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Masterarbeit

Relative Lubin-Tate Groups and Rigid Character Varieties in p-adic Fourier Theory

vorgelegt von

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Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Masterarbeit selbstständig verfasst und keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt habe.

Ort, Datum

(Milan Malčić)

Abstract

In this thesis, we study and generalize some results of the article [ST01] by Peter Schneider and Jeremy Teitelbaum. Schneider and Teitelbaum studied a certain rigid analytic group variety $\mathfrak{X}$ that parametrizes the locally L-analytic characters on o_L, where L is a finite extension of $\mathbb{Q}_p$. One of their main results in [ST01] states that $\mathfrak{X}$ becomes isomorphic to the open unit disk $\mathfrak{B}$ after base extension to $\mathbb{C}_p$. They proved this by using Lubin-Tate theory to make $\mathfrak{B}$ a group variety and p-adic Hodge theory to construct an isomorphism $\kappa\colon \mathfrak{B}/\mathbb{C}_p \longrightarrow \mathfrak{X}/\mathbb{C}_p$ of rigid group varieties. They raised the question whether this isomorhism can be generalized to the situation where $\mathfrak{B}$ is endowed with the group structure given by a *relative* Lubin-Tate formal group law (a generalization of the usual Lubin-Tate groups, introduced in [dS85]). We succeed in verifying a weaker version of this, namely that there is a group isomorphism $\mathfrak{B}(\mathbb{C}_p) \longrightarrow \mathfrak{X}(\mathbb{C}_p)$ on the level of $\mathbb{C}_p$-points that generalizes $\kappa(\mathbb{C}_p)$. Furthermore, in the last section of [ST01], Schneider and Teitelbaum applied their results to construct p-adic L-functions for supersingular elliptic curves. We study the question how the existence of their L-function implies congruences between the special values.

Zusammenfassung

In dieser Arbeit untersuchen und verallgemeinern wir Ergebnisse aus dem Artikel [ST01] von Peter Schneider und Jeremy Teitelbaum. Schneider und Teitelbaum haben eine gewisse rigid-analytische Gruppenvarietät $\mathfrak{X}$ untersucht, welche die lokal L-analytischen Charaktere auf o_L parametrisiert, wobei L eine endliche Erweiterung von $\mathbb{Q}_p$ ist. Eines ihrer Hauptergebnisse in [ST01] besagt, dass $\mathfrak{X}$ nach Basiserweiterung zu $\mathbb{C}_p$ isomorph zur offenen Einheitsdisk $\mathfrak{B}$ wird. Um dies zu beweisen, haben sie mittels Lubin-Tate-Theorie $\mathfrak{B}$ mit der Struktur einer Gruppenvarietät versehen. Im Anschluss haben sie mit Hilfe von p-adischer Hodge Theorie einen Isomorphismus $\kappa\colon \mathfrak{B}/\mathbb{C}_p \longrightarrow \mathfrak{X}/\mathbb{C}_p$ rigider Gruppenvarietäten konstruiert. Sie haben die Frage gestellt, ob sich obiger Isomorphismus auf die Situation verallgemeinern lässt, dass $\mathfrak{B}$ mit derjenigen Gruppenstruktur ausgestattet ist, die von einem *relativen* Lubin-Tate formalen Gruppengesetz induziert wird (relative Lubin-Tate-Gruppen sind eine von Ehud de Shalit in [dS85] eingeführte Verallgemeinerung der üblichen Lubin-Tate-Gruppen). Wir beantworten eine schwächere Version obiger Frage, indem wir zeigen, dass ein Gruppenisomorphismus $\mathfrak{B}(\mathbb{C}_p) \longrightarrow \mathfrak{X}(\mathbb{C}_p)$ auf der Ebene der $\mathbb{C}_p$-Punkte existiert, der $\kappa(\mathbb{C}_p)$ verallgemeinert. Darüber hinaus haben Schneider und Teitelbaum im letzten Abschnitt von [ST01] ihre Ergebnisse angewendet, um p-adische L-Funktionen zu elliptischen Kurven zu konstruieren. Wir untersuchen die Frage, wie sich aus der Existenz ihrer L-Funktion Kongruenzen zwischen den speziellen Werten herleiten lassen.

Contents

Introduction

The theory of L-functions plays a central role in modern number theory. For instance, two of the seven Millennium Problems are questions about L-functions. One of these two problems is the famous Riemann hypothesis, which asserts that all the non-trivial zeros of the Riemann Zeta function lie on the line with real part $1/2$. The Riemann Zeta function can be thought of as the archetypical L-function. Indeed, many classical L-functions are defined in association to an arithmetic object, by a formula that in some way mimics the formula of the Riemann Zeta function. Examples of such L-functions are Dedekind Zeta functions and Hasse-Weil L-functions. They are associated to algebraic number fields and elliptic curves, respectively. A remarkable thing about these L-functions is that their values at the integers ("the special values") contain information concerning the arithmetic of the associated objects. A conjectural example of this phenomenon is the second of the two Millennium Problems we mentioned: the Birch and Swinnerton-Dyer conjecture. Given an elliptic curve E over a number field K, the conjecture asserts that the rank of the abelian group $E(K)$ of K-valued points of E is equal to the order of the zero of $L_E(s)$ at $s = 1$, where L_E is the Hasse-Weil L-function associated to E. To give another example, due to Kummer, we first recall that the values of the Riemann Zeta function ζ at the negative integers are rational numbers, as has been known since Euler. Kummer has proven that an odd prime number p divides the order of the ideal class group of $\mathbb{Q}(e^{2\pi i/p})$ if and only if p divides the numerator of at least one of $\zeta(-1), \zeta(-3), \ldots, \zeta(2-p)$. An important ingredient in Kummer's proof of this theorem are certain congruences modulo p^n between the values of the Zeta function at the negative integers. From a modern point of view, these congruences are equivalent to the existence of a (necessarily unique) continuous function ζ_p on the ring $\mathbb{Z}_p$ of p-adic integers, such that the values of ζ_p at the negative integers essentially coincide with those of ζ. The function ζ_p is called the Kubota-Leopoldt p-adic Zeta function. When we write ζ_p, we mean "the first branch" $\zeta_{p,1}$ from [Col02] Théorème VI.2.1. (In particular, we have cheated in not mentioning that ζ_p has a pole at $1 \in \mathbb{Z}_p$ and that ζ_p satisfies the interpolation property only at the negative integers that are congruent to 1 modulo $p-1$.) For a precise statement about the various branches $\zeta_{p,i}$ and their interpolation properties, see [Col02] Théorème VI.2.1 and the subsequent Remarque VI.2.2.

The Kubota-Leopoldt p-adic Zeta function is historically the first example of a p-adic L-function. p-adic L-functions are analytic functions of a p-adic variable that interpolate special values of classical L-functions. They are of great interest to number theorists. One reason for this is that, in order to link the properties of a classical L-function to the arithmetic properties of the associated object, one often has to consider the p-adic scenario. But p-adic interpolation (i.e. showing the existence of a p-adic L-function) is an intricate matter. This is where the theory of p-adic distributions plays a crucial role. To roughly illustrate this on the example of the Kubota-Leopoldt p-adic Zeta function ζ_p, we recall that a locally analytic distribution on $\mathbb{Z}_p$ can be thought of as a generalized measure, against which locally $\mathbb{Q}_p$-analytic functions on $\mathbb{Z}_p$ can be integrated. For the moment, we only consider distributions with values in $\mathbb{Q}_p$, to keep things simple. The space $D(\mathbb{Z}_p, \mathbb{Q}_p)$ of all such distributions is a ring, with multiplication given by the convolution product. For a distribution $\lambda \in D(\mathbb{Z}_p, \mathbb{Q}_p)$, one can define (the minus-first branch of) its Mellin transform Mel_λ. It is an analytic function on $\mathbb{Z}_p$ (see [Col02] Corollaire VI.2.7) satisfying

$$\mathrm{Mel}_\lambda(n) = \int_{\mathbb{Z}_p^\times} x^n \lambda(x)$$

for all $n \in \mathbb{N}$ such[1] that $n \equiv -1 \mod p-1$. So if one finds a distribution μ such that $\int_{\mathbb{Z}_p^\times} x^n \mu(x)$ essentially coincides with $\zeta(-n)$, one may define ζ_p to be $s \longmapsto \mathrm{Mel}_\mu(-s)$ up to a few factors. Thus, thanks to the formalism of the Mellin transform, the p-adic Zeta function can be identified with a distribution μ satisfying the interpolation property.

Since the distribution ring $D(\mathbb{Z}_p, \mathbb{Q}_p)$ is (a priori) a very complicated ring, it is not at all clear that the matter becomes simpler when reduced to finding a distribution μ that satisfies the interpolation property. Fortunately, Yvette Amice discovered an isomorphism between the distribution ring $D(\mathbb{Z}_p, \mathbb{Q}_p)$ and the ring of power series in one variable over $\mathbb{Q}_p$ converging on the open unit disk in $\mathbb{C}_p$. Given a distribution λ, we have

$$\int_{\mathbb{Z}_p^\times} x^n \lambda(x) = \left(\frac{d}{dt}\right)^n A_\lambda(\exp(T)-1)\Big|_{T=0} \tag{$*$}$$

where A_λ is the power series corresponding to λ under Amice's isomorphism. So the problem of constructing μ (and thus ζ_p) amounts to finding a power series in $\mathbb{Q}[[T]]$ that converges on the unit disk in $\mathbb{C}_p$ and whose n-th derivative at the origin (as on the right-hand side of $(*)$) is essentially equal to $\zeta(-n)$. But it is not too hard to find such a power series, see [Col02] Proposition VI.1.2. This completes our outline of the construction of ζ_p via distribution theory. We review the Amice isomorphism in Section 1.1.

In view of Amice's result, it is natural to ask whether there is a similar description of the ring $D(o_L, K)$ of K-valued locally L-analytic distributions on o_L, when $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ is a tower of fields with L finite over $\mathbb{Q}_p$ and K complete and o_L the ring of integers in L. In their article [ST01], Peter Schneider and Jeremy Teitelbaum studied this question. They introduced a certain rigid analytic group variety $\mathfrak{X}$, whose K-valued points parametrize

[1] We have tacitly assumed that p is odd. If $p = 2$, then the equality holds for all $n \in \mathbb{N}$ such that $n \equiv -1 \mod 2$.

K-valued locally L-analytic characters on o_L. Hence $\mathfrak{X}$ is called the rigid character variety (associated to o_L). Schneider and Teitelbaum then produced an isomorphism ("the Fourier isomorphism") between $D(o_L, K)$ and $\mathcal{O}(\mathfrak{X}/K)$, where $\mathfrak{X}/K$ is the base extension of $\mathfrak{X}$ to K and $\mathcal{O}(\mathfrak{X}/K)$ is the ring of rigid functions on $\mathfrak{X}/K$. In the case $L = \mathbb{Q}_p = K$, the variety $\mathfrak{X}$ can be identified with the rigid open unit disk $\mathfrak{B}$, $\mathcal{O}(\mathfrak{B})$ is the ring of power series converging on $\mathfrak{B}(\mathbb{C}_p) = \{z \in \mathbb{C}_p \colon |z| < 1\}$, and the Fourier isomorphism coincides with the Amice isomorphism. However, if $L \neq \mathbb{Q}_p$ then $\mathfrak{X}$ is not isomorphic to the unit disk $\mathfrak{B}$ and $D(o_L, L)$ is generally not a ring of convergent power series, see [ST01] Corollary 3.8 and Lemma 3.9. Fortunately, the varieties $\mathfrak{B}$ and $\mathfrak{X}$ do become isomorphic after base extension to $\mathbb{C}_p$. To prove this, Schneider and Teitelbaum endowed the $\mathbb{C}_p$-valued points $\mathfrak{B}(\mathbb{C}_p)$ with the group structure provided by a Lubin-Tate group law associated to L. They then used a result of Tate's in [Tat67] to construct a group isomorphism $\kappa(\mathbb{C}_p) \colon \mathfrak{B}(\mathbb{C}_p) \xrightarrow{\sim} \mathfrak{X}(\mathbb{C}_p)$ on the level of $\mathbb{C}_p$-points, see [ST01] Proposition 3.1. The isomorphism $\kappa(\mathbb{C}_p)$ depends on the choice of a period of the Lubin-Tate group. Finally, with the help of Fontaine's work on p-adic Hodge theory, they showed that the group isomorphism $\kappa(\mathbb{C}_p)$ derives from an isomorphism $\kappa \colon \mathfrak{B}/\mathbb{C}_p \xrightarrow{\sim} \mathfrak{X}/\mathbb{C}_p$ between rigid $\mathbb{C}_p$-analytic varieties, see [ST01] Theorem 3.6. One importance of this result is that it gives an isomorphism $\mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \xrightarrow{\sim} \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$. Composing with the Fourier isomorphism yields an isomorphism

$$D(o_L, \mathbb{C}_p) \xrightarrow{\sim} \mathcal{O}(\mathfrak{B}/\mathbb{C}_p). \qquad (**)$$

Thus, $D(o_L, \mathbb{C}_p)$ is the ring of power series over $\mathbb{C}_p$ converging on the open unit disk $\mathfrak{B}(\mathbb{C}_p)$. In Section 5 of [ST01], Schneider and Teitelbaum exploited this fact to construct a p-adic L-function for a CM elliptic curve at a supersingular prime. Their approach is analogous to the one we described for the p-adic Zeta function: the isomorphism $(**)$ reduces the problem to finding a convergent power series $F \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ such that the values $\partial^n F(T)|_{T=0}$ essentially coincide with the special values of the L-function of the elliptic curve. Here ∂ denotes the invariant differential of the Lubin-Tate group. We note that it is not easy to find such a power series F. For this, Schneider and Teitelbaum rely on the machinery of Coleman power series and elliptic units from [dS87].

In Section 5 of [ST01], Schneider and Teitelbaum mention that their method for constructing the p-adic L-function might generalize to the situation of so-called relative Lubin-Tate groups. These are formal groups generalizing Lubin-Tate groups, and were introduced by Ehud de Shalit in [dS85]. A relative Lubin-Tate group is attached to a finite unramified extension E/L and to an element $\xi \in L$ of valuation[2] $[E : L]$. In the case $E = L$, one recovers the usual Lubin-Tate groups. The question of extending the results of [ST01] to the situation of relative Lubin-Tate groups can now be stated more precisely: does the isomorphism of rigid group varieties $\kappa \colon \mathfrak{B}/\mathbb{C}_p \longrightarrow \mathfrak{X}/\mathbb{C}_p$ generalize to the case when $\mathfrak{B}$ is endowed with a relative Lubin-Tate group law attached to a finite unramified extension E/L? A positive answer would allow for construction of the p-adic L-function in a more general setting (see Example 1.30).

[2] Here we mean the valuation on L that is normalized so that the valuation of a uniformizer is 1.

In this thesis, we answer a weaker version of the question above by proving that if $\mathfrak{B}(\mathbb{C}_p)$ is endowed with a relative Lubin-Tate group law associated to a finite unramified extension E/L, then there is a group isomorphism on the level of $\mathbb{C}_p$-points $\mathfrak{B}(\mathbb{C}_p) \xrightarrow{\sim} \mathfrak{X}(\mathbb{C}_p)$ that generalizes $\kappa(\mathbb{C}_p)$ (see Theorem 3.20). Our proof is a direct generalization of the proof of [ST01] Proposition 3.1, which uses results of [Tat67]. In Section 3.2, we survey these results of Tate's and verify that they hold for relative Lubin-Tate groups. We do not assume that the reader is familiar with relative Lubin-Tate groups, and we review them in Section 3.1. The above is an outline of Chapter 3.

In Chapter 1, we review the first three sections of [ST01], preparing the way for Chapter 3. In Chapter 2, we establish some results that complement the last two sections of [ST01]. Although these results are interesting in themselves, they were envisaged as tools to help derive congruences for the special values of Schneider and Teitelbaum's L-function. We suggest an idea for an approach for deriving congruences in Section 2.3.

Chapter 2 and Chapter 3 are independent of each other. Whereas the goal of Chapter 3 is to generalize to the relative Lubin-Tate case, Chapter 2 stays in the realm of classical Lubin-Tate groups and is concerned with analyzing the construction of Schneider and Teitelbaum's L-function and deriving consequences from its existence.

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Notations and preliminaries

Throughout the thesis, we use the following notations.

The letter p denotes a fixed prime number. We fix fields $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ such that $L/\mathbb{Q}_p$ is a finite extension and K is complete. We always use the absolute value $|\cdot|$ on $\mathbb{C}_p$ normalized by $|p| = p^{-1}$. The corresponding additive valuation $v_p\colon \mathbb{C}_p^\times \longrightarrow\mathrel{\mkern-14mu}\rightarrow \mathbb{Q}$ is then given by $v_p(x) = -\log|x| / \log(p)$, so that $|x| = p^{-v_p(x)}$. For any intermediate field $\mathbb{Q}_p \subseteq F \subseteq \mathbb{C}_p$, we denote the valuation ring $\{a \in F\colon |a| \leq 1\}$ by o_F, its unique maximal ideal $\{a \in F\colon |a| < 1\}$ by $\mathfrak{m}_F$ and its residue field $o_F/\mathfrak{m}_F$ by k_F. For L, let $d := [L : \mathbb{Q}_p], o := o_L$ and $\pi := \pi_L$ a fixed prime element of o. We denote the cardinality of the residue field $k := k_L = o/\pi$ by $q = p^f$ and the ramification index of $L/\mathbb{Q}_p$ by e.

p-adic analysis

We will work with objects of p-adic analysis, specifically with (paracompact) locally L-analytic manifolds. For the definitions and basic properties of such objects, see [Sch11]. For a locally L-analytic manifold M we let $C^{an}(M, K)$ denote the K-vector space of all K-valued locally analytic functions on M. Then $C^{an}(M, K)$ is a locally convex topological K-vector space in a natural way. It is a Hausdorff space and if $(M_i)_{i\in I}$ is a cover of M by pairwise disjoint open subsets, then there is a topological isomorphism $C^{an}(M, K) \cong \prod_{i\in I} C^{an}(M_i, K)$. We will give a brief description of the topology on $C^{an}(M, K)$ in a special case in Section 1.2, the general details may be found in Section 10 of Chapter II in [Sch11].
If $M = G$ is a locally L-analytic group (sometimes also called a Lie group over L), we let $\widehat{G}(K) \subseteq C^{an}(G, K)$ denote the group of locally analytic K-valued characters of G. By a K-valued character we mean a continuous group homomorphism $\chi\colon G \longrightarrow K^\times$. Note that for compact G, the image of every K-valued character χ is contained in $o_K = \{z \in K\colon |z| \leq 1\}$. Indeed, a $z \in K^\times \setminus o_K$ cannot be a member of the image $\mathrm{im}(\chi)$, because $|z^n| = |z|^n$ gets arbitrarily large whereas $\mathrm{im}(\chi)$ is a compact and hence bounded subset of $K^\times$.

Let G be a locally L-analytic group. The continuous dual $D(G, K) := C^{an}(G, K)'$ is a K-algebra with multiplication given by the convolution product $*$ (see [ST02], Proposition 2.3). It is called the algebra of K-valued distributions on G. Equipped with the strong dual topology, it is a locally convex K-vector space. See [Sch13] for the basics on continuous dual spaces. In the case when G is compact, the convolution product is continuous, i.e. $D(G, K)$ is a topological K-algebra. In the case when G is commutative, $D(G, K)$ is commutative as a ring. It follows that $D(o, K)$ is a commutative topological K-algebra. In Section 1.2.1, we will see that $D(o, K)$ is a K-Fréchet space. Distribution algebras are interesting for many reasons. For instance, they play an important role in the theory of locally analytic representations of p-adic Lie groups (see [ST02]). However, our focus will be on the algebra $D(o, K)$ and its role in the construction of p-adic L-functions.

Although locally analytic distributions will be the primary objects of study, there is another type of distributions that we will make use of in Chapter 2. These are the so-called continuous distributions. They are sometimes also called "measures" in the literature. For topological spaces X and Y, let $C^{cont}(X, Y)$ denote the set of all continuous functions $X \longrightarrow Y$. For a compact X, we endow $C^{cont}(X, K)$ with the supremum norm $\|f\|_{\sup} = \sup_{x \in X} |f(x)|$, which makes it into a K-Banach space (cf. Example 2 after Lemma 2.3 in [Sch11]). The elements of the continuous dual $D^{cont}(X, K) := C^{cont}(X, K)'$ are called K-valued continuous distributions on X. In the case when $X = G$ is a compact locally L-analytic group, Proposition 12.1 in [ST] (together with the preceding comments there) explains how $D^{cont}(G, K)$ is a K-algebra. We remark that there are two different natural topologies with which one can equip $D^{cont}(G, K)$. One is the topology given by the usual operator norm. For a description of the other, see the discussion around Definition 11.2 in [ST].

Rigid analytic geometry

We also consider objects of rigid analytic geometry, the theory of which is presented rather extensively in [BRG84]. The notes [Sch98] provide an excellent short overview of the theory.

If $\mathfrak{Y}$ is a rigid analytic variety over L, let $\mathcal{O}(\mathfrak{Y})$ denote the ring of global sections on $\mathfrak{Y}$. This ring is also referred to as the ring of holomorphic (or rigid) functions on $\mathfrak{Y}$.

Suppose that $\mathfrak{Y}$ is reduced. Then a function $f \in \mathcal{O}(\mathfrak{Y})$ is called bounded if there exists a real constant $C > 0$ such that $|f(y)| \leq C$ for any $\mathbb{C}_p$-point $y \in \mathfrak{Y}(\mathbb{C}_p)$. On the subring $\mathcal{O}^b(\mathfrak{Y}) := \{f \in \mathcal{O}(\mathfrak{Y}) \colon f \text{ is bounded}\}$ we have the supremum norm $\|f\|_{\mathfrak{Y}} = \sup_{y \in \mathfrak{Y}(\mathbb{C}_p)} |f(y)|$, which makes $\mathcal{O}^b(\mathfrak{Y})$ into a Banach K-algebra (as mentioned in the preliminaries in [BSX15]).

If $\mathfrak{Y}$ is separated, base change to K is possible (see [BRG84] 9.3.6), and we denote it by $\mathfrak{Y}/K$. This is e.g. the case when $\mathfrak{Y}$ is affinoid, i.e. $\mathfrak{Y} = \mathrm{Sp}(A)$ is the affinoid space associated to some affinoid L-algebra A. Then $\mathfrak{Y}/K := \mathfrak{Y}\widehat{\otimes}_L K := \mathrm{Sp}(A\widehat{\otimes}_L K)$, where $\widehat{\otimes}$ is the completed tensor product. It is worth remarking that base change is a special case of the fiber product only if K is finite algebraic over L, because only in this case can K be viewed as an affinoid L-algebra, allowing one to write $\mathfrak{Y}\widehat{\otimes}_L K = \mathfrak{Y} \times_{\mathrm{Sp}(L)} \mathrm{Sp}(K)$. In general, $\mathfrak{Y}/K$ is constructed by gluing the base change of affinoids.

The Tate algebra in n variables over K is denoted by $K\langle T_1,\dots,T_n\rangle$. It consists of all power series $f=\sum_{\nu\in\mathbb{N}^n} a_\nu T^\nu \in K[[T_1,\dots,T_n]]$ such that $\lim_{|\nu|\to\infty} a_\nu = 0$, and is equipped with the Gauss norm $\|f\| = \max|a_\nu|$. Recall that a K-algebra A is called affinoid if there exists an epimorphism of K-algebras $\alpha\colon K\langle T_1,\dots,T_n\rangle \twoheadrightarrow A$ for some $n\in\mathbb{N}$. Affinoid algebras are always equipped with the following topology. The Gauss norm induces a residue norm $\|\|_\alpha$ on A via $\|\alpha(f)\|_\alpha = \inf_{g\in\ker\alpha}\|f-g\|$. All residue norms on A are equivalent and all K-algebra-homomorphisms between two affinoid algebras are continuous. As an example, for $c\in L$ with $|c|\geq 1$ the affinoid L-algebra $L\langle c^{-1}T\rangle$ is defined as the image of the morphism $L\langle T\rangle \longrightarrow L\langle T\rangle, T\longmapsto c^{-1}T$. This morphism is well-defined, since the condition $|c|\geq 1$ ensures that $c^{-1}T\in L\langle T\rangle$ is power bounded (cf. [Bos14] 3.1 Lemma 19). Since the morphism is injective, the norm of an element $\sum_{n=0}^\infty a_n c^{-n}T^n \in L\langle c^{-1}T\rangle$ is given by the Gauss norm of its unique preimage $\sum_{n=0}^\infty a_n T^n\in L\langle T\rangle$. Thus, by setting $b_n := a_n c^{-n}$, $L\langle c^{-1}T\rangle$ can be identified with the algebra $\{f=\sum_{n=0}^\infty b_nT^n \in L[[T]]\colon \lim_{n\to\infty} b_nc^n = 0\}$ equipped with the norm $\|f\| = \max|b_nc^n|$. Note that $L\langle T\rangle \longrightarrow L\langle c^{-1}T\rangle, T\longmapsto c^{-1}T$ is an isomorphism of K-Banach-algebras, whereas the inclusion $L\langle c^{-1}T\rangle \hookrightarrow L\langle T\rangle$ is not a topological embedding in general. Otherwise its image, i.e. $L\langle c^{-1}T\rangle$ equipped with the Gauss norm of $L\langle T\rangle$, would also be a Banach-algebra and hence closed in $L\langle T\rangle$. But this is not the case in general. To see this, note that $L\langle c^{-1}T\rangle$ is dense in $L\langle T\rangle$, since $L[T]\subseteq L\langle c^{-1}T\rangle$ and $L[T]$ is dense in $L\langle T\rangle$. If $L\langle c^{-1}T\rangle$ were moreover closed in $L\langle T\rangle$, we could conclude $L\langle c^{-1}T\rangle = L\langle T\rangle$. This is obviously not true in general, as there may exist power series that converge on the closed unit disk but not on the closed disk of radius $|c|$, if $|c|>1$.

$\mathfrak{B}_1$, resp. $\mathfrak{B}$ will always denote the rigid L-analytic open disk of radius one around the point $1\in L$, resp. around $0\in L$. Its K-points are $\mathfrak{B}_1(K)=\{z\in K\colon |z-1|<1\}$, resp. $\mathfrak{B}(K)=\{z\in K\colon |z|<1\}$.

For $x\in\mathbb{Z}_p$ and $n\in\mathbb{N}$, we let $\binom{x}{n} = \frac{x(x-1)\cdots(x-(n-1))}{n!}$, which is easily seen to be an element of $\mathbb{Z}_p$. The multiplicative abelian group $\mathfrak{B}_1(K)\subseteq K^\times$ becomes a $\mathbb{Z}_p$-module via $\mathbb{Z}_p\times\mathfrak{B}_1(K)\longrightarrow\mathfrak{B}_1(K)$, $(a,z)\longmapsto z^a := \sum_{n=0}^\infty\binom{a}{n}(z-1)^n$. The notation z^a is adequate because $z^a = z\cdots z$ (a times) if $a\in\mathbb{N}$. Furthermore, if $(a_n)_n$ is a sequence of natural numbers converging to a in $\mathbb{Z}_p$, then z^{a_n} converges to z^a in $\mathfrak{B}_1(K)$. It follows that $\log(z^a) = a\log(z)$ for all $a\in\mathbb{Z}_p, z\in\mathfrak{B}_1(K)$.

