})}{\Gamma}$ is finite, so $\mathrm{H}^0(\frac{\mathbb{P}^1(\mathbb{Q})}{\Gamma}, \pi_*^\Gamma \underline{M})$ is a finite sum of copies of M, which is finitely generated, and $\mathrm{H}^1(\Gamma, M)$ is finitely generated by assumption. ☐

Note that the groups $\Gamma = \Gamma_1(N)$ are torsion free for $N \geq 4$, so they fulfil the requirements in proposition 1.2.

2. Classical modular symbols and complex error terms

We now choose special instances of the data from the previous section to define classical modular symbols. Let R be a commutative ring. Fix $N, k \in \mathbb{N}$ with $N \geq 4$ and $k \geq 2$. Let $\Gamma = \Gamma_1(N)$ and $\Delta = \Delta_1(N)^\flat$. Then $\mathcal{H}_R(\Delta, \Gamma)$ is the standard Hecke algebra $\mathcal{H}(N)_R$ of level N by corollary I.1.57. For the representation we use the symmetric tensor linear representation $\mathrm{Sym}^{k-2} R^2$ of $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$, which is discussed in appendix A.1.

Definition 2.1: The right $\mathcal{H}(N)_R$-module of *classical modular symbols of weight k and level N* is defined as

$$\mathrm{MS}_k(N, R) := \mathrm{MSymb}(\Gamma_1(N), \mathrm{Sym}^{k-2} R^2).$$

From the definition it is clear that we have

$$\mathrm{MS}_k(N, S) = \mathrm{MS}_k(N, R) \underset{R}{\otimes} S$$

if S is a flat R-algebra.

Proposition 2.2: *(a) Write $f\colon E_1(N)^{\mathrm{an}} \longrightarrow Y_1(N)^{\mathrm{an}}$ for the universal analytic elliptic curve. Then there is a canonical $\mathcal{H}(N)_R$-linear isomorphism*

$$\mathrm{MS}_k(N, R) \cong \mathrm{H}^1_c(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}_{\underline{R}} \mathrm{R}^1 f_* \underline{R}).$$

Moreover, this isomorphism respects the Atkin-Lehner endomorphism, which on the left side is given by the standard Atkin-Lehner element of level N from definition I.1.60 as explained in section I.1.6 and on the right side is given as in section II.3.4.

(b) The R-module $\mathrm{MS}_k(N, R)$ is free of finite rank. In particular $\mathrm{H}^1_c(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}_{\underline{R}} \mathrm{R}^1 f_* \underline{R})$ is free of finite rank over R.

Proof: In this proof we abbreviate $\Gamma = \Gamma_1(N)$.

We first prove statement (a). By theorem II.1.13, the right hand side is isomorphic to $\mathrm{H}^1_\mathrm{c}(\Gamma\backslash\mathfrak{h}, \mathrm{Sym}^{k-2}_{\underline{R}}\,\mathrm{R}^1 f_*\underline{R})$, where f now is the map $\Gamma\backslash E_{\mathrm{Lat}} \longrightarrow \Gamma\backslash\mathfrak{h}$. Then by lemma II.2.2, $\mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{R}$ on $\Gamma\backslash\mathfrak{h}$ is isomorphic to $\pi^\Gamma_*\underline{\mathrm{Sym}^{k-2} R^2}$. Thus proposition 1.2 completes the proof of the isomorphism. The statement about the Atkin-Lehner endomorphism follows easily from lemma II.3.7. This completes the proof of statement (a).

We now turn to statement (b). By similar arguments as above (using corollary I.1.43) we get isomorphisms

$$\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2} R^2) \cong \mathrm{H}^1(\tfrac{\mathfrak{h}}{\Gamma}, \pi^\Gamma_*\underline{\mathrm{Sym}^{k-2} R^2}) \cong \mathrm{H}^1(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}_{\underline{R}}\,\mathrm{R}^1 f_*\underline{R}).$$

By base change it suffices to prove the claim for $R = \mathbb{Z}$. With lemma II.2.3 we conclude that $\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{Z}^2)$ is finitely generated. Then by proposition 1.2 (c) we know that $\mathrm{MS}_k(N,R)$ is finitely generated. Since it is obviously torsion free as a subset of the Hom set, the claim follows. □

Remark 2.3: By lemma II.4.25, we now know the Hecke eigenalgebra of classical modular symbols: we have canonically

$$\mathbf{T}_R^{(\Delta_1(N),\Gamma_1(N))}(\mathrm{MS}_k(N,R)) \cong \mathbf{T}_k(N,R).$$

The above result allows us to relate modular symbols to the Betti realisation of ${}^N_k\mathcal{W}$. Consider the composition

$$\begin{aligned}\mathrm{MS}_k(N,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^1_\mathrm{c}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}}) &\twoheadrightarrow \mathrm{H}^1_\mathrm{p}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}}) \\ &\hookrightarrow \mathrm{H}^1_\mathrm{p}(Y(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Q}}) = {}^N_k\mathcal{W}_\mathrm{B} \qquad (2.1)\end{aligned}$$

where the first map is from proposition 2.2 (a), the second is tautological and the last one comes from the morphism $Y(N) \longrightarrow Y_1(N)$. This composition is Hecke equivariant and $\mathrm{G}_\mathbb{R}$-equivariant by proposition 1.2 (a) and lemma II.2.2. Moreover, the second map is by definition surjective and the last one is injective since it is so after tensoring with $\mathbb{C}$, by the description of the analytification of $Y(N)$ in remark II.1.14 (b) and the complex Eichler-Shimura isomorphism (theorem II.6.3).

Lemma 2.4: *For each newform $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, K)$ with coefficients in a number field K, the map (2.1) induces isomorphisms*

$$\mathrm{MS}_k(N,K)^\pm[f] \xrightarrow{\sim} \mathcal{M}(f)^\pm_\mathrm{B}.$$

Proof: By Hecke- and $\mathrm{G}_\mathbb{R}$-equivariance it is clear that we get a map between the spaces in the statement. If we look at Hecke eigenspaces in (2.1), we get

$$\begin{aligned}\mathrm{H}^1_\mathrm{c}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{K})[f] &\longrightarrow \mathrm{H}^1_\mathrm{p}(Y_1(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{K})[f] \\ &\longrightarrow \mathrm{H}^1_\mathrm{p}(Y(N)^{\mathrm{an}}, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{K})[f].\end{aligned}$$

That this composition is an isomorphism can be checked after tensoring with $\mathbb{C}$, and then we can use the Eichler-Shimura isomorphisms (theorem II.6.3). If we do so, we see first that all spaces involved here are two-dimensional (using corollary II.4.27 (a)) and further that the first map is surjective. The right map is injective since it is the restriction of an injective map. Hence the composition is an isomorphism. ☐

Definition 2.5: Fix $f \in \mathrm{S}_k(X_1(N), K)$, where K is a subfield of $\mathbb{C}$.

(a) Define a group homomorphism[2]

$$\xi_f\colon \operatorname{Div}^0(\mathbb{P}^1(\mathbb{Q})) \longrightarrow \operatorname{Sym}^{k-2}\mathbb{C}^2, \quad (x)-(y) \longmapsto (2\pi\mathrm{i})^{k-1}\int_y^x (zX+Y)^{k-2} f(z)\mathrm{d}z.$$

One can check that ξ_f is in fact invariant under the action of $\Gamma_1(N)$, so we have

$$\xi_f \in \mathrm{MS}_k(N, \mathbb{C})$$

and ξ_f is called the *modular symbol attached to* f. Moreover one can check that if f is a Hecke eigenform, then

$$\xi_f \in \mathrm{MS}_k(N, \mathbb{C})[f].$$

(b) We call the endomorphism $\mathcal{E} \in \mathcal{H}(N)$ (defined in (I.1.6)) of $\mathrm{MS}_k(N, \mathbb{C})$ also *complex conjugation* and define in this way an action of $\mathrm{G}_\mathbb{R}$ on $\mathrm{MS}_k(N, \mathbb{C})$. Let $\xi_f^\pm$ be the image of ξ_f in the respective part of the decomposition

$$\mathrm{MS}_k(N, \mathbb{C}) = \mathrm{MS}_k(N, \mathbb{C})^+ \oplus \mathrm{MS}_k(N, \mathbb{C})^-.$$

Note that if f is a Hecke eigenform, then $\xi_f^\pm \in \mathrm{MS}_k(N, \mathbb{C})^\pm[f]$.

Lemma 2.6: *Consider the composition*

$$\begin{aligned}\mathrm{MS}_k(N, \mathbb{C}) &\xrightarrow{\sim} \mathrm{H}^1_\mathrm{c}(Y_1(N)^\mathrm{an}, \operatorname{Sym}^{k-2}\mathrm{R}^1 f_*\underline{\mathbb{C}}) \\ &\twoheadrightarrow \mathrm{H}^1_\mathrm{p}(Y_1(N)^\mathrm{an}, \operatorname{Sym}^{k-2}\mathrm{R}^1 f_*\underline{\mathbb{C}}) \xrightarrow{\sim} \mathrm{S}_k(X_1(N), \mathbb{C}) \oplus \overline{\mathrm{S}_k(X_1(N), \mathbb{C})}\end{aligned}$$

where the first map is from proposition 2.2 (a), the second one is tautological and the last one is the (inverse) Eichler-Shimura isomorphism. This composition maps ξ_f to $f = f \oplus 0$ and

$$\xi_f^\pm \longmapsto \frac{1}{2}(f \oplus (\pm\overline{f^*})).$$

Proof: The first assertion is easy to see using the definition of ξ_f, proposition 1.2 (b) and proposition II.6.4. The second assertion follows from the first one using lemma II.6.6 and the obvious identity

$$\xi_f^\pm = \frac{1}{2}(\xi_f \pm \operatorname{Frob}_\infty(\xi_f))$$

since the composition is $\mathrm{G}_\mathbb{R}$-equivariant by lemmas II.2.2 and II.6.6. ☐

[2] There are different conventions regarding the definition of ξ_f in the literature, differing in the exponent of $2\pi\mathrm{i}$. We use an exponent $k-1$. For a comparison of these conventions and an explanation why we use this power, see appendix A.3.

We can now define complex error terms. Fix a number field K and an embedding $K \hookrightarrow \mathbb{C}$, and let $\mathcal{O}_K$ be its ring of integers. Further fix a normalised Hecke eigenform $f \in \mathrm{S}_k(X_1(N), K)$.

Proposition 2.7: *The $\mathcal{O}_K$-modules $\mathrm{MS}_k(N, \mathcal{O}_K)^{\pm}[f]$ are free of rank 1.*

Proof: [Kit94, Prop. 3.3] □

Definition 2.8: Choose $\mathcal{O}_K$-bases $\eta_f^{\pm}$ of $\mathrm{MS}_k(N, \mathcal{O}_K)^{\pm}[f]$. Because

$$\mathrm{MS}_k(N, \mathcal{O}_K) \underset{\mathcal{O}_K}{\otimes} \mathbb{C} = \mathrm{MS}_k(N, \mathbb{C}),$$

there exist unique $\mathcal{E}_\infty(f, \eta_f^{\pm}) \in \mathbb{C}^\times$ such that

$$\xi_f^{\pm} = \mathcal{E}_\infty(f, \eta_f^{\pm})\eta_f^{\pm}$$

in the right hand side. They are called the *complex error terms attached to f*.

Remark 2.9: These error terms are often called “periods”. We want to reserve the word “period” for a number defined using some kind of comparison isomorphism coming from the theory of motives. We will later show that our error term is in fact (more or less) a period in this sense, but until then we will call it an error term. Also, the error terms are often denoted by Ω, but we want to reserve this symbol for periods.

We finally study refinements of modular symbols. For this we let K be a number field with ring of integers $\mathcal{O}$, $p \nmid N$ a prime and $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, \mathcal{O})$ an eigenform away from the level. Let α and β be the roots of its p-th Hecke polynomial and assume the they lie in $\mathcal{O}$.

Proposition 2.10: *There exists a canonical morphism $\mathrm{Ref}_\alpha \colon \mathrm{MS}_k(N, K) \longrightarrow \mathrm{MS}_k(Np, K)$ such that the following hold:*

(a) It induces isomorphisms

$$\mathrm{Ref}_\alpha \colon \mathrm{MS}_k(N, K)^{\pm}[f] \xrightarrow{\sim} \mathrm{MS}_k(Np, K)^{\pm}[f_\alpha].$$

If f is ordinary at a prime $\mathfrak{p} \mid p$ of K and α is the unit root of the p-th Hecke polynomial, then it induces

$$\mathrm{Ref}_\alpha \colon \mathrm{MS}_k(N, \mathcal{O})^{\pm}[f] \xrightarrow{\sim} \mathrm{MS}_k(Np, \mathcal{O})^{\pm}[f_\alpha].$$

(b) The diagram

$$\begin{tikzcd} \mathrm{MS}_k(N, K) \arrow[r, "\mathrm{Ref}_\alpha"] \arrow[d] & \mathrm{MS}_k(Np, K) \arrow[d] \\ {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \otimes_{\mathbb{Q}} K \arrow[r, "\mathrm{Ref}_\alpha"] & {}^{Np}_{k}\mathcal{W}_{\mathrm{B}} \otimes_{\mathbb{Q}} K \end{tikzcd}$$

commutes. Here the vertical arrows are the maps (2.1) and the bottom map is induced by the motivic refinement morphism from corollary II.7.7.

(c) Let f_α be the refinement of f at α. Then $\mathrm{Ref}_\alpha(\xi_f) = \xi_{f_\alpha}$.

Proof: Using the isomorphism from proposition 2.2 (a) we simply *define* Ref_α in the same way as the motivic refinement morphism from corollary II.7.7, i. e. as $\sigma^*_{Np,N} + p^{-k}\beta\theta^*_{Np,N}$ where $\sigma^*_{Np,N}$ resp. $\theta^*_{Np,N}$ denote the maps induced in cohomology by the morphism $\sigma_{Np,N}$ resp. $\theta_{Np,N}$ between modular curves from section II.7.2. Then (b) is trivial and (c) follows directly from corollary II.7.7 and lemma 2.6.

Alternatively we could also define the above morphism more directly as

$$\mathrm{Ref}_\alpha\colon \mathrm{MS}_k(N,K) \longrightarrow \mathrm{MS}_k(Np,K), \quad \mathrm{Ref}_\alpha = \mathrm{id} - \beta p^{1-k}\begin{bmatrix} p & \\ & 1 \end{bmatrix}.$$

That this gives the same map can either be proved directly by a similar reasoning as in the proof of proposition II.7.6, or one just uses the Eichler-Shimura isomorphism (theorem II.6.3) to see that both maps are equal after tensoring with $\mathbb{C}$. From lemma II.7.1 it then follows that we get a map as in (a), and it is an isomorphism since it is so after tensoring with $\mathbb{C}$, which follows from (c). If f is ordinary and α is the unit root, then $v_{\mathfrak{p}}(\beta) = k-1$, so in this case we get a map over $\mathcal{O}_K$. $\square$

Let f be ordinary at $\mathfrak{p}$. If we now choose $\eta_f^\pm \in \mathrm{MS}_k(N,\mathcal{O})^\pm[f]$ as above, we may take $\eta^\pm_{f_\alpha} := \mathrm{Ref}_\alpha(\eta_f^\pm)$ as a basis of $\mathrm{MS}_k(Np,\mathcal{O})^\pm[f_\alpha]$. The following is then clear.

Corollary 2.11: *If f is ordinary at $\mathfrak{p}$ and α is the unit root, then $\mathcal{E}_\infty(f,\eta_f^\pm) = \mathcal{E}_\infty(f_\alpha,\eta^\pm_{f_\alpha})$.*

3. Hida families

In this section, we recall the most important facts about Hida families and fix the notation for the next sections. During this we prove some statements that are well-known but whose proofs are not easy to find in the literature.

We first fix the notation that should be in place throughout the whole rest of this chapter. Let $\mathcal{O}$ be the ring of integers in a finite extension L of $\mathbb{Q}_p$ and write $\Gamma^{\mathrm{wt}} := 1 + p\mathbb{Z}_p$, $\Gamma^{\mathrm{wt}}_r := 1 + p^r\mathbb{Z}_p \subseteq \Gamma^{\mathrm{wt}}$ and $\Lambda^{\mathrm{wt}} := \mathcal{O}[\![\Gamma^{\mathrm{wt}}]\!]$. We further fix an integer N prime to p such that $Np \geq 4$ and regard Γ^{wt} as a subgroup as well as a quotient of $\mathbb{Z}^\times_{p,N}$. On finite levels, we regard $\Gamma^{\mathrm{wt}}/\Gamma^{\mathrm{wt}}_r$ as a subgroup as well as a quotient of $(\mathbb{Z}/Np^r)^\times$. The integers Np^r for varying r will now play the role of the level, so the N here is a different N than in the previous chapter. This N is often called the *tame level.*

If ε is a character of Γ^{wt}, we regard it also as a character of $\mathbb{Z}^\times_{p,N}$ via the projection to Γ^{wt}. If ε is a character of Γ^{wt} of finite order, factoring over $\Gamma^{\mathrm{wt}}/\Gamma^{\mathrm{wt}}_r$, then note that the notation $\mathrm{S}_k(X_1(Np^r), \Gamma^{\mathrm{wt}}/\Gamma^{\mathrm{wt}}_r, \varepsilon, \mathcal{O})$ is well-defined (see definition II.4.18).

The *weight space* is defined as $\mathcal{X}^{\mathrm{wt}} := \operatorname{Spec}\Lambda^{\mathrm{wt}}$. If $\mathcal{K}$ is a finite extension of the fraction field of Λ^{wt} and $\mathcal{I}$ is the integral closure of Λ^{wt} in $\mathcal{K}$, then we write $\mathcal{X}^{\mathrm{wt}}_{\mathcal{I}} := \operatorname{Spec}\mathcal{I}$.

Write κ_{wt} for the canonical embedding

$$\kappa_{\mathrm{wt}}\colon \Gamma^{\mathrm{wt}} \hookrightarrow \mathcal{O}^\times.$$

For each $k \in \mathbb{Z}$ and each $\mathcal{O}^\times$-valued character ε of Γ^{wt} of finite order, we let

$$\phi_{k,\varepsilon}\colon \Lambda^{\mathrm{wt}} \longrightarrow \mathcal{O}$$

be the $\mathcal{O}$-algebra morphism induced by

$$\Gamma^{\mathrm{wt}} \longrightarrow \mathcal{O}^\times, \quad \gamma \longmapsto \varepsilon(\gamma)\kappa_{\mathrm{wt}}(\gamma)^k$$

and we write $P_{k,\varepsilon}$ for its kernel, which is then an element of $\mathcal{X}^{\mathrm{wt}}$.

We put

$$\mathcal{X}^{\mathrm{arith}} := \{P_{k,\varepsilon} : k \geq 2,\ \varepsilon \colon \Gamma^{\mathrm{wt}} \longrightarrow \mathcal{O}^{\times} \text{ character of finite order}\} \subseteq \mathcal{X}^{\mathrm{wt}}$$

and call its elements the *arithmetic points in the weight space.* If we have fixed $\mathcal{I}$ as above, let $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$ be the preimage of $\mathcal{X}^{\mathrm{arith}}$ under the natural map $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}} \longrightarrow \mathcal{X}^{\mathrm{wt}}$. We say that $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{wt}}$ is of type (k, ε, r) if $P \cap \Lambda^{\mathrm{wt}} = P_{k,\varepsilon}$ with k and ε as above and $\ker \varepsilon = \Gamma_r^{\mathrm{wt}}$. Finally define the ideals

$$\omega_{k,r} := \prod_{\varepsilon} P_{k,\varepsilon}$$

for each fixed $k \in \mathbb{Z}$, $r \geq 0$, where ε runs through all $\mathcal{O}^{\times}$-valued characters of $\Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}$.

Lemma 3.1: *(a) The map $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}} \longrightarrow \mathcal{X}^{\mathrm{wt}}$ is surjective and open.*

(b) Any infinite subset $U \subseteq \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$ is Zariski dense in $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}}$.

Proof: (a) The morphism $\operatorname{Spec} \mathcal{I} \longrightarrow \operatorname{Spec} \Lambda^{\mathrm{wt}}$ is obviously finite. Hence it is proper, so its image is closed, and since the image contains the generic point, the morphism is surjective. Moreover the morphism is finitely presented and flat, hence open.

(b) By (a) it suffices to prove this for $\mathcal{I} = \Lambda^{\mathrm{wt}}$. It is enough to prove that for any $f \in \Lambda^{\mathrm{wt}}$, $f \neq 0$, the basic open set $D(f) = \{\mathfrak{p} \in \operatorname{Spec} \Lambda^{\mathrm{wt}} : f \notin \mathfrak{p}\}$ contains some $P_{k,\varepsilon} \in U$, which is the case precisely when $\phi_{k,\varepsilon}(f) \neq 0$.

Fix a topological generator γ of Γ^{wt}. We use the identification $I \colon \mathcal{O}[\![T]\!] \xrightarrow{\sim} \Lambda^{\mathrm{wt}}$ of the Iwasawa algebra with the power series ring given by $T \longmapsto \gamma - 1$, see [NSW13, Prop. 5.3.5]. The composite $\phi_{k,\varepsilon} \circ I$ is given by

$$f \longmapsto f(\varepsilon\kappa_{\mathrm{wt}}^{k}(\gamma) - 1)$$

and since a non-zero power series can have only finitely many zeros by the Weierstraß preparation theorem [NSW13, (5.3.4)], the claim follows. ☐

We will often look at $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ or similar objects. The elements are by definition certain $\mathcal{O}$-algebra morphisms $\mathcal{I} \longrightarrow \mathcal{O}$ and if we identify them with their kernels, we can view $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ as a subset of $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$ as usual. Sometimes however it is important to distinguish the morphisms and the kernels. We will typically denote morphisms as ϕ and prime ideals as P, so for example if we write $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ we mean the kernel and not the morphism. If we want to make clear which morphism belongs to which prime ideal, we will use notations like ϕ_P and P_ϕ.

3.1. Hida's big Hecke algebra and control theory

Recall lemma 1.1.55 (a), which allows us to identify the abstract standard Hecke algebras $\mathcal{H}(Np^r)$ for $r \geq 1$ with $\mathcal{H}(Np)$. In the following we will often use this lemma without further comment.

Definition 3.2: (a) For a power series $f = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{O}[\![q]\!]$, define its norm as

$$\|f\| := \sup_n |a_n| .$$

(b) For (at the moment fixed) $r, i \in \mathbb{N}$, we put

$$\mathrm{M}^i(Np^r, \mathcal{O}) := \bigoplus_{k=1}^{i} \mathrm{M}_k(X_1(Np^r)^{\mathrm{arith}}, \mathcal{O}).$$

By q-expansions, we can view each $\mathrm{M}_k(X_1(Np^r)^{\mathrm{arith}}, \mathcal{O})$ and $\mathrm{M}^i(Np^r, \mathcal{O})$ as $\mathcal{O}$-submodules of $\mathcal{O}[\![q]\!]$ (see corollary II.4.17 and note that $\mathcal{O}$ is flat over $\mathbb{Z}$). We endow them with the norm induced thereby. Put

$$\mathcal{M}(Np^\infty, \mathcal{O}) := \varinjlim_{i,r} \mathrm{M}^i(Np^r, \mathcal{O}),$$

where the limit is taken along the natural inclusions. Since these inclusions are compatible with the norm, this colimit still carries a norm. Define the $\mathcal{O}$-Banach module $\overline{\mathcal{M}}(Np^\infty, \mathcal{O})$ to be the completion of $\mathcal{M}(Np^\infty, \mathcal{O})$ with respect to this norm.

(c) For again fixed i, r we write

$$\mathbf{T}^i(Np^r, \mathcal{O}) := \mathbf{T}_{\mathcal{O}}^{(\Delta_1(Np^r), \Gamma_1(Np^r))}(\mathrm{M}^i(Np^r, \mathcal{O}))$$

for the Hecke eigenalgebra of $\mathrm{M}^i(Np^r, \mathcal{O})$ (see definition I.1.64) and

$$\mathbf{T}^i(Np^r, \mathcal{O})^\iota := \mathbf{T}_{\mathcal{O}}^{(\Delta_1(Np^r)^\iota, \Gamma_1(Np^r))}(\mathrm{M}^i(Np^r, \mathcal{O}))$$

for the corresponding adjoint Hecke eigenalgebra. As explained in remark I.1.58, the natural inclusions between the $\mathrm{M}^i(Np^r, \mathcal{O})$ are Hecke equivariant, so we have natural restriction maps between these eigenalgebras and can form the limit

$$\mathbf{T}(Np^\infty, \mathcal{O}) := \varprojlim_{i,r} \mathbf{T}^i(Np^r, \mathcal{O}),$$

which we call the *big Hida Hecke algebra of level Np^∞ for modular forms.* The *adjoint big Hida Hecke algebra of level Np^∞ for modular forms* is defined as

$$\mathbf{T}^\iota(Np^\infty, \mathcal{O}) := \varprojlim_{i,r} \mathbf{T}^i(Np^r, \mathcal{O})^\iota.$$

By continuity, they both act on $\overline{\mathcal{M}}(Np^\infty, \mathcal{O})$.

(d) One can make analogous definitions with cusp forms instead of modular forms. We denote the resulting objects by $\mathcal{S}(Np^\infty, \mathcal{O})$, $\overline{\mathcal{S}}(Np^\infty, \mathcal{O})$, $\mathbf{t}(Np^\infty, \mathcal{O})$ and $\mathbf{t}^\iota(Np^\infty, \mathcal{O})$.

The perfect pairings from theorem II.4.28 induce a perfect pairing

$$\langle \cdot, \cdot \rangle : \mathbf{t}(Np^\infty, \mathcal{O}) \times \overline{\mathcal{S}}(Np^\infty, \mathcal{O}) \longrightarrow \mathcal{O} \tag{3.1}$$

of $\mathcal{O}$-Banach modules, so one can identify $\mathbf{t}(Np^\infty, \mathcal{O})$ with the $\mathcal{O}$-Banach dual of $\overline{\mathcal{S}}(Np^\infty, \mathcal{O})$ and vice versa, see [Hid88, Thm. 1.3]. A similar statement also holds for modular forms instead of cusp forms, see [Oht99, Rem. 2.5.5], but we will not need it.

The $\mathcal{O}$-algebras $\mathbf{T}(Np^\infty, \mathcal{O})$ and $\mathbf{t}(Np^\infty, \mathcal{O})$ are canonically algebras over $\mathcal{O}[\![\mathbb{Z}_{p,N}^\times]\!]$. We explain this for $\mathbf{T}(Np^\infty, \mathcal{O})$, the case $\mathbf{t}(Np^\infty, \mathcal{O})$ works analogously. For $\ell \in \mathbb{Z}$ coprime to Np, we let ℓ act on $\mathrm{M}^i(Np^r, \mathcal{O})$ as the endomorphism $\ell^2 S_\ell$, which is an element of $\mathbf{T}^i(Np^r, \mathcal{O})$ and is easily seen to be invertible (it is essentially a diamond operator). Since such ℓ are dense in $\mathbb{Z}_{p,N}^\times$, this induces a morphism $\mathbb{Z}_{p,N}^\times \longrightarrow \mathbf{T}^i(Np^r, \mathcal{O})^\times$. It is clear that these morphisms are compatible with the transition maps used to form the limit, so this induces further morphisms

$$\mathbb{Z}_{p,N}^\times \longrightarrow \mathbf{T}(Np^\infty, \mathcal{O})^\times \tag{3.2}$$

and thus

$$\mathcal{O}[\![\mathbb{Z}_{p,N}^\times]\!] \longrightarrow \mathbf{T}(Np^\infty, \mathcal{O}),$$

which makes $\mathbf{T}(Np^\infty, \mathcal{O})$ an algebra over $\mathcal{O}[\![\mathbb{Z}_{p,N}^\times]\!]$. In particular, $\mathbf{T}(Np^\infty, \mathcal{O})$ and $\mathbf{t}(Np^\infty, \mathcal{O})$ are Λ^{wt}-algebras.

Remark 3.3: The isomorphism $\mathcal{H}_+(Np) \xrightarrow{\sim} \mathcal{H}_+(Np)^\iota$ from lemma I.1.35 (b) induces isomorphisms $\mathbf{T}(Np^\infty, \mathcal{O}) \xrightarrow{\sim} \mathbf{T}^\iota(Np^\infty, \mathcal{O})$ and $\mathbf{t}(Np^\infty, \mathcal{O}) \xrightarrow{\sim} \mathbf{t}^\iota(Np^\infty, \mathcal{O})$. We regard each of the adjoint Hecke algebras as Λ^{wt}-algebras via the map from Λ^{wt} to the non-adjoint Hecke algebra followed by the respective isomorphism, such that these isomorphisms become isomorphisms of Λ^{wt}-algebras.

Definition 3.4: The *ordinary projection* $e \in \mathbf{t}(Np^\infty, \mathcal{O})$ is defined as $e = (e_{i,r})_{i,r}$ with

$$e_{i,r} := \varprojlim_{n\to\infty} T_p^{n!} \in \mathbf{t}^i(Np^r, \mathcal{O}).$$

The *adjoint ordinary projection* $e^\iota \in \mathbf{t}^\iota(Np^\infty, \mathcal{O})$, also called *anti-ordinary projection*, is defined as $e^\iota = (e^\iota_{i,r})_{i,r}$ with

$$e^\iota_{i,r} := \varprojlim_{n\to\infty} (T_p^\iota)^{n!} \in \mathbf{t}^i(Np^r, \mathcal{O})^\iota.$$

Both are well-defined and idempotent by [HidLFE, §7.2, Lem. 1]. If M is a module over $\mathbf{t}(Np^\infty, \mathcal{O})$, we write $M^{\mathrm{ord}} := eM$. We write $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ instead of $\mathbf{t}(Np^\infty, \mathcal{O})^{\mathrm{ord}}$, and similarly for other modules. Further, if M is a module over $\mathbf{t}^\iota(Np^\infty, \mathcal{O})$, we write $M^{\iota\text{-ord}} := e^\iota M$, and again similarly for other modules.

Note that the isomorphisms from remark 3.3 induce isomorphisms $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \xrightarrow{\sim} \mathbf{T}^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$ and $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \xrightarrow{\sim} \mathbf{t}^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$.

An important property of Hida's ordinary Hecke algebras is that one could also take the limit only over either just the weight or just the level:

Proposition 3.5 (Hida, Ohta)**:** *If we define*

$$\mathbf{T}_k(Np^\infty, \mathcal{O}) = \varprojlim_r \mathbf{t}_k(Np^r, \mathcal{O}), \quad \mathbf{T}(Np^r, \mathcal{O}) = \varprojlim_j \mathbf{t}^j(Np^r, \mathcal{O}),$$
$$\mathbf{t}_k(Np^\infty, \mathcal{O}) = \varprojlim_r \mathbf{t}_k(Np^r, \mathcal{O}), \quad \mathbf{t}(Np^r, \mathcal{O}) = \varprojlim_j \mathbf{t}^j(Np^r, \mathcal{O}),$$

then there are canonical isomorphisms

$$\mathbf{T}^{\mathrm{ord}}(Np^r, \mathcal{O}) \cong \mathbf{T}_k^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \cong \mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}),$$
$$\mathbf{t}^{\mathrm{ord}}(Np^r, \mathcal{O}) \cong \mathbf{t}_k^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \cong \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$$

for all $k \geq 2$ and $r \geq 1$, with the ordinary idempotents e defined appropriately in each of the cases, and analogously for the anti-ordinary parts.

Proof: See [Hid86a, Thm. 1.1] for $\mathbf{t}$ and [Oht99, Thm. 1.5.7 (i)] for $\mathbf{T}$. □

Proposition 3.6 (Hida, Ohta)**:** *The ordinary Hecke algebras* $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ *and* $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ *are free* Λ^{wt}*-modules of finite rank.*

Proof: See [Hid86b, Thm. 3.1] for $\mathbf{t}$ and [Oht99, Thm. 1.5.7 (ii)] for $\mathbf{T}$. □

The following is Hida's control theorem, which lies at the heart of the whole theory.

Theorem 3.7 (Hida)**:** *Let* $k \geq 2$ *and* $\varepsilon\colon \Gamma^{\mathrm{wt}} \longrightarrow \mathcal{O}^\times$ *be a character of finite order. Then there are canonical isomorphisms of* $\mathcal{O}$*-algebras*

$$\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\Lambda^{\mathrm{wt}} \middle/ P_{k,\varepsilon}\right) \cong \mathbf{t}_k^{\mathrm{ord}}(Np^r, \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}, \varepsilon, \mathcal{O}),$$
$$\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\Lambda^{\mathrm{wt}} \middle/ \omega_{k,r}\right) \cong \mathbf{t}_k^{\mathrm{ord}}(Np^r, \mathcal{O}).$$

Analogous statements hold for $\mathbf{T}$ *instead of* $\mathbf{t}$.

Proof: The first statement (for $\mathbf{t}$) is [Hid86a, Thm. 1.2]. There the Hecke algebra

$$\mathbf{t}_k^{\mathrm{ord}}(Np^r, \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}, \varepsilon, \mathcal{O})$$

is defined using classical cusp forms for the congruence subgroup $\Phi_r := \Gamma_1(Np) \cap \Gamma_0(p^r) \subseteq \mathrm{SL}_2(\mathbb{Z})$. The quotient $\Phi_r/\Gamma_1(Np^r)$ is isomorphic to $\Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}$, the isomorphism being induced by

$$\Phi_r \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \in 1 + Np\mathbb{Z} \subseteq 1 + p\mathbb{Z}_p = \Gamma^{\mathrm{wt}} \longrightarrow\!\!\!\!\!\rightarrow \Gamma^{\mathrm{wt}} \middle/ \Gamma_r^{\mathrm{wt}} .$$

Therefore classical cusp forms with coefficients in $\mathcal{O}$ for the congruence subgroup Φ_r with nebentype ε are isomorphic to $\mathrm{S}_k(X_1(Np^r), \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}, \varepsilon, \mathcal{O})$ and the statement in [Hid86a, Thm. 1.2] is equivalent to our statement.

The second statement can easily be derived from the first one, see [Hid86a, p. 553]. For the analogous statements for $\mathbf{T}$ instead of $\mathbf{t}$ see [Oht99, Thm. 1.5.7 (iii)]. □

3.2. Hida families and coefficient rings of the big Hecke algebra

Let $\mathcal{Q}$ be the quotient field of Λ^{wt}, and fix an algebraic closure $\overline{\mathcal{Q}}$ of it.

The meaning of the notion "Hida family" varies in the literature, as depending on what one wants to do with them certain definitions may be more convenient than others. For us it will mean the following.

Definition 3.8: A *Hida family* is a morphism of Λ^{wt}-algebras

$$F\colon \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \overline{\mathcal{Q}}.$$

It will become clear in a moment why such a morphism is called Hida family. Note that it is clear that Hida families exist.

First observe that since $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ is free of finite rank by proposition 3.6, the image of a Hida family generates a finite field extension of $\mathcal{Q}$, say $\mathcal{K}$, and the image even lies in the integral closure of Λ^{wt} inside $\mathcal{K}$, which we call $\mathcal{I}$. Moreover note that since the kernel of F contains a minimal prime ideal, there are only finitely many $\mathcal{I}$ that can occur in this way as long as N and $\mathcal{O}$ are fixed. By [Hid88, Lem. 3.1] each such $\mathcal{I}$ is free of finite rank over Λ^{wt}.

Definition 3.9: If F is a Hida family, then we call the ring $\mathcal{I}$ from above the *coefficient ring* of F. We call the finitely many $\mathcal{I}$ that can occur the *coefficient rings* of $\mathbf{t}^{\mathrm{ord}}(Np^{\infty}, \mathcal{O})$.

The above definition of coefficient rings and Hida families is not totally standard in the literature. Note that in our definition a Hida family will in general not surject onto its coefficient ring (because the image need not be integrally closed). Sometimes Hida families are defined as irreducible components of $\operatorname{Spec} \mathbf{t}^{\mathrm{ord}}(Np^{\infty}, \mathcal{O})$, whose underlying rings are then used as coefficient rings. For us it will be more convenient to work with integrally closed coefficient rings, see remark 3.19 below.

Before we continue, we study some ring-theoretic properties of the coefficient rings. First note that our situation is just the one described in section 1.2.5 (the "algebraic" case there), so we have all the properties of $\mathcal{I}$ stated there, for example we know that $\mathcal{I}$ is a local ring.

Lemma 3.10: *By possibly enlarging L (and thus $\mathcal{O}$), one can assume that $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ is Zariski dense in $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}}(\overline{\mathbb{Q}}_p)$ (which we both view as subsets of $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}} = \operatorname{Spec} \mathcal{I}$).*

Proof: By lemma 3.1 (b) $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$ is Zariski dense in $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}}$, so in particular $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\overline{\mathbb{Q}}_p)$ is dense in $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}}(\overline{\mathbb{Q}}_p)$. It remains to see that already $\mathcal{O}$-valued points in $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$ are dense (after possibly enlarging $\mathcal{O}$). For this it suffices to see that the field which is generated over L by the images of all $f \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\overline{\mathbb{Q}}_p)$ (which are morphisms $f\colon \mathcal{I} \longrightarrow \overline{\mathbb{Q}}_p$) is finite over L. Then we can replace L by this field and the claim follows.

So take a morphism $f\colon \mathcal{I} \longrightarrow \overline{\mathbb{Q}}_p$ which lies over a morphism $g\colon \mathcal{O}[\![T]\!] \longrightarrow \mathcal{O}$ and let $\mathcal{O}'$ be its image. We look at the pushout

$$\begin{array}{ccc} \mathcal{O}[\![T]\!] & \hookrightarrow & \mathcal{I} \\ {\scriptstyle g}\downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O} \otimes_{\mathcal{O}[\![T]\!]} \mathcal{I}. \end{array}$$

By its universal property, the morphism f factors through $\mathcal{O} \otimes_{\mathcal{O}[\![T]\!]} \mathcal{I}$ and is surjective from there onto $\mathcal{O}'$. Hence $\mathcal{O}'$ is a quotient of $\mathcal{O} \otimes_{\mathcal{O}[\![T]\!]} \mathcal{I}$. Since $\mathcal{O} \otimes_{\mathcal{O}[\![T]\!]} \mathcal{I}$ is a finite rank $\mathcal{O}$-module by base change and its rank is independent of f, the rank of $\mathcal{O}'$ as an $\mathcal{O}$-module is bounded independently of f. Since there are only finitely many extensions of L of bounded degree, we are done. ☐

From now on we assume that we have the density from above. For later reference, let us summarise the notations and assumptions we are now using.

Situation 3.11: We have fixed a finite extension L of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, an integer N prime to p such that $Np \geq 4$ and a coefficient ring $\mathcal{I}$ of $\mathbf{t}^{\mathrm{ord}}(Np^{\infty}, \mathcal{O})$. We assume L and $\mathcal{O}$ large enough such that $\Sigma := \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ is Zariski dense in $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}}(\overline{\mathbb{Q}}_p)$. Further we assume that L is the maximal subfield inside $\mathcal{K}$ which is algebraic over $\mathbb{Q}_p$.

Some of the statements that follow are valid without these assumptions, but as we will need them in the end anyway, we just assume them throughout for simplicity. If we consider again the situation from section 1.2.5 and use $\Sigma = \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ as a set of specialisations, then all the rings denoted $\mathcal{O}_{\phi}$ there are equal to $\mathcal{O}$. Therefore we have the following.

Remark 3.12: For any $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ there is a commutative diagram

$$\begin{tikzcd} \Lambda^{\mathrm{wt}} \arrow[r] \arrow[d, hook] & \Lambda^{\mathrm{wt}}/P \cap \Lambda^{\mathrm{wt}} \arrow[d, "\sim"] \\ \mathcal{I} \arrow[r] & \mathcal{I}/P \end{tikzcd}$$

with an isomorphism on the right. Hence for any Λ^{wt}-module M we have canonically

$$M \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\mathcal{I}\middle/P\right) \cong \left(M \underset{\Lambda^{\mathrm{wt}}}{\otimes} \mathcal{I}\right) \underset{\mathcal{I}}{\otimes} \left(\mathcal{I}\middle/P\right) \cong M \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\mathcal{I} \underset{\mathcal{I}}{\otimes} \left(\mathcal{I}\middle/P\right)\right) \cong M \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\Lambda^{\mathrm{wt}}\middle/P \cap \Lambda^{\mathrm{wt}}\right).$$

Definition 3.13: Let $F\colon \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \mathcal{I}$ be a Hida family and let

$$\psi\colon \left(\mathbb{Z}\middle/Np\right)^\times \longrightarrow \mathcal{I}^\times$$

be the character obtained as the composition

$$\left(\mathbb{Z}\middle/Np\right)^\times \hookrightarrow \mathbb{Z}_{p,N}^\times \hookrightarrow \mathcal{O}[\![\mathbb{Z}_{p,N}]\!]^\times \longrightarrow \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})^\times \xrightarrow{F} \mathcal{I}^\times.$$

We call this ψ the *nebentype of F*.

Fix a Hida family $F\colon \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \mathcal{I}$ and $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r). If we reduce F modulo P we get a morphism $\mathbf{t}_k^{\mathrm{ord}}(Np^r, \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}, \varepsilon, \mathcal{O}) \longrightarrow \mathcal{O}$ by theorem 3.7 and remark 3.12. By the perfect pairing from theorem II.4.28, this morphism corresponds uniquely to a cusp form $F_P \in \mathrm{S}_k(X_1(Np^r), \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}, \varepsilon, \mathcal{O})$. If ψ is the nebentype of F, then using the definition of the Λ^{wt}-algebra structure on $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ one can even show that in fact $F_P \in \mathrm{S}_k(X_1(Np^r), \varepsilon\psi\omega^{-k}, \mathcal{O})$, where ω is the Teichmüller character. Hence F gives rise to a whole family of cusp forms parametrised by the points in $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$, which justifies the name "Hida family". What's more, it is clear from theorem 3.7 that any cusp form lives in some Hida family.

When we view the elements of $\mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ as morphisms instead of ideals, we shall also write F_ϕ instead of F_P for $\phi = \phi_P$.

