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Masterarbeit

Euler-Poincaré formula and Tate duality for (φ, Γ)-modules over the Robba ring with an additional coefficient field

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Abstract

In this thesis we follow the work of Liu in [10], where Liu proves the Euler-Poincaré formula and Tate duality for (φ, Γ)-modules over the Robba ring. We pick up an idea of Liu in [10] to extend these results to (φ, Γ)-modules over the Robba ring with an additional coefficient field.

We first show that given an L-representation of the absolute Galois group of a local field, we can retrieve the Galois cohomology from the étale (φ, Γ)-module associated to the representation. We use this fact to show that there exists an analogue to the Euler-Poincaré formula and the local Tate duality for the cohomology of étale (φ, Γ)-modules over the Robba ring with an additional coefficient field.

The main result of this thesis is to extend the Euler-Poincaré formula for the category of étale (φ, Γ)-modules to the category of generalised (φ, Γ)-modules and the Tate duality for the category of étale (φ, Γ)-modules to the category of (φ, Γ)-modules.

Abstrakt

In dieser Arbeit folgen wir der Arbeit [10] von Liu, welche die Euler-Poincaré Formel und die lokale Tate Dualität für (φ, Γ)-Moduln über gewissen Robba Ringen beweißt. Wir greifen dabei eine Idee Lius aus [10] auf und verallgemeinern diese Resultate auf (φ, Γ)-Moduln über dem Robba Ring mit einem zusätzlichen Koeffizientenkörper.

Zunächst zeigen wir, dass wir aus einer L-Darstellung der abslouten Galoisgruppe eines lokalen Körpers und dem dazugehörigen étalen (φ, Γ)-Modul, die Galoiskohomologie des lokalen Körpers zurückgewinnen können. Daraus folgern wir die Existenz eines Analogons der Euler-Poincaré Formel und der lokalen Tate Dualität, welche man aus der Galoiskohomologie kennt, für die Kohomologie von étalen (φ, Γ)-Moduln über dem gewöhnlichen Robba Ring mit zusätzlichem Koeffizientenkörper.

Das Hauptresultat der Arbeit ist die Euler Poincaré Formel und die Tate Dualität für étale (φ, Γ)-Moduln auf die Kategorie der verallgemeinerten (φ, Γ)-Moduln, im Fall der Euler Poincaré Formel und die Kategorie der (φ, Γ)-Moduln, im Fall der Tate Dualität, zu verallgemeinern.

Contents

1 Introduction

In [10] Liu proved that the Euler-Poincaré characteristic formula and Tate's local duality from the theory of Galois cohomology over a local field can be generalised to larger categories than the categpory of p-adic representations, namely the category of (φ, Γ)-modules over the Robba ring $\mathcal{R}_K$ and the category of generalized (φ, Γ)-modules over the Robba ring $\mathcal{R}_K$. Liu statet in [10] that these results can be generalised to modules over the Robba ring $\mathcal{R}_K$ with additional coefficient field L, where L is a finite extension of $\mathbb{Q}_p$ and the φ and Γ-action act trivially on L. In this thesis we give a proof of this generalisation.

We first prove the Euler-Poincaré formula and Tate duality for the category of étale (φ, Γ)-modules by showing it is equivalent to the category of L-representations of $G_K = \mathrm{Gal}(\overline{K}/K)$. Then the results follow from the Euler-Poincaré formula and Tate duality for Galois cohomology. We will then introduce generalised (φ, Γ)-modules, which only need to be finitely presented and compute the cohomology for such torsion modules, which will show that the Euler-Poincaré formula holds for these torsion modules.

We can then prove the main results of this thesis, which we will state here:

Theorem (Euler-Poincaré formula). *For any generalised (φ, Γ)-module D over $\mathcal{R}^L_{\mathbb{Q}_p}$, we get that*
(i) $\dim_L H^i(D)$ *is finite for all* $i = 0, 1, 2$
(ii) $\chi(D) = \sum_{i=0}^{2} (-1)^i \dim_L H^i(D) = -\mathrm{rank} D.$

Theorem (Tate duality). *For any (φ, Γ)-module D over $\mathcal{R}^L_K$ the compositum*

$$H^i(D) \times H^{2-i}(D^\vee(\omega)) \to H^2((D \otimes D^\vee)(\omega)) \to H^2(\mathcal{R}^L_K(\omega)) \cong L$$

is a perfect pairing.

1.1 p-adic Hodge theory

Let p be a fixed natural prime number and K be a finite extension of $\mathbb{Q}_p$. Let k be the residue field of K and write $W(k)$ for the Witt vectors with coefficients in k. Set $K_0 = W(k)[1/p]$ to be the maximal unramified subfield of K. For $n \geq 1$ set $K_n = K(\mu_{p^n})$ for μ_{p^n} the group of p^n-th roots of unity and set $K_\infty = \cup_{n \geq 1} K_n$. We write $G_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of K. The cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ has the kernel $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$. So we get that the group $\Gamma_K = G_K/H_K$ is an open subgroup of $\mathbb{Z}_p^\times$. We will write simply Γ for Γ_K, if there is no confusion about which field is being considered.

Furthermore let $\mathbb{C}_p$ be the p-adic completion of $\overline{\mathbb{Q}_p}$ and set

$$\widetilde{E} = \varprojlim \mathbb{C}_p = \left\{ (x^{(i)})_{i\in\mathbb{N}} \;\middle|\; (x^{(i+1)})^p = x^{(i)} \right\}$$

where the connecting maps are given by taking p-th power.

The following properties of $\widetilde{E}$ can be found in [12, Section 4.1] and [3, Section 4.1.1]. One has a ring structure on $\widetilde{E}$ where addition and multiplication for $x, y \in \widetilde{E}$ are given by

$$(x+y)^{(i)} = \lim_{j\to\infty} (x^{(i+j)} + y^{(i+j)})^{p^j}$$

and

$$(xy)^{(i)} = x^{(i)} y^{(i)}$$

for any $i \in \mathbb{N}$. With this ring structure $\widetilde{E}$ becomes an algebraically closed field of characterstic p. Furthermore $\widetilde{E}$ is perfect, so the Frobenius map φ is an automorphism on $\widetilde{E}$. We can define a valuation $v_{\widetilde{E}}$ on $\widetilde{E}$ by setting $v_{\widetilde{E}}(x) = v_p(x^{(0)})$ for $x = (x^{(i)})_{i\in\mathbb{N}} \in \widetilde{E}$ and $\widetilde{E}$ is complete with respect to the topology induced by this valuation. We set $\widetilde{E}^+ = \{x \in \widetilde{E} \mid x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}\}$ to be the ring of integers of $\widetilde{E}$. Also note that we can endow $\widetilde{E}$ with a $G_{\mathbb{Q}_p}$-action, which preserves the valuation $v_{\widetilde{E}}$. This $G_{\mathbb{Q}_p}$-action is defined by $g((x^{(i)})_{i\in\mathbb{N}}) = (g(x^{(i)}))_{i\in\mathbb{N}}$ for any $g \in G_{\mathbb{Q}_p}$. We choose a sequence $(\varepsilon^{(n)})_{n\in\mathbb{N}}$ with $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ for all $n \in \mathbb{N}$. Then $\varepsilon = (\varepsilon^{(n)})_{n\in\mathbb{N}}$ can be viewed as an element of $\widetilde{E}^+$ as well as a generator of $\mathbb{Z}_p^\times$. We set $E_{K_0} = k((\varepsilon - 1))$ and take E to be the seperable closure of E_{K_0} in $\widetilde{E}$ and set $E_K = E^{H_K}$. Note that E_K also carries a discrete valuation and a Γ_K-action induced from $\widetilde{E}$. Let K_0' denote the maximal unramified extension of K_0 in K_∞ and k' be the residue field of K_0'. Then the discrete valutaion ring $E_K^+ = (\widetilde{E}^+ \cap E)^{H_K}$ is simply $k'[[\overline{\pi}_K]]$, where $\overline{\pi}_K$ is a uniformizer (see [12, Remarks 2.3.3 and Section 4.2]).

Now set $\widetilde{A} = W(\widetilde{E})$ to be the ring of Witt vectors with coefficients in $\widetilde{E}$. Since $\widetilde{E}$ is perfect, we can write elements of $\widetilde{A}$ as convergent p-adic series, which means we get

$$\widetilde{A}^+ = \left\{ \sum_{k=0}^{\infty} p^k [x_k] \mid x_k \in \widetilde{E} \right\}.$$

We can define a valuation v_p on $\widetilde{(A)}$, by setting $v_p(x) = \min_{k\in\mathbb{N}} \left\{ k \mid [x_k] \neq 0 \right\}$ for $x = \sum_{k=0}^\infty p^k [x_k]$ a nonzero element in $\widetilde{A}$. We will refer to the topology on $\widetilde{A}$ induced by this valuation v_p as the p-adic topology.

The Frobenius operation φ of $\widetilde{E}$ lifts to a Frobenius operation φ on $\widetilde{A}$ and we get that
$$\varphi(\sum_{k=0}^{\infty} p^k[x_k]) = \sum_{k=0}^{\infty} p^k[\varphi(x_k)].$$
The $G_{\mathbb{Q}_p}$-action lifts in the same way. We set $\pi = [\varepsilon] - 1 \in \widetilde{A}$ and $q = \varphi(\pi)/\pi \in \widetilde{A}$.

Next we define A_{K_0} to be the completion of $\mathcal{O}_{K_0}[[\pi]][\pi^{-1}]$ with respect to the p-adic topology in $\widetilde{A}$. We get that A_{K_0} is a Cohen ring with residue field E_{K_0}. Set $B_{K_0} = A_{K_0}[1/p]$ and define B as the p-adic completion of the maximal unramified extension of B_{K_0}. Furthermore we define $A = \widetilde{A} \cap B$. These rings inherit a Frobenius and $G_{\mathbb{Q}_p}$-action from the ring $\widetilde{B}$.