We briefly recall how the the rigid structure on $\mathfrak{B}$ is defined by gluing an increasing sequence of closed disks. This is analogous to the construction of the rigid affine space in Section 5.4 of [Bos14]. For $c\in L$ with $r=|c|\geq 1$, let $\overline{\mathfrak{B}}(r)$ be the closed disk of radius r around 0. This is an affinoid rigid L-variety, associated to the affinoid L-algebra $L\langle c^{-1}T\rangle$. As mentioned, $L\langle c^{-1}T\rangle$ can be identified with $\{f=\sum_{n=0}^\infty a_nT^n\in L[[T]]\colon \lim_{n\to\infty} a_nc^n = 0\} = \left\{f\in L[[T]]\colon f \text{ converges on } \overline{\mathfrak{B}}(r)(\mathbb{C}_p)\right\}$. More generally, we also have closed rigid L-disks of any radius $r\in p^{\mathbb{Q}} = \left|\mathbb{C}_p^\times\right|$. Indeed, for $r=p^{-m/n}$ with $m\in\mathbb{Z}$ and $n\in\mathbb{N}$, let $A := L\langle X,Y\rangle/(p^mY-X^n)$ and $\overline{\mathfrak{B}}(r) := \mathrm{Sp}(A)$ the associated affinoid space. Its K-points are $\overline{\mathfrak{B}}(r)(K) = \{(z,z^n/p^m)\in K^2\colon |z|\leq 1 \text{ and } |z^n/p^m|\leq 1\}\cong\{z\in K\colon |z|\leq p^{-m/n}=r\}$. Interpreting the chain $\overline{\mathfrak{B}}(p^{-1})\subseteq\overline{\mathfrak{B}}(p^{-1/2})\subseteq\overline{\mathfrak{B}}(p^{-1/3})\dots$ as a gluing datum, we obtain the rigid space $\mathfrak{B}$ equipped with the admissible covering $\mathfrak{B} = \bigcup_n \overline{\mathfrak{B}}(p^{-1/n})$. The ring of global sections $\mathcal{O}(\mathfrak{B}/K)$ of $\mathfrak{B}/K$ is the ring of power series $f = \sum_{n=0}^{\infty} a_n T^n$ over K which converge on $\mathfrak{B}(\mathbb{C}_p)$.

In general, by an open disk we always mean a disk defined by a strict inequality in the appropriate setting. A closed disk, on the other hand, is defined by the condition that the distance to the center be less than or equal to the radius of the disk. Almost always, any disk (whether open or closed in the sense just explained) will be both open and closed topologically in the ambient space, since we are dealing with ultrametric spaces.

Other conventions

Our convention is that rings have multiplicative identity elements, and that homomorphisms of rings respect the identity elements. Throughout, "ring" means "not necessarily commutative ring" unless explicit mention is made to the contrary. Nevertheless, almost all of the rings appearing in this thesis will be commutative.

Chapter 1

Review of p-adic Fourier theory

In Sections 1.2 and 1.3, we review the article [ST01] by Schneider and Teitelbaum. Along the way, we provide proofs for facts that are stated without proof or reference in [ST01]. In Section 1.1, we review a classical theorem by Amice that inspired Schneider and Teitelbaum's Fourier theory. Amice's theorem not only serves the purpose of showcasing and motivating the goals of Sections 1.2 and 1.3, but is also an important ingredient in the proof of the main theorem of Fourier theory (Theorem 1.16).

1.1 The Amice transform as a motivational example

In the 1960s, french mathematician Amice formulated a complete description of the ring $D(\mathbb{Z}_p, \mathbb{Q}_p)$ by showing that it is isomorphic to a certain ring of convergent power series, namely the ring $\mathcal{O}(\mathfrak{B})$ defined below. We will state two formulations of this result ((1.B) and (1.D) below) and then discuss how p-adic Fourier theory generalizes them.
We assume $L = \mathbb{Q}_p = K$ for the whole section. Recall that $\mathfrak{B}$ then denotes the rigid $\mathbb{Q}_p$-analytic open disk of radius one around the point $0 \in \mathbb{Q}_p$. Its ring of global sections is

$$\begin{aligned}\mathcal{O}(\mathfrak{B}) &= \left\{ f = \sum_{n=0}^{\infty} a_n T^n \in \mathbb{Q}_p[[T]] \colon v_p(a_n) + rn \to +\infty \text{ for all } r > 0 \right\} \\ &= \left\{ f \in \mathbb{Q}_p[[T]] \colon f \text{ converges on } \mathfrak{m}_{\mathbb{C}_p} = \mathfrak{B}(\mathbb{C}_p) \right\}.\end{aligned}$$

Example 1.1. Any power series $f \in \mathbb{Q}_p[[T]]$ whose coefficients are bounded lies in $\mathcal{O}(\mathfrak{B})$. There is an obvious isomorphism between the ring of power series with bounded coefficients and $\mathbb{Q}_p \otimes \mathbb{Z}_p[[T]]$. An example for a power series that lies in $\mathcal{O}(\mathfrak{B})$ but whose coefficients are not bounded is the logarithm $\log(1+T) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} T^n$.

The Amice transform of a distribution $\lambda \in D(\mathbb{Z}_p, \mathbb{Q}_p)$ is defined as the formal power series

$$A_\lambda(T) = \sum_{n=0}^{\infty} T^n \int_{\mathbb{Z}_p} \binom{y}{n} \lambda(y) = \int_{\mathbb{Z}_p} (1+T)^y \lambda(y).$$

The last identity is purely formal here and the expression $\int_{\mathbb{Z}_p} f(y)\lambda(y)$ for a function $f \in C^{an}(\mathbb{Z}_p, \mathbb{Q}_p)$ is just notation for $\lambda(f)$. Nevertheless, Lemme V.3.10 in [Col02] tells us that $A_\lambda \in \mathcal{O}(\mathfrak{B})$ and

$$A_\lambda(z) = \int\limits_{\mathbb{Z}_p} (1+z)^y \lambda(y) \quad \text{for all } z \in \mathfrak{B}(\mathbb{C}_p). \tag{1.A}$$

Theorem 1.2. *The map*

$$\begin{aligned} D(\mathbb{Z}_p, \mathbb{Q}_p) &\xrightarrow{\sim} \mathcal{O}(\mathfrak{B}) \\ \lambda &\longmapsto A_\lambda \end{aligned} \tag{1.B}$$

is an isomorphism of $\mathbb{Q}_p$-Fréchet algebras.

Proof. The following is an elaboration of the proof of [Col02] Thèoréme V.3.11. The proof relies on Thèoréme V.3.3, which is a version of Mahler's theorem on expansions. This theorem says that the functions $[\frac{k}{p^n}]!\binom{\cdot}{k}$ for $k \in \mathbb{N}$ form a so-called Banach basis of the Banach space $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$. We define the space $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$ in Section 1.2.1. Colmez denotes this space by LA_n. He also denotes the floor function (taking the integer part of a real number x) by $[x]$. By the definition of a Banach basis, we have an isometric isomorphism

$$\begin{aligned} c_0(\mathbb{N}, \mathbb{Q}_p) &\xrightarrow{\sim} \mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p) \\ (a_k)_k &\longmapsto \sum_{k\in\mathbb{N}} a_k \, [k/p^n]! \binom{\cdot}{k} \end{aligned}$$

where $c_0(\mathbb{N}, \mathbb{Q}_p)$ is the Banach space of all zero sequences $(a_k)_{k\in\mathbb{N}}$ in $\mathbb{Q}_p$ equipped with the maximum norm $\|(a_k)_k\| = \max_k |a_k|$. The dual space of $c_0(\mathbb{N}, \mathbb{Q}_p)$ is well-understood: [Sch11] Lemma 2.6 says that

$$\begin{aligned} c_0(\mathbb{N}, \mathbb{Q}_p)' &\xrightarrow{\sim} \ell^\infty(\mathbb{N}, \mathbb{Q}_p) \\ \ell &\longmapsto (\ell(e_k))_{k\in\mathbb{N}} \end{aligned}$$

is an isometric isomorphism onto the Banach space $\ell^\infty(\mathbb{N}, \mathbb{Q}_p)$ of all bounded sequences in $\mathbb{Q}_p$ equipped with the maximum norm. Here, $e_k \in c_0(\mathbb{N}, \mathbb{Q}_p) \subseteq \ell^\infty(\mathbb{N}, \mathbb{Q}_p)$ denotes the "k-th unit vector" and $c_0(\mathbb{N}, \mathbb{Q}_p)'$ is equipped with the operator norm (which induces the strong dual topology, cf. [Sch13] Remark 6.7). Using the linear embedding $\ell^\infty(\mathbb{N}, \mathbb{Q}_p) \hookrightarrow \mathbb{Q}_p[[T]], e_k \longmapsto T^k$, we conclude that the mapping

$$\begin{aligned} \mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)' &\longrightarrow \mathbb{Q}_p[[T]] \\ \ell &\longmapsto \sum_{k=0}^{\infty} b_k T^k \end{aligned} \tag{1.C}$$

with

$$b_k := \ell\left(\binom{\cdot}{k}\right) = \ell\left([k/p^n]!\binom{\cdot}{k}\right)([k/p^n]!)^{-1}$$

induces a bijection onto its image, which consists of all power series $\sum_{k=0}^{\infty} c_k T^k$ for which $(c_k \cdot [k/p^n]!)_k$ is bounded. This boundedness condition implies (see Lemma 1.3 (i)) that $\sum_{k=0}^{\infty} c_k T^k$ converges on the open disk of radius $p^{-1/(p-1)p^n}$.
Now, one may obtain the Amice transform A_λ of a distribution $\lambda \in D(\mathbb{Z}_p, \mathbb{Q}_p)$ in the following way: compose $\lambda\colon C^{an}(\mathbb{Z}_p, \mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ with the continuous inclusion $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p) \longrightarrow \varinjlim_m \mathcal{F}_m(G, K) = C^{an}(\mathbb{Z}_p, \mathbb{Q}_p)$ to get an element of $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)'$ and then apply (1.C) to it. This shows that A_λ converges on the open disk of radius $p^{-1/(p-1)p^n}$. Since this is true for all n, it follows that $A_\lambda \in \mathcal{O}(\mathfrak{B})$.
To describe the inverse of (1.B), let $\sum_{k=0}^{\infty} b_k T^k \in \mathcal{O}(\mathfrak{B})$. The corresponding distribution λ is constructed as follows. Any $f \in C^{an}(\mathbb{Z}_p, \mathbb{Q}_p)$ belongs to an $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$ for some n and can hence be written as

$$f = \sum_{k=0}^{\infty} v_k(f,n)\,[k/p^n]!\binom{\cdot}{k}$$

with a unique zero sequence $(v_k(f,n))_k \in c_0(\mathbb{N}, \mathbb{Q}_p)$. We define

$$\lambda(f) := \sum_{k=0}^{\infty} v_k(f,n)\,[k/p^n]!\,b_k.$$

The term on the right-hand side converges because $(b_k \cdot [k/p^n]!)_k$ is bounded (see Lemma 1.3 (ii)) and $(v_k(f,n))_k$ is a zero sequence. Moreover, it is independent of the choice of n: as the functions $\binom{\cdot}{k}$ for $k \in \mathbb{N}$ form a Banach basis of $C^{cont}(\mathbb{Z}_p, \mathbb{Q}_p)$ (cf. [Col02] Corollaire V.2.3.), f has a unique series representation $f = \sum_{k=0}^{\infty} a_k(f)\binom{\cdot}{k}$ in $C^{cont}(\mathbb{Z}_p, \mathbb{Q}_p)$. Together with Lemma 1.3 (iii), this implies that

$$(v_k(f,n) \cdot [k/p^n]!)_k = (a_k(f))_k = (v_k(f,m)\,[k/p^m]!)_k$$

holds for any $n, m \in \mathbb{N}$ such that $f \in \mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$ and $f \in \mathcal{F}_m(\mathbb{Z}_p, \mathbb{Q}_p)$. For any n, the linear form λ is clearly continuous on $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$, as it is the element of $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)' \cong \ell^\infty(\mathbb{N}, \mathbb{Q}_p)$ corresponding to the bounded sequence $(b_k \cdot [k/p^n]!)_k$. Therefore it is also continuous on $\varinjlim_m \mathcal{F}_m(G, K) = C^{an}(\mathbb{Z}_p, \mathbb{Q}_p)$.
We omit the proof of continuity and multiplicativity of the map (1.B) and refer to [Col02] V.4 v). $\square$

Lemma 1.3. (i) *Let $n \in \mathbb{N}$ and let $g = \sum_{k=0}^{\infty} c_k T^k \in \mathbb{Q}_p[[T]]$. Suppose that the sequence $(c_k \cdot [k/p^n]!)_k$ is bounded. Then g converges on the open disk of radius $p^{-1/(p-1)p^n}$.*

(ii) *Let $\sum_{k=0}^{\infty} b_k T^k \in \mathcal{O}(\mathfrak{B})$. Then the sequence $(b_k \cdot [k/p^n]!)_k$ is bounded for all $n \in \mathbb{N}$.*

(iii) *Let $n \in \mathbb{N}$ and let $f \in C^{an}(\mathbb{Z}_p, \mathbb{Q}_p)$. Suppose that there exist functions $f_k \in \mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$ and scalars $r_k \in \mathbb{Q}_p$ such that $f = \sum_{k=0}^{\infty} r_k f_k$ in $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$ (i.e. the series converges to f with respect to the norm of $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p)$). Then $f = \sum_{k=0}^{\infty} r_k f_k$ in $C^{cont}(\mathbb{Z}_p, \mathbb{Q}_p)$ (i.e. the series converges to f with respect to the norm of $C^{cont}(\mathbb{Z}_p, \mathbb{Q}_p)$).*

Proof. (i) Since $(c_k \cdot [k/p^n]!)_k$ is bounded, there is a $C \in \mathbb{R}$ such that $v_p(c_k \cdot [k/p^n]!) \geq C$ holds for all k. We need to show that $v_p(c_k z^k) \to \infty$ for all $z \in \mathbb{C}_p$ that satisfy $v_p(z) > 1/(p-1)p^n$. We have $v_p(c_k z^k) = v_p(c_k) + k v_p(z) \geq C - v_p([k/p^n]!) + k v_p(z)$. But $k v_p(z) - v_p([k/p^n]!) \to \infty$ according to the proof of [Col02] Lemme V.3.10 (there stated in the form $z^k/[k/p^n]! \to 0$).

(ii) We have $k/p^{n+1} - 1 \leq v_p([k/p^n]!)$, as mentioned in the proof of [Col02] Thèoréme V.3.11. Hence $v_p(b_k) + v_p([k/p^n]!) + 1 \geq v_p(b_k) + k/p^{n+1}$. But $v_p(b_k) + k/p^{n+1} \to \infty$ because $\sum_{k=0}^{\infty} b_k T^k \in \mathcal{O}(\mathfrak{B})$. It follows that $v_p(b_k \cdot [k/p^n]!)$ is bounded from below, which is equivalent to the assertion (ii).

(iii) This follows immediately from the fact that the inclusion $\mathcal{F}_n(\mathbb{Z}_p, \mathbb{Q}_p) \xrightarrow{\subseteq} C^{cont}(\mathbb{Z}_p, \mathbb{Q}_p)$ is continuous (see Lemma 1.5 in Section 1.2).

$\square$

Remark 1.4. Note that the Amice transform of the Dirac distribution δ_1 is $A_{\delta_1} = 1 + T$. The definition of the Dirac distribution $\delta_x \in D(M, K)$, for a general locally L-analytic manifold M and $x \in M$, is given by $\delta_x(f) := f(x)$.

We discuss the Fréchet algebra structures of $D(\mathbb{Z}_p, \mathbb{Q}_p)$ and $\mathcal{O}(\mathfrak{B})$ in a more general situation in Section 1.2. Before we address the significance of the Amice isomorphism above, we wish to give an alternative formulation of it. We consider $\mathbb{Z}_p$ as a locally $\mathbb{Q}_p$-analytic group and use the bijections from (1.M) in Section 1.2:

$$\begin{aligned} \widehat{\mathbb{Z}_p}(\mathbb{Q}_p) \longleftrightarrow \mathfrak{B}_1(\mathbb{Q}_p) &\longleftrightarrow \mathfrak{B}(\mathbb{Q}_p) \\ \chi &\longmapsto \chi(1) \\ [a \mapsto z^a] &\longleftarrow z. \end{aligned}$$

The second one is given by $z \longmapsto z - 1$. This allows us to consider $\widehat{\mathbb{Z}_p}(\mathbb{Q}_p)$ as the $\mathbb{Q}_p$-points of a rigid $\mathbb{Q}_p$-analytic variety $\mathfrak{X}$, which is called the rigid character variety of the locally $\mathbb{Q}_p$-analytic group $\mathbb{Z}_p$. Now the Amice isomorphism (1.B) may be restated as

$$\begin{aligned} D(\mathbb{Z}_p, \mathbb{Q}_p) &\xrightarrow{\sim} \mathcal{O}(\mathfrak{X}) \\ \lambda &\longmapsto F_\lambda, \end{aligned} \tag{1.D}$$

where F_λ is the rigid function corresponding to $A_\lambda \in \mathcal{O}(\mathfrak{B})$ under the above identification of $\mathfrak{B}$ and $\mathfrak{X}$. To determine the value of F_λ at a character $\chi \in \widehat{\mathbb{Z}_p}(\mathbb{Q}_p)$, let $z = \chi(1)$. We obtain

$$F_\lambda(\chi) = A_\lambda(z - 1) = \lambda(\chi),$$

where the last equality follows from (1.A). The function F_λ is called the Fourier transform of λ.
Now, the first formulation (1.B) of the theorem gives a very useful concrete description of the ring $D(\mathbb{Z}_p, \mathbb{Q}_p)$ through the well-understood ring of power series $\mathcal{O}(\mathfrak{B})$. It allows us, among other things, to construct integrals which interpolate given special values and therefore give rise to p-adic L-functions. See Chapitre VI of [Col02] for a construction of the Kubota-Leopoldt p-adic zeta function using this approach.
On the other hand, it is the second formulation (1.D) of the isomorphism that allows a generalization to arbitrary L and K (such that $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ with $L/\mathbb{Q}_p$ a finite extension and K complete). Schneider and Teitelbaum discovered the analogue of the rigid character variety $\mathfrak{X}$ in this setting, and we present their construction in Section 1.2. There we also review their generalization of (1.D).
For general L, the character variety $\mathfrak{X}$ is no more simply the open unit disk. In fact, if $L \neq \mathbb{Q}_p$, it is not isomorphic to the unit disk over any discretely valued field K. So, a priori, we do not have a good understanding of $\mathfrak{X}$. However, using the theory of Lubin-Tate formal groups, Schneider and Teitelbaum showed that $\mathfrak{X}/\mathbb{C}_p$ becomes isomorphic to the open unit disk over $\mathbb{C}_p$. This gives a nice description of $\mathcal{O}(\mathfrak{X}/\mathbb{C}_p)$ and thus, via descent, of $\mathcal{O}(\mathfrak{X})$. We treat these facts in Section 1.3.

1.2 Fourier theory for $G = o_L$

Let $G := o$ denote the additive group $o = o_L$ viewed as a locally L-analytic group. One of the main results of [ST01] states that the ring $D(G, K)$ is isomorphic to the ring of rigid functions on $\mathfrak{X}/K$, where $\mathfrak{X}$ is a certain rigid group variety over L, called the rigid character variety of G. This is a generalization of Amice's description of the ring $D(\mathbb{Z}_p, \mathbb{Q}_p)$. In this section, we first recall some facts about $C^{an}(G, K)$ and $D(G, K)$ and the "restriction of scalars", then we describe the construction of $\mathfrak{X}$ and exhibit the Fourier isomorphism $\mathscr{F} : D(G, K) \xrightarrow{\sim} \mathcal{O}(\mathfrak{X}/K)$.

Before we begin, we need one more bit of notation. For a locally L-analytic manifold M, let M_0 denote M but viewed as a locally $\mathbb{Q}_p$-analytic manifold. We say that M_0 is obtained from M by restriction of scalars.
The Lie algebra of G can naturally be identified with the Lie algebra of G_0, and we denote it by $\mathfrak{g}$. We have that $\mathfrak{g} \cong L$ and the exponential map $\mathfrak{g} \dashrightarrow G$ is given by the identity map $L \dashrightarrow o$. We use the dashed arrow to indicate that the map is defined only on a neighbourhood of $0 \in \mathfrak{g}$.

1.2.1 Facts about $C^{an}(G, K), D(G, K)$ and the restriction of scalars

Here we state the results of Section 1 of [ST01] for $G = o$, even though they hold more generally for any compact locally L-analytic group. This makes some of our formulae

simpler, for instance note how in general (1.J) involves the exponential map $\mathfrak{g} \dashrightarrow G$, which in our case is just the identity $L \dashrightarrow o$. Also note that G is a one-dimensional Lie group over L and G_0 a d-dimensional Lie group over $\mathbb{Q}_p$.
To describe the topology on $C^{an}(G, K)$, we start with the standard fundamental system $G = o \supseteq \pi o \supseteq \ldots \supseteq \pi^n o \ldots$ of open subgroups of G. Note that $\pi^n o = \overline{\mathfrak{B}}(p^{-n/e})(L)$ is a disk in L of radius $|\pi^n| = p^{-n/e}$. To simplify notation, we write $H_n := \pi^n o$. By taking the cosets of H_n as an open cover of G, we obtain for each n a so-called index $\mathfrak{I}_n$ for G (see [Sch11], Section 10). For such an index, we have the Banach space

$$\mathcal{F}_n(G, K) := \prod_{g \in G/H_n} \mathcal{F}_{(g+H_n)}(K) \subseteq \prod_{g \in G/H_n} C^{an}(g + H_n, K) = C^{an}(G, K),$$

where $\mathcal{F}_{(g+H_n)}(K)$ is the K-Banach space of K-valued, globally L-analytic functions on the disk $g + H_n$, i.e.

$$\mathcal{F}_{(g+H_n)}(K) = \left\{ f \colon g + H_n \longrightarrow K \colon f(x) = \sum_{i=0}^{\infty} c_i (x-g)^i, c_i \in K, c_i \pi^{ni} \to 0 \right\}.$$

The norm on $\mathcal{F}_{(g+H_n)}(K)$ is given by

$$\| \sum_{i=0}^{\infty} c_i (x-g)^i \|_{g,n} = \max_i \left| c_i \pi^{ni} \right|,$$

and so $\mathcal{F}_{(g+H_n)}(K)$ is isomorphic to the K-Banach space of r_n-convergent power series with coefficients in K (defined in Section 5 of [Sch11]), with $r_n = |\pi^n|$. In other words, $\mathcal{F}_{(g+H_n)}(K)$ is isomorphic to the Tate algebra $\mathcal{O}(\overline{\mathfrak{B}}(r_n)/K)$ of rigid functions on the affinoid disk $\overline{\mathfrak{B}}(r_n)/K$. In particular, by the maximum principle, the norm on $\mathcal{F}_{(g+H_n)}(K)$ has the alternative description

$$\|f\|_{g,n} = \max_x |f(x)| \tag{1.E}$$

where the maximum is taken over all elements $x \in g + \overline{\mathfrak{B}}(r_n)(\mathbb{C}_p) = g + \pi^n o_{\mathbb{C}_p}$. Note that this implies that $\|f\|_{g,n}$ doesn't depend on the choice of representative of $g + H_n$. Alternatively, this can be seen directly using Corollary 5.5 in [Sch11]. There are many equivalent norms that induce the product topology on $\mathcal{F}_n(G, K)$. We choose the maximum norm, which is given by the maximum of the norms of the components.
Finally, the space $C^{an}(G, K)$ is the locally convex inductive limit

$$C^{an}(G, K) = \varinjlim_n \mathcal{F}_n(G, K). \tag{1.F}$$

This limit topology on $C^{an}(G, K)$ is the same as the limit topology on $C^{an}(G, K) = \varinjlim_{\mathfrak{I}} \mathcal{F}_{\mathfrak{I}}(K)$ from Section 10 of [Sch11], where $\mathfrak{I}$ runs over all indices for G. The reason for this is the fact that the family $(\mathfrak{I}_n)_n$ of indices obtained above from $(H_n)_n$ is a cofinal subfamily of the family of all indices for G.

Lemma 1.5. *The inclusion $C^{an}(G,K) \xrightarrow{\subseteq} C^{cont}(G,K)$ is continuous.*

Proof. Since $C^{an}(G,K) = \varinjlim_m \mathcal{F}_m(G,K)$ carries the limit topology, it suffices to show that the inclusion $\mathcal{F}_n(G,K) \longrightarrow C^{cont}(G,K)$ is continuous for any n. This is a linear map between normed spaces, so we have to show that there is a constant $c > 0$ such that $\sup_{y \in G} |f(y)| \leq c \cdot \max_{g \in G/H_n} \|f\|_{g,n}$ holds for all $f \in \mathcal{F}_n(G,K)$. The identity (1.E) implies that $c = 1$ does the job. $\square$

Remark 1.6. In general, the inclusion $\mathcal{F}_n(G,K) \longrightarrow C^{an}(G,K)$ is not a topological embedding. More precisely, the topology on $\mathcal{F}_n(G,K)$ in general is strictly finer than the subspace topology induced by $\mathcal{F}_{n+1}(G,K) \supseteq \mathcal{F}_n(G,K)$. Indeed, if the inclusion $\mathcal{F}_n(G,K) \longrightarrow \mathcal{F}_{n+1}(G,K)$ were a topological embedding, the image would be a Banach space and hence closed in $\mathcal{F}_{n+1}(G,K)$. But, if $L = \mathbb{Q}_p = K$, Mahler's theorem on expansions (cf. proof of Theorem 1.2) tells us that any $f \in \mathcal{F}_{n+1}(\mathbb{Z}_p, \mathbb{Q}_p)$ has a unique representation as a series in $\binom{\cdot}{k} \in \mathcal{F}_0(\mathbb{Z}_p,\mathbb{Q}_p) \subseteq \mathcal{F}_n(\mathbb{Z}_p,\mathbb{Q}_p)$, so the image of $\mathcal{F}_n(\mathbb{Z}_p,\mathbb{Q}_p)$ in $\mathcal{F}_{n+1}(\mathbb{Z}_p,\mathbb{Q}_p)$ is dense and therefore not closed.