Definition 3.14: Let $F\colon \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \overline{\mathcal{Q}}$ be a Hida family. Then we call F *new* if there does not exist a proper divisor $M \mid N$ and a Hida family $G\colon \mathbf{t}^{\mathrm{ord}}(Mp^\infty, \mathcal{O}) \longrightarrow \overline{\mathcal{Q}}$ such that $F(T_\ell) = G(T_\ell)$ for almost all primes ℓ.[3]

Note the similarity of this definition to the definition of a newform (definition II.4.31).

Theorem 3.15 (Hida)**:** *Let $F\colon \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \mathcal{I}$ be a Hida family of nebentype ψ. Then the following are equivalent:*

(i) F is new.

(ii) For some $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$, F_P is new.

[3] Instead of "new", it is also common to call such forms "primitive". By [Hid87, Thm. 2.3] there is always a unique smallest N for each Hida family F such that F is primitive for this N, which is then called the *conductor* of F.

(iii) For infinitely many $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$, F_P is new.

(iv) F_P is new for all $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r) with $r > 1$.

(v) F_P is new for all $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r) such that the p-part of $\varepsilon\psi\omega^{-k}$ is nontrivial.

Now assume that the above equivalent statements hold and $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r) is such that the p-part of $\varepsilon\psi\omega^{-k}$ is trivial (in particular $r = 1$). We can then view $\varepsilon\psi\omega^{-k}$ as a character of $(\mathbb{Z}/N)^\times$. In this situation $F_P \in \mathrm{S}_k(X_1(Np), \varepsilon\psi\omega^{-k})$ can either be new (in which case $k = 2$), or F_P is the unique ordinary refinement of an ordinary newform $F_P^0 \in \mathrm{S}_k(X_1(N), \varepsilon\psi\omega^{-k})$ (see remark II.7.4).

Proof: Clearly we have implications (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii), so it remains to see (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (v). For these implications see [Hid87, Thm. 2.4] and for the final statement see [Hid88, Thm. 4.1]. $\square$

Definition 3.16: Let $\mathcal{I}$ be a coefficient ring of $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ and fix a Hida family F which is new. Then by theorem 3.15, for almost all $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ the form F_P is new, and for the P such that F_P is not new, there exists a newform F_P^0 such that F_P is a refinement of F_P^0. Let us write F_P^{new} to mean either F_P if F_P itself is new, or F_P^0 if F_P is not new.

If F is a new Hida family, then from theorem 3.15 and the proof of lemma 3.10, it is clear that by possibly enlarging $\mathcal{O}$ we can assume that the points $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ such that F_P is a newform are Zariski dense in $\mathcal{X}_{\mathcal{I}}^{\mathrm{wt}}$.

Definition 3.17: Let $\mathcal{I}$ be a coefficient ring of $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$. We define the module of *$\mathcal{I}$-adic cusp forms of level Np^∞* as

$$\mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) := \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{I})$$

(here we mean morphisms of Λ^{wt}-*modules*). Let the Hecke algebra $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ act on this module by duality, i. e. $(TF)(X) = F(TX)$ for $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$, $T, X \in \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$.

So Hida families are special $\mathcal{I}$-adic cusp forms. Note that from the definition and proposition 3.6, it is clear that $\mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) = \mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}$ as $\mathcal{I}$-modules. Observe that this equality does not say anything meaningful about Hida families since we cannot detect Λ^{wt}-algebra homomorphisms on the right hand side.

By construction there is a perfect $\mathcal{I}$-bilinear pairing

$$\mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) \times (\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \underset{\Lambda^{\mathrm{wt}}}{\otimes} \mathcal{I}) \longrightarrow \mathcal{I}. \tag{3.3}$$

For a fixed $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$, we define F-eigenspace

$$\mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})[F] := \{G \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) : \forall T \in \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})\colon TG = F(T)G\}.$$

It is then clear that $\mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})[F]$ is free of rank 1 over $\mathcal{I}$ (this can be seen similarly as corollary II.4.27 (a)).

3.3. Big Galois representations attached to Hida families

We keep the setting described in situation 3.11.

Theorem 3.18 (Hida): *(a) Fix a Hida family $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$ which is new. Then there is a unique (up to isomorphism) free $\mathcal{I}$-module $\mathcal{T}$ of rank 2 and a continuous (in the sense of definition I.2.32 (b)) odd irreducible Galois representation unramified outside $Np\infty$*

$$\rho_F\colon \mathrm{G}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{I}}(\mathcal{T})$$

such that for each $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$, the reduction of ρ_F modulo P is equivalent to the Galois representation attached to F_P^{new}.

(b) There is a free rank 1 $\mathcal{I}$-direct summand $\mathcal{T}^0$ of $\mathcal{T}$ which is an unramified $\mathrm{G}_{\mathbb{Q}_p}$-subrepresentation.

Proof: Statement (a) is a variation of [Hid86a, Thm. 2.1]. Our statement can easily be obtained from the form stated there using lemma I.2.33. Statement (b) follows from [Gou90, Thm. 4]. More precisely: The representation from the theorem there is by uniqueness the same as ours, and the theorem states that it is ordinary in the sense of [Gou90, Def. 1]. It is easy to see that this definition of ordinariness implies that we have $\mathcal{T}^0$ as claimed. The restriction to $p \geq 7$ there can be removed by [Böc01, p. 991]. ☐

Remark 3.19: Recall that our ring $\mathcal{I}$ is integrally closed by definition. This convention is not standard in the literature. In fact many texts use as coefficient rings such $\mathcal{I}$ for which $\mathrm{Spec}\,\mathcal{I}$ is an irreducible component of $\mathrm{Spec}\,\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ (which need not be normal). In this case, the image of ρ_F does in general not lie in $\mathrm{GL}_2(\mathcal{I})$ but only in $\mathrm{GL}_2(\mathrm{Quot}(\mathcal{I}))$ (after choosing a basis), and one needs extra assumptions on $\mathcal{I}$ to have it in $\mathrm{GL}_2(\mathcal{I})$, such as $\mathcal{I}$ being a unique factorisation domain or the residual representation $\overline{\rho}_F$ being absolutely irreducible. See [Hid15, §9] for a discussion of these issues. We chose to take $\mathcal{I}$ always as integrally closed, which is maybe not so directly related to the geometry of $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$, but allows us to work with representations into $\mathrm{GL}_2(\mathcal{I})$, which seems more suitable for our purpose.

Corollary 3.20: *Let $\mathcal{T}$ be as above at let $\mathcal{M}(F_\phi^{\mathrm{new}})$ be the motive attached to the newform F_ϕ^{new} for $\phi \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$. Then for each such ϕ there is a map*

$$\mathcal{T}_\phi := \mathcal{T} \underset{\mathcal{I},\phi}{\otimes} \mathcal{O} \hookrightarrow \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathfrak{p}}$$

and the left side is a $\mathrm{G}_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice in the right side. Hence $\mathcal{T}$ is a p-adic family of motives in the sense of definition I.3.35 parametrised by the set of specialisations $\Sigma = \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$. The family of motives defined by $\mathcal{T} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1)$ satisfies the strong Dabrowski-Panchishkin condition with $(\mathcal{T} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1))^{\mathrm{DP}} = \mathcal{T}^0 \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1)$.

Proof: The first statement is obvious from the definitions. The last statement follows from theorem 3.18 (b). The Tate twist is just to make the motives in the family critical. ☐

Remark 3.21: In remark II.5.5 we said that when we work with the motives attached to modular forms it will do no harm to assume that the ring we are working over contains the roots of unity of order given by the level of the modular forms. It is important to note that this remains true even when we work with families. The reason for this is that for $\phi \in \Sigma$ to be of type (k, ε, r) means in particular that ε is a *primitive* character of $\Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}} \cong \mathbb{Z}/p^{r-1}$, so the existence of such a character implies that we have a primitive p^{r-1}-st root of unity in $\mathcal{O}$. Said differently, the choice of $\mathcal{O}$ bounds the possible r's that may appear.

4. Modular symbols for Hida families

We continue to use the notation introduced at the beginning of section 3 and consider the setting described in situation 3.11. In this section we introduce modular symbols for Hida families following [Kit94] and prove some important properties of them. This will enable us to define Kitagawa's p-adic error term.

4.1. $\mathcal{I}$-adic modular symbols

We introduce $\mathcal{I}$-adic modular symbols, which are the modular symbols pendant of $\mathcal{I}$-adic cusp forms. We proceed in several steps, which will be motivated afterwards.

Definition 4.1: We put

$$\mathcal{MS}_k(Np^\infty, \mathcal{O}) := \varinjlim_r \mathrm{MS}_k(Np^r, \mathcal{O}),$$

where the maps are induced from the canonical maps $Y_1(Np^s) \longrightarrow Y_1(Np^r)$ on modular curves for $s \geq r \geq 0$. We write $\overline{\mathcal{MS}}_k(Np^\infty, \mathcal{O})$ for the p-adic completion of $\mathcal{MS}_k(Np^\infty, \mathcal{O})$. We can define the same with $\mathcal{O}$ replaced by $\mathcal{O}/p^t$ for some $t \geq 0$. This will be used in section 4.2.

From the Hecke action on each of the modules $\mathrm{MS}_k(Np^r, \mathcal{O})$ and using remark 2.3, we get a $\mathbf{T}(Np^\infty, \mathcal{O})$-module structure on these modules. So in particular, we get a $\mathcal{O}[[\mathbb{Z}_{p,N}^\times]]$-module structure and a Λ^{wt}-module structure. Moreover, it is also clear that the transition maps used to form the limit are compatible with the action of $\mathfrak{d} \in \mathrm{GL}_2(\mathbb{Z})$ since $\mathfrak{d}$ describes the action of complex conjugation on the modular curves. Hence $\mathfrak{d}$ acts on $\overline{\mathcal{MS}}_k(Np^\infty, \mathcal{O})$ in a well-defined way.

Definition 4.2: The module of *universal p-adic modular symbols* is defined as

$$\mathcal{UM}(Np^\infty, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}_2(Np^\infty, \mathcal{O}), \mathcal{O}).$$

Here we mean the $\mathcal{O}$-Banach dual, i. e. continuous homomorphisms.

There is then an obvious perfect pairing

$$\overline{\mathcal{MS}}_2(Np^\infty, \mathcal{O}) \times \mathcal{UM}(Np^\infty, \mathcal{O}) \longrightarrow \mathcal{O}. \tag{4.1}$$

The Hecke action is the dual Hecke action coming from the action on $\overline{\mathcal{MS}}_2(Np^\infty, \mathcal{O})$, that is, $(T\alpha)(\xi) = \alpha(T\xi)$ for $\alpha \in \mathcal{UM}(Np^\infty, \mathcal{O})$, $\xi \in \overline{\mathcal{MS}}_2(Np^\infty, \mathcal{O})$ and $T \in \mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$. In particular, this makes $\mathcal{UM}(Np^\infty, \mathcal{O})$ a Λ^{wt}-module.

From this definition of the Hecke action, it is easy to see that

$$\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O}) \tag{4.2}$$

(this is an easy calculation using the fact that if R is a ring and $e \in R$ is idempotent, then $eR \cong R/(1-e)R$).

Proposition 4.3 (Kitagawa): *$\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ is free of finite rank over Λ^{wt}.*

Proof: [Kit94, Prop. 5.7] □

Definition 4.4: The $\mathcal{I}$-module of *$\mathcal{I}$-adic ordinary modular symbols* is defined as

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) := \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{I}).$$

The Hecke action is again the dual action of the action on $\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$.

From proposition 4.3 it is clear that $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) = \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}$.

To motivate these definitions, recall that philosophically modular symbols and modular forms are two incarnations of the same phenomenon, as suggested by the Eichler-Shimura isomorphisms (complex or p-adic). The definition of $\overline{\mathcal{MS}}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ parallels in some way the definition of $\overline{\mathcal{S}}(Np^\infty, \mathcal{O})$. By the perfect pairing (3.1), $\mathcal{I}$-adic cusp forms are $\mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{S}}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O}), \mathcal{I})$, while by the perfect pairing (4.1) $\mathcal{I}$-adic modular symbols are $\mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O}), \mathcal{I})$, so these definitions have some analogy. From this point of view, $\mathcal{UM}(Np^\infty, \mathcal{O})$ is "something like a Hecke algebra" (however, it has no algebra structure and this observation will not be important).

For an $\mathcal{I}$-algebra morphism $F\colon \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I} \longrightarrow \mathcal{I}$ (that is, an $\mathcal{I}$-adic eigenform), we denote the induced morphism $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I} \longrightarrow \mathcal{I}$ still by F, by abuse of notation. Then $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F]$ is well-defined. An important condition on this module we will need to impose later is the following.

Condition 4.5: $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F]$ is free of rank 1 over $\mathcal{I}$.

Remark 4.6: There are several conditions which are known to imply condition 4.5, among them the condition that $\mathcal{I}$ be factorial (which is satisfied for example for $\mathcal{I} = \Lambda^{\mathrm{wt}}$). We do not list the other conditions, see [Kit94, Lem. 5.11] for this.

4.2. Twists of modular symbols and comparison of different weights

This section contains a technical statement which is only important as an ingredient in a crucial step in the proof of the main result of the next section. Since its proof is not so well-documented in the literature, we expose it here in detail.

We use the abstract Hecke theory for the group $\Delta_0(Np^r)^\flat$. This is the only place in this work where we will need the more general situation where we have a module with an action of just $\Delta_0(Np^r)^\flat$ and not of the surrounding ι-stable semigroup $\Sigma = \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$, as explained in remark I.1.25.

Fix integers $r \geq t \geq 0$ and put $R := \mathcal{O}/p^t$. It is easy to see from the definition of $\Delta_0(Np^r)^\flat$ that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a \bmod p^t \in R$$

defines a character

$$\chi\colon \Delta_0(Np^r)^\flat \longrightarrow R^\times.$$

For $n \in \mathbb{Z}$ let $R(\chi^n)$ denote R seen as a free R-module of rank 1 with a left action of $\Delta_0(Np^r)^\flat$ via the character χ^n. Alternatively, we could of course define it as R with a right action of $(\Delta_0(Np^r)^\flat)^\iota$ defined by $\chi^n \circ \iota$. Any left action of $\Delta_0(Np^r)^\flat$ occurring in the following can also be seen as a right action of $(\Delta_0(Np^r)^\partial)^\iota$ in the same manner. We formulate everything using left actions.

On the R-module $\operatorname{Sym}^{k-2} R^2$ we also have an action of $\Delta_0(Np^r)^\partial$ from the left, by left multiplication. We look at the R-linear map

$$\operatorname{Sym}^{k-2} R^2 \longrightarrow R(\chi^{k-2}), \quad f \longmapsto f(1,0),$$

where we view f as a homogeneous polynomial in two variables X, Y of degree $k-2$, so this map is the projection onto the coefficient of the monomial X^{k-2}. It is then an easy calculation to check that this map is $\Delta_0(Np^r)^\partial$-equivariant. In particular it is $\Delta_1(Np^r)^\partial$-equivariant.

We apply the functor $\operatorname{MSymb}(\Gamma_1(Np^r), -)$ to this map, which is (in this case) a functor from R-modules with a left action of $\Delta_1(Np^r)$ to right $\mathcal{H}(Np^r)_R$-modules. Clearly, as an R-module $\operatorname{MSymb}(\Gamma_1(Np^r), R(\chi^{k-2}))$ coincides with $\mathrm{MS}_2(Np^r, R)$. We therefore denote it by $\mathrm{MS}_2(Np^r, R)\{\chi^{k-2}\}$; we use here curly brackets because this is not a usual twist, as we will study below. Thus we get an $\mathcal{H}(Np^r)_R$-linear map

$$\mathrm{MS}_k(Np^r, R) \longrightarrow \mathrm{MS}_2(Np^r, R)\{\chi^{k-2}\}. \tag{4.3}$$

Note that the action of the operators T_p on $\mathrm{MS}_2(Np^r, R)\chi^{k-2}$ is just the same as on $\mathrm{MS}_2(Np^r, R)$, since χ vanishes on the coset representatives from lemma 1.1.54. The action of the S_ℓ (or the diamond operators) is changed, and we will study the change below.

Similar objects and maps are studied in [Kit94, §5.2]. Using remark 1.1.25 (b) it is easy to see that they are essentially the same as introduced here, and we can cite the results proved there.

Proposition 4.7 (Kitagawa)**:** *After restricting to the ordinary part, the above map induces an isomorphism*

$$\mathrm{MS}_k^{\mathrm{ord}}(Np^r, R) \xrightarrow{\sim} \mathrm{MS}_2^{\mathrm{ord}}(Np^r, R)\{\chi^{k-2}\}.$$

Proof: [Kit94, Cor. 5.2] □

We now take the colimit for $r \to \infty$ (while t is still fixed). This gives us an isomorphism

$$\begin{aligned}\mathcal{MS}_k^{\mathrm{ord}}(Np^\infty, R) := \varinjlim_{r\in\mathbb{N}} \mathrm{MS}_k^{\mathrm{ord}}(Np^r, R) &\xrightarrow{\sim} \\ &\mathcal{MS}_2^{\mathrm{ord}}(Np^\infty, R)\{\chi^{k-2}\} := \varinjlim_{r\in\mathbb{N}} \mathrm{MS}_2^{\mathrm{ord}}(Np^r, R)\{\chi^{k-2}\}.\end{aligned}$$

Now taking the limit for $t \to \infty$, we get the following result comparing modular symbols of different weights, which is [Kit94, Thm. 5.3]. Here the object on the right side is defined to be the limit $\varprojlim_{t\in\mathbb{N}} \mathcal{MS}_2^{\mathrm{ord}}(Np^\infty, R)\{\chi^{k-2}\}$.

Corollary 4.8: *There is a canonical* $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$*-linear isomorphism*

$$\overline{\mathcal{MS}}_k^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \xrightarrow{\sim} \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})\{\chi^{k-2}\}$$

which is compatible with the action of ∂.

We now use the canonical morphism $\mathbb{Z}_{p,N}^\times \longrightarrow \mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})^\times$ defined by the diamond operators. More precisely, take $z \in \mathbb{Z}_{p,N}^\times$ and write $z \bmod Np^r$ for its image under the projection $\mathbb{Z}_{p,N}^\times \twoheadrightarrow (\mathbb{Z}/Np^r)^\times$. For fixed r and i, we associate to z the element in $\mathrm{End}_{\mathcal{O}}(\mathrm{M}^i(Np^r))$ given by the diamond operator $\langle z \bmod Np^r\rangle$. Then we know that this element lies in fact in $\mathbf{T}^i(Np^r)^\times$. Moreover for varying r and i these elements are compatible with the restriction maps between the Hecke algebras, such that we get a well-defined morphism $\mathbb{Z}_{p,N}^\times \longrightarrow \mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})^\times$. Beware that this map is *not* the same as the map (3.2) used to define the Λ^{wt}-algebra structure on $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$! For reasons that will become clear later, we want to use just the diamond operator action here.

In particular, this defines two actions of $\Gamma^{\mathrm{wt}} \subseteq \mathbb{Z}_{p,N}^\times$ on both modules $\overline{\mathcal{MS}_k}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ and $\overline{\mathcal{MS}_2}^{\mathrm{ord}}(Np^\infty, \mathcal{O})(\chi^{k-2})$: one via the morphism we just constructed and one via the morphism (3.2). We use here mainly the first action, which we call the action via diamond operators, while we call the other one the action through the Hecke algebra. By construction the isomorphism from corollary 4.8 is equivariant for the diamond operator action.

If we denote the diamond operator action by $(\gamma, \xi) \longmapsto \gamma\xi$ and the action of Γ^{wt} through the Hecke algebra by $(\gamma, \xi) \longmapsto \xi|\gamma$, then the two actions are related by

$$\xi|\gamma = \kappa_{\mathrm{wt}}^k(\gamma)\gamma\xi \tag{4.4}$$

for $\gamma \in \Gamma^{\mathrm{wt}}$ and $\xi \in \overline{\mathcal{MS}_k}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ or $\xi \in \overline{\mathcal{MS}_2}^{\mathrm{ord}}(Np^\infty, \mathcal{O})(\chi^{k-2})$. To see this, recall that on finite level by definition of the action through the Hecke algebra an $\ell \in \mathbb{Z}$, $(\ell, Np^r) = 1$, acts as $\ell^2 S_\ell$ and that $S_\ell = \ell^{k-2}\langle\ell\rangle$ since the matrix $\left(\begin{smallmatrix} \ell & \\ & \ell \end{smallmatrix}\right)$ acts as ℓ^{k-2}.

We will later be interested in eigenspaces for this action. So fix a character $\varepsilon\colon \Gamma^{\mathrm{wt}} \longrightarrow \mathcal{O}^\times$. Of course, by equivariance, we get a canonical isomorphism of $\mathcal{O}$-modules

$$\overline{\mathcal{MS}_k}^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon] \xrightarrow{\sim} \overline{\mathcal{MS}_2}^{\mathrm{ord}}(Np^\infty, \mathcal{O})\{\chi^{k-2}\}[\varepsilon].$$

We want to better understand the right hand side.

We therefore look again at the $\mathcal{H}(Np^r)_R$-module $\mathrm{MSymb}(\Gamma_1(Np^r), R(\chi^n))$ (for $r, t, n \in \mathbb{Z}$, $r \geq t \geq 0$ fixed and $R = \mathcal{O}/p^t$), which as an R-module is just $\mathrm{MSymb}(\Gamma_1(Np^r), R)$. Let

$$\varphi\colon \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \longrightarrow R$$

be a $\Gamma_1(Np^r)$-invariant homomorphism of abelian groups. Then we can view φ at the same time as an element of $\mathrm{MSymb}(\Gamma_1(Np^r), R)$ and $\mathrm{MSymb}(\Gamma_1(Np^r), R(\chi^n))$. For $\ell \in \mathbb{Z}$ with $(\ell, Np) = 1$ the action of the diamond operator $\langle\ell\rangle$ is given by any matrix $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ satisfying $\sigma_\ell \equiv \left(\begin{smallmatrix} \ell^{-1} & \\ & \ell \end{smallmatrix}\right) \bmod Np^r$. If we view φ as an element in $\mathrm{MSymb}(\Gamma_1(Np^r), R)$ we therefore have

$$\varphi\langle\ell\rangle(q) = \varphi(\sigma_\ell \bullet q)[\sigma_\ell] \quad (q \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))),$$

while if we view it as an element in $\mathrm{MSymb}(\Gamma_1(Np^r), R(\chi^n))$ we have

$$\varphi\langle\ell\rangle(q) = \chi^n(\sigma_\ell)\varphi(\sigma_\ell \bullet q)[\sigma_\ell] \quad (q \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))).$$

Since $\chi(\sigma_\ell) = \ell^{-1}$ this shows that the the diamond operators act on $\mathrm{MSymb}(\Gamma_1(Np^r), R(\chi^n))$ as on $\mathrm{MSymb}(\Gamma_1(Np^r), R)$ twisted by the character $\kappa_{Np^r}^{-n}$, where $\kappa_{Np^r}\colon (\mathbb{Z}/Np^r)^\times \longrightarrow R^\times$ is the canonical inclusion.

If we now take the limits and colimits as above and look just at the subgroup Γ^{wt} of $\mathbb{Z}_{p,N}^{\times}$, the limit of the characters κ_{Np^r} becomes $\kappa_{\mathrm{wt}}\colon \Gamma^{\mathrm{wt}} \longrightarrow \mathcal{O}^{\times}$. This shows that as $\mathcal{O}$-modules with Γ^{wt}-action, we have an isomorphism

$$\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})\{\chi^{k-2}\} \cong \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})(\kappa_{\mathrm{wt}}^{2-k})$$

where the $(\kappa_{\mathrm{wt}}^{2-k})$ now indeed means a twist of the Γ^{wt}-action, which is why we denote it using usual brackets. If we now take the ε-eigenspace of the right hand side, an easy calculation shows

$$\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})(\kappa_{\mathrm{wt}}^{2-k})[\varepsilon] = \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}].$$

We therefore have proved the following lemma.

Lemma 4.9: *The isomorphism from corollary 4.8 induces an isomorphism of $\mathcal{O}$-modules*

$$\overline{\mathcal{MS}}_k^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[\varepsilon] \xrightarrow{\sim} \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}].$$

This map commutes with the T_p operators and with the action of ə.

4.3. Control theory for $\mathcal{I}$-adic modular symbols

We keep the notations from the previous sections. Moreover we fix a $\phi \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k,ε,r) throughout the section and write $P = P_{\phi}$ for its kernel.

If M is some $\mathcal{O}$-module with an action of Γ^{wt} and ε is an $\mathcal{O}^{\times}$-valued character of Γ^{wt}, we write $M[\varepsilon]$ for the submodule where the action of Γ^{wt} is given by ε. When we apply this to a module like $\overline{\mathcal{MS}}^{\mathrm{ord}}(Np^{\infty},\mathcal{O})$, we will always mean the action of Γ^{wt} *via diamond operators, not through the Hecke algebra*, as in the previous section.

The purpose of this section is to prove the following control theorem for $\mathcal{I}$-adic modular symbols.

Theorem 4.10 (Kitagawa):

(a) *There is a canonical isomorphism of $\mathbf{T}^{\mathrm{ord}}(Np^{\infty},\mathcal{O})\otimes_{\Lambda^{\mathrm{wt}}}\mathcal{I}$-modules*

$$\mathbb{MS}^{\mathrm{ord}}(Np^{\infty},\mathcal{I}) \underset{\mathcal{I}}{\otimes} \left(\mathcal{I}\big/P\right) \xrightarrow{\sim} \mathrm{MS}_k^{\mathrm{ord}}(Np^r,\mathcal{O})[\varepsilon]$$

which is compatible with the action of ə.

(b) *There is a canonical isomorphism of $\mathcal{O}$-modules*

$$\mathbb{MS}^{\mathrm{ord}}(Np^{\infty},\Lambda^{\mathrm{wt}}) \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\Lambda^{\mathrm{wt}}\big/\omega_{k,r}\right) \xrightarrow{\sim} \mathrm{MS}_k^{\mathrm{ord}}(Np^r,\mathcal{O}).$$

(c) *Fix an $\mathcal{I}$-adic eigenform $F \in \mathbb{S}^{\mathrm{ord}}(Np^{\infty},\mathcal{I})$ and write F_P for the member at P of the Hida family associated to F. Assume further that condition 4.5 is satisfied. Then there is a canonical isomorphism of $\mathcal{O}$-modules*

$$\mathbb{MS}^{\mathrm{ord}}(Np^{\infty},\mathcal{I})^{\pm}[F] \underset{\mathcal{I}}{\otimes} \left(\mathcal{I}\big/P\right) \xrightarrow{\sim} \mathrm{MS}_k^{\mathrm{ord}}(Np^r,\mathcal{O})^{\pm}[F_P].$$

(d) *Let $\Xi \in \mathbb{MS}^{\mathrm{ord}}(Np^{\infty},\mathcal{I}) = \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{I})$, $u \in \mathcal{UM}^{\mathrm{ord}}(Np^{\infty},\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{O})$ and let Ξ_{ϕ} be the image of Ξ in the right hand side in statement (a). Then*

$$\phi(\Xi(u)) = u(\Xi_{\phi}).$$

The statement in theorem 4.10 (c) is claimed without proof in [Och05, Prop. 4.3].

The proof of the theorem will be divided into three lemmas, the combination of which immediately gives the result. More precisely, the results in theorem 4.10 (a), (c) will immediately follow from lemmas 4.11 to 4.13 below. The claim in theorem 4.10 (b) follows from the claim in theorem 4.10 (a) by definition of the ideals $\omega_{k,r}$. The claim in theorem 4.10 (d) will be clear by the construction of the maps.

In the following, we regard $\mathcal{O}$ as an $\mathcal{I}$-algebra via ϕ; in particular, we regard $\mathcal{O}$ as a Λ^{wt}-algebra. Moreover, we consider modules over $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}$ or similar rings having an additional action of $\mathrm{G}_{\mathbb{R}} \cong \mathrm{G}_\partial$ as $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}[\mathrm{G}_{\mathbb{R}}]$-modules

Lemma 4.11: *There are canonical isomorphisms of $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}[\mathrm{G}_{\mathbb{R}}]$-modules induced by ϕ*

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) \underset{\mathcal{I}}{\otimes} \left(\mathcal{I} \big/ P \right) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O})$$

and, if we assume that condition 4.5 is satisfied, also an isomorphism of $\mathcal{O}$-modules

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^{\pm}[F] \underset{\mathcal{I}}{\otimes} \left(\mathcal{I} \big/ P \right) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O})^{\pm}[F].$$

Proof: By remark 3.12 we have

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) \underset{\mathcal{I}}{\otimes} \left(\mathcal{I} \big/ P \right) \cong \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\Lambda^{\mathrm{wt}} \big/ P \cap \Lambda^{\mathrm{wt}} \right),$$

so for the first statement we can assume without loss of generality that $\mathcal{I} = \Lambda^{\mathrm{wt}}$ and hence $\phi = \phi_{k,\varepsilon}$.

We apply the functor $\mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), -)$ to the exact sequence of Λ^{wt}-modules

$$0 \longrightarrow P \longrightarrow \Lambda^{\mathrm{wt}} \xrightarrow{\ \phi\ } \mathcal{O} \longrightarrow 0.$$

Since $\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ is free of finite rank over Λ^{wt} by proposition 4.3, this functor is exact and we obtain an exact sequence of $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\mathrm{G}_{\mathbb{R}}]$-modules

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), P) \longrightarrow \mathbb{MS}^{\mathrm{ord}}&(Np^\infty, \Lambda^{\mathrm{wt}}) \\ &\longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O}) \longrightarrow 0. \end{aligned}$$

Since P is a prime ideal of Λ^{wt} of height 1, it is a principal ideal by [NSW13, Lem. 5.3.7]. This implies

$$\begin{aligned} \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), P) &= P \cdot \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \Lambda^{\mathrm{wt}}) \\ &\quad (= P\, \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})) \end{aligned}$$

and proves the first isomorphism.

Now we let $\mathcal{I}$ and ϕ and P be again as in the general case (because otherwise F may not exist). Similarly as before we get an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), P) \longrightarrow \mathbb{MS}^{\mathrm{ord}}&(Np^\infty, \mathcal{I}) \\ &\longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O}) \longrightarrow 0 \end{aligned}$$

in which all maps are compatible with the actions of $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}$ and $ə \in \mathrm{GL}_2(\mathbb{Z})$, so we can add $(-)^\pm[F]$ to each module in the sequence. Under condition 4.5 we see without requiring P to be principal that

$$\begin{aligned}\mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), P)^\pm[F] &= P\,\mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{I})^\pm[F] \\ &\quad (= P\,\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F]\ ). \qquad \square\end{aligned}$$

We put

$$\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[F_\phi] \coloneqq \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\phi \circ F],$$

by which we mean the submodule of $\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ where the action of the Hecke algebra $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}$ is given by the character $\phi \circ F\colon \mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I} \longrightarrow \mathcal{O}$.

Lemma 4.12: *The canonical biduality map*

$$\begin{aligned}\Phi\colon \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O}) &\longrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{UM}(Np^\infty, \mathcal{O}), \mathcal{O}) \\ &\qquad = \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}_2(Np^\infty, \mathcal{O})), \mathcal{O}) \\ \xi &\longmapsto [f \longmapsto f(\xi)]\end{aligned}$$

induces an isomorphism of $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}[\mathrm{G}_{\mathbb{R}}]$*-modules*

$$\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}] \xrightarrow{\ \sim\ } \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O}).$$

We further have an inclusion

$$\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[F_\phi] \subseteq \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}]$$

and the restriction of the above isomorphism gives an isomorphism of $\mathcal{O}$*-modules*

$$\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})^\pm[F_\phi] \xrightarrow{\ \sim\ } \mathrm{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O})^\pm[F].$$

Proof: Φ is injective because the pairing (4.1) is perfect. Further, from the equality (4.2) we see immediately that the image of Φ lies in $\mathrm{Hom}_{\mathcal{O}}(\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O})$.

We first show that $\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[F_\phi] \subseteq \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}]$. Take $\gamma \in \Gamma^{\mathrm{wt}}$ and $\xi \in \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[F_\phi]$, and let us assume without loss of generality that $\xi \in \mathrm{MS}_2^{\mathrm{ord}}(Np^s, \mathcal{O})$ for some $s \in \mathbb{N}$. By (4.4), we have

$$\gamma\xi = \kappa_{\mathrm{wt}}^{-2}(\gamma)\xi|\gamma.$$

Since ξ is in the F_ϕ-eigenspace, the action of the Hecke algebra on ξ is given by the character $\phi \circ F$ corresponding to F_ϕ. Hence the action of Λ^{wt} on ξ (through the Hecke algebra) is given by the character which is the composition

$$\Lambda^{\mathrm{wt}} \longrightarrow \mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \underset{\Lambda^{\mathrm{wt}}}{\otimes} \mathcal{I} \xrightarrow{\ F\ } \mathcal{I} \xrightarrow{\ \phi\ } \mathcal{O},$$

and since F is Λ^{wt}-linear this composition is just the restriction of ϕ to Λ^{wt}. The action of Γ^{wt} on ξ is hence described by the restriction of ϕ to Γ^{wt}, which by definition is just $\varepsilon\kappa_{\mathrm{wt}}^{k}$. Thus we have $\xi|\gamma = \varepsilon(\gamma)\kappa_{\mathrm{wt}}^{k}(\gamma)\xi$ whence $\gamma\xi = \varepsilon\kappa_{\mathrm{wt}}^{k-2}(\gamma)\xi$ as claimed.

Using this, we check that for $\xi \in \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}]$ the morphism $\Phi(\xi)$ is in fact Λ^{wt}-linear. So let $f \in \operatorname{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{O})$ and $\lambda \in \Lambda^{\mathrm{wt}}$. Without loss of generality, assume that $\lambda \in \Gamma^{\mathrm{wt}}$. We now compute

$$\begin{aligned}(\lambda f)(\xi) = f(\xi|\lambda) = f(\kappa_{\mathrm{wt}}^{2}(\lambda)\lambda\xi) = \kappa_{\mathrm{wt}}^{2}(\lambda)f(\varepsilon\kappa_{\mathrm{wt}}^{k-2}(\lambda)\xi)&\\ = \varepsilon\kappa_{\mathrm{wt}}^{k}(\lambda)f(\xi) = \phi(\lambda)f(\xi) = \lambda(f(\xi))&\end{aligned}$$

using the definition of the Λ^{wt}-module structure on $\mathcal{UM}(Np^{\infty},\mathcal{O})$ (by duality and through the Hecke algebra), (4.4), the fact that ξ is in the $\varepsilon\kappa_{\mathrm{wt}}^{k-2}$-eigenspace and finally the definition of the Λ^{wt}-module structure on $\mathcal{O}$. This proves the Λ^{wt}-linearity of $\Phi(\xi)$.

Now assume that $\xi \in \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})^{\pm}[F_\phi]$. Using again that the action of the Hecke algebra $\mathbf{T}^{\mathrm{ord}}(Np^{\infty},\mathcal{O})\otimes_{\Lambda^{\mathrm{wt}}}\mathcal{I}$ on ξ is given by the character $\phi\circ F$, that the $\mathcal{I}$-module structure on $\mathcal{O}$ is defined via ϕ and how the various Hecke actions are defined, we compute for $f \in \operatorname{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{O})$ and $T \in \mathbf{T}^{\mathrm{ord}}(Np^{\infty},\mathcal{O})\otimes_{\Lambda^{\mathrm{wt}}}\mathcal{I}$

$$\begin{aligned}(T\Phi(\xi))(f) = (\Phi(\xi))(Tf) = (Tf)(\xi) = f(T\xi)&\\ = f(\phi(F(T))\xi) = \phi(F(T))f(\xi) = F(T)f(\xi) = (F(T)\Phi(\xi))(f),&\end{aligned}$$

which shows that $\Phi(\xi)$ in fact lies in $\operatorname{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{O})[F]$.

So now we have well-defined injective maps

$$\begin{aligned}\Phi\colon \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}] &\hookrightarrow \operatorname{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{O}),\\ \Phi\colon \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[F_\phi] &\hookrightarrow \operatorname{Hom}_{\Lambda^{\mathrm{wt}}}(\mathcal{UM}^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{O})[F],\end{aligned}$$

and it remains to prove their surjectivity. If $\varphi \in \operatorname{Hom}_{\mathcal{O}}(\mathcal{UM}^{\mathrm{ord}}(Np^{\infty},\mathcal{O}),\mathcal{O})$, by (4.2) there is a $\xi \in \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},O)$ with $\varphi = \langle\xi,\cdot\rangle = \Phi(\xi)$. Now by totally analogous arguments and computations as above, one checks that if φ is Λ^{wt}-linear instead of just $\mathcal{O}$-linear, then in fact $\xi \in \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}]$ and if further φ is in the F-eigenspace, then $\xi \in \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[F_\phi]$. We omit the details.

Finally, the compatibility with the actions of $\mathbf{T}^{\mathrm{ord}}(Np^{\infty},\mathcal{O})\otimes_{\Lambda^{\mathrm{wt}}}\mathcal{I}$ and $\partial \in \mathrm{GL}_2(\mathbb{Z})$ is clear. $\square$

Lemma 4.13: *There are canonical isomorphisms of* $\mathbf{T}^{\mathrm{ord}}(Np^{\infty},\mathcal{O})\otimes_{\Lambda^{\mathrm{wt}}}\mathcal{I}[\mathrm{G}_{\mathbb{R}}]$*-modules*

$$\mathrm{MS}_k^{\mathrm{ord}}(Np^{r},\mathcal{O})[\varepsilon] \xrightarrow{\sim} \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}]$$

and of $\mathcal{O}$*-modules*

$$\mathrm{MS}_k^{\mathrm{ord}}(Np^{r},\mathcal{O})^{\pm}[F_\phi] \xrightarrow{\sim} \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^{\infty},\mathcal{O})^{\pm}[F_\phi].$$

Proof: We know that the map $\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O}) \longrightarrow \mathcal{MS}_k^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ is compatible with the action of $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}[\mathrm{G}_\mathbb{R}]$, so in particular with the action of Γ^{wt}, and therefore induces a map

$$\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})[\varepsilon] \longrightarrow \overline{\mathcal{MS}}_k^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon].$$

By [Kit94, Thm. 5.5 (1)][4] this map is in fact an isomorphism. The first isomorphism follows therefore from lemma 4.9. Now since P is of type (k, ε, r), we have

$$\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})[F_\phi] \subseteq \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})[\varepsilon],$$
$$\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[F_\phi] \subseteq \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}].$$

The second inclusion comes from lemma 4.12. Since the first isomorphism from the statement is compatible with the Hecke action as well as with the action of $\text{ə} \in \mathrm{GL}_2(\mathbb{Z})$, it induces the second isomorphism. □

4.4. The p-adic error term

We keep the notations from the previous sections. Fix an eigenform $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$ and assume that condition 4.5 is satisfied.

We are now ready to define Kitagawa's p-adic error term. For this we choose an $\mathcal{I}$-basis $\Xi^\pm$ of $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F]$ and $\mathcal{O}$-bases $\eta_\phi^\pm$ of $\mathrm{MS}_k(Np^r, \mathcal{O})^\pm[F_\phi]$ for each $\phi \in \mathcal{X}_\mathcal{I}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r). Let

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F] \underset{\Lambda^{\mathrm{wt}}}{\otimes} \left(\mathcal{I} \big/ \phi\right) \xrightarrow{\sim} \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})^\pm[F_\phi]$$

be the canonical isomorphism from theorem 4.10 (c). Both sides are free $\mathcal{O}$-modules of rank 1 by proposition 2.7. For $\phi \in \mathcal{X}_\mathcal{I}^{\mathrm{arith}}(\mathcal{O})$ write $\Xi_\phi^\pm$ for the image of $\Xi^\pm \in \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F]$ in $\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})^\pm[F_\phi]$ under the above isomorphism.

Definition 4.14: For each $\phi \in \mathcal{X}_\mathcal{I}^{\mathrm{arith}}(\mathcal{O})$, let $\mathcal{E}_\mathrm{p}(\Xi^\pm, \eta_\phi^\pm) \in \mathcal{O}$ be the unique element such that

$$\Xi_\phi^\pm = \mathcal{E}_\mathrm{p}(\Xi^\pm, \eta_\phi^\pm)\eta_\phi^\pm.$$

This element is called the *p-adic error term at ϕ*.

Proposition 4.15 (Kitagawa): *One has always $\mathcal{E}_\mathrm{p}(\Xi^\pm, \eta_\phi^\pm) \neq 0$.*

Proof: [Kit94, Prop. 5.12] □

4.5. The Galois action on modular symbols

From étale cohomology we get an action of $\mathrm{G}_\mathbb{Q}$ on modular symbols.

Proposition 4.16: *(a) There is a canonical isomorphism of $\mathcal{O}$-modules*

$$\mathrm{MS}_k(Np^r, \mathcal{O}) \cong \mathrm{H}^1_{\text{ét,c}}(Y_1(Np^r) \underset{\mathbb{Z}}{\times} \overline{\mathbb{Q}}, \mathrm{Sym}^{k-2}\, \mathrm{R}^1 f_* \underline{\mathcal{O}})$$

commuting with the Hecke action. Hence $\mathrm{MS}_k(Np^r, \mathcal{O})$ carries a canonical $\mathcal{O}$-linear action of $\mathrm{G}_\mathbb{Q}$ which commutes with the Hecke action.

[4] In [Kit94, §5.3], the character ε is assumed to have kernel Γ_s^{wt} instead of Γ_r^{wt} (for some $s \le r$). This must be a typo since in this case the claim "$L_n(A) = L_n(\varepsilon, A)$ as $\Gamma_1(Np^r)$-module" there is not true.

(b) *The refinement morphism (tensored with L)*

$$\mathrm{Ref}_\alpha\colon \mathrm{MS}_k(N,L) \longrightarrow \mathrm{MS}_k(Np,L)$$

from proposition 2.10 is $\mathrm{G}_{\mathbb{Q}}$*-equivariant. It thus induces a* $\mathrm{G}_{\mathbb{Q}}$*-equivariant isomorphism*

$$\mathrm{MS}_k(N,L)[f] \xrightarrow{\sim} \mathrm{MS}_k(Np,L)[f_\alpha].$$

Both sides are isomorphic to Deligne's Galois representation attached to f.