For any ring S that has a $G_{\mathbb{Q}_p}$-action we will set $S_K = S^{H_K}$. Note that this ring is then endowed with a Γ_K-action.

To define the Robba ring we need to introduce the ring of overconvergent elements, which for an $r \in \mathbb{R}_{\geq 0}$ is given by
$$\widetilde{B}^{\dagger,r} = \left\{ \sum_{k>-\infty}^{\infty} p^k[x_k] \in \widetilde{B} | \lim_{k\to\infty} v_{\widetilde{E}}(x_k) + kpr/(p-1) = \infty \right\}.$$

We set $\widetilde{B}^{\dagger} = \cup_{r\geq 0}\widetilde{B}^{\dagger,r}$, $B^{\dagger,r} = \widetilde{B}^{\dagger,r} \cap B$ and $B^{\dagger} = \cup_{r\geq 0}B^{\dagger,r}$. Furthermore we set
$$\widetilde{A}^{\dagger,r} = \left\{ \sum_{k>-\infty}^{\infty} p^k[x_k] \in \widetilde{B}^{\dagger,r} | v_{\widetilde{E}}(x_k) + kpr/(p-1) \geq 0 \text{ for any } k \right\} \cap \widetilde{A},$$

as well as $\widetilde{A}^{\dagger} = \cup_{r\geq 0}\widetilde{A}^{\dagger,r}$, $A^{\dagger,r} = \widetilde{A}^{\dagger,r} \cap A$ and $A^{\dagger} = \cup_{r\geq 0}A^{\dagger,r}$. We now choose an element $\pi_K \in A_K^{\dagger}$, which has image $\overline{\pi}_K$ modulo p. Also we let e_K denote the ramification index of K_∞/K_0. One can show for r large enough, that the ring $B_K^{\dagger,r}$ is defined by

$$B_K^{\dagger,r} = \Big\{ f(\pi_K) = \sum_{k=-\infty}^{\infty} a_k \pi_K^k | a_k \in K_0' \text{ and } f(T) \text{ is convergent and}$$

$$\text{bounded on } p^{-1/e_K r} \leq |T| < 1 \Big\}.$$

One can show that the element $t = \log([\varepsilon]) \in B_K^{\dagger,r}$. For $s_1, s_2 \in [p^{-1/e_k r}, 1)$ the supremum norms on closed annuli $\{x \mid s_1 \leq |x| \leq s_2 < 1\}$ form a family of norms on $B_K^{\dagger,r}$. The Fréchet completion with respect to this family of norms is

$$B_{\mathrm{rig},K}^{\dagger,r} = \Big\{ f(\pi_K) = \sum_{k=-\infty}^{\infty} a_k \pi_K^k | a_k \in K_0' \text{ and } f(T) \text{ is convergent}$$

$$\text{on } p^{-1/e_K r} \leq |T| < 1 \Big\}.$$

The union of these rings

$$B_{\mathrm{rig},K}^{\dagger} = \cup_{r \geq 0} B_{\mathrm{rig},K}^{\dagger,r}$$

is called the *Robba ring*. We will simply write $\mathcal{R}_K$ to denote this ring. We note that $B_K^{\dagger} = (B^{\dagger})^{H_K}$ are the bounded functions of $B_{\mathrm{rig},K}^{\dagger}$.

Another way to define the Robba ring is stated in [8, Section 1.1]. Let $s \in \mathbb{R}_{[0,1]}$ and $F/\mathbb{Q}_p$ a finite extension of fields and let $\mathcal{R}_{[s,1)}^F$ be the ring of Laurent series in T with coefficients in F converging on the annulus $s \leq |T| < 1$. Note that by [5, Prop 4.6] this ring is a Bézout domain (a Bézout ring is a ring where every finitely generated ideal is principal). Then by mapping π_K to T we can identify the ring $B_{\mathrm{rig},K}^{\dagger,r}$ with $\mathcal{R}_{[p^{1/(e_K r)},1)}^{K_0'}$, the ring of rigid analytic functions on the annulus $p^{-1/e_K r} \leq |T| < 1$. This allows us to identify the Robba ring with $\cup_{r\geq 0} \mathcal{R}_{[r,1)}^{K_0'}$, the set of holomorphic functions on the boundary of the open unit disc.

Note that if K is unramified, we can explicitly describe the φ and Γ_K-action for any series $f(\pi_K) = \sum_{i\in\mathbb{Z}} a_n \pi_K^n \in \mathcal{R}_K$ by

$$\varphi(f(\pi_K)) = \sum_{i\in\mathbb{Z}} \varphi(a_n)((1+\pi_K)^p - 1)^n$$

and

$$g(f(\pi_K)) = \sum_{i\in\mathbb{Z}} g(a_n)((1+\pi)^{\chi(g)} - 1)^n$$

for any $g \in \Gamma_K$.

In this thesis we will also discuss (φ, Γ)-modules over the Robba ring with an additional coefficient field. For this let $L/\mathbb{Q}_p$ be a fixed, finite extension. For any of the following rings $S \in \{B, B^{\dagger}, B^{\dagger,r}, B_K, B^{\dagger}_{\mathbb{Q}_p}, B^{\dagger,r}_K, B^{\dagger}_{\mathrm{rig},K}, B^{\dagger,r}_{\mathrm{rig},K}\}$ we define $S^L = S \otimes_{\mathbb{Q}_p} L$. We call $B^{\dagger,L}_{\mathrm{rig},K}$ the Robba ring with additional coefficient field L. Similarly for any $S \in \{A, A_K, A^{\dagger}_K\}$ we can define $S^L = (S \otimes_{\mathbb{Z}_p} \mathcal{O}_L)$. All these rings are endowed with a φ and Γ-action that acts trivially on L (resp. $\mathcal{O}_L$). We set $\mathcal{R}^L_K = B^{\dagger,L}_{\mathrm{rig},K}$.

For $K = \mathbb{Q}_p$, recall that we can identify $B^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}$ with $\mathcal{R}^{\mathbb{Q}_p}_{[p^{1/r},1)}$. We can then identify $B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}$ with $\mathcal{R}^L_{[p^{1/r},1)}$ by simply mapping $\sum_{n\in\mathbb{Z}} a_n \pi^n_{\mathbb{Q}_p} \otimes l$ to $\sum_{n\in\mathbb{Z}} a_n l T^n$. This map is clearly well-defined and injective. To see that it is also surjective, take $f(T) = \sum_{n\in\mathbb{Z}} a_n T^n$ a Laurent series with coefficients in L and converging on the annulus $p^{-1/r} \leq |T| < 1$. Let $l_1, \dots l_m$ be a $\mathbb{Q}_p$-basis of L. We can then write $a_n = \sum_{i=1}^m a_{ni} l_i$ for some $a_{ni} \in \mathbb{Q}_p$ for all i and claim that $\sum_{n\in\mathbb{Z}} a_{ni} T^n \in B^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}$ for all $i = 1, \dots, m$. For any $y \in \mathbb{C}_p$ with $|y| \in [p^{-1/r}, 1)$ we have that

$$0 = \lim_{n\to\pm\infty} a_n y^n = \lim_{n\to\pm\infty} \sum_{i=1}^m a_{ni} l_i y^n = \sum_{i=1}^m (\lim_{n\to\pm\infty} a_{n_i} y^n) l_i.$$

Then since the l_i's are a basis we get that $\lim_{n\to\pm\infty} a_{ni} y^n = 0$ for all $i = 1, \dots, m$. Hence we have $\sum_{n\in\mathbb{Z}} a_{ni} T^n \in \mathcal{R}_{\mathbb{Q}_p}$. This means that $\sum_{i=1}^m (\sum_{n\in\mathbb{Z}} a_{ni} T^i \otimes l_i)$ gets mapped to $f(T)$, which proves the surjectivity. Note that the induced φ- and Γ-action on the Laurent series act trivially on the coefficients. This is important since it allows us to define the slope of a φ-module over $\mathcal{R}^L_{\mathbb{Q}_p}$ in Section 1.3.

We can get a similar result for general K. Note that K'_0 is a seperable extension and hence by the primitive element theorem we have that $K'_0 \cong \mathbb{Q}_p[T]/(f(T))$ for some irreducible polynomial $f \in \mathbb{Q}_p[T]$. Then by the chinese remainder theorem we get $K'_0 \otimes_{\mathbb{Q}_p} L \cong L[T]/(f(T)) \cong \oplus_{i=1}^m L^{(i)}$ for some $m \in \mathbb{N}$ and finite extensions $L^{(i)}/\mathbb{Q}_p$. Hence we have an isomorphism $\alpha : K'_0 \otimes_{\mathbb{Q}_p} L \to \oplus_{i=1}^m L^{(i)}$. Recall that we can identify $B^{\dagger,r}_{\mathrm{rig},K}$ with $\mathcal{R}^{K'_0}_{[p^{1/(e_K r)}}$. Thus we can indentify $B^{\dagger,r}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} L$ with $\oplus_{i=1}^m \mathcal{R}^{L^{(i)}}_{[p^{1/(e_K r)},1)}$, by mapping $\sum_{k\in\mathbb{Z}} a_k \pi^k_K \otimes l$ to $(\sum_{k\in\mathbb{Z}} \alpha_i(a_k \otimes l) T^k)_{1\leq i\leq m}$, where α_i denotes the projection $K'_0 \otimes_{\mathbb{Q}_p} L \hookrightarrow L^{(i)}$ induced by α. The proof for this is similar to the one for the case $K = \mathbb{Q}_p$. Therefore $\mathcal{R}^L_K$ can be identified with a finite direct sum of holomorphic functions on the boundary of the open unit disc.

1.2 Étale (φ, Γ)-modules and representations

Let S be any of the rings $\{B_K^L, B_K^{\dagger,L}, \mathcal{R}_K^L\}$. In this section we define (φ, Γ)-modules over S and show what it means for a (φ, Γ)-module over S to be étale. Following the papers [6],[2],[8] we will show that the category of étale (φ, Γ)-modules over $\mathcal{R}_K^L$ is equivalent to the category of L-representations of G_K. From this equivalence of categories we can then derive that the Euler-Poincaré formula and Tate local duality hold for all étale (φ, Γ)-modules over the Robba ring $\mathcal{R}_K^L$.