By dualizing (1.F), we obtain an identification

$$D(G,K) = \varprojlim_n \mathcal{F}_n(G,K)'. \tag{1.G}$$

of K-vector spaces. The strong dual topology on $D(G,K)$ coincides with the projective limit topology. In particular, $D(G,K)$ is a K-Fréchet algebra. Moreover, from (1.G) we can see that there is a natural strict inclusion

$$D(G,K) \hookrightarrow D(G,\mathbb{C}_p).$$

Because, as we will see in the next subsection, $\mathfrak{X}$ is constructed as a subvariety of a variety $\mathfrak{X}_0$ associated to the $\mathbb{Q}_p$-analytic group G_0, it is important to consider the relation between the L-analytic distributions $D(G,K)$ and the $\mathbb{Q}_p$-analytic distributions $D(G_0,K)$. Since the obvious injective K-linear map

$$C^{an}(G,K) \hookrightarrow C^{an}(G_0,K) \tag{1.H}$$

is in fact a homeomorphism onto its image ([ST01] Lemma 1.2), the dual map

$$D(G_0,K) \twoheadrightarrow D(G,K) \tag{1.I}$$

is shown, by an application of the Hahn-Banach theorem in the appropriate setting, to be surjective. By the open mapping theorem, it follows then that (1.I) is an open quotient map. Lemma 1.1 in [ST01] describes the image of (1.H) by using the action of $\mathfrak{g} = L$ on $C^{an}(G,K)$ and $C^{an}(G_0,K)$. In particular, we can characterize the L-analytic characters among the $\mathbb{Q}_p$-analytic ones: define

$$\begin{aligned} \mathrm{d}\colon \widehat{G_0}(K) &\longrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(L,K) \\ \chi &\longmapsto \mathrm{d}\chi, \end{aligned}$$

by

$$\begin{aligned} \mathbb{d}\chi \colon L &\longrightarrow K \\ \mathfrak{r} &\longmapsto \frac{d}{dt}\chi(t\mathfrak{r})|_{t=0}. \end{aligned} \tag{1.J}$$

Note that $\mathbb{d} \colon \widehat{G_0}(K) \longrightarrow \operatorname{Hom}_{\mathbb{Q}_p}(L, K)$ is clearly a group homomorphism. By [ST01] Lemma 1.3, we have

$$\begin{aligned} \widehat{G}(K) &= \{\chi \in \widehat{G_0}(K) \colon \mathbb{d}\chi \text{ is } L\text{-linear}\} \\ &= \{\chi \in \widehat{G_0}(K) \colon \mathbb{d}\chi(t) = t \cdot \mathbb{d}\chi(1) \text{ for all } t \in L\}. \end{aligned} \tag{1.K}$$

An equivalent way of formulating (1.K) is to say that the diagram

$$\begin{tikzcd} \widehat{G}(K) \arrow[r, "\subseteq"] \arrow[d, "\mathbb{d}"'] & \widehat{G_0}(K) \arrow[d, "\mathbb{d}"] \\ \operatorname{Hom}_L(L, K) \arrow[r, "\subseteq"] & \operatorname{Hom}_{\mathbb{Q}_p}(L, K) \end{tikzcd} \tag{1.L}$$

is cartesian. Note the philosophical similarity to complex analysis, where a function $f \colon \mathbb{C} \longrightarrow \mathbb{C}$ is complex analytic if and only if it is real analytic (i.e. both of its component functions $f_1, f_2 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ are real analytic) and its differential is $\mathbb{C}$-linear.

1.2.2 The rigid character variety $\mathfrak{X}$ and the Fourier isomorphism

Recall that $\mathfrak{B}_1(K)$ is a $\mathbb{Z}_p$-module and that $a \in \mathbb{Z}_p$ acts on $z \in \mathfrak{B}_1(K)$ by sending it to its a-th power z^a. As a stepping stone to defining $\mathfrak{X}$, we first introduce the rigid L-analytic variety

$$\mathfrak{X}_0 := \mathfrak{B}_1 \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p).$$

Since $\operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module of rank d, the variety $\mathfrak{X}_0$ is non-canonically isomorphic to a rigid d-dimensional open polydisk over L:

$$\mathfrak{X}_0 \cong \mathfrak{B}_1^d.$$

Since the rigid open unit disk is constructed as an admissible increasing union of closed disks of radius $p^{-1/n}$ over all $n \geq 1$, we deduce by using the sheaf property that $\mathcal{O}(\mathfrak{X}_0)$ is a projective limit of affinoid algebras. In particular, it is in a natural way an L-Fréchet algebra.

The rigid variety $\mathfrak{X}_0$ is designed so that its K-points parametrize locally analytic characters on G_0. To prove this, we need the following results.

Lemma 1.7. *We have*

$$\operatorname{Hom}^{cont}_{\mathbb{Z}}(\mathbb{Z}_p, K^\times) = \operatorname{Hom}^{cont}_{\mathbb{Z}}(\mathbb{Z}_p, \mathfrak{B}_1(K)),$$

i.e. any continuous character $\mathbb{Z}_p \longrightarrow K^\times$ factors over $\mathfrak{B}_1(K) \subseteq K^\times$.

Proof. Let $\chi\colon \mathbb{Z}_p \longrightarrow K^\times$ be a continuous character. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, the image of χ is contained in the closure of $\chi(\mathbb{Z})$. Since $\mathfrak{B}_1(K) \subseteq K^\times$ is a closed subset, the assertion will follow if we show that $\chi(\mathbb{Z}) \subseteq \mathfrak{B}_1(K)$. Because $p^n \to 0$ in $\mathbb{Z}_p$, it follows that $\chi(1)^{p^n} \to 1$ in $\mathbb{C}_p$, i.e. $\chi(1)^{p^n} - 1 \to 0$. For large $N \in \mathbb{N}$, we obtain in $o_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p}$the equation $0 = \chi(1)^{p^N} - 1 = (\chi(1) - 1)^{p^N}$, which implies $\chi(1) \in (1 + \mathfrak{m}_{\mathbb{C}_p}) \cap K = \mathfrak{B}_1(K)$. It follows that $\chi(\mathbb{Z}) \subseteq \mathfrak{B}_1(K)$. $\square$

Lemma 1.8. *We have*

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, \mathfrak{B}_1(K)) \subseteq \widehat{\mathbb{Z}_p}(K),$$

i.e. any $\mathbb{Z}_p$-linear map $\mathbb{Z}_p \longrightarrow \mathfrak{B}_1(K)$ is locally $\mathbb{Q}_p$-analytic.

Proof. Any $\mathbb{Z}_p$-linear map $f\colon \mathbb{Z}_p \longrightarrow \mathfrak{B}_1(K)$ is completely determined by $f(1)$. If we set $z := f(1)$, then $f(a) = z^a$ for all $a \in \mathbb{Z}_p$. Hence, we need to argue that z^a is locally analytic in a. First consider those $a \in \mathbb{Z}_p$ that are close to zero, i.e. such that $|a|$ is small. For such a, we have that $\exp(a \cdot \log(z))$ converges and that $z^a = \exp(a \cdot \log(z))$. Indeed, the last equality obviously holds for $a \in \mathbb{N}$ close to zero, and these are dense around zero. Now consider a general $a \in \mathbb{Z}_p$, close to some $a_0 \in \mathbb{Z}_p$. Then $z^a = z^{a-a_0+a_0} = z^{a-a_0} \cdot z^{a_0}$ and z^{a-a_0} is a power series in $a - a_0$ by the preceding case, since $a - a_0$ is close to zero. This completes the proof. $\square$

Proposition 1.9. *We have*

$$\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(K)) = \widehat{G_0}(K).$$

Proof. We will in fact show that

$$\widehat{G_0}(K) = \mathrm{Hom}_{\mathbb{Z}}^{cont}(o, K^\times) = \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(K)).$$

Clearly we have $\widehat{G_0}(K) \subseteq \mathrm{Hom}_{\mathbb{Z}}^{cont}(o, K^\times)$. To prove that $\mathrm{Hom}_{\mathbb{Z}}^{cont}(o, K^\times) \subseteq \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(K))$, consider a $\chi \in \mathrm{Hom}_{\mathbb{Z}}^{cont}(o, K^\times)$. Since we have an isomorphism

$$o \cong \mathbb{Z}_p^d$$

of topological groups, the isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}^{cont}(\mathbb{Z}_p, K^\times)^d &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}^{cont}(\mathbb{Z}_p^d, K^\times) \\ (\varphi_i)_i &\longmapsto [(a_i)_i \longmapsto \varphi_1(a_1) \cdots \varphi_d(a_d)] \end{aligned}$$

and Lemma 1.7 together imply that the image of χ is contained in $\mathfrak{B}_1(K)$. Since the map $\chi\colon o \longrightarrow \mathfrak{B}_1(K)$ is $\mathbb{Z}$-linear and continuous, we deduce by the density of $\mathbb{Z}$ in $\mathbb{Z}_p$ that χ is $\mathbb{Z}_p$-linear, completing the proof of the desired inclusion. Finally, since o is isomorphic to $\mathbb{Z}_p^d$ as a locally $\mathbb{Q}_p$-analytic group, the inclusion $\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(K)) \subseteq \widehat{G_0}(K)$ follows similarly from Lemma 1.8. $\square$

As an immediate consequence of Proposition 1.9 (in the case $L = \mathbb{Q}_p$) we obtain the mutually inverse isomorphisms that we have used in Section 1.1:

$$\begin{aligned} \widehat{\mathbb{Z}_p}(K) &\longleftrightarrow \mathfrak{B}_1(K) \\ \chi &\longmapsto \chi(1) \\ [a \mapsto z^a] &\longleftarrow z. \end{aligned} \tag{1.M}$$

More generally, we obtain:

Corollary 1.10. *The map*

$$\begin{aligned} \mathfrak{B}_1(K) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) &\xrightarrow{\sim} \widehat{G_0}(K) \\ z \otimes \beta &\longmapsto [g \longmapsto z^{\beta(g)}] =: \chi_{z \otimes \beta} \end{aligned}$$

is an isomorphism of $\mathbb{Z}_p$-modules.

Proof. In general, for a commutative ring A and A-modules M and N, we have the homomorphism

$$\begin{aligned} \mathrm{Hom}_A(M, A) \otimes_A N &\longrightarrow \mathrm{Hom}_A(M, N) \\ \beta \otimes n &\longmapsto [m \longmapsto \beta(m) \cdot n]. \end{aligned}$$

This map is clearly an isomorphism if M is a free A-module. For $A = \mathbb{Z}_p, M = o$ and $N = \mathfrak{B}_1(K)$, this yields the isomorphism

$$\begin{aligned} \mathfrak{B}_1(K) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(K)) \\ z \otimes \beta &\longmapsto \chi_{z \otimes \beta} \end{aligned}$$

which, together with Proposition 1.9, establishes the assertion. $\square$

Remark 1.11. Let $t_1, \ldots, t_d$ be a $\mathbb{Z}_p$-basis of o and let $\beta_1, \ldots, \beta_d$ be the corresponding dual basis of $\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$. We may identify $\mathfrak{B}_1(K)^d$ with $\mathfrak{B}_1(K) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$ via $(z_1, \ldots, z_d) \longmapsto \sum_{i=1}^d z_i \otimes \beta_i$. Then Corollary 1.10 says that the characters $\chi \in \widehat{G_0}$ are precisely those of the form $\chi(a) = z_1^{a_1} \cdots z_d^{a_d}$ where $a = \sum_{i=1}^d a_i t_i$.

In view of Corollary 1.10 we have $\mathfrak{X}_0(K) = \widehat{G_0}(K)$, and in this sense the variety $\mathfrak{X}_0$ represents the character group $\widehat{G_0}$. We will now define the Fourier morphism $\mathscr{F}$ for G_0 and $\mathfrak{X}_0$. In this situation it is immediately seen, by reducing to the case of Amice, to be well-defined and an isomorphism.

Definition 1.12. The Fourier transform of a distribution $\lambda \in D(G_0, K)$ is the function

$$\begin{aligned} F_\lambda \colon \widehat{G_0}(\mathbb{C}_p) &\longrightarrow \mathbb{C}_p \\ \chi &\longmapsto \lambda(\chi). \end{aligned}$$

The Fourier transform is indeed well-defined for any K, because we have a strict inclusion $D(G_0, K) \hookrightarrow D(G_0, \mathbb{C}_p)$, as already mentioned. By a several-variable-version of the Amice isomorphism (which is described at the beginning of the proof of Theorem 4.4 in [Koh11]), the map

$$\begin{aligned} \mathscr{F} : D(G_0, K) &\xrightarrow{\sim} \mathcal{O}(\mathfrak{X}_0/K) \\ \lambda &\longmapsto F_\lambda. \end{aligned}$$

is a well-defined isomorphism of K-Fréchet algebras. The isomorphism $\mathscr{F}$ plays an important role in the proof of the Fourier isomorphism for G and $\mathfrak{X}$ (Theorem 1.16 below).

In order to define $\mathfrak{X}$, we show that $\widehat{G}$ is cut out in $\widehat{G_0}$ by analytic equations. For this, we will need the following lemma.

Lemma 1.13. *The following formula holds for all $z \in \mathfrak{B}_1(K)$ and all $\beta \in \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$:*

$$\mathbb{d}\chi_{z\otimes\beta} = \log(z) \cdot \beta.$$

Proof. For $x \in o$ sufficiently close to zero, we have $z^{\beta(x)} = \exp(\beta(x) \cdot \log(z))$. Consequently, $\mathbb{d}\chi_{z\otimes\beta}(a)$ is just the derivative of (the germ of) the function

$$t \longmapsto \exp(\beta(a \cdot t) \cdot \log(z))$$

at $t = 0$. Since this function is the composition of $t \longmapsto \beta(a \cdot t)$ with $x \longmapsto \exp(x \cdot \log(z))$, the assertion follows by the chain rule and the equality $\frac{d}{dt}\beta(a \cdot t)|_{t=0} = \beta(a)$. To see this last equality, note that $\frac{d}{dt}\beta(a \cdot t)|_{t=0}$ is, by the chain rule, just the differential of β at the origin applied to a. But β, being linear, is equal to its own differential at any point. $\square$

This lemma, combined with the description of $\widehat{G}(K)$ stated in (1.K), implies that $\widehat{G}$ is cut out by the equations

$$(\beta(t) - t \cdot \beta(1)) \cdot \log(z) = 0 \qquad \text{for all } t \in L. \tag{1.N}$$

Definition 1.14. The analytic subset of $\mathfrak{X}_0$ cut out by the equations above in (1.N) can be endowed with a (unique) structure of a reduced closed rigid analytic subvariety of $\mathfrak{X}_0$, see [BRG84] 9.5.3 Proposition 4. The variety thus obtained is denoted by $\mathfrak{X}$ and called the rigid character variety of G. Note that $\mathfrak{X}$ can also be defined by any finite subfamily of equations of (1.N) parametrized by a $\mathbb{Q}_p$-basis $t_1, \ldots, t_d$ of L.

The following description of $\mathfrak{X}$ indicates that $\mathfrak{X}$ is one-dimensional (cf. also Corollary 1.19 later on).

Proposition 1.15. *Let $t_1, \ldots, t_d$ be a $\mathbb{Z}_p$-basis of o and let $\beta_1, \ldots, \beta_d$ be the corresponding dual basis of $\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$. If one uses the basis $\beta_1, \ldots, \beta_d$ to identify $\mathfrak{X}_0$ with $\mathfrak{B}_1^d$, then $\mathfrak{X}$ identifies with*

$$\left\{ (z_1, \ldots, z_d) \in \mathfrak{B}_1^d \colon \log(z_j) = \frac{t_j}{t_1} \log(z_1) \text{ for all } j = 1, \ldots, d \right\}.$$

Proof. A point $(z_1, \ldots, z_d)$ corresponds to $\sum_{i=1}^d z_i \otimes \beta_i$ under the stated identification $\mathfrak{B}_1^d = \mathfrak{X}_0$. The condition that the differential $\mathbb{d}$ of the character corresponding to $\sum_{i=1}^d z_i \otimes \beta_i$ satisfies (1.N) for all $t_1, \ldots, t_d$ becomes

$$\sum_{i=1}^{d} (\beta_i(t_j) - t_j \beta_i(1)) \cdot \log(z_i) = 0 \qquad \text{for all } j = 1, \ldots, d.$$

But since $\sum_{i=1}^d \beta_i(t_j) \cdot \log(z_i) = \log(z_j)$, the condition above is equivalent to

$$\log(z_j) = \sum_{i=1}^{d} t_j \beta_i(1) \log(z_i) \qquad \text{for all } j = 1, \ldots, d.$$

If this condition is satisfied, it follows that $\frac{1}{t_j} \log(z_j)$ is independent of j and therefore equal to $\frac{1}{t_1} \log(z_1)$. Conversely, if the condition in the statement of the proposition is satisfied, then

$$\sum_{i=1}^{d} \log(z_i) \beta_i(1) = \sum_{i=1}^{d} \frac{t_i}{t_1} \log(z_1) \beta_i(1) = \frac{\log(z_1)}{t_1} \sum_{i=1}^{d} t_i \beta_i(1) = \frac{\log(z_1)}{t_1} \cdot 1 = \frac{\log(z_j)}{t_j}$$

holds for all $j = 1, \ldots, d$. □

By definition, $\mathfrak{X}$ represents $\widehat{G}$ in the sense that $\mathfrak{X}(K) = \widehat{G}(K)$. Moreover, by construction, the structure sheaf satisfies $\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}_0}/\mathcal{J}$, where $\mathcal{J}$ is the sheaf of ideals in $\mathcal{O}_{\mathfrak{X}_0}$ consisting of all germs of functions vanishing on $\widehat{G}(\mathbb{C}_p)$. Since $\mathfrak{X}_0$ is a Stein space, the global section functor is exact on coherent sheaves. This remains true after base change to K, so we conclude that

$$\mathcal{O}(\mathfrak{X}/K) = \mathcal{O}(\mathfrak{X}_0/K)/J(o),$$

where $J(o)$ is the ideal of all global holomorphic functions vanishing on $\widehat{G}(\mathbb{C}_p)$. This allows us to equip $\mathcal{O}(\mathfrak{X}/K)$ with a K-Fréchet algebra structure in a natural way.

Theorem 1.16. *The Fourier transform*

$$\begin{aligned} \mathscr{F} : D(G, K) &\xrightarrow{\sim} \mathcal{O}(\mathfrak{X}/K) \\ \lambda &\longmapsto F_\lambda \end{aligned}$$

is a well-defined isomorphism of K-Fréchet algebras.

Proof. Considering the diagram

$$\begin{tikzcd} D(G_0, K) \arrow[r, "\sim", "\mathscr{F}"'] \arrow[d, two heads] & \mathcal{O}(\mathfrak{X}_0/K) \arrow[d, two heads] \\ D(G, K) \arrow[r, dotted] & \mathcal{O}(\mathfrak{X}/K) \end{tikzcd}$$

where both vertical maps are open quotient maps (the left one being the one from (1.I)), one only needs to check that $J(o) = \mathscr{F}(I(o))$, where $I(o)$ denotes the kernel of the left vertical map. This is done in Corollary 1.5 in [ST01]. □

We end this section by collecting some facts about the rigid character variety $\mathfrak{X}$. Before we state and prove them, we recall the relevant concepts of rigid analytic geometry.

Definition 1.17. (i) A rigid analytic variety $\mathfrak{Y}$ is called reduced, normal or smooth if for every $x \in \mathfrak{Y}$, the local ring $\mathcal{O}_{\mathfrak{Y},x}$ is reduced, normal or regular, respectively.

(ii) The dimension of a rigid analytic variety $\mathfrak{Y}$ at a point $x \in \mathfrak{Y}$ is defined as the Krull dimension of the local ring $\mathcal{O}_{\mathfrak{Y},x}$. The dimension of $\mathfrak{Y}$ is the supremum of the dimensions at the points of $\mathfrak{Y}$.

(iii) A rigid analytic variety $\mathfrak{Y}$ is called quasi-Stein if it has an admissible covering $\mathfrak{Y} = \bigcup_n \mathfrak{Y}_n$ by an increasing sequence $\mathfrak{Y}_1 \subseteq \ldots \subseteq \mathfrak{Y}_n \subseteq \ldots$ of affinoid subdomains such that the restriction maps $\mathcal{O}(\mathfrak{Y}_{n+1}) \longrightarrow \mathcal{O}(\mathfrak{Y}_n)$ all have dense image.

(iv) A morphism $f\colon \mathfrak{Y} \longrightarrow \mathfrak{Z}$ of rigid analytic varieties is called étale if for every $y \in \mathfrak{Y}$ the induced homomorphism of local rings $\mathcal{O}_{\mathfrak{Z},f(y)} \longrightarrow \mathcal{O}_{\mathfrak{Y},y}$ is flat and unramified. The latter means that $\mathcal{O}_{\mathfrak{Y},y}/\mathfrak{m}\mathcal{O}_{\mathfrak{Y},y}$ is a finite separable field extension of $\mathcal{O}_{\mathfrak{Z},f(x)}/\mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_{\mathfrak{Z},f(x)}$.

Proposition 1.18. *Choose a $\mathbb{Z}_p$-basis $t_1, \ldots, t_d$ of o and let $\beta_1, \ldots, \beta_d$ be the corresponding dual basis of $\operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$. We have the cartesian diagram of rigid L-analytic varieties*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\subseteq"] \arrow[d, "\mathrm{ev}_1 \circ \mathrm{d}"'] & \mathfrak{B}_1 \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \cong \mathfrak{B}_1^d \arrow[d, "\log \otimes id"] \\
\mathbb{A}^1 \arrow[r] & \mathbb{A}^1 \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \cong \mathbb{A}^d
\end{tikzcd} \tag{1.O}$$

where the lower horizontal arrow is the map $a \longmapsto \sum_{i=1}^d at_i \otimes \beta_i$. Moreover, the horizontal arrows are closed immersions and the vertical arrows are étale.

Proof. The cartesian diagram (1.L) and Lemma 1.13 together imply that the diagram

$$\begin{tikzcd}
\widehat{G}(K) \arrow[r, "\subseteq"] \arrow[d, "\mathrm{d}"'] & \widehat{G_0}(K) = \mathfrak{B}_1(K) \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \arrow[d, "\log \otimes \mathrm{id}"] \\
\operatorname{Hom}_L(L, K) \arrow[r, "\subseteq"] & \operatorname{Hom}_{\mathbb{Q}_p}(L, K) = K \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)
\end{tikzcd}$$

is cartesian. Since $\operatorname{Hom}_L(L, K) \cong K$ via evaluation at $1 \in L$, we obtain the cartesian diagram (1.O). For the assertion about the lower horizontal arrow, we need to show that $\sum_{i=1}^d at_i \otimes \beta_i$ corresponds to the multiplication-by-a map in $\operatorname{Hom}_{\mathbb{Q}_p}(L, K)$ under the identification $\operatorname{Hom}_{\mathbb{Q}_p}(L, K) = K \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$. But $\sum_i at_i \otimes \beta_i$ corresponds to the map $x \longmapsto \sum_i \beta_i(x) \cdot at_i$ under the identification, and $\sum_i \beta_i(x)at_i = a\sum_i \beta_i(x)t_i = ax$. This proves that the lower horizontal arrow is the map $a \longmapsto \sum_i at_i \otimes \beta_i$. This map is a closed immersion. In fact, if we use the basis $\beta_1, \ldots, \beta_d$ to identify the space $\mathbb{A}^1 \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$

with $\mathbb{A}^d$, this map becomes the closed embedding $a \longmapsto (at_1, \dots, at_d)$. The upper horizontal arrow in (1.O) is a closed immersion by the definition of $\mathfrak{X}$. The vertical arrows are étale according to [ST01] paragraph after Theorem 2.3. $\square$

Corollary 1.19. *The rigid character variety $\mathfrak{X}$ is smooth, one-dimensional and quasi-Stein. For every $x \in \mathfrak{X}$, the local ring $\mathcal{O}_{\mathfrak{X},x}$ is a discrete valuation ring.*

Proof. In general, by [FVdP12] paragraph after Definition 4.10.1, for a closed immersion $\mathfrak{Z} \longrightarrow \mathfrak{Y}$ of rigid analytic varieties one has that for any admissible affinoid $U \subseteq \mathfrak{Y}$, the preimage $f^{-1}(U)$ is an admissible affinoid of $\mathfrak{Z}$ and $\mathcal{O}_{\mathfrak{Y}}(U) \longrightarrow \mathcal{O}_{\mathfrak{Z}}(f^{-1}(U))$ is surjective. This immediately implies that a closed rigid subvariety of a quasi-Stein variety is quasi-Stein. As $\mathfrak{B}_1$ is obviously quasi-Stein by construction, we conclude that $\mathfrak{X}$ is quasi-Stein.
We now argue that the stalks of the affine space $\mathbb{A}^1$ are discrete valuation rings. We have an admissible open covering $\mathbb{A}^1 = \cup_n \overline{\mathfrak{B}}(|c|^n)$ for some $c \in L$ with $|c| > 1$. Since $\overline{\mathfrak{B}}(|c|^n) = \mathrm{Sp}(L\langle c^{-n}T\rangle)$ and each $L\langle c^{-n}T\rangle$ is isomorphic to $L\langle T\rangle$, it suffices to consider the stalks of $\mathrm{Sp}(L\langle T\rangle) =: Y$. These stalks are closely related to the localizations of $L\langle T\rangle$. To make this statement precise, let $y \in Y$ be a point corresponding to the maximal ideal $\mathfrak{m} \subseteq L\langle T\rangle$. According to [BRG84] 7.3.2 Proposition 8, $\mathcal{O}_{Y,y}$ is a discrete valuation ring if and only if $L\langle T\rangle_{\mathfrak{m}}$ is a discrete valuation ring. Hence it suffices to show that every $L\langle T\rangle_{\mathfrak{m}}$ is a discrete valuation ring. But this is true because $L\langle T\rangle$, being a principal ideal domain and not a field, is a Dedekind domain. This completes the argument.
Now, we show that $\mathfrak{X}$ is one-dimensional and smooth. In fact, we show the stronger statement that the stalks of $\mathfrak{X}$ are discrete valuation rings. Let $x \in \mathfrak{X}$. The étale morphism $\mathbb{d}\colon \mathfrak{X} \longrightarrow \mathbb{A}^1$ induces a flat unramified homomorphism

$$A := \mathcal{O}_{\mathbb{A}^1,\mathbb{d}(x)} \xrightarrow{\varphi} \mathcal{O}_{\mathfrak{X},x} =: B$$

of local rings. Any flat local ring homomorphism of local rings fulfills condition (iii) in [Liu02] Corollary 2.20 and is therefore faithfully flat. Hence φ is faithfully flat. Lemma 1.20 (i) below tells us that φ is injective. Denote the maximal ideal of A by $\mathfrak{m}$ and the maximal ideal of B by $\mathfrak{n}$. As we have shown above, A is a discrete valuation ring, so $\mathfrak{m}$ is a principal ideal. Because φ is unramified, we have $\varphi(\mathfrak{m})B = \mathfrak{n}$, so in particular $\mathfrak{n}$ is also principal ideal. Since the local ring B is noetherian (as stalks of rigid analytic varieties always are, cf. [BRG84] 7.3.2 Proposition 7), it suffices to show that the Krull dimension of B is greater than zero. Indeed, [Mat89] Theorem 11.2 then implies that B is a discrete valuation ring and we are done.
Suppose that the dimension of B is zero. The variety $\mathfrak{X}$ is reduced, so B is a reduced ring. Thus Lemma 1.20 (ii) applies, to show that B is a field. But B cannot be a field since its maximal ideal is $\mathfrak{n} = \varphi(\mathfrak{m})B$ and $\varphi(\mathfrak{m}) \neq 0$ (because $\mathfrak{m} \neq 0$ and φ is injective), so we have arrived at a contradiction. $\square$

Lemma 1.20. (i) *Any faithfully flat ring homomorphism $R \longrightarrow S$ between two commutative rings is injective.*

(ii) *Let R be a commutative zero-dimensional reduced local ring. Then R is a field.*

Proof. Let $f\colon R \longrightarrow S$ be a faithfully flat ring homomorphism and let $I := \ker(f)$. We want to show that the canonical map $\pi\colon R \longrightarrow R/I$ is injective. It suffices to show that the homomorphism $\pi \otimes \mathrm{id}\colon R \otimes_R S \longrightarrow (R/I) \otimes_R S$ is injective. The canonical isomorphisms $R \otimes_R S \cong S$ and $(R/I) \otimes_R S \cong S/(IS) = S/0 = S$ fit into the commutative diagram

$$\begin{tikzcd} R \otimes_R S \arrow[r, "\pi\otimes\mathrm{id}"] \arrow[d, "\cong"'] & (R/I) \otimes_R S \arrow[d, "\cong"] \\ S \arrow[r, "\mathrm{id}"] & S \end{tikzcd}$$

which implies that $\pi \otimes \mathrm{id}$ is an isomorphism. This proves assertion (i).
Assertion (ii) can be proven very nicely with use of algebraic geometry. The affine scheme $\mathrm{Spec}(R)$ is reduced, because R is a reduced ring. As R is a zero-dimensional local ring, $\mathrm{Spec}(R)$ consists of a single point and is hence irreducible. Being irreducible and reduced, the scheme $\mathrm{Spec}(R)$ is integral, which means that R is an integral domain. But any zero-dimensional integral domain is a field. $\square$

For any rigid variety satisfying the properties stated in Corollary 1.19, there is a divisor theory (see [BSX15] Section 1.1). One can use this to show that $\mathcal{O}(\mathfrak{X})$ enjoys many nice properties, for instance that it is a Prüfer domain (i.e. the classes of closed, finitely generated and invertible ideals in this ring coincide). The same is true for $\mathcal{O}(\mathfrak{X}/K)$ (see the discussion after Lemma 3.10 in [ST01]).