Proof: Statement (a) follows from proposition 2.2 (a) and proposition II.3.4 and the considerations in section II.3.3. Statement (b) is clear from the proof of proposition 2.10 because the refinement morphism is induced in étale cohomology by maps between modular curves which are defined over $\mathbb{Q}$. □

During the proof of the next theorem we will need the following result on elliptic curves, which we prove here due to lack of a reference. In the statement we identify the scheme $E[p^t]$, which is finite étale over S by proposition II.1.1, with the étale sheaf it represents.

Proposition 4.17: *Let S be a $\mathbb{Q}$-scheme, $f\colon E \longrightarrow S$ an elliptic curve and $t \in \mathbb{N}$. Then $\mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t}$ and $E[p^t]$ are canonically $\mathbb{Z}/p^t$-dual to each other as étale sheaves on S.*

Proof: Fix a geometric point $s\colon \operatorname{Spec}\overline{\mathbb{Q}} \longrightarrow S$ on S. We use that the category of locally constant constructible sheaves on S is equivalent to finite discrete continuous $\pi_1^{\text{ét}}(S,s)$-modules via sending a sheaf to its stalk at s, see [Fu11, Thm. 3.2.12] and [Con09, Thm. 5.1.2.1, Rem. 5.1.2.2]. The stalk $(\mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t})_s$ is isomorphic to $\mathrm{H}^1_{\text{ét}}(E_s, \mathbb{Z}/p^t)$ by the proper base change theorem, while the dual of the sheaf defined by $E[p^t]$ has stalk $\operatorname{Hom}(E_s[p^t](\overline{\mathbb{Q}}), \mathbb{Z}/p^t)$ at s. We need to find a canonical isomorphism between these abelian groups which is equivariant for the $\pi_1^{\text{ét}}(S,s)$-action.

Let us elaborate on the origin of the $\pi_1^{\text{ét}}(S,s)$-actions, which are explained in [FK88, §§A I.4–7]. The action on $E_s[p^t](\overline{\mathbb{Q}})$ is easy to describe: since $E[p^t]$ is a finite étale cover of S, we have a canonical surjection $\pi_1^{\text{ét}}(S,s) \twoheadrightarrow \operatorname{Aut}_s(E[p^t]/S)$, where the subscript "$s$" should mean automorphisms respecting the geometric point s. Each such automorphism clearly induces an automorphism of $E_s[p^t](\overline{\mathbb{Q}})$. On the other hand, if $\mathcal{F}$ is any étale sheaf on S, then the stalk at s is expressed as $\mathcal{F}_s = \varinjlim_U \mathcal{F}(U)$, where U runs over étale covers of S containing s, and without loss of generality we may restrict to such U which are Galois over S. For each such U we have a surjection $\pi_1^{\text{ét}}(S,s) \twoheadrightarrow \operatorname{Aut}_s(U/S)$ which induces an action of $\pi_1(S,s)$ on $\mathcal{F}(U)$ and thus, by compatibility of these surjections, an action on $\mathcal{F}_s$.

From [Fu11, Prop. 5.7.20] (see also [Con09, Thm. 5.2.2.1]) we have a canonical isomorphism $\mathrm{H}^1_{\text{ét}}(E_s, \mathbb{Z}/p^t) \cong \operatorname{Hom}(\pi_1^{\text{ét}}(E_s,s), \mathbb{Z}/p^t)$. Since this isomorphism is functorial in the space under consideration, it is compatible with the action of $\pi_1^{\text{ét}}(S,s)$. Obviously we may replace here $\pi_1^{\text{ét}}(E_s,s)$ by $(\pi_1^{\text{ét}}(E_s,s))^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^t$. Any étale cover of E_s is easily seen to be an isogeny from another elliptic curve (after choosing a lift of the origin). We can precompose any such isogeny with its dual isogeny and see that in the limit defining $\pi_1^{\text{ét}}(E_s,s)$ we may restrict to covers by multiplication-by-n maps for all $n \in \mathbb{N}$ (see [KM85, p. 79–81] or [Sil86, §III.6]). The automorphisms of such a cover may be identified with $E_s[n](\overline{\mathbb{Q}})$. Since these are non-canonically isomorphic to $(\mathbb{Z}/n)^2$, we see that $\pi_1^{\text{ét}}(E_s,s) \cong \widehat{\mathbb{Z}}^2$ and that $(\pi_1^{\text{ét}}(E_s,s))^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^t$ are just the automorphisms of the multiplication-by-p^t cover, i. e. $E_s[p^t](\overline{\mathbb{Q}})$.

We now see that there is a canonical isomorphism $\mathrm{H}^1_{\text{ét}}(E_s, \mathbb{Z}/p^t) \cong \operatorname{Hom}(E_s[p^t](\overline{\mathbb{Q}}), \mathbb{Z}/p^t)$. By our previous description, on both sides the action of $\pi_1(S,s)$ comes from the natural action of the quotient $\operatorname{Aut}_s(E[p^t]/S)$, so the isomorphism is equivariant for this action. □

For $s \geq r \geq 0$, the maps $\mathrm{MS}_k(Np^r, \mathcal{O}) \longrightarrow \mathrm{MS}_k(Np^s, \mathcal{O})$ are $\mathrm{G}_{\mathbb{Q}}$-equivariant since they are induced from the canonical maps of curves $Y_1(Np^s) \longrightarrow Y_1(Np^r)$, which are defined over $\mathbb{Q}$. Hence by definition of $\overline{\mathcal{MS}}_k(Np^\infty, \mathcal{O})$ we can extend the action of $\mathrm{G}_{\mathbb{Q}}$ to this $\mathcal{O}$-module. Of course it still commutes with the Hecke action. We then endow $\mathcal{UM}(Np^\infty, \mathcal{O})$ with the dual $\mathrm{G}_{\mathbb{Q}}$-action and $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$ again with the dual $\mathrm{G}_{\mathbb{Q}}$-action. It is clear that the $\mathrm{G}_{\mathbb{Q}}$-action on $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$ is then $\mathcal{I}$-linear and still commutes with the Hecke action.

By $\mathcal{I}$-linearity, we thus get a $\mathrm{G}_{\mathbb{Q}}$-action on the reduction of $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$ at arithmetic points. We want to show that the control theory isomorphisms are $\mathrm{G}_{\mathbb{Q}}$-equivariant.

Theorem 4.18: *Fix a $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r). The $\mathcal{O}$-linear isomorphism*

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) \underset{\mathcal{I}}{\otimes} \left(\mathcal{I} \middle/ P\right) \xrightarrow{\ \sim\ } \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})[\varepsilon]$$

from theorem 4.10 (a) is $\mathrm{G}_{\mathbb{Q}}$*-equivariant.*

Proof: The isomorphism we study was defined as the composition of three maps: the reduction map from lemma 4.11, the biduality map from lemma 4.12 and the map which compares modular symbols of different weights from lemma 4.13. By construction and the definition of the Galois actions, it is clear that the first two respect the action of $\mathrm{G}_{\mathbb{Q}}$. The third one was defined using the canonical isomorphism

$$\overline{\mathcal{MS}}_k^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon] \xrightarrow{\ \sim\ } \overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})[\varepsilon\kappa_{\mathrm{wt}}^{k-2}]$$

from lemma 4.9, which came from the map

$$\mathrm{MS}_k(Np^r, \mathcal{O}/p^t) \longrightarrow \mathrm{MS}_2(Np^r, \mathcal{O}/p^t)(\chi^{k-2})$$

from (4.3) (with $r \geq t \geq 0$ fixed), and it remains to prove that this map is $\mathrm{G}_{\mathbb{Q}}$-equivariant. If we use the description of modular symbols as in proposition 1.2 (a) we see that the map (4.3) comes from a map of sheaves on $Y_1(Np^r)^{\mathrm{an}}$

$$\mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_* \underline{\mathcal{O}/p^t} \cong \pi_*^{\Gamma_1(Np^r)} \underline{\mathrm{Sym}^{k-2}(\mathcal{O}/p^t)^2} \longrightarrow \pi_*^{\Gamma_1(Np^r)} \underline{\mathcal{O}/p^t(\chi^{k-2})} = \underline{\mathcal{O}/p^t}.$$

Here the right equality follows easily from lemma I.1.38, noting that $\Gamma_1(Np^r)$ acts trivially on $R(\chi^{k-2})$. Further it is easy to see that the map is Sym^{k-2} of a map $\mathrm{R}^1 f_* \underline{\mathcal{O}/p^t} \longrightarrow \underline{\mathcal{O}/p^t}$. With respect to our fixed trivialisation of $\mathrm{R}^1 f_* \underline{\mathbb{Z}}$ on $\mathfrak{h}$ (by choice of a basis, see section II.2.1), it comes from projection onto the first coordinate. We note here that by theorem II.1.13, in fibres of $E_1(N)^{\mathrm{an}} \longrightarrow Y_1(N)^{\mathrm{an}}$ the point of order N lies in the *second* coordinate with respect to this basis. This will be used below.

Since the category of locally constant torsion sheaves with finite fibres on $Y_1(N)^{\mathrm{an}}$ is equivalent to the category of locally constant constructible torsion sheaves étale on $Y_1(N) \times_{\mathbb{Z}} \mathbb{C}$ by [SGA4.3, Exp. XI, Thm. 4.4 (i)], the above morphism of sheaves corresponds to a morphism of such étale sheaves. To see $\mathrm{G}_{\mathbb{Q}}$-equivariance, we need to show that this map of étale sheaves already exists over $\mathbb{Q}$, i. e. it comes from a map of étale sheaves defined over $\mathbb{Q}$

$$\mathrm{R}^1 f_* \underline{\mathcal{O}/p^t} \longrightarrow \underline{\mathcal{O}/p^t}$$

on $Y_1(Np^r)_{/\mathbb{Q}}$. We construct such a map which after base change to $\mathbb{C}$ gives back the morphism from before.

We now work over $\mathbb{Q}$. Take an étale open U in $Y_1(Np^r)_{/\mathbb{Q}}$. Then on $E_1(Np^r)_{/\mathbb{Q}} \times_{Y_1(Np^r)_{/\mathbb{Q}}} U =: E_U$ the level structure gives us a point of exact order p^r, i. e. a morphism of group schemes

$$\alpha\colon \underline{\mathbb{Z}/p^r}_{/U} \hookrightarrow E_U[p^r].$$

If we compose α with the map $E_U[p^r] \longrightarrow E_U[p^t]$ which is multiplication by p^{r-t}, then it is easy to see that it factors through $\mathbb{Z}/p^t$ and gives a point of exact order p^t

$$\beta\colon \underline{\mathbb{Z}/p^t}_{/U} \hookrightarrow E_U[p^t]$$

(here we just applied the change of level morphism σ_{p^r,p^t} from section II.7.2). Dualising β and using proposition 4.17 we obtain a surjection

$$\mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t} \twoheadrightarrow \underline{\mathbb{Z}/p^t} \tag{$*$}$$

of étale sheaves on U.

By propositions II.1.1 and 4.17, $\mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t}$ is étale locally on $Y_1(Np^r)_{/\mathbb{Q}}$ isomorphic to the constant sheaf $\underline{(\mathbb{Z}/p^t)^2}$. Now assume U is small enough and choose an isomorphism of sheaves on U

$$\psi_U\colon \mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t} \xrightarrow{\sim} \underline{(\mathbb{Z}/p^t)^2}$$

such that the composition $\pi_1 \circ \psi_U$ is the map $(*)$ (where $\pi_1, \pi_2\colon \underline{(\mathbb{Z}/p^t)^2} \longrightarrow \underline{(\mathbb{Z}/p^t)}$ are the projections on the first and second factor, respectively). We then define a morphism of sheaves on U

$$\varphi_U\colon \mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t} \longrightarrow \underline{\mathbb{Z}/p^t}$$

as $\varphi_U := \pi_2 \circ \psi_U$ and claim that this globalises to a morphism of sheaves $\mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t} \longrightarrow \underline{\mathbb{Z}/p^t}$ on $Y_1(Np^r)_{/\mathbb{Q}}$. To check this, take two small étale open sets U_1, U_2 as above. Then $\psi_{U_2}^{-1}|_{U_1 \cap U_2} \circ \psi_{U_1}|_{U_1 \cap U_2}$ is an automorphism of the constant sheaf $(\mathbb{Z}/p^t)^2$ on $U_1 \cap U_2$, so it may be described by a tuple of matrices in $\mathrm{GL}_2(\mathbb{Z}/p^t)$. But since it has to respect the point of order p^t each matrix has to be upper unitriangular. So the automorphism does not change the projection onto the second factor, i. e.

$$\varphi_{U_1}|_{U_1 \cap U_2} = \pi_2|_{U_1 \cap U_2} \circ \psi_{U_1}|_{U_1 \cap U_2} = \pi_2|_{U_1 \cap U_2} \circ \psi_{U_2}|_{U_1 \cap U_2} = \varphi_{U_2}|_{U_1 \cap U_2}$$

and we indeed get a morphism of sheaves $\mathrm{R}^1 f_* \underline{\mathbb{Z}/p^t} \longrightarrow \underline{\mathbb{Z}/p^t}$ on $Y_1(Np^r)_{/\mathbb{Q}}$. By construction and our previous considerations, it is clear that after tensoring with $\mathcal{O}/p^t$ and base change to $\mathbb{C}$ we get the morphism $\mathrm{R}^1 f_* \underline{\mathcal{O}/p^t} \longrightarrow \underline{\mathcal{O}/p^t}$ from before. $\square$

Corollary 4.19: *Let $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$ be an $\mathcal{I}$-adic eigenform and assume that condition 4.5 is satisfied. Then the eigenspace $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})[F]$ is as an $\mathcal{I}$-linear representation of $\mathrm{G}_\mathbb{Q}$ isomorphic to Hida's big representation ρ_F from theorem 3.18.*

Proof: It follows from condition 4.5 that the eigenspace $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})[F]$ is free of rank 2 over $\mathcal{I}$. Moreover, under our assumptions the map

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})[F] \underset{\mathcal{I}}{\otimes} \left(\mathcal{I}\big/P\right) \xrightarrow{\sim} \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})[F_P] = \mathrm{MS}_k(Np^r, \mathcal{O})[F_P]$$

from theorem 4.10 (c) is $G_{\mathbb{Q}}$-equivariant by theorem 4.18 and the fact that the Hecke and Galois actions commute. By proposition 4.16 there is a canonical $\mathcal{O}$-linear $G_{\mathbb{Q}}$-equivariant isomorphism

$$\mathrm{MS}_k(Np^r, \mathcal{O})[F_P] \cong \mathrm{H}^1_{\text{ét,c}}(Y_1(Np^r) \underset{\mathbb{Z}}{\times} \overline{\mathbb{Q}}, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathcal{O}})[F_P].$$

After tensoring this isomorphism with L (the quotient field of $\mathcal{O}$) we get a two-dimensional vector space by proposition 2.7. If F_P is a newform, by theorem II.5.12 and the definition of $\mathcal{M}(f)$ this vector space is isomorphic to Deligne's Galois representation attached to F_P. Otherwise F_P is the unique ordinary refinement of a newform F_P^{new}, and then by proposition 4.16 (b) the vector space is isomorphic to Deligne's Galois representation attached to F_P^{new}. Hence $\mathbb{MS}^{\text{ord}}(Np^\infty, \mathcal{I})[F]$ has the same properties as Hida's big Galois representation ρ_F from theorem 3.18, and since ρ_F is unique with these properties the claim follows. □

5. Families of p-adic Eichler-Shimura isomorphisms

We keep the setting described in situation 3.11 and continue to use the notation introduced at the beginning of section 3. The goal of this section is to explain the fact that Faltings' p-adic Eichler-Shimura isomorphisms from theorem II.6.9 can be interpolated in a Hida family. This is formulated in terms of a map relating $\mathcal{I}$-adic modular symbols and $\mathcal{I}$-adic cusp forms. Before we can cite this result, we need to study trace-compatible systems of cusp forms and modular symbols, which comprise modules that are isomorphic to $\mathcal{I}$-adic cusp forms resp. $\mathcal{I}$-adic modular symbols. This is because the $\mathcal{I}$-adic Eichler-Shimura isomorphism is formulated in the literature in terms of these trace-compatible systems.

5.1. Trace-compatible projective systems

The techniques in this section are heavily inspired from [Oht95, §2.3] and [Wak14].

In [Oht95] it is proved that Λ^{wt}-adic modular forms can be described also as projective limits of cusp forms. More precisely, $\mathbb{S}^{\text{ord}}(Np^\infty, \Lambda^{\text{wt}})$ is shown to be isomorphic to the projective limit of certain spaces of cusp forms along trace maps. The Eichler-Shimura philosophy suggests that a similar statement should hold for Λ^{wt}-adic modular symbols. This is in fact true, as we prove in this section.

The proof works in great parts analogous to the proof given in [Oht95, §2.3]. While there many calculations are omitted, we perform these here in some detail to make sure that they still work in the modular symbols setting.

5.1.1. Formal properties

Let $\hat{\Gamma}^{\text{wt}}_{\text{f}}$ be the group of finite order $\mathcal{O}^\times$-valued characters of Γ^{wt} and let $\hat{\Gamma}^{\text{wt}}_{\text{f},r}$ be the subgroup of characters that factor over $\Gamma^{\text{wt}}/\Gamma^{\text{wt}}_r$.

Let M be a $\mathbb{C}_p$-linear representation of (Σ, ι) (with $\Sigma = \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$). Throughout the section we assume that the scalar matrices $\left(\begin{smallmatrix} q & \\ & q \end{smallmatrix}\right)$ for all primes q act injectively on M (later this will be satisfied trivially). We will here use mostly the right representative of the action. Then for $r \geq 0$ we have a well-defined trace map

$$\mathrm{tr}_r \colon M^{\Gamma_1(Np^{r+1})} \longrightarrow M^{\Gamma_1(Np^r)}, \quad m \longmapsto \sum_i m[\beta_i],$$

where the β_i comprise a set of representatives of left cosets of $\Gamma_1(Np^{r+1})$ in $\Gamma_1(Np^r)$, i. e.

$$\Gamma_1(Np^r) = \bigsqcup_i \Gamma_1(Np^{r+1})\beta_i.$$

An explicit representation of the trace map is the following: if we define maps

$$M^{\Gamma_1(Np^{r+1})} \xrightarrow{\mathrm{tr}_{r,1}} M^{\Gamma_1(Np^r)\cap\Gamma_0(p^{r+1})} \xrightarrow{\mathrm{tr}_{r,2}} M^{\Gamma_1(Np^r)}$$

by

$$\mathrm{tr}_{r,1}(m) = \sum_{\alpha\in\Gamma_r^{\mathrm{wt}}/\Gamma_{r+1}^{\mathrm{wt}}} m\langle\alpha\rangle, \qquad \mathrm{tr}_{r,2}(m) = \sum_{j=0}^{p-1} m\begin{pmatrix}1 & 0\\ a & c\end{pmatrix} tionNp^r j1,$$

then $\mathrm{tr}_r = \mathrm{tr}_{r,2}\circ\mathrm{tr}_{r,1}$. This is shown in [Oht95, (2.3.1)].

Let now $M^0 \subseteq M$ be an $\mathcal{O}$-submodule (not necessarily stable under the action of Σ) and for each $r \geq 0$ put $M_r^0 := M^0 \cap M^{\Gamma_1(Np^r)}$, and require that M_r^0 is stable under the action of $\Delta_1(Np^r)$. Then define an $\mathcal{O}$-module

$$D_r(M, M^0) := \{m \in M^{\Gamma_1(Np^r)} : m[w_{Np^r}] \in M_r^0\}$$

and a map

$$W_{Np^r} : D_r(M, M^0) \longrightarrow M_r^0, \quad m \longmapsto m[w_{Np^r}]$$

for each $r \geq 0$ (here $[w_{Np^r}]$ is the Atkin-Lehner endomorphism). We assume that W_{Np^r} is an isomorphism of $\mathcal{O}$-modules.

The condition that M_r^0 be stable under the action of $\Delta_1(Np^r)$ makes $D_r(M, M^0)$ stable under the action of $\Delta_1(Np^r)^\iota$. This implies that for any $m \in D_r(M, M^0)$ and any $T \in \mathcal{H}(Np)$ there is an $m' \in D_r(M, M^0)$ such that $m[w_{Np^r}][T] = m'[w_{Np^r}]$, so $D_r(M, M^0)$ is stable under the action of $\mathcal{H}(Np)^\iota$. Note that we used lemma I.1.55 (a) here to identify the $\mathcal{H}(Np^r)$ and $\mathcal{H}(Np^s)$ with $\mathcal{H}(Np)$. On the other hand, since M_r^0 is stable under the action of $\Delta_1(Np^r)$, it is a module over $\mathcal{H}(Np)$. Let $\mathbf{T}$ be the Hecke eigenalgebra of M_r^0 and $\mathbf{T}^\iota$ the adjoint Hecke eigenalgebra of $D_r(M, M^0)$.

Lemma 5.1: *The isomorphism $\mathcal{H}(Np) \xrightarrow{\sim} \mathcal{H}(Np)^\iota$ from lemma I.1.35 (b) induces an isomorphism $\mathbf{T} \xrightarrow{\sim} \mathbf{T}^\iota$. The isomorphism W_{Np^r} is such that the action of a $T \in \mathcal{H}(Np)^\iota$ on $D_r(M, M^0)$ corresponds under W_{Np^r} to the action of its image in $\mathcal{H}(Np)$ on M_r^0.*

Proof: If for $\varphi \in \mathrm{End}_{\mathcal{O}}(M_r^0)$ and $m \in D_r(M, M^0)$ there is a unique $m' \in D_r(M, M^0)$ such that $m'[w_{Np^r}] = \varphi(m[Np^r])$, then $m \longmapsto m'$ gives a well-defined element of $\mathrm{End}_{\mathcal{O}}(D_r(M, M^0))$. Since W_{Np^r} is injective, by our previous observations this gives us a morphism $\mathbf{T} \longrightarrow \mathbf{T}^\iota$, which we denote by $T \longmapsto T^\iota$. From the commutative diagram

$$\begin{tikzcd} \mathcal{H}(Np) \arrow[r, two heads] \arrow[d, "\sim"'] & \mathbf{T} \arrow[d] \\ \mathcal{H}(Np)^\iota \arrow[r, two heads] & \mathbf{T}^\iota \end{tikzcd}$$

we see that it is in fact surjective. Finally an element T is in the kernel if and only if $m[T^\iota] = 0$ for all $m \in D_r(M, M^0)$. But then for any $x \in M_r^0$ we have $x[T][w_{Np^r}] = x[w_{Np^r}][T^\iota] = 0$, so $T = 0$ by since W_{Np^r} is injective. The final statement is clear. $\square$

Using the argument in [Oht95, Dfn.-Lem. 2.3.4], one can show that the trace map tr_r maps $D_{r+1}(M, M^0)$ into $D_r(M, M^0)$. Denote the composite map

$$M^0_{r+1} \xrightarrow{W^{-1}_{Np^{r+1}}} D_{r+1}(M, M^0) \xrightarrow{\mathrm{tr}_r} D_r(M, M^0) \xrightarrow{W_{Np^r}} M^0_r$$

by $\tilde{\mathrm{tr}}_r$ and define

$$M^0_\infty = \varprojlim_r M^0_r, \quad D_\infty(M, M^0) = \varprojlim_r D_r(M, M^0)$$

with the limits being taken along the respective trace maps $\tilde{\mathrm{tr}}$ resp. tr. It is clear that the W_{Np^r} induces an isomorphism of $\mathcal{O}$-modules

$$W_{Np^\infty}\colon D_\infty(M, M^0) \xrightarrow{\sim} M^0_\infty.$$

Lemma 5.2: *The trace map* tr_r *commutes with the action of the Hecke algebra* $\mathcal{H}(Np)^\iota$. *Hence* $D_\infty(M, M^0)$ *is a right* $\mathcal{H}(Np)^\iota$*-module. Consequently, the trace map* $\tilde{\mathrm{tr}}_r$ *commutes with the action of the Hecke algebra* $\mathcal{H}(Np)$*, so* M^0_∞ *is a right* $\mathcal{H}(Np)$*-module. The isomorphism* W_{Np^∞} *is such that the action of a* $T \in \mathcal{H}(Np)^\iota$ *on* $D_\infty(M, M^0)$ *corresponds under* W_{Np^∞} *to the action of its image in* $\mathcal{H}(Np)$ *on* M^0_∞.

Proof: The identity

$$w_{Np^{r+1}} \begin{pmatrix} 1 & -j \\ & p \end{pmatrix} w^{-1}_{Np^r} = \begin{pmatrix} p & \\ & p \end{pmatrix} \begin{pmatrix} 1 & \\ Np^r j & 1 \end{pmatrix} \quad \text{for all } j \in \mathbb{Z}$$

shows together with lemma I.1.54 that we have the relation

$$\mathrm{tr}_{r,2}(m) \begin{bmatrix} p & \\ & p \end{bmatrix} = m[w_{Np^{r+1}}][T_p][w_{Np^r}]^{-1},$$

which is an analogue of [Oht95, (2.3.3), second line]. From this and the relations from corollary I.1.63 it can be seen that $\left[\begin{smallmatrix} p & \\ & p \end{smallmatrix}\right] \circ \mathrm{tr}_{r,2}$ is compatible with the action of any $T^\iota \in \mathcal{H}(Np)^\iota$, hence $\mathrm{tr}_{r,2}$ is compatible since we assumed that $\left(\begin{smallmatrix} p & \\ & p \end{smallmatrix}\right)$ acts injectively. For $\mathrm{tr}_{r,1}$ this is clear from the definition, so the claim follows. □

Remark 5.3: In (II.7.2) we defined the change of level morphisms

$$\Sigma_{Np^{r+1}, Np^r}, \Theta_{Np^{r+1}, Np^r}\colon Y_1(Np^{r+1}) \longrightarrow Y_1(Np^r).$$

If we look at complex points, we see that the map $Y_1(Np^{r+1})^{\mathrm{an}} \longrightarrow Y_1(Np^r)^{\mathrm{an}}$ coming from Σ_{Np^{r+1}, Np^r} can be described using the isomorphism $Y_1(Np^{r+1})^{\mathrm{an}} \cong \Gamma_1(Np^{r+1})\backslash\mathfrak{h}$ from theorem II.1.13 as the canonical projection $Y_1(Np^{r+1})^{\mathrm{an}} \longrightarrow Y_1(Np^r)^{\mathrm{an}}$ coming from the inclusion $\Gamma_1(Np^{r+1}) \subseteq \Gamma_1(Np^r)$ (see the proof of proposition II.7.6 or [KLZ17, §2.4]). So in particular it is a finite covering map of topological spaces. Therefore, as explained in remark I.1.46, we have a trace (or corestriction) map

$$\mathrm{H}^q(\Gamma_1(Np^{r+1}), M) \longrightarrow \mathrm{H}^q(\Gamma_1(Np^r), M), \quad q \geq 0.$$

For $q = 0$ this recovers our map tr_r.

One can check that the two morphisms are interchanged by the Atkin-Lehner involutions on $Y_1(Np^{r+1})$ and $Y_1(Np^r)$, i.e. we have $\Sigma_{Np^{r+1}, Np^r} \circ w_{Np^{r+1}} = w_{Np^r} \circ \Theta_{Np^{r+1}, Np^r}$. Thus the trace map in group cohomology coming from Θ_{Np^{r+1}, Np^r} recovers the map $\tilde{\mathrm{tr}}_r$. This is discussed in [Wak14, Appendix A].

5.1.2. Trace-compatible systems of cusp forms

We now cite the above-mentioned result for cusp forms, which can be found at [Oht95, §2.3–4].

Put[5]

$$\mathrm{S}_k^{\iota}(Np^r, \mathcal{O}) := \{\xi \in \mathrm{S}_k(X_1(Np^r)^{\mathrm{arith}}, \mathbb{C}_p) : \xi[w_{Np^r}] \in \mathrm{S}_k(X_1(Np^r)^{\mathrm{arith}}, \mathcal{O})\}$$

and

$$\mathfrak{S}_k^{\iota}(Np^\infty, \mathcal{O}) := \varprojlim_r \mathrm{S}_k^{\iota}(Np^r, \mathcal{O}).$$

By the abstract properties described before,[6] the adjoint Hecke eigenalgebra of $\mathrm{S}_k^{\iota}(Np^r, \mathcal{O})$ is $\mathbf{t}_k^{\iota}(Np^r, \mathcal{O})$, so we can consider $\mathfrak{S}_k^{\iota}(Np^\infty, \mathcal{O})$ as a module over $\mathbf{t}^{\iota}(Np^\infty, \mathcal{O})$ and via this also as a Λ^{wt}-module. This big Hecke algebra contains the adjoint ordinary projection e^{ι}, so we can consider the anti-ordinary part $\mathfrak{S}_k^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$.

Recall from remark 3.3 that there is an isomorphism $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \xrightarrow{\sim} \mathbf{t}^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$. In the following, if $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})$, we write $F_{k,\varepsilon}$ for its specialisation at $P_{k,\varepsilon} \in \mathcal{X}^{\mathrm{arith}}$.

Theorem 5.4 (Ohta)**:** *For any $k \geq 2$ there is a canonical isomorphism of Λ^{wt}-modules*

$$\mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \xleftrightarrow{\quad\sim\quad} \mathfrak{S}_k^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$$

$$F \longmapsto (f_r)_r \text{ with } f_r = \frac{1}{p^{r-1}}\left(\sum_{\varepsilon \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}} F_{k,\varepsilon}[T_p^{-r}]\right)[w_{Np^r}]^{-1}$$

the unique F such that

$$F_{k,\varepsilon} = \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) f_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1} \longleftarrow (f_r)_r.$$

Under this isomorphism, a Hecke operator from $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ on the left side corresponds to its image in $\mathbf{t}^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$ on the right side.

Proof: [Oht95, Thm. 2.3.6] □

Using lemma 5.2, we obtain the following corollary.

Corollary 5.5: *There is a canonical isomorphism of $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$-modules*

$$\mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \xrightarrow{\sim} \varprojlim_r \mathrm{S}_k^{\mathrm{ord}}(X_1(Np^r), \mathcal{O}),$$

$$F \longmapsto (f_r)_r \text{ with } f_r = \frac{1}{p^{r-1}}\left(\sum_{\varepsilon \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}} F_{k,\varepsilon}[T_p^{-r}]\right)[w_{Np^r}]^{-1}$$

where the limit is taken along the maps $\tilde{\mathrm{tr}}_r$ introduced before.

[5] We have to use the arithmetic model of the modular curve here because Ohta defines modular forms with coefficients in a ring R as the tensor product of the space of classical modular forms with Fourier coefficients in $\mathbb{Z}$ with R; see corollary II.4.17.

[6] This does not directly fit with the abstract setting we described before, but if one chooses an isomorphism $\mathbb{C} \cong \mathbb{C}_p$ one can see this as a special case of the abstract setting, so we can apply the statements there. Anyways, this is not too important since all the statements can be proved directly in the more concrete setting with exactly the same proofs.

5.1.3. Trace-compatible systems of modular symbols

Now we specialise the abstract definitions to modular symbols and prove a theorem analogous to theorem 5.4.

Fix $k \geq 2$. Let

$$M := \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Div}^0(\mathbb{P}^1(\mathbb{Q})), \operatorname{Sym}^{k-2} \mathbb{C}_p^2), \ M^0 := \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Div}^0(\mathbb{P}^1(\mathbb{Q})), \operatorname{Sym}^{k-2} \mathcal{O}^2).$$

Then $M_r^0 = \mathrm{MS}_k(Np^r, \mathcal{O})$ and

$$D_r(M, M_0) = \{\xi \in \mathrm{MS}_k(Np^r, \mathbb{C}_p) : \xi[w_{Np^r}] \in \mathrm{MS}_k(Np^r, \mathcal{O})\} =: \mathrm{MS}_k^\iota(Np^r, \mathcal{O})$$

for $r \geq 0$. Put

$$\mathfrak{MS}_k(Np^\infty, \mathcal{O}) := D_\infty(M, M^0) = \varprojlim_r \mathrm{MS}_k^\iota(Np^r, \mathcal{O}).$$

By lemmas II.4.25 and 5.1, the adjoint Hecke eigenalgebra of $\mathrm{MS}_k^\iota(Np^r, \mathcal{O})$ is $\mathbf{T}_k^\iota(Np^r, \mathcal{O})$. So we can consider $\mathfrak{MS}_k(Np^\infty, \mathcal{O})$ as a module over $\mathbf{T}^\iota(Np^\infty, \mathcal{O})$ and via this also as a Λ^{wt}-module, and we have again an anti-ordinary part $\mathfrak{MS}_k^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$.

As a preparation to the proof of the main theorem of this section, we need some lemmas.

Lemma 5.6: *Fix $u \in \mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ and $(x_r)_r \in \mathfrak{MS}_2^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$. Then there is a unique $X(u) \in \Lambda^{\mathrm{wt}}$ such that*

$$X(u) \bmod P_{2,\varepsilon} = \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) u(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1})$$

for all $\varepsilon \in \hat{\Gamma}_{\mathrm{f}}^{\mathrm{wt}}$.

Proof: For $\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}$, abbreviate $u_\alpha := u(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1})$. Define a map

$$F \colon \hat{\Gamma}_{\mathrm{f}}^{\mathrm{wt}} \longrightarrow \mathcal{O}, \quad \varepsilon \longmapsto \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) u_\alpha.$$

Fix $\alpha_0 \in \Gamma^{\mathrm{wt}}$. Then we calculate

$$\begin{aligned}
\sum_{\varepsilon \in \hat{\Gamma}_{\mathrm{f},r}^{\mathrm{wt}}} \varepsilon(\alpha_0)^{-1} F(\varepsilon) &= \sum_{\varepsilon \in \hat{\Gamma}_{\mathrm{f},r}^{\mathrm{wt}}} \varepsilon(\alpha_0)^{-1} \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) u_\alpha \\
&= \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \sum_{\varepsilon \in \hat{\Gamma}_{\mathrm{f},r}^{\mathrm{wt}}} \varepsilon(\alpha_0^{-1}\alpha) u_\alpha \\
&= p^{r-1} u_{\alpha_0} + \sum_{\alpha \neq \alpha_0} u_\alpha \sum_\varepsilon \varepsilon(\alpha_0^{-1}\alpha).
\end{aligned}$$

Since $\sum_\varepsilon \varepsilon(\alpha_0^{-1}\alpha) = 0$ if $\alpha \neq \alpha_0$, it follows that

$$\sum_{\varepsilon \in \hat{\Gamma}_{\mathrm{f},r}^{\mathrm{wt}}} \varepsilon(\alpha_0)^{-1} F(\varepsilon) \in p^{r-1}\mathcal{O}.$$

Then the claim follows from [Oht95, Lem. 2.4.2]. □

In the following, for $X \in \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})$ we denote its image in $\mathrm{MS}_2(Np^r, \mathcal{O})[\varepsilon]$ under the morphism from theorem 4.10 (a) by $X_{2,\varepsilon}$ (with $k = 2$ there).

Lemma 5.7: *Fix* $(x_r)_r \in \mathfrak{MS}_2^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$.

(a) The map

$$X\colon \mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \Lambda^{\mathrm{wt}}, \quad u \longmapsto X(u)$$

with $X(u)$ *as in lemma 5.6 is* Λ^{wt}*-linear.*

(b) X *is the unique element in* $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})$ *such that if* $X_{2,\varepsilon}$ *is its image mod* $P_{2,\varepsilon}$ *in* $\mathrm{MS}_2(Np^r, \mathcal{O})[\varepsilon]$ *under the isomorphism from theorem 4.10 (a), then*

$$X_{2,\varepsilon} = \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) x_r [w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}.$$

Proof: (a) First, let $u, v \in \mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$. Then $X(u) + X(v)$ has the property that

$$\begin{aligned} X(u) + X(v) \bmod P_{2,\varepsilon} &= \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) u(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}) \\ &\qquad + \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) v(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}) \\ &= \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha)(u+v)(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}) \end{aligned}$$

for all $\varepsilon \in \hat{\Gamma}_{\mathrm{f}}^{\mathrm{wt}}$. Since $X(u+v)$ was defined to be the unique element in Λ^{wt} with this property, it follows $X(u+v) = X(u) + X(v)$.

Now let $u \in \mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ and $\lambda \in \Lambda^{\mathrm{wt}}$. Without loss of generality, assume $\lambda \in \Gamma^{\mathrm{wt}}$. Then

$$\begin{aligned} \lambda X(u) \bmod P_{2,\varepsilon} &= \varepsilon\kappa_{\mathrm{wt}}^2(\lambda) \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) u(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}) \\ &= \kappa_{\mathrm{wt}}^2(\lambda) \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\lambda\alpha) u(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}), \end{aligned}$$

while

$$\begin{aligned} X(\lambda u) \bmod P_{2,\varepsilon} &= \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) u(x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}\langle\lambda\rangle\kappa_{\mathrm{wt}}^2(\lambda)) \\ &= \kappa_{\mathrm{wt}}^2(\lambda) \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) u(x_r[w_{Np^r}][T_p^r]\langle\alpha\lambda^{-1}\rangle^{-1}). \end{aligned}$$

Replacing α in the second calculation by $\lambda\alpha$, which then also travels through all elements in $\Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}$, shows that the two expressions are equal.

(b) Since by (4.2) and the perfectness of the pairing (4.1) the element $X_{2,\varepsilon}$ is determined by its images under u for all $u \in \mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$, it suffices to prove that

$$u(X_{2,\varepsilon}) = u\Big(\sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma_r^{\mathrm{wt}}} \varepsilon(\alpha) x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}\Big)$$

for all $u \in \mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$. But by theorem 4.10 (d) $u(X_{2,\varepsilon}) = X(u) \bmod P_{2,\varepsilon}$, so the claim follows from lemma 5.6. ☐

Now we can prove the main theorem of this section, which is an analogue of theorem 5.4. Recall from remark 3.3 that there is an isomorphism $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \xrightarrow{\sim} \mathbf{T}^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$

Theorem 5.8: *There is a canonical isomorphism of Λ^{wt}-modules*

$$\begin{aligned}
\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) &\xleftrightarrow{\quad\sim\quad} \mathfrak{MS}_2^{\iota\text{-ord}}(Np^\infty, \mathcal{O}) \\
X &\longmapsto (x_r)_r \text{ with } x_r = \frac{1}{p^{r-1}}\Bigg(\sum_{\varepsilon \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}} X_{2,\varepsilon}[T_p^{-r}]\Bigg)[w_{Np^r}]^{-1} \\
X \text{ as in lemma 5.7} &\longleftarrow\!\shortmid (x_r)_r.
\end{aligned}$$

Under this isomorphism, a Hecke operator from $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ on the left side corresponds to its image in $\mathbf{T}^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$ on the right side.

Proof: First, that $(x_r)_r$ as in the statement forms a compatible system for the trace maps can be shown exactly in the same manner as in the proof given in [Oht95, §2.4]. Hence we know that both maps are well-defined. They are obviously $\mathcal{O}$-linear, and that they are in fact Λ^{wt}-linear follows by definition of the Λ^{wt}-module structure from the final statement about Hecke operators.

We sketch the calculations that show that the two maps are inverse to each other. First, for $X \in \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})$, let Y be the image of X under the composition of the two maps. Then for $\varepsilon_0 \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}$

$$\begin{aligned}
Y_{2,\varepsilon_0} &= \sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma^{\mathrm{wt}}_r} \varepsilon_0(\alpha) \frac{1}{p^{r-1}}\Big(\sum_{\varepsilon \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}} X_{2,\varepsilon}[T_p^{-r}]\Big)[w_{Np^r}^{-1}][w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1} \\
&= \frac{1}{p^{r-1}}\sum_\alpha \varepsilon_0(\alpha)\sum_\varepsilon X_{2,\varepsilon}\langle\alpha\rangle^{-1} = \frac{1}{p^{r-1}}\sum_\alpha \varepsilon_0(\alpha)\sum_\varepsilon \varepsilon(\alpha^{-1})X_{2,\varepsilon} \\
&= \frac{1}{p^{r-1}}\sum_\alpha \varepsilon_0(\alpha)\varepsilon_0(\alpha^{-1})X_{2,\varepsilon_0} + \frac{1}{p^{r-1}}\sum_\alpha \varepsilon_0(\alpha)\sum_{\varepsilon \neq \varepsilon_0} \varepsilon(\alpha^{-1})X_{2,\varepsilon} \\
&= X_{2,\varepsilon_0} + \frac{1}{p^{r-1}}\sum_{\varepsilon \neq \varepsilon_0}\Big(\sum_\alpha \varepsilon_0\varepsilon^{-1}(\alpha)\Big)X_{2,\varepsilon}.
\end{aligned}$$

Since $\sum_\alpha \varepsilon_0\varepsilon^{-1}(\alpha) = 0$ for $\varepsilon \neq \varepsilon_0$, it follows $Y_{2,\varepsilon_0} = X_{2,\varepsilon_0}$. By the Zariski density of arithmetic points in $\operatorname{Spec}\Lambda^{\mathrm{wt}}$ (see lemma 3.1 (b)) this shows $X = Y$.

On the other hand, for $(x_r)_r \in \mathfrak{MS}_2^{\iota\text{-ord}}(Np^\infty, \mathcal{O})$, let $(y_r)_r$ be the image of $(x_r)_r$ under the composition of the two maps. Then

$$\begin{aligned}
y_r &= \frac{1}{p^{r-1}}\sum_{\varepsilon \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}}\sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma^{\mathrm{wt}}_r} \varepsilon(\alpha)x_r[w_{Np^r}][T_p^r]\langle\alpha\rangle^{-1}[T_p^{-r}][w_{Np^r}^{-1}] \\
&= \frac{1}{p^{r-1}}\sum_{\varepsilon \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}}\sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma^{\mathrm{wt}}_r} \varepsilon(\alpha)x_r\langle\alpha\rangle \\
&= \frac{1}{p^{r-1}}\sum_{\alpha \in \Gamma^{\mathrm{wt}}/\Gamma^{\mathrm{wt}}_r}\Bigg(\sum_{\varepsilon \in \hat{\Gamma}^{\mathrm{wt}}_{\mathrm{f},r}} \varepsilon(\alpha)\Bigg)x_r\langle\alpha\rangle.
\end{aligned}$$

Since $\sum_\varepsilon \varepsilon(\alpha)$ is p^{r-1} if $\alpha = 1$ and 0 otherwise, it follows $y_r = x_r$.