Definition. We say an S-module D is a (φ, Γ)-module if the following holds

(i) D is a finite free S-module

(ii) D is equipped with a semi-linear φ-action $\varphi : D \to D$, such that the induced linear map $\varphi^* D = S \otimes_{\varphi, S} D \to D, a \otimes x \mapsto a\varphi(x)$ is an isomorphism.

(iii) D is equipped with a continuous semi-linear Γ_K-action, which commutes with the φ-action.

Note that semi-linear in this context means that $\varphi(ax) = \varphi(a)\varphi(x)$ and $\gamma(ax) = \gamma(a)\gamma(x)$ for any $\gamma \in \Gamma_K, a \in S$ and $x \in D$.

Remark. The category of (φ, Γ)-modules over $\mathcal{R}_K^L$ admits tensor products and taking duals. The φ- and Γ-actions of such modules are defined as follows.

For two (φ, Γ)-modules D_1, D_2 over $\mathcal{R}_K^L$ we define a φ and Γ-action on $D_1 \otimes_{\mathcal{R}_K^L} D_2$ by setting $\varphi(a_1 \otimes a_2) = \varphi(a_1) \otimes \varphi(a_2)$ and $\gamma(a_1 \otimes a_2) = \gamma(a_1) \otimes \gamma(a_2)$ for any $\gamma \in \Gamma$.

For any (φ, Γ)-module D over $\mathcal{R}_K^L$ write $D^\vee = \mathrm{Hom}_{\mathcal{R}_K^L}(D, \mathcal{R}_K^L)$ for the dual module. Take $f \in D^\vee$, then for $x = \sum_{i=1}^n a_i \varphi(x_i)$ we set $(\varphi f)x = \sum_{i=1}^n a_i \varphi(f(x_i))$ with $a_i \in \mathcal{R}_K^L$ and $x_i \in D$. For $\gamma \in \Gamma_K$ and $x \in D$ we set $(\gamma f)x = \gamma(f(\gamma^{-1}x))$.

An L-representation V is a finitely dimensional L-vector space with a continous linear action on G_K. The dimension of the representation V is simply the dimension of V as an L-vector space. We will write $\dim(V) = d$. We define $D(V) = (B^L \otimes_L V)^{H_K}$, which is an B_K^L-vectorspace and carries a φ- and Γ_K-action. For T a lattice of V, we define $D(T) = (A^L \otimes_{\mathcal{O}_L} T)^{H_K}$, which is a free A_K^L-module of rank d. We call a (φ, Γ)-module D over B_K^L étale if there is a free A_K^L-submodule T of D, that is stable under φ and Γ-action and $T \otimes_{A_K^L} B_K^L = D$ holds. Hence $D(V)$ is an étale (φ, Γ)-module for any L-representation V and we can adapt a result from Fontaine in [6] to get the following theorem.

Theorem 1.1. *There is an equivalence of categories between the category of L-representations of G_K and the category of étale (φ, Γ)-modules over B_K^L.*

The functor between this categories is given by $V \mapsto D(V)$ and the inverse functor is $D \mapsto V(D) = (B^L \otimes_{B_K^L} D)^{\varphi=1}$.

We can extend this statement to étale (φ, Γ)-modules over $B_K^{\dagger,L}$. Again we define a (φ, Γ)-module D over $B_K^{\dagger,L}$ to be étale if D has a φ and Γ-stable $A_K^{\dagger,L}$-submodule T, that satisfies $B_K^{\dagger,L} \otimes_{A_K^{\dagger,L}} T$. For an L-representation V of G_K we can define $D^{\dagger,r}(V) = (B^{\dagger,r,L} \otimes_L V)^{H_K}$ and $D^{\dagger}(V) = \cup_{r\geq 0} D^{\dagger,r}(V) = (B^{\dagger,L} \otimes_L V)^{H_K}$. Now we can adapt a result of Cherbonnier and Colmez in [2] to get the following theorem.

Theorem 1.2. *For any L-representation V of G_K there exists $r(V)$, such that $D(V) = B_K^L \otimes_{B_K^{\dagger,r,L}} D^{\dagger,r}(V)$ for $r \geq r(V)$.*

This means that $D^{\dagger}(V)$ is a d-dimensional, étale (φ, Γ)-module over $B_K^{\dagger,L}$ and hence the functor $V \mapsto D^{\dagger}(V)$ gives us an equivalence between the category of L-representations of G_K and the category of étale (φ, Γ)-modules over $B_K^{\dagger,L}$.

We will now extend this statement once more to étale (φ, Γ)-modules over the Robba ring $B_{\mathrm{rig},K}^{\dagger,L}$. We say a (φ, Γ)-module D over $B_{\mathrm{rig},K}^{\dagger,L}$ is étale if it has a $B_K^{\dagger,L}$-submodule D', which is étale as a (φ, Γ)-module over $B_K^{\dagger,L}$ with the restricted φ and Γ actions and for which $D' \otimes_{B_K^{\dagger,L}} B_{\mathrm{rig},K}^{\dagger,L} = D$ holds. Again for an L-representation V of G_K, we set $D_{\mathrm{rig}}^{\dagger,r}(V) = D_{\mathrm{rig}}^{\dagger}(V) \otimes_{B_K^{\dagger,r,L}} B_{\mathrm{rig},K}^{\dagger,L}$ and $D_{\mathrm{rig}}^{\dagger}(V) = \cup_{r\geq 0} D_{\mathrm{rig}}^{\dagger,r}(V) = D^{\dagger}(V) \otimes_{B_K^{\dagger,L}} B_{\mathrm{rig},K}^{\dagger,L}$. Then by a result of Kedlaya in [8] we get the following theorem.

Theorem 1.3. *The functor $D \mapsto B_{\mathrm{rig},K}^{\dagger,L} \otimes_{B_K^{\dagger,L}} D$ gives us an equivalence between the category of étale (φ, Γ)-modules over $B_K^{\dagger,L}$ and the category of étale (φ, Γ)-modules over $B_{\mathrm{rig},K}^{\dagger,L}$.*

Remark. Combining the three previous theorems gives us an equivalence of categories between the category of L-representations of G_K and the category of étale (φ, Γ)-modules over $B_{\mathrm{rig},K}^{\dagger,L}$, given by $V \mapsto D_{\mathrm{rig},K}^{\dagger}(V)$.

1.3 Slope theory for φ-modules

In this section we will discuss some basics about the slope theory of φ-modules over the ring $R \in \{\mathcal{R}_{\mathbb{Q}_p}^L, \mathcal{R}_K\}$. We have seen earlier that the ring R can be indentified with the set of holomorphic functions on the boundary of the open unit disc and hence we can define the slope of a φ-module over this ring as in the paper [8] by Kedlaya. Keeping with the notation of [8] we write R^{bd} for the functions in R with bounded coefficients. R^{bd} is a discretely valued field, where the valuation w is given by $w(\sum_{n\in\mathbb{Z}} a_n T^n) = \inf_{n\in\mathbb{Z}} v_p(a_n)$ and we write R^{int} for its ring of integers.

Definition. A φ-module is a finitely generated free module M over R, equipped with a Frobenius action φ, such that for $\varphi^* M := M \otimes_{\varphi,R} R$ the induced linear map $\varphi^* M \to M$, $m \otimes r \mapsto \varphi(m)r$, for $m \in M, r \in R$, is an isomorphism.

We say a φ-module M over R is étale, if M has a free φ-stable R^{int}-submodule M', such that $\varphi^* M' \cong M'$ and $M' \otimes_{R^{\text{int}}} R = M$.

We can also interpret a φ-module as a left-module over the twisted polynomial ring $R\{T\}$, which is finite free over R and the twisted polynomial ring is defined as $R\{T\} := \left\{ \sum_{i=0}^{\infty} r_i T^i | r_i \in R \right\}$, where the multiplication is noncommutative and satisfies $Tr = \varphi(r)T$ for any $r \in R$. Then for any $a \in \mathbb{N}$ we can define the a-pushforward functor $[a]_*$ from φ-modules to φ^a-modules along the inclusion $R\{T^a\} \hookrightarrow R\{T\}$.

For a φ-module M with rank $M = n$ the nth exterior power $\wedge^n M$ has rank 1. Let v be a generator of $\wedge^n M$ then we can choose $\lambda \in (R)^\times \subseteq R^{\text{bd}}$, such that $\varphi(v) = \lambda v$ and set the *degree* of M to be $\deg(M) := w(\lambda)$. If M is not trivial we can define the *slope* of M by setting $\mu(M) = \deg(M)/\text{rank} M$. We will write $M^\vee = \text{Hom}_R(M, R)$ for the dual module. We will now give some basic properties about the degree and slope of a φ-module over R.

Lemma 1.4. *Let M, M_1, M_2 be φ-modules, then the following holds:*

(i) *For an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ we have* $\deg(M) = \deg(M_1) + \deg(M_2)$.

(ii) *For the tensor product of φ-modules we have* $\mu(M_1 \otimes M_2) = \mu(M_1) + \mu(M_2)$.

(iii) *For the dual module we have* $\deg(M^\vee) = -\deg(M)$ *and* $\mu(M^\vee) = -\mu(M)$.

(iv) *If $M_1 \subseteq M$ and M, M_1 have the same rank, then $\mu(M_1) \geq \mu(M)$ and $\mu(M_1) = \mu(M)$ if and only if $M_1 = M$.*

Proof. [8, 1.4.5 and 1.4.10] $\square$

Definition.

(i) We say a φ-module M over R of slope $\mu(M) = c/d$, $(c, d$ coprime$)$ is *pure* if there exists a rank 1 φ-module N of degree $-c$, such that $([d])_* M) \otimes N$ is an étale φ-module over R.