1.3 An isomorphism $\mathfrak{X}/\mathbb{C}_p \cong \mathfrak{B}/\mathbb{C}_p$ via Lubin-Tate theory

If $L \neq \mathbb{Q}_p$, the rigid varieties $\mathfrak{B}$ and $\mathfrak{X}$ are not isomorphic. For instance, the ring $\mathcal{O}(\mathfrak{B}) = \{f \in L[[T]]\colon f \text{ converges on } \mathfrak{B}(\mathbb{C}_p)\}$ is a Bezout ring (i.e. the classes of closed, finitely generated and principal ideals all coincide), whereas in $\mathcal{O}(\mathfrak{X})$ there is a finitely generated ideal that is not principal ([ST01] Lemma 3.10). If K is discretely valued, the same argument applies to show that $\mathfrak{B}$ and $\mathfrak{X}$ are not isomorphic even after base change to K. But, remarkably, $\mathfrak{X}$ and $\mathfrak{B}$ become isomorphic over $\mathbb{C}_p$. If one endows $\mathfrak{B}(\mathbb{C}_p)$ with a Lubin-Tate group law associated to L and to π, then there is an explicit isomorphism of rigid group varieties

$$\kappa\colon \mathfrak{B}/\mathbb{C}_p \xrightarrow{\sim} \mathfrak{X}/\mathbb{C}_p$$

that depends on the choice of a period of the Lubin-Tate group (i.e. on the choice of a generator of the o-module T' from Section 1.3.2). Schneider and Teitelbaum referred to this result as the "uniformization of $\mathfrak{X}$ by the open unit disk $\mathfrak{B}$". Accordingly, we refer to the isomorphism κ as the "uniformization isomorphism". With κ, we obtain an isomorphism

$$\mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \xrightarrow{\sim} \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$$

which allows for many applications of the Fourier isomorphism $\mathscr{F}$ from Theorem 1.16. The goal of this section is to present the construction of the uniformization isomorphism on the level on $\mathbb{C}_p$-points:

$$\kappa(\mathbb{C}_p)\colon \mathfrak{B}(\mathbb{C}_p) \xrightarrow{\sim} \widehat{G}(\mathbb{C}_p). \tag{1.P}$$

We do this in Section 1.3.2, following the approach in Section 3 of [ST01]. Our ultimate goal, which we achieve in Chapter 3, is of course to show that the isomorphism (1.P) generalizes to the case of relative Lubin-Tate groups. Many of the details we provide here in Section 1.3.2 will later play an important role in the relative case.
In Section 1.3.1, we summarize the necessary Lubin-Tate theory needed for Section 1.3.2. Much of this can be seen as a special case of the results of Section 3.1 in Chapter 3, where we will review relative Lubin-Tate groups. Nevertheless, this separate discussion of the properties of classical Lubin-Tate groups and their role in the proof of (1.P) will make clear which properties we will want to expect of relative Lubin-Tate laws later. Moreover, it facilitates the reading of Chapter 2, which stays in the classical Lubin-Tate case.

1.3.1 A brief summary of Lubin-Tate theory

To motivate our intentions, let us consider the case $L = \mathbb{Q}_p$ for the moment. Then $G = G_0 = \mathbb{Z}_p$ and (1.M) gives a group isomorphism $\widehat{G}(\mathbb{C}_p) \cong \mathfrak{B}_1(\mathbb{C}_p)$. As we wish to have an isomorphism between $\widehat{G}(\mathbb{C}_p)$ and $\mathfrak{B}(\mathbb{C}_p)$, we transport the group structure of $\mathfrak{B}_1(\mathbb{C}_p)$ onto $\mathfrak{B}(\mathbb{C}_p)$ via the bijection $z \longmapsto z-1$. This defines the following group law on $\mathfrak{B}(\mathbb{C}_p)$:

$$z +_{\hat{\mathbb{G}}_m} z' := (z+1)(z'+1) - 1 = z + z' + zz'.$$

Note that the this group law is given by a formal power series $\hat{\mathbb{G}}_m(X,Y) = X + Y + XY \in \mathbb{Z}_p[[X,Y]]$. This is an instance of the following general principle.

Definition 1.21. Let A be a commutative ring. A one-dimensional, commutative formal group law over A is a formal power series $F \in A[[X,Y]]$ in two variables such that

(i) $F(X,Y) = X + Y +$ terms of degree ≥ 2,

(ii) $F(X, F(Y,Z)) = F(F(X,Y),Z)$,

(iii) $F(X,Y) = F(Y,X)$, and

(iv) there exists a unique $\iota_F(T) \in TA[[T]]$ such that $F(X, \iota_F(X)) = 0$.

Since we consider no other in this chapter, we shall refer to one-dimensional commutative formal group laws simply as formal group laws.
A homomorphism $h : F \longrightarrow G$ of formal group laws F and G over A is a power series $h \in TA[[T]]$ such that $h(F(X,Y)) = G(h(X), h(Y))$.

By taking $Y = Z = 0$ in axioms (i) and (ii), one deduces that $F(X,0) = X$ and $F(0,Y) = Y$. For any formal group laws F and G, the set $\mathrm{Hom}_A(F,G)$ of homomorphisms from F to G becomes an abelian group under the addition $f +_G g := G(f(T), g(T))$. The abelian group $\mathrm{End}_A(F)$ of endomorphisms of F becomes a (not necessarily commutative) ring under the multiplication $f \circ g$.

Definition 1.22. (i) Let F be a formal group over a field of characteristic p. Then $[p]_F(X) = X +_F \ldots +_F X$ (p times) is a power series in X^r with $r = p^h$ for some $h \geq 1$ (cf. [Haz78] 18.3.1-18.3.3). If $[p]_F \neq 0$, the largest possible h is called the height of F. Otherwise, we say that F is of infinite height.

(ii) Let B be a commutative local ring with residue field $B/\mathfrak{m}$ of characteristic p and let $F(X,Y)$ be a formal group over B. We define the height of F as the height of the reduction of F over $B/\mathfrak{m}$.

Let $A = o$ and let $F = \sum_{i,j}^{\infty} a_{ij} X^i Y^j \in o[[X,Y]]$ be a formal group law. For every $x, y \in \mathfrak{B}(K)$, the series $F(x,y)$ converges to an element $x +_F y \in \mathfrak{B}(K)$, because $a_{ij} x^i y^j \to 0$ as $(i,j) \to \infty$ and K is a complete non-Archimedean field. Because of the axioms imposed on F, $\mathfrak{B}(K)$ becomes an abelian group $(\mathfrak{B}(K), +_F)$. Any endomorphism $h : F \longrightarrow F$ induces a group endomorphism $\mathfrak{B}(K) \longrightarrow \mathfrak{B}(K), z \longmapsto h(z)$.

Example 1.23. $\hat{\mathbb{G}}_m(X,Y) = X + Y + XY = (1+X)(1+Y) - 1$ is called the multiplicative formal group law. We have already seen the reason for this, namely the isomorphism

$$(\mathfrak{B}(K), +_{\hat{\mathbb{G}}_m}) \xrightarrow{\sim} \mathfrak{B}_1(K), z \longmapsto z + 1$$

onto the multiplicative group $\mathfrak{B}_1(K)$. The polynomial $f(T) = (1+T)^p - 1$ is an endomorphism of $\hat{\mathbb{G}}_m$:

$$\hat{\mathbb{G}}_m(f(X), f(Y)) = (1+X)^p(1+Y)^p - 1 = f(\hat{\mathbb{G}}_m(X,Y)).$$

Under our identification of $(\mathfrak{B}(K), +_{\hat{\mathbb{G}}_m})$ and $\mathfrak{B}_1(K)$, f corresponds to the endomorphism $z \longmapsto z^p$, i.e. the diagram

$$\begin{tikzcd}
\mathfrak{B}(K) \arrow[r, "f"] \arrow[d, "z \mapsto z+1"] & \mathfrak{B}(K) \arrow[d, "z \mapsto z+1"] \\
\mathfrak{B}_1(K) \arrow[r, "z \mapsto z^p"] & \mathfrak{B}_1(K)
\end{tikzcd}$$

is commutative. Note that we clearly have $[p]_{\hat{\mathbb{G}}_m}(T) = f(T) = (1+T)^p - 1$. Hence the reduction of $[p]_{\hat{\mathbb{G}}_m}(T)$ modulo p is equal to X^p, which means that the height of $\hat{\mathbb{G}}_m$ is one.

Example 1.24. Let E be an elliptic curve over an arbitrary field $\mathcal{K}$ determined by a non-singular Weierstrass equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

$a_i \in \mathcal{K}$. Let R be a subring of $\mathcal{K}$ containing all the Weierstrass coefficients a_i. Chapter IV in [Sil09] explains how one associates a formal group law $\widehat{E} = \widehat{E}(T_1, T_2) \in R[[T_1, T_2]]$ to E. First, one constructs formal Laurent series $x(T), y(T) \in \mathcal{K}((T)) =: \mathcal{L}$ such that $(x(T) : y(T) : 1)$ provides a formal solution to the Weierstrass equation, i.e. such that $(x(T) : y(T) : 1) \in E(\mathcal{L})$. Then one constructs the power series $\widehat{E}$ formally giving the addition law on $E(\mathcal{L})$.
If $\mathcal{K}$ is of characteristic p, then we can consider the height of the formal group $\widehat{E}$ associated to E. One can show that the height of $\widehat{E}$ is either one or two according as E is ordinary or supersingular.
If $\mathcal{K}$ is the fraction field of a commutative complete local integral domain $(R, \mathfrak{m})$, then

$$\begin{aligned} \mathfrak{m} &\hookrightarrow E(\mathcal{K}) \\ z &\longmapsto (x(z) : y(z) : 1) \end{aligned} \tag{1.Q}$$

is an injective group homomorphism from $\mathfrak{m}$ (endowed with the group structure provided by $\widehat{E}$) into $E(\mathcal{K})$ (endowed with the group law of the elliptic curve E), cf. [Sil09] Example IV.3.1.3 and the discussion on page 119 in IV.1.
For $\mathcal{K} = L$ and $R = o$, [Sil09] Proposition VII.2.2 describes the image of the monomorphism (1.Q).

After these general preliminaries on formal group laws, we turn towards Lubin-Tate formal group laws. Once a uniformizer $\pi \in o$ has been fixed, these are exactly the formal group laws defined over o that admit a so-called Frobenius power series as an endomorphism.

Definition 1.25. A Frobenius power series for π is a formal power series $\phi \in o[[X]]$ such that

(i) $\phi(X) = \pi X +$ terms of degree ≥ 2,

(ii) $\phi(X) \equiv X^q \mod \pi o[[X]]$.

The set of all Frobenius power series for π is denoted by $\mathcal{F}_\pi$.

To avoid confusion later on, we remark now that we usually denote such a Frobenius power series by ϕ, whereas we reserve the letter φ for denoting the Frobenius element $\varphi \in \mathrm{Gal}(E/L)$ of an unramified extension E/L.
The following are examples of Frobenius power series.

Example 1.26. (i) The polynomial $\phi(X) = \pi X + X^q$ lies in $\mathcal{F}_\pi$.

(ii) If $L = \mathbb{Q}_p$ and $\pi = p$ then the polynomial

$$\phi(X) = (1 + X)^p - 1 = pX + \binom{p}{2} X^2 + \dots pX^{p-1} + X^p$$

lies in $\mathcal{F}_p$.

Theorem 1.27. *For any $\phi \in \mathcal{F}_\pi$, there is a unique formal group law F_ϕ with coefficients in o admitting ϕ as an endomorphism. It is called the Lubin-Tate formal group law of the Frobenius power series ϕ. Moreover, for $\phi, \psi \in \mathcal{F}_\pi$ and $a \in o$, there is a unique element $[a]_{\phi,\psi} \in o[[X]]$ such that*

$$\begin{aligned} &[a]_{\phi,\psi}(X) = aX + \text{ terms of degree } \geq 2, \text{ and} \\ &\phi \circ [a]_{\phi,\psi} = [a]_{\phi,\psi} \circ \psi. \end{aligned}$$

Such an $[a]_{\phi,\psi}$ is necessarily a group homomorphism $F_\psi \longrightarrow F_\phi$. We also have

$$[a+b]_{\phi,\psi} = [a]_{\phi,\psi} +_{F_\phi} [b]_{\phi,\psi}, \text{ and} \tag{1.R}$$

$$[ab]_{\tau,\psi} = [a]_{\tau,\phi} \circ [b]_{\phi,\psi} \tag{1.S}$$

for any $a, b \in o$ and $\tau \in \mathcal{F}_\pi$.

Proof. All of the following refers to [Sch17]. See Proposition 1.3.4 for the existence and uniqueness of F_ϕ. The existence and uniqueness of $[a]_{\phi,\psi}$ follows from Lemma 1.3.3 as the special case $n = 1$ and $F_1 = aX$. The proof of Proposition 1.3.6 shows that $[a]_{\phi,\psi}$ is a homomorphism $F_\phi \longrightarrow F_\psi$. Finally, (1.R) and (1.S) are true because the power series on the right obviously satisfies the two defining conditions of the power series on the left, in each case. □

Corollary 1.28. *For $\phi, \psi \in \mathcal{F}_\pi$, any choice of a unit $u \in o^\times$ gives rise to an isomorphism $[u]_{\phi,\psi}\colon F_\psi \xrightarrow{\sim} F_\phi$ of formal groups. In particular, we have the canonical isomorphism $[1]_{\phi,\psi}\colon F_\psi \xrightarrow{\sim} F_\phi$.*

By setting $[a]_\phi := [a]_{\phi,\phi}$, we see that there is a unique ring homomorphism

$$\begin{aligned} o &\longrightarrow \mathrm{End}_o(F_\phi) \\ a &\longmapsto [a]_\phi \end{aligned}$$

such that

(a) $[a]_\phi = aX+$ terms of degree ≥ 2, and

(b) $[a]_\phi$ commutes with ϕ.

Because of (a), the homomorphism $o \longrightarrow \mathrm{End}_o(F_\phi)$ is injective. We also have $[\pi]_\phi = \phi$, because ϕ clearly satisfies (a) and (b) for π. The formal group isomorphisms $[u]_{\phi,\psi}$ from Corollary 1.28 are actually isomorphisms of formal o-modules, i.e. they commute with the actions of o on F_ψ and F_ϕ. Indeed,

$$[a]_\phi \circ [u]_{\phi,\psi} = [au]_{\phi,\psi} = [ua]_{\phi,\psi} = [u]_{\phi,\psi} \circ [a]_\psi$$

holds by (1.S).
It follows that the abelian group $(\mathfrak{B}(K), +_{F_\phi})$ has a natural o-module structure. If $\psi \in \mathcal{F}_\pi$, then any unit $u \in o^\times$ gives rise to an o-module isomorphism $[u]_{\phi,\psi}\colon (\mathfrak{B}(K), +_{F_\psi}) \longrightarrow (\mathfrak{B}(K), +_{F_\phi})$.

Example 1.29. For $L = \mathbb{Q}_p$, we have seen that $\phi = (1+X)^p - 1 \in \mathcal{F}_p$ is an endomorphism of $\hat{\mathbb{G}}_m$, cf. Example 1.23. This implies $F_\phi = \hat{\mathbb{G}}_m$. Recall that, for $a \in \mathbb{Z}_p$, the formal power series $(1+X)^a \in \mathbb{Z}_p[[X]]$ is defined as

$$(1+X)^a = \sum_{n=0}^{\infty} \binom{a}{n} X^n.$$

If $a \in \mathbb{N}$, we clearly have $(1+X)^a = (1+X)\cdots(1+X)$ (a times). We claim that

$$[a]_\phi(X) = (1+X)^a - 1$$

holds for all $a \in \mathbb{Z}_p$. Certainly, $(1+X)^a - 1 = aX+$ terms of degree ≥ 2. Moreover,

$$\phi \circ ((1+X)^a - 1) = (1+X)^{ap} - 1 = ((1+X)^a - 1) \circ \phi$$

holds if $a \in \mathbb{N}$, which implies that it holds for all $a \in \mathbb{Z}_p$, since $\mathbb{N}$ is dense in $\mathbb{Z}_p$. The claim is thus proven. Note that it follows that the isomorphism $z \longmapsto z+1$ transforms the action of $[a]_\phi$ on $(\mathfrak{B}(K), +_{\hat{\mathbb{G}}_m})$ into the map sending an element of $\mathfrak{B}_1(K)$ to its a-th power.

Example 1.30. Let $\mathcal{K}$ be an imaginary quadratic field, let $\mathcal{M}$ be a finite extension of $\mathcal{K}$, and let E be an elliptic curve defined over $\mathcal{M}$ with complex multiplication by the full ring of integers of $\mathcal{K}$. Let $\widehat{E}$ be the formal group law associated to E (see Example 1.24). Let $\mathfrak{p}$ be a prime of $\mathcal{K}$ and $\mathfrak{P}$ a prime of $\mathcal{M}$ dividing $\mathfrak{p}$. Assume that E has good reduction at $\mathfrak{P}$ and that $\mathfrak{p}$ splits completely in $\mathcal{M}/\mathcal{K}$. Then $\widehat{E}$, as a formal group law defined over $o_\mathfrak{p}$ (the ring of integers in the completion $\mathcal{K}_\mathfrak{p} = \mathcal{M}_\mathfrak{P}$), is a Lubin-Tate law. If we replace the condition that $\mathfrak{p}$ splits completely in $\mathcal{M}/\mathcal{K}$ by the weaker condition that $\mathfrak{P}$ is not ramified in $\mathcal{M}/\mathcal{K}$, then $\widehat{E}$ is a so-called relative Lubin-Tate group with respect to the unramified extension $\mathcal{M}_\mathfrak{P}/\mathcal{K}_\mathfrak{p}$, see [dS85] Example 9.

Any Lubin-Tate law F_ϕ over o is of height d. This is easily verified using the following three facts: (i) $p = u\pi^e$ for some $u \in o^\times$; (ii) $\mathrm{ht}[u\pi^e]_\phi = \mathrm{ht}([u]_\phi) + e\,\mathrm{ht}([\pi]_\phi) = e\,\mathrm{ht}([\pi]_\phi)$; and (iii) one may assume that $[\pi]_\phi(X) = \pi X + X^q$. Fact (ii) is true because the height function ht defines a valuation on the ring $\mathrm{End}_k(F)$, see [Haz78] 18.3.2.

We fix a Frobenius power series $\phi \in \mathcal{F}_\pi$ and write $F := F_\phi$ and $[.] := [.]_\phi$. For any $n \geq 1$, we have the o-submodule $\mathcal{W}_n \subseteq \mathfrak{B}(\mathbb{C}_p)$ defined by

$$\mathcal{W}_n := \ker[\pi^n] = \{z \in \mathfrak{B}(\mathbb{C}_p) : [\pi^n](z) = 0\}.$$

Since it is annihilated by π^n, $\mathcal{W}_n$ is naturally an o/π^n-module. If $\psi \in \mathcal{F}_\pi$ and $\mathcal{W}'_n := \ker[\pi^n]_\psi$, then $\mathcal{W}'_n$ and $\mathcal{W}_n$ are isomorphic as o/π^n-modules. Indeed, any $u \in o^\times$ gives rise to an isomorphism $[u]_{\phi,\psi} : \mathcal{W}'_n \xrightarrow{\sim} \mathcal{W}_n$. By [Sch17] Remark 1.3.8, we have the equality $L(\mathcal{W}_n) = L(\mathcal{W}'_n)$ of field extensions. By [Sch17] Corollary 1.3.11, $\mathcal{W}_n$ consists of algebraic elements, i.e. $\mathcal{W}_n = \{z \in \mathfrak{B}(\mathbb{Q}_p^{\mathrm{alg}}) : [\pi^n](z) = 0\}$. Finally, we define the π-adic Tate module $T = T_\pi(F)$ by

$$T = \varprojlim(\ldots \xrightarrow{[\pi](.)} \mathcal{W}_{n+1} \xrightarrow{[\pi](.)} \mathcal{W}_n \xrightarrow{[\pi](.)} \ldots \xrightarrow{[\pi](.)} \mathcal{W}_1).$$

Proposition 1.31. *Let $n \geq 1$.*

(i) *$\mathcal{W}_n$ is a free o/π^n-module of rank one. If $z \in \mathcal{W}_n \setminus \mathcal{W}_{n-1}$, then*

$$\begin{aligned} o/\pi^n &\xrightarrow{\sim} \mathcal{W}_n \\ a + \pi^n o &\longmapsto [a](z) \end{aligned}$$

is an isomorphism.

(ii) *If $z \in \mathcal{W}_n$ generates $\mathcal{W}_n$ over o/π^n, then $[\pi](z)$ generates $\mathcal{W}_{n-1}$ over o/π^{n-1}. The converse is true for $n \geq 2$.*

Proof. See [Sch17] Proposition 1.3.10 and its proof for (i). The assertion (ii) is obvious for $n = 1$, since $\mathcal{W}_0 = 0$. If $n \geq 2$, it follows from (i) and the equivalence

$$z \in \mathcal{W}_n \setminus \mathcal{W}_{n-1} \iff [\pi](z) \in \mathcal{W}_{n-1} \setminus \mathcal{W}_{n-2}.$$

□

Corollary 1.32. *Any choice of a sequence $(z_n)_n$ with $z_n \in \mathcal{W}_n \setminus \mathcal{W}_{n-1}$ and $[\pi](z_n) = z_{n-1}$ gives rise to an isomorphism*

$$\begin{tikzcd} \dots \arrow[r, "\mathrm{pr}"] & o/\pi^{n+1} \arrow[r, "\mathrm{pr}"] \arrow[d, "{[.](z_{n+1})}", "\cong"'] & o/\pi^n \arrow[r, "\mathrm{pr}"] \arrow[d, "{[.](z_n)}", "\cong"'] & \dots \arrow[r, "\mathrm{pr}"] & o/\pi \arrow[d, "{[.](z_1)}", "\cong"'] \\ \dots \arrow[r, "{[\pi](.)}"] & \mathcal{W}_{n+1} \arrow[r, "{[\pi](.)}"] & \mathcal{W}_n \arrow[r, "{[\pi](.)}"] & \dots \arrow[r, "{[\pi](.)}"] & \mathcal{W}_1 \end{tikzcd}$$

of projective systems. In particular, the Tate module T is free of rank one over o.

Proof. Such a sequence $(z_n)_n$ exists since the projective limit $\varprojlim_n (\mathcal{W}_n \setminus \mathcal{W}_{n-1})$ is non-empty. Indeed, a projective limit of a projective system of non-empty finite sets is non-empty, cf. [RZ10] Proposition 1.1.4. Everything else follows immediately from Proposition 1.31 and the fact that $\varprojlim_n o/\pi^n \cong o$. □

1.3.2 The uniformization isomorphism

Let $\mathcal{G} = \mathcal{G}_\pi$ be a Lubin-Tate formal group law for π. We identify $\mathcal{G}$ with $\mathfrak{B}$, so $\mathfrak{B}$ becomes an o-module object. The o-action on $\mathfrak{B}$, denoted by $(a, z) \longmapsto [a](z)$, is given by formal power series $[a](Z) \in Zo[[Z]]$. We denote by $T' = T_p(\mathcal{G}')$ the Tate module of the p-divisible dual $\mathcal{G}'$ of $\mathcal{G}$, i.e. the $\mathbb{Z}_p$- module

$$T' = \mathrm{Hom}_{o_{\mathbb{C}_p}}(\mathcal{G}, \hat{\mathbb{G}}_m)$$

of all formal group homomorphisms, defined over $o_{\mathbb{C}_p}$, from $\mathcal{G}$ to $\hat{\mathbb{G}}_m$. T' inherits its structure as an abelian group, or rather even its structure as a $\mathbb{Z}_p$-module, from $\hat{\mathbb{G}}_m$. For

an element $t' \in T'$, we often also write $F_{t'}$ to emphasize that we are regarding it as a formal power series $F_{t'} \in Zo_{\mathbb{C}_p}[[Z]]$.
We will always denote the leading coefficient of $F_{t'} \in Zo_{\mathbb{C}_p}[[Z]]$ by $\Omega_{t'}$, i.e. $\Omega_{t'} = \frac{d}{dZ}F_{t'}(0)$ and so $F_{t'}(Z) = \Omega_{t'}Z+$ terms of degree ≥ 2. Since the coefficients of the power series $F_{t'}$ lie in $o_{\mathbb{C}_p}$, we have that $F_{t'} \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$.
On the other hand, we consider

$$\mathscr{H} := \{f \in o_{\mathbb{C}_p}[[Z]] \colon f(X +_{\mathcal{G}} Y) = f(X)f(Y) \text{ and } f(0) = 1\}.$$

Lemma 1.33. *We have the following bijection*

$$\begin{aligned} \operatorname{Hom}_{o_{\mathbb{C}_p}}(\mathcal{G}, \hat{\mathbb{G}}_m) &\xrightarrow{\sim} \mathscr{H} \\ f &\longmapsto f+1 \end{aligned}$$

of sets. Let $\dotplus$ *and* $\odot$ *denote the unique operations of addition and scalar multiplication on* $\mathscr{H}$ *making the above bijection an isomorphism of* $\mathbb{Z}_p$*-modules. Then we have*

$$\begin{aligned} (f \dotplus g)(Z) &= f(Z)g(Z) \text{ and} \\ (a \odot f)(Z) &= f(Z)^a \end{aligned}$$

for any $f, g \in \mathscr{H}$ *and* $a \in \mathbb{Z}_p$*, where* $f(Z)^a$ *denotes the power series* $\sum_{n\geq 0} \binom{a}{n}(f(Z) - 1)^n$ *(cf. Example 1.29). Moreover, we have the equality*

$$f(Z)^a = f \circ [a](Z), \tag{1.T}$$

where $[a]$ *is, as always, the power series giving the action of* $a \in \mathbb{Z}_p \subseteq o$ *on the Lubin-Tate group* $\mathcal{G}$*.*

Proof. Let $f \in \mathscr{H}$. To show that $f - 1 \in T'$, we need to argue that $(f - 1)(X +_{\mathcal{G}} Y) = ((f-1)(X)+1)((f-1)(Y)+1)-1$. The latter term is obviously equal to $f(X)f(Y)-1$ and the former term is equal to $f(X +_{\mathcal{G}} Y) - 1 = f(X)f(Y) - 1$. Conversely, let $g \in T'$ and let $f := g+1$. We need to show that $f(X +_{\mathcal{G}} Y)$ is equal to $f(X)f(Y) = (g(X)+1)(g(Y)+1)$. But $f(X +_{\mathcal{G}} Y) = g(X +_{\mathcal{G}} Y) + 1 = (g(X) + 1)(g(Y) + 1) - 1 + 1$. This proves that the bijection in the lemma is well-defined.
Next, we prove the statement about the resulting module structure on $\mathscr{H}$. For $f, g \in \mathscr{H}$, we have $f \dotplus g = (f - 1) +_{\hat{\mathbb{G}}_m} (g - 1) + 1 = fg - 1 + 1 = fg$. To determine $a \odot f$ for an $a \in \mathbb{Z}_p$, we first let a act on $f - 1$ in T'. This gives $((f - 1) + 1)^a - 1 = f^a - 1$. Translating back to $\mathscr{H}$, we obtain $a \odot f = f^a$. It remains to prove that $f^a = f \circ [a]$. This is true for $a \in \mathbb{N}$, which we can see from $f([2](X)) = f(X +_{\mathcal{G}} X) = f(X)f(X)$ and induction. The general case now follows by continuity, since $\mathbb{N}$ is dense in $\mathbb{Z}_p$. ☐

We use the special symbols $\dotplus$ and $\odot$ in order to avoid confusion with the natural addition and scalar multiplication in $o_{\mathbb{C}_p}[[Z]] \supseteq \mathscr{H}$. On the other hand, $+_{\hat{\mathbb{G}}_m}$ denotes the module addition in T'.
If $t \in \mathscr{H}$ and $a \in o$, then obviously also $t \circ [a] \in \mathscr{H}$, and so (1.T) implies that the $\mathbb{Z}_p$-module

structure on $\mathscr{H}$ extends to an o-module structure. We now make preparations for the proof of the vital fact that $\mathscr{H}$ is a free o-module of rank one.