The last claim about the Hecke operators follows with an easy calculation from the relations in corollary I.1.63. ☐

Again using lemma 5.2, we obtain the following corollary.

Corollary 5.9: *For each k there is a canonical isomorphism of $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$-modules*

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \xrightarrow{\sim} \varprojlim_r \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O}),$$

$$X \longmapsto (x_r)_r \text{ with } x_r = \frac{1}{p^{r-1}}\left(\sum_{\varepsilon\in\hat{\Gamma}_{\mathrm{f},r}^{\mathrm{wt}}} X_{2,\varepsilon}[T_p^{-r}]\right),$$

where the limit is taken along the maps $\tilde{\mathrm{tr}}_r$ introduced before.

To end this section we study how this isomorphism behaves with respect to Galois actions. In section 4.5 we introduced an $\mathcal{O}$-linear action of $\mathrm{G}_\mathbb{Q}$ on $\mathrm{MS}_k(Np^r, \mathcal{O})$ and a Λ^{wt}-linear action on $\mathbb{MS}(Np^\infty, \Lambda^{\mathrm{wt}})$. We endow the limit $\varprojlim_r \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})$ with the limit of the actions of $\mathrm{G}_\mathbb{Q}$ on each term.

Proposition 5.10: *The isomorphism from corollary 5.9 is $\mathrm{G}_\mathbb{Q}$-equivariant.*

Proof: This follows directly from theorem 4.18 and the formula in corollary 5.9. ☐

5.2. The $\mathcal{I}$-adic Eichler-Shimura isomorphism

The next theorem states a consequence of what is called the $\mathcal{I}$-adic Eichler-Shimura isomorphism, where $\mathcal{I}$ is a coefficient ring of $\mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ which we fix.

In the statement we view $\mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})$ as a $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$-module via the natural map $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \mathbf{t}^{\mathrm{ord}}(Np^\infty, \mathcal{O})$.

Theorem 5.11 (Ohta, Kato, Loeffler/Kings/Zerbes)**:** *There is a canonical $\mathbf{T}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{I}$-linear surjection (called the $\mathcal{I}$-adic Eichler-Shimura map)*

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) \twoheadrightarrow \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$$

such that the following hold.

(a) If we reduce it modulo the ideal $\omega_{k,r}$, the resulting $\mathbf{T}_k^{\mathrm{ord}}(Np^r, \mathcal{O})$-linear surjection[7]

$$\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O}) \twoheadrightarrow \mathrm{S}_k^{\mathrm{ord}}(X_1(Np^r), \mathcal{O})$$

fits into a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^1_{\text{ét},\mathrm{c}}(Y_1(Np^r) \times_\mathbb{Z} \overline{\mathbb{Q}}_p, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_* \underline{\mathcal{O}})^{\mathrm{ord}} & \twoheadrightarrow & \mathrm{S}_k^{\mathrm{ord}}(X_1(Np^r), \mathcal{O}) \\
\downarrow & & \downarrow \\
\mathrm{H}^1_{\text{ét},\mathrm{p}}(Y_1(Np^r) \times_\mathbb{Z} \overline{\mathbb{Q}}_p, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_* \underline{\mathbb{C}_p}) & \twoheadrightarrow & \mathrm{S}_k(X_1(Np^r), \mathbb{C}_p),
\end{array}$$

[7] Here we use theorem 4.10 (b) and theorem 3.7.

where the bottom row is the p-adic Eichler-Shimura isomorphism from theorem II.6.9 composed with the projection onto the first factor and the vertical maps are the natural ones.[8]

(b) *The kernel is the submodule* $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^{\mathrm{I}_p}$ *fixed under the inertia group, using the Galois action on modular symbols introduced in section 4.5.*

Proof: Since $\mathcal{I}$ is flat over Λ^{wt} and the formation of both $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, -)$ and $\mathbb{S}^{\mathrm{ord}}(Np^\infty, -)$ is compatible with base change from Λ^{wt} to $\mathcal{I}$, we can assume that $\mathcal{I} = \Lambda^{\mathrm{wt}}$. Similarly we can assume that $\mathcal{O} = \mathbb{Z}_p$.

The claim in (a) is obviously equivalent to the existence of a commutative diagram

$$\begin{tikzcd} \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \arrow[r, two heads] \arrow[d] & \mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}}) \arrow[d] \\ \mathrm{H}^1_{\text{ét},\mathrm{p}}(Y_1(Np^r) \times_{\mathbb{Z}} \overline{\mathbb{Q}}_p, \mathrm{Sym}^{k-2} \mathrm{R}^1 f_* \underline{\mathbb{C}_p}) \arrow[r, two heads] & \mathrm{S}_k(X_1(Np^r), \mathbb{C}_p), \end{tikzcd} \tag{$*$}$$

where the top map is the $\mathcal{I}$-adic Eichler-Shimura map, the bottom one is the p-adic comparison isomorphism and the vertical ones are the reduction maps. The existence of this diagram is essentially [KLZ17, Thm. 9.5.2], but there some different notations and normalisations are used. We explain these differences.

A useful discussion of these differences can also be found in [Wak14, Appendix A]. There are the following four possible conventions to describe the situation:

(1) Use the modular curve $Y_1(Np^r)^{\mathrm{naive}}$ and look at ordinary parts,

(2) use the modular curve $Y_1(Np^r)^{\mathrm{arith}}$ and look at ordinary parts,

(3) use the modular curve $Y_1(Np^r)^{\mathrm{arith}}$ and look at anti-ordinary parts,

(4) use the modular curve $Y_1(Np^r)^{\mathrm{naive}}$ and look at anti-ordinary parts.

Our reference [KLZ17] uses the same conventions as [FK12], which is case 4, while we want to use case 1. Other important works on this topic are [Oht95; Oht00], which use case 3. In [Wak14, Appendix A] it is described how to transform these cases into each other, and we apply this to the result from [KLZ17].

First, the modules involved in our reference also carry a Galois action, but since this action commutes with the Hecke action in all cases and we do not need it (we are just interested in the Hecke action), we ignore it throughout. Therefore we omit in our citations everything that only changes Galois actions, such as Tate twists. Also, in the texts the functor $\mathbf{D}$ is used, which is defined for a $\mathbb{Z}_p$-module T with a continuous unramified $\mathrm{G}_{\mathbb{Q}_p}$-action as $\mathbf{D}(T) := (T \hat{\otimes}_{\mathbb{Z}_p} \mathrm{W}(\overline{\mathbb{F}_p}))^{\mathrm{Frob}_p = 1}$, see [FK12, §1.7.4]. But as a $\mathbb{Z}_p$-module, each such T is canonically isomorphic to its $\mathbf{D}(T)$, more precisely: $\mathbf{D}$ is an equivalence of categories from $\mathbb{Z}_p$-modules with continuous unramified $\mathrm{G}_{\mathbb{Q}_p}$-action to just $\mathbb{Z}_p$-modules with the forgetful functor as a quasi-inverse: see [FK12, Prop. 1.7.6]. Since we are not interested in the Galois actions, we omit also every $\mathbf{D}$ in our citations.

[8] Here we use proposition 4.16, and we omitted the Tate twist from theorem II.6.9 since we are not interested in the Galois action at this point.

Following the definition in [KLZ17, Prop. 7.2.1 (1)], let $\mathrm{H}^1_{\mathrm{ord}} := e' \varprojlim_r \mathrm{H}^1_{\text{ét}}(Y_1(Np^r) \times_{\mathbb{Z}} \overline{\mathbb{Q}}, \mathbb{Z}_p)$ and $\mathrm{H}^1_{\mathrm{ord,p}} := e' \varprojlim_r \mathrm{H}^1_{\text{ét,p}}(Y_1(Np^r) \times_{\mathbb{Z}} \overline{\mathbb{Q}}, \mathbb{Z}_p)$. The limit is taken here over the trace maps along the change of level morphism Σ_{Np^{r+1},Np^r} from (II.7.2) (see [SGA4.3, exp. XVII, §6.2] for the trace map in étale cohomology). Then the combination of the diagrams in [KLZ17, Thm. 9.5.2] and [KLZ17, Thm. 7.2.3] yields a diagram

$$\begin{tikzcd} \mathrm{H}^1_{\mathrm{ord}}(Np^\infty) \arrow[r, two heads] \arrow[d] & e'\mathfrak{M}_2'(N, \mathbb{Z}_p) \arrow[d] \\ e'\mathrm{H}^1_{\text{ét}}(Y_1(Np^r) \times_{\mathbb{Z}} \overline{\mathbb{Q}}_p, \mathrm{TSym}^{k-2}\, \mathrm{R}^1 f_* \underline{\mathbb{Z}_p}) \arrow[r, two heads] & \mathrm{M}_k'(Np^r, \mathbb{Z}_p), \end{tikzcd} \tag{$**$}$$

which commutes modulo the Eisenstein subspace of $\mathrm{M}_k'(Np^r, \mathbb{Z}_p)$ and in which the bottom map is the comparison isomorphism (see appendix A.1 for a discussion of TSym). Here the notation in the right column is taken from there. The module $\mathrm{M}_k'(Np^r, \mathbb{Z}_p)$ there is defined in [KLZ17, §7.4] (see also [KLZ17, §2.6]); in our notation it would be

$$\mathrm{M}_k'(Np^r, \mathbb{Z}_p) = \{f \in \mathrm{M}_k(X_1(N)^{\mathrm{naive}}, \mathbb{Q}_p) : w_{Np^r}^{-1}(f) \in \mathrm{M}_k(X_1(N)^{\mathrm{arith}}, \mathbb{Z}_p)\}.$$

Further $\mathfrak{M}_2'(N, \mathbb{Z}_p)$ is defined in [KLZ17, §7.4] as $\mathfrak{M}_2'(N, \mathbb{Z}_p) = \varprojlim_r \mathrm{M}_2'(Np^r, \mathbb{Z}_p)$ (the limit again taken over the trace along $\mathrm{pr}_{1,r}$).

The map in the first row in ($**$) (which is the Λ^{wt}-adic Eichler-Shimura map) was originally constructed by Ohta in [Oht00]; it is just cited in this version in [KLZ17] (using the transformation between the cases (1)–(4) mentioned above). There is also a version for cusp forms instead of modular forms which was constructed by Ohta in the previous paper [Oht95]. Therefore by restriction we have a map $\mathrm{H}^1_{\mathrm{ord,p}} \twoheadrightarrow e'\mathfrak{S}_2'(N, \mathbb{Z}_p)$ if we define $\mathfrak{S}_2'(N, \mathbb{Z}_p)$ analogously. Therefore by restriction of the diagram ($**$) we get a diagram

$$\begin{tikzcd} \mathrm{H}^1_{\mathrm{ord,p}}(Np^\infty) \arrow[r, two heads] \arrow[d] & e'\mathfrak{S}_2'(N, \mathbb{Z}_p) \arrow[d] \\ e'\mathrm{H}^1_{\text{ét,p}}(Y_1(Np^r) \times_{\mathbb{Z}} \overline{\mathbb{Q}}_p, \mathrm{TSym}^{k-2}\, \mathrm{R}^1 f_* \underline{\mathbb{Z}_p}) \arrow[r, two heads] & \mathrm{S}_k'(Np^r, \mathbb{Z}_p), \end{tikzcd} \tag{$+$}$$

which now really commutes.

We now apply the inverse Atkin-Lehner endomorphism to the whole diagram (+). This interchanges the ordinary projections e and e' and we obtain

$$\begin{tikzcd} \varprojlim_r \mathrm{H}^1_{\text{ét,p}}(Y_1(Np^r) \times_{\mathbb{Z}} \overline{\mathbb{Q}}_p, \mathbb{Z}_p)^{\mathrm{ord}} \arrow[r, two heads] \arrow[d] & \varprojlim_r \mathrm{S}_2^{\mathrm{ord}}(X_1(Np^r)^{\mathrm{arith}}, \mathbb{Z}_p) \arrow[d] \\ \mathrm{H}^1_{\text{ét,p}}(Y_1(Np^r) \times_{\mathbb{Z}} \overline{\mathbb{Q}}_p, \mathrm{TSym}^{k-2}\, \mathrm{R}^1 f_* \underline{\mathbb{Z}_p})^{\mathrm{ord}} \arrow[r, two heads] & \mathrm{S}_k(X_1(Np^r)^{\mathrm{arith}}, \mathbb{Z}_p). \end{tikzcd}$$

Further we have to take the limit now along the morphisms Θ_{Np^{r+1},Np^r} (which we denoted also $\tilde{\mathrm{tr}}_r$, see remark 5.3).

Now in the bottom line, we can replace $\mathbb{Z}_p$ by $\mathbb{C}_p$; this allows us also to replace TSym by Sym (see appendix A.1). Of course the diagram as above still exists if we replace $\mathrm{H}^1_{\text{ét,p}}$ by $\mathrm{H}^1_{\text{ét,c}}$ in the left column. Then the upper left object is isomorphic to $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})$ by corollary 5.9 and proposition 4.16 and the upper right object is isomorphic to $\mathbb{S}^{\mathrm{ord}}(Np^\infty, \Lambda^{\mathrm{wt}})$ by corollary 5.5. We arrive at the diagram ($*$), which completes the proof of statement (a).

For statement (b), we cite that the claim is true if we use case 3 above, which is the original statement proved by Ohta, see [Oht95, Thm., p. 50]. Using [Wak14, §A.2.3, Thm. A.2] we transform this to case 1 and see that it remains true; we use here proposition 5.10 to see that the Galois action defined in [Wak14] coincides with the one we defined in section 4.5. $\square$

Chapter IV.

Periods and p-adic L-functions

In this chapter we can finally put everything together. The close study of modular curves and motives for modular forms on the one hand and modular symbols on the other hand allows us to compute the periods for these motives and to express them in terms of the error terms. It turns out that the complex period and the complex error term essentially match, while the p-adic versions differ by a constant which is a unit.

We can then modify Kitagawa's p-adic L-function by this unit and some other elements that deal with the remaining terms in the interpolation formula to get the p-adic L-function we want.

1. Complex L-functions of modular forms and their twists

So far we have not yet considered L-functions of modular forms. In this section we give the necessary background on these, both from a classical and a motivic point of view. Throughout the section we fix $N \geq 4$ and $k \geq 2$.

1.1. Classical complex L-functions, twists and refinements

Definition 1.1: The *complex* L*-function of a cusp form* $f \in \mathrm{S}_k(X_1(N), \mathbb{C})$ having Fourier expansion $q(f) = \sum_{n=1}^{\infty} a_n q^n$ is defined as

$$\mathrm{L}(f, s) := \sum_{n=1}^{\infty} a_n n^{-s}.$$

More generally, for a Dirichlet character $\chi\colon (\mathbb{Z}/c)^{\times} \longrightarrow \mathbb{C}^{\times}$ one defines the *twisted* L-*function* as

$$\mathrm{L}(f, \chi, s) := \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}.$$

The Dirichlet series introduced above converge for $\mathrm{Re}(s) > \frac{k}{2} + 1$ and the holomorphic function they define admits a holomorphic continuation to the whole of $\mathbb{C}$ and satisfy a functional equation [Shi71, Thm. 3.66]. Moreover, if (and only if) f is a normalised eigenform, there is an Euler product expansion

$$\mathrm{L}(f, \chi, s) = \prod_{\ell \text{ prime}} (1 - \chi(\ell) a_{\ell} p^{-s} + \psi \chi^2(\ell) \ell^{k-1-2s})^{-1},$$

see [DS05, Thm. 5.9.2]. Note that due to the convention that the values of χ at integers not coprime to c is 0, the Euler factors in the above product are 1 for primes $\ell \mid c$.

Proposition 1.2: *Let $f \in \mathrm{S}_k(X_1(N), \mathbb{C})$ have Fourier expansion $q(f) = \sum_{n=1}^{\infty} a_n q^n$ and let $\chi\colon (\mathbb{Z}/c)^\times \longrightarrow \mathbb{C}^\times$ is a primitive Dirichlet character, then the function f_χ defined by*

$$f_\chi(\tau) := \sum_{n=1}^{\infty} \chi(n) a_n q^n \quad (\tau \in \mathfrak{h})$$

is a modular form in $\mathrm{S}_k(X_1(M), \mathbb{C})$ for some multiple M of N. If f has nebentype ψ of conductor d, then $M = \operatorname{lcm}(N, c^2, cd)$ and f_χ has nebentype $\psi\chi^2$. One has

$$f_\chi(\tau) = \frac{1}{\mathrm{G}(\chi^*)} \sum_{j=1}^{c-1} \chi^*(j) f(\tau + \tfrac{j}{c}) \quad (\tau \in \mathfrak{h}).$$

Moreover, f_χ is a normalised eigenform if and only if f is a normalised eigenform.

Proof: See [Shi71, Prop. 3.64] and its proof for everything except the last statement. We thus know $\mathrm{L}(f, \chi, s) = \mathrm{L}(f_\chi, s)$. The last claim then follows from the fact that f is a normalised eigenform if and only if $\mathrm{L}(f, s)$ admits an Euler product, together with the observation that $\mathrm{L}(f, s)$ admits an Euler product if and only if $\mathrm{L}(f, \chi, s)$ does. □

A very useful fact is that the L-function can be expressed in terms of the Mellin transform of f as follows.

Proposition 1.3: *We have for all $s \in \mathbb{C}$*

$$\mathrm{L}(f, s) = -\frac{(-2\pi \mathrm{i})^s}{\Gamma(s)} \int_0^{\mathrm{i}\infty} f(z) z^{s-1} \mathrm{d}z.$$

In particular, for each $n \in \mathbb{N}$ we have

$$\int_0^{\mathrm{i}\infty} f(z) z^n \mathrm{d}z = (-1)^n \frac{n!}{(2\pi \mathrm{i})^{n+1}} \mathrm{L}(f, n+1).$$

Proof: In general, if $(a_n)_{n \in \mathbb{N}}$ is any sequence of complex numbers and we define functions

$$F(z) = \sum_{n=1}^{\infty} a_n z^n, \quad L(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (z \text{ resp. } s \in \mathbb{C} \text{ such that the series converge}),$$

then the relation

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} F(\mathrm{e}^{-t}) t^{s-1} \mathrm{d}t$$

holds whenever the convergence is such that this makes sense, see [Zag81, §3, (17)]. The claim follows from this by an easy substitution. □

Now fix a prime p and let $f \in \mathrm{S}_k(X_1(N), \psi, \mathbb{C})$ be a normalised eigenform (whose Fourier coefficients then lie in a number field). We study how L-functions behave under refinements. For this let $\alpha \in \mathbb{C}$ be a root of the p-th Hecke polynomial and f_α the corresponding refinement, as in section II.7.1. Then the L-functions of f and f_α are connected in the following way.

Proposition 1.4: *Assume that f is a normalised eigenform and that $\alpha \neq 0$. We have for all $s \in \mathbb{C}$ and all Dirichlet characters χ*

$$\mathrm{L}(f_\alpha, \chi, s) = (1 - \alpha^{-1}\chi\psi(p)p^{k-s-1})\mathrm{L}(f, \chi, s)$$

(where $\chi\psi$ is the product of the Dirichlet characters χ and ψ in the naive sense, i. e. $\chi\psi(p) = \chi(p)\psi(p)$).

Proof: In the untwisted situation, an easy calculation (see [Bel10, p. 131]) using the definitions of f_α and the complex L-function, the observation $\mathrm{L}(f\left[\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right], s) = p^{k-1-s}\mathrm{L}(f, s)$ and the relation $\alpha\beta = \psi(p)p^{k-1}$ shows that

$$\mathrm{L}(f_\alpha, s) = (1 - \alpha^{-1}\psi(p)p^{k-s-1})\mathrm{L}(f, s). \tag{$*$}$$

We now use proposition 1.2 to reduce the twisted case to this formula. Let f_χ be as in proposition 1.2, note that f_χ is again a normalised eigenform and recall that $\mathrm{L}(f, \chi, s) = \mathrm{L}(f_\chi, s)$. The p-th Hecke polynomial of f_χ is

$$X^2 - a_p\chi(p)X + \psi\chi^2(p)p^{k-1}, \tag{$**$}$$

and since α is a root of $X^2 - a_pX + \psi(p)p^{k-1}$, it is immediate that $\chi(p)\alpha$ is a root of $(**)$. Inserting this into $(*)$, we get $(1 - \alpha^{-1}\chi\psi(p)p^{k-s-1})\mathrm{L}(f_\chi, s) = \mathrm{L}((f_\chi)_{\chi(p)\alpha}, s)$ and it remains to see that $(f_\chi)_{\chi(p)\alpha} = (f_\alpha)_\chi$. This follows directly from corollary II.7.2 since both forms have the same Hecke eigenvalues. ☐

1.2. Complex L-functions and twists from the motivic point of view

We now turn to motives, so for the rest of the section fix a number field K, an embedding $K \hookrightarrow \overline{\mathbb{Q}}$, a newform $f \in \mathrm{S}_k(X_1(N), K)$ and a Dirichlet character χ.

Definition 1.5: The modular form f_χ from proposition 1.2 will in general not be new, but by theorem II.4.33 there exists a unique newform of some level dividing M almost all of whose Fourier coefficients are the same as the ones of f_χ. We call this newform the *twist of f by χ* and denote it by $f \otimes \chi$.

Although f_χ will not be new in general, it can be. See [AL78, Cor. 3.1] for a characterisation of when this happens.

Proposition 1.6: *We have an equality of complex L-functions*

$$\mathrm{L}(\mathcal{M}(f), s) = \mathrm{L}(f, s) \quad (s \in \mathbb{C})$$

of the motivic L-function as defined in section I.3.3 and the classical L-function attached to f. More generally, if χ is a Dirichlet character, then

$$\mathrm{L}(\mathcal{M}(f)(\chi^*), s) = \mathrm{L}(f \otimes \chi, s) \quad (s \in \mathbb{C}).$$

In particular, conjecture I.3.15 holds for $\mathcal{M}(f)$.

Proof: The first statement follows from theorem II.5.12 (b). From theorem I.2.26, Chebotarev's density theorem and the definition of $f \otimes \chi$ it follows easily that for each place $\mathfrak{p}$ of K the $\mathrm{G}_\mathbb{Q}$-representations $\mathcal{M}(f)_\mathfrak{p} \otimes \chi$ and $\mathcal{M}(f \otimes \chi)_\mathfrak{p}$ have isomorphic semisimplifications (see also example I.3.16 (c)). The second claim then follows from remark I.3.14. ☐

Now fix a primitive Dirichlet character χ of conductor $C \in \mathbb{N}$. As the above result shows, the L-functions $\mathrm{L}(\mathcal{M}(f)(\chi^*), s)$ and $\mathrm{L}(f, \chi, s)$ will *differ* in general. More precisely:

Proposition 1.7: *We have for $s \in \mathbb{C}$*

$$\mathrm{L}(\mathcal{M}(f)(\chi^*), s) = \mathrm{L}(f, \chi, s) \cdot \prod_{\ell | (N,C)} \mathrm{P}_\ell(\mathcal{M}(f)(\chi^*), s), \ell^{-s})^{-1}.$$

Proof: Obviously the function $\mathrm{L}(f, \chi, s)$ has trivial Euler factors at the primes dividing C. It is further easy to see that if ℓ is a prime dividing at most one of N and C, then the polynomial $\mathrm{P}_\ell(\mathcal{M}(f)(\chi^*)_\mathfrak{p}, T) = \mathrm{P}_\ell(\mathcal{M}(f)_\mathfrak{p} \otimes \chi^*, T)$ (for any prime $\mathfrak{p} \nmid \ell$) is just the Euler factor of $\mathrm{L}(f, \chi, s)$ at ℓ. This proves the first claim. $\square$

If f is ordinary at the place $\lambda \mid \ell$ (for a prime $\ell \mid N$) defined by our fixed embeddings $K \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ then we can explicitly compute the Euler factor of $f \otimes \chi$ at ℓ. To be consistent with other sections in this work, we slightly change our notation and denote the primes now by p and $\mathfrak{p}$ instead of ℓ and λ.

So assume that f is ordinary at $\mathfrak{p}$, $p \mid N$, and write Dp^m ($p \nmid D$, $m > 0$) for the conductor of χ. Let $\mathcal{M}(f)^0_\mathfrak{p}$ and δ be as in theorem II.5.15 and recall that $\delta(\mathrm{Frob}_p) = a_p$ since $p \mid N$. Further, if η is any Dirichlet character we write $\widetilde{\eta}$ for the associated primitive character.

Lemma 1.8: *With the above notation, we have*

$$\mathrm{P}_p(\mathcal{M}(f)(\chi^*), T) = 1 - \alpha^{-1}\widetilde{\chi\psi}(p)p^{k-1}T.$$

Proof: We choose a basis of $\mathcal{M}(f)_\mathfrak{p}$ with respect to which the representation has the form $\left(\begin{smallmatrix} \delta & * \\ & \varepsilon \end{smallmatrix}\right)$ with a character ε of $\mathrm{G}_\mathbb{Q}$. Looking at the determinant and using theorem II.5.12 (c), we get $\delta\varepsilon = \psi^*\kappa_{\mathrm{cyc}}^{1-k}$. We have an exact sequence

$$0 \longrightarrow \delta\chi^* \longrightarrow \mathcal{M}(f)(\chi^*)_\mathfrak{p} \longrightarrow \varepsilon\chi^* \longrightarrow 0$$

and we claim that the sequence

$$0 \longrightarrow \mathrm{D_{cris}}(\delta\chi^*) \longrightarrow \mathrm{D_{cris}}(\mathcal{M}(f)(\chi^*)_\mathfrak{p}) \longrightarrow \mathrm{D_{cris}}(\varepsilon\chi^*) \longrightarrow 0 \qquad (*)$$

is exact. It is clear that this sequence is left exact.

We tensor the first sequence with $\chi\varepsilon^*$ and obtain

$$0 \longrightarrow \delta\varepsilon^* \longrightarrow \mathcal{M}(f)(\varepsilon^*)_\mathfrak{p} \longrightarrow L \longrightarrow 0,$$

where L stands for the trivial representation. This is an extension of $\delta\varepsilon^*$ by the trivial representation, thus it defines a class $x \in \mathrm{H}^1(\mathbb{Q}_p, \delta\varepsilon^*)$. Since all representations in the sequence are de Rham, the sequence

$$0 \longrightarrow \mathrm{D_{dR}}(\delta\varepsilon^*) \longrightarrow \mathrm{D_{dR}}(\mathcal{M}(f)(\varepsilon^*)_\mathfrak{p}) \longrightarrow \mathrm{D_{dR}}(L) \longrightarrow 0$$

is still exact. By lemma I.2.46 (d) we therefore have $x \in \mathrm{H}^1_\mathrm{g}(\mathbb{Q}_p, \delta\varepsilon^*)$ and by lemma I.2.45 we have $\mathrm{H}^1_\mathrm{g}(\mathbb{Q}_p, \delta\varepsilon^*) = \mathrm{H}^1_\mathrm{f}(\mathbb{Q}_p, \delta\varepsilon^*)$ in this situation, whence by lemma I.2.46 (c) the sequence

$$0 \longrightarrow \mathrm{D_{cris}}(\delta\varepsilon^*) \longrightarrow \mathrm{D_{cris}}(\mathcal{M}(f)(\varepsilon^*)_\mathfrak{p}) \longrightarrow \mathrm{D_{cris}}(L) \longrightarrow 0$$

is exact. We now tensor this sequence with $\mathrm{D_{cris}}(\varepsilon\chi^*)$ and obtain a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{D_{cris}}(\delta\varepsilon^*)\otimes\mathrm{D_{cris}}(\varepsilon\chi^*) & \longrightarrow & \mathrm{D_{cris}}(\mathcal{M}(f)(\varepsilon^*)_{\mathfrak{p}})\otimes\mathrm{D_{cris}}(\varepsilon\chi^*) & \longrightarrow & \mathrm{D_{cris}}(\varepsilon\chi^*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{D_{cris}}(\delta\chi^*) & \longrightarrow & \mathrm{D_{cris}}(\mathcal{M}(f)(\chi^*)_{\mathfrak{p}}) & \longrightarrow & \mathrm{D_{cris}}(\varepsilon\chi^*) & & \end{array}$$

where the vertical arrows come from remark I.2.16 (they are even injective, but we do not need this). From this diagram we see that the lower right map is surjective, hence the sequence $(*)$ is exact.

Since δ is unramified and χ is ramified, we have $\mathrm{D_{cris}}(\delta\chi^*) = 0$. From the sequence $(*)$ we get then $\mathrm{D_{cris}}(\mathcal{M}(f)(\chi^*)_{\mathfrak{p}}) \cong \mathrm{D_{cris}}(\varepsilon\chi^*)$ and we need to determine when the character $\varepsilon\chi^*$ is crystalline, which depends only on its restriction to the inertia group I_p. Decompose the characters χ and ψ into p-power and prime-to-p conductor parts $\chi = \chi_p\chi_{\mathrm{nr}}$ and $\psi = \psi_p\psi_{\mathrm{nr}}$. We have $\varepsilon|_{\mathrm{I}_p} = (\delta\varepsilon)|_{\mathrm{I}_p} = (\psi^*\kappa_{\mathrm{cyc}}^{1-k})|_{\mathrm{I}_p}$ because δ is unramified, so $(\varepsilon\chi^*)|_{\mathrm{I}_p} = ((\psi\chi)^*\kappa_{\mathrm{cyc}}^{1-k})|_{\mathrm{I}_p}$ and we see that the character is crystalline if and only if $\psi_p = \chi_p^*$. In this case, by lemma I.2.20 φ_{cris} acts on $\mathrm{D_{cris}}(\delta\varepsilon\chi^*) = \mathrm{D_{cris}}((\psi_{\mathrm{nr}}\chi_{\mathrm{nr}})^*\kappa_{\mathrm{cyc}}^{1-k})$ by $\psi_{\mathrm{nr}}\chi_{\mathrm{nr}}(p)p^{k-1}$, hence it acts on $\mathrm{D_{cris}}(\varepsilon\chi^*)$ as $\delta(\mathrm{Frob}_p)^{-1}\psi_{\mathrm{nr}}\chi_{\mathrm{nr}}(p)p^{k-1} = \alpha^{-1}\psi_{\mathrm{nr}}\chi_{\mathrm{nr}}(p)p^{k-1}$ as claimed. Otherwise we get $\mathrm{D_{cris}}(M_{\mathfrak{p}}) = 0$ and the Euler factor is 1. □

Even without assuming ordinariness these Euler factors can be written down explicitly using the theory of automorphic forms, which we summarise briefly in theorem 5.4 below. This is not difficult, but since we will not need their precise value in the general case we omit this.

2. Some intermediate observations

In this section we compute some expressions that appear in the conjectural interpolation formula (apart from the periods) and prove some related lemmas.

In this whole section, fix a number field $K \subseteq \overline{\mathbb{Q}}$, integers $N \geq 4$ and $k \geq 2$, a newform $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, K)$, an integer n with $1 \leq n \leq k-1$, a prime $p \geq 3$ and a Dirichlet character χ of conductor Dp^m with $p \nmid D$. Further assume that f is ordinary at $\mathfrak{p}$, where $\mathfrak{p}$ is the place of K lying above p which is fixed by our embedding $K \subseteq \overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_p$. Write α for the unit root of the p-th Hecke polynomial (see definition II.5.16). Observe that $\alpha = a_p$ if $p \mid N$, where a_p is the p-th Hecke eigenvalue of f.

We look at the motive $M \coloneqq \mathcal{M}(f)(\chi^*)(n)$. It is critical by proposition II.8.7 and satisfies the strong Dabrowski-Panchishkin condition by theorem II.5.15 (b) and lemma I.3.30 (a).

Proposition 2.1: *Let $V \coloneqq M_{\mathfrak{p}}$. If k is even, $n = \frac{k}{2}$ and χ is nontrivial, assume that* $\mathrm{L}(f,\chi,n) \neq 0$. *Then* $\mathrm{H}^i_{\mathrm{f}}(\mathbb{Q}, V) = \mathrm{H}^i_{\mathrm{f}}(\mathbb{Q}, V^*(1)) = 0$ *for* $i = 0, 1$.

Proof: Using lemma I.2.48 it is clear that we have $\mathrm{H}^0_{\mathrm{f}}(\mathbb{Q}, V) = \mathrm{H}^0_{\mathrm{f}}(\mathbb{Q}, V^*(1)) = 0$ because V and $V^*(1)$ are irreducible by theorem II.5.12 (b). Since we know already that M is critical, we have

$$\dim_L \left(\mathrm{D_{dR}}(V|_{\mathrm{G}_{\mathbb{Q}_p}}) \big/ \mathrm{fil}^0\,\mathrm{D_{dR}}(V|_{\mathrm{G}_{\mathbb{Q}_p}}) \right) = \dim_L \mathrm{H}^0(\mathbb{R}, V)\ (= 1),$$

and the dimension formula from lemma I.2.51 tells us that $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V)$ vanishes if and only if $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V^*(1))$ does so. By proposition I.2.50, $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V) = 0$ if $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, V/T)$ is finite, where T is an $\mathcal{O}$-lattice in V. This finiteness result is proved in [Kato4, Thm. 14.2] under the hypothesis as in the statement. □

We compute the local correction factor which was introduced in definition I.3.39.

Lemma 2.2: *Let ψ be the nebentype of f. Then the local correction factor at p is*

$$\mathrm{LF}_p(M) = (1 - \alpha^{-1}\chi^*(p)p^{n-1})(1 - \alpha^{-1}\widetilde{\chi\psi}(p)p^{k-n-1}).$$

Here $\widetilde{\chi\psi}$ denotes again the primitive character associated to $\chi\psi$.

Proof: Let $\mathcal{M}(f)^0_{\mathfrak{p}}$, δ and α be as in theorem II.5.15. Then the action of $\mathrm{G}_{\mathbb{Q}_p}$ on $M^{\mathrm{DP}}_{\mathfrak{p}}$ is given by the character $\delta \otimes \chi^* \otimes \kappa^n_{\mathrm{cyc}}$. Recall from theorem II.5.12 (b) that

$$\mathrm{P}_p(\mathcal{M}(f)_{\mathfrak{p}}, T) = 1 - a_pT + \psi(p)p^{k-1}T^2 \tag{$*$}$$

and note that the roots of this polynomial are the inverses of the roots of the Hecke polynomial, in particular α^{-1} is a root.

We distinguish two cases.

(1) We first assume that $m = 0$, i. e. χ is unramified at p. As a first step we compute the expression

$$\frac{\mathrm{P}_p(M_{\mathfrak{p}}, T)}{\mathrm{P}_p(M^{\mathrm{DP}}_{\mathfrak{p}}, T)}$$

occurring in the definition of the local factor. For this we distinguish again two cases.

(i) First let $p \nmid N$. From $(*)$ we see that $\mathcal{M}(f)_{\mathfrak{p}}$ is crystalline in this case since $\psi(p) \neq 0$, and hence also $M_{\mathfrak{p}}$ is crystalline. We have then $\mathrm{D}_{\mathrm{cris}}(M_{\mathfrak{p}}) = \mathrm{D}_{\mathrm{cris}}(\mathcal{M}(f)_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(\chi^*) \otimes_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(\mathbb{Q}_p(n))$, and from lemma I.2.20 we get

$$\mathrm{P}_p(M_{\mathfrak{p}}, T) = 1 - a_p\chi(p)p^{-n}T + \psi\chi^2(p)p^{k-2n-1}T^2.$$

On the other hand, by the above description of $M^{\mathrm{DP}}_{\mathfrak{p}}$ and again lemma I.2.20 we know that

$$\mathrm{P}_p(M^{\mathrm{DP}}_{\mathfrak{p}}, T) = 1 - \alpha\chi(p)p^{-n}T.$$

We claim that the quotient of these polynomials is

$$\frac{\mathrm{P}_p(M_{\mathfrak{p}}, T)}{\mathrm{P}_p(M^{\mathrm{DP}}_{\mathfrak{p}}, T)} = 1 - \alpha^{-1}\psi\chi(p)p^{k-n-1}T.$$

Indeed, multiplying we get

$$(1 - \alpha^{-1}\psi\chi(p)p^{k-n-1}T)(1 - \alpha\chi(p)p^{-n}T) = \\ 1 - \chi(p)p^{-n}(\alpha + \alpha^{-1}\psi(p)p^{k-1})T + \psi\chi^2(p)p^{k-2n-1}T^2$$

and since α^{-1} is a zero of $\mathrm{P}_p(\mathcal{M}(f)_{\mathfrak{p}}, T)$, it follows that $\alpha + \alpha^{-1}\psi(p)p^{k-1} = a_p$.

(ii) Now we let $p \mid N$. Then $\psi(p) = 0$ and $(*)$ tells us that $\mathcal{M}(f)_{\mathfrak{p}}$ is not crystalline, so $M_{\mathfrak{p}}$ is not crystalline either (since χ and $\kappa_{\mathrm{cyc}}^{n}$ are crystalline). But $\mathcal{M}(f)_{\mathfrak{p}}^{0}$ is crystalline by theorem II.5.15 (b) and lemma I.2.19 (b), hence so is $M_{\mathfrak{p}}^{\mathrm{DP}}$ and we get $\mathrm{D}_{\mathrm{cris}}(M_{\mathfrak{p}}) = \mathrm{D}_{\mathrm{cris}}(M_{\mathfrak{p}}^{\mathrm{DP}})$ for dimension reasons. Thus the expression

$$\frac{\mathrm{P}_p(M_{\mathfrak{p}}, T)}{\mathrm{P}_p(M_{\mathfrak{p}}^{\mathrm{DP}}, T)}$$

appearing in the definition of LF_p is equal to 1 in this case.

It remains to compute $\mathrm{P}_p((M_{\mathfrak{p}}^{\mathrm{DP}})^*(1), T)$, still assuming that χ is unramified at p. We have $(M_{\mathfrak{p}}^{\mathrm{DP}})^*(1) = (\delta \otimes \chi^* \otimes \kappa_{\mathrm{cyc}}^{n})^*(1) = \delta^{-1} \otimes \chi \otimes \kappa_{\mathrm{cyc}}^{1-n}$, so

$$\mathrm{P}_p((M_{\mathfrak{p}}^{\mathrm{DP}})^*(1), T) = 1 - \alpha^{-1}\chi^*(p)p^{n-1}T$$

by lemma I.2.20.

(2) Now we let $m > 0$. Since χ is then ramified at p, the one-dimensional representations $M_{\mathfrak{p}}^{\mathrm{DP}}$ and $(M_{\mathfrak{p}}^{\mathrm{DP}})^*(1)$ are both not crystalline, so $\mathrm{P}_p(M_{\mathfrak{p}}^{\mathrm{DP}}, T) = \mathrm{P}_p((M_{\mathfrak{p}}^{\mathrm{DP}})^*(1), T) = 1$. The value of the remaining expression $\mathrm{P}_p(M_{\mathfrak{p}}, 1)$ was computed in lemma 1.8. ☐

Remark 2.3: In [MTT86, §14], Mazur, Tate and Teitelbaum introduce an expression they call the "p-adic multiplier". They consider the same situation as we do here, and their p-adic multiplier is (up to a power of α which we ignore here) defined as

$$e_p(M) := (1 - \alpha^{-1}\chi^*(p)p^{n-1})(1 - \alpha^{-1}\chi\psi(p)p^{k-n-1}), \tag{2.1}$$

so by our above calculation it essentially equals the local correction factor, except that it contains $\chi\psi(p) = \chi(p)\psi(p)$ instead of the value at p of the associated primitive character. We will later need to know when the two expressions differ. Recall that we always assumed χ to be primitive (while ψ may be imprimitive). Write χ_p and ψ_p for the p-parts of χ and ψ, respectively. Then we have $e_p(M) \neq \mathrm{LF}_p(M)$ if and only if $\chi_p\psi_p(p) \neq \widetilde{\chi_p\psi_p}(p)$, and it is elementary to check that this happens precisely when ψ_p and χ_p are nontrivial and inverse to each other. In these cases, we have

$$\begin{aligned} e_p(M) &= (1 - \alpha^{-1}\chi^*(p)p^{n-1}), \\ \mathrm{LF}_p(M) &= (1 - \alpha^{-1}\chi^*(p)p^{n-1})(1 - \alpha^{-1}\widetilde{\chi\psi}(p)p^{k-n-1}). \end{aligned}$$

Next we calculate the ε-factor introduced in definition I.3.23.

Lemma 2.4: *Write the Dirichlet character χ as a product $\chi = \chi_{\mathrm{nr}}\chi_p$ with χ_{nr} of conductor D and χ_p of conductor p^m. Then the ε-factor is*

$$\varepsilon(M_{\mathfrak{p}}^{\mathrm{DP}}) = \alpha^m \chi_{\mathrm{nr}}(p)^m p^{-nm}\mathrm{G}(\chi_p).$$

Proof: Since the action of $\mathrm{G}_{\mathbb{Q}_p}$ on $M_{\mathfrak{p}}^{\mathrm{DP}}$ is given by the character $\delta \otimes \chi^* \otimes \kappa_{\mathrm{cyc}}^{n}$ and $\delta(\mathrm{Frob}_p) = \alpha$, this follows directly from proposition I.3.25. ☐

Finally we calculate the Hodge invariant.

Lemma 2.5: *The Hodge invariant is* $\mathrm{t_H}(M_\mathfrak{p}^{\mathrm{DP}}) = -n$.