(ii) We say a φ-module M over R is called *semistable*, if for every nontrivial φ-submodule N of M, we have $\mu(N) \geq \mu(M)$.

An important fact of [8] is that a φ-module M is pure if and only if it is semistable ([8, Theorem 1.7.1]). With this one can show the following:

(i) A φ-module is pure of slope 0 if and only if it is étale.

(ii) If M is a pure φ-module of slope s, then $M^\vee$ is pure of slope $-s$.

(iii) If M_1, M_2 are pure φ-modules of slopes s_1 and s_2, then $M_1 \otimes M_2$ is pure of slope $s_1 + s_2$.

Note that the φ-action for tensor products and duals of φ-modules is defined in the same way as the φ-actions for tensor products and duals for (φ, Γ)-modules in Section 1.2.

Furthermore we say a submodule M' of a finite free module M over R is *saturated* if $M' = M \cap (M' \otimes_R \mathrm{Frac}\ R)$. Note that since R is a Bézout domain by [5, Prop 4.9] this means that both M' and M/M' are also free modules. With this we can state the slope filtration theorem which will be important later.

Theorem 1.5 (Slope filtration theorem). *Every φ-module M over R admits a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ of saturated φ-submodules, such that all the quotients $M_1/M_0, \ldots, M_l/M_{l-1}$ are pure and have increasing slopes, i.e. $\mu(M_1/M_0) < \cdots < \mu(M_l/M_{l-1})$. Note that if M is a (φ, Γ)-module, all the subquotients are also (φ, Γ)-modules.*

Proof. [8, Theorem 1.7.1] $\square$

2 Cohomology of (φ, Γ)-modules

In this section we will define cohomology for (φ, Γ)-modules over the ring $\mathcal{R}_K^L$ and then show that we can use the equivalences of categories between étale (φ, Γ)-modules and L-representations of G_K obtained in Section 1.2 to prove the Euler-Poincaré formula and local Tate duality for étale (φ, Γ)-modules over $\mathcal{R}_K^L$. To prepare for the general case we will then study the cohomology of certain rank 1 (φ, Γ)-modules over $\mathcal{R}_K^L$. Lastly let E/K be a finite extension of fields. Then we can define for any (φ, Γ)-module D over $\mathcal{R}_E^L$ the induced (φ, Γ)-module $\mathrm{Ind}_E^K D$ over $\mathcal{R}_K^L$, such that there exists an isomorphism between the cohomology of the two modules.

2.1 Definition of cohomology for (φ, Γ)-modules

To define cohomology for a (φ, Γ)-module D over $\mathcal{R}_K^L$, we take a p-torsion subgroup Δ_K of Γ_K, such that Γ_K/Δ_K is procyclic. Note that for $p \neq 2$ the group Γ_K itself is always procyclic, and for $p = 2$ the group Δ_K is at most of order 2. We set $D' = D^{\Delta_K}$ and define the projection $p_\Delta = (1/|\Delta_K|) \sum_{\delta \in \Delta_K} \delta$ from D to D'. We take a topological generator γ of Γ_K/Δ_K and to define the following complex:

$$C^\bullet_{\varphi,\gamma} : 0 \to D' \xrightarrow{d_1} D' \oplus D' \xrightarrow{d_2} D' \to 0,$$

where $d_1(x) = ((\gamma - 1)x, (\varphi - 1)x)$ and $d_2(x, y) = ((\varphi - 1)x - (\gamma - 1)y)$. Since for $\mathcal{R}_K^L$ the φ- and Γ-action on the coefficient field is trivial, we can show that the cohomology is well defined, i.e. independent of the choice of Δ_K as in [10, Section 2.1].

Remark. For a (φ, Γ)-Module D over $\mathcal{R}_K^L$, we have that $H^1(D)$ classifies all extensions of $\mathcal{R}_K^L$ by D.

We can now define cup products for two (φ, Γ)-modules M, N by setting

$$H^0(M) \times H^0(N) \to H^0(M \otimes N), (x, y) \mapsto x \otimes y$$

$$H^0(M) \times H^1(N) \to H^1(M \otimes N), (x, (\overline{y}, \overline{z})) \mapsto (\overline{x \otimes y}, \overline{x \otimes z})$$

$$H^0(M) \times H^2(N) \to H^2(M \otimes N), (x, \overline{y}) \mapsto \overline{x \otimes y}$$

$$H^1(M) \times H^1(N) \to H^2(M \otimes N), (\overline{x}, \overline{y}), (\overline{z}, \overline{t})) \mapsto \overline{y \otimes \gamma(z) - x \otimes \varphi(t)}.$$

2.2 Euler-Poincaré formula and Tate duality for étale (φ, Γ)-modules

In [10, Cor 2.9] Liu has proven the Euler-Poincaré formula and Tate duality for étale (φ, Γ)-modules over $\mathcal{R}_K$. In this section we will make slight adaptations to the proof to show that the Euler-Poincaré formula and Tate duality is also true for étale (φ, Γ)-modules over the ring $\mathcal{R}_K^L$.

In section 1.2 we established an equivalence of categories between L-representations of G_K and étale (φ, Γ)-modules over $\mathcal{R}_K^L$. We will now use these results to show that the Euler-Poincaré formula and Tate local duality hold for such modules. Note that since the field B is an extension of degree p of $\varphi(B)$, we can define an operator $\psi : B \to B, x \mapsto (1/p)\varphi^{-1}(\mathrm{Tr}_{B/\varphi(B)}(x))$, which is surjective, commutes with the Galois action and satisfies $\psi(\varphi(x)) = x$ for any $x \in B$, as well as $\psi(A) \subseteq A$ hold, see [3, Section 5.3.1] for the construction. Since the φ and Γ-action on the additional coefficient field is trivial, we can extend this operator to B^L, such that ψ is surjective, commutes with the Galois action and satisfies $\psi(A^L) \subseteq A^L$, as well as $\psi(\varphi(x))$. Furthermore ψ can be extended (φ, Γ)-modules $D(V)$ and $D^\dagger(V)$ for any L-representation V of G_K, so that it is still surjective, commutes with the Galois operation and satisfies $\psi(\varphi(x)) = x$ for all $x \in D(V)$ (resp.$(D^\dagger(V))$.

Theorem 2.1. *Let V be an $\mathcal{O}_L$-representation of G_K. Then for $i = 0, 1, 2$ there exist isomorphisms*

$$H^i(D(V)) \cong H^i(G_K, V)$$

which are functorial in V and compatible with cup products.

Proof. As in [10, Thm. 2.3], for V of finite length we can adapt the proof of [3, Thm. 5.2.2] for V to the case where Γ does not have to be procyclic by replacing H_K with the preimage of a p-torsion group Δ_K in G_K and $D(V)$ by $D(V)'$. Note that the exact sequence in [3, Thm. 5.2.2] in this case is

$$0 \longrightarrow \mathcal{O}_L \longrightarrow A^L \xrightarrow{\varphi-1} A^L \longrightarrow 0\,.$$

$\square$

Lemma 2.2. *The morphism $\gamma - 1 : ((D^\dagger(V))')^{\psi=0} \to ((D^\dagger(V))')^{\psi=0}$ has a continuous inverse.*

Proof. We have that $\chi(\Gamma_{K_1}) \subseteq 1 + p\mathbb{Z}_p$ is procyclic, so we can choose a topological generator γ' of Γ_{K_1} such that $\gamma' = \gamma^m$ in Γ_K/Δ_K for some $m \in \mathbb{N}$.

We get the following commutative diagramm

$$\begin{tikzcd}
D^\dagger(V)^{\psi=0} \arrow[r, "\gamma'-1"] \arrow[d, "p_{\Delta_{\mathbb{Q}_p}}"] & D^\dagger(V)^{\psi=0} \arrow[d, "p_{\Delta_{\mathbb{Q}_p}}"] \\
(D^\dagger(V)')^{\psi=0} \arrow[r, "\gamma^m-1"] & (D^\dagger(V)')^{\psi=0}
\end{tikzcd}$$

From [2, Prop. 2.6.1] we know that the first map $\gamma' - 1 : D^\dagger(V)^{\psi=0} \to D^\dagger(V)^{\psi=0}$ has a continuous inverse. And since p_{Δ_K} is idempotent, the map $\gamma^m - 1 : (D^\dagger(V)')^{\psi=0} \to (D^\dagger(V)')^{\psi=0}$ has a continuous inverse as well and therefore we get that $\gamma - 1$ has a continuous inverse, which is given by $(\gamma^m - 1)^{-1}(\mathrm{id} + \gamma + \ldots \gamma^{m-1})$. □

Let $C^\bullet_{\psi,\gamma}(D^\dagger(V))$ be the complex

$$0 \longrightarrow (D^\dagger(V))' \xrightarrow{\widetilde{d_1}} (D^\dagger(V))' \oplus (D^\dagger(V))' \xrightarrow{\widetilde{d_2}} (D^\dagger(V))' \longrightarrow 0$$

where $\widetilde{d_1}(x) = ((\gamma - 1)x, (\psi - 1)x)$ and $\widetilde{d_2}(x, y) = (\psi - 1)x - (\gamma - 1)y$. We can now get the following commutative diagram of complexes

$$\begin{tikzcd}
C^\bullet_{\varphi,\gamma}D^\dagger(V)) : 0 \arrow[r] & D^\dagger(V)' \arrow[r, "d_1"] \arrow[d, "\mathrm{id}"] & D^\dagger(V)' \oplus D^\dagger(V)' \arrow[r, "d_2"] \arrow[d, "-\psi\oplus\mathrm{id}"] & D^\dagger(V)' \arrow[r] \arrow[d, "-\psi"] & 0 \\
C^\bullet_{\psi,\gamma}(D^\dagger(V)) : 0 \arrow[r] & D^\dagger(V)' \arrow[r, "\widetilde{d_1}"] & D^\dagger(V)' \oplus D^\dagger(V)' \arrow[r, "\widetilde{d_2}"] & D^\dagger(V)' \arrow[r] & 0
\end{tikzcd}$$

Lemma 2.3. *The commutative diagram of complexes above induces an isomorphism on the cohomology.*

Proof. Recall that the map ψ is surjective and hence the cokernel complex of the above diagram of complexes is trivial. Furthermore the kernel complex is just

$$0 \longrightarrow 0 \longrightarrow ((D^\dagger(V))')^{\psi=0} \xrightarrow{\gamma-1} ((D^\dagger(V))')^{\psi=0} \longrightarrow 0,$$

which has trivial cohomology by Lemma 2.2. Hence the cohomology of the complexes in the above diagram is isomorphic. □

Lemma 2.4. *For any G_K-stable $\mathcal{O}_L$-lattice T of V the natural morphism $D^\dagger(T)/(\psi - 1) \to D(T)/(\psi - 1)$ is an isomorphism.*

Proof. Since the φ- and Γ-action act trivially on the additional coefficient field this lemma can be proven the same way as in [10, Lemma 2.6]. □

Proposition 2.5. *Let V be an L-representation of G_K. Then for $i = 0, 1, 2$ the natural morphisms*

$$H^i(D^\dagger(V)) \xrightarrow{\alpha_i} H^i(D^\dagger_{\text{rig}}(V))$$

$$H^i(D^\dagger(V)) \xrightarrow{\beta_i} H^i(D(V))$$

are all isomorphisms.