Recall that the p-adic Tate module $T_p\mathcal{G}$ of $\mathcal{G}$ is defined by

$$T_p\mathcal{G} = \varprojlim(\ldots \xrightarrow{[p](.)} \ker[p^{n+1}] \xrightarrow{[p](.)} \ker[p^n] \xrightarrow{[p](.)} \ldots \xrightarrow{[p](.)} \ker[p]),$$

where $\ker[p^n] = \{z \in \mathfrak{B}(\mathbb{C}_p) \colon [p^n](z) = 0\}$. Note that the elements of $\ker[p^n]$ are all algebraic over L, since $p = u\pi^e$ for some $u \in o^\times$ and the π^k-torsion points are algebraic for all $k \in \mathbb{N}$ by Corollary 1.3.11 in [Sch17].
Similarly, the Tate module $T_p\hat{\mathbb{G}}_m$ of $\hat{\mathbb{G}}_m$ is the projective limit over the p^n-torsion points of $\hat{\mathbb{G}}_m$. Since $\hat{\mathbb{G}}_m$ is the Lubin-Tate group law for $\mathbb{Q}_p$ and the uniformizer $p \in \mathbb{Z}_p$, Corollary 1.32 says that there is an isomorphism

$$T_p\hat{\mathbb{G}}_m \cong \mathbb{Z}_p \tag{1.U}$$

of $\mathbb{Z}_p$-modules.

Lemma 1.34. *$T_p\mathcal{G}$ is a free module of rank d over $\mathbb{Z}_p$.*

Proof. We claim that we have an isomorphism

$$\begin{tikzcd} \ldots \arrow[r, "\mathrm{pr}"] & (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^d \arrow[r, "\mathrm{pr}"] \arrow[d, "\cong"'] & (\mathbb{Z}_p/p^n\mathbb{Z}_p)^d \arrow[r, "\mathrm{pr}"] \arrow[d, "\cong"'] & \ldots \arrow[r, "\mathrm{pr}"] & (\mathbb{Z}_p/p\mathbb{Z}_p)^d \arrow[d, "\cong"'] \\ \ldots \arrow[r, "{[p](.)}"] & \ker[p^{n+1}] \arrow[r, "{[p](.)}"] & \ker[p^n] \arrow[r, "{[p](.)}"] & \ldots \arrow[r, "{[p](.)}"] & \ker[p] \end{tikzcd} \tag{1.V}$$

of projective systems. Observe that

$$\ker[p^n] = \ker[\pi^{en}] \cong o/\pi^{en} = o/p^n \cong \mathbb{Z}_p^d/p^n(\mathbb{Z}_p)^d = \mathbb{Z}_p^d/(p^n\mathbb{Z}_p)^d = (\mathbb{Z}_p/p^n\mathbb{Z}_p)^d$$

as a $\mathbb{Z}_p$-module, where $\ker[\pi^{en}] \cong o/\pi^{en}$ by a choice of a $z_{en} \in \mathcal{W}_{en} \setminus \mathcal{W}_{en-1}$ as in Corollary 1.32. Note that $[p] = [\pi] \circ \ldots \circ [\pi]$ (e times) and that the projection pr factors as

$$(\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^d = o/\pi^{e(n+1)} \xrightarrow{\mathrm{pr}} o/\pi^{e(n+1)-1} \xrightarrow{\mathrm{pr}} \ldots \xrightarrow{\mathrm{pr}} o/\pi^{e(n+1)-e} = (\mathbb{Z}_p/p^n\mathbb{Z}_p)^d.$$

Hence our claim follows at once from Corollary 1.32, i.e. the corresponding statement for the projective system $(\ker[\pi^n])_n$. □

Remark 1.35. The statement of Lemma 1.34 holds more generally if $\mathcal{G}$ is any formal group over o of height d. This follows from the the theory of p-divisible groups as developed in [Tat67], since the category of finite height formal groups over o is equivalent to the category of connected p-divisible groups over o. We will elaborate on this in Section 3.2.1.

Every $f \in T' = \mathrm{Hom}_{o_{\mathbb{C}_p}}(\mathcal{G}, \hat{\mathbb{G}}_m)$ induces a homomorphism of the Tate modules, so we have a mapping

$$T' \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_p\mathcal{G}, T_p\hat{\mathbb{G}}_m) \tag{1.W}$$

which is an isomorphism of $\mathbb{Z}_p$-modules by a result of [Tat67], as mentioned on page 6 in [Box86]. That (1.W) is an isomorphism is also true more generally if $\mathcal{G}$ is any finite height formal group law over o, cf. Section 3.2.2.
Together with Lemma 1.34 and (1.U), the isomorphism (1.W) implies that T' is free of rank d over $\mathbb{Z}_p$.

Corollary 1.36. *$\mathcal{H}$ is a free o-module of rank one.*

Proof. We know that $\mathcal{H}$ is finitely generated over o, since it is already finitely generated over the subring $\mathbb{Z}_p \subseteq o$. The structure theorem for finitely generated modules over a principal ideal domain tells us that $\mathcal{H} \cong o^r \oplus \mathcal{H}_{\mathrm{tors}}$ as an o-module, where $\mathcal{H}_{\mathrm{tors}} \cong o/\pi^{k_1} \oplus \ldots \oplus o/\pi^{k_s}$ for some integers $k_i \geq 1$. In particular, the underlying set of $\mathcal{H}_{\mathrm{tors}}$ is finite. But $o^r \oplus \mathcal{H}_{\mathrm{tors}} \cong \mathbb{Z}_p^{dr} \oplus \mathcal{H}_{\mathrm{tors}}$ as a $\mathbb{Z}_p$-module, where $\mathcal{H}_{\mathrm{tors}}$, having finite underlying set, is a $\mathbb{Z}_p$-torsion module. Since we know that $\mathcal{H}$ is free of rank d over $\mathbb{Z}_p$, we conclude that $\mathcal{H}_{\mathrm{tors}} = 0$ and $r = 1$. $\square$

Another consequence of the isomorphism (1.W) is the following fact, which is stated here for use in the next chapter.

Lemma 1.37. *For each non-zero element $\eta \in \ker[p]$, there exists an element $t \in \mathcal{H}$ such that $t(\eta)$ is a primitive p-th root of unity.*

Proof. Denote the group of p-th roots of unity in $\mathbb{C}_p$ by μ_p. Any $f \in T'$ maps $\ker[p]$ into the p-torsion of $\hat{\mathbb{G}}_m$, which is precisely $\{\xi - 1 \colon \xi \in \mu_p\}$. It follows that $t(z) \in \mu_p$ for any $z \in \ker[p]$ and any $t \in \mathcal{H}$. Hence we have to show that there doesn't exist a non-zero $\eta \in \ker[p]$ such that $t(\eta) = 1$ holds for all $t \in \mathcal{H}$. Suppose that there exists such an element η. If ω is a lift of η to $T_p\mathcal{G}$ (i.e. an element of $\varprojlim \ker[p^n] = T_p\mathcal{G}$ whose first component is equal to η), then all homomorphisms in $\mathrm{Hom}_{\mathbb{Z}_p}(T_p\mathcal{G}, T_p\hat{\mathbb{G}}_m)$ are trivial on the first component of ω. Indeed, this follows from the fact that (1.W) is an isomorphism and that $f(\eta) = 0$ for all $f \in T'$. By taking the constant lift of η in the upper row of the diagram (1.V) from the proof of Lemma 1.34, we obtain an element $\tilde{\eta} \in \varprojlim(\mathbb{Z}_p/p^n\mathbb{Z}_p)^d$ such that all homomorphisms $\varprojlim(\mathbb{Z}_p/p^n\mathbb{Z}_p)^d \longrightarrow T_p\hat{\mathbb{G}}_m = \varprojlim(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ are trivial on the first component of $\tilde{\eta}$. In particular, because this is true for the d projections, it follows that the first component of $\tilde{\eta}$ is $0 \in (\mathbb{Z}_p/p\mathbb{Z}_p)^d$, a contradiction. $\square$

We fix a generator t_1 of the o-module $\mathcal{H}$. Then every $t \in \mathcal{H}$ is of the form $t_1 \circ [a]$ for a unique $a \in o$. We write t_a for $t_1 \circ [a]$. Using our identification of the o-modules T' and $\mathcal{H}$, we obtain a generator t'_0 of T' by setting

$$F_{t'_0} := t_1 - 1.$$

We write Ω for $\Omega_{t'_0}$, the leading coefficient of $F_{t'_0}$. Then we obviously have

$$t_a(Z) = 1 + \Omega a Z + \text{ terms of degree } \geq 2.$$

Lemma 1.38. *The pairing*

$$\begin{aligned} T' \times \mathbb{C}_p &\longrightarrow \mathbb{C}_p \\ (t', x) &\longmapsto \Omega_{t'} x \end{aligned}$$

is o-bilinear.

Proof. The pairing is clearly o-linear in the second component. To show that it is o-linear in the first component, let $t', s' \in T'$. For shortness, write H for $F_{t'} + 1$, G for $F_{s'} + 1$ and H' for the formal derivative $\frac{d}{dZ} H$. Since $F_{t'} +_{\hat{\mathbb{G}}_m} F_{s'}$ and $(F_{t'} +_{\hat{\mathbb{G}}_m} F_{s'}) + 1 = H \dotplus G$ have the same derivative at the origin, we obtain

$$\Omega_{t' +_{\hat{\mathbb{G}}_m} s'} = (H \dotplus G)'(0) = (H \cdot G)'(0) = H'(0)G(0) + H(0)G'(0) = \Omega_{t'} + \Omega_{s'}$$

using $H(0) = 1 = G(0)$. For $a \in o$, we have

$$\Omega_{at'} = (H \circ [a])'(0) = H'([a](0)) \cdot [a]'(0) = H'(0) \cdot a = \Omega_{t'} \cdot a \tag{1.X}$$

since $[a](0) = 0$ and $[a]'(0) = a$. The assertion now follows immediately. □

A key ingredient in our considerations to come will be the pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon T' \times \mathfrak{B}(\mathbb{C}_p) &\longrightarrow \mathfrak{B}_1(\mathbb{C}_p) \\ (t', z) &\longmapsto 1 + F_{t'}(z), \end{aligned}$$

or, equivalently

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathscr{H} \times \mathfrak{B}(\mathbb{C}_p) &\longrightarrow \mathfrak{B}_1(\mathbb{C}_p) \\ (t, z) &\longmapsto t(z). \end{aligned}$$

This pairing is $\mathbb{Z}$-bilinear:

$$\begin{aligned} \langle t \dotplus s, z \rangle = (t \dotplus s)(z) = t(z)s(z) = \langle t, z \rangle \langle s, z \rangle, \text{ and} \\ \langle t, z +_{\mathcal{G}} z' \rangle = t(z +_{\mathcal{G}} z') = t(z)t(z') = \langle t, z \rangle \langle t, z' \rangle \end{aligned}$$

hold for all $t, s \in \mathscr{H}, z, z' \in \mathfrak{B}(\mathbb{C}_p)$. Note that this is indeed bilinearity, since $\mathfrak{B}_1(\mathbb{C}_p)$ is a $\mathbb{Z}_p$-module in such a way that the module addition is the multiplication in $\mathfrak{B}_1(\mathbb{C}_p) \subseteq \mathbb{C}_p^\times$. Moreover, the pairing is o-invariant:

$$\langle a \odot t, z \rangle = \langle t \circ [a], z \rangle = t([a](z)) = \langle t, [a](z) \rangle \tag{1.Y}$$

holds for all $a \in o, t \in \mathscr{H}, z \in \mathfrak{B}(\mathbb{C}_p)$. Note that the pairing cannot be o-bilinear, because the target is $\mathfrak{B}_1(\mathbb{C}_p)$, which only has the structure of a $\mathbb{Z}_p$-module and not of an o-module. Consider the map

$$\begin{aligned} \kappa_{t' \otimes z} \colon G &\longrightarrow \mathfrak{B}_1(\mathbb{C}_p) \\ a &\longmapsto \langle t', [a](z) \rangle, \end{aligned}$$

for fixed $t' \in T'$ and $z \in \mathfrak{B}(\mathbb{C}_p)$. The $\mathbb{Z}$-bilinearity of the pairing implies that this map is a character on $o = G$. Since $\kappa_{t'\otimes z}$ is also continuous, it is automatically $\mathbb{Z}_p$-linear and hence $\mathbb{Q}_p$-analytic by Proposition 1.9. The proof of Proposition 3.1 in [ST01] shows that $\kappa_{t'\otimes z}$ is even L-analytic, i.e. $\kappa_{t'\otimes z} \in \widehat{G}(\mathbb{C}_p) = \mathfrak{X}(\mathbb{C}_p)$.
Consequently, we have the pairing

$$\begin{aligned} T' \times \mathfrak{B}(\mathbb{C}_p) &\longrightarrow \widehat{G}(\mathbb{C}_p) \\ (t', z) &\longmapsto [a \longmapsto \langle t', [a](z)\rangle]. \end{aligned}$$

Being $\mathbb{Z}$-bilinear and o-invariant as well, it factors through $\otimes_o$, giving rise to a group homomorphism

$$\begin{aligned} \kappa(\mathbb{C}_p)\colon T' \otimes_o \mathfrak{B}(\mathbb{C}_p) &\xrightarrow{\sim} \widehat{G}(\mathbb{C}_p) \\ t' \otimes z &\longmapsto \kappa_{t'\otimes z}, \end{aligned}$$

which is in fact a group isomorphism by [ST01] Proposition 3.1. We will prove this proposition in a more general setting in Section 3.3. Since T' is a free o-module of rank one and we have fixed a generator t'_0, we obtain the group isomorphism

$$\begin{aligned} \kappa(\mathbb{C}_p)\colon (\mathfrak{B}(\mathbb{C}_p), +_{\mathcal{G}}) &\xrightarrow{\sim} \widehat{G}(\mathbb{C}_p) \\ z &\longmapsto \kappa_z, \end{aligned} \tag{1.Z}$$

where $\kappa_z = \kappa_{t'_0\otimes z}$, i.e. $\kappa_z(a) = 1 + F_{t'_0}([a](z)) = t_a(z)$.

Example 1.39. If $L = \mathbb{Q}_p$ and $\pi = p \in \mathbb{Z}_p$, then we can choose $\mathcal{G} = \hat{\mathbb{G}}_m$ (cf. Example 1.29). Therefore, the identity mapping is a generator of the $\mathbb{Z}_p$-module $\mathrm{Hom}_{\mathbb{Z}_p}(T_p\mathcal{G}, T_p\hat{\mathbb{G}}_m)$, and we can choose t'_0 to be the corresponding generator of T', i.e. $F_{t'_0}(Z) = Z$. Then $\kappa_z(a) = 1 + [a](z) = 1 + ((1+z)^a - 1) = (1+z)^a$. Thus, if we identify $(\mathfrak{B}(\mathbb{C}_p), +_{\hat{\mathbb{G}}_m})$ with $\mathfrak{B}_1(\mathbb{C}_p)$, $\kappa(\mathbb{C}_p)$ is the isomorphism from (1.M).

In Theorem 3.6 in [ST01], an isomorphism

$$\kappa\colon \mathfrak{B}/\mathbb{C}_p \xrightarrow{\sim} \mathfrak{X}/\mathbb{C}_p$$

of rigid $\mathbb{C}_p$-varieties is constructed, which on the level of $\mathbb{C}_p$-points is given exactly by $\kappa(\mathbb{C}_p)$. This is the isomorphism we refer to as the uniformization isomorphism. With κ, we obtain an isomorphism $\mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \xrightarrow{\sim} \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$.

The remainder of this thesis ramifies in two separate directions. In Chapter 2, we analyze the applications of the previous results in constructing p-adic L-functions, following [ST01] Section 5. On the other hand, Chapter 3 is concerned with proving our main result, namely that (1.Z) generalizes to an isomorphism in the relative Lubin-Tate case.
We end the current chapter with two remarks on further applications of the uniformization isomorphism, which we will not pursue in this thesis.

Remark 1.40. Let L^{alg} be the algebraic closure of L in $\mathbb{C}_p$. Since the absolute Galois group $G_L := \mathrm{Gal}(L^{\mathrm{alg}}/L)$ of L acts naturally on $\mathcal{O}(\mathfrak{X}/\mathbb{C}_p)$ and we have $\mathcal{O}(\mathfrak{X}) = \mathcal{O}(\mathfrak{X}/\mathbb{C}_p)^{G_L}$, the isomorphism between $\mathcal{O}(\mathfrak{X}/\mathbb{C}_p)$ and $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ restricts to an isomorphism of $\mathcal{O}(\mathfrak{X})$ and the subring $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)^{G_L,*}$ of Galois-fixed elements in the power series ring $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ with respect to a "twisted" action of G_L on $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$. This twisted action is by definition precisely the one that makes the isomorphism $\mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \xrightarrow{\sim} \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ Galois-equivariant. See [ST01] Corollary 3.8 and the surrounding discussion for more details.

Remark 1.41. In Section 1.1, we have seen how the Amice isomorphism is proven with the theory of Mahler expansions for functions in $C^{an}(\mathbb{Z}_p, \mathbb{Q}_p)$. Conversely, Schneider and Teitelbaum applied their Fourier theory to obtain a generalization of the Mahler expansion for functions in $C^{an}(G, \mathbb{C}_p)$ (see Section 4 in [ST01]). If one were able to first prove the generalized Mahler expansion, one could then prove Theorem 1.16 by identifying $\mathcal{O}(\mathfrak{X}/\mathbb{C}_p)$ with $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ and constructing the inverse $\mathscr{F}^{-1}\colon \mathcal{O}(\mathfrak{B}/\mathbb{C}_p) \longrightarrow D(G, \mathbb{C}_p)$ in a way analogous to the proof of Theorem 1.2.

Chapter 2

Applications to p-adic interpolation

In this chapter, we analyze in detail the ingredients with which Schneider and Teitelbaum proved their interpolation result ([ST01] Proposition 5.1). Given a $\mathbb{C}_p$-valued locally L-analytic distribution λ on o that is supported on $o^\times$, the mentioned interpolation result provides a link between the values $\int_o x^n \lambda(x)$ (which are called the moments of λ) on the one hand and the Fourier transform F_λ on the other hand. Schneider and Teitelbaum used this link to construct a distribution that gives rise to a p-adic L-function, such that the special values of the L-function agree with the moments of the distribution. An early goal of this thesis was to investigate what congruences between the special values are implied by the existence of this L-function. The idea would be to deduce information about the values

$$\left| \int_o (x^n - x^m)\lambda(x) \right| \tag{2.A}$$

in dependence of: the coefficients of F_λ, the numerical invariants of L and the relation between the natural numbers n and m. I later became aware that Kenichi Bannai and Shinichi Kobayashi had already found a solution to this problem (see [BK16] Theorem 1.1 or Section 2.3 of this thesis for a precise statement of their result). Thereupon I shifted the goal of the current chapter to providing proofs for facts that are stated without proof or reference in the last section of [ST01]. For instance, Section 2.2 provides a proof of analyticity of the Mellin transform. Section 2.3 gives a sketch of what would have been my approach to deriving an inequality involving the values (2.A) in the case when the Fourier transform has bounded coefficients. An auxiliary result which is interesting in itself is Proposition 2.14.

This chapter and the next are independent of each other. Chapter 3 can therefore be read directly after Chapter 1.

2.1 The integration pairing

We keep the notations of the previous chapter. Thus, G denotes o viewed as a locally L-analytic group, the Lubin-Tate formal group is again denoted by $\mathcal{G} = \mathcal{G}_\pi$ and the dual Tate module $\operatorname{Hom}_{o_{\mathbb{C}_p}}(\mathcal{G}, \hat{\mathbb{G}}_m)$ by T'. Finally, with a fixed generator t_0' of T', we have the isomorphism $\kappa\colon \mathfrak{B}/\mathbb{C}_p \xrightarrow{\sim} \mathfrak{X}/\mathbb{C}_p$ given on $\mathbb{C}_p$-points by the group isomorphism

$$\begin{aligned}(\mathfrak{B}(\mathbb{C}_p), +_{\mathcal{G}}) &\xrightarrow{\sim} \widehat{G}(\mathbb{C}_p)\\ z &\longmapsto \kappa_z := \kappa_{t_0' \otimes z}\end{aligned}$$

from Section 1.3.2. The Fourier transform and the uniformization isomorphism κ allow us to identify $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ with the continuous dual $D(G, \mathbb{C}_p)$ of $C^{an}(G, \mathbb{C}_p)$, giving rise to an "integration pairing"

$$\{\cdot, \cdot\}\colon \mathcal{O}(\mathfrak{B}/\mathbb{C}_p) \times C^{an}(G, \mathbb{C}_p) \longrightarrow \mathbb{C}_p.$$

If a power series $F \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ corresponds to a distribution $\lambda \in D(G, \mathbb{C}_p)$ under the identification above (i.e. if $\lambda = \{F, \cdot\}$), we write $F = F_\lambda$. For $z \in \mathfrak{B}(\mathbb{C}_p)$, the formula

$$F_\lambda(z) = \lambda(\kappa_z)$$

holds. Schneider and Teitelbaum's p-adic L-functions arise as locally analytic distributions on Galois groups that are naturally isomorphic to multiplicative, rather than additive groups. Proposition 2.4 below, which characterizes when a distribution on o is supported in $o^\times$, is therefore of technical importance. The goal of this section is to provide a proof of this proposition. For this, we first prove a few lemmata.

Lemma 2.1. *Given $f \in C^{an}(G, \mathbb{C}_p)$ and $F \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$, the equation*

$$\{F, \kappa_z f\} = \{F(z +_{\mathcal{G}} .), f\}$$

holds for all $z \in \mathfrak{B}(\mathbb{C}_p)$.

Proof. Let $\lambda \in D(G, \mathbb{C}_p)$ be the unique distribution such that $F = F_\lambda$. Define the distribution $\lambda_z \in D(G, \mathbb{C}_p)$ by $\lambda_z\colon C^{an}(G, \mathbb{C}_p) \longrightarrow \mathbb{C}_p, g \longmapsto \lambda(\kappa_z g)$. Then we have

$$F_{\lambda_z}(w) = \lambda_z(\kappa_w) = \lambda(\kappa_z \kappa_w) = \lambda(\kappa_{z +_{\mathcal{G}} w}) = F_\lambda(z +_{\mathcal{G}} w)$$

for any $w \in \mathfrak{B}(\mathbb{C}_p)$, i.e. $F(z +_{\mathcal{G}} .) = F_{\lambda_z}$. It follows that

$$\{F(z +_{\mathcal{G}} .), f\} = \lambda_z(f) = \lambda(\kappa_z f) = \{F, \kappa_z f\}.$$

□

Lemma 2.2. *We have the equality*

$$\sum_{[\pi](z)=0} \kappa_z = q \cdot \mathbb{1}_{\pi o}$$

of functions on o, where $\mathbb{1}_{\pi o}$ is the indicator function given by 1 if $x \in \pi o$ and 0 if $x \in o \backslash \pi o$.

Proof. Let $\mathfrak{B}^{(\pi)} := \ker[\pi] = \{z \in \mathfrak{B}(\mathbb{C}_p) \colon [\pi](z) = 0\}$. By Proposition 1.31, we have an isomorphism

$$\mathfrak{B}^{(\pi)} \cong o/\pi \tag{2.B}$$

of o-modules.

For a fixed $\alpha \in o$, consider the group homomorphism

$$\begin{aligned} t_\alpha \colon \mathfrak{B}^{(\pi)} &\longrightarrow \mu_p \subseteq \mathbb{C}_p^\times \\ z &\longmapsto t_\alpha(z) = \kappa_z(\alpha), \end{aligned}$$

where μ_p denotes the group of p-th roots of unity. To convince ourselves that t_α is well-defined, observe that because of (2.B) and the fact that $o/\pi = k$ is of characteristic p, every $0 \neq z \in \mathfrak{B}^{(\pi)}$ has additive order equal to p. As $\kappa(\mathbb{C}_p) \colon \mathfrak{B}^{(\pi)} \longrightarrow \widehat{G}(\mathbb{C}_p)$ is a group homomorphism, we conclude that κ_z^p is the trivial character for all $z \in \mathfrak{B}^{(\pi)}$. Hence $\kappa_z(\alpha)$ is a p-th root of unity.

Now, if $\alpha \in \pi o$, we have $[\alpha](z) = 0$ and hence

$$\kappa_z(\alpha) = t_\alpha(z) = t_1([\alpha](z)) = t_1(0) = 1,$$

from which we conclude

$$\sum_{z \in \mathfrak{B}^{(\pi)}} \kappa_z(\alpha) = q.$$

On the other hand, if $\alpha \in o^\times$, we claim that the image of t_α is not trivial. Suppose that we have shown this. As the only non-trivial subgroup of μ_p is μ_p itself, it follows that t_α is surjective. Hence, for any $x \in \mu_p$ the fiber $t_\alpha^{-1}(x)$ has the same cardinality as $\ker(t_\alpha)$, i.e. equal to $q/p = p^{f-1}$. Therefore,

$$\sum_{z \in \mathfrak{B}^{(\pi)}} \kappa_z(\alpha) = p^{f-1} \cdot \sum_{\xi \in \mu_p} \xi = p^{f-1} \cdot 0 = 0$$

and we are done. To see that $\sum_\xi \xi = 0$, pick a primitive $\xi_0 \in \mu_p$ and observe that $0 = \xi_0^p - 1 = (\xi_0 - 1)(\xi_0^{p-1} + \xi_0^{p-2} + \ldots + \xi_0 + 1) = (\xi_0 - 1)\sum_\xi \xi$ and $\xi_0 - 1 \neq 0$.