Proof: Since $\dim_L \mathrm{D_{dR}}(M_\mathfrak{p}^{\mathrm{DP}}) = \dim_L \mathrm{fil}^0 \mathrm{D_{dR}}(M_\mathfrak{p}) = 1$ and

$$\mathrm{D_{dR}}(M_\mathfrak{p}^{\mathrm{DP}}) \xrightarrow{\sim} \mathrm{D_{dR}}(M_\mathfrak{p}) \big/ \mathrm{fil}^0 \mathrm{D_{dR}}(M_\mathfrak{p}) ,$$

we must have $\mathrm{D_{dR}}(M_\mathfrak{p}^{\mathrm{DP}}) \cap \mathrm{fil}^0 \mathrm{D_{dR}}(M_\mathfrak{p}) = 0$. Further the functor $\mathrm{D_{dR}}$ is exact on de Rham representations by [FO08, Thm. 5.28] and $\mathrm{D_{dR}}(M_\mathfrak{p}^{\mathrm{DP}})$ is a subobject of $\mathrm{D_{dR}}(M_\mathfrak{p})$ as a filtered vector space. It follows that $\mathrm{t_H}(M_\mathfrak{p}^{\mathrm{DP}})$ must be the unique $i \in \mathbb{Z}$ such that $\mathrm{fil}^i \mathrm{D_{dR}}(M_\mathfrak{p}) = \mathrm{D_{dR}}(M_\mathfrak{p})$ and $\mathrm{fil}^{i+1} \mathrm{D_{dR}}(M_\mathfrak{p}) \neq \mathrm{D_{dR}}(M_\mathfrak{p})$. It is easy to see that this means $\mathrm{t_H}(M_\mathfrak{p}^{\mathrm{DP}}) = -n$. ☐

Corollary 2.6: *The pairs (χ^*, n), where χ is any Dirichlet character and $1 \leq n \leq k-1$ are an appropriate pair for $\mathcal{M}(f)$ in the sense of definition I.3.40, except possibly in the following cases:*

(1) k is even, $n = \frac{k}{2}$ and $\mathrm{L}(f, \chi, n) = 0$, in which we may have $\mathrm{H}^1_\mathrm{f}(\mathbb{Q}, M_\mathfrak{p}) \neq 0$;

(2) $k = 2$, $n = 1$, $p \parallel N$, $m = 0$ and $\chi^*(p) = a_p$, in which case we have $\mathrm{LF}_p(M) = 0$.

Proof: The motive $\mathcal{M}(f)(\chi^*)(n)$ is then critical, and by proposition 2.1 the necessary Galois cohomology groups vanish. It remains to see that the local correction factor, which we computed in lemma 2.2, does not vanish. Since α is a p-adic unit, one sees immediately that it can only possibly vanish if $n = 1$ or $n = k-1$ and α is a root of unity. To sort this out, we use an argumentation inspired from [MTT86, §15, p. 22]. We distinguish two cases:

(1) Assume $p \mid N$ (so that $\alpha = a_p$). Then [Miy89, Thm. 4.6.17] shows that there are two possible cases that can occur (since $a_p \neq 0$): either the archimedean absolute value of a_p is $p^{(k-1)/2}$ or we have $a_p^2 = \widetilde{\psi}(p)p^{k-2}$, $p \parallel N$ and the p-part of ψ is trivial. The first case is impossible because in order for a_p to be a root of unity we would need $k = 1$, which we excluded. In the other case we would need $k = 2$ for the same reason, and the local factor then vanishes if and only if $a_p = \chi^*(p)$ or $a_p = \widetilde{\chi\psi}(p)$. Since χ is primitive and the p-part of ψ is trivial, we have $\widetilde{\chi\psi}(p) = \chi(p)\widetilde{\psi}(p)$, and inserting $a_p^2 = \widetilde{\psi}(p)$ we see that $a_p = \widetilde{\chi\psi}(p)$ is in fact equivalent to $a_p = \chi^*(p)$. Hence we are in the situation (2) above.

(2) Assume $p \nmid N$. Then by the generalised Ramanujan conjecture proved by Deligne ([Del69, Thm. 5.1]; see also [Con09, Lem. 0.0.0.3 and §5.4]), all archimedean absolute values of α are $p^{(k-1)/2}$, so as in the previous case we would need $k = 1$ for α to be a root of unity, which we excluded. ☐

3. A problem with interpolation formulas in the literature

Before we continue, we insert a section where we observe that there are contradicting interpolation formulas for p-adic L-functions for modular forms, both in conjectures and in actual constructions. To exhibit this, we look at the construction by Mazur, Tate and Teitelbaum on the one hand and of Kitagawa on the other hand.

Since this paragraph only aims to point out this contradiction, we work in a simplified setting. Fix $p > 2$, $k \geq 2$, $N \geq 4$, a number field K, an embedding $K \hookrightarrow \overline{\mathbb{Q}}$ fixing a prime $\mathfrak{p} \mid p$, and let L be the completion of K at $\mathfrak{p}$, $\mathcal{O}$ the ring of integers of L and put $\Lambda^{\mathrm{cyc}} := \mathcal{O}[\![\mathrm{G_{cyc}}]\!]$.

Let $f \in \mathrm{S}_k(X_1(N)^{\mathrm{arith}}, K)$ be a $\mathfrak{p}$-ordinary newform and $a_p \in \mathcal{O}^\times$ its p-th Hecke eigenvalue. We assume that $p \mid N$, so that a_p is the unit root of the p-th Hecke polynomial of f (see theorem II.5.15 (c)).

We make some further simplifying assumptions. In the following, both L-functions $\mathrm{L}(f, \chi, s)$ and $\mathrm{L}(\mathcal{M}(f)(\chi^*), s)$ (where χ is a Dirichlet character) will appear, and by proposition 1.7 these differ by finitely many Euler factors for the primes dividing both the level N of f and the conductor of χ. To minimise this discrepancy, we will only consider nontrivial Dirichlet characters of p-power conductor, so that the only such prime is p. We will further assume that f has trivial nebentype. By lemma 1.8, the Euler factor of $\mathrm{L}(\mathcal{M}(f)(\chi^*), s)$ is then also trivial, which ensures that in this case have do have an equality of L-functions $\mathrm{L}(\mathcal{M}(f)(\chi^*), s) = \mathrm{L}(f, \chi, s)$.

3.1. Contradicting interpolation formulas

We cite the aforementioned interpolation formulas, in which $n \in \mathbb{N}$ should satisfy $0 \leq n \leq k-2$ and χ is a Dirichlet character of conductor p^m which we assume to be nontrivial for simplicity.

Mazur, Tate and Teitelbaum [MTT86, §14] construct a measure $\mu_{\mathrm{MTT}} \in \Lambda^{\mathrm{cyc}}$ such that for all such χ and n

$$\int_{\mathrm{G}_{\mathrm{cyc}}} \chi^* \kappa_{\mathrm{cyc}}^n \mathrm{d}\mu_{\mathrm{MTT}} = \frac{p^{m(n+1)} n!}{a_p^m (-2\pi \mathrm{i})^n \mathrm{G}(\chi) \mathcal{E}_\infty(f)} \mathrm{L}(f, \chi, n+1).$$

On the other hand, Kitagawa [Kit94, Thm. 1.1, Thm. 4.8] constructs a measure $\mu_{\mathrm{Kit}} \in \Lambda^{\mathrm{cyc}}$ such that for all such χ and n

$$\int_{\mathrm{G}_{\mathrm{cyc}}} \chi^* \kappa_{\mathrm{cyc}}^n \mathrm{d}\mu_{\mathrm{Kit}} = \frac{(-1)^n p^{nm} \mathrm{G}(\chi^*) n!}{a_p^m (2\pi \mathrm{i})^n \mathcal{E}_\infty(f)} \mathrm{L}(f, \chi, n+1).$$

Here $\mathcal{E}_\infty(f)$ is a the complex error term similar as in definition III.2.8 which depends of course on a choice of $\eta_f^\pm$ as there, but since this will not be important in what follows, we suppress it in the notation.[1]

If we divide the two expressions above we obtain

$$\frac{\text{MTT's value}}{\text{Kitagawa's value}} = \frac{p^m}{\mathrm{G}(\chi)\mathrm{G}(\chi^*)} = \chi(-1) \tag{3.1}$$

by the classical Gauß sum relation $\mathrm{G}(\chi)\mathrm{G}(\chi^*) = \chi(-1)p^m$ (see [Miy89, Lem. 3.1.1 (2)]).

The following lemma shows that only one of the two functions $\mu_{\mathrm{MTT}} \in \Lambda^{\mathrm{cyc}}$ and $\mu_{\mathrm{Kit}} \in \Lambda^{\mathrm{cyc}}$ can exist (recall that we assumed $k > 2$), so one of the two articles must contain an error.

Lemma 3.1: *Let $Q = \operatorname{Quot} \Lambda^{\mathrm{cyc}}$ be the total ring of fractions. Fix a subset $A \subseteq \mathbb{Z}$ containing at least one even and one odd number. There cannot exist an element $\mu \in Q$ such that for each finite order character χ of $\mathrm{G}_{\mathrm{cyc}}$ and each $n \in A$ we have*

$$\int_{\mathrm{G}_{\mathrm{cyc}}} \chi^* \kappa_{\mathrm{cyc}}^n \mathrm{d}\mu = \chi(-1)$$

[1] To be precise, Mazur, Tate and Teitelbaum do not use any error terms, but rather they identify $\mathbb{C}$ with $\mathbb{C}_p$, but dividing their function by the error terms yields μ_{MTT} as above. On the other hand one should remark that Kitagawa constructs a two-variable p-adic L-function for a Hida family; the element μ_{Kit} above is the specialisation of this function to one form in the family.

(whenever this integral is defined).

Proof: For χ a finite order character of $\mathrm{G}_{\mathrm{cyc}}$ and $n \in A$, let $\Phi_{n,\chi}\colon \Lambda^{\mathrm{cyc}} \longrightarrow \overline{\mathbb{Q}}_p$ be the morphism induced by $\chi^*\kappa_{\mathrm{cyc}}^n$ and put $P_{n,\chi} := \ker \Phi_{n,\chi} \in \operatorname{Spec} \Lambda^{\mathrm{cyc}}$. By [NSW13, Prop. 5.3.5] the ring Λ^{cyc} is isomorphic to a direct sum of $p-1$ copies of the power series ring $\mathcal{O}[\![T]\!]$, and since $\overline{\mathbb{Q}}_p$ is a domain, each $\Phi_{n,\chi}$ will factor over one of these copies. Decomposing $\mathrm{G}_{\mathrm{cyc}} \cong \mathbb{F}_p{}^\times \times (1 + p\mathbb{Z}_p)$, each character $\chi^*\kappa_{\mathrm{cyc}}^n$ determines a character of $\mathbb{F}_p{}^\times$, which is a power $\omega^{i(n,\chi)}$ of the Teichmüller character ω for some exponent $i(n,\chi) \in \{1,\ldots,p-1\}$ depending on n and χ. This exponent determines the copy of $\mathcal{O}[\![T]\!]$ over which $\Phi_{n,\chi}$ factors, or phrasing it more geometrically, the connected component of $\operatorname{Spec} \Lambda^{\mathrm{cyc}}$ that $P_{n,\chi}$ lies on. It follows from lemma III.3.1 (b) that on each connected component of $\operatorname{Spec} \mathcal{O}[\![T]\!]$, any infinite set of $P_{n,\chi}$ contained in it is Zariski dense. But for each fixed $i \in \{1,\ldots,p-1\}$ one can find infinitely many pairs (n,χ) such that $i(n,\chi) = i$ and $\chi(-1) = 1$ and also infinitely many pairs (n,χ) such that $i(n,\chi) = i$ and $\chi(-1) = -1$. Therefore such a μ cannot exist in Λ^{cyc}.

By the Weierstraß preparation theorem [NSW13, (5.3.4)] any element of Λ^{cyc} can have only finitely many zeros in $\operatorname{Spec} \Lambda^{\mathrm{cyc}}$. Therefore if we had such a μ in Q, then the integral in the statement would be defined for all but finitely many pairs (χ, n). Since excluding finitely many pairs obviously does not destroy the argument from before, such an element also cannot exist in Q. $\square$

The problem also occurs with other texts. For each of the texts listed below, one can check that its interpolation formula may be transformed by some easy steps into either Mazur-Tate-Teitelbaum's one or Kitagawa's one. The following table gives an overview of which text contains which version. Probably one can find more examples for each version.

same as Mazur-Tate-Teitelbaum	**same as Kitagawa**
Vishik [Vis76]	Amice-Velu [AV75]
Pollack-Stevens [PS11], [PS13]	Kato [Kat04]
Bellaïche [Bel11]	Hida [HidLFE]
Delbourgo [Del08]	

In appendix B we reproduce Kitagawa's construction. The final result we obtain (theorem B.1.11) *differs* from our citation of Kitagawa's result above, in fact it coincides with the result by Mazur, Tate and Teitelbaum. This suggests that the latter construction is the correct one, and indeed there seems to be a mistake in Kitagawa's paper.

To point out this error, note that the formula in [Kit94, Prop. 4.7] (or alternatively, [Kit94, Thm. 6.2]) contains a factor $(-\Delta)^\nu$, where Δ is the non-p part of the conductor of χ, so it is 1 in our special case (the ν there is what we called n). The same formula contains an expression $A(\xi, \chi, \nu+1)$. The meaning of this latter expression is defined in [Kit94, §4.1], where the defining formula also contains a factor $(-1)^\nu$. Hence if one inserts the definition of $A(\xi, \chi, \nu+1)$ into the formula in [Kit94, Prop. 4.7] (or alternatively, [Kit94, Thm. 6.2]), the two factors $(-1)^\nu$ should disappear. However, in the final formula in [Kit94, Thm. 1.1], the factor $(-\Delta)^\nu$ is still present, while $(-1)^\nu$ is not. It is this formula which we cited above, and by our explanations it seems to be wrong. Apparently the corresponding statements in the texts in the right column of the above table are then also wrong.

3.2. Comparison with the general conjectures

We want to compare the interpolation formulas from the previous section to general conjectures about p-adic L-functions for motives. One such conjecture is of course the one by Fukaya and Kato, which we cited in conjecture 1.3.41. On the other hand there are also older conjectures due to Coates, Perrin-Riou [CP89; Coa89]. In the following we show that one conjecture predicts the Mazur-Tate-Teitelbaum function, while the other one predicts Kitagawa's function, so these conjectures also seem to contradict each other.

3.2.1. The conjecture of Coates and Perrin-Riou

We follow the text [Coa89].[2] The conjecture there assumes that one has a critical motive M which is good ordinary at p. Let ρ be one of $\pm\mathrm{i} \in \mathbb{C}$. Then there should be a p-adic L-function μ_{CP} such that for all Dirichlet characters χ of p-power order and all $n \in \mathbb{Z}$ such that $M(\chi^*)(n)$ is still critical and such that $\chi(-1) = (-1)^n$, we have[3]

$$\int_{\mathrm{G_{cyc}}} \chi^* \kappa_{\mathrm{cyc}}^n \mathrm{d}\mu_{\mathrm{CP}} = \frac{\Lambda^{(\rho)}_{(p,\infty)}(M(n)(\chi^*))}{\Omega^{(\rho)}(M)},$$

where for a motive N

$$\Lambda^{(\rho)}_{(p,\infty)}(N) = \frac{\mathcal{L}^{(\rho)}_\infty(N)\mathcal{L}^{(\rho)}_p(N)}{L^{(\rho)}_\infty(N)L^{(\rho)}_p(N)}\Lambda(N,0)$$

and

$$\Omega^{(\rho)}(M) = \Omega_\infty(M)(2\pi\rho)^{r(M)};$$

see below for the remaining unexplained expressions. Here $\Omega_\infty(M)$ is the complex period of M (see definition 1.3.18; we ignore here the dependence on a choice of a basis) and $\Lambda(N,s)$ is the completed complex L-function of N.

We now apply this to $M = \mathcal{M}(f)(1)$. Fix a character χ of conductor p^m and $n \in \{0,\dots,k-2\}$. To simplify the notation we set $r = n+1$, such that $\mathcal{M}(f)(1)(n) = \mathcal{M}(f)(r)$. Again for simplicity we assume χ to be non-trivial, i. e. $m > 0$. Following [Coa89, p. 109], we define the Gauß sum

$$\mathrm{G}_\rho(\chi) = \sum_{x \bmod p^m} \chi(x)\, \mathrm{e}^{-2\pi\rho x p^{-m}},$$

such that $\mathrm{G}_{-\mathrm{i}}(\chi)$ is the usual Gauß sum.

We will not repeat here the definitions of the remaining expressions from above. Rather we give a table (on page 188) that lists their values in our particular situation, together with a brief explanation how to obtain them. The references in the table to pages, statements or objects we did not define here refer to [Coa89].

[2] Note that the earlier version of the conjecture from [CP89] seemed to contain an error, as explained by Coates at the beginning of the article [Coa89], whose purpose is mainly to correct this error. We follow here the later, corrected version.

[3] In comparison to [Coa89], we have replaced χ by χ^*. The reason for this lies in the normalisation of class field theory: in [Coa89], the reciprocity map sends primes to *arithmetic* Frobenii, while L-functions are defined by taking characteristic polynomials of *geometric* Frobenii (just as in this work). Since we want to consider the L-function $\mathrm{L}(f,\chi,s) = \mathrm{L}(\mathcal{M}(f)(\chi^*),s)$, we therefore have to twist by χ^* instead of χ.

expression	**definition in [Coa89]**	**value**
$h(j,k)$ (for the motive $\mathcal{M}(f)(\chi^*)(r)$)	bottom of p. 103	$h(j,k) = 1$ if $j = -r$ or $j = k-1-r$ and $h(j,k) = 0$ otherwise, by proposition II.5.9
$\tau(\mathcal{M}(f)(\chi^*)(r))$	eqn. (12), p. 106	$-r$
$\dfrac{\mathcal{L}_\infty^{(\rho)}(\mathcal{M}(f)(\chi^*)(r))}{L_\infty^{(\rho)}(\mathcal{M}(f)(\chi^*)(r))}$	p. 103/104	In the definition, we have a product over certain U, but in our case there is only one such U, and we are in the situation of (a) at the bottom of p. 103. Then the j there is $-r$, the k there is $k-1-r$ (the latter k being the weight of f) and $h(j,k) = 1$, such that the expression is equal to ρ^{-r}.
$P(\mathcal{M}(f)(r))$	middle of p. 108	The polynomial $\det(1-\mathrm{Frob}_p X, \mathcal{M}(f)_\ell)$ is equal to $1 - a_p X$ by theorem II.5.12 (b). Thus $\det(1-\mathrm{Frob}_p X, \mathcal{M}(f)(r)_\ell) = 1 - a_p p^{-r} X$. The set $P(\mathcal{M}(f)(r))$ is the set of inverse roots of the latter polynomial, thus it contains one element $a_p p^{-r}$.
$h_p(\mathcal{M}(f)(r))$	p. 109, before Lemma 3	By the above description of $P(\mathcal{M}(f)(r))$, this is equal to 1.
$\dfrac{\mathcal{L}_p^{(\rho)}(\mathcal{M}(f)(\chi^*)(r))}{L_p^{(\rho)}(\mathcal{M}(f)(\chi^*)(r))}$	eqn. (18), p. 109	We use [Coa89, Lem. 3 (ii)] with the M there being $\mathcal{M}(f)(r)$. By our above descriptions of $h_p(\mathcal{M}(f)(r))$ and $P(\mathcal{M}(f)(r))$ we get $\mathrm{G}_\rho(\chi)^{-1}(a_p p^{-r})^{-m}$.
$\mathrm{L}_\infty(\mathcal{M}(f)(\chi^*)(r))$	bottom of p. 103 and bottom of p. 104	In the definition, we have a product over certain U, but in our case there is only one such U, and j, k and $h(j,k)$ there are as before. Hence the expression equals $\Gamma_\mathbb{C}(r) = 2(2\pi)^{-r}\Gamma(r) = 2(2\pi)^{-r}(r-1)!$.
$\Lambda(\mathcal{M}(f)(\chi^*)(r))$		By definition, this is $\mathrm{L}_\infty(\mathcal{M}(f)(\chi^*)(r)) \cdot \mathrm{L}(\mathcal{M}(f)(\chi^*)(r)) = 2(2\pi)^{-r}(r-1)!\mathrm{L}(f,\chi,r)$.

Using the calculations from the table, we obtain the following conjectural interpolation formula: If $\chi(-1) = (-1)^n$ we should have (recall that $r = n + 1$)

$$\int_{\mathrm{G}_{\mathrm{cyc}}} \chi^* \kappa_{\mathrm{cyc}}^n \mathrm{d}\mu_{\mathrm{CP}} = \frac{p^{m(n+1)} 2(n-1)!}{a_p^m (2\pi\rho)^n \mathrm{G}_\rho(\chi) \Omega_\infty(\mathcal{M}(f)(1))} \mathrm{L}(f, \chi, n+1).$$

It can be shown that the Deligne period $\Omega_\infty(\mathcal{M}(f)(1))$ is equal, under suitable normalisations, to the appropriate error term $\mathcal{E}_\infty(f)$ from before (this follows from theorem 4.1 below; see also appendix A.3). Hence if we choose $\rho = -\mathrm{i}$, then up to a factor 2 this is precisely the value of the function of Mazur-Tate-Teitelbaum.

3.2.2. The conjecture of Fukaya and Kato

We now look at the conjecture by Fukaya and Kato, here we use the formulation from [FK06, Thm. 4.2.26]. We choose M there to be $\mathcal{M}(f)(1)$ and ρ there to be a non-trivial Dirichlet character χ of conductor p^m. Further we fix again n (there called j) with $0 \le n \le k-2$ and assume $\chi(-1) = (-1)^n$ for simplicity. We point out that in [FK06], just as in [Coa89], the normalisation of class field theory is via the arithmetic convention and Euler factors are defined using characteristic polynomials of geometric Frobenii. The table on page 190 contains the values of the relevant expressions in the interpolation formula. (We later study this in greater detail.)

Putting all this together, we see that the conjecture predicts a μ_{FK} such that[4]

$$\int_{\mathrm{G}_{\mathrm{cyc}}} \chi^* \kappa_{\mathrm{cyc}}^n \mathrm{d}\mu_{\mathrm{FK}} = \frac{p^{m(n+1)}(n-1)!}{a_p^m \Omega_\infty(\mathcal{M}(f)(1))(2\pi\mathrm{i})^n \mathrm{G}(\chi)} \mathrm{L}(f, \chi, n+1).$$

This equals the value from Mazur-Tate-Teitelbaum's interpolation formula *up to a factor* $(-1)^n$, so by (3.1) it equals the value from Kitagawa's interpolation formula up to a factor $(-1)^n\chi(-1)$, which is 1 in our case. If we choose instead χ and n such that $\chi(-1) = -(-1)^n$, then the calculation still works, so that in both cases Fukaya-Kato's conjectural value equals Kitagawa's value up to a factor $\chi(-1)(-1)^n$ (we omit the details). Of course there is an element in Λ^{cyc} whose integrals are this factor (namely the complex conjugation in $\mathrm{G}_{\mathrm{cyc}}$).

3.3. Conclusion

We have seen that the conjectures of Coates and Perrin-Riou on the one hand and of Fukaya and Kato on the other hand seem to contradict each other, and that the former seems to be the correct one (at least in the setting of modular forms) and the latter yields a p-adic L-function whose construction contains an error. Therefore we cannot hope to prove Fukaya's and Kato's conjecture in our situation, i. e. to construct a p-adic L-function having exactly their predicted interpolation behaviour, but only up to a factor of $\chi(-1)$. At least this can be achieved, as we show in the following sections.

A solution to this problem that was suggested by Y. Zaehringer [Zae17, Ex. 9.2.14] is to replace the character ψ in the table on page 190 by its inverse. This indeed makes the unwanted factor $\chi(-1)$ disappear. This is equivalent to changing a convention we made at the very beginning: on page xvii we required $\mathbb{C}$ and $\mathbb{C}_p$ to be "oriented compatibly" (see

[4] Actually the integral there is over $\chi\kappa_{\mathrm{cyc}}^{-n}$, but the involution on Λ^{cyc} induced by inversion on $\mathrm{G}_{\mathrm{cyc}}$ transforms this one into the integral here.

expression	definition in [FK06]	value
$d(\chi, n, \pm)$	in Thm. 4.2.26	We have $d(\chi, n, +) = 1$ and $d(\chi, n, -) = 0$ because $\chi(-1) = (-1)^n$.
$\Omega(\gamma^+, \delta)$	in Thm. 4.2.26	This is by definition the Deligne period of $\mathcal{M}(f)(1)$, which we called $\Omega_\infty(\mathcal{M}(f)(1))$.
Υ	§4.2.13, p. 62	$\varnothing$
$f_p(\rho)$	in Thm. 4.2.26	m
$h(l)$ for $l \in \mathbb{Z}$	p. 58, in Thm. 4.1.12 (2)	This is 1 if $l = -n$ or $l = k - 1 - n$ and 0 otherwise, by proposition II.5.9.
$P_{L,p}(\cdots)$	§4.2.19, p. 66	This is 1 by lemma 2.2.
M_λ^0	§4.2.3, comes from the Dabrowski-Panchishkin condition	This is $\mathcal{M}(f)_\mathrm{p}^0(1)$, where $\mathcal{M}(f)_\mathrm{p}^0$ is the 1-dimensional unramified $\mathrm{G}_{\mathbb{Q}_p}$-subrepresentation $\mathcal{M}(f)_\mathrm{p}^0$ of $\mathcal{M}(f)_\mathrm{p}$ from theorem II.5.15 on which the geometric Frobenius at p acts as a_p.
ν	in Thm. 4.2.26	This is the eigenvalue of the crystalline Frobenius on $\mathrm{D}_\mathrm{cris}(M_\lambda^0)$. By the above description, this is equal to $a_p p^{-1}$.
ψ	p. 52, beginning of chapter 4	$\psi\colon \mathbb{Q}_p \longrightarrow \overline{\mathbb{Q}}_p^\times$ is the character with kernel $\mathbb{Z}_p$ such that $\psi(p^{-l}) = \mathrm{e}^{2\pi \mathrm{i} p^{-l}}$ for all $l > 0$ under the chosen embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$
$\varepsilon(\mathbb{Q}_p, \chi^*, \psi)$	§3.2.2, p. 40	By property (7) in §3.2.2 on p. 41 (or alternatively lemma 2.4), this equals the usual Gauß sum $\mathrm{G}(\chi)$.
d	in Thm. 4.2.26	This is $\dim_K \mathrm{t}_{\mathcal{M}(f)(1)} = 1$.

footnote 1 there), i. e. the system ξ of p-power roots of unity in $\mathbb{C}_p$ should be identified with $(e^{2\pi i p^{-n}})_n$ by our pair of embeddings. If we reverse this (thus using $(e^{-2\pi i p^{-n}})_n$ instead), the effect is the same and the factor $\chi(-1)$ disappears. The same choice is also made by Perrin-Riou, see [Per00, top of p. 91]. In [FK06, beginning of §4] it is written explicitly that the system $(\mathrm{e}^{2\pi i p^{-n}})_n$ should be used, but this could be wrong. In order to figure out whether this can be a satisfactory solution, one should check which consequences this has for other motives, such as e. g. the motives attached to Dirichlet characters.

It is thus a very important task to figure out how Fukaya's and Kato's conjecture should be changed in order to predict the correct interpolation formulas. It might seem a good idea to look for a version that gives back the formulas of Coates and Perrin-Riou in their situation. At least this would produce the correct result for modular forms. But this issue lies outside the scope of this work.

4. Comparing periods and error terms

4.1. Choosing good bases

In this section we fix a number field $K \subseteq \overline{\mathbb{Q}}$, integers $N \geq 4$ and $k \geq 2$ and a newform $f \in \mathrm{S}_k(X_1(N), K)$, and we assume that K contains the N-th roots of unity (so that we may identify the modular curves $X_1(N)^{\mathrm{naive}}$ and $X_1(N)^{\mathrm{arith}}$). We choose bases of the tangent space and the $\mathrm{G}_{\mathbb{R}}$-invariant subspace of the Betti realisation of the critical twists of $\mathcal{M}(f)$. With respect to these bases we will later compute periods.

Fix an integer n with $1 \leq n \leq k-1$ and a Dirichlet character χ of arbitrary conductor. Then $\mathcal{M}(f)(\chi^*)(n)$ is critical by proposition II.8.7. Further we know from lemma I.3.10 (b) that

$$(\mathcal{M}(f)(\chi^*)(n))_{\mathrm{B}}^+ = \mathcal{M}(f)_{\mathrm{B}}^{\chi(-1)(-1)^n} \otimes K(n)_{\mathrm{B}} \otimes \mathcal{M}(\chi^*)_{\mathrm{B}}$$

and

$$\mathrm{t}_{\mathcal{M}(f)(\chi^*)(n)} = \mathrm{gr}^0\, \mathcal{M}(f)_{\mathrm{dR}} \otimes K(n)_{\mathrm{dR}} \otimes \mathcal{M}(\chi^*)_{\mathrm{dR}}.$$

We use the notation from section I.3.2 for the canonical bases of the realisations of the Tate resp. Dirichlet motives.

By corollary II.8.5 we can choose any $\delta_0 \in \mathrm{S}_k(X(N), K)^\vee$ such that $\delta_0(w_N f) = 1$ (where w_N is the Atkin-Lehner endomorphism) and use its image in $\mathrm{gr}^0\, \mathcal{M}(f)_{\mathrm{dR}}$ (which we denote again by δ_0) as a basis of this space. Note that δ_0 is then unique with this property, since $\mathrm{gr}^0\, \mathcal{M}(f)_{\mathrm{dR}}$ is one-dimensional. So

$$\delta := \delta_0 \otimes (b_{\mathrm{dR}}^{\mathbb{Q}(1)})^{\otimes n} \otimes b_{\mathrm{dR}}^{\chi} \in \mathrm{t}_{\mathcal{M}(f)(\chi^*)(n)}$$

is a basis for the tangent space.

We now turn to the Betti side and recall lemma III.2.4, which gives us isomorphisms

$$\mathrm{MS}_k(N,K)^{\pm}[f] \xrightarrow{\sim} \mathcal{M}(f)_{\mathrm{B}}^{\pm}$$

Further we use the modified pairing $\langle\cdot,\cdot\rangle_{\mathrm{B}}^{\iota}$ from section II.8.1 and the fact that it induces a perfect pairing between $\mathcal{M}(f)_{\mathrm{B}}^{\pm}$ and $\mathcal{M}(f)_{\mathrm{B}}^{\mp}$ by lemma II.8.6. Fix a basis $\eta^+ := \eta_f^+ \in \mathrm{MS}_k(N, \mathcal{O}_K)^+[f]$ as in definition III.2.8, and by abuse of notation denote its image in $\mathcal{M}(f)_{\mathrm{B}}^+$

under the above map still by η^+. By the pairing there exists a unique $\eta^- = \eta_f^- \in \mathcal{M}(f)_{\mathrm{B}}^-$, coming from an $\eta^- \in \mathrm{MS}_k(N, \mathcal{O}_K)^-[f]$, such that

$$\left\langle \eta^\pm, \eta^\mp \right\rangle_{\mathrm{B}}^\iota = b_{\mathrm{B}}^{\mathbb{Q}(1-k)}.$$

We choose then

$$\gamma := \eta^{\chi(-1)(-1)^n} \otimes (b_{\mathrm{B}}^{\mathbb{Q}(1)})^{\otimes n} \otimes b_{\mathrm{B}}^{\chi} \in (\mathcal{M}(f)(\chi^*)(n))_{\mathrm{B}}^+$$

as a basis of the Betti side.[5]

As a side remark, note that the choice of η^+ is the only non-canonical choice we ever made in this whole process.

We stress that both γ and δ of course depend on n and χ, but we omit this from their notation. This dependence should be always clear from the context. Also every element introduced in this section of course depends on f, and we will also often omit this from the notation. Though, later we will consider families of motives of modular forms parametrised by some set of specialisations Σ, and we will then put a subscript "ϕ" to all of the elements introduced here to indicate their dependence on $\phi \in \Sigma$.

4.2. Complex periods

We now compute Deligne's complex period of the critical twists of $\mathcal{M}(f)$ with respect to our chosen bases, as defined in section I.3.3. We use the elements and notations introduced in section 4.1.

We remark that a similar idea for calculating the complex periods appears in [Och06, §6.1, §6.3], although there many details are omitted.

Theorem 4.1: *We have for the complex period*

$$\Omega_\infty^{\gamma,\delta}(\mathcal{M}(f)(\chi^*)(n)) = \frac{(2\pi\mathrm{i})^{n+1-k}}{\mathrm{G}(\chi^*)} \mathcal{E}_\infty(f, \eta^s)$$

with $s = -\chi(-1)(-1)^n$.

Proof: Recall that we identified $\eta^\pm$ with their images under the map (III.2.1). By our choices of $\eta^\pm$ we have $\langle \eta^{-s}, w_N \eta^s \rangle_{\mathrm{B}} = b_{\mathrm{B}}^{\mathbb{Q}(1-k)}$, and since the pairing $\langle \cdot, \cdot \rangle_{\mathrm{B}}^\iota$ vanishes on $\mathcal{M}(f)_{\mathrm{B}}^\pm \times \mathcal{M}(f)_{\mathrm{B}}^\pm$, we have further $\langle \eta^{-s}, w_N \eta^s \rangle_{\mathrm{B}} = \langle \eta^{-s}, w_N(\eta^s + x) \rangle_{\mathrm{B}}$ for any $x \in \mathcal{M}(f)_{\mathrm{B}}^{-s}$. Therefore (by the definition of the complex error term)

$$\mathcal{E}_\infty(f, \eta^s) \cdot b_{\mathrm{B}}^{\mathbb{Q}(1-k)} = \langle \eta^{-s}, w_N \mathcal{E}_\infty(f, \eta^s) \eta^s \rangle_{\mathrm{B}} = \langle \eta^{-s}, w_N \xi^s \rangle_{\mathrm{B}} = \langle \eta^{-s}, w_N \xi \rangle_{\mathrm{B}}, \qquad (*)$$

where ξ and $\xi^\pm$ are as in definition III.2.5 (which we again identify with their images under the map (III.2.1)).

By the compatibility of the comparison isomorphism with the Eichler-Shimura map (see corollary II.6.8) and lemma III.2.6, we have that the image of ξ under the map

$$\mathrm{MS}_k(N, \mathbb{C}) \longrightarrow {}_k^N\mathcal{W}_{\mathrm{B}} \otimes \mathbb{C} \xrightarrow{\sim} {}_k^N\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C}$$

[5] We could also start with a choice of bases $\eta^\pm := \eta_f^\pm \in \mathrm{MS}_k(N, K)^\pm[f]$ as in definition III.2.8 and then use the unique $\gamma_0^\pm \in \mathcal{M}(f)_{\mathrm{B}}^\pm$ such that $\langle \gamma_0^\pm, \eta^\mp \rangle_{\mathrm{B}}^\iota = b_{\mathrm{B}}^{\mathbb{Q}(1-k)}$ to define $\gamma := \gamma_0^{\chi(-1)(-1)^n} \otimes (b_{\mathrm{B}}^{\mathbb{Q}(1)})^{\otimes n} \otimes b_{\mathrm{B}}^{\chi} \in (\mathcal{M}(f)(\chi^*)(n))_{\mathrm{B}}^+$ as a basis of the Betti side. It is thus not necessary to take the $\eta^\pm$ dual to each other, but this will later simplify our notation, so we use this choice.

is the image of f under the inclusion $\mathrm{S}_k(X(N),K) \hookrightarrow {}^N_k\mathcal{W}_{\mathrm{dR}} \otimes K$ coming from the Hodge filtration, so we denote this image by f. Now let $\rho \in {}^N_k\mathcal{W}_{\mathrm{dR}} \otimes \mathbb{C}$ be the image of $\eta^{-s} \in \mathcal{M}(f)_{\mathrm{B}}$ under the comparison isomorphism. Since the comparison isomorphism identifies the pairings $\langle\cdot,\cdot\rangle_{\mathrm{dR}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{B}}$ (see proposition II.8.3), $(*)$ is equivalent to

$$\langle\rho, w_N f\rangle_{\mathrm{dR}} = (2\pi\mathrm{i})^{1-k}\mathcal{E}_\infty(f,\eta^s)\cdot b_{\mathrm{dR}}^{\mathbb{Q}(1-k)}.$$

This means that the image of ρ in $\mathrm{gr}^0\,\mathcal{M}(f)_{\mathrm{dR}}$ is $(2\pi\mathrm{i})^{1-k}\mathcal{E}_\infty(f,\eta^s)\delta_0$.

Altogether, we see that the isomorphism

$$(\mathcal{M}(f)(\chi^*)(n))_{\mathrm{B}}^+ \xrightarrow{\sim} \mathrm{t}_{\mathcal{M}(f)(\chi^*)(n)}$$

maps $\gamma = \eta^{-s}\otimes(b_{\mathrm{B}}^{\mathbb{Q}(1)})^{\otimes n}\otimes b_{\mathrm{B}}^{\chi}$ to

$$((2\pi\mathrm{i})^{1-k}\mathcal{E}_\infty(f,\eta^s)\delta_0)\otimes(2\pi\mathrm{i}\,b_{\mathrm{dR}}^{\mathbb{Q}(1)})^{\otimes n}\otimes(\mathrm{G}(\chi^*)^{-1}b_{\mathrm{dR}}^{\chi}).$$

This completes the proof. □

4.3. p-adic periods

We now use again the setup for Hida families as described in situation III.3.11 and the notations introduced at the beginning of section III.3. For our fixed finite extension L of $\mathbb{Q}_p$ we let K be the number field $\overline{\mathbb{Q}}\cap L$ and $\mathfrak{p}$ the place of K such that $L = K_{\mathfrak{p}}$. Further we fix a Hida family $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty,\mathcal{I})$ which is new.

Throughout this section, we assume that condition III.4.5 is satisfied.

Let $\rho_F\colon \mathrm{G}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{I}}(\mathcal{T})$ be the big Galois representation attached to F from theorem III.3.18. By corollary III.3.20, it is a family of motives satisfying the strong Dabrowski-Panchishkin condition. Hence conjecture I.3.42 should apply to it.[6] To look closer at this conjecture, we need an isomorphism β as in section I.3.7.1. We can say something meaningful about p-adic periods only if we can choose this β in a "more or less canonical" way, for which we will need an extra condition which we now explain. Let $\mathcal{T}^0$ be as in theorem III.3.18.

Lemma 4.2: *Consider the following conditions.*

(i) For some choice of an $\mathcal{I}$-basis of $\mathcal{T}$ the image of ρ_F contains $\mathrm{SL}_2(\mathcal{I})$.

(ii) For any choice of an $\mathcal{I}$-basis of $\mathcal{T}$ the image of ρ_F contains $\mathrm{SL}_2(\mathcal{I})$.

(iii) There exists $\sigma \in \mathrm{im}\,\rho_F \subseteq \mathrm{Aut}_{\mathcal{I}}(\mathcal{T})$ such that $\sigma(\mathcal{T}^+) = \mathcal{T}^0$.

Then (i) and (ii) are equivalent and they imply (iii).

Proof: That (i) and (ii) are equivalent is obvious. To see that they imply (iii), we choose a submodule $\mathcal{T}' \subseteq \mathcal{T}$ complementary to $\mathcal{T}^0$ and isomorphisms $b_1\colon \mathcal{T}^+ \xrightarrow{\sim} \mathcal{T}^0$ and $b_2\colon \mathcal{T}^- \xrightarrow{\sim} \mathcal{T}'$, which gives us an automorphism $b_1\oplus b_2$ of $\mathcal{T} = \mathcal{T}^+\oplus\mathcal{T}^- = \mathcal{T}^0\oplus\mathcal{T}'$. Write $u \in \mathcal{I}^\times$ for the determinant of $b_1\oplus b_2$ and change one of b_1 or b_2 by u^{-1}. Then the determinant using the new choices will be 1 and the resulting automorphism σ lies in the image of ρ_F. □

The above conditions allow us to perform the following trick, which is inspired from [Hid96, §3.3].

[6] More precisely, the family should be Tate-twisted once so that its motives become critical.

Proposition 4.3: *If the condition from lemma 4.2 (iii) holds, then after possibly changing the complex embedding ι_∞, we can assume that $\mathcal{T}^+ = \mathcal{T}^0$.*

Proof: Let $\Phi := b_1 \oplus b_2 \in \mathrm{Aut}_{\mathcal{I}}(\mathcal{T})$ be as in lemma 4.2 (iii). Then the elements of $\mathcal{T}^0$ remain fixed under $\Phi^{-1}\rho_F(\mathrm{Frob}_\infty)\Phi$. Take a $\tau \in \mathrm{G}_{\mathbb{Q}}$ such that $\rho_F(\tau) = \Phi$. Then if we replace ι_∞ by $\iota_\infty \circ \tau$, the complex conjugation with respect to the new pair of embeddings is $\Phi^{-1}\rho_F(\mathrm{Frob}_\infty)\Phi$, so that then $\mathcal{T}^+ = \mathcal{T}^0$. □

From now on we assume that the following holds, which can be achieved under any of the conditions from lemma 4.2.

Condition 4.4: The embeddings (ι_∞, ι_p) are chosen such that $\mathcal{T}^+ = \mathcal{T}^0$.

Remark 4.5: Condition 4.4 seems to be a moderate condition. As explained in proposition 4.3 and lemma 4.2, we are safe if the image of ρ_F contains SL_2. There are results on when this happens: in [MW86, §10] Mazur and Wiles show that if $\mathcal{O} = \mathbb{Z}_p$, $\mathcal{I} = \Lambda^{\mathrm{wt}}$ and the image of the residual representation contains SL_2, then so does the image of ρ_F. There is work by J. Lang [Lan16] that can be used to extend this to more general $\mathcal{I}$ and to relax the condition on the residual representation. Hence there are quite some cases in which we know that the validity of condition 4.4 can be achieved.

Note that by reducing modulo $\phi \in \Sigma$ this implies that $\mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{p}}^+ = \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{p}}^0$ for any ϕ, where $\mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{p}}^0$ is the $\mathrm{G}_{\mathbb{Q}_p}$-stable subspace in $\mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{p}}$ from theorem II.5.15 (a).