Proof. We first proof the case $i = 1$. We know that the groups $H^1(D^\dagger(V)), (H^1(D(V))$ and $H^1(D^\dagger_{\text{rig}}(V)))$ classifiy all extensions of étale (φ, Γ)-modules of $B^{\dagger,L}_K, B^L_K$ and $\mathcal{R}^L_K$ by $D^\dagger(V), D(V)$ and $D^\dagger_{\text{rig}}(V)$. We know from the theorems 1.1, 1.2, 1.3, that the categories of étale (φ, Γ)-modules over $B^{\dagger,L}_K, B^L_K$ and $\mathcal{R}^L_K$ are equivalent to the category of L-representations of $G_{\mathbb{Q}_p}$. Hence the maps α_1 and β_1 are isomorphisms.

Next we will deal with α_0 and α_2. From [8, Prop 1.5.4] we know that the natural maps $D^\dagger(V)^{\varphi=1} \to D^\dagger_{\text{rig}}(V)^{\varphi=1}$ and $D^\dagger(V)/(\varphi-1) \to D^\dagger_{\text{rig}}(V)/(\varphi-1)$ are bijective. Taking Δ_K-invariants in the first map yields that $(D^\dagger_{\text{rig}}(V)')^{\varphi=1} \to (D^\dagger(V)')^{\varphi=1}$ is also bijective. The operator $p_{\Delta_K} : D(V) \to D(V)'$ induces the following commutative diagram

$$\begin{tikzcd}
D^\dagger(V)/(\varphi - 1) \arrow[r] \arrow[d, "p_{\Delta_K}"] & D^\dagger_{\text{rig}}(V)/(\varphi - 1) \arrow[d, "p_{\Delta_K}"] \\
D^\dagger(V)'/(\varphi - 1) \arrow[r] & D^\dagger_{\text{rig}}(V)'/(\varphi - 1)
\end{tikzcd}$$

Since p_{Δ_K} is idempotent, we get that also $D^\dagger(V)'/(\varphi-1) \to D^\dagger_{\text{rig}}(V)'/(\varphi-1)$ is an isomorphism. And hence also α_0 and α_2 are isomorphisms.

Note that $H^0(D^\dagger(V)) = V^{\Gamma_K} = H^0(D(V))$, and hence β_0 is an isomorphism.

Now by Lemma 2.3 and 2.4 we get that

$$H^2(D^\dagger(V)) \cong (D^\dagger(V))'/(\psi-1, \gamma-1) \cong (D(V))'/(\psi-1, \gamma-1) \cong H^2(D(V)).$$

And hence also β_2 is an isomorphism. □

Combining the isomorphisms above with the isomorphisms of Theorem 2.1, we get that for any L-representation of G_K, that for $i = 0, 1, 2$ there are isomorphisms

$$H^i(D^\dagger(V)) \cong H^i(G_K, V)$$
$$H^i(D^\dagger_{\text{rig}}(V)) \cong H^i(G_K, V)$$

which are functorial and compatible with cup products. With this result we can prove the following theorem.

Theorem 2.6. *The Euler-Poincaré formula and the local Tate duality hold for étale (φ, Γ)-modules over the Robba ring $\mathcal{R}_K^L$.*

Proof. We have seen that $H^2(D_{\mathrm{rig}}^{\dagger}(L(1)))$ is isomorphic to $H^2(L(1))$ and hence the Euler-Poincaré formula and Tate duality for étale (φ, Γ)-modules over the Robba ring $\mathcal{R}_K^L$ follow from the Euler-Poincaré formula and Tate duality for Galois cohomology. $\square$

2.3 (φ, Γ)-modules of rank 1

In this section we will show how to construct rank 1 (φ, Γ)-modules using continuous characters and show that the cohomology H^0 is trivial for certain rank 1 modules, which will be used later in the proof of the Euler-Poincaré formula and Tate duality.

Let δ be a continuous character from $\mathbb{Q}_p^\times$ to $L^\times$. Then we can associate a (φ, Γ)-module $\mathcal{R}_{\mathbb{Q}_p}^L(\delta)$ to δ by defining

$$\varphi(xv) = \delta(p)\varphi(x)v \text{ and } \gamma(xv) = \delta(\chi(\gamma))\gamma(x)v,$$

where v is a basis for $\mathcal{R}_{\mathbb{Q}_p}^L(\delta)$ and $x \in \mathcal{R}_{\mathbb{Q}_p}^L$. For D a (φ, Γ)-module of any rank, we define $D(x) = D \otimes_{\mathcal{R}_{\mathbb{Q}_p}^L} \mathcal{R}_{\mathbb{Q}_p}^L(x)$.

Remark. We will now give some examples for rank 1 (φ, Γ)-modules.

(i) Let $x : \mathbb{Q}_p^\times \to L^\times$ be the character induced from the inclusion $\mathbb{Q}_p \hookrightarrow L$. Then the φ and Γ actions of $\mathcal{R}(x)$ are defined by $\varphi(v) = pv$ and $\gamma(v) = \chi(\gamma)v$.
We can now compute $H^0(\mathcal{R}_{\mathbb{Q}_p}^L(x))$. Let $av \in H^0(\mathcal{R}_{\mathbb{Q}_p}^L(x))$ then $\varphi(a) = a/p$. But then $\varphi(at) = at$ and therefore a is a constant and hence $a = 0$. This implies $H^0(\mathcal{R}_{\mathbb{Q}_p}^L(x)) = 0$.

(ii) Let $|x| : \mathbb{Q}_p^\times \to L^\times, x \mapsto p^{-v_p(x)}$ be a character, then the φ and Γ actions of $\mathcal{R}_{\mathbb{Q}_p}^L(|x|)$ are defined by $\varphi(v) = v/p$ and $\gamma(v) = v$, since the image of χ lies in $\mathbb{Z}_p^\times$.
Again we can compute $H^0(\mathcal{R}_{\mathbb{Q}_p}^L(|x|))$. Let $av \in H^0(\mathcal{R}_{\mathbb{Q}_p}^L(|x|))$ then $\gamma(a) = a$, so a is a constant. But we have $\varphi(a) = pa$ and hence $a = 0$. This implies $H^0(\mathcal{R}_{\mathbb{Q}_p}^L(|x|)) = 0$.

(iii) Lastly let $\omega = x|x|$, then the φ- and Γ-actions of $\mathcal{R}_{\mathbb{Q}_p}^L(\omega)$ are $\varphi(v) = v$ and $\gamma(v) = \chi(\gamma)p^{-v_p(\chi(\gamma))}$.

Next we will prove some facts about the modules $\mathcal{R}_{\mathbb{Q}_p}^L(x)$ and $\mathcal{R}_{\mathbb{Q}_p}^L(|x|)$ and their duals, which will be useful for the proof of the Tate duality in the last section.

Lemma 2.7. *We have the following formulas*

(i) $\mathcal{R}^L_{\mathbb{Q}_p}(|x|)^\vee \cong \mathcal{R}^L_{\mathbb{Q}_p}(x\omega^{-1}) = \mathcal{R}^L_{\mathbb{Q}_p}(|x|^{-1})$

(ii) $\mathcal{R}^L_{\mathbb{Q}_p}(x)^\vee \cong \mathcal{R}^L_{\mathbb{Q}_p}(|x|\omega^{-1}) = \mathcal{R}^L_{\mathbb{Q}_p}(x^{-1})$

Proof. For (i) let us assume that v denotes a generator of $\mathcal{R}^L_{\mathbb{Q}_p}(|x|)$, then v^* is a generator of $\mathcal{R}^L_{\mathbb{Q}_p}(|x|)^\vee$ defined by $v^*(v) = 1$. Then we can write $v = p\varphi(v)$ and hence have $\varphi(v^*)(v) = p\varphi(v^*(v)) = p$. Also we have for any $\gamma \in \Gamma$ that $\gamma(v^*)(v) = \gamma(v^*(\gamma^{-1}(v)) = 1$. If we compare these operations to the operations of $\mathcal{R}^L_{\mathbb{Q}_p}(|x|^{-1})$, see example above, we get that they are the same on generators and hence (i) holds.