It remains to show that the image of t_α is not trivial if $\alpha \in o^\times$. Pick a non-zero element $w \in \mathfrak{B}^{(\pi)}$. Then $[\alpha](w)$ is also non-zero, since $\alpha \in o^\times$. Hence we may apply Lemma 1.37 to $\eta = [\alpha](w)$, which tells us that there exists a $t \in \mathscr{H}$ such that $t(\eta)$ is a primitive p-th root of unity. Recall that t_1 generates the o-module $\mathscr{H}$, so t is of the form $t = t_c = t_1 \circ [c]$ for some $c \in o$. The equality $t_c \circ [\alpha] = t_{c\alpha} = t_{\alpha c} = t_\alpha \circ [c]$ then implies that the image of t_α contains the primitive p-th root of unity $t(\eta) = t_\alpha([c](w))$. □

Let H denote $o^\times$ viewed as a Lie group over L. As a locally L-analytic manifold, H is an open submanifold of G. In particular, there is a mapping $D(G, K) \longrightarrow D(H, K)$ given

by "restriction", which we now explain.
In general, we may consider any open subset $X \subseteq o$ as a locally L-analytic submanifold of $G = o$. Suppose moreover that X is closed in G (we call such a set "clopen"). Then the following two operators are well-defined:

$$\begin{aligned} C^{an}(G,K) &\longrightarrow C^{an}(G,K) \\ f &\longmapsto \mathbb{1}_X f \end{aligned} \quad \text{and} \quad \begin{aligned} C^{an}(X,K) &\longrightarrow C^{an}(G,K) \\ h &\longmapsto h_{ex,0}, \end{aligned}$$

where $h_{ex,0}$ denotes the extension of h by zero, i.e. $h_{ex,0}(x) = h(x)$ for $x \in X$ and $h_{ex,0}(x) = 0$ for $x \in G \setminus X$.
For $\lambda \in D(G,K)$, we define $\mathrm{Res}_X(\lambda) \in D(G,K)$ by

$$\mathrm{Res}_X(\lambda)(f) := \lambda(\mathbb{1}_X f).$$

The following is common notation for $\mathrm{Res}_X(\lambda)(f)$:

$$\int\limits_G f(x)\,\mathrm{Res}_X(\lambda)(x) = \mathrm{Res}_X(\lambda)(f) = \int\limits_X f(x)\lambda(x).$$

On the other hand, for $\lambda \in D(G,K)$ we also define $\overline{\mathrm{Res}}_X(\lambda) \in D(X,K)$ by

$$\overline{\mathrm{Res}}_X(\lambda)(h) := \lambda(h_{ex,0}).$$

Observe that, for an $f \in C^{an}(G,K)$, the extension of $f|_X$ by zero is equal to $\mathbb{1}_X f$ and hence

$$\overline{\mathrm{Res}}_X(\lambda)(f|_X) = \mathrm{Res}_X(\lambda)(f). \tag{2.C}$$

Definition 2.3. We say that a distribution $\lambda \in D(G,K)$ is supported in X if λ vanishes on functions with support in $G \setminus X$.

It is especially useful to consider $\overline{\mathrm{Res}}_X$ for a distribution $\lambda \in D(G,K)$ with support in X. The reason for this is the following. If $\tilde{h} \in C^{an}(G,K)$ is any extension of $h \in C^{an}(X,K)$ (in the sense that $\tilde{h}|_X = h$), then $\tilde{h} - h_{ex,0}$ is supported in $G \setminus X$, and so $\lambda(h_{ex,0}) = \lambda(\tilde{h})$. Therefore, we have $\lambda(f) = \overline{\mathrm{Res}}_X(\lambda)(f|_X)$ for any $f \in C^{an}(G,K)$. Together with (2.C), this implies that $\lambda(f) = \mathrm{Res}_X(\lambda)(f)$ (which is

$$\int\limits_G f(x)\lambda(x) = \int\limits_X f(x)\lambda(x)$$

in the notation of integrals) holds for all $f \in C^{an}(G,K)$, if λ is supported in X. From now on, we will mostly work with the notation of integrals.
We are now ready to prove the following result.

Proposition 2.4. *A distribution $\lambda \in D(G, \mathbb{C}_p)$ is supported on $o^\times$ if and only if*

$$\sum_{[\pi](z)=0} F_\lambda(. +_{\mathcal{G}} z) = 0. \tag{2.D}$$

Proof. Using Lemma 2.1 and Lemma 2.2, we deduce

$$\begin{aligned}
\{\sum_{[\pi](z)=0} F_\lambda(. +_{\mathcal{G}} z), f\} &= \sum_{[\pi](z)=0} \{F_\lambda(. +_{\mathcal{G}} z), f\} \\
&= \sum_{[\pi](z)=0} \{F_\lambda, \kappa_z f\} \\
&= \{F_\lambda, \sum_{[\pi](z)=0} \kappa_z f\} \\
&= \{F_\lambda, q\mathbb{1}_{\pi o} f\} \\
&= q \cdot \{F_\lambda, \mathbb{1}_{\pi o} f\}
\end{aligned}$$

for any $f \in C^{an}(G, \mathbb{C}_p)$. Suppose now that (2.D) holds and let $f \in C^{an}(G, \mathbb{C}_p)$ be a function with support in πo. Then $f = \mathbb{1}_{\pi o} f$ and consequently

$$0 = \{0, f\} = \{\sum_{[\pi](z)=0} F_\lambda(. +_{\mathcal{G}} z), f\} = q \cdot \{F_\lambda, \mathbb{1}_{\pi o} f\} = q \cdot \{F_\lambda, f\} = q \cdot \lambda(f),$$

which means that $\lambda(f) = 0$. Hence λ vanishes on functions supported in πo.
The converse follows similarly. $\square$

2.2 The Mellin transform

In this section, we define the Mellin transform of a distribution and show that it is a rigid analytic function. The analyticity of the Mellin transform will imply the analyticity of the interpolating function (the p-adic L-function) in Proposition 2.8.
Let us briefly recall the structure of $o^\times$. We have the split exact sequence

$$1 \longrightarrow 1 + \pi o \longrightarrow o^\times \longrightarrow k^\times \longrightarrow 1.$$

A section of this short exact sequence is given by the Teichmüller character $\omega\colon k^\times \longrightarrow o^\times$. Recall that for $x \in k^\times = (o/\pi)^\times$, $\omega(x)$ is defined as the unique $(q-1)$-th root of unity in L that is congruent to x modulo π. With this definition, ω is obviously multiplicative and induces an isomorphism

$$\omega\colon k^\times \xrightarrow{\sim} \mu_{q-1} \subseteq o^\times$$

onto the group of $(q-1)$-th roots of unity. Recall that H denotes $o^\times$ viewed as an L-analytic Lie group, and let H_1 denote the open subgroup $1+\pi o \subseteq o^\times$. Then we have the isomorphism

$$\begin{aligned} H &\xrightarrow{\sim} H_1 \times k^\times \\ x &\longmapsto (\langle x\rangle, x+\pi o) \end{aligned}$$

where $\langle x\rangle := x \cdot \omega(x+\pi o)^{-1}$. In the following, we often write $\omega(x)$ for $\omega(x+\pi o)$.
From now on, until the rest of this section, we assume that the ramification index e of $L/\mathbb{Q}_p$ satisfies $e < p-1$. With this assumption we can apply Satz 5.5 in Kapitel II of [Neu06], which tells us that the power series of exp and log converge and give rise to mutually inverse isomorphisms

$$\pi o \underset{\log}{\overset{\exp}{\rightleftarrows}} 1+\pi o.$$

Therefore $\ell := \pi^{-1} \cdot \log$ defines an L-analytic isomorphism

$$\ell\colon H_1 \xrightarrow{\sim} G$$

that in turn induces isomorphisms

$$\begin{aligned} D(H_1,K) &\xrightarrow{\sim} D(G,K) \\ \lambda &\longmapsto \ell_*\lambda \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathfrak{B}(\mathbb{C}_p) &\xrightarrow{\sim} \widehat{H_1}(\mathbb{C}_p) \\ z &\longmapsto \psi_z, \end{aligned}$$

with $\ell_*(\lambda)(f) := \lambda(f \circ \ell)$ and $\psi_z := \kappa_z \circ \ell$. Since $\widehat{k^\times}$ is a cyclic group of order $q-1$ generated by ω, we further conclude that

$$\widehat{H}(\mathbb{C}_p) = \{\omega^i \psi_z \colon z \in \mathfrak{B}(\mathbb{C}_p), 1 \leq i \leq q-1\}.$$

Definition 2.5. Given a distribution $\lambda \in D(H,K)$, its Mellin transform is defined as the function

$$M_\lambda(z,\omega^i) := \lambda(\omega^i \psi_z)$$

of $z \in \mathfrak{B}(\mathbb{C}_p)$ and $i \in \{1,\dots,q-1\}$.

Proposition 2.6. *Let $\lambda \in D(H,K)$. For each fixed value of i, $M_\lambda(-,\omega^i)$ is a rigid analytic function in $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$.*

Proof. Our proof is inspired by the proof of Proposition VI.2.6 in [Col02]. The goal is to construct a distribution $\Gamma_\lambda^{(i)} \in D(G,K)$ such that

$$M_\lambda(z,\omega^i) = \int_G \kappa_z(x) \Gamma_\lambda^{(i)}(x).$$

Once this is done, it follows that we have an equality

$$M_\lambda(-,\omega^i) = F_{\Gamma_\lambda^{(i)}}$$

between the i-th Mellin transform of λ and the Fourier transform of $\Gamma_\lambda^{(i)}$. Since the latter is a rigid function in $\mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$, this proves the proposition.

To obtain $\Gamma_\lambda^{(i)}$, we will need the following way of constructing new distributions from given ones. If $\mu \in D(H,K)$ and $a \in o^\times$, we define the distribution $\delta_a \star \mu$ by

$$\int_{o^\times} f(x)(\delta_a \star \mu)(x) = \int_{o^\times} f(ax)\mu(x). \tag{2.E}$$

Denote the group of $(q-1)$-th roots of unity by $\mu \subseteq L^\times$. Let

$$\Gamma_\lambda^{(i)} := \ell_* \overline{\mathrm{Res}}_{1+\pi o}\Big(\sum_{\varepsilon\in\mu} \varepsilon^i \delta_{\varepsilon^{-1}} \star \lambda\Big).$$

Then

$$\int_o \kappa_z(y)\Gamma_\lambda^{(i)}(y) = \sum_{\varepsilon\in\mu} \varepsilon^i \int_{1+\pi o} \psi_z(\langle x\rangle)(\delta_{\varepsilon^{-1}} \star \lambda)(x).$$

We have used the fact that $\kappa_z \circ \ell(x) = \psi_z(x) = \psi_z(\langle x\rangle)$. We have also passed from $\overline{\mathrm{Res}}$ to Res by virtue of (2.C).

Recall that we use the notation

$$\int_{1+\pi o} \psi_z(\langle x\rangle)(\delta_{\varepsilon^{-1}} \star \lambda)(x) = \int_{o^\times} \psi_z(\langle x\rangle)\,\mathrm{Res}_{1+\pi o}(\delta_{\varepsilon^{-1}} \star \lambda)(x).$$

Next, using the fact that $\mathrm{Res}_{1+\pi o}(\delta_{\varepsilon^{-1}} \star \lambda) = \delta_{\varepsilon^{-1}} \star \mathrm{Res}_{\varepsilon+\pi o}(\lambda)$ by Lemma 2.7 below and that $\langle \varepsilon x\rangle = \langle x\rangle$ holds for all $x \in o^\times$, we see that

$$\begin{aligned}
\int_{o^\times} \psi_z(\langle x\rangle)\,\mathrm{Res}_{1+\pi o}(\delta_{\varepsilon^{-1}} \star \lambda)(x) &= \int_{o^\times} \psi_z(\langle \varepsilon^{-1}x\rangle)\,\mathrm{Res}_{\varepsilon+\pi o}(\lambda)(x) \\
&= \int_{\varepsilon+\pi o} \psi_z(\langle \varepsilon^{-1}x\rangle)\lambda(x) \\
&= \int_{\varepsilon+\pi o} \psi_z(\langle x\rangle)\lambda(x).
\end{aligned}$$

Now we use the fact that $\omega(x) = \varepsilon$ if $x \in \varepsilon + \pi o$ to conclude that

$$\begin{aligned}\int_{o} \kappa_z(y)\Gamma_\lambda^{(i)}(y) &= \sum_{\varepsilon\in\mu} \varepsilon^i \int_{\varepsilon+\pi o} \psi_z(\langle x\rangle)\lambda(x) \\ &= \sum_{\varepsilon\in\mu} \int_{\varepsilon+\pi o} \omega(x)^i\psi_z(\langle x\rangle)\lambda(x) \\ &= \int_{o^\times} \omega(x)^i\psi_z(\langle x\rangle)\lambda(x),\end{aligned}$$

which completes the proof. $\square$

Lemma 2.7. *Let X be a clopen subset of $o^\times$, $\lambda \in D(H,K)$ a distribution and $\alpha \in o^\times$. Then we have the following equality of distributions (which are defined according to (2.E)):*

$$\mathrm{Res}_X(\delta_\alpha \star \lambda) = \delta_\alpha \star \mathrm{Res}_{\alpha^{-1}X}(\lambda).$$

Proof. Let $f \in C^{an}(H,K)$. Using the equality $\mathbb{1}_X(\alpha x) = \mathbb{1}_{\alpha^{-1}X}(x)$, we compute:

$$\begin{aligned}\int_{o^\times} f(x)\,\mathrm{Res}_X(\delta_\alpha\star\lambda)(x) &= \int_{o^\times} \mathbb{1}_X(x)f(x)(\delta_\alpha\star\lambda)(x) = \int_{o^\times} \mathbb{1}_X(\alpha x)f(\alpha x)\lambda(x) \\ &= \int_{o^\times} \mathbb{1}_{\alpha^{-1}X}(x)f(\alpha x)\lambda(x) = \int_{o^\times} f(\alpha x)\,\mathrm{Res}_{\alpha^{-1}X}(\lambda)(x) \\ &= \int_{o^\times} f(x)(\delta_\alpha\star\mathrm{Res}_{\alpha^{-1}X}(\lambda))(x).\end{aligned}$$

$\square$

Recall that we have fixed a generator t_0' of the o-module T' and that Ω denotes the leading coefficient of t_0'. Let $\log_{\mathcal{G}}$ (resp. $\exp_{\mathcal{G}}$) denote the logarithm (resp. the exponential) of the formal group $\mathcal{G}$ and let ∂ denote the invariant differential of $\mathcal{G}$. By [Lan12] §8.6 Lemma 4, $\log_{\mathcal{G}}$ and $\exp_{\mathcal{G}}$ induce mutually inverse group isomorphisms

$$(\mathfrak{B}(\rho), +_{\mathcal{G}}) \underset{\exp_{\mathcal{G}}}{\overset{\log_{\mathcal{G}}}{\rightleftarrows}} (\mathfrak{B}(\rho), +_{\hat{\mathbb{G}}_a})$$

where $\rho := p^{-1/e(q-1)}$, $\mathfrak{B}(\rho) = \mathfrak{B}(\rho)(\mathbb{C}_p) = \{z \in \mathbb{C}_p\colon\ |z| < \rho\}$ and $\hat{\mathbb{G}}_a$ is the additive formal group law $\hat{\mathbb{G}}_a(X,Y) = X+Y$. Furthermore, for $z \in \mathfrak{B}(\rho)$ we have $|z| = |\log_{\mathcal{G}}(z)| = |\exp_{\mathcal{G}}(z)|$.

Proposition 2.8. *Let λ be a distribution in $D(G,\mathbb{C}_p)$ that is supported on H, let F_λ be its Fourier transform and M_λ its Mellin transform. Suppose that $n \in \mathbb{N}$ satisfies $n \equiv i \mod q-1$. Then*

$$M_\lambda(\exp_{\mathcal{G}}(n\pi/\Omega), \omega^i) = \int_{o^\times} x^n\lambda(x) = \Omega^{-n}(\partial^n F_\lambda(z)|_{z=0}). \tag{2.F}$$

Proof. [ST01] Proposition 5.1. □

By [ST01] Lemma 3.4, we have $|\Omega| = p^{-s}$ with $s = \frac{1}{p-1} - \frac{1}{e(q-1)}$. Hence the hypothesis $e < p-1$ guarantees that $|x\pi/\Omega| < \rho$, i.e. that $\exp_{\mathcal{G}}(x\pi/\Omega)$ converges for all $x \in o$. In view of Proposition 2.6, it follows that the left-hand side of the equations in (2.F) gives a (globally) L-analytic interpolation of the values on the right side.

2.3 Towards congruences

Schneider and Teitelbaum's approach for constructing a p-adic L-function for a CM elliptic curve at a supersingular prime was the following. They made use of the machinery of Coleman power series and elliptic units from [dS87] to produce a power series $F \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ such that $\Omega^{-n}(\partial^n F(z)|_{z=0})$ essentially coincides with the n-th special value that is to be interpolated. Since they made sure that F satisfies $\sum_{[\pi](z)=0} F(.+_{\mathcal{G}} z) = 0$, the corresponding distribution $\lambda := \{F, \cdot\}$ is supported on H (by Proposition 2.4). Finally, they applied their interpolation result (Proposition 2.8 above) to obtain the L-function. A nice aspect of their construction is that the n-th special value is equal to $\int_{o^\times} x^n \lambda(x)$. This motivates the following general question: given a power series $\psi \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ such that the corresponding distribution $\mu := \{\psi, \cdot\}$ is supported on H, what can be said about the values

$$\left|\int_{o^\times} x^n \mu(x)\right| \quad \text{and} \quad \left|\int_{o^\times} (x^m - x^n)\mu(x)\right| ?$$

Of course, one would also need to identify reasonable conditions that need to be imposed on n (resp. $m-n$) and on the coefficients of ψ in order to ensure that the question may be answered in a satisfactory way.

We note again that [BK16] provides a solution for the above problem. To describe it, we will need the following notation. For $k \in \mathbb{N}$, we let

$$\overline{\rho}(k) := \max_{m \geq k} |m!/\Omega^m| \,.$$

For $\psi = \sum_{k=0}^\infty c_k T^k \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ and $N \in \mathbb{N}$, we let

$$\|\psi\|_N := \max_k \left\{ |c_k|\, \overline{\rho}\left(\left[\frac{k}{q^N}\right]\right)\right\}.$$

The main theorem of [BK16] is the following one.

Theorem 2.9. *Let $g \in o$ and $f \in \mathcal{F}_{(g+\pi^N o)}(K)$. Let $\psi \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ and $\mu = \{\psi, \cdot\}$. Then*

$$\left|\int_{g+\pi^N o} f(x)\mu(x)\right| \leq \overline{\rho}(0) \left|\frac{\pi}{q}\right|^N \|f\|_{g,N} \|\psi\|_N.$$

Moreover, if $e \leq p-1$, then

$$\left|\int_{g+\pi^N o} f(x)\mu(x)\right| \leq \left|\frac{\pi}{q}\right|^N \|f\|_{g,N} \|\psi\|_N.$$

Proof. [BK16] Theorem 4.3 ii). □

In [BK16] Section 5, this theorem is used to prove the following proposition.

Proposition 2.10. *Let L be the unramified quadratic extension of $\mathbb{Q}_p$, let $\psi \in o_{\mathbb{C}_p}[[T]]$ and let $\mu = \{\psi, \cdot\}$. If $m \equiv n \mod p^l(q-1)$, then*

$$\left| \int_{o^\times} (x^m - x^n)\mu(x) \right| \leq p^{-l+\frac{p}{q-1}}.$$

Proof. [BK16] Proposition 5.3. □

Using this, Bannai and Kobayashi go on to prove congruences between the Bernoulli-Hurwitz numbers.

We will now sketch an idea for an alternative approach involving measures (i.e. continuous distributions).
Recall that $D^{cont}(G, K)$ is the continuous dual of $C^{cont}(G, K)$. It is a Banach space with the usual operator norm

$$\|\nu\| = \sup_f \frac{|\nu(f)|}{\|f\|}.$$

The presence of a norm allows us to deduce the following statement.

Lemma 2.11. *Let $\nu \in D^{cont}(G, K)$ with $\|\nu\| \leq 1$. Let $k \in \mathbb{N}_{\geq 1}$ and $m, n \in \mathbb{N}$ be such that $m \equiv n \mod q^{k-1}(q-1)$ and $m, n \geq k$. Then*

$$\left| \int_o (x^m - x^n)\nu(x) \right| \leq \left|\pi^k\right|.$$

Proof. We have

$$\left| \int_o (x^m - x^n)\nu(x) \right| \leq \|\nu\| \|f_{m,n}\| \leq \|f_{m,n}\|$$

where $f_{m,n}$ denotes the function $x \longmapsto x^m - x^n$. Hence we need to show that $x^m \equiv x^n \mod \pi^k$ for all $x \in o$. This is obviously true for $x \in \pi o$, since $m, n \geq k$. On the other hand, if $x \in o^\times$, then its residue modulo π^k (which we again denote simply by x) lies in $(o/\pi^k)^\times$. Using the fact that $m - n$ is a multiple of $q^{k-1}(q-1)$ and that $q^{k-1}(q-1)$ is the cardinality of $(o/\pi^k)^\times$, we obtain the following equations in $(o/\pi^k)^\times$:

$$x^m = x^n \cdot x^{m-n} = x^n \cdot 1 = x^n,$$

i.e. $x^m \equiv x^n \mod \pi^k$. □

Returning to the given $\psi \in \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ and $\mu = \{\psi, \cdot\}$, we see that it is useful to ask under what circumstances the distribution μ extends to a measure. Such an extension, if it exists, is necessarily unique:

Lemma 2.12. *The restriction map $D^{cont}(G,K) \longrightarrow D(G,K)$ is well-defined and injective.*

Proof. The restriction of an $\ell \in D^{cont}(G,K)$ to $C^{an}(G,K)$ is a continuous linear form on $C^{an}(G,K)$ by Lemma 1.5. Hence the restriction map $D^{cont}(G,K) \longrightarrow D(G,K)$ is well-defined. It is moreover injective, since $C^{an}(G,K)$ is dense in $C^{cont}(G,K)$. We remark that already the space of all locally constant functions is dense in $C^{cont}(G,K)$ (cf. the proof of [BSX15] Lemma 1.2.1). $\square$

In the case $L = \mathbb{Q}_p$, we know that μ extends to a measure precisely when the coefficients of ψ are bounded. In fact, it follows from Mahler's expansion theorem for functions in $C^{cont}(\mathbb{Z}_p,K)$ (compare the proof of Theorem 1.2) that the map associating to a measure $\nu \in D^{cont}(\mathbb{Z}_p,K)$ its Amice transform A_ν is an isometric isomorphism of $D^{cont}(\mathbb{Z}_p,K)$ onto

$$K[[T]]_{bc} := \{f \in K[[T]] \colon \text{the coefficients of } f \text{ are bounded}\}$$

equipped with the supremum norm of the coefficients. We obtain the commutative diagram

$$\begin{tikzcd} D^{cont}(\mathbb{Z}_p,K) \arrow[r] \arrow[d, "\cong"'] & D(\mathbb{Z}_p,K) \arrow[d, "\cong"] \\ K[[T]]_{bc} \arrow[r, "\subseteq"] & \mathcal{O}(\mathfrak{B}/K) \end{tikzcd} \tag{2.G}$$

where the upper horizontal map is the restriction map and the vertical maps are given by the Amice transform. Because the left vertical map in is an isometry, we see that a distribution $\lambda \in D(\mathbb{Z}_p,K)$ extends to a measure $\lambda \in D^{cont}(\mathbb{Z}_p,K)$ satisfying $\|\lambda\| \leq 1$ if and only if its Amice transform A_λ lies in $o_K[[T]]$. In that case we may apply Lemma 2.11. See [Col02] §VI.1 for a proof of the famous Kummer congruences using this argument.

For general $L/\mathbb{Q}_p$, composing the Fourier isomorphism $\mathscr{F}\colon D(G_0,K) \longrightarrow \mathcal{O}(\mathfrak{X}_0/K)$ with the restriction map $D^{cont}(G,K) \longrightarrow D(G_0,K)$ (which is injective by the same argument as in Lemma 2.11) yields an isometric isomorphism

$$D^{cont}(G,K) \xrightarrow{\sim} \mathcal{O}^b(\mathfrak{X}_0/K)$$

(cf. the proof of [BSX15] Lemma 1.2.1). For $K = \mathbb{C}_p$, we obtain the commutative diagram

$$\begin{tikzcd} D^{cont}(G,\mathbb{C}_p) \arrow[r] \arrow[d, "\cong"', "\mathscr{F}"] & D(G_0,\mathbb{C}_p) \arrow[r, two heads] \arrow[d, "\cong"', "\mathscr{F}"] & D(G,\mathbb{C}_p) \arrow[d, "\cong"', "\mathscr{F}"] & \\ \mathcal{O}^b(\mathfrak{X}_0/\mathbb{C}_p) \arrow[r, "\subseteq"] & \mathcal{O}(\mathfrak{X}_0/\mathbb{C}_p) \arrow[r, two heads] & \mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \arrow[r, "\kappa", "\cong"'] & \mathcal{O}(\mathfrak{B}/\mathbb{C}_p). \end{tikzcd} \tag{2.H}$$

In particular, since the map obtained by composing all the maps in the upper row is the injective map $D^{cont}(G,K) \longrightarrow D(G,K)$, the map $\mathcal{O}^b(\mathfrak{X}_0,\mathbb{C}_p) \longrightarrow \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$ obtained by composing all the maps in the lower row is also injective.

Question 2.13. Is there a nice description of the image of the map $\mathcal{O}^b(\mathfrak{X}_0, \mathbb{C}_p) \longrightarrow \mathcal{O}(\mathfrak{B}/\mathbb{C}_p)$? Does the image contain $\mathcal{O}^b(\mathfrak{B}/\mathbb{C}_p)$? If so, can the norm of the preimage of a given $f \in \mathcal{O}^b(\mathfrak{B}/\mathbb{C}_p)$ be estimated by the norm of f?

At the moment, we do not know the answers to these questions. Suppose that the answers are positive. If $\psi \in \mathcal{O}^b(\mathfrak{B}/\mathbb{C}_p)$, then $\mu = \{\psi, \cdot\}$ extends to a measure. Moreover, the norm of ψ (i.e. the supremum norm of its coefficients, cf. Proposition 2.14 below) estimates the norm of μ in $D^{cont}(G, \mathbb{C}_p)$, so that we can apply Lemma 2.11 to μ. Hence Question 2.13 is an interesting problem for the future.