Now fix $D \in \mathbb{N}$ with $p \nmid D$ and let $F_\infty := \mathbb{Q}(\mu_{Dp^\infty})$, which is then a Galois extension of $\mathbb{Q}$ satisfying the requirements in section I.3.7.1. We assume that $\mathcal{O}$ contains the D-th roots of unity. Let further $G := \mathrm{Gal}(F_\infty/\mathbb{Q})$, $\Lambda := \mathcal{I}[\![G]\!]$. Define $\mathbb{T} := \Lambda \otimes_{\mathcal{I}} \mathcal{T}$ and $\mathbb{T}^0 := \Lambda \otimes_{\mathcal{I}} \mathcal{T}^0$, similarly as we did in section I.3.7.1, and let $\mathbb{T}(1) := \mathbb{T} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1)$, $\mathbb{T}^0(1) := \mathbb{T}^0 \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1) = \mathbb{T}(1)^{\mathrm{DP}}$. We have to choose an isomorphisms of Λ-modules

$$\beta\colon \mathbb{T}(1)^+ \xrightarrow{\sim} \mathbb{T}(1)^{\mathrm{DP}}.$$

Since we have

$$\begin{aligned}
\mathbb{T}(1)^+ &= (\Lambda \underset{\mathcal{I}}{\otimes} \mathcal{T} \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p(1))^+ \\
&= \left(\Lambda^+ \underset{\mathcal{I}}{\otimes} \mathcal{T}^- \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p(1)\right) \oplus \left(\Lambda^- \underset{\mathcal{I}}{\otimes} \mathcal{T}^+ \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p(1)\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{T}(1)^{\mathrm{DP}} &= \mathbb{T}^0(1) = (\Lambda \underset{\mathcal{I}}{\otimes} \mathcal{T}^0 \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p(1)) \\
&= \left(\Lambda^+ \underset{\mathcal{I}}{\otimes} \mathcal{T}^0 \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p(1)\right) \oplus \left(\Lambda^- \underset{\mathcal{I}}{\otimes} \mathcal{T}^0 \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p(1)\right),
\end{aligned}$$

we see that any choice of an isomorphism of $\mathcal{I}$-modules

$$\beta_0\colon \mathcal{T}^+ \xrightarrow{\sim} \mathcal{T}^-$$

gives us an isomorphism β as above (recall that $\mathcal{T}^+ = \mathcal{T}^0$).

We now replace the abstract family of motives $\mathcal{T}$ by the $\mathcal{I}$-module $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})[F]$. By corollary III.4.19 it is isomorphic to $\mathcal{T}$ as a $\mathrm{G}_{\mathbb{Q}}$-representation, so we can assume without loss of generality that $\mathcal{T}$ is of this concrete form.

Situation 4.6: Here we choose some elements that should be fixed for the following. We have already fixed $D \in \mathbb{N}$ with $p \nmid N$ and defined $G = \mathrm{Gal}(\mathbb{Q}(\mu_{Dp^\infty}/\mathbb{Q})$. We choose $\mathcal{I}$-bases $\Xi^\pm$ of $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F]$ as in section III.4.4. Having done so, we define β_0 as the isomorphism

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^+[F] \xrightarrow{\sim} \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^-[F]$$

that sends Ξ^+ to Ξ^- and write

$$\beta\colon \mathbb{T}(1)^+ \xrightarrow{\sim} \mathbb{T}(1)^{\mathrm{DP}}$$

for the isomorphism induced by it. For each $\phi \in \Sigma$ of type (k, ε, r) we fix bases $\eta_\phi^\pm \in \mathrm{MS}_k(Np^r, \mathcal{O}_K)^\pm[F_\phi]$ and $\eta_{\phi,\mathrm{new}}^\pm \in \mathrm{MS}_k(Np^r, \mathcal{O}_K)^\pm[F_\phi^{\mathrm{new}}]$. We assume that $\eta_\phi^\pm = \eta_{\phi,\mathrm{new}}^\pm$ whenever $F_\phi = F_\phi^{\mathrm{new}}$ and that $\eta_\phi^\pm$ and $\eta_{\phi,\mathrm{new}}^\pm$ are connected via refinement as at the end of section 4.2 – more precisely that $\mathrm{Ref}_\alpha(\eta_{\phi,\mathrm{new}}^\pm) = \eta_\phi^\pm$, where α is the unique unit root of the p-th Hecke polynomial of F_ϕ^{new} (see definition II.5.16) and Ref_α is as in proposition III.2.10 (a). Moreover we assume that they are dual to each other under the pairing $\langle \cdot, \cdot \rangle_{\mathrm{B}}^\iota$, as in section 4.1. Finally we choose $\delta_{0,\phi} \in \mathrm{S}_k(X(Np^r), K)^\vee$ such that $\delta_{0,\phi}(w_{Np^r} F_\phi^{\mathrm{new}}) = 1$ as in section 4.1 to obtain bases of $\mathrm{gr}^0 \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{dR}}$.

Definition 4.7: (a) Let $\phi \in \Sigma$ be of type (k, ε, r) and let $P = P_\phi$. Reducing β_0 modulo P gives an isomorphism

$$\beta_{0,\phi}\colon \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{B}}^+ \underset{K}{\otimes} L \cong \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathfrak{p}}^+ = \mathcal{T}_\phi^+ \xrightarrow{\sim} \mathcal{T}_\phi^- = \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathfrak{p}}^- \cong \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{B}}^- \underset{K}{\otimes} L.$$

The elements $\eta_{\phi,\mathrm{new}}^\pm$ are bases of $\mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{B}}^\pm$, respectively. Define

$$C(\beta_{0,\phi}) := \Bigl(\det_{\eta_{\phi,\mathrm{new}}^+, \eta_{\phi,\mathrm{new}}^-} \beta_{0,\phi} \Bigr)^{-1} \in L^\times.$$

(b) Let $\Psi^\pm$ be the images of $\Xi^\pm$ under the $\mathcal{I}$-adic Eichler-Shimura map

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I}) \longrightarrow \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$$

from theorem III.5.11. Since this map is Hecke equivariant, we have in fact $\Psi^\pm \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})[F]$. Since the latter space is free of rank 1 over $\mathcal{I}$ and F is a basis, there are unique $U^\pm \in \mathcal{I}$ such that

$$\Psi^\pm = U^\pm F. \tag{4.1}$$

For $\phi \in \Sigma$, write $U_\phi^\pm \in \mathcal{O}$ for the reduction of $U^\pm$ modulo P.

Lemma 4.8: *Under our choices, we have*

$$\mathcal{E}_{\mathfrak{p}}(\Xi^+, \eta_\phi^+) = C(\beta_{0,\phi}) \mathcal{E}_{\mathfrak{p}}(\Xi^-, \eta_\phi^-).$$

Proof: Recall that β_0 was defined as

$$\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^+[F] \xrightarrow{\sim} \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^-[F], \quad \Xi^+ \longmapsto \Xi^-.$$

By theorem III.4.10 (c) the reduction of this map modulo P

$$\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})^+[F_\phi] \xrightarrow{\sim} \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})^-[F_\phi]$$

sends Ξ_ϕ^+ to Ξ_ϕ^-, where $\Xi_\phi^\pm$ denotes the images of $\Xi^\pm$ in $\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})^\pm[F_\phi]$. By definition III.4.14 we have $\Xi_\phi^\pm = \mathcal{E}_\mathrm{p}(\Xi^\pm, \eta_\phi^\pm)\eta_\phi^\pm$. Thus by definition of $C(\beta_{0,\phi})$ it follows

$$C(\beta_{0,\phi}) = \frac{\mathcal{E}_\mathrm{p}(\Xi^+, \eta_\phi^+)}{\mathcal{E}_\mathrm{p}(\Xi^-, \eta_\phi^-)}$$

(in the case where $F_\phi \neq F_\phi^{\mathrm{new}}$ we have to take the refinement maps into account, but by our choices of $\eta_{\phi,\mathrm{new}}^\pm$ and $\eta_\phi^\pm$ the relation still holds). □

Before we compute the p-adic period, we prove the following important result about the constants $U^\pm$.

Theorem 4.9: *We have $U^+ = 0$, while $U^- \in \mathcal{I}^\times$. In particular, $U_\phi^- \in \mathcal{O}^\times$ for each $\phi \in \Sigma$.*

Proof: By theorem III.3.18 (b) the representation $\mathcal{T}^0$ is an unramified direct summand of $\mathcal{T}$, and since the whole representation $\mathcal{T}$ is ramified at p, we must have $\mathcal{T}^0 = \mathcal{T}^{\mathrm{I}_p}$ for dimension reasons. Since we further assumed that $\mathcal{T}^0 = \mathcal{T}^+$ and $\mathcal{T} = \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$, we conclude that $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^{\mathrm{I}_p} = \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^+$.

By theorem III.5.11 (b) the kernel of the $\mathcal{I}$-adic Eichler-Shimura map is $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^{\mathrm{I}_p}$, so it follows immediately from the definition of U^+ that $U^+ = 0$.

On the other hand, fix $\phi \in \Sigma$. By a similar reasoning as above, it follows from theorem II.5.15 (a) that $\mathrm{MS}_k(Np^r, \mathcal{O})^+[F_\phi] = \mathrm{MS}_k(Np^r, \mathcal{O})[F_\phi]^{\mathrm{I}_p}$. We now use that the morphism $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F] \longrightarrow \mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})^\pm[F_\phi]$ is $\mathrm{G}_\mathbb{Q}$-equivariant by theorem III.4.18. This tells us that Ξ_ϕ^- is not fixed under the inertia group I_p (note that we have $\Xi_\phi^- \neq 0$ by proposition III.4.15). Further it implies that the kernel of the map $\mathrm{MS}_k^{\mathrm{ord}}(Np^r, \mathcal{O})[F_\phi] \longrightarrow \mathrm{S}_k^{\mathrm{ord}}(X_1(Np^r), \mathcal{O})[F_\phi]$ contains $\mathrm{MS}_k(Np^r, \mathcal{O})[F_\phi]^{\mathrm{I}_p}$, hence it equals $\mathrm{MS}_k(Np^r, \mathcal{O})[F_\phi]^{\mathrm{I}_p}$, again for dimension reasons. Therefore we get $\Psi_\phi^- \neq 0$ (with $\Psi^\pm$ as in definition 4.7 (b) and $\Psi_\phi^\pm$ the reduction modulo P_ϕ) and thus $U_\phi^- \neq 0$.

So $U^- \in \mathcal{I}$ defines a global section on $\operatorname{Spec} \mathcal{I}$ that does not vanish at any point $P_\phi \in \Sigma$. Since Σ is Zariski dense in $\operatorname{Spec} \mathcal{I}$ (since it is Zariski dense in $\operatorname{Spec} \mathcal{I}(\overline{\mathbb{Q}}_p)$ and $\operatorname{Spec} \mathcal{I}$ is 2-dimensional) and supports of sections are closed by [Stacks, Tag 01AU], U^- is a nowhere vanishing section, hence U^- is a unit. □

We can now compute the p-adic period.

Theorem 4.10: *Assume that conditions III.4.5 and 4.4 are satisfied. Fix $\phi \in \Sigma$ of type (k, ε, r), an integer n with $1 \leq n \leq k$ and a Dirichlet character χ of conductor Dp^m which we write as a product $\chi = \chi_{\mathrm{nr}}\chi_p$ with χ_{nr} of conductor D and χ_p of conductor p^m. The choices of $\eta_\phi^\pm$ and $\delta_{0,\phi}$ determine bases γ_ϕ and δ_ϕ of $\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*)(n)_\mathrm{B}^+$ resp. $\mathrm{t}_{\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*)(n)}$ as in section 4.1. Let $\alpha_{p,\phi}$ be the unit root of the p-th Hecke eigenvalue of F_ϕ^{new} and $s = -\chi(-1)(-1)^n$.*

Then we have for the p-adic period

$$\Omega_{\mathrm{p}}^{\gamma_\phi,\delta_\phi,\beta_\phi(\chi^*,n)}(\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*)(n)) = -\frac{\alpha_{p,\phi}^{-m}p^{nm}\mathcal{E}_{\mathrm{p}}(\Xi^s,\eta_\phi^s)}{U_\phi^-\chi_{\mathrm{nr}}(p)^m\mathrm{G}(\chi_p)\mathrm{G}(\chi^*)}.$$

Proof: We first assume that ϕ is such that F_ϕ is a newform, i. e. $F_\phi = F_\phi^{\mathrm{new}}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{B}_{\mathrm{dR}}\otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{B}}^+ \arrow[r, "\mathrm{cp}_{\text{ét}}", "\sim"'] \arrow[d, hook] & \mathrm{B}_{\mathrm{dR}}\otimes {}^{Np^r}_{k}\mathcal{W}_p^+ \arrow[r, equal] \arrow[d, hook] & \mathrm{B}_{\mathrm{dR}}\otimes {}^{Np^r}_{k}\mathcal{W}_p^0 \arrow[r, "\sim"] \arrow[d, hook] & {} \\
\mathrm{B}_{\mathrm{dR}}\otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{B}} \arrow[r, "\mathrm{cp}_{\text{ét}}", "\sim"'] & \mathrm{B}_{\mathrm{dR}}\otimes {}^{Np^r}_{k}\mathcal{W}_p \arrow[r, equal] & \mathrm{B}_{\mathrm{dR}}\otimes {}^{Np^r}_{k}\mathcal{W}_p \arrow[r, "\sim"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\sim"] & \mathrm{B}_{\mathrm{dR}}\otimes \mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_p^0\right) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{B}_{\mathrm{dR}}\otimes\left(\dfrac{\mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_p\right)}{\mathrm{fil}^1\,\mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_p\right)}\right) \arrow[r, "\mathrm{cp}_{\mathrm{dR}}", "\sim"'] & \mathrm{B}_{\mathrm{dR}}\otimes \mathrm{gr}^0\,{}^{Np^r}_{k}\mathcal{W}_{\mathrm{dR}} \\
{} \arrow[r, "\sim"] & \mathrm{B}_{\mathrm{dR}}\otimes \mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_p\right) \arrow[r, equal] & \mathrm{B}_{\mathrm{dR}}\otimes \mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_p\right) \arrow[r, "\mathrm{cp}_{\mathrm{dR}}", "\sim"'] \arrow[u, two heads] & \mathrm{B}_{\mathrm{dR}}\otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{dR}}. \arrow[u, two heads]
\end{tikzcd} \tag{4.2}$$

On the spaces in the lower row, we have the pairings $\langle\cdot,\cdot\rangle_?$ with ? being "B", "dR" and "p", see equations (II.8.1), (II.8.3) and (II.8.4), and all the maps in the lower row respect these pairings. Recall from proposition II.8.3 the concrete meaning of the latter statement.

We denote the images of elements of $\mathrm{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}} {}^{Np^r}_{k}\mathcal{W}_{\mathrm{B}}$ in $\mathrm{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} {}^{Np^r}_{k}\mathcal{W}_p$ by the same symbol, by abuse of notation. If we reduce (4.1) modulo ϕ we get $U_\phi^\pm F_\phi = \Psi_\phi^\pm$, where $\Psi_\phi^\pm$ denotes the reduction of $\Psi^\pm$ modulo ϕ. Using theorem III.5.11 and the definition of the p-adic error term (definition III.4.14), this means that the p-adic comparison isomorphism (see theorem II.6.9)

$$\begin{aligned}
{}^{Np^r}_{k}\mathcal{W}_p \underset{\mathbb{Q}_p}{\otimes} \mathbb{C}_p = \mathrm{H}^1_{\mathrm{p}}(Y(Np^r)\underset{\mathbb{Z}}{\times}\overline{\mathbb{Q}}_p, \mathrm{Sym}^{k-2}\,\mathrm{R}^1 f_*\underline{\mathbb{Z}}_p) \underset{\mathbb{Z}_p}{\otimes} \mathbb{C}_p &\xrightarrow{\sim} \\
\mathrm{S}_k(X(Np^r),\mathbb{C}_p)(1-k)\ \oplus\ \mathrm{H}^1(X(Np^r),\omega_{X(N)}^{2-k})&\underset{\mathcal{O}}{\otimes}\mathbb{C}_p \\
\twoheadrightarrow \mathrm{S}_k(X(Np^r),\mathbb{C}_p)(1-k) = \mathrm{gr}^{k-1}\,{}^{Np^r}_{k}&\mathcal{W}_{\mathrm{dR}} \underset{\mathbb{Q}}{\otimes}\mathbb{C}_p(1-k)
\end{aligned}$$

maps $\mathcal{E}_{\mathrm{p}}(\Xi^\pm,\eta_\phi^\pm)\eta_\phi^\pm \longmapsto F_\phi\otimes U_\phi^\pm \mathrm{t}_{\mathrm{dR}}^{1-k}$, where we identify $\mathbb{C}_p(1-k)$ with $\mathbb{C}_p\cdot\mathrm{t}_{\mathrm{dR}}^{1-k}\subseteq \mathrm{B}_{\mathrm{dR}}$.[7]

Let $\rho^\pm \in \mathrm{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}} {}^{Np^r}_{k}\mathcal{W}_{\mathrm{dR}}$ be the images of $\eta_\phi^\pm \in \mathrm{B}_{\mathrm{dR}}\otimes_K \mathcal{M}(F_\phi^{\mathrm{new}})_{\mathrm{B}} \subseteq \mathrm{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}} {}^{Np^r}_{k}\mathcal{W}_{\mathrm{B}}$ under the map in the lower row in diagram (4.2). We know by definition of $\eta_\phi^\pm$ that

$$\left\langle \eta_\phi^\pm, w_{Np^r}\mathcal{E}_{\mathrm{p}}(\Xi^\mp,\eta_\phi^\mp)\mathrm{t}_{\mathrm{dR}}^{k-1}\eta_\phi^\mp\right\rangle_p = \mathcal{E}_{\mathrm{p}}(\Xi^\mp,\eta_\phi^\mp)\mathrm{t}_{\mathrm{dR}}^{k-1}\cdot b_p^{\mathbb{Q}(1-k)}.$$

[7] Recall that we ignored the Galois actions on the various modules in theorem III.5.11. We use t_{dR} as a basis of $\mathbb{C}_p(1)$ to identify $\mathbb{C}_p(1-k)$ with $\mathbb{C}_p$. This is where the factor $\mathrm{t}_{\mathrm{dR}}^{1-k}$ comes from.

Hence since the comparison isomorphism respects the pairings (see proposition II.8.3) and maps $\mathcal{E}_{\mathrm{p}}(\Xi^{\mp}, \eta_{\phi}^{\mp})\mathrm{t}_{\mathrm{dR}}^{k-1}\eta_{\phi}^{\mp} \longmapsto U_{\phi}^{\pm}F_{\phi}$ it follows

$$\left\langle \rho^{+}, w_{Np^r}F_{\phi} \right\rangle_{\mathrm{dR}} = \frac{\mathcal{E}_{\mathrm{p}}(\Xi^{-}, \eta_{\phi}^{-})}{U_{\phi}^{-}}.$$

This means that the map in the upper row in diagram (4.2) maps

$$\eta_{\phi}^{+} \longmapsto \frac{\mathcal{E}_{\mathrm{p}}(\Xi^{-}, \eta_{\phi}^{-})}{U_{\phi}^{-}}\delta_{0,\phi};$$

note that although this latter map is defined over B_{dR}, our calculations over $\mathbb{C}_p$ suffice for seeing this, as explained in remark I.3.34.

We now look at the following variant of diagram (4.2), which differs only by the extra map $\beta_{0,\phi}^{-1}$:

$$\begin{tikzcd}
\mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{B}}^{-} \arrow[r, "\mathrm{cp}_{\text{ét}}", "\sim"'] \arrow[d, hook] & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{p}}^{-} \arrow[r, "\beta_{0,\phi}^{-1}", "\sim"'] \arrow[d, hook] & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{p}}^{+} \arrow[r, equal] \arrow[d, hook] & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{p}}^{0} \arrow[r, "\sim"] \arrow[d, hook] & {} \\
\mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{B}} \arrow[r, "\mathrm{cp}_{\text{ét}}", "\sim"'] & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{p}} \arrow[r, "\beta_{0,\phi}^{-1}", "\sim"'] & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{p}} \arrow[r, equal] & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{p}} \arrow[r, "\sim"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\sim"] & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_{p}^{0}\right) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{B}_{\mathrm{dR}} \otimes \left(\frac{\mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_{p}\right)}{\mathrm{fil}^{1}\mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_{p}\right)}\right) \arrow[r, "\mathrm{cp}_{\mathrm{dR}}", "\sim"'] & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{gr}^{0}\, {}^{Np^r}_{k}\mathcal{W}_{\mathrm{dR}} \\
{} \arrow[r, "\sim"] & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_{p}\right) \arrow[r, equal] & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}\left({}^{Np^r}_{k}\mathcal{W}_{p}\right) \arrow[r, "\mathrm{cp}_{\mathrm{dR}}", "\sim"'] \arrow[u, two heads] & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np^r}_{k}\mathcal{W}_{\mathrm{dR}}. \arrow[u]
\end{tikzcd} \tag{4.3}$$

Here the maps in the lower row are no longer compatible with the pairings because we introduced the map $\beta_{0,\phi}^{-1}$. Anyway, by definition of $C(\beta_{0,\phi})$ and our previous calculations, we know that the map in the upper row in diagram (4.3) maps

$$\eta_{\phi}^{-} \longmapsto \frac{C(\beta_{0,\phi})\mathcal{E}_{\mathrm{p}}(\Xi^{-}, \eta_{\phi}^{-})}{U_{\phi}^{-}}\delta_{0,\phi} = \frac{\mathcal{E}_{\mathrm{p}}(\Xi^{+}, \eta_{\phi}^{+})}{U_{\phi}^{-}}\delta_{0,\phi},$$

where the equality above comes from lemma 4.8.

Now let $s = -\chi(-1)(-1)^{n}$. We know

$$\begin{aligned}
\mathcal{M}(F_{\phi})(\chi^{*})(n)_{\mathrm{B}}^{+} &= \mathcal{M}(F_{\phi})_{\mathrm{B}}^{-s} \otimes \mathcal{M}(\chi^{*})_{\mathrm{B}} \otimes K(n)_{\mathrm{B}}, \\
\mathrm{t}_{\mathcal{M}(F_{\phi})(\chi^{*})(n)}^{+} &= \mathrm{gr}^{0}\, \mathcal{M}(F_{\phi})_{\mathrm{dR}}^{s} \otimes \mathcal{M}(\chi^{*})_{\mathrm{dR}} \otimes K(n)_{\mathrm{dR}}, \\
\mathcal{M}(F_{\phi})(\chi^{*})(n)_{\mathrm{p}}^{+} &= \mathcal{M}(F_{\phi})_{\mathrm{p}}^{-s} \otimes \mathcal{M}(\chi^{*})_{\mathrm{p}} \otimes K(n)_{\mathrm{p}}, \\
\mathcal{M}(F_{\phi})(\chi^{*})(n)_{\mathrm{p}}^{\mathrm{DP}} &= \mathcal{M}(F_{\phi})_{\mathrm{p}}^{0} \otimes \mathcal{M}(\chi^{*})_{\mathrm{p}} \otimes K(n)_{\mathrm{p}}.
\end{aligned}$$

We have to look at the determinant of the composition

$$\begin{aligned}
\mathrm{B}_{\mathrm{dR}} \underset{K}{\otimes} \mathcal{M}(F_\phi)(\chi^*)(n)^+_{\mathrm{B}} &\xrightarrow{\mathrm{cp}_{\text{ét}}} \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathcal{M}(F_\phi)(\chi^*)(n)^+_{\mathrm{p}} \\
&\xrightarrow{\beta_\phi(\chi^*,\,n)} \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathcal{M}(F_\phi)(\chi^*)(n)^{\mathrm{DP}}_{\mathrm{p}} \\
&\xrightarrow{\alpha^{-1}} \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathrm{D}_{\mathrm{dR}}(\mathcal{M}(F_\phi)(\chi^*)(n)^{\mathrm{DP}}_{\mathrm{p}}) \\
&\xrightarrow{\mathrm{dp}} \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \left(\mathrm{D}_{\mathrm{dR}}(\mathcal{M}(F_\phi)(\chi^*)(n)_{\mathrm{p}}) \middle/ \mathrm{fil}^0\, \mathrm{D}_{\mathrm{dR}}(\mathcal{M}(F_\phi)(\chi^*)(n)_{\mathrm{p}}) \right) \\
&\xrightarrow{\mathrm{cp}_{\mathrm{dR}}} \mathrm{B}_{\mathrm{dR}} \underset{K}{\otimes} \mathrm{t}_{\mathcal{M}(F_\phi)(\chi^*)(n)}.
\end{aligned}$$

From our definition of $\beta\colon \mathbb{T}(1)^+ \xrightarrow{\sim} \mathbb{T}(1)^{\mathrm{DP}}$ and the proofs of lemmas 1.3.37 and 1.3.38, one can see that the map

$$\beta_\phi(\chi^*, n)\colon\ \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathcal{M}(F_\phi)^{-s}_{\mathrm{p}} \otimes \mathcal{M}(\chi^*)_{\mathrm{p}} \otimes K(n)_{\mathrm{p}} \longrightarrow \mathrm{B}_{\mathrm{dR}} \underset{L}{\otimes} \mathcal{M}(F_\phi)^{0}_{\mathrm{p}} \otimes \mathcal{M}(\chi^*)_{\mathrm{p}} \otimes K(n)_{\mathrm{p}}$$

is the identity if $s = -1$ and is $\beta^{-1}_{0,\phi}$ (tensored with the identity) if $s = +1$. We omit the details of the verification of this. Hence the above composition is the tensor product of the upper row of the diagram (4.2) or (4.3) with the comparison isomorphisms for the motives $\mathcal{M}(\chi^*)$ and $\mathbb{Q}(n)$, according to $s = -1$ or $s = +1$. Therefore (see facts 1.3.6 and 1.3.8) this composition maps

$$\gamma_\phi = \eta_\phi^{\chi(-1)(-1)^n} \otimes (b_{\mathrm{B}}^{\mathbb{Q}(1)})^{\otimes n} \otimes b_{\mathrm{B}}^{\chi} \longmapsto \left(\frac{\mathcal{E}_{\mathrm{p}}(\Xi^s, \eta^s_\phi)}{U^-_\phi} \delta_{0,\phi} \right) \otimes \left(\mathrm{t}_{\mathrm{dR}} b_{\mathrm{dR}}^{\mathbb{Q}(1)} \right)^{\otimes n} \otimes \left(\mathrm{G}(\chi^*)^{-1} b^{\chi}_{\mathrm{dR}} \right).$$

The claim now follows from proposition 1.3.33 and lemmas 2.4 and 2.5, which completes the proof in the case where F_ϕ is a newform.

In the case where F_ϕ is not a newform (then automatically of level Np), we replace diagram (4.2) by

$$\begin{tikzcd}
& \mathrm{B}_{\mathrm{dR}} \otimes {}^{N}_{k}\mathcal{W}^{+}_{\mathrm{B}} \arrow[rr, "\mathrm{cp}_{\text{ét}}", "\sim"'] \arrow[dl] \arrow[dd, hook] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{N}_{k}\mathcal{W}^{+}_{p} \arrow[rr, equal] \arrow[dl] \arrow[dd, hook] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{N}_{k}\mathcal{W}^{0}_{p} \arrow[r, "\sim"] \arrow[dl] \arrow[dd, hook] & {} \\
\mathrm{B}_{\mathrm{dR}} \otimes {}^{Np}_{k}\mathcal{W}^{+}_{\mathrm{B}} \arrow[rr] \arrow[dd, hook] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np}_{k}\mathcal{W}^{+}_{p} \arrow[rr, equal] \arrow[dd, hook] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np}_{k}\mathcal{W}^{0}_{p} \arrow[r] \arrow[dd, hook] & {} & \\
& \mathrm{B}_{\mathrm{dR}} \otimes {}^{N}_{k}\mathcal{W}_{\mathrm{B}} \arrow[rr] \arrow[dl] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{N}_{k}\mathcal{W}_{p} \arrow[rr, equal] \arrow[dl] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{N}_{k}\mathcal{W}_{p} \arrow[r, "\sim"] \arrow[dl] & {} \\
\mathrm{B}_{\mathrm{dR}} \otimes {}^{Np}_{k}\mathcal{W}_{\mathrm{B}} \arrow[rr, "\mathrm{cp}_{\text{ét}}", "\sim"'] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np}_{k}\mathcal{W}_{p} \arrow[rr, equal] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np}_{k}\mathcal{W}_{p} \arrow[r, "\sim"] & {} &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, "\sim"] & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}({}^{N}_{k}\mathcal{W}^{0}_{p}) \arrow[rr, "\sim"] \arrow[dl] \arrow[dd, hook] & & \mathrm{B}_{\mathrm{dR}} \otimes \left(\frac{\mathrm{D}_{\mathrm{dR}}({}^{N}_{k}\mathcal{W}_{p})}{\mathrm{fil}^{1}\,\mathrm{D}_{\mathrm{dR}}({}^{N}_{k}\mathcal{W}_{p})}\right) \arrow[rr, "\mathrm{cp}_{\mathrm{dR}}", "\sim"'] \arrow[dl] & & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{gr}^{0}\,{}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \arrow[dl] \\
\mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}({}^{Np}_{k}\mathcal{W}^{0}_{p}) \arrow[rr] \arrow[dd, hook] & & \mathrm{B}_{\mathrm{dR}} \otimes \left(\frac{\mathrm{D}_{\mathrm{dR}}({}^{Np}_{k}\mathcal{W}_{p})}{\mathrm{fil}^{1}\,\mathrm{D}_{\mathrm{dR}}({}^{Np}_{k}\mathcal{W}_{p})}\right) \arrow[rr] & & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{gr}^{0}\,{}^{Np}_{k}\mathcal{W}_{\mathrm{dR}} & \\
{} \arrow[r] & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}({}^{N}_{k}\mathcal{W}_{p}) \arrow[rr, equal] \arrow[dl] & & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}({}^{N}_{k}\mathcal{W}_{p}) \arrow[rr] \arrow[uu, two heads] \arrow[dl] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{N}_{k}\mathcal{W}_{\mathrm{dR}} \arrow[uu, two heads] \arrow[dl] \\
\mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}({}^{Np}_{k}\mathcal{W}_{p}) \arrow[rr, equal] & & \mathrm{B}_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}({}^{Np}_{k}\mathcal{W}_{p}) \arrow[rr, "\mathrm{cp}_{\mathrm{dR}}", "\sim"'] \arrow[uu, two heads] & & \mathrm{B}_{\mathrm{dR}} \otimes {}^{Np}_{k}\mathcal{W}_{\mathrm{dR}} \arrow[uu, two heads] &
\end{tikzcd}$$

where the maps from the back layer to the front layer are induced from the motivic refinement morphism $\mathrm{Ref}_{\alpha_\phi}\colon {}^{N}_{k}\mathcal{W} \otimes_{\mathbb{Q}} K \longrightarrow {}^{Np}_{k}\mathcal{W} \otimes_{\mathbb{Q}} K$ from corollary II.7.7, and similarly also for diagram (4.3) (we omit drawing this). Here the front and back faces are just the diagrams from before, which clearly commute, and the top and bottom faces as well as the vertical ones commute since they come from a morphism of motives. By our choices of $\eta^{\pm}_{\phi}$ and $\eta^{\pm}_{\phi,\mathrm{new}}$ and the commutativity, it does not matter if we take the determinant in the front or back layer, and by a similar reasoning as before we obtain the result also in this case. $\square$

5. p-adic L-functions with motivic periods for Hida families

In this final section we put everything together. We keep situation III.3.11, so let L, $\mathcal{O} = \mathcal{O}_L$ and $\mathcal{I}$ be as before, let K be the number field $\overline{\mathbb{Q}} \cap L$ and $\mathfrak{p}$ the place of K such that $L = K_{\mathfrak{p}}$, and fix a new Hida family $F \in \mathbb{S}^{\mathrm{ord}}(Np^{\infty}, \mathcal{I})$. Further let any choices of any basis elements and so on be as before in section 4.3, especially situation 4.6. We continue to assume that conditions III.4.5 and 4.4 are satisfied. Further fix $D \in \mathbb{N}$ prime to p and let $G = \mathrm{Gal}(\mathbb{Q}(\mu_{Dp^{\infty}})/\mathbb{Q})$.

Before we begin we prove an easy lemma.

Lemma 5.1: *Let R be a profinite ring and $A \subseteq \mathbb{Z}$ any subset. For each finite order character χ of G and any $n \in A$ let $a_{n,\chi} \in R$ be given. Then the following are equivalent:*

(i) There exists an element $\mu \in R[\![G]\!]$ such that for any χ and n as above

$$\int_G \chi^{*}\kappa_{\mathrm{cyc}}^{n}\,\mathrm{d}\mu = a_{n,\chi}.$$

(ii) For each $l \in \mathbb{Z}$ there exists an element $\mu_l \in R[\![G]\!]$ such that for any χ and n as above

$$\int_G \chi^{*}\kappa_{\mathrm{cyc}}^{n+l}\,\mathrm{d}\mu_l = a_{n,\chi}.$$

(iii) For each $l \in \mathbb{Z}$ there exists an element $\mu'_l \in R[\![G]\!]$ such that for any χ and n as above

$$\int_G \chi\kappa_{\mathrm{cyc}}^{l-n}\,\mathrm{d}\mu'_l = a_{n,\chi}.$$

If any of the above elements μ, μ_l or μ'_l is a unit, then the other ones are also units.

Proof: Assume that (i) holds. Let $\Phi_l\colon R[\![G]\!] \longrightarrow R[\![G]\!]$ be the endomorphism induced by $G \longrightarrow (R[\![G]\!])^\times$, $g \longmapsto \kappa_{\mathrm{cyc}}^l(g)g$. Then the diagram

$$\begin{tikzcd} R[\![G]\!] \arrow[rr, "\Phi_l"] \arrow[dr, "\chi^*\kappa_{\mathrm{cyc}}^n"'] & & R[\![G]\!] \arrow[dl, "\chi^*\kappa_{\mathrm{cyc}}^{n+l}"] \\ & R & \end{tikzcd}$$

commutes and therefore $\mu_l \coloneqq \Phi_l(\mu)$ satisfies the required property from (ii). This shows the equivalence of (i) and (ii) since the other implication is trivial.

The equivalence of (ii) and (iii) can be shown by a similar argument using the endomorphism of $R[\![G]\!]$ induced by the inversion endomorphism $g \longmapsto g^{-1}$ of G and the equivalence of (i) and (ii).

The final statement about units is clear. $\square$

5.1. The explicit form of the conjectural interpolation formula

We have now computed all the expressions that occur in the conjectural interpolation formula by Fukaya and Kato. The general formula was given in conjecture I.3.42. We use our computations of the periods in theorems 4.1 and 4.10 and the local factor in lemma 2.2 as well as corollary III.2.11. Inserting all this into the general formula, the conjectural value of the p-adic L-function evaluated at $(\phi, \chi\kappa_{\mathrm{cyc}}^{-n})$ becomes

$$\begin{aligned} &-(n-1)!(1-\alpha_\phi^{-1}\chi^*(p)p^{n-1})(1-\alpha_\phi^{-1}\widetilde{\chi\varepsilon\psi\omega^{-k}}(p)p^{k-n-1}) \\ &\qquad\cdot \frac{\alpha_{p,\phi}^{-m}p^{nm}\mathcal{E}_{\mathrm{p}}(\Xi^s, \eta_\phi^s)}{(2\pi\mathrm{i})^{n+1-k}U_\phi^-\chi_{\mathrm{nr}}(p)^m\mathrm{G}(\chi_p)\mathcal{E}_\infty(F_\phi, \eta_\phi^s)}\mathrm{L}(F_\phi^{\mathrm{new}} \otimes \chi, n). \end{aligned} \tag{5.1}$$

Here $\phi \in \Sigma$ is of type (k, ε, r), χ is a primitive Dirichlet character of conductor Dp^m with $m \geq 0$ and $p \nmid D$, $1 \leq n \leq k-1$ and $s = -\chi(-1)(-1)^n$.

We want to express this using the naively twisted L-function $\mathrm{L}(F_\phi^{\mathrm{new}}, \chi, n)$ instead of $\mathrm{L}(F_\phi^{\mathrm{new}} \otimes \chi, n)$. In proposition 1.7 we described the relation between these two twisted L-functions, which is given by

$$\mathrm{L}(F_\phi^{\mathrm{new}} \otimes \chi, n) = \mathrm{L}(F_\phi^{\mathrm{new}}, \chi, n) \cdot \prod_{\ell \mid (Np^{r'}, Dp^m)} \mathrm{P}_\ell(\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*), \ell^{-n})^{-1},$$

where r' below the product sign should mean either r or 0, depending on whether $F_\phi = F_\phi^{\mathrm{new}}$ or not (so that $Np^{r'}$ is the level of F_ϕ^{new}). In the product, the prime ℓ can be our fixed prime p if and only if $r' > 0$ and $m > 0$, and by lemma 1.8 the corresponding factor then equals

$$\mathrm{P}_p(\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*), p^{-n}) = (1-\alpha_\phi^{-1}\widetilde{\chi\varepsilon\psi\omega^{-k}}(p)p^{k-n-1}).$$

This factor is nontrivial if and only if the p-parts of χ and $\varepsilon\psi\omega^{-k}$ are inverse to each other (and nontrivial, since $m > 0$ and χ is primitive).[8] By remark 2.3, the cases in which we have

[8] The case that $n = k-1$ and $\alpha_\phi = \widetilde{\chi\varepsilon\psi\omega^{-k}}(p)$ cannot occur, see corollary 2.6 (2) (and its proof).

a nontrivial Euler factor at p are therefore precisely the cases in which

$$(1 - \alpha_\phi^{-1}\widetilde{\chi\varepsilon\psi\omega^{-k}}(p)p^{k-n-1}) \neq (1 - \alpha_\phi^{-1}\chi\varepsilon\psi\omega^{-k}(p)p^{k-n-1}),$$

and in these cases the first expression (coming from the local correction factor in the interpolation formula) just cancels the Euler factor at p and the second expression is 1. This discussion shows that

$$\begin{aligned}(1 - \alpha_\phi^{-1}\widetilde{\chi\varepsilon\psi\omega^{-k}}(p)p^{k-n-1})\mathrm{L}(F_\phi^{\mathrm{new}} \otimes \chi, n) = \\ (1 - \alpha_\phi^{-1}\chi\varepsilon\psi\omega^{-k}(p)p^{k-n-1})\mathrm{L}(F_\phi^{\mathrm{new}}, \chi, n) \cdot \prod_{\substack{\ell \mid (Np^{r'}, Dp^m) \\ \ell \neq p}} \mathrm{P}_\ell(\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*), \ell^{-n})^{-1}\end{aligned}$$

in each case. To simplify the formulas below, we set

$$\mathrm{EEF}(F_\phi^{\mathrm{new}}, \chi, n) := \prod_{\ell \mid (N,D)} \mathrm{P}_\ell(\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*), \ell^{-n})^{-1}$$

(where "EEF" stands for "extra Euler factors"). In conclusion, the value of the conjectural p-adic L-function at $(\phi, \chi\kappa_{\mathrm{cyc}}^{-n})$ from (5.1) becomes

$$\begin{aligned}-(n-1)!(1 - \alpha_\phi^{-1}\chi^*(p)p^{n-1})(1 - \alpha_\phi^{-1}\chi\varepsilon\psi\omega^{-k}(p)p^{k-n-1})\,\mathrm{EEF}(F_\phi^{\mathrm{new}}, \chi, n) \\ \cdot \frac{\alpha_{p,\phi}^{-m}p^{nm}\mathcal{E}_{\mathrm{p}}(\Xi^s, \eta_\phi^s)}{(2\pi\mathrm{i})^{n+1-k}U_\phi^{-}\chi_{\mathrm{nr}}(p)^m\mathrm{G}(\chi_p)\mathcal{E}_\infty(F_\phi, \eta_\phi^s)}\mathrm{L}(F_\phi^{\mathrm{new}}, \chi, n).\end{aligned}$$

We now take Kitagawa's p-adic L-function for F, whose construction is recalled in appendix B. In the appendix, we worked with measures on $\mathbb{Z}_{p,D}^\times$, but we can identify $\mathbb{Z}_{p,D}^\times$ with G via class field theory. Then the map which was denoted κ in appendix B becomes κ_{cyc}. So we can copy the interpolation formula from corollary B.2.2. However we change the evaluation point to make it the same as Fukaya-Kato's one, which we can do thanks to lemma 5.1.

Therefore we have an element $\mu_F^{\mathrm{Kit}} \in \mathcal{I}[\![G]\!]$ whose interpolation behaviour is

$$\begin{aligned}\phi\left(\int_{\mathbb{Z}_{p,D}^\times} \chi\kappa^{-n}\mathrm{d}\mu_F\right) = (n-1)!(1 - \alpha_\phi^{-1}\chi^*(p)p^{n-1})(1 - \alpha_\phi^{-1}\chi\varepsilon\psi\omega^{-k}(p)p^{k-n-1}) \\ \cdot \frac{D^{n-1}p^{m(n-1)}\mathrm{G}(\chi^*)\mathcal{E}_{\mathrm{p}}(\Xi^\pm, \eta_\phi^\pm)}{\alpha_\phi^m(2\pi\mathrm{i})^{n+1-k}\mathcal{E}_\infty(F_\phi, \eta_\phi^s)}\mathrm{L}(F_\phi^{\mathrm{new}}, \chi, n),\end{aligned}$$

with ϕ, χ, n and s as above.

Let us calculate the quotient of these two interpolation formulas to see what the difference is. In the calculation below, we use two classical relations for Gauß sums:

(a) $\mathrm{G}(\chi_p)\mathrm{G}(\chi_p^*) = \chi_p(-1)p^m$, see [Miy89, Lem. 3.1.1 (2)],

(b) $\mathrm{G}(\chi^*) = \chi_p^*(D)\chi_{\mathrm{nr}}^*(p^m)\mathrm{G}(\chi_p^*)\mathrm{G}(\chi_{\mathrm{nr}}^*)$, see [Miy89, Lem. 3.1.2].

So if we divide Kitagawa's value by Fukaya-Kato's value (at $(\phi, \chi\kappa_{\mathrm{cyc}}^{-n})$, respectively), we get

$$\begin{aligned}\frac{\text{Kitagawa's value}}{\text{Fukaya-Kato's value}} &= -\frac{U_\phi^- D^{n-1} p^{-m} \mathrm{G}(\chi^*)\mathrm{G}(\chi_p)\chi_{\mathrm{nr}}(p)^m}{\mathrm{EEF}(F_\phi^{\mathrm{new}}, \chi, n)} \\ &= -\frac{U_\phi^- D^{n-1} p^{-m} \chi_p^*(D)\chi_{\mathrm{nr}}^*(p)^m \mathrm{G}(\chi_p^*)\mathrm{G}(\chi_{\mathrm{nr}}^*)\mathrm{G}(\chi_p)\chi_{\mathrm{nr}}(p)^m}{\mathrm{EEF}(F_\phi^{\mathrm{new}}, \chi, n)} \\ &= -\frac{\chi_p(-1) U_\phi^- D^{n-1}\chi_p^*(D)\mathrm{G}(\chi_{\mathrm{nr}}^*)}{\mathrm{EEF}(F_\phi^{\mathrm{new}}, \chi, n)}.\end{aligned} \tag{5.2}$$

We discuss now how to interpolate (most of) the expressions in this quotient.