For (ii) let us assume that v denotes a generator of $\mathcal{R}^L_{\mathbb{Q}_p}(x)$, then v^* is agenerator of $\mathcal{R}^L_{\mathbb{Q}_p}(x)^\vee$ defined by $v^*(v) = 1$. Then we can write $v = \varphi(v)/p$ and hence we have $\varphi(v^*)(v) = \varphi(v^*(v))/p = 1/p$. Also we have for any $\gamma \in \Gamma$ that $\gamma(v^*)(v) = \gamma(v^*(\gamma^{-1}(v)) = \gamma(v^*(\chi(\gamma)^{-1}v)) = 1$. If we compare these operations to the operations of $\mathcal{R}^L_{\mathbb{Q}_p}(x^{-1})$, see example above, we get that they are the same on generators and hence (ii) holds. $\square$

2.4 Induced (φ, Γ)-modules

In this section we will prove Shapiros's Lemma, which compares cohomology of (φ, Γ)-modules over $\mathcal{R}^L_K$ with (φ, Γ)-modules over $\mathcal{R}^L_E$, when E/K is a finite extension of fields. For this we will introduce the notion of induced modules and show that it is well behaved with taking duals.

Definition. Let D be a (φ, Γ_E)-module and define

$$\mathrm{Ind}^{\Gamma_K}_{\Gamma_E} D = \{f : \Gamma_K \to D \mid f(hg) = h \cdot f(g) \text{ for } h \in \Gamma_E\},$$

which has an $\mathcal{R}^L_K$-module structure with a φ- and Γ_K-action that can be defined by $(af)(g) = g(a)f(g)$ and $(\varphi(f))(g) = \varphi(f(g))$ and $(hf)(g) = f(hg)$ for any $a \in \mathcal{R}^L_K$ and $g, h \in \Gamma_K$. With this, $\mathrm{Ind}^{\Gamma_K}_{\Gamma_E} D$ becomes a (φ, Γ_K)-module, which we will call the *induced* (φ, Γ)-module of D from E to K. To simplify notation, we write $\mathrm{Ind}^K_E D$ for $\mathrm{Ind}^{\Gamma_K}_{\Gamma_E} D$.

Note that $[E : K] = [E_\infty : K_\infty][\Gamma_K : \Gamma_E] = [\mathcal{R}^L_E : \mathcal{R}^L_K][\Gamma_K : \Gamma_E]$, where the last equality follows from [11, Section 2.1]. Hence we have $\mathrm{rank}_{\mathcal{R}^L_E} D = [E : K]\mathrm{rank}_{\mathcal{R}^L_K} \mathrm{Ind}^K_E D$.

Theorem 2.8 (Shapiro's Lemma)**.** *Let E be a finite extension of K. Then for a (φ, Γ)-module D over $\mathcal{R}^L_E$, there exist isomorphisms*

$$H^i(D) \cong H^i(\mathrm{Ind}^K_E D)$$

for $i = 0, 1, 2$.

Proof. First we prove the theorem in the case that Γ_K and Γ_E are both procyclic. Write e for the neutral element of Γ_K. We assume $[\Gamma_K : \Gamma_E] = m$ and set γ_K to be a topological generator of Γ_K, then γ_K^m is a topological generator of Γ_E. Note that this means any element in $\mathrm{Ind}_E^K D$ can by uniquely determined by the images of $\{e, \gamma_K, \gamma_K^2, \dots, \gamma_K^{m-1}\}$, a set of representatives of the cosets of Γ_K/Γ_E. We define a map $Q : D \to \mathrm{Ind}_E^K D$ by setting $(Q(x))(e) = x$ and $(Q(x))(\gamma_K^i) = 0$ for $1 \leq i \leq m-1$ for any $x \in D$. We show that Q is well-defined, injective and φ and Γ_E-equivariant morphism of $\mathcal{R}_K^L$-modules, which will become evident by the following: Let $x, y \in D$, $g \in \Gamma_E$ and $r \in \mathcal{R}_K^L$. Since $Q(x)$ is defined on a set of representatives of the cosets of Γ_K/Γ_E, we in fact have $Q(x) \in \mathrm{Ind}_E^K D$ for any $x \in D$. We have that $(Q(rx+y))(e) = rx + y = e(r)(Q(x))(e) + (Q(y))(e) = (rQ(x) + Q(y))(e)$, hence Q is a morphism of $\mathcal{R}_K^L$-modules. Assume that $Q(x) = Q(y)$, then we have that $x = (Q(x))(e) = (Q(y))(e) = y$ and hence Q is injective. Also $\varphi(Q(x))(e) = \varphi((Q(x)(e)) = \varphi(x) = Q(\varphi(x))(e)$, therefore $\varphi(Q(x)) = Q(\varphi(x))$ and hence Q is φ-equivariant. Finally the Γ_E-equivariance of Q follows from the definition of $\mathrm{Ind}_E^K D$.

We now claim that Q induces a φ-equivariant isomorphism $\overline{Q} : D/(\gamma_E - 1) \to (\mathrm{Ind}_E^K D)/(\gamma_K - 1)$.

First we need to show that $\overline{Q}$ is well-defined. For this take any $x \in D$ and now show that $Q((\gamma_E - 1)x) \in (\gamma_K - 1)\mathrm{Ind}_E^K D$. Take $f \in \mathrm{Ind}_E^K D$ with $f(e) = x$ and $f(\gamma_K^i) = \gamma_E x$ for $1 \leq i \leq m-1$. Then we have that $(\gamma_K - 1)f(e) = f(\gamma_K) - f(e) = \gamma_E x - x$ and $(\gamma_K - 1)f(\gamma_K^i) = f(\gamma_K^{i+1}) - f(\gamma_K^i) = 0$ for $1 \leq i \leq m-1$, since $f(\gamma_K^m) = \gamma_E f(e) = \gamma_E x$. Therefore $Q((\gamma_E - 1)x) = (\gamma_K - 1)f$ and hence $\overline{Q}$ is well-defined.

Next we show $\overline{Q}$ is injective, for this assume we have $x \in D$ such that $Q(x) \in (\gamma_K - 1)\mathrm{Ind}_E^K D$, so there is $f \in \mathrm{Ind}_E^K D$ such that $(\gamma_K - 1)f = Q(x)$, then for $1 \leq i \leq m-1$ we have

$$x = Q(x)(e) = f(\gamma_K) - f(e)$$

$$0 = Q(x)(\gamma_K^i) = f(\gamma_K^{i+1}) - f(\gamma_K^i).$$

Summing these equalities we obtain

$$x = \sum_{i=0}^{m-1} f(\gamma_K^{i+1}) - f(\gamma_K^i) = f(\gamma_E) - f(e) = (\gamma_E - 1)f(e)$$

and therefore $x \in (\gamma_E - 1)D$, so $\overline{Q}$ is injective.

Next we show $\overline{Q}$ is surjective. For this take $f \in \mathrm{Ind}_E^K D$ and set $x_j = f(\gamma_K^j)$ for $0 \leq j \leq m-1$. We claim that

$f \equiv Q(x) \bmod (\gamma_K - 1)\mathrm{Ind}_E^K D$ for $x = (\gamma_E)^{-1}(\sum_{i=1}^m x_{m-i})$. Now define $g \in \mathrm{Ind}_E^K D$ by setting $g(e) = (\gamma_E)^{-1}(\sum_{j=1}^m x_{m-j})$ and $g(\gamma_K^i) = \sum_{j=0}^{i-1} x_j$. Then we have

$$(f - Q(x))(e) = x_0 - (\gamma_E)^{-1}(\sum_{j=0}^{m} x_{m-j}) = g(\gamma_K) - g(e) = (\gamma_K - 1)g(e)$$

and for $1 \le i \le m-1$ we have

$$(f - Q(x))(\gamma_K^i) = x_i = g(\gamma_K^{i+1}) - g(\gamma_K^i) = (\gamma_K - 1)g(\gamma_K^i),$$

since $g(\gamma_K^m) = \gamma_E g(e) = \sum_{i=1}^m x_{m-i}$.

Hence we have $Q(x) \equiv f \bmod (\gamma_K - 1)$ and therefore $\overline{Q}$ is surjective. The φ-equivariance of $\overline{Q}$ follows directly from the φ-equivariance of Q.
For any $g \in \Gamma_K$, we can define a map Q^g by setting $Q^g(x) = g(Q(x))$ for $x \in D$. Note that Q^g in fact is a morphism of $\mathcal{R}_K^L$-modules, since for $x, y \in D$ and $r \in \mathcal{R}_K^L$ we have

$$\begin{aligned}(rQ^g(x) + Q^g(y))(\gamma) &= r(Q^g(x))(\gamma) + (Q^g(y))(\gamma)\\ &= r(Q(x))(g\gamma) + Q(y)(g\gamma)\\ &= (rQ(x) + Q(y))(g\gamma)\\ &= Q(rx + y)(g\gamma)\\ &= Q^g(rx + y)(\gamma).\end{aligned}$$

Now we define $\widetilde{Q} = \sum_{i=0}^{m-1} Q^{\gamma_K^i}$. The morphism $\widetilde{Q}$ is also Γ_E- and φ-equivariant since the φ and Γ_K-action commute. Now we show that $\widetilde{Q}$ is injective. Assume that $x \in D$ such that $\widetilde{Q}(x) = 0$. Then $\widetilde{Q}(x)(e) = Q(x)(e) = x = 0$, since $Q^{\gamma_K^i}(y)(e) = Q(y)(\gamma_K^i) = 0$ for all $i = 1, \ldots, m-1$ and all $y \in D$. Hence $\widetilde{Q}$ is in fact injective.

Next we claim that $\widetilde{Q}$ induces a φ-equivariant isomorphism $\widehat{Q} : D^{\Gamma_E} \to (\mathrm{Ind}_E^K D)^{\Gamma_K}$.

We first check that $\widehat{Q}$ is well-defined. For this let $x \in D^{\Gamma_E}$, then for any $0 \le j \le m-1$ we get that

$$\gamma_K \widetilde{Q}(x)(\gamma_K^j) = \widetilde{Q}(x)(\gamma_K^{1+j}) = Q^{\gamma_K^{m-1+j}}(x)(\gamma_K^{1+j}) = \gamma_E x = x.$$

And hence $\gamma_K \widetilde{Q}(x) = \widetilde{Q}(x)$, so $\widehat{Q}(x)$ is well-defined.