We end this chapter by proving the following result which, in particular, says that (2.H) complies with (2.G) in the case $L = \mathbb{Q}_p$:

Proposition 2.14. *We have an equality*

$$\mathcal{O}^b(\mathfrak{B}/K) = K[[T]]_{bc}$$

of rings. Moreover, the identity map is an isometry:

$$\sup_{z \in \mathfrak{B}(\mathbb{C}_p)} |f(z)| = \sup_{n \in \mathbb{N}} |a_n|$$

holds for all $f = \sum_{n=0}^{\infty} a_n T^n \in K[[T]]_{bc} = \mathcal{O}^b(\mathfrak{B}/K)$.

Proof. For any $f = \sum_{n=0}^{\infty} a_n T^n \in K[[T]]_{bc}$, the inequality

$$\sup_{z \in \mathfrak{B}(\mathbb{C}_p)} |f(z)| \leq \sup_{n \in \mathbb{N}} |a_n| \tag{2.I}$$

is clear and implies $f \in \mathcal{O}^b(\mathfrak{B}/K)$. Conversely, let $f = \sum_{n=0}^{\infty} a_n T^n \in \mathcal{O}^b(\mathfrak{B}/K)$. If $f = 0$, then the conclusion is trivial, so we may assume that f is non-zero. The radius of convergence of f is the extended real number $0 \leq r_f \leq \infty$ defined by

$$r_f = \sup\{r \geq 0 \colon |a_n| r^n \to 0\}.$$

The fact that f converges on the open unit disk implies $r_f \geq 1$.
Let us make a brief digression to introduce the so-called growth modulus of f (cf. [Rob00] 6.1.4), a notion that we will need to complete our proof. The growth modulus of a power series $g = \sum_{n=0}^{\infty} b_n T^n \in K[[T]]$ with radius of convergence $r_g > 0$ is defined by

$$M_r(g) = \max_{n \geq 0} |b_n| r^n \quad (0 \leq r < r_g)$$

so that $r \longmapsto M_r(g)$ is a positive increasing real-valued function on $[0, r_g)$. We say that $r \in [0, r_g)$ is a regular radius for g if the equality $M_r(g) = |b_n| r^n$ holds for one index $n = n(r)$ only. We have the inequality

$$|g(z)| \leq M_r(g) \quad \text{if } |z| = r < r_g$$

which is an equality

$$|g(z)| = M_r(g)$$

for all regular radii r. The "Classical Lemma" in [Rob00] 6.1.4 says that the set $\mathrm{Reg}(g)$ of regular radii of a non-zero g is dense in $[0, r_g)$. More precisely, it says that any non-zero g has only finitely many non-regular radii smaller than any given value $r < r_g$. This also implies that $\mathrm{Reg}(g)$ is open in $[0, r_g)$. As the value group $\left|\mathbb{C}_p^\times\right|$ is dense in $\mathbb{R}_{>0}$, we may conclude that the intersection $\mathrm{Reg}(g) \cap \left|\mathbb{C}_p^\times\right|$ is dense in $[0, r_g)$. Indeed, the density and openness of $\mathrm{Reg}(g)$ imply that $V \cap \mathrm{Reg}(g)$ is non-empty and open for any non-empty open $V \subseteq [0, r_g)$. The density of $\left|\mathbb{C}_p^\times\right|$ then implies that $V \cap \mathrm{Reg}(g) \cap \left|\mathbb{C}_p^\times\right|$ is non-empty, completing the argument that $\mathrm{Reg}(g) \cap \left|\mathbb{C}_p^\times\right|$ is dense in $[0, r_g)$.
We now continue our proof where we left off. Since $f \in \mathcal{O}^b(\mathfrak{B}/K)$, we have that

$$C := \sup_{z \in \mathfrak{B}(\mathbb{C}_p)} |f(z)| < \infty.$$

We claim that $|a_n| \leq C$ holds for all $n \in \mathbb{N}$. To show this, let $m \in \mathbb{N}$ be arbitrary. Let $\varepsilon > 0$. By continuity of the function $y \longmapsto y^m$, there exists a $\delta > 0$ such that

$$1 - y^m < \varepsilon$$

holds for all $y \in (0,1)$ with $1 - y < \delta$. Since $\mathrm{Reg}(f) \cap \left|\mathbb{C}_p^\times\right|$ is dense in $[0,1) \subseteq [0, r_f)$, we can choose a $z_0 \in \mathfrak{B}(\mathbb{C}_p)$ such that $R := |z_0| \in (0,1)$ is a regular radius for f satisfying $1 - R < \delta$. Then

$$|a_m| - |a_m| R^m < |a_m| \varepsilon. \tag{2.J}$$

On the other hand, we also have

$$|a_m| R^m \leq M_R(f) = |f(z_0)| \leq C.$$

Plugging this into (2.J) implies that

$$|a_m| < C + |a_m| \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$|a_m| \leq C,$$

which completes the proof.

□

Remark 2.15. For $r < 1$, let $\mathfrak{B}(r)$ denote the rigid open disk of radius r, so that we have $\mathfrak{B}(r)(K) = \{z \in K\colon\ |z| < r\}$. The inequality

$$\sup_{z \in \mathfrak{B}(r)(\mathbb{C}_p)} |f(z)| \leq \sup_{n \in \mathbb{N}} |a_n| r^n \leq \sup_{n \in \mathbb{N}} |a_n|$$

holds for any $f = \sum_{n=0}^{\infty} a_n T^n \in K[[T]]_{bc}$ and implies the inclusion $K[[T]]_{bc} \subseteq \mathcal{O}^b(\mathfrak{B}(r)/K)$. However, the reverse inclusion does not hold in general. Indeed, for $r_0 = p^{-1/(p-1)}$, the power series $\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!}$ defines an isometric isomorphism

$$\exp\colon \mathfrak{B}(r_0)(\mathbb{C}_p) \xrightarrow{\sim} 1 + \mathfrak{B}(r_0)(\mathbb{C}_p)$$

of groups (cf. [Rob00] 5.4.2, Corollary after Proposition 3). In particular, its image is bounded even though its coefficients aren't. But, it is easy to see that our proof may be adapted in an obvious way to show the following generalization of Proposition 2.14:

$$\mathcal{O}^b(\mathfrak{B}(r)/K) = K[[T]]_{bc,r}$$

where $K[[T]]_{bc,r}$ is the ring of power series $f = \sum_{n=0}^{\infty} a_n T^n$ for which $\{|a_n|\, r^n\}$ is bounded. To verify this in the case of exp, observe that we have $r_0 = \left|p^{1/(p-1)}\right|$ for any chosen $(p-1)$-th root $p^{1/(p-1)} \in \mathbb{C}_p$ of p. The values $|a_n|\, r_0^n = \left|p^{n/(p-1)}/n!\right|$ are then indeed bounded (from above), since their additive valuations are bounded from below:

$$v_p\left(\frac{p^{n/(p-1)}}{n!}\right) = v_p(p^{n/(p-1)}) - v_p(n!) = \frac{n}{p-1} - \frac{n - S_p(n)}{p-1} \geq 0.$$

Here $S_p(n)$ is the sum of the coefficients appearing in the p-adic expansion of n.

Chapter 3

Uniformization with relative Lubin-Tate groups

We start this chapter by introducing the relative Lubin-Tate groups of [dS85]. These are certain formal groups that generalize Lubin-Tate groups. They are "relative" to a finite unramified extension of L.
We therefore fix an $m \in \mathbb{N}$ and let $E \subseteq \mathbb{C}_p$ be the unique unramified extension of L of degree m. We also fix a prime element $\varpi \in o_E$, so that $\mathfrak{m}_E = \varpi o_E$ and $k_E = o_E/\varpi$. Of course, we have $\varpi = u\pi$ for some $u \in o_E^\times$. Since the restriction map induces an isomorphism

$$\mathrm{Gal}(E/L) \xrightarrow{\sim} \mathrm{Gal}(k_E/k),$$

there is a unique $\varphi \in \mathrm{Gal}(E/L)$ (called the Frobenius automorphism of E/L) such that $\varphi(a) \equiv a^q \mod \varpi$ for all $a \in o_E$. The group $\mathrm{Gal}(E/L)$ is cyclic and φ is a generator.

3.1 Relative Lubin-Tate group laws

Generalizing the starting point of Lubin-Tate theory, we first define a family of relative Frobenius power series in $o_E[[X]]$ for ϖ. Any element ϕ of this family is going to play the role of a homomorphism lifting $X \longmapsto X^q$. But ϕ will only be an "endomorphism up to a transformation by φ". To make this precise, we introduce the following notation.

For any $\nu \in \mathbb{Z}$ and any formal power series $F(X_1, \ldots, X_n) = \sum_{i_1,\ldots i_n \geq 0} c_{i_1,\ldots i_n} X_1^{i_1} \cdots X_n^{i_n}$ in $o_E[[X_1, \ldots, X_n]]$ we define the formal power series

$${}^{\varphi^\nu}F(X_1, \ldots, X_n) = \sum_{i_1,\ldots i_n \geq 0} \varphi^\nu(c_{i_1,\ldots i_n}) X_1^{i_1} \cdots X_n^{i_n}.$$

Let $F, G \in o_E[[X_1, \ldots, X_n]]$. Let $H_1, \ldots, H_n$ be power series over o_E in any fixed number of variables. Assume that the constant terms of the H_i are all zero, so that $F(H_1, \ldots, H_n)$ is defined. The following facts are easily verified:

- ${}^{\varphi^\nu}(F+G) = {}^{\varphi^\nu}F + {}^{\varphi^\nu}G$.
- ${}^{\varphi^\nu}(F(H_1,\ldots,H_n)) = {}^{\varphi^\nu}F({}^{\varphi^\nu}H_1,\ldots,{}^{\varphi^\nu}H_n)$.
- If F is a formal group law, then ${}^{\varphi^\nu}F$ is also a formal group law.

Definition 3.1. A relative Frobenius power series for ϖ is a formal power series $\phi \in o_E[[X]]$ such that

(i) $\phi(X) = \varpi X +$ terms of degree ≥ 2,

(ii) $\phi(X) \equiv X^q \mod \varpi o_E[[X]]$.

Theorem 3.2. *For any relative Frobenius power series $\phi(X)$ there is a unique formal group law $F_\phi(X,Y)$ over o_E such that $\phi\colon F_\phi \longrightarrow {}^{\varphi}(F_\phi)$ is a homomorphism of formal group laws.*

Proof. See [dS85] Theorem 1 for the statement, proofs are provided in Chapter I of [dS87]. □

F_ϕ is called the relative (to the extension E/L) Lubin-Tate formal group law of the Frobenius power series ϕ.
If $E = L$, we are in the situation of Section 1.3.1 and F_ϕ is a Lubin-Tate group. In this case, $\phi \in \mathrm{End}_o(F_\phi)$. Moreover, as we have seen in Section 1.3.1, if ϕ and ψ are Frobenius power series for the same prime element π (i.e. $\phi,\psi \in \mathcal{F}_\pi$), then there is an o-isomorphism $F_\psi \xrightarrow{\sim} F_\phi$. In this sense, F_ϕ depends only on π.
For a general E/L and a relative Frobenius power series ϕ for ϖ, the relative Lubin-Tate law F_ϕ depends only on $\mathrm{Norm}_{E/L}(\varpi)$, as we will see in Corollary 3.5 (ii) and (iii). It is therefore convenient to introduce the set

$$\mathcal{F}_\xi^{rel} = \{f \in o_E[[X]] \colon f = cX + \text{higher terms}, \mathrm{Norm}_{E/L}(c) = \xi \text{ and } f \equiv X^q \mod \varpi o_E[[X]]\}$$

where ξ is a fixed element of o such that $v_p(\xi) = \frac{m}{e}$. As $\mathrm{Norm}_{E/L}(c)$ is the product over all the Galois-conjugates of c and each Galois-conjugate has valuation equal to $v_p(c)$, we conclude $v_p(c) = \frac{1}{e}$. This means that c is a prime element of o_E. Thus, we obtain the following alternative description of $\mathcal{F}_\xi^{rel}$:

$$\mathcal{F}_\xi^{rel} = \bigcup_{\mathrm{Norm}_{E/L}(\varpi')=\xi} \{\text{all relative Frobenius power series for } \varpi'\}.$$

Example 3.3. $\phi(X) = \pi X + X^q$ is a relative Frobenius power series for π and $\phi \in \mathcal{F}_\xi^{rel}$ for $\xi = \pi^m$.

Theorem 3.4. *Let ϕ and ψ be relative Frobenius power series for the prime elements ϖ and ϖ', respectively. Let $o_E^{\varpi,\varpi'}$ denote the additive subgroup $\{a \in o_E \colon \varpi a = \varpi'\varphi(a)\}$. There is a unique map*

$$[.]_{\phi,\psi} \colon o_E^{\varpi,\varpi'} \longrightarrow (X) \subseteq o_E[[X]]$$

such that

$$[a]_{\phi,\psi}(X) = aX + \text{higher terms} \qquad \text{and} \qquad \phi \circ [a]_{\phi,\psi} = {}^{\varphi}[a]_{\phi,\psi} \circ \psi.$$

Moreover, any $[a]_{\phi,\psi}$ is necessarily a group homomorphism $F_\psi \longrightarrow F_\phi$ and $[.]_{\phi,\psi}$ is actually an isomorphism

$$[.]_{\phi,\psi} \colon o_E^{\varpi,\varpi'} \xrightarrow{\sim} \mathrm{Hom}_{o_E}(F_\psi, F_\phi)$$

of additive groups. We also have

$$[ab]_{\tau,\psi} = [a]_{\tau,\phi} \circ [b]_{\phi,\psi} \tag{3.A}$$

for any relative Frobenius power series τ for ϖ'', $a \in o_E^{\varpi,\varpi'}$ and $b \in o_E^{\varpi',\varpi''}$.

Proof. See [dS85] Theorem 2. Compare [dS87] Chapter I Proposition 1.5. Note that we have $o_E^{\varpi,\varpi'} \cdot o_E^{\varpi',\varpi''} \subseteq o_E^{\varpi,\varpi''}$, so that the left-hand side of (3.A) is well-defined. □

Corollary 3.5. (i) *The map $[.]_\phi := [.]_{\phi,\phi} \colon o \longrightarrow \mathrm{End}_{o_E}(F_\phi)$ is a ring isomorphism. In particular, $(F_\phi, [.]_\phi)$ is a formal o-module.*

(ii) *If $u \in o_E^{\varpi,\varpi'} \cap o_E^\times$, then $[u]_{\phi,\psi} \colon F_\psi \xrightarrow{\sim} F_\phi$ is an isomorphism of formal o-modules with inverse $[u^{-1}]_{\psi,\phi}$.*

(iii) *The set $o_E^{\varpi,\varpi'} \cap o_E^\times$ is non-empty if and only if $\mathrm{Norm}_{E/L}(\varpi) = \mathrm{Norm}_{E/L}(\varpi')$.*

Proof. The assertions (i) and (ii) follow immediately from the theorem. For (iii), note that the set $o_E^{\varpi,\varpi'} \cap o_E^\times$ is non-empty if and only if there exists a $u \in o_E^\times$ satisfying $\varpi/\varpi' = \varphi(u)/u$. By Hilbert's theorem 90, such a u exists precisely when $\mathrm{Norm}_{E/L}(\varpi/\varpi') = 1$, i.e. when $\mathrm{Norm}_{E/L}(\varpi) = \mathrm{Norm}_{E/L}(\varpi')$. □

Fix a relative Frobenius power series ϕ for ϖ and let $\mathfrak{G} = F_\phi$ and $[.] = [.]_\phi$. The height of $\mathfrak{G}$ is equal to d (cf. the discussion following the proof of Theorem 1.3 in Chapter I of [dS87]). Another important fact is that the p-adic Tate module $T_p\mathfrak{G}$ is free of rank d over $\mathbb{Z}_p$. This can be proven using the o-modules

$$\begin{aligned} \mathcal{W}_n^{rel} &:= \{z \in \mathfrak{B}(\mathbb{C}_p) \colon [a](z) = 0 \text{ for all } a \in \pi^n o\} \\ &= \{z \in \mathfrak{B}(\mathbb{C}_p) \colon [\pi^n](z) = 0\}. \end{aligned}$$

If $z \in \mathcal{W}_n^{rel} \setminus \mathcal{W}_{n-1}^{rel}$, then $a \longmapsto [a](z)$ gives an isomorphism $o/\pi^n \xrightarrow{\sim} \mathcal{W}_n^{rel}$ (cf.[dS85] Proposition 1). Therefore, the same argument as in the proof of Lemma 1.34 can be used to show that $T_p\mathfrak{G}$ is free of rank d over $\mathbb{Z}_p$.
We remark that the elements of $\mathcal{W}_n^{rel}$ are algebraic over L, see [dS87] Chapter I, Proposition 1.8.

3.2 p-divisible groups

The purpose of this section is to prepare the proof of Theorem 3.20 (uniformization on points in the relative Lubin-Tate case). We briefly review definitions and facts from the theory of p-divisible groups (or Barsotti-Tate groups). Then we apply these results to relative Lubin-Tate group laws. This is possible because there is an equivalence of categories between the category of "divisible" formal group laws over o and the category of connected p-divisible groups over o.
For the applications to relative Lubin-Tate groups, we keep the setting and the notations of the previous section.
All the general rings appearing in this section (R, Λ etc.) are assumed to be commutative.

3.2.1 General theory

Definition 3.6. Let R be a ring and h a non-negative integer. A p-divisible group G of height h over R is a sequence of commutative affine group schemes $G_\nu = \mathrm{Spec}(A_\nu), \nu \geq 0$, together with morphisms of group schemes $i_\nu \colon G_\nu \longrightarrow G_{\nu+1}$ such that

(i) G_ν is finite flat of order $p^{\nu h}$ over R (i.e. the R-algebra A_ν is a finitely generated module over R and the localization $(A_\nu)_\mathfrak{p}$ is free of rank $p^{\nu h}$ over $R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A_\nu)$), and

(ii) the sequence $0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{p^\nu} G_{\nu+1}$ is exact (in the sense of [Tat67] (1.3)), where p^ν is the homomorphism "adding an element to itself p^ν times".

As a consequence of (ii), the multiplication map $p\colon G_{\nu+1} \longrightarrow G_{\nu+1}$ lands in the kernel of $p^\nu \colon G_{\nu+1} \longrightarrow G_{\nu+1}$, which is equal to $i_\nu(G_\nu)$. Thus there is a unique homomorphism $j_\nu \colon G_{\nu+1} \longrightarrow G_\nu$ such that $i_\nu \circ j_\nu = p$.

Definition 3.7. Let G be a p-divisible group of height h over an integral domain R of characteristic zero. Let $\mathcal{K}$ be the field of fractions of R and let $\mathcal{K}^{\mathrm{alg}}$ be an algebraic closure of $\mathcal{K}$. The Tate module of G is

$$T_p(G) := \varprojlim_\nu G_\nu(\mathcal{K}^{\mathrm{alg}})$$

where the limit is taken over the maps j_ν.

Lemma 3.8. *Keep the setting of Definition 3.7. The Tate module is a free $\mathbb{Z}_p$-module of rank h.*

Proof. Since the abelian group $G_\nu(\mathcal{K}^{\mathrm{alg}})$ is annihilated by p^ν, it is a $\mathbb{Z}/p^\nu\mathbb{Z}$-module. Hence $\varprojlim_\nu G_\nu(\mathcal{K}^{\mathrm{alg}}) = T_p(G)$ is a module over $\varprojlim_\nu \mathbb{Z}/p^\nu\mathbb{Z} = \mathbb{Z}_p$. Tate mentions at the bottom of page 167 in [Tat67] that $T_p(G)$ is free of rank h. We remark that the proof of this relies on the following three facts: (i) every finite flat group scheme over a field of characteristic zero is finite étale; (ii) if $\mathcal{F}$ is an algebraically closed field of characteristic zero, then taking $\mathcal{F}$-points induces an equivalence between the category of finite étale group schemes over $\mathcal{F}$ and the category of finite abelian groups; (iii) given a tower of finite abelian groups $N_1 \subseteq N_2 \subseteq \ldots$ such that N_ν is the kernel of multiplication by p^ν in $N_{\nu+1}$ and the cardinality of N_ν is equal to $p^{\nu h}$, group theory shows that $N_\nu \cong (\mathbb{Z}/p^\nu\mathbb{Z})^h$.

□

Definition 3.9. Let R be a complete noetherian local ring. A finite flat group scheme $\mathrm{Spec}(B)$ over R is called connected if and only if B is a local ring, cf. [Tat67] (1.4). A p-divisible group G over R is called connected if and only if each G_ν is connected.

Now we describe the relation between formal groups and p-divisible groups, following [Tat67] (2.2). Let R be a complete noetherian local ring with residue field of characteristic p. Let $F(X,Y)$ be a (commutative) n-dimensional formal group over R, which is essentially the same as an n-dimensional formal Lie group as defined on page 162 of [Tat67]. This means that F is a family of n power series in $2n$ variables over R such that the axioms (i)-(iv) from Definition 1.21 are satisfied when we write $F(X,Y), X$ and Y as column vectors (cf. [Haz78] 9.1). Consider $[p]_F(X) := X +_F \ldots +_F X$ (p times). It is an n-tuple of power series in n variables: $[p]_F(X) = (H_1(X), \ldots, H_n(X))$. The formal group F is called "divisible" if the ring $R[[X]] = R[[X_1, \ldots, X_n]]$ is a finitely generated free module over the subring $R[[H_1, \ldots, H_n]]$. If this is the case, the rank of $R[[X_1, \ldots, X_n]]$ over $R[[H_1, \ldots, H_n]]$ is necessarily p^h for some $h \in \mathbb{N}$, as Tate explains in the discussion preceding Proposition 1 in [Tat67]. We call h the degree of the isogeny $[p]_F^*$. For a one-dimensional formal group over R, the property of being divisible is equivalent to the property of having finite height, as we will see in Proposition 3.13.

Remark 3.10. In Section 1.3.1, we defined the height of a one-dimensional formal group law. Some authors define height for higher-dimensional formal group laws as well, in such a way that having finite height basically corresponds to what we call "being divisible", cf. [Haz78] 18.3.8 and 18.3.9. See also [Haz78] Appendix B.2.

Suppose that F is divisible with degree of isogeny equal to h. For $\nu \in \mathbb{N}$, let J_ν be the ideal of $R[[X]] = R[[X_1, \ldots, X_n]]$ generated by the n power series of the n-tuple $[p^\nu]_F$. Then $R[[X]]/J_\nu$ is free of rank $p^{\nu h}$ over R and the comultiplication $R[[X]] \longrightarrow R[[X]]\widehat{\otimes}_R R[[X]]$ defined by $F(X,Y)$ induces a comultiplication $R[[X]]/J_\nu \longrightarrow R[[X]]/J_\nu \otimes_R R[[X]]/J_\nu$ which makes $F(p)_\nu := \mathrm{Spec}(R[[X]]/J_\nu)$ into a connected finite commutative group scheme over R. The $F(p)_\nu$ combine to define a connected p-divisible group $F(p)$.

Theorem 3.11. *Let R be a complete noetherian local ring whose residue field is of characteristic p. Then $F \longmapsto F(p)$ is an equivalence of categories between the category of divisible formal groups over R and the category of connected p-divisible groups over R.*

Proof. See [Tat67] Proposition 1. Compare [Haz78] Appendix B.2, for we have followed the approach taken there to describe $F \longmapsto F(p)$. We note that Tate's construction of $F(P)_\nu$ is actually the same as the one we described. He uses the ideals $\psi^\nu(I)R[[X]]$, where $I \subseteq R[[X]]$ is the ideal generated by the variables X_i and $\psi = [p]_F^* \colon R[[X]] \longrightarrow R[[X]], P \longmapsto P \circ [p]_F = P(H_1, \ldots, H_n)$. We clearly have $\psi^\nu(I)R[[X]] = J_\nu$. □

For the proof of Proposition 3.13, we will need the Weierstrass Preparation Theorem, which we now state.

Theorem 3.12 (Weierstrass Preparation). *Let Λ be a complete local ring and let $f = \sum_{i=0}^{\infty} c_i T^i \in \Lambda[[T]]$ have first unit coefficient in degree n (i.e. all the c_i with $i < n$ are non-units in Λ and $a_n \in \Lambda^\times$). Then there is a unique pair (U, g) such that $U \in \Lambda[[T]]^\times$, $g \in \Lambda[T]$ is a Weierstrass (or "distinguished") polynomial of degree n (i.e. g is monic and reduces to X^n modulo the maximal ideal of Λ) and*

$$f = Ug.$$

Proof. [Lan02] Chapter IV, Theorem 9.2. □

Proposition 3.13. *Let R be a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk$ of characteristic p. A one-dimensional formal group law F over R is of finite height h if and only if it is divisible with degree of isogeny equal to h.*

Proof. Suppose that F is of finite height h. This means that the first unit coefficient of $H := [p]_F \in R[[X]]$ appears in degree $r = p^h$. Here $R[[X]]$ denotes the ring of formal power series in one variable over R. Let $a_i \in R$ denote the i-th coefficient of H, so that $H = \sum_{i=0}^{\infty} a_i X^i$. We have to show that $R[[X]]$ is free of rank r over its subring $R[[H]]$. Note that the variable X is obviously not free over $R[[H]]$, as it satisfies the non-trivial relation $\sum_{i=0}^{\infty} a_i X^i - H = 0$. In a sense, this is the only non-trivial relation X satisfies over $R[[H]]$. To make this statement precise, let T be a free variable over $R[[H]]$ and consider the homomorphism of $R[[H]]$-algebras

$$\begin{aligned} R[[H]][[T]] \Big/ \left(\textstyle\sum_i a_i T^i - H\right) &\xrightarrow{\Theta} R[[X]] \\ T &\longmapsto X. \end{aligned} \tag{3.B}$$

This homomorphism is well-defined because $R[[X]]$ is complete for the (X)-adic topology, so the universal property of $R[[H]][[T]]$ says that we can define a homomorphism by plugging in any element of (X) for T. Moreover, Θ is clearly surjective. To see that it is injective, we define the homomorphism of R-algebras $\theta\colon R[[X]] \longrightarrow R[[H]][[T]]/(\sum_i a_i T^i - H), X \longmapsto T$. We obviously have $\theta \circ \Theta = \mathrm{id}$, which implies the injectivity of Θ. Now we want to apply the Weierstrass Preparation Theorem 3.12 with $\Lambda = R[[H]]$, $f = \sum_i a_i T^i - H$ and $n = r$. This is possible because $R[[H]]$, being isomorphic to $R[[X]]$ via the isomorphism of R-algebras $R[[X]] \xrightarrow{\sim} R[[H]]$ sending X to H, is a complete local ring[1] with maximal ideal $(\mathfrak{m}, H)$. Thus, there is a Weierstrass polynomial $g \in R[[H]][T]$ of degree r such that left-hand side of (3.B) is equal to

$$R[[H]][[T]]/(g).$$

[1] $R[[X]]$ itself is a complete local ring because it is a power series ring over a complete local ring (see [Lan02] Chapter IV, the discussion preceding Theorem 9.1).