5.2. Interpolation of the Euler factors away from p

We explain how to interpolate the Euler factors occurring in $\mathrm{EEF}(F_\phi^{\mathrm{new}}, \chi, n)$. Fix a prime $\ell \mid D$ (so in particular $\ell \neq p$) and write $D = \ell^\nu D'$ with $\ell \nmid D'$. We assume that $\mathcal{O}$ is large enough to contain the values of all characters $(\mathbb{Z}/\ell^\nu)^\times \longrightarrow \overline{\mathbb{Q}}_p^\times$. If H is a profinite group, we write $[\cdot]$ for the canonical map $H \longrightarrow \mathcal{I}[\![H]\!]^\times$.

We use the map

$$\begin{aligned}\Psi\colon \bigoplus_{\substack{\eta \text{ Char. of} \\ (\mathbb{Z}/\ell^\nu)^\times}} \mathcal{I}[\![\mathbb{Z}_{p,D'}^\times]\!] &\longrightarrow \mathcal{I}[\![\mathbb{Z}_{p,D}^\times]\!] = \mathcal{I}[\![\mathbb{Z}_{p,D'}^\times]\!][(\mathbb{Z}/\ell^\nu)^\times], \\ (\lambda_\eta)_\eta &\longmapsto \sum_{a\in(\mathbb{Z}/\ell^\nu)^\times} \sum_\eta \eta^{-1}(a)\lambda_\eta[a],\end{aligned}$$

where η runs over all characters $(\mathbb{Z}/\ell^\nu)^\times \longrightarrow \mathcal{O}^\times$.

Lemma 5.2: *Let $\tau\colon \mathbb{Z}_{p,D}^\times \longrightarrow \mathcal{I}^\times$ be a character which we write as a product $\tau = \tau_\ell\tau'$ of characters of $(\mathbb{Z}/\ell^\nu)^\times$ and $\mathbb{Z}_{p,D'}^\times$, respectively. Then for all $\lambda = (\lambda_\eta)_\eta \in \bigoplus_\eta \mathcal{I}[\![\mathbb{Z}_{p,D'}^\times]\!]$ we have*

$$\int_{\mathbb{Z}_{p,D}^\times} \tau \,\mathrm{d}\Psi(\lambda) = \varphi(\ell^\nu)\int_{\mathbb{Z}_{p,D'}} \tau' \,\mathrm{d}\lambda_{\tau_\ell},$$

where φ is Euler's totient function, i. e. $\varphi(\ell^\nu) = \ell^{\nu-1}(\ell-1)$.

Proof: We have

$$\begin{aligned}\int_{\mathbb{Z}_{p,D}^\times} \tau \,\mathrm{d}\Psi(\lambda) &= \int_{\mathbb{Z}_{p,D}^\times} \tau \,\mathrm{d}\left(\sum_a\sum_\eta \eta^{-1}(a)\lambda_\eta[a]\right) \\ &= \sum_a\sum_\eta \eta^{-1}(a)\tau_\ell(a)\int_{\mathbb{Z}_{p,D'}^\times} \tau' \,\mathrm{d}\lambda_\eta \\ &= \sum_\eta\left(\sum_a \eta^{-1}(a)\tau_\ell(a)\right)\int_{\mathbb{Z}_{p,D'}^\times} \tau' \,\mathrm{d}\lambda_\eta.\end{aligned}$$

Therefore the claim follows from the orthogonality relation for characters

$$\sum_a\sum_\eta \eta^{-1}(a)\tau_\ell(a) = \begin{cases}\varphi(\ell^\nu) & \text{if } \eta = \tau_\ell, \\ 0 & \text{if } \eta \neq \tau_\ell.\end{cases} \qquad \square$$

For a character $\eta_0\colon (\mathbb{Z}/\ell^\nu)^\times \longrightarrow \mathcal{O}^\times$ we define

$$\delta_{\eta_0} := \Psi(0,\ldots,0,[\ell],0,\ldots,0) \in \mathcal{I}[\![\mathbb{Z}_{p,D}^\times]\!], \tag{5.3}$$

with $(0,\ldots,0,[\ell],0,\ldots,0) \in \bigoplus_\eta \mathcal{I}[\![\mathbb{Z}_{p,D'}^\times]\!]$ having its non-zero entry in the η_0-component. The following then follows immediately from the above lemma.

Corollary 5.3: *With the above notation, we have*

$$\int_{\mathbb{Z}_{p,D}^\times} \tau \,\mathrm{d}\delta_{\eta_0} = \begin{cases} \varphi(\ell^\nu)\tau'(\ell) & \text{if } \tau_\ell = \eta_0, \\ 0 & \text{if } \tau_\ell \neq \eta_0. \end{cases}$$

We now use this to interpolate Euler factors of Hida families away from p. We identify $G = \mathrm{Gal}(\mathbb{Q}(\mu_{Dp^\infty})/\mathbb{Q})$ with $\mathbb{Z}_{p,D}^\times$ to make the notation clearer. Note that then $\tau^*(\mathrm{Frob}_\ell) = \tau(\ell)$ if τ is a character of G of conductor prime to ℓ.

We insert at this point a brief summary of some facts from the theory of automorphic representations, which we will need to formulate the next theorem.

Theorem 5.4: *Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$.*

(a) For each newform $f \in \mathrm{S}_k(X_1(N),\mathbb{C})$ there is a canonical attached automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A})$. (The property by which this representation is characterised is given in part (d) below.)

(b) Each automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ is in a unique way a restricted tensor product $\pi = \bigotimes_v' \pi_v$ of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_v)$ for all places v of $\mathbb{Q}$.

(c) The isomorphism types of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ for a prime ℓ can be described by the following disjoint classes:

Principle series representations *These are parametrised as follows: For each pair of characters $\eta_1,\eta_2\colon \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times$ such that $\eta_1\eta_2^{-1} \neq |\cdot|, |\cdot|^{-1}$ there is a representation $\mathrm{B}(\eta_1,\eta_2)$ and $\mathrm{B}(\eta_1,\eta_2) \cong \mathrm{B}(\eta_1',\eta_2')$ if and only if $\{\eta_1,\eta_2\} = \{\eta_1',\eta_2'\}$.*

Special representations *These are parametrised as follows: For each character $\eta\colon \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times$ there is a representation $\eta\,\mathrm{St}_\ell$, and two of these are isomorphic if and only if the characters are the same.*

Supercuspidal representations *These can also be classified, but this will not be important here.*

(d) For an automorphic representation π with local components π_v, define Euler factors for each prime ℓ as follows:

(1) $\mathrm{L}(\pi_\ell,s) := (1-\alpha_1 p^{-s})^{-1}(1-\alpha_2 p^{-s})^{-1}$, if $\pi_\ell \cong \mathrm{B}(\eta_1,\eta_2)$ is a principle series representation, where $\alpha_i := \eta_i(p)$ if $\eta_i|_{\mathbb{Z}_\ell^\times}$ is trivial and $\alpha_i = 0$ otherwise;[9]

(2) $\mathrm{L}(\pi_\ell,s) := (1-\beta p^{-s+1/2})^{-1}$ if $\pi_\ell \cong \eta\,\mathrm{St}_\ell$ is special, where $\beta = \eta(p)$ if $\eta|_{\mathbb{Z}_\ell^\times}$ is trivial and $\beta = 0$ otherwise;

[9] The character η_i of $\mathbb{Q}_\ell^\times$ is called *unramified* if it is trivial on $\mathbb{Z}_\ell^\times$. In fact, α_i is defined as the value of η_i at a uniformis, and one needs unramifiedness in order for this to be well-defined.

(3) $\mathrm{L}(\pi_\ell, s) := 1$ *if* π_ℓ *is supercuspidal.*

Define the L*-function of* π *as*

$$\mathrm{L}(\pi, s) := \prod_\ell \mathrm{L}(\pi_\ell, s) \qquad (s \in \mathbb{C}).$$

Then $\mathrm{L}(f, s) = \mathrm{L}(\pi_f, s - \frac{k-1}{2})$ *for each newform* $f \in \mathrm{S}_k(X_1(N), \mathbb{C})$.

(e) *For each automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A})$ *and each Dirichlet character* χ*, one can twist* π *by* χ *to obtain another automorphic representation* $\pi \otimes \chi$ *of* $\mathrm{GL}_2(\mathbb{A})$*. We have*

$$\pi_f \otimes \chi = \pi_{f \otimes \chi}.$$

The local automorphic types are stable under twisting, i. e. if π_ℓ *is a principle series representation (resp. a special representation, resp. supercuspidal), then so is* $(\pi \otimes \chi)_\ell$*, for any prime* ℓ*.*

Proof: All this is well-known, so we only briefly indicate references. To be precise, the term "automorphic representation of $\mathrm{GL}_2(\mathbb{A})$" here means an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{A})$ that is a subquotient of the space of L^2-automorphic forms, see [Bum97, p. 300].

For (a), a description of how to obtain the automorphic representation π_f from f is given in [Bum97, beg. of §3.6, esp. Thm. 3.6.1]. Statement (b) is the tensor product theorem from the theory of automorphic representations, see [Bum97, Thm. 3.3.3]. For the definition of the local Euler factors as in (d) see [Bum97, p. 516/517], [Gel75, Table on p. 113]. The equality of the L-functions is proved in [Gel75, Prop. 6.21].

We are left to prove (c) and (e). The principal series representations $\mathrm{B}(\eta_1, \eta_2)$ are introduced in [Bum97, p. 470/471]; see also [Gel75, §4.2]. They are (in a suitable sense) induced from the representation of the subgroup of upper triangular matrices given by $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \longmapsto \eta_1(a)\eta_2(d)$. If $\eta_1\eta_2^{-1} \neq |\cdot|, |\cdot|^{-1}$ they are irreducible; this and the characterisation of when two of them are isomorphic are shown in [Bum97, Thm. 4.5.1–3].

Let us explain at this point how to twist an irreducible admissible representation π of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ by a character $\chi\colon \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times$: the resulting representation $\pi \otimes \chi$ is given by $(\chi \circ \det) \cdot \pi$, i. e. $g \in \mathrm{GL}_2(\mathbb{Q}_\ell)$ acts as $\chi(\det(g))\pi(g)$. Using the decomposition $\mathbb{Q}_\ell^\times \cong \mathbb{Z} \times \mathbb{Z}_\ell^\times$ we can view any Dirichlet character of ℓ-power order as a character of $\mathbb{Q}_\ell^\times$ (trivial on $\mathbb{Z}$) and thereby also twist by Dirichlet characters.

If $\eta_1\eta_2^{-1} = |\cdot|$ we may write $\eta_1 = \eta\,|\cdot|^{1/2}$ and $\eta_2 = \eta\,|\cdot|^{-1/2}$ with a unique character $\eta\colon \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times$. The special representation $\eta\,\mathrm{St}_\ell$ is then a subspace of $\mathrm{B}(\eta_1, \eta_2)$ (which is no longer irreducible); if η is the trivial character then the representation is denoted St_ℓ and is called the *Steinberg representation*, and in general it equals $\eta \otimes \mathrm{St}_\ell$ with the twist as introduced above (this explains the notation). If $\eta_1\eta_2^{-1} = |\cdot|^{-1}$, then $\mathrm{B}(\eta_1, \eta_2)$ is also not irreducible, it then has a quotient which is again isomorphic to $\eta\,\mathrm{St}_\ell$ for $\eta_1 = \eta\,|\cdot|^{-1/2}$ and $\eta_2 = \eta\,|\cdot|^{1/2}$. For more details see [Bum97, p. 482] and [Gel75, §4.2].

For a definition of supercuspidal representations see [Gel75, Def. 4.20]. That these three types make up all irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ and that they are disjoint is shown in [Gel75, Thm. 4.21]. This completes the proof of (c).

We now explain (e). We can write any Dirichlet character χ as a product of characters χ_ℓ for all primes ℓ (almost all of which are trivial). Using the decomposition as a restricted tensor product from (b) and the local twist explained above, we can then define the twist of an automorphic representation $\pi = \bigotimes_v' \pi_v$ of $\mathrm{GL}_2(\mathbb{A})$ as the restricted tensor product of $\pi_\ell \otimes \chi_\ell$ for all primes ℓ, and with the archimedean part unchanged.

From the definitions of principal series and special representations and the twist, it is easy to see that $\mathrm{B}(\eta_1, \eta_2) \otimes \chi = \mathrm{B}(\eta_1\chi, \eta_2\chi)$ and $\eta \,\mathrm{St}_\ell \otimes \chi = \eta\chi \,\mathrm{St}_\ell$. This shows that the classes of principal series and special representations are stable under twisting, respectively. It follows from the classification that the class of supercuspidal representations must therefore also be stable under twisting.

We are left to show that $\pi_f \otimes \chi = \pi_{f\otimes\chi}$ for a newform $f \in \mathrm{S}_k(X_1(N), \mathbb{C})$. By the Multiplicity One Theorem for automorphic representations [Bum97, Thm. 3.3.6] it suffices to show that $\pi_{f,\ell} \otimes \chi_\ell = \pi_{f\otimes\chi,\ell}$ for all primes $\ell \nmid N \operatorname{cond}\chi$. For automorphic forms there is a notion of a conductor, which in our case is an ideal in $\mathbb{Z}$, and the conductor of π_f is N. From the explicit description of the conductors of the local components in [Gel75, (4.20), p. 73] we see that $\pi_{f,\ell}$ must be a principal series representations for these primes ℓ, and the characters η_1, η_2 are trivial on $\mathbb{Z}_\ell^\times$. Hence they are uniquely determined by their values at p, whence by their Euler factor $\mathrm{L}(\pi_{f,\ell}, s)$. It therefore suffices to show the equality of Euler factors $\mathrm{L}(\pi_{f,\ell} \otimes \chi_\ell, s) = \mathrm{L}(\pi_{f\otimes\chi,\ell}, s)$. But this is clear from the definition of the twist and part (d). ☐

The Euler factors we want to interpolate are described by the following theorem. Here the character $[\kappa_{\mathrm{cyc}}]\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathcal{I}^\times$ means the composition

$$[\kappa_{\mathrm{cyc}}]\colon \mathrm{G}_{\mathbb{Q}_\ell} \xrightarrow{\kappa_{\mathrm{cyc}}} \mathbb{Z}_p^\times \twoheadrightarrow 1 + p\mathbb{Z}_p = \Gamma^{\mathrm{wt}} \xrightarrow{[\cdot]} (\Lambda^{\mathrm{wt}})^\times \longrightarrow \mathcal{I}^\times.$$

Theorem 5.5: *Let F be a new Hida family as before.*

(a) The automorphic types of F are rigid in the following sense: For any prime q, if F_{P_0} for some $P_0 \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$ is of a certain automorphic type (principal series, special, supercuspidal) at q, then F_P is of that automorphic type at q for any $P \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$. We then say that F is of that automorphic type at q.

(b) Let $\ell \mid N$ be a prime. Then the restriction to $\mathrm{G}_{\mathbb{Q}_\ell}$ of the big Galois representation ρ_F attached to F can be described as follows:

(i) If F is in the principal series at ℓ, then

$$\rho_F|_{\mathrm{G}_{\mathbb{Q}_\ell}} \cong \alpha\xi_1\kappa_{\mathrm{cyc}}^{1/2}[\kappa_{\mathrm{cyc}}]^{-1/2} \oplus \alpha^{-1}\xi_2\kappa_{\mathrm{cyc}}^{1/2}[\kappa_{\mathrm{cyc}}]^{-1/2}$$

with $\alpha\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathcal{I}^\times$ unramified and $\xi_1, \xi_2\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathcal{O}^\times$ of finite order.

(ii) If F is special at ℓ, then

$$\rho_F|_{\mathrm{G}_{\mathbb{Q}_\ell}} \cong \begin{pmatrix} \xi\kappa_{\mathrm{cyc}}[\kappa_{\mathrm{cyc}}]^{-1/2} & * \\ & \xi[\kappa_{\mathrm{cyc}}]^{-1/2} \end{pmatrix}$$

with $\alpha\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathcal{I}^\times$ unramified and $\xi\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathcal{O}^\times$ of finite order.

(iii) If F is supercuspidal at ℓ, then

$$\rho_F|_{\mathrm{G}_{\mathbb{Q}_\ell}} \cong \rho_0 \otimes [\kappa_{\mathrm{cyc}}]^{-1/2}$$

with $\rho_0\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathrm{GL}_2(\mathcal{O})$ an irreducible Artin representation.

Proof: See [Hsi17, §3.2] and the references given there. □

We now fix an unramified character $\alpha\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathcal{I}^\times$, a character $\xi\colon \mathrm{G}_{\mathbb{Q}_\ell} \longrightarrow \mathcal{O}^\times$ of finite order and further $i, j \in \frac{1}{2}\mathbb{Z}$. As ξ is of finite order, it factors through $\mathrm{Gal}(\mathbb{Q}_p(\mu_M)/\mathbb{Q}_p)$ for some $M \in \mathbb{N}$ and we can write it as a product $\xi = \xi_\ell \xi'$ as before. Assume that M is chosen minimally (i. e. $M = \operatorname{cond}\xi$) and let $\mu := \operatorname{ord}_\ell M$. If $\nu \geq \mu$ we can and do view ξ as a character of $\mathbb{Z}_{p,D}^\times$.

Attached to this data we define

$$\mu(\alpha, \xi, i, j) := \begin{cases} \ell^{-i}\alpha\xi'[\kappa_{\mathrm{cyc}}]^j(\mathrm{Frob}_\ell)\delta_{\xi_\ell} \in \mathcal{I}[\![\mathbb{Z}_{p,D}^\times]\!] & \text{if } \nu \geq \mu, \\ 0 & \text{otherwise} \end{cases}$$

(where δ_{ξ_ℓ} is as in (5.3)).

Lemma 5.6: *For each finite order character $\chi\colon \mathbb{Z}_{p,D}^\times \longrightarrow \mathcal{O}^\times$ which we write as a product $\chi = \chi_\ell \chi'$ and each $n \in \mathbb{Z}$ we have*

$$\int_{\mathbb{Z}_{p,D}^\times} \chi\kappa_{\mathrm{cyc}}^{-n}\,\mathrm{d}\mu(\alpha, \xi, i, j) = \varphi(\ell^\nu)\ell^{-i-n}\alpha\widetilde{\xi\chi^*}[\kappa_{\mathrm{cyc}}]^j(\mathrm{Frob}_\ell).$$

Proof: Observe that $\widetilde{\xi\chi^*}(\mathrm{Frob}_\ell) = \xi'\chi'^*(\mathrm{Frob}_\ell)\widetilde{\xi_\ell\chi_\ell^*}(\mathrm{Frob}_\ell)$ and

$$\widetilde{\xi_\ell\chi_\ell^*}(\mathrm{Frob}_\ell) = \begin{cases} 1 & \text{if } \xi_\ell = \chi_\ell, \\ 0 & \text{if } \xi_\ell \neq \chi_\ell. \end{cases}$$

The first case can never occur if $\nu < \mu = \operatorname{ord}_\ell \operatorname{cond}\xi$, so the statement is trivial in this case and we may assume $\nu \geq \mu$. Then the claim follows directly from corollary 5.3 and the definition of $\mu(\alpha, \xi, i, j)$. □

We can now define the measures interpolating the Euler factors of F away from p.

Proposition 5.7: *Let $\ell \mid N$ be a prime. Define*

$$\mu_\ell(F) := \begin{cases} (\varphi(\ell^\nu) - \mu(\alpha, \xi_1, \frac{1}{2}, -\frac{1}{2}))(\varphi(\ell^\nu) - \mu(\alpha^{-1}, \xi_2, \frac{1}{2}, -\frac{1}{2})) & \textit{if } F \textit{ in the principal series at } \ell, \\ (\varphi(\ell^\nu) - \mu(1, \xi, 1, -\frac{1}{2}))(\varphi(\ell^\nu) - \mu(1, \xi, 0, -\frac{1}{2})) & \textit{if } F \textit{ special at } \ell, \\ \varphi(\ell^\nu)^2 & \textit{if } F \textit{ supercuspidal at } \ell, \end{cases}$$

where α, ξ_1, ξ_2, ξ are as in theorem 5.5 (b). Then for each $\phi \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}$, each finite order character $\chi\colon \mathbb{Z}_{p,D}^\times \longrightarrow \mathcal{O}^\times$ and each $n \in \mathbb{Z}$ we have

$$\phi\left(\int_{\mathbb{Z}_{p,D}^\times} \chi\kappa_{\mathrm{cyc}}^{-n}\,\mathrm{d}\mu_\ell(F)\right) = \varphi(\ell^\nu)^2\,\mathrm{P}_\ell(\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*)(n), 0).$$

Proof: The supercuspidal case is clear by theorem 5.4 (d) and (e) and theorem 5.5 (a). In the other cases we have

$$\det(1-\mathrm{Frob}_\ell,(\rho_F|_{\mathrm{G}_{\mathbb{Q}_\ell}}\otimes\chi^*\otimes\kappa_{\mathrm{cyc}}^n)^{\mathrm{I}_\ell}) = \begin{cases}(1-\ell^{-n-1/2}\alpha[\kappa_{\mathrm{cyc}}]^{-1/2}\widetilde{\xi_1\chi^*}(\mathrm{Frob}_\ell))(1-\ell^{-n-1/2}\alpha^{-1}[\kappa_{\mathrm{cyc}}]^{-1/2}\widetilde{\xi_2\chi^*}(\mathrm{Frob}_\ell)) \\ \qquad \text{if } F \text{ is of principal series type at } \ell, \\ \\ (1-\ell^{-n}[\kappa_{\mathrm{cyc}}]^{-1/2}\widetilde{\xi\chi^*}(\mathrm{Frob}_\ell))(1-[\kappa_{\mathrm{cyc}}]^{-1/2}\widetilde{\xi\chi^*}(\mathrm{Frob}_\ell)) \\ \qquad \text{if } F \text{ is of special type at } \ell\end{cases}$$

by theorem 5.5 (b). Then lemma 5.6 says that

$$\varphi(\ell^\nu)^2\det(1-\mathrm{Frob}_\ell,(\rho_F|_{\mathrm{G}_{\mathbb{Q}_\ell}}\otimes\chi^*\otimes\kappa_{\mathrm{cyc}}^n)^{\mathrm{I}_\ell}) = \int_{\mathbb{Z}_{p,D}^\times}\chi\kappa_{\mathrm{cyc}}^n\,\mathrm{d}\mu_\ell(F)$$

by definition of the $\mu_\ell(F)$. The claim then follows by applying ϕ to both sides, using that

$$\begin{aligned}\phi(\det(1-\mathrm{Frob}_\ell,(\rho_F|_{\mathrm{G}_{\mathbb{Q}_\ell}}\otimes\chi^*\otimes\kappa_{\mathrm{cyc}}^n)^{\mathrm{I}_\ell})) &= \det(1-\mathrm{Frob}_\ell,(\rho_{F_\phi^{\mathrm{new}}}|_{\mathrm{G}_{\mathbb{Q}_\ell}}\otimes\chi^*\otimes\kappa_{\mathrm{cyc}}^n)^{\mathrm{I}_\ell}) \\ &= \mathrm{P}_\ell(\mathcal{M}(F_\phi^{\mathrm{new}})(\chi^*)(n),0).\end{aligned}$$ □

5.3. Finish

To interpolate the remaining factors from the quotient (5.2) we show the following.

Lemma 5.8: *Let R be a profinite ring such that $D\in R^\times$ and R contains the D-th roots of unity. Then there is a unit $\mu_D\in(R[\![G]\!])^\times$ such that for each $n\in\mathbb{Z}$ and Dirichlet character χ of conductor Dp^m for some $m\in\mathbb{N}_0$, which we write as a product $\chi=\chi_{\mathrm{nr}}\chi_p$ of characters of conductors D and p^m, we have*

$$\int_G\chi\kappa_{\mathrm{cyc}}^{-n}\mathrm{d}\mu_D = D^{n-1}\chi_p^*(D)\mathrm{G}(\chi_{\mathrm{nr}}^*).$$

Proof: Identify G with $\mathbb{Z}_{p,D}^\times$. We first define an element of the ring $R[(\mathbb{Z}/D)^\times]$. For $a\in(\mathbb{Z}/D)^\times$ write again $[a]$ for the corresponding element in $R[(\mathbb{Z}/D)^\times]$. Fix a primitive D-th root of unity $\zeta\in R^\times$, set

$$\mu' := \sum_{a\in(\mathbb{Z}/D)^\times}[a^{-1}]\zeta^a \in R[(\mathbb{Z}/D)^\times]$$

and define μ_1 as the image of μ' under the canonical map $R[(\mathbb{Z}/D)^\times]\longrightarrow R[\![\mathbb{Z}_{p,D}^\times]\!]$. We claim that μ_1 is a unit. If we define

$$\mu'' := \sum_{a\in(\mathbb{Z}/D)^\times}[a]\zeta^a \in R[(\mathbb{Z}/D)^\times]$$

then one can show that $\mu'\cdot\mu''=[-1]D$, which is a unit in $R[(\mathbb{Z}/D)^\times]$. This can be seen by a calculation similar to the one that proves the Gauß sum relation (a) above, see the proof of [Shi71, Lem. 3.63]. Hence μ_1 is also a unit.

We now define an element $\mu'' \in R[\mathbb{Z}_p^\times]$ as $\mu'' := [D^{-1}]$. Obviously μ'' is a unit, hence so is its image under the canonical map $R[\mathbb{Z}_p^\times] \longrightarrow R[\![\mathbb{Z}_{p,D}^\times]\!]$, which we call μ_2.

We now put $\mu_D := \mu_1\mu_2$. By construction it is then clear that the integrals of μ_D are as claimed up to a shift in n, but this can be fixed by applying lemma 5.1. □

We now have all ingredients ready to arrive at our main result. To state it more clearly, we introduce the following notations for the technical conditions we need.

Condition 5.9: We consider the following conditions on F and primes $\ell \mid (N, C)$:

$$p \nmid \ell - 1 \tag{nd_ℓ}$$

$$\left.\begin{array}{c} F \text{ is in the principal series at } \ell \text{ and } \operatorname{ord}_\ell \operatorname{cond} \xi_i < \operatorname{ord}_\ell D \text{ for } i = 1, 2, \text{ or} \\ F \text{ is special at } \ell \text{ and } \operatorname{ord}_\ell \operatorname{cond} \xi < \operatorname{ord}_\ell D, \text{ or} \\ F \text{ is supercuspidal at } \ell \end{array}\right\} \tag{triv_ℓ}$$

Here, ξ, ξ_1, ξ_2 are as in theorem 5.5 (b).

If (nd_ℓ) holds for ℓ, then $\varphi(\ell^\nu)$ is a unit in $\mathcal{O}$. If (triv_ℓ) holds for ℓ, then $\mu_\ell(F) = \varphi(\ell^\nu)^2$. This follows directly from the definition of $\mu_\ell(F)$.

Theorem 5.10: *Fix coefficient rings as in situation III.3.11, a Hida family F which is new and basis elements as in situation 4.6. Assume that conditions III.4.5 and 4.4 hold.*

(a) There exists a p-adic L*-function $\mu_F^{\mathrm{FK}} \in \Lambda[\frac{1}{p}] = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{I}[\![G]\!]$ such that for each $\phi \in \Sigma$ of type (k, ε, r), each primitive Dirichlet character χ of conductor Dp^m for some $m \geq 0$ and each $1 \leq n \leq k-1$ the evaluation*

$$\phi\left(\int_G \chi\kappa_{\mathrm{cyc}}^{-n} \mathrm{d}\mu_F^{\mathrm{FK}}\right)$$

is the value predicted by conjecture I.3.42 up to a factor $\chi_p(-1)$.

(b) If we assume in addition that for each prime $\ell \mid (N, D)$ at least one of (nd_ℓ) and (triv_ℓ) holds, then $\mu_F^{\mathrm{FK}} \in \Lambda$. In this case the ideals generated by μ_F^{FK} and μ_F^{Kit} differ by $\prod_{\ell \mid (N,C)} \mu_\ell(F)$.

(c) Assume that for each prime $\ell \mid (N, D)$ both (nd_ℓ) and (triv_ℓ) hold. Then μ_F^{FK} and μ_F^{Kit} generate the same ideal in $\mathcal{I}[\![G]\!]$.

Proof: We just define

$$\mu_F^{\mathrm{FK}} := -\frac{\mu_F^{\mathrm{Kit}}}{U^- \cdot \mu_D} \prod_{\ell \mid (N,C)} \frac{\mu_\ell(F)}{\varphi(\ell^{\nu_\ell})^2}$$

where μ_D is as in lemma 5.8, $\mu_\ell(F)$ are as in proposition 5.7 and $\nu_\ell = \operatorname{ord}_\ell D$. This is an element of Λ if (nd_ℓ) or (triv_ℓ) holds for each $\ell \mid (N, D)$ and of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$ otherwise. The claims follow from our calculation (5.2) of the quotient of the two values, proposition 5.7 about the Euler factors at the primes ℓ, lemma 5.8 about the remaining expressions and the fact that U_ϕ^- comes by definition from an element $U^- \in \mathcal{I}^\times$ which is a unit by theorem 4.9. □

We close with some remarks about our result.

Remark 5.11: (a) The conditions in theorem 5.10 that (nd_ℓ) or (triv_ℓ) or even both of them hold for all $\ell \mid (N, D)$ are the most general ones we could get for these results to hold, but of course they are bit ugly. Each of the following conditions implies them:

(1) F is supercuspidal at all primes dividing (N, D),

(2) F is supercuspidal at all primes dividing N,

(3) $(N, D) = 1$,

(4) $N = 1$,

(5) $D = 1$.

(b) Note that by corollary 2.6 the evaluation points we use are appropriate pairs, apart from two exceptional cases; but even in these cases the statement in theorem 5.10 (a) is true (although not predicted by conjecture I.3.42) – it then holds trivially since both values vanish.

(c) In theorem 5.10 (a) we need the Dirichlet character χ to be primitive, i. e. its "D-part" away from p must be primitive. This is because the interpolation formula in Kitagawa's construction is known to hold only in this case. Of course we can perform the construction for each D, but there does not seem to be an easy relation between the measures obtained for different D, see [MTT86, "Warning" on p. 19]. Conjecture I.3.42 predicts a similar interpolation behaviour also for imprimitive characters, but we do not know how to prove this. Of course this problem disappears if we set $D = 1$.

Remark 5.12: At a first glance, the fact that μ_F^{Kit} and μ_F^{FK} do not always generate the same ideal looks problematic with regard to the Main Conjecture, but in fact this is consistent. Since this work focuses on the analytic side of Iwasawa Theory, we do not go into detail here, but let us sketch the reason.

Let Σ be a finite set of places of $\mathbb{Q}$ not containing p. Then one can consider Σ-primitive p-adic L-functions, which should satisfy similar interpolation properties as above but with the Euler factors of the complex L-function for the primes in Σ omitted. In our case, since we can interpolate the Euler factors away from p by proposition 5.7, we know that such Σ-primitive p-adic L-functions exist for any Σ. For example, Kitagawa's p-adic L-function is such a Σ-primitive one for Σ being the set of primes dividing (N, D).

On the algebraic side, the Iwasawa modules one considers are (duals of) Selmer groups. We omit defining these in detail, we just state that they are subgroups of an H^1 in continuous Galois cohomology whose elements satisfy certain local conditions, similar as $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, -)$ in section I.2.7. As on the analytic side, there is also a notion of Σ-primitivity here: we can form Σ-primitive Selmer groups by omitting the local conditions for the places in Σ.

Obviously, the usual Selmer group (the one with $\Sigma = \varnothing$) is then contained in the Σ-primitive one for any nonempty Σ, and the cokernel can be described in terms of the local conditions at the primes in Σ. In accordance with the analytic side, the characteristic ideal of this cokernel is generated by a product of elements interpolating the Euler factors at primes in Σ. See [GV00, §2, esp. Prop. 2.4, Cor. 2.3] for these facts and for a treatment of Σ-primitive Selmer groups.

In their proof of the Iwasawa Main Conjecture for (certain) modular forms and families, Skinner and Urban a priori use those Σ-primitive objects on both the algebraic and analytic side for Σ containing all primes dividing N (among others); using the techniques mentioned above they deduce from this the Main Conjecture for any set Σ. See [SU14, §3, esp. §3.6 and the proof of Thm. 3.29].

In Fukaya's and Kato's work, the Selmer group (or rather Selmer complex) which is related to the p-adic L-function from conjecture I.3.42 via the Main Conjecture is the usual one (i. e. $\Sigma = \varnothing$), as it should be, since the p-adic L-function from conjecture I.3.42 has all Euler factors away from p, i. e. it is $\varnothing$-primitive. Hence our result and the fact that μ_F^{Kit} and μ_F^{FK} may generate different ideals in Λ is consistent with the Main Conjecture proved by Skinner and Urban.

Let us finally remark that Fukaya and Kato also formulate a version of their conjecture involving Σ-primitive objects. They formulate this in terms of an open subset U of $\operatorname{Spec}\mathbb{Z}$ which plays the role of the complement of Σ. See [FK06, §4.1.2 resp. §4.2.11] for the definitions of the Σ-primitive resp. $\varnothing$-primitive Selmer complexes and [FK06, Thm. 4.2.22, case with "(resp. ...)"] for the conjectural interpolation formula and the Main Conjecture in the Σ-primitive case. Since we can interpolate all Euler factors away from p, we can show with our methods that also these p-adic L-functions exist for any U (of course up to the factor $\chi_p(-1)$), and this is still consistent with the Main Conjecture.

6. Outlook: Overconvergent families and further generalisations

It should be possible to obtain similar results also for overconvergent families instead of Hida families. In this final section, we briefly describe this idea.

An important object in the context of overconvergent families is the Coleman-Mazur-Buzzard *eigencurve*. The eigencurve of tame level N, where N is an integer prime to p, is a separated rigid analytic space C_N over $\mathbb{Q}_p$ which is equidimensional of dimension 1 and has a canonical flat and locally finite map $\kappa\colon C_N \longrightarrow \mathcal{W}_N$ to the *weight space* $\mathcal{W}_N$, which is the unique rigid analytic space $\mathcal{W}_N$ over $\mathbb{Q}_p$ such that $\mathcal{W}_N(L) = \operatorname{Hom}(\mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times, L^\times)$ for each complete $\mathbb{Q}_p$-algebra L. Its points can be interpreted as overconvergent modular eigenforms, and a dense set of points (the *classical points*) correspond to classical modular eigenforms.

The eigencurve was originally introduced by Coleman and Mazur in [CM98] (there only of tame level 1). The construction was axiomatised and generalised by Buzzard in [Buz07], where he introduced his "eigenvariety machine". In the original construction, this machine is fed with spaces of overconvergent modular forms. It was Stevens who realised that these spaces can be replaced by spaces of overconvergent modular symbols, which are easier to handle – this is another incarnation of the Eichler-Shimura philosophy. This approach of defining the eigencurve using Buzzard's eigenvariety machine and Stevens's overconvergent modular symbols was worked out in detail by Bellaïche in [Bel10]; see [Bel10, §III.5.1, §IV.1.4, §IV.3] for proofs of the above statements.

The use of modular symbols instead of modular forms in the construction has several advantages, one of which is that it gives in a natural way a family of Galois representations. Let us sketch the idea.

The eigencurve is built out of affinoid pieces which are glued together. These affinoid pieces live over affinoid pieces of the weight space. If $\operatorname{Sp} A \subseteq \mathcal{W}_N$ is an affinoid piece (of a certain kind), then one builds the affinoid piece of the eigencurve lying over $\operatorname{Sp} A$ using overconvergent modular symbols with coefficients in A, denoted by $\mathrm{MSymb}(\Gamma_1(N) \cap \Gamma_0(p), D_A)$. Here D_A is a certain A-linear representation of an appropriate Hecke pair, so this is an A-module of modular symbols in the sense of definition III.1.1, and we thus have Hecke operators. This allows one to cut out the submodule where T_p acts by eigenvalues of valuation $< k-1$, which is denoted $\mathrm{MSymb}(\Gamma_1(N) \cap \Gamma_0(p), D_A)^{<k-1}$ (and similarly on other modules). This submodule has a Hecke eigenalgebra, which we denote by $\mathbf{T}_A^{<k-1}$, and the affinoid piece of the eigencurve over $\operatorname{Sp} A$ is then $\operatorname{Sp} \mathbf{T}_A^{<k-1}$. We do not define the module D_A here; in fact there are several different definitions in the literature. The only thing we want to remark is that if A is a finite extension of $\mathbb{Q}_p$, then there is a canonical isomorphism $\mathrm{MSymb}(\Gamma_1(N) \cap \Gamma_0(p), D_A)^{<k-1} \cong \mathrm{MS}_k(N, A)^{<k-1}$: this is Stevens's control theorem. For proofs and details see [Bel10, §III.6, Thm. III.6.36]; see also [Bel11, §3], [AIS15, §3], [Ste94; Pol14; PS13; PS11].[10]

Now the modules $\mathrm{MSymb}(\Gamma_1(N) \cap \Gamma_0(p), D_A)^{<k-1}$ carry a canonical action of $\mathrm{G}_{\mathbb{Q}}$. This allows us to interpret them (or the coherent sheaf they define on the eigencurve) as an analytic family of Galois representations in the sense of definition I.2.27. If we specialise the family to some point $x \in C_N$ corresponding to a classical eigenform f_x defined over a finite extension L of $\mathbb{Q}_p$, then the stalk at this point is the base change of $\mathrm{MSymb}(\Gamma_1(N) \cap \Gamma_0(p), D_A)^{<k-1}$ to L, and by Stevens's control theorem and proposition III.4.16, it is canonically isomorphic to the Galois representation attached to f_x, i. e. to the p-adic realisation of the motive $\mathcal{M}(f_x)$. Hence we get a family of motives in the sense of definition I.3.35, where Σ is the set of classical points on C_N lying over $\operatorname{Sp} A$.

By theorem II.5.17, each motive in this family satisfies the weak Dabrowski-Panchishkin condition. It seems therefore reasonable to expect that the whole family satisfies the weak Dabrowski-Panchishkin condition (see condition I.3.36 (b)). A result in this direction was proved by Kedlaya, Pottharst and Xiao [KPX14]: they show that the family is globally trianguline (after going to a desingularisation of the eigenvariety); the weak Dabrowski-Panchishkin condition should follow from this similarly as in theorem II.5.17. Moreover it seems reasonable to conjecture the existence of a p-adic L-function for the whole family similar to conjecture I.3.42. Some evidence for this is provided by [Zae17], which generalises the work of Fukaya and Kato to the case of a single motive satisfying only the weak Dabrowski-Panchishkin condition.

There is a construction of a candidate for such a p-adic L-function by Bellaïche. His result [Bel11, Thm. 3] is cited below. It uses the concept of a decent newform, whose definition we do not repeat here, see [Bel11, Def. 1]. As explained there, this condition is rather mild. Let us remark that the eigencurve is smooth at all points corresponding to such forms by [Bel11, Thm. 4], so locally around such points the Galois representation is trianguline by the above result of Kedlaya, Pottharst and Xiao.

Theorem 6.1 (Bellaïche)**:** *Let $x \in C_N$ be a point on the eigencurve corresponding to a decent refined newform. Then there is an affinoid neighbourhood $V \subseteq C_N$ of x and a two-variable p-adic* L*-function* L_p *defined on* $V \times \operatorname{Hom}(\mathrm{G}_{\mathrm{cyc}}, \mathbb{C}_p^\times)$ *interpolating the one-variable p-adic* L*-functions in the following sense: for each $y \in V$ corresponding to a refined newform f there exists a constant $c_y \neq 0$ such that* $\mathrm{L}_p(y, \cdot) = c_y L_p(f, \cdot)$*, where* $\mathrm{L}_p(f, \cdot)$ *is the one-variable p-adic*

[10] Note that in these texts the symmetric tensor representation used to define modular symbols is introduced in terms of inhomogeneous polynomials as in lemma A.1.2 (b).

L-*function for* f. *The constant* c_y *depends on* y, *but not on the second variable.*

The constant c_y from the above theorem is of a non-canonical nature and may thus be called an error term. In this aspect Bellaïche's result is very similar to Kitagawa's: his function does interpolate the one-variable functions, but there is an error term depending on a non-canonical choice. In fact, the whole construction is somewhat similar to Kitagawa's, it replaces Kitagawa's $\mathcal{I}$-adic modular symbols by overconvergent ones. In particular the definition of Bellaïche's error term (see [Bel11, Prop. 4.14, Def. 4.15]) is similar to Kitagawa's definition. Therefore our methods should apply to this case as well.

In order to apply our method and to define a constant U^- similarly as in definition 4.7 (b), we need to compare overconvergent modular forms and overconvergent modular symbols. We thus need an overconvergent version of the $\mathcal{I}$-adic Eichler-Shimura map from section III.5.2, which we used to define U^- in the Hida family setting. Such a map was constructed by Andreatta, Iovita and Stevens in [AIS15]. They proved an overconvergent analogue of theorem III.5.11 which says in particular that their overconvergent Eichler-Shimura map does indeed interpolate Faltings's comparison isomorphisms, see [AIS15, Thm. 1.3].

It should therefore be possible to apply the same methods to define a constant U^- and to divide Bellaïche's function by this constant to find a p-adic L-function having an interpolation behaviour with motivic periods (under an assumption similar to condition 4.4). The details of this construction remain to be checked.

As further generalisations, it should be possible to apply the same methods to more general automorphic representations, such as Hilbert modular forms over totally real fields. For these Carayol [Car86] constructed associated Galois representations and Blasius and Rogawski [BR93] showed that there is an underlying Grothendieck motive. In the setting of Hida families Dimitrov [Dim13] constructed a two-variable p-adic L-function. For overconvergent families of Hilbert modular forms Bellaïche's construction of a two-variable p-adic L-function was generalised very recently by J. Bergdall and D. Hansen. It is expected that the construction of the p-adic Eichler-Shimura isomorphism in families extends to this situation, see e. g. [CHJ17] and unpublished work in progress by A. Betina. There are even overconvergent Eichler-Shimura isomorphisms in more general settings: The case of Shimura curves over $\mathbb{Q}$ is recent work by D. Barrera and S. Gao [BG17], the case of Shimura curves over totally real fields was done by the same authors building on Gao's thesis [Gao16] and the case of certain unitary Shimura varieties is a work in progress by D. Barrera and R. Brasca.