Clearly $\widehat{Q}$ is also injective. To see it is also surjective take $f \in (\mathrm{Ind}_E^K D)^{\Gamma_K}$ and set $x = f(e)$. Because of the Γ_K-invariance of f we then have $f(g) = gf(e) = f(e) = x$ for any $g \in \Gamma_K$. But if $g \in \Gamma_E$ we also have that $f(g) = g(f(e)) = gx$, which implies that x is Γ_E-invariant and therefore $x \in D^{\Gamma_E}$. Furthermore since $\widetilde{Q}(x)(\gamma_K^i) = Q^{\gamma_K^{m-i}}(\gamma_K^i) = \gamma_E x = x$ for all $0 \leq i \leq m-1$ we have that $\widetilde{Q}(x) = f$ and hence $\widehat{Q}$ is surjective. We will now consider the following diagramm

$$\begin{tikzcd}
C^{\bullet}_{\varphi,\gamma_E} : 0 \arrow[r] & D \arrow[r] \arrow[d, "\widetilde{Q}"] & D \oplus D \arrow[r] \arrow[d, "Q\oplus\widetilde{Q}"] & D \arrow[r] \arrow[d, "Q"] & 0 \\
C^{\bullet}_{\varphi,\gamma_K} : 0 \arrow[r] & \mathrm{Ind}_E^K D \arrow[r] & \mathrm{Ind}_E^K D \oplus \mathrm{Ind}_E^K D \arrow[r] & \mathrm{Ind}_E^K D \arrow[r] & 0
\end{tikzcd}$$

To see that this diagram commutes we need to check that $(\gamma_K - 1)\widetilde{Q} = Q(\gamma_E - 1)$. This holds since for $x \in D$ we have

$$(\gamma_K - 1)\widetilde{Q}(x) = \sum_{i=0}^{m-1} Q^{\gamma_K^{i+1}}(x) - Q^{\gamma_K^{i}}(x) = Q^{\gamma_E}(x) - Q(x) = Q((\gamma_E - 1)x).$$

This induces morphisms $\alpha^i : H^i(D) \to H^i(\mathrm{Ind}_E^K D)$ for $0 \leq i \leq 2$. We will now show that these are the required isomorphisms.

For H^0, recall that we have already shown that $\widetilde{Q}$ induces a φ-equivariant isomorphism $\widehat{Q} : D^{\Gamma_E} \to (\mathrm{Ind}_E^K D)^{\Gamma_K}$. By taking φ-invariants on both sides, we obtain that the map α^0 is also an isomorphism.

For H^2, recall that we have shown that Q induces a φ-equivariant isomorphism $\overline{Q} : D/(\gamma_E - 1) \to (\mathrm{Ind}_E^K D)/(\gamma_K - 1)$ and hence α_2 is an isomorphism. For H^1, consider the following short exact sequence

$$0 \longrightarrow (D^{\Gamma_E})/(\varphi - 1) \xrightarrow{\beta_1} H^1(D) \xrightarrow{\beta_2} (D/(\gamma_E - 1))^{\varphi=1} \longrightarrow 0$$

where $\beta_1(x) = (0, x)$ and $\beta_2(y, x) = y$. We check now that the diagram is in fact commutative.

We can see that β_1 is well-defined, since for $x \in D^{\Gamma_E}$ we have that $\beta_1((\varphi - 1)x) = (0, (\varphi - 1)x) = ((\gamma_E - 1)x, (\varphi - 1)x) \in \mathrm{im}(d_1)$ and $\beta_1(x) \in \ker(d_2)$ since $d_2(0, x) = (\gamma_E - 1)x = 0$. We have that β_1 is injective. To see this take a $y \in (D^{\Gamma_E})$ such that $\beta_1(y) = (0, y) = 0$. Then there exists $x \in D$ such that $d_1(x) = ((\gamma_E - 1)x, (\varphi - 1)x) = (0, y)$, so $y \in (\varphi - 1)D^{\Gamma_E})$

We can see that β_2 is well-defined since for $(y, x) \in H^1(D)$ we get that $0 = (\varphi-1)y + (\gamma_E-1)x$ and therefore $\varphi(x) = x$ in $D/(\gamma_E-1)$.We have that β_2 is surjective, since for any $y \in (D/(\gamma_E-1))^{\varphi=1}$, we have that $(y, 0) \in H^1(D)$, because $d_2(y, 0) = (\varphi - 1)y = 0$ and $\beta_2(y, 0) = y$.

We get the same short exact sequence for the induced module and then obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D^{\Gamma_E}/(\varphi - 1) \arrow[r] \arrow[d, "\widetilde{Q}"] & H^1(D) \arrow[r] \arrow[d, "\alpha^1"] & (D/(\gamma_E - 1))^{\varphi=1} \arrow[r] \arrow[d, "Q"] & 0 \\
0 \arrow[r] & (\mathrm{Ind}_E^K D)^{\Gamma_K}/(\varphi - 1) \arrow[r] & H^1(\mathrm{Ind}_E^K D) \arrow[r] & (\mathrm{Ind}_E^K D)/(\gamma_K - 1))^{\varphi=1} \arrow[r] & 0
\end{tikzcd}$$

We have seen that $\widetilde{Q}$ and Q are isomorphisms, so by the Five lemma we get that α^1 is also an isomorphism, which concludes the proof for the case that Γ_E and Γ_K are procyclic.

For the general case, let Δ_E and Δ_K be the torsion subgroups of Γ_E and Γ_K respectively. Then Γ_E/Δ_E is a subgroup of Γ_K/Δ_K and both groups are procyclic. We take a topological generator γ_K of Γ_K/Δ_K. We assume $m = [(\Gamma_K/\Delta_K) : (\Gamma_E/\Delta_E)]$, then $\gamma_E = \gamma_K^m$ is a topological generator of Γ_E/Δ_E. We can define $Q' : D' \to (\mathrm{Ind}_E^K D)'$ by setting for any $x \in D'$, $Q'(x)(e) = x$ and $Q'(x)(y) = 0$ for any non-trivial $y \in \Gamma_K/\Gamma_E$. Similarly we can define $\widetilde{Q}' = \sum_{i=0}^{m-1} \gamma_K^i Q'$. Now by replacing Q by Q' and $\widetilde{Q}$ by $\widetilde{Q}'$ in the above argument, one can show the statement for the non-procyclic case in the same way. □

Next we will show that the induced modules are compatible with taking duals.

Theorem 2.9. *Let E/K be a finite field extension and let D be a (φ, Γ)-module over $\mathcal{R}_E^L$, then* $\mathrm{Ind}_E^K(D^\vee) \cong (\mathrm{Ind}_E^K D)^\vee$.

Proof. Let $(d^k)_{k\in I}$ be an $\mathcal{R}_K^L$-basis of D and $(\gamma_j)_{j\in J}$ be a system of representatives of Γ_E/Γ_K.

Next for any $j \in J$ and $k \in I$ define $h_{jk} \in \mathrm{Ind}_K^E D$ by setting

$$h_{jk}(\gamma_i) = \begin{cases} d^k & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

for any $i \in J$.

Then $(h_{jk})_{k\in I, j\in J}$ is a $\mathcal{R}_K^L$-basis of $\mathrm{Ind}_E^K D$. Write $(h_{jk}^*)_{k\in I, j\in J}$ for the corresponding dual basis.

Write $((d^k)^*)_{k\in I}$ for the $\mathcal{R}_K^L$-dual basis of $(d^k)_{k\in I}$. Furthermore for any $j \in J, k \in I$ define $f_{jk} \in \mathrm{Ind}_E^K(D^\vee)$ by setting

$$f_{jk}(\gamma_i) = \begin{cases} (d^k)^* & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

for any $i \in J$.

Then $(f_{jk})_{k\in I, j\in J}$ is a $\mathcal{R}_K^L$-basis of $\mathrm{Ind}_E^K(D^\vee)$. This allows us to define the following isomorphism of modules

$$\begin{array}{rccc} \Phi : & (\mathrm{Ind}_E^K D)^\vee & \to & \mathrm{Ind}_E^K(D^\vee) \\ & h_{jk}^* & \mapsto & f_{jk} \end{array}$$

We still need to check that Φ respects φ and Γ-actions.

We will start with checking the Γ-action. Take $\gamma \in \Gamma_K$, then for any $j \in J$ there exists one $m_j \in J$ such that $\gamma\gamma_{m_j} = \gamma_j\gamma_j^E$ for some $\gamma_j^E \in \Gamma_E$. We write $(\gamma_j^E)^{-1}(d^k) = \sum_{n\in I} b_{jn}^k d^n$ for some $b_{jn}^k \in \mathcal{R}_K^L$. Now fix $i \in J$ and $l \in I$ and compute γh_{il}^*. We have

$$\gamma h_{il}^*(h_{jk}) = \gamma(h_{il}^*(\gamma^{-1}h_{jk})$$

and

$$\gamma^{-1}h_{jk}(\gamma_i) = h_{jk}(\gamma^{-1}\gamma_i) = h_{jk}(\gamma_{m_i}(\gamma_i^E)^{-1})$$

Note that this means if $j \neq m_i$, then $h_{il}^*(h_{jk}) = 0$.

So now assume that $j = m_i$, then we have

$$\gamma^{-1}h_{jk}(\gamma_i) = h_{jk}(\gamma_j(\gamma_i^E)^{-1}) = (\gamma_i^E)^{-1}(d^k) = \sum_{n\in I} b_{in}^k d^n$$

Hence we have $\gamma^{-1}h_{jk} = \sum_{n\in I}\gamma_i^{-1}(b_{in}^k)h_{in}$.
Then

$$\gamma h_{il}^*(h_{jk}) = \gamma(h_{il}^*(\sum_{n\in I}\gamma_i^{-1}(b_{in}^k)h_{in})) = \gamma\gamma_i^{-1}(b_{il}^k).$$

And so $\gamma h_{il}^* = \sum_{k\in I}\gamma\gamma_i^{-1}(b_{il}^k)h_{jk}^*$.