Since $R[[H]]$ is complete and g is a Weierstrass polynomial, the natural map

$$R[[H]][T]/(g) \xrightarrow{\sim} R[[H]][[T]]/(g)$$

is an isomorphism of $R[[H]]$-algebras (cf. [Ell14] Lemma 3.5(2)). Hence $R[[H]][[T]]/(g)$ is free of rank r over $R[[H]]$, with basis $\{1, T, \ldots, T^{r-1}\}$. Together with the isomorphism (3.B), this proves that $R[[X]]$ is free of rank r over $R[[H]]$.
Conversely, suppose that F is divisible with degree of isogeny equal to h and let $H = [p]_F$. Then there is an $R[[H]]$-basis of $R[[X]]$ consisting of $s = p^h$ elements $P_1, \ldots, P_s \in R[[X]]$. Let $\widetilde{P}$ denote the power series obtained from $P \in R[[X]]$ by reducing the coefficients modulo $\mathfrak{m}$. Then $\{\widetilde{P}_1, \ldots, \widetilde{P}_s\}$ generates $\mathscr{k}[[X]]$ over $\mathscr{k}[[\widetilde{H}]]$. Since $\mathscr{k}[[X]]$ is infinite-dimensional as a vector space over $\mathscr{k}$, we have $\widetilde{H} \neq 0$. In particular, H is of finite height. Let h' denote the height of H and let $r = p^{h'}$. The first part of the proof then shows that $R[[X]]$ is free of rank r over $R[[H]]$. It follows that $r = s$ and hence $h' = h$. This completes the proof. $\square$

We now review some objects and concepts related to p-divisible groups. Many of these are introduced and studied in [Tat67] for arbitrary p-divisible groups over a complete discrete valuation ring R of characteristic zero whose residue field is of characteristic p and perfect. We restrict our exposition to the case of a connected p-divisible group over $R = o$.

Let G be a connected p-divisible group over o. Let F be the formal group corresponding to G and let n be the dimension of F.

- The group $G(o_K)$ of points of G with values in o_K is defined in [Tat67] (2.4). It can be identified (non-canonically) with $(\mathfrak{B}(K)^n, +_F)$, as is explained at the bottom of page 167 in [Tat67].
- The tangent space t_G of G is, by definition, the tangent space of F, see [Tat67] page 168. Its K-points $t_G(K)$ form an n-dimensional vector space over K.
- The logarithm map $\log_G \colon G(o_K) \longrightarrow t_G(K)$ is defined at the bottom of page 167 in [Tat67]. It is a group homomorphism. The kernel of $\log_G$ is the torsion subgroup $G(o_K)_{\mathrm{tors}}$ of $G(o_K)$ and its cokernel is a torsion group. If $K = \mathbb{C}_p$, then $\log_G \colon G(o_{\mathbb{C}_p}) \twoheadrightarrow t_G(\mathbb{C}_p)$ is surjective.
- The p-divisible dual G' of G is the p-divisible group

  $$G_0^* \xrightarrow{j_1^*} G_1^* \xrightarrow{j_2^*} \ldots$$

  where G_ν^* is the dual of G_ν in the sense of [Tat67] (1.2) and j_n^* is the dual of the map j_n. If $G_\nu = \mathrm{Spec}(A_\nu)$, then $G_\nu^* = \mathrm{Spec}(\mathrm{Hom}_{o\text{-modules}}(A_\nu, o))$. There is a canonical isomorphism $G_\nu \xrightarrow{\sim} (G_\nu^*)^*$. For any o-algebra S, we have a natural group isomorphism

  $$G_\nu^*(S) \xrightarrow{\sim} \mathrm{Hom}_S(G_\nu \otimes_o S, \mathbb{G}_{m,S}) \tag{3.C}$$

  onto the group of morphisms of group schemes over S from $G_\nu \otimes_o S := G_\nu \times_{\mathrm{Spec}(o)} \mathrm{Spec}(S)$ into $\mathbb{G}_{m,S} = \mathrm{Spec}(S[T, T^{-1}])$.

The following results are obtained from [Tat67] page 177 by taking $R = o, K = L, C = \mathbb{C}_p$ and $D = o_{\mathbb{C}_p}$. Since $G_\nu(D) = G_\nu(\mathbb{C}_p) = G_\nu(L^{\mathrm{alg}})$ holds by the theory of finite flat group schemes over an algebraically closed field of characteristic zero, we have $T_p(G') = \varprojlim_\nu G_\nu^*(D)$. Using (3.C) and passing to the projective limit, we see that there is a natural isomorphism

$$T_p(G') = \varprojlim_{\nu} G_\nu^*(D) \xrightarrow{\sim} \mathrm{Hom}_D(G \mathbin{\widehat{\otimes}_o} D, \mathbb{G}_{m,D}(p)) \tag{3.D}$$

where $\mathbb{G}_{m,D}(p)$ is the p-divisible group attached to $\mathbb{G}_{m,D}$ and Hom_D denotes morphisms of p-divisible groups over D. By functoriality, any map of p-divisible groups induces a map on points and tangent spaces. Consequently, (3.D) gives us pairings

$$\langle \cdot, \cdot \rangle \colon T_p(G') \times G(D) \longrightarrow \mathbb{G}_{m,D}(p)(D) \cong \mathfrak{B}_1(\mathbb{C}_p) \tag{3.E}$$

and

$$(\cdot, \cdot) \colon T_p(G') \times t_G(\mathbb{C}_p) \longrightarrow t_{\mathbb{G}_{m,D}(p)}(\mathbb{C}_p) \cong \mathbb{C}_p. \tag{3.F}$$

These pairings induce the homomorphisms of $\mathbb{Z}_p$-modules

$$\begin{aligned} \alpha \colon G(D) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mathfrak{B}_1(\mathbb{C}_p)) \\ z &\longmapsto \langle \cdot, z \rangle \end{aligned}$$

and

$$\begin{aligned} d\alpha \colon t_G(\mathbb{C}_p) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mathbb{C}_p) \\ x &\longmapsto (\cdot, x) \end{aligned}$$

that fit into the commutative diagramm

$$\begin{tikzcd} G(D) \arrow[r, two heads, "\log_G"] \arrow[d, "\alpha"'] & t_G(\mathbb{C}_p) \arrow[d, "d\alpha"] \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mathfrak{B}_1(\mathbb{C}_p)) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mathbb{C}_p) \end{tikzcd} \tag{3.G}$$

where the lower horizontal map is $\mathrm{Hom}(T_p(G'), \log)$ (i.e. the map obtained by applying the functor $\mathrm{Hom}(T_p(G'), \cdot)$ to the usual logarithm $\log \colon \mathfrak{B}_1(\mathbb{C}_p) \longrightarrow \mathbb{C}_p$).
We have the following:

Proposition 3.14. *α and $d\alpha$ are both injective.*

Proof. [Tat67] Proposition 11. □

Lemma 3.15. *The commutative diagram (3.G) is cartesian.*

Proof. Let us simplify the notation and write T' for $T_p(G')$, U for $\mathfrak{B}_1(\mathbb{C}_p)$, and ℓ for the map $\mathrm{Hom}(T', \log)$. Given a commutative diagram of the form

$$\begin{tikzcd} M \arrow[drr, bend left, "g"] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "f"'] & & \\ & G(D) \arrow[r, "\log_{\mathfrak{G}}"] \arrow[d, "\alpha"] & t_G(\mathbb{C}_p) \arrow[d, "d\alpha"] \\ & \mathrm{Hom}_{\mathbb{Z}_p}(T', U) \arrow[r, "\ell"] & \mathrm{Hom}_{\mathbb{Z}_p}(T', \mathbb{C}_p) \end{tikzcd} \tag{3.H}$$

it suffices to show that the image of f lies in the image of α. Indeed, suppose that we have shown this. Then we may use the fact that α is injective to define $\psi := \alpha^{-1} \circ f$. We claim that this ψ is the unique homomorphism making (3.H) commute. The only non-obvious part of this claim is the equality $\log_{\mathfrak{G}} \circ \psi = g$. But this equality follows from the equality $d\alpha \circ \log_{\mathfrak{G}} \circ \psi = \ell \circ \alpha \circ \psi = \ell \circ f = d\alpha \circ g$ and the injectivity of $d\alpha$. This completes the proof that (3.G) is cartesian, under the assumption that the image of f lies in the image of α.

To see that the image of f lies in the image of α, we note that the diagram (3.G) is part of the larger commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & G(D)_{\mathrm{tors}} \arrow[r] \arrow[d, "\alpha_0"] & G(D) \arrow[r, "\log_G"] \arrow[d, "\alpha"] & t_G(\mathbb{C}_p) \arrow[r] \arrow[d, "d\alpha"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T', U_{\mathrm{tors}}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T', U) \arrow[r, "\ell"] & \mathrm{Hom}_{\mathbb{Z}_p}(T', \mathbb{C}_p) \arrow[r] & 0 \end{tikzcd} \tag{3.I}$$

whose rows are exact (see $(*)$ on page 177 in [Tat67]). Here α_0 denotes the restriction of α to $G(D)_{\mathrm{tors}}$. By [Tat67] Proposition 11, α_0 is bijective. Therefore, the exactness of the bottom row implies that the kernel of ℓ lies in the image of α:

$$\ker(\ell) \subseteq \mathrm{im}(\alpha). \tag{3.J}$$

Let $m \in M$. Since $\log_G$ is surjective, we may choose a $z \in G(D)$ such that $\log_G(z) = g(m)$. Then $\ell(f(m)) = d\alpha(g(m)) = d\alpha(\log_G(z)) = \ell(\alpha(z))$. Hence $f(m)$ lies in $\alpha(z) + \ker(\ell)$, which by (3.J) is a subset of the image of α. Since $m \in M$ was arbitrary, this establishes our claim. $\square$

We need one more fact about the dual Tate module $T_p(G')$. We have the homomorphism

$$G_\nu^*(\mathbb{C}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}_p}(G_\nu \otimes_o \mathbb{C}_p, \mathbb{G}_{m,\mathbb{C}_p}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(G_\nu(\mathbb{C}_p), \mu_{p^\nu}) \tag{3.K}$$

where the first arrow is the one from (3.C), the second arrow is the one sending a map to the induced map on points[2], and μ_{p^ν} denotes the group of p^ν-th roots of unity. The homomorphism (3.K) gives a duality pairing

$$G_\nu^*(\mathbb{C}_p) \times G_\nu(\mathbb{C}_p) \longrightarrow \mu_{p^\nu}.$$

[2]Note that $(G_\nu \otimes_o \mathbb{C}_p)(\mathbb{C}_p) = G_\nu(\mathbb{C}_p)$ since $\mathrm{Hom}_{\mathbb{C}_p\text{-algebras}}(A_\nu \otimes_o \mathbb{C}_p, \mathbb{C}_p) = \mathrm{Hom}_{o\text{-algebras}}(A_\nu, \mathbb{C}_p)$.

As explained in Step 1 of the proof of Proposition 11 in [Tat67], this pairing is perfect for each ν. This means that the homomorphism $G_\nu^*(\mathbb{C}_p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(G_\nu(\mathbb{C}_p), \mu_{p^\nu})$ from (3.K) is in fact an isomorphism. By passing to the projective limit, we obtain a natural isomorphism

$$T_p(G') \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), H) \tag{3.L}$$

where $H = T_p(\mathbb{G}_{m,o}(p)) = \varprojlim_\nu \mu_{p^\nu}$.

3.2.2 Applications to relative Lubin-Tate groups

Let Γ be a one-dimensional formal group law over o of finite height h. Consider the $\mathbb{Z}_p$-module

$$\mathrm{Hom}_{o_{\mathbb{C}_p}}(\Gamma, \hat{\mathbb{G}}_m).$$

It can be identified with

$$\mathscr{H}(\Gamma) := \{f \in o_{\mathbb{C}_p}[[Z]] \colon f(X +_\Gamma Y) = f(X)f(Y) \text{ and } f(0) = 1\},$$

cf. Section 1.3.2. As Katz mentions on page 58 of [Kat77], $\mathrm{Hom}_{o_{\mathbb{C}_p}}(\Gamma, \hat{\mathbb{G}}_m)$ is free of rank h over $\mathbb{Z}_p$. This assertion can be verified as follows. Proposition 3.13 ensures that we can apply Theorem 3.11 to Γ to obtain a corresponding p-divisible group G of height h over o. The p-divisible dual G' of G is also of height h, cf. [Tat67](2.3). By Lemma 3.8, $T' = T_p(G')$ is free of rank h over $\mathbb{Z}_p$. On the other hand, the natural isomorphism $T' \longrightarrow \mathrm{Hom}_D(G \widehat{\otimes}_o D, \mathbb{G}_{m,D}(p))$ from (3.D), combined with the isomorphism $\mathrm{Hom}_D(G \widehat{\otimes}_o D, \mathbb{G}_{m,D}(p)) \cong \mathrm{Hom}_{o_{\mathbb{C}_p}}(\Gamma, \hat{\mathbb{G}}_m)$ obtained by the equivalence of categories from Theorem 3.11, gives an isomorphism $T' \longrightarrow \mathrm{Hom}_{o_{\mathbb{C}_p}}(\Gamma, \hat{\mathbb{G}}_m)$. This proves the assertion.
Another important result is that the natural map $\mathrm{Hom}_{o_{\mathbb{C}_p}}(\Gamma, \hat{\mathbb{G}}_m) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p\Gamma, T_p\hat{\mathbb{G}}_m)$ is an isomorphism. We have already used this for $\Gamma = \mathcal{G}$ in the discussion preceding Corollary 1.36. We are now in a position to see that this fact is immediately obtained by applying the equivalence of categories to the isomorphism $T' = \mathrm{Hom}_D(G \widehat{\otimes}_o D, \mathbb{G}_{m,D}(p)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), H)$ from (3.L).

Consider now a relative Lubin-Tate law $\mathfrak{G}$ for $\varpi \in o_E$. As established at the end of Section 3.1, $\mathfrak{G}$ is a formal group over o_E of height d. In Subsection 3.2.1, we have worked with the valuation ring $o = o_L$ of an arbitrary finite extension $L/\mathbb{Q}_p$ as the base ring. Hence, if we let E play the role of L, we can apply the above observations to $\mathfrak{G}$. In particular,

$$\mathcal{T}' := \mathrm{Hom}_{o_{\mathbb{C}_p}}(\mathfrak{G}, \hat{\mathbb{G}}_m)$$

is free of rank d over $\mathbb{Z}_p$. Recall that $\mathfrak{G}$ is a formal o-module via $[.]\colon o \longrightarrow \mathrm{End}(\mathfrak{G}) \subseteq Z o_E[[Z]]$. Furthermore, we again write $F'_{t'}$ for $t' \in \mathcal{T}'$ when we want to emphasize that we are regarding t' as a formal power series.

Lemma 3.16. *The $\mathbb{Z}_p$-action on $\mathcal{T}'$ extends via $o \times \mathcal{T}' \longrightarrow \mathcal{T}', (a, t') \longmapsto F_{t'} \circ [a]$ to an o-action on $\mathcal{T}'$, making $\mathcal{T}'$ a free o-module of rank one.*

Proof. The proof of Lemma 1.33 and the proof of Corollary 1.36 apply verbatim here. □

We also write $a \odot t' := F_{t'} \circ [a]$. In the current setting, the pairings (3.E) and (3.F) correspond to

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathcal{T}' \otimes_o \mathfrak{B}(\mathbb{C}_p) &\longrightarrow \mathfrak{B}_1(\mathbb{C}_p) \\ t' \otimes z &\longmapsto 1 + F_{t'}(z) \end{aligned} \qquad \text{and} \qquad \begin{aligned} (\cdot, \cdot) \colon \mathcal{T}' \otimes_o \mathbb{C}_p &\longrightarrow \mathbb{C}_p \\ t' \otimes x &\longmapsto \Omega_{t'} x, \end{aligned}$$

where $\Omega_{t'} = \frac{d}{dZ} F_{t'}(0)$ is the leading coefficient of $F_{t'}$.

Proposition 3.17. *The pairing $\langle \cdot, \cdot \rangle$ is $\mathbb{Z}$-bilinear and o-invariant, and the pairing $(\cdot, \cdot)$ is o-bilinear.*

Proof. Both pairings are $\mathbb{Z}$-bilinear by design. The o-invariance of $\langle \cdot, \cdot \rangle$ may be proven exactly as in (1.Y). Finally, (1.X) from the proof of Lemma 1.38 also applies here to show that $(\cdot, \cdot)$ is o-bilinear. Indeed, the computation in (1.X) uses the properties $[a](0) = 0$ and $[a]'(0) = a$ for all $a \in o$, which are also true for the current $[.]$ by Theorem 3.4. □

The following lemma is trivial to prove.

Lemma 3.18. *Let M be an o-module. Then, for any $\mathbb{Z}_p$-module N, the set $\mathrm{Hom}_{\mathbb{Z}_p}(M, N)$ is an o-module under pointwise addition and scalar multiplication defined by $(c, \psi) \longmapsto c\psi$, where $(c\psi)(m) = \psi(cm)$ for $c \in o, \psi \in \mathrm{Hom}_{\mathbb{Z}_p}(M, N)$ and $m \in M$. The map*

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_p}(M, N) \otimes_o M &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(o, N) \\ \psi \otimes m &\longmapsto [a \longmapsto \psi(am)] \end{aligned} \tag{3.M}$$

is a well-defined homomorphism of o-modules. If M is free of rank one over o, then (3.M) is an isomorphism.

Proof. We only mention that, if $M \cong o$, then the last assertion follows immediately from the standard isomorphism $P \otimes_o o \xrightarrow{\sim} P, p \otimes c \longmapsto cp$ which holds for any o-module P. □

We tensor the diagram (3.G) by $\mathcal{T}'$ and apply Lemma 3.18 to the bottom row to obtain the commutative diagram

$$\begin{tikzcd} \mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}' \arrow[r, "\log_{\mathfrak{G}} \otimes \mathrm{id}"] \arrow[d, "\alpha"'] & \mathbb{C}_p \otimes_o \mathcal{T}' \arrow[d, "d\alpha"] \\ \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(\mathbb{C}_p)) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{C}_p) \end{tikzcd} \tag{3.N}$$

where the lower horizontal arrow is $\mathrm{Hom}(\mathcal{T}', \log)$ and $\log_{\mathfrak{G}}$ is the logarithm map of the formal group $\mathfrak{G}$. The left (resp. right) vertical arrow is actually $\alpha \otimes \mathrm{id}$ (resp. $d\alpha \otimes \mathrm{id}$)

composed with the isomorphism from Lemma 3.18. Abusing notation, we denote it again by α (resp. $d\alpha$). Thus, the image of an element $z \otimes t'$ under the map α in (3.N) is the map $g \longmapsto \langle g \odot t', z\rangle$, for $g \in o$. The same is true for $d\alpha$ if we replace $\langle \cdot, \cdot \rangle$ by $(\cdot, \cdot)$. The maps α and $d\alpha$ are injective by Proposition 3.14 and because $\mathcal{T}'$ is a free (and hence flat) o-module.

Lemma 3.19. *The commutative diagram (3.N) is cartesian.*

Proof. Since $\mathcal{T}'$ is flat, the rows of the commutative diagram (3.I) from the proof of Lemma 3.15 remain exact after tensoring with $\mathcal{T}'$ over o. Hence we may conclude with the same arguments as in the proof Lemma 3.15. □

This completes our preparations for Section 3.3.

3.3 Generalized uniformization $\mathfrak{X}(\mathbb{C}_p) \cong \mathfrak{B}(\mathbb{C}_p)$ on the level of points

In the previous sections, we have used the letter G to denote p-divisible groups in various situations. From now on, G denotes o viewed as a Lie group over L, just as in Chapters 1 and 2. Accordingly, G_0 denotes the Lie group over $\mathbb{Q}_p$ obtained from G by restriction of scalars. But otherwise we keep the setting and the notations of Sections 3.1 and 3.2.2. Thus, E/L is an unramified extension of degree m, $\mathfrak{G}$ is a relative Lubin-Tate group law for a prime element $\varpi \in o_E$ and $\mathcal{T}' = \mathrm{Hom}_{o_{\mathbb{C}_p}}(\mathfrak{G}, \hat{\mathbb{G}}_m)$. For fixed $t' \in \mathcal{T}'$ and $z \in \mathfrak{B}(\mathbb{C}_p)$, define the map

$$\begin{aligned} \kappa_{z\otimes t'} \colon o &\longrightarrow \mathfrak{B}_1(\mathbb{C}_p) \\ g &\longmapsto \langle t', [g](z)\rangle. \end{aligned}$$

Note that $\langle t', [g](z)\rangle = 1 + F_{t'}([g](z)) = \langle g \odot t', z\rangle = \alpha(z \otimes t')(g)$ for all $g \in o$, i.e.

$$\kappa_{z\otimes t'} = \alpha(z \otimes t'). \tag{3.O}$$

In particular, $\kappa_{z\otimes t'}$ is $\mathbb{Z}_p$-linear. Therefore, we have the group homomorphism

$$\begin{aligned} \kappa(\mathbb{C}_p) \colon \mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}' &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(\mathbb{C}_p)) \\ z \otimes t' &\longmapsto \kappa_{z\otimes t'}, \end{aligned}$$

where the o-module structure on $\mathfrak{B}(\mathbb{C}_p)$ is provided by $(+_{\mathfrak{G}}, [.])$. Note that $\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(\mathbb{C}_p))$ coincides with $\widehat{G_0}(\mathbb{C}_p)$ by Proposition 1.9. We arrive at the main theorem:

Theorem 3.20. *The map*

$$\begin{aligned} \kappa(\mathbb{C}_p) \colon \mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}' &\xrightarrow{\sim} \widehat{G}(\mathbb{C}_p) \\ z \otimes t' &\longmapsto \kappa_{z\otimes t'} \end{aligned}$$

is a well-defined isomorphism of groups.

Proof. We use the diagram

$$
\begin{tikzcd}
\mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}' \arrow[rr, "\log_{\mathfrak{G}} \otimes \mathrm{id}"] \arrow[dr, dashed, "\kappa"] \arrow[dd, "\alpha"] & & \mathbb{C}_p \otimes_o \mathcal{T}' \arrow[dr, "\underline{d\alpha}"] \arrow[dd, "d\alpha"] & \\
& \widehat{G}(\mathbb{C}_p) \arrow[rr, "\mathbb{d}"] \arrow[dd, "\subseteq"] & & \mathrm{Hom}_L(L, \mathbb{C}_p) \arrow[dd, "\subseteq"] \\
\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(\mathbb{C}_p)) \arrow[rr, "\mathrm{Hom}(o,\log)"] \arrow[dr, "\iota"] & & \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{C}_p) \arrow[dr, "j"] & \\
& \widehat{G_0}(\mathbb{C}_p) \arrow[rr, "\mathbb{d}"] & & \mathrm{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)
\end{tikzcd}
$$

to prove the assertion in a series of steps. First we discuss all the maps and argue that each face of the cube is commutative. To simplify notation, we will write κ in place of $\kappa(\mathbb{C}_p)$.

The front face is the cartesian diagram (1.L) from Section 1.2.

In the bottom face, the map ι is just the inclusion $\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(\mathbb{C}_p)) \xrightarrow{\subseteq} \widehat{G_0}(\mathbb{C}_p)$. A crucial fact is that ι is actually an isomorphism (cf. Proposition 1.9). To explain the map j, note that any $\mathbb{Z}_p$-basis of o is also a $\mathbb{Q}_p$-basis of L. Therefore, any element of $f \in \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{C}_p)$ has a unique $\mathbb{Q}_p$-linear extension $j(f) \in \mathrm{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)$. Clearly j is an isomorphism. The maps ι and j being inclusions, showing the commutativity of the bottom face of the cube amounts to showing that

$$\mathbb{d}\chi_{z\otimes\beta} = \log \circ \chi_{z\otimes\beta}$$

holds for all $z \in \mathfrak{B}_1(\mathbb{Z}_p)$ and $\beta \in \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{C}_p)$ (cf. Corollary 1.10). But this is clear since $\mathbb{d}\chi_{z\otimes\beta} = \log(z) \cdot \beta$ by Lemma 1.13 and $\log \circ \chi_{z\otimes\beta}(a) = \log(z^{\beta(a)}) = \beta(a)\log(z)$ for all $a \in o$.

The rear face of the cube is the cartesian diagram (3.N) from Lemma 3.19.

In the right face of the cube, the map $\underline{d\alpha}$ is defined by $\underline{d\alpha} := j \circ d\alpha$. We claim that the image of $\underline{d\alpha}$ lies in $\mathrm{Hom}_L(L, \mathbb{C}_p)$. This follows from the following two statements, taken together: (i) any map in the image of $d\alpha$ is o-linear; (ii) if $f \in \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{C}_p)$ is o-linear, then $j(f)$ is L-linear. To prove statement (i), it suffices to check that $d\alpha(x \otimes t')$ is o-linear for any $x \in \mathbb{C}_p, t' \in \mathcal{T}'$. But this is clear, since $d\alpha(x \otimes t')$ is the map $a \longmapsto (a \odot t', x)$ and $(a \odot t', x) = (t', ax) = a \cdot (t', x)$ holds for all $a \in o$ by Proposition 3.17. To prove statement (ii), observe that it is equivalent to the statement: (ii)' any map $f \in \mathrm{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)$ whose restriction to o is o-linear, is L-linear. Finally, statement (ii)' is clearly true because $L = o[p^{-1}]$.

Next we claim that $\underline{d\alpha}\colon \mathbb{C}_p \otimes_o \mathcal{T}' \longrightarrow \mathrm{Hom}_L(L, \mathbb{C}_p)$ is an isomorphism. By counting dimensions, we find that it suffices to show that $\underline{d\alpha}$ is injective. But $\underline{d\alpha} = j \circ d\alpha$ is injective, since j and $d\alpha$ are both injective.

We are now in a position to show that the image of κ lies in $\widehat{G}(\mathbb{C}_p)$. As each of the mentioned faces of the cube is commutative, the maps

$$\mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}' \xrightarrow{f_1 := \iota\circ\alpha} \widehat{G_0}(\mathbb{C}_p) \quad \text{and} \quad \mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}' \xrightarrow{f_2 := d\alpha\circ(\log_{\mathfrak{G}} \otimes \mathrm{id})} \mathrm{Hom}_L(L, \mathbb{C}_p)$$

induce a map $(f_1, f_2)\colon \mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}' \longrightarrow \widehat{G}(\mathbb{C}_p)$ by the universal property of the front face of the cube. This is the unique map making the top face and the left face (and thus the whole cube) commute. By the commutativity of the left face and (3.O), we conclude that $(f_1, f_2) = \kappa$. In particular, the image of κ lies in $\widehat{G}(\mathbb{C}_p)$, i.e. κ is well-defined.

To obtain a homomorphism inverse to κ, we recall that ι and $\underline{d\alpha}$ are isomorphisms, and that the rear face of the cube is cartesian. Denote the inclusion $\widehat{G}(\mathbb{C}_p) \xrightarrow{\subseteq} \widehat{G_0}(\mathbb{C}_p)$ by i. Using the commutativity of the cube, it is easy to see that the maps

$$\widehat{G}(\mathbb{C}_p) \xrightarrow{g_1 := \iota^{-1}\circ i} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathfrak{B}_1(\mathbb{C}_p)) \quad \text{and} \quad \widehat{G}(\mathbb{C}_p) \xrightarrow{g_2 := \underline{d\alpha}^{-1}\circ \mathrm{d}} \mathbb{C}_p \otimes_o \mathcal{T}'$$

induce a map $(g_1, g_2)\colon \widehat{G}(\mathbb{C}_p) \longrightarrow \mathfrak{B}(\mathbb{C}_p) \otimes_o \mathcal{T}'$ by the universal property of the rear face of the cube. Now it is straightforward to verify that (g_1, g_2) is inverse to κ. $\square$

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