Appendix

Appendix A.

Comparison of conventions in the literature

For many objects introduced in this work there are different conventions to define them in the literature. In this appendix we list and compare some of them.

1. Symmetric powers and symmetric tensors

As a preliminary for the next section, we collect some facts on symmetric powers and symmetric tensors. Let $n \in \mathbb{N}_0$ be fixed for this section.

Definition 1.1: Let R be a commutative ring and M be an R-module. Then the symmetric group $\mathfrak{S}_n$ acts on the n-th tensor power $M^{\otimes n}$ by permuting the factors. We write $\operatorname{TSym}_R^n M$ for the invariants under this action and $\operatorname{Sym}_R^n M$ for the coinvariants. This is clearly functorial in M. If R is clear from the context, we may omit the subscript "R".

The functor Sym is compatible with base change in the following sense: If S is an R-algebra, then we have

$$(\operatorname{Sym}_R^n M) \underset{R}{\otimes} S = \operatorname{Sym}_S^n(M \underset{R}{\otimes} S).$$

For the functor TSym this is wrong in general, but if M is a free R-module the analogous equality for TSym is true.

There is a canonical map $\operatorname{TSym}_R^n M \longrightarrow \operatorname{Sym}_R^n M$ which is obviously injective. If $n!$ is invertible in R this map is an isomorphism of R-modules (this is well-known; it will also follow from the proof of lemma 1.3).

We will use this mainly just for the R-module $M = R^2$. The semigroup $\mathrm{M}_2(R)$ acts on R^2 by multiplication from the left, and this induces a (diagonal) action on $(R^2)^{\otimes n}$ and thus on $\operatorname{TSym}^n R^2$ and $\operatorname{Sym}^n R^2$. The canonical map $\operatorname{TSym}_R^n M \longrightarrow \operatorname{Sym}_R^n M$ is equivariant for this action.

The representation $\operatorname{Sym}^n \mathbb{Z}^2$ plays a central role in this work. Over $\mathbb{Q}$ it is isomorphic to $\operatorname{TSym}^n \mathbb{Q}^2$, and some texts in the literature use $\operatorname{TSym}^n \mathbb{Z}^2$ instead. The importance of this representation is explained by the fact that over $\mathbb{Q}$, any irreducible $\mathbb{Q}$-linear representation of $\mathrm{GL}_2(\mathbb{Z})$ is isomorphic to $\operatorname{Sym}^n \mathbb{Q}^2$ for some $n \geq 0$ up to twists by the determinant; we do not use this fact.

There are various ways to describe this more explicitly. In the following we write $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for the standard basis of $\mathbb{Z}^2$.

Lemma 1.2: *(a)* $\operatorname{Sym}^n \mathbb{Z}^2$ *is canonically isomorphic to the group of homogeneous polynomials of degree n in two variables* X, Y *with integer coefficients,* $\mathbb{Z}[X, Y]_{\deg=n}$. *A matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma$ *acts by sending* X *to* $aX + cY$ *and* Y *to* $bX + dY$.

(b) $\operatorname{Sym}^n \mathbb{Z}^2$ *is canonically isomorphic to the group of polynomials in one variable* Z *of degree* $\leq n$, $\mathbb{Z}[Z]_{\deg \leq n}$. *The action of a matrix* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *on such a polynomial* P *is given by*

$$(\gamma P)(z) = (a + cz)^n P\left(\frac{b + dz}{a + cz}\right).$$

Proof: (a) We can take the equivalence classes of

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{i \text{ times}} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{n-i \text{ times}}, \qquad \text{for } i = 0, \ldots, n$$

as a basis of $\operatorname{Sym}^n \mathbb{Z}^2$. We may thus identify $\operatorname{Sym}^n \mathbb{Z}^2$ with $\mathbb{Z}[X, Y]_{\deg = n}$ by identifying the i-th basis vector from above with the monomial $X^i Y^{n-i}$. It is a straightforward calculation to check that the action is as claimed.

(b) Define a map

$$\mathbb{Z}[X, Y]_{\deg = k} \longrightarrow \mathbb{Z}[Z]_{\deg \leq k}, \quad X \longmapsto 1, \ Y \longmapsto Z.$$

It is obviously an isomorphism of abelian groups, and it follows again from an easy calculation that this transforms the action of Σ to what is claimed above. $\square$

Lemma 1.3: *The representation* $\operatorname{TSym}^n \mathbb{Z}^2$ *can be described as follows. Let* $L_n(\mathbb{Z}) := \mathbb{Z}^{n+1}$ *and define a map*

$$[\cdot]^n \colon \mathbb{Z}^2 \longrightarrow L_n(\mathbb{Z}), \qquad \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u^n \\ u^{n-1}v \\ \vdots \\ un^{n-1} \\ v^n \end{pmatrix}.$$

Then there is a natural action of Σ *on* $L_n(\mathbb{Z})$ *satisfying*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} u \\ v \end{bmatrix}^n = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right]^n \quad \textit{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma, \ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}^2 \tag{1.1}$$

and $L_n(\mathbb{Z})$ *with this action is canonically isomorphic to* $\operatorname{TSym}^n \mathbb{Z}^2$.

Proof: There is a canonical Σ-equivariant injection $\Psi \colon \operatorname{TSym}^n \mathbb{Z}^2 \hookrightarrow \operatorname{Sym}^n \mathbb{Z}^2$. We view $\operatorname{Sym}^n \mathbb{Z}^2$ as homogeneous polynomials via lemma 1.2 (a). Define an injection

$$\Phi \colon L_n(\mathbb{Z}) \hookrightarrow \operatorname{Sym}^n \mathbb{Z}^2, \qquad \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \sum_{i=0}^{n} \binom{n}{i} a_i X^i Y^{n-i}.$$

We will prove that the images of Ψ and Φ inside $\operatorname{Sym}^n \mathbb{Z}^2$ coincide. This implies in particular that the image of Φ is stable under the action of Σ, so this induces an action of Σ on $L_n(\mathbb{Z})$.

The canonical map $\mathbb{Z}^2 \longrightarrow \operatorname{Sym}^n \mathbb{Z}^2$ mapping $w \in \mathbb{Z}^2$ to the equivalence class of $w \otimes \cdots \otimes w$ is described in terms of homogeneous polynomials by

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \sum_{i=0}^{n} \binom{n}{i} (uX)^i (vY)^{n-i} = (uX + vY)^n \qquad \text{for} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}^2$$

and is therefore equal to the composition $\Phi \circ [\cdot]^n$. From this follows that the action of Σ on $L_n(\mathbb{Z})$ satisfies (1.1) (using again lemma 1.2 (a)).

We can choose as a basis of $(\mathbb{Z}^2)^{\otimes n}$ the elements

$$e_I := b_1 \otimes \cdots \otimes b_n, \quad I \subseteq \{1, \ldots, n\}, \quad \text{with } b_i = \begin{cases} e_1, & i \in I, \\ e_2, & i \notin I. \end{cases}$$

If some linear combination

$$\sum_{I \subseteq \{1,\ldots,n\}} a_I e_I$$

of these lies in $\operatorname{TSym}^n \mathbb{Z}^2$, we must have $a_I = a_{\sigma(I)}$ for all $\sigma \in \mathfrak{S}_n$. Since the orbit of an $I \subseteq \{1, \ldots, n\}$ under $\mathfrak{S}_n$ are all subsets of $\{1, \ldots, n\}$ with the same cardinality as I, the coefficient a_I depends only on this cardinality, so we put $a_I =: a_{\#I}$. We can then write each element of $\operatorname{TSym}^n \mathbb{Z}^2$ uniquely as

$$\sum_{s=0}^{n} a_s \Bigg(\sum_{\substack{I \subseteq \{1,\ldots,n\} \\ \#I = s}} e_I \Bigg),$$

so the elements

$$E_s := \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ \#I = s}} e_I, \quad s = 0, \ldots, n$$

form a basis of $\operatorname{TSym}^n \mathbb{Z}^2$. For these basis elements we have obviously

$$\Psi(E_s) = \binom{n}{s} X^s Y^{n-s}.$$

Therefore the image of Ψ coincides with the image of Φ. $\square$

Definition 1.4: Let

$$\operatorname{Sym}_n^\vee(\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Sym}^n \mathbb{Z}^2, \mathbb{Z})$$

be the dual representation in the sense of remark I.1.4, and $\operatorname{Sym}_n^\vee(R) = \operatorname{Sym}_n^\vee(\mathbb{Z}) \otimes_{\mathbb{Z}} R$.

Of course, $\operatorname{Sym}_n^\vee(\mathbb{Z})$ is isomorphic to $\operatorname{Sym}^n \mathbb{Z}^2$ as a representation of $\mathrm{GL}_2(\mathbb{Z})$, but the isomorphism is not canonical. We can see this explicitly as follows. Using the canonical isomorphism of abelian groups $\operatorname{Hom}(\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z}^2$ coming from the standard basis e_1, e_2 of $\mathbb{Z}^2$, we can identify $\operatorname{Sym}_n^\vee(\mathbb{Z})$ with $\operatorname{Sym}^n \mathbb{Z}^2$ where a matrix now acts as its transpose. If we compose this with the automorphism of $\operatorname{Sym}^n \mathbb{Z}^2$ induced by conjugation with $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$, this action is transformed into the original action on $\operatorname{Sym}^n \mathbb{Z}^2$. We can therefore imagine $\operatorname{Sym}_n^\vee(\mathbb{Z})$ as a group isomorphic to $\operatorname{Sym}^n \mathbb{Z}^2$ but with a different action. Concretely, we view elements of $\operatorname{Sym}_n^\vee(\mathbb{Z})$ also as homogeneous polynomials in two variables $X^\vee$, $Y^\vee$ which correspond to the dual basis of the standard basis of $\mathbb{Z}^2$, and a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma$ then acts as $X^\vee \longmapsto dX^\vee - bY^\vee$, $Y^\vee \longmapsto -cX^\vee + aY^\vee$.

2. Conventions regarding symmetric powers and symmetric tensors

In our definition of classical modular symbols (definition III.2.1) we used the representation $\mathrm{Sym}^n \mathbb{Z}^2$. This is also relevant in the context of the Eichler-Shimura isomorphism if it is described in terms of group cohomology of this representation, as in proposition II.6.4. Some other texts also use this representation, sometimes defined in terms of homogeneous or inhomogeneous polynomials as in lemma 1.2. It is also common to use $\mathrm{TSym}^n \mathbb{Z}^2$ (often in the form $L_n(\mathbb{Z})$ as described in lemma 1.3) or $\mathrm{Sym}_n^\vee(\mathbb{Z})$ instead. Since over a field of characteristic zero all of them are isomorphic, for most purposes this gives equivalent theories of modular symbols. Further some texts use a right action which arises from the left action via the main involution ι. By the abstract Hecke theory in section I.1, it is clear that this also does not change the theory of modular symbols.

We list some texts for each convention. Using the concrete descriptions given above, one can see that the definitions of modular symbols in each of this text is equivalent to ours.

Texts that use $\mathrm{Sym}^n \mathbb{Z}^2$: [Pol14], [PS11], [Ste94], [Del08], [Con09], [Kat04]

Texts that use $\mathrm{TSym}^n \mathbb{Z}^2$: [KLZ17], [Hid81], [Hid86b], [Hid86a],[Oht95], [Oht99], [Oht00], [Kit94], [Shi71]

Texts that use $\mathrm{Sym}_n^\vee(\mathbb{Z})$: [HidLFE], [Bel11], [PS13], [AIS15]

The text [Bel10] uses yet another representation (but it is also isomorphic to $\mathrm{Sym}^n \mathbb{Z}^2$).

Now look again at the description of the Eichler-Shimura isomorphism in terms of group cohomology in proposition II.6.4. The question which representation of Σ appears here is in fact determined by another choice: namely the choice of a trivialisation of the local system $\mathrm{R}^1 f_* \mathbb{Z}$ on $\mathfrak{h}$. In section II.2.1 we chose such a trivialisation by choosing the ordered basis $(\tau, 1)$ of the homology group $\mathrm{H}_1(E_\tau, \mathbb{Z})$ for $\tau \in \mathfrak{h}$. Lemma II.2.1 then showed that this forces the representation to be $\mathrm{Sym}^n \mathbb{Z}^2$. If we had chosen the basis $(-1, \tau)$ or $(\tau, -1)$ instead, this would have resulted in the representation $\mathrm{Sym}_n^\vee(\mathbb{Z})$.

3. Conventions regarding powers of $2\pi\mathrm{i}$ in complex error terms

In definition III.2.5, we defined the modular symbol ξ_f attached to an eigenform f by a formula with a factor $c := (2\pi\mathrm{i})^{k-1}$ in front of it. This factor c is not standard in the literature.

We chose to define ξ_f with this factor appearing because the same factor also appears in proposition II.6.4 (where no choice is involved, so it appears naturally). The definition of ξ_f affects the definition of the complex error term $\mathcal{E}_\infty(f, \eta^\pm)$ (see definition III.2.8), i. e. if we change the factor to some c', the complex error term also has to be changed by a factor c'/c and hence also the interpolation formula in theorem B.1.11. The complex period $\Omega_\infty^{\gamma,\delta}(\mathcal{M}(f)(\chi^*)(n))$ of course does not change, in the formula in theorem IV.4.1 a factor c/c' will appear, cancelling out the change in $\mathcal{E}_\infty(f, \eta^\pm)$. So all results remain valid (with these changes).

We list some texts in the literature and how they define ξ_f.

One of our main references, Kitagawa's [Kit94], has no factor at all, i. e. $c = 1$. Therefore the power of $2\pi\mathrm{i}$ in Kitagawa's original interpolation formula is different from ours. This convention seems to be not very common.

In the following texts, a factor $c = 2\pi\mathrm{i}$ is used: [MTT86],[1] [Del08], [HidLFE], [Pol14], [PS11], [PS13], [Bel10].

Finally, [Kat04] uses the same convention as we do. More precisely, Kato defines complex error terms in [Kat04, §7.6] with a factor $c = (2\pi\mathrm{i})^{k-1}$ appearing (but without using modular symbols). His definition is not completely the same as our definition of the error term, but one can see easily that they differ only by an algebraic factor. In [Kat04, (7.13.6)], his error term is interpreted in terms of the pairing $\langle\cdot,\cdot\rangle_{\mathrm{B}}$, similarly as we did in the proof of theorem IV.4.1. Again this is not precisely the same expression as here, but what Kato denotes "per" is the Eichler-Shimura map and his element $\delta_{1,N}(k, j)$ is a class in the Betti cohomology over $\mathbb{Q}$, so we see again that his error term differs from our one by an algebraic factor. This convention fits best to the motivic situation, which is why we prefer it.

[1] We remark that the original text [MTT86], which first constructed the p-adic L-function for a single modular form via modular symbols, does not use any complex periods at all. Instead, they directly use the modular symbol ξ_f to construct the p-adic L-function (rather than η_f) and use a fixed isomorphism $\mathbb{C} \cong \mathbb{C}_p$ to view ξ_f as a p-adic modular symbol. In their definition of ξ_f, they use the constant $c = 2\pi\mathrm{i}$.

Appendix B.

Kitagawa's construction of p-adic L-functions

In this appendix we briefly sketch Kitagawa's construction of p-adic L-functions, using our previous work. Kitagawa uses slightly different conventions as we did, and further there are some errors in his article (see section IV.3.1). Since this thesis relies heavily on Kitagawa's work, it seemed reasonable to reproduce this here in order to avoid errors in the interpolation formulas. Furthermore, we include the case where p does not divide the level, which is omitted by Kitagawa.

1. The p-adic L-function for a single cusp form

We first construct the measure for a single cusp form. This is in fact due to Mazur, Tate and Teitelbaum [MTT86], although we present it here in a more modern way, following Kitagawa.

1.1. Some preliminary calculations with modular symbols

Let $N \geq 3$, $k \geq 2$ and fix a field K of characteristic 0. We define a map

$$Q\colon \mathrm{MS}_k(N,K) \longrightarrow \mathrm{Maps}(\mathbb{Q}/\mathbb{Z}, \mathrm{Sym}^{k-2} K^2), \quad \xi \longmapsto Q_\xi$$

with Q_ξ defined by

$$\begin{aligned} Q_\xi(x) &= \xi \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} ((\infty) - (0)) \qquad (x \in \mathbb{Q}) \\ &= \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \bullet \xi((\infty) - (x)). \end{aligned}$$

Here "$[\cdot]$" and "$\bullet$" denote the matrix actions. More precisely, we use the standard left action of $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ on $\mathrm{Sym}^{k-2} K^2$, denoted by "$\bullet$" and then use remark I.1.4 to get an action of $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ on maps $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \longrightarrow \mathrm{Sym}^{k-2} K^2$, which we normalise as a right action and denote by "$[\cdot]$". Note that we have a priori only actions of $\Sigma = \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$, but since we use coefficients in the field K, they extend to well-defined actions of $\mathrm{GL}_2(\mathbb{Q})$. Furthermore, since the elements of $\mathrm{MS}_k(N,K)$ are by definition invariant under the action of $\Gamma_1(N)$ and $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) \in \Gamma_1(N)$, the map Q_ξ is indeed well-defined as a map on $\mathbb{Q}/\mathbb{Z}$. It is easy to see that the map Q is a homomorphism of K-vector spaces if we endow $\mathrm{Maps}(\mathbb{Q}/\mathbb{Z}, \mathrm{Sym}^{k-2} K^2)$ with the canonical K-vector space structure.

Definition 1.1: For $\xi \in \mathrm{MS}_k(N,K)$ and a primitive Dirichlet character $\chi\colon (\mathbb{Z}/c)^\times \longrightarrow K^\times$ we write

$$A(\xi,\chi) := \sum_{j=0}^{c} \chi^*(j)\, Q_\xi\left(\frac{j}{c}\right) \in \mathrm{Sym}^{k-2} K^2.$$

Note that if $c = 1$ then we have $A(\xi, \chi) = Q_\xi(0)$ and that the association $\xi \longmapsto A(\xi, \chi)$ for fixed χ is a K-linear map $\mathrm{MS}_k(N, K) \longrightarrow \mathrm{Sym}^{k-2} K^2$.

We define $A(\xi, \chi, n) \in K$ ($n = 0, \ldots, k-2$) to be the coefficients of this element when we see it as a homogeneous polynomial as in lemma A.1.2 (a), i. e. these are such that

$$A(\xi, \chi) = \sum_{n=0}^{k-2} A(\xi, \chi, n) X^n Y^{k-2-n}.$$

Lemma 1.2: *We have for $n = 0, \ldots, k-2$*

$$A(\xi[ə], \chi, n) = (-1)^n \chi(-1) A(\xi, \chi, n).$$

Proof: We first note that for any $x \in \mathbb{Q}$ the relation

$$ə \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} ə$$

holds. Using this, the definition of Q_ξ and the definition of the action of matrices on modular symbols, it is easy to check that for any $x \in \mathbb{Q}$ we have

$$Q_{\xi[ə]}(x) = ə^\iota \bullet Q_\xi(-x).$$

We now further use the fact that the map $Q_{\xi[ə]}$ is defined on $\mathbb{Q}/\mathbb{Z}$ and compute

$$\begin{aligned} A(\xi[ə], \chi) &= \sum_{j=0}^{c} \chi^*(j) Q_{\xi[ə]}\left(\frac{j}{c}\right) \\ &= \sum_{j=0}^{c} \chi^*(-j) Q_{\xi[ə]}\left(\frac{-j}{c}\right) \\ &= \chi(-1) ə^\iota \bullet \sum_{j=0}^{c} \chi^*(j) Q_\xi\left(\frac{j}{c}\right). \end{aligned}$$

Since $ə^\iota$ acts on $\mathrm{Sym}^{k-2} K^2$ as $(-1)^n$ on the n-th component, the claim follows. □

1.2. The measure attached to a modular symbol

We continue to use the notation from the previous section. Now let R be a domain of characteristic 0 and K its field of fractions, and fix a prime p.

Lemma 1.3: *Let $\xi \in \mathrm{MS}_k(N, R)$. Then we have for all $x \in \mathbb{Q}/\mathbb{Z}$:*

$$Q_{T_p \xi}(x) = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \bullet \sum_{j=0}^{p-1} Q_\xi\left(\frac{x+j}{p}\right).$$

Proof: See [Kit94, Lem. 4.2]. The proof comes down to a calculation that essentially uses only the definition of T_p and its explicit description from lemma 1.1.54. We point out that the proof of [Kit94, Lem. 4.2] contains an error: in the fourth line from below, the matrix $\left(\begin{smallmatrix} p & j \\ 0 & 1 \end{smallmatrix}\right)$ has to be replaced by $\left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)$. ☐

Now fix $D \in \mathbb{N}$ prime to p. For $a \in R$ we write $\mathrm{MS}_k(N,R)^{T_p=a}$ for the R-submodule of modular symbols ξ for which $T_p\xi = a\xi$. To each such modular symbol we want to attach a measure on $\mathbb{Z}_{p,D}^\times$ with values in $\mathrm{Sym}^{k-2} R^2$. Such a measure is completely described if we specify the measure of the sets $z + Dp^m\mathbb{Z}_{p,D}$ for each $z \in \mathbb{Z}$ with $(z, Dp) = 1$ and $m \in \mathbb{N}$ since these sets form a fundamental system of open sets in $\mathbb{Z}_{p,D}^\times$. Of course these values cannot be assigned arbitrarily, they have to obey the distribution law. We write $\mathrm{D}^\mathrm{b}(\mathbb{Z}_{p,D}^\times, \mathrm{Sym}^{k-2} R^2)$ for the R-module of such measures (where D^b should mean "bounded distributions").

Now fix $\xi \in \mathrm{MS}_k(N,R)^{T_p=a}$, and assume that $a \in R^\times$. We define the measure by

$$\begin{aligned}\tilde{\mu}_\xi(z + Dp^m\mathbb{Z}_{p,D}) &:= a^{-m}\begin{pmatrix} p^m & \\ & 1\end{pmatrix} \bullet Q_\xi\left(\frac{z}{Dp^m}\right) \\ &= a^{-m}\begin{pmatrix} p^m & -z/D \\ & 1\end{pmatrix} \bullet \xi\left((\infty) - \left(\frac{z}{Dp^m}\right)\right), \quad z \in \mathbb{Z},\ (z, Dp) = 1,\ m \in \mathbb{N}.\end{aligned}$$

It is clear that the right hand side lies in $\mathrm{Sym}^{k-2} R^2$.

Lemma 1.4: *(a) $\tilde{\mu}_\xi$ satisfies the distribution relation.*

(b) The association $\xi \longmapsto \tilde{\mu}_\xi$ is a well-defined R-linear map

$$\mathrm{MS}_k(N,R)^{T_p=a} \longrightarrow \mathrm{D}^\mathrm{b}(\mathbb{Z}_{p,D}^\times, \mathrm{Sym}^{k-2} R^2).$$

Proof: The proof of the first assertion works analogous to [Kit94, Lem. 4.4] (using lemma 1.3). The second assertion is clear by construction. ☐

Lemma 1.5: *Let $m \geq 0$ and $\chi\colon (\mathbb{Z}/Dp^m)^\times \longrightarrow R^\times$ be a primitive Dirichlet character (which we view as a character of $\mathbb{Z}_{p,D}^\times$). Then*

$$\int_{\mathbb{Z}_{p,D}^\times} \chi^* \mathrm{d}\tilde{\mu}_\xi = \begin{cases} \left(1 - a^{-1}\begin{pmatrix} p & \\ & 1\end{pmatrix}\right) \bullet A(\xi,\chi), & m = 0, \\ a^{-m}\begin{pmatrix} p^m & \\ & 1\end{pmatrix} \bullet A(\xi,\chi), & m > 0.\end{cases}$$

With the convention that $\chi(p) = 0$ if $m > 0$ this can be written more compactly as

$$\int_{\mathbb{Z}_{p,D}^\times} \chi^* \mathrm{d}\tilde{\mu}_\xi = a^{-m}\left(\begin{pmatrix} 1 & \\ & p^m\end{pmatrix}\left(1 - a^{-1}\chi^*(p)\begin{pmatrix} 1 & \\ & p\end{pmatrix}\right)\right) \bullet A(\xi,\chi).$$

Proof: For $m > 0$ we have

$$\int_{\mathbb{Z}_{p,D}^\times} \chi^* \mathrm{d}\tilde{\mu}_\xi = \sum_{z=0}^{Dp^m} \chi^*(z)\tilde{\mu}_\xi(z + Dp^m\mathbb{Z}_{p,D}).$$

Inserting the definition of $\tilde{\mu}_\xi$ here immediately gives the result. For $m = 0$ this can be proved as in [Kit94, Lem. 4.5]. □

Since $\mathrm{Sym}^{k-2} R^2$ is a free R-module of rank $k-1$, we can view each $\mathrm{Sym}^{k-2} R^2$-valued measure as a $(k-1)$-tuple of R-valued measures. Viewing the elements of $\mathrm{Sym}^{k-2} R$ as homogeneous polynomials, we write the projection $\mathrm{Sym}^{k-2} R \longrightarrow R$ to the last component as $f \longmapsto f(0,1)$.

From now on we need to specialise our ring R to be the ring of integers $\mathcal{O}$ in a finite extension of $\mathbb{Q}_p$. We have then a canonical character $\kappa\colon \mathbb{Z}_{p,D}^\times \twoheadrightarrow \mathbb{Z}_p^\times \hookrightarrow \mathcal{O}^\times$.

We now define a measure μ_ξ on $\mathbb{Z}_{p,D}^\times$ with values in $\mathcal{O}$ by the formula

$$\mu_\xi(z + Dp^m\mathbb{Z}_{p,D}) := a^{-m}\xi\left((\infty) - \left(\frac{z}{Dp^m}\right)\right)(0,1), \quad z \in \mathbb{Z},\ (z, Dp) = 1,\ \ m \in \mathbb{N}.$$

This is just the last entry in the $(k-1)$-tuple of $\mathcal{O}$-valued measures coming from $\tilde{\mu}_\xi$, as one can easily see from the relation $\left(\begin{smallmatrix} p^m & -z/D \\ & 1 \end{smallmatrix}\right) \bullet f(0,1) = f(0,1)$ for $f \in \mathrm{Sym}^{k-2}\mathcal{O}^2$, $z \in \mathbb{Z}$ and $m \in \mathbb{N}$. More generally, we have the following important statement, which is known as "Manin's Lemma" or "Manin's trick":

Lemma 1.6: *We have*

$$\mathrm{d}\tilde{\mu}_\xi(z) = \left(-\frac{\kappa(z)}{D}X + Y\right)^{k-2} \mathrm{d}\mu_\xi(z) \quad (z \in \mathbb{Z}_{p,D}^\times).$$

Proof: Analogous to the proof of [Kit94, Lem. 4.6]. □

We can express the meaning of the above lemmas more explicitly as follows.

Corollary 1.7: *Let $\chi\colon \mathbb{Z}_{p,D}^\times \longrightarrow \mathcal{O}^\times$ be a primitive character of conductor Dp^m and $n \in \{0, \dots, k-2\}$. Then*

$$\int_{\mathbb{Z}_{p,D}^\times} \chi^*\kappa^n \mathrm{d}\mu_\xi = \binom{k-2}{n}^{-1} (-D)^n a^{-m} p^{mn}(1 - a^{-1}\chi^*(p)p^n)A(\xi, \chi, n).$$

Proof: Write the integral

$$\int_{\mathbb{Z}_{p,D}^\times} \chi^* \mathrm{d}\tilde{\mu}_\xi \in \mathrm{Sym}^{k-2}\mathcal{O}$$

as a $(k-1)$-tuple $(a_0, \dots, a_{k-2})$ with $a_i \in \mathcal{O}$. Then

$$a_n = (-D)^{-n}\binom{k-2}{n} \int_{\mathbb{Z}_{p,D}^\times} \chi^*\kappa^n \mathrm{d}\mu_\xi$$

by lemma 1.6. The claim then follows from lemma 1.5. □

1.3. The measure attached to a cusp form

Up to now a large part of our discussion was independent of a concrete choice of a modular symbol and partly also of the ring R. We now first look at the case $R = K = \mathbb{C}$. Fix a normalised eigenform $f \in \mathrm{S}_k(X_1(N), \mathbb{C})$ and let $\xi_f \in \mathrm{MS}_k(N, \mathbb{C})$ be the modular symbol attached to f, as defined in definition III.2.5. The key to relating special values of the complex L-function attached to f to the modular symbol ξ_f comes from the fact that the L-function can be expressed in terms of the Mellin transform of f, as stated in proposition IV.1.3. This together with proposition IV.1.2 allows us to compute $A(\xi_f, \chi)$.

Proposition 1.8: *Let $\chi\colon (\mathbb{Z}/c)^\times \longrightarrow \mathbb{C}^\times$ be a primitive Dirichlet character and $0 \le n \le k-2$. Then*

$$A(\xi_f, \chi, n) = \binom{k-2}{n}(-1)^n n!\mathrm{G}(\chi^*)(2\pi\mathrm{i})^{k-2-n}\mathrm{L}(f, \chi, n+1).$$

Proof: This is a calculation that uses propositions IV.1.2 and IV.1.3, together with an easy substitution and the definitions of the objects:

$$\begin{aligned}
A(\xi_f, \chi) &= \sum_{j=0}^{c-1} \chi^*(j) Q_\xi\left(\frac{j}{c}\right) \\
&= \sum_{j=0}^{c-1} \chi^*(j)\, \xi_f\left((\infty) - \left(\frac{j}{c}\right)\right) \begin{bmatrix} 1 & \frac{j}{c} \\ & 1 \end{bmatrix} \\
&= (2\pi\mathrm{i})^{k-1} \sum_{j=0}^{c-1} \chi^*(j) \left(\int_{\frac{j}{c}}^{\infty} f(z)(zX+Y)^{k-2}\mathrm{d}z\right) \begin{bmatrix} 1 & \frac{j}{c} \\ & 1 \end{bmatrix} \\
&= (2\pi\mathrm{i})^{k-1} \sum_{j=0}^{c-1} \chi^*(j) \int_{\frac{j}{c}}^{\infty} f(z)((z-\tfrac{j}{c})X+Y)^{k-2}\mathrm{d}z \\
&= (2\pi\mathrm{i})^{k-1} \int_0^\infty \left(\sum_{j=0}^{c-1} \chi^*(j) f(z+\tfrac{j}{c})\right)(zX+Y)^{k-2}\mathrm{d}z \\
&= (2\pi\mathrm{i})^{k-1}\mathrm{G}(\chi^*) \int_0^\infty f_\chi(z)(zX+Y)^{k-2}\mathrm{d}z \\
&= (2\pi\mathrm{i})^{k-1}\mathrm{G}(\chi^*) \sum_{l=0}^{k-2} \binom{k-2}{l} \left(\int_0^\infty f_\chi(z) z^l \mathrm{d}z\right) X^l Y^{k-2-l} \\
&= (2\pi\mathrm{i})^{k-1}\mathrm{G}(\chi^*) \sum_{l=0}^{k-2} \binom{k-2}{l} \left((-1)^l \frac{l!}{(2\pi\mathrm{i})^{l+1}}\mathrm{L}(f, \chi, n+1)\right) X^l Y^{k-2-l}. \qquad \square
\end{aligned}$$

Now let K be the number field generated by the Fourier coefficients of f and R its ring of integers. Fix an embedding $K \hookrightarrow \overline{\mathbb{Q}}$, which fixes an embedding of K into $\mathbb{C}$.

Let $\xi_f^\pm$ be the projections of ξ_f to the ±1-eigenspaces for the involution ə. Fix bases $\eta_f^\pm \in \mathrm{MS}_k(N, R)^\pm[f]$ bases and let $\mathcal{E}_\infty(f, \eta_f^\pm) \in \mathbb{C}^\times$ be the associated error terms (see definition III.2.8).

Corollary 1.9: *Let* $\chi\colon (\mathbb{Z}/c)^\times \longrightarrow \mathbb{C}^\times$ *be a primitive Dirichlet character and* $0 \leq n \leq k-2$. *Then*

$$A(\eta_f^\pm, \chi, n) = \frac{1}{2}\binom{k-2}{n}\frac{1}{\mathcal{E}_\infty(f, \eta_f^\pm)}(1 \pm \chi(-1)(-1)^n)(-1)^n n!\mathrm{G}(\chi^*)(2\pi\mathrm{i})^{k-2-n}\mathrm{L}(f, \chi, n+1).$$

Proof: Using the relation $\xi_f^\pm = \frac{1}{2}(\xi_f \pm \xi_f[\partial])$ and the fact that $\xi \longmapsto A(\xi, \chi)$ is K-linear, this follows immediately from proposition 1.8 and lemma 1.2. □

Up to now we have not yet fixed a prime p. We now do so, but here we do not assume that it divides N (in contrast to section 1.2). Our fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}$ then fixes an embedding of K into $\overline{\mathbb{Q}}_p$ and a place $\mathfrak{p}$ of K. Let $\mathcal{O}$ be the ring of integers of the completion of K at $\mathfrak{p}$.

We can now define the p-adic L-function for f. For this we assume that f is ordinary, i. e. that $T_p f = a_p f$ with $a_p \in \mathcal{O}^\times$. Since we want to use the results from section 1.2, which need that p divides the level of the involved modular form, we need to go over to a refinement. More precisely, let ψ be the nebentype of f and $\alpha \in \mathcal{O}$ be the unit root of the p-th Hecke polynomial (see definition II.5.16) and let f_α be the corresponding refinement (see definition II.7.3). Then f_α is itself ordinary (see remark II.7.4). If $p \mid N$, then we have $f_\alpha = f$, while f_α is a form of level Np if $p \nmid N$. Recall that $\alpha = a_p$ if $p \mid N$ and that in any case the p-th Hecke eigenvalue of f_α is α. We apply the theory from section 1.2 to f_α.

Definition 1.10: Define the p-adic L-function attached to f as the measure on $\mathbb{Z}_{p,D}^\times$ with values in $\mathcal{O}$ defined by

$$\mu_f := \mu_{\eta_{f_\alpha}^+ + \eta_{f_\alpha}^-} = \mu_{\eta_{f_\alpha}^+} + \mu_{\eta_{f_\alpha}^-}.$$

Theorem 1.11 (Mazur/Tate/Teitelbaum)**:** *For* $n = 0, \ldots, k-2$ *and each primitive Dirichlet character* $\chi\colon (\mathbb{Z}/Dp^m)^\times \longrightarrow \mathbb{C}^\times$ *we have*

$$\int_{\mathbb{Z}_{p,D}^\times} \chi^*\kappa^n \mathrm{d}\mu_f = n!(1-\alpha^{-1}\chi^*(p)p^n)(1-\alpha^{-1}\chi\psi(p)p^{k-n-1})\frac{D^n p^{mn}\mathrm{G}(\chi^*)}{\alpha^m(2\pi\mathrm{i})^{n+2-k}\mathcal{E}_\infty(f_\alpha, \eta_{f_\alpha}^s)}\mathrm{L}(f, \chi, n+1),$$

where $s = (-1)^n\chi(-1)$.

Proof: By corollaries 1.7 and 1.9, we have

$$\int_{\mathbb{Z}_{p,D}^\times} \chi^*\kappa^n \mathrm{d}\mu_{\eta_{f_\alpha}^\pm} = \frac{1}{2}(1\pm\chi(-1)(-1)^n)(-1)^n(-D)^n\alpha^{-m}p^{mn}(1-\alpha^{-1}\chi^*(p)p^n)\frac{n!\mathrm{G}(\chi^*)}{(2\pi\mathrm{i})^{n+2-k}\mathcal{E}_\infty(f_\alpha, \eta_{f_\alpha}^\pm)}\mathrm{L}(f_\alpha, \chi, n+1).$$

The result then follows from proposition IV.1.4. □

Note that the two Euler factor-like expressions are of a different nature: the first comes from the fact that we need to go over to a refinement in order to construct the p-adic L-function, while the second is a technical necessity forced by lemma 1.5.

2. The p-adic L-function for a Hida family

The step from the p-adic L-function for a single modular form to that for a Hida family is easy to describe using our previous work. We use the setup for Hida families: Let L, $\mathcal{O}$ and $\mathcal{I}$ be as in situation III.3.11, let K be the number field $\overline{\mathbb{Q}} \cap L$ and $\mathfrak{p}$ the place of K such that $L = K_{\mathfrak{p}}$, and fix an $\mathcal{I}$-adic cusp form $F \in \mathbb{S}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$. We need to assume that condition III.4.5 is satisfied.

Let $\Xi \in \mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})$. Then Ξ is a map $\mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \longrightarrow \mathcal{I}$, and we have a canonical map $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \longrightarrow \mathcal{UM}^{\mathrm{ord}}(Np^\infty, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{MS}}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O}), \mathcal{O})$ given by

$$(x) - (y) \longmapsto \left[\xi \longmapsto \xi((x) - (y))\right]$$

(of course it has to be checked that this is well-defined). Hence by precomposing with this map we can regard Ξ as a map $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \longrightarrow \mathcal{I}$. We can thus define

$$Q_\Xi(x) := \Xi((\infty) - (x)), \quad x \in \mathbb{Q}.$$

Totally analogously as in section 1.2, one can show that this defines a map $Q_\Xi \colon \mathbb{Q}/\mathbb{Z} \longrightarrow \mathcal{I}$ and that

$$aQ_\Xi(x) = \sum_{j=0}^{p-1} Q_\Xi\left(\frac{x+j}{p}\right)$$

if $T_p\Xi = a\Xi$ for some $a \in \mathcal{I}^\times$. We then define

$$\mu_\Xi(z + Dp^m\mathbb{Z}_p) := a^{-m} Q_\Xi\left(\frac{z}{p^m}\right), \quad z \in \mathbb{Z},\ (z, Dp) = 1,\ m \in \mathbb{N}.$$

Using the above property, we can show that this defines an $\mathcal{I}$-valued measure on $\mathbb{Z}_{p,D}^\times$ again analogously as in section 1.2.

We now choose $\mathcal{I}$-bases $\Xi^\pm$ of $\mathbb{MS}^{\mathrm{ord}}(Np^\infty, \mathcal{I})^\pm[F]$ and let $\mathcal{E}_{\mathfrak{p}}(\Xi^\pm, \eta_\phi^\pm) \in \mathcal{O}$ be the p-adic error term for each $\phi \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$, as defined in definition III.4.14. By construction, it is then clear that for any character ψ of $\mathbb{Z}_{p,D}^\times$ and any $\phi \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$, we have

$$\phi\left(\int_{\mathbb{Z}_{p,D}^\times} \psi \mathrm{d}\mu_{\Xi^\pm}\right) = \mathcal{E}_{\mathfrak{p}}(\Xi^\pm, \eta_\phi^\pm) \int_{\mathbb{Z}_{p,D}^\times} \psi \mathrm{d}\mu_{\eta_\phi^\pm}.$$

Theorem 2.1 (Kitagawa)**:** *There is an $\mathcal{I}$-valued measure μ_F on $\mathbb{Z}_{p,D}^\times$ such that for each $\phi \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r), each primitive Dirichlet character $\chi\colon (\mathbb{Z}/Dp^m)^\times \longrightarrow \mathbb{C}^\times$ and each $n = 0, \ldots, k-2$ we have*

$$\phi\left(\int_{\mathbb{Z}_{p,D}^\times} \chi^* \kappa^n \mathrm{d}\mu_F\right) = n!(1 - a_{p,\phi}^{-1}\chi^*(p)p^n)\frac{D^n p^{mn} \mathrm{G}(\chi^*)\mathcal{E}_{\mathfrak{p}}(\Xi^\pm, \eta_\phi^\pm)}{a_{p,\phi}^m (2\pi \mathrm{i})^{n+2-k}\mathcal{E}_\infty(F_\phi, \eta_\phi^s)} \mathrm{L}(F_\phi, \chi, n+1),$$

where $s = (-1)^n\chi(-1)$.

Proof: Put $\mu_F := \mu_{\Xi^+} + \mu_{\Xi^-}$. Then the result follows from the above and theorem 1.11. □

At the end we now assume that the Hida family F is new and has nebentype ψ. Then we can formulate the statement also in the following slightly different form. Recall from theorem III.3.15 that F_ϕ is often a newform, and in the case where it is not, there exists a newform of level N such that F_ϕ is its unique ordinary refinement. As introduced in definition III.3.16 we write F_ϕ^{new} for the newform at ϕ. For each ϕ of type (k, ε, r) let α_ϕ denote the unit root of the Hecke polynomial

$$X^2 - a_{p,\phi}^{\mathrm{new}} X + \varepsilon\psi\omega^{-k}(p)p^{k-1}$$

of F_ϕ^{new}, where $a_{p,\phi}^{\mathrm{new}}$ is the p-th Hecke eigenvalue of F_ϕ^{new}. Note that in the cases where F_ϕ is already new, we just have $\alpha_\phi = a_{p,\phi}$. Using again proposition IV.1.4, we can derive the following version of theorem 2.1.

Corollary 2.2: *There is an $\mathcal{I}$-valued measure μ_F on $\mathbb{Z}_{p,D}^\times$ such that for each $\phi \in \mathcal{X}_{\mathcal{I}}^{\mathrm{arith}}(\mathcal{O})$ of type (k, ε, r), each primitive Dirichlet character $\chi\colon (\mathbb{Z}/Dp^m)^\times \longrightarrow \mathbb{C}^\times$ and $n = 0, \dots, k-2$ we have*

$$\phi\left(\int_{\mathbb{Z}_{p,D}^\times} \chi^* \kappa^n \mathrm{d}\mu_F\right) =$$

$$n!(1 - \alpha_\phi^{-1}\chi^*(p)p^n)(1 - \alpha_\phi^{-1}\chi\varepsilon\psi\omega^{-k}(p)p^{k-n-1})\frac{D^n p^{mn} \mathrm{G}(\chi^*)\mathcal{E}_{\mathrm{p}}(\Xi^\pm, \eta_\phi^\pm)}{\alpha_\phi^m (2\pi\mathrm{i})^{n+2-k}\mathcal{E}_\infty(F_\phi, \eta_\phi^s)}\mathrm{L}(F_\phi^{\mathrm{new}}, \chi, n+1),$$

where $s = (-1)^n\chi(-1)$.

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List of symbols

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