Next we will compute γf_{il}.
For $j \in J$ we have $\gamma f_{il}(\gamma_j) = f_{il}(\gamma\gamma_j) = 0$, if $j \neq m_i$, since then $\gamma\gamma_j \notin \gamma_i\Gamma_E$. So assume that $j = m_i$, then

$$\gamma f_{il}(\gamma_j) = f_{il}(\gamma\gamma_j) = f_{il}(\gamma_i\gamma_i^E) = \gamma_i^E(d^l)^*.$$

We have for $k \in I$ that

$$(\gamma_i^E(d^l)^*)(d^k) = \gamma_i^E((d^l)^*((\gamma_i^E)^{-1}(d^k)) = \gamma_i^E((d^l)^*(\sum_{n\in I} b_{in}^k d^n)) = \gamma_i^E(b_{il}^k).$$

This gives us

$$\gamma f_{il}(\gamma_j) = \gamma_i^E((d^l)^*) = \sum_{k\in I}\gamma_i^E(b_{il}^k)(d^k)^*.$$

And hence

$$\gamma f_{il} = \sum_{k\in I}\gamma_j^{-1}\gamma_i^E(b_{il}^k)f_{jk} = \sum_{k\in I}\gamma\gamma_i^{-1}(b_{il}^k)f_{jik}.$$

Therefore $\Phi(\gamma h_{il}^*) = \gamma(\Phi(h_{il}))$ holds for all $i \in J, l \in I$ and hence Φ respects the Γ-action.

Now we check the φ-action. First note that for any $k \in I$ we can write $d^k = \sum_{n\in K} b_{nk}\varphi(d^n)$ for some $b_{nk} \in \mathcal{R}_K^L$. Now fix $i \in J$ and $l \in I$ and compute φf_{il}.

Clearly $\varphi f_{il}(\gamma_j) = \varphi(f_{il}(\gamma_j)) = 0$, if $j \neq i$. And $\varphi f_{il}(\gamma_i) = \varphi((d^l)^*)$. For $k \in I$ we have

$$\varphi((d^l)^*)(d^k) = \varphi((d^l)^*)(\sum_{n\in I} b_{nk}\varphi(d^n)) = \sum_{n\in I} b_{nk}\varphi((d^l)^*(d^n)) = b_{lk}.$$

This means $\varphi f_{il}(\gamma_i) = \sum_{k\in I} b_{lk}(d^k)^*$ and therefore

$$\varphi f_{il} = \sum_{k\in I} \gamma_i^{-1}(b_{lk}) f_{ik}.$$

And lastly we will compute φh_{il}^*.

First note that for $k \in I$ we have $\varphi h_{il}^*(h_{jk}) = 0$ if $j \neq i$, since then $h_{jk}(\gamma_i) = 0$. Hence it suffices to compute $\varphi h_{il}^*(h_{ik})$.

Note that

$$h_{ik}(\gamma_i) = d^k = \sum_{n\in I} b_{nk}\varphi(d^n) = \sum_{n\in I} b_{nk}\varphi h_{in}(\gamma_i).$$

Therefore $h_{ik} = \sum_{n\in I} \gamma_i^{-1}(b_{nk})\varphi h_{in}$ and hence

$$\varphi h_{il}^*(h_i k) = \varphi h_{il}^*(\sum_{n\in I} \gamma_i^{-1}(b_{nk})\varphi h_{in}) = \sum_{n\in I} \gamma_i^{-1}(b_{nk})\varphi(h_{il}^*(h_{in})) = \gamma_i^{-1}(b_{lk}).$$

This gives us

$$\varphi h_{il}^* = \sum_{k\in I} \gamma_i^{-1}(b_{lk}) h_{ik}^*.$$

These computations show that $\varphi(\Phi(h_{il}^*)) = \Phi(\varphi f_{il})$ for any $i \in J, l \in I$ and hence Φ respects the φ-action and defines an isomorphism of (φ, Γ)-modules.

$\square$

3 Generalized (φ, Γ)-modules

In this section we will define generalized (φ, Γ)-modules over the ring $\mathcal{R}_K^L$ and first study some properties of torsion (φ, Γ)-modules over $\mathcal{R}_{\mathbb{Q}_p}^L$. With these results and Shapiro's lemma we can then show that the Euler-Poincaré formula holds for torsion (φ, Γ)-modules over $\mathcal{R}_K^L$.

To define generalised (φ, Γ)-modules over $\mathcal{R}_K^L$, we first need to show that $\mathcal{R}_K^L$ is a Bézout ring. Recall from Section 1.1, that we can write $\mathcal{R}_K^L$ as a finite direct sum of Bézout domains. Hence $\mathcal{R}_K^L$ is no longer a domain, but in the following lemma it can be seen that it maintains the Bézout property:

Lemma 3.1. *Let R be a commutative ring, such that $R = \oplus_{i=1}^m R_i$ for some $m \in \mathbb{N}$ and some R_i commutative Bézout domains. Then R is a Bézout ring.*

Proof. Let $I = (f_1, \dots f_l)$ be a finitely generated ideal in R. Write $f_j = (f_j^{(1)}, \dots, f_j^{(m)})$ for $j = 1, \dots l$. Since R_i are Bézout, we get that for $i = 1, \dots, m$, there is $g^{(i)} \in R_i$, such that $(g^{(i)}) = (f_1^{(i)}, \dots, f_l^{(i)})$. We now claim that for $g = (g^{(1)}, \dots, g^{(m)})$ we have that $I = (g)$. We have

$$\begin{aligned} x \in I \iff & x = \sum_{j=1}^{l} r_j f_j = \sum_{j=1}^{l} (r_j^{(1)} f_j^{(1)}, \dots, r_j^{(m)} f_j^{(m)}) \\ & \text{for some } r_j = (r_j^{(1)}, \dots, r_j^{(m)}) \in R \\ \iff & x = (r^{(1)} g^{(1)}, \dots, r^{(m)} g^{(m)}) = rg \\ & \text{for some } r = (r^{(1)}, \dots, r^{(m)}) \in R \\ \iff & x \in (g) \end{aligned}$$

Hence R is in fact a Bézout ring. □

Definition. We say a generalized (φ, Γ)-module over $\mathcal{R}_K^L$ is a finitely presented $\mathcal{R}_K^L$-module D with commuting φ and Γ-actions, such that $\varphi^* D \to D$ is an isomorphism. By the previous Lemma $\mathcal{R}_K^L$ is a Bézout ring and hence also a coherent ring, hence the generalized (φ, Γ)-modules over $\mathcal{R}_K^L$ form an abelian category. We say a generalized (φ, Γ)-module is a torsion (φ, Γ)-module if it has $\mathcal{R}_K^L$-torsion. We say a generalized (φ, Γ)-module is a pure t^k-torsion (φ, Γ)-module, if it is a free $\mathcal{R}_K^L/(t^k \otimes 1)$-module.

We can also define the rank of a generalized (φ, Γ)-module D. For this we take the torsion submodule S of D, which is a torsion (φ, Γ)-module. Then by [5, Prop. 4.8] we get that D/S is a free (φ, Γ)-module and we can set $\operatorname{rank} D = \operatorname{rank} D/S$.

We can define the cohomology for a generealized (φ, Γ)-module using the same complex that was used for the usual (φ, Γ)-modules in Section 2.1.

We will now compute the cohomology of torsion (φ, Γ)-modules over $\mathcal{R}^L_{\mathbb{Q}_p}$.

Lemma 3.2. *For any principal ideal I of $\mathcal{R}^L_{\mathbb{Q}_p}$, which is stable under $\Gamma_{\mathbb{Q}_p}$, there exist $j_n \in \mathbb{N}$ such that $I = (\pi^{j_0}_{\mathbb{Q}_p} \prod_{n=1}^{\infty} (\varphi^{n-1}(q)/p)^{j_n} \otimes 1)$. Furthermore, if $\varphi(I) \subseteq I$, we get that the $(j_n)_n$ form a decreasing sequence.*

Proof. Recall that we can identify the ring $\mathcal{R}^L_{\mathbb{Q}_p}$ with the set of holomorphic functions on the boundary of the open unit disc.

We choose a Laurent series $f(T) \in \mathcal{R}^L_{\mathbb{Q}_p}$ which generates I. We know that the series $f(T)$ converges on some annulus $p^{-1/r} \leq |T| < 1$. Let $V(I)$ denote the set of zeros of I. Then by [9, Prop. 4 bis and Section 4] we know that for any $\delta \geq 0$ the set $V_\delta(I) = V(I) \cap \{z \in \mathbb{C}_p \big| p^{-1/r} + \delta < |z| < 1 - \delta\}$ is finite.

Since I is invariant under $\Gamma_{\mathbb{Q}_p}$ and since $g(T) = (1+T)^{\chi(g)} - 1$ for all $g \in \Gamma_{\mathbb{Q}_p}$, we get that the transformation $z \mapsto (1+z)^g - 1$ maps $V_\delta(I)$ onto itself for all $g \in \Gamma_{\mathbb{Q}_p}$. And since $V_\delta(I)$ is finite and $\Gamma_{\mathbb{Q}_p}$ is infinte we get that there is a $g \in \Gamma_{\mathbb{Q}_p} \setminus \{1\} \subseteq \mathbb{Z}_p^\times \setminus \{1\}$, such that $(1+z)^g - 1 = z$ for all $z \in V_\delta(I)$. This however implies that $(z+1)^{g-1} = 1$, and hence we get that $z = 0$ or $z + 1 \in \mu_{p^n} \setminus \mu_{p^{n-1}}$ for some $n \in \mathbb{N}$. In case $z = 0$ we get the minimal polynomial $P_0(T) = T$ for z and in case $z + 1 \in \mu_{p^n} \setminus \mu_{p^{n-1}}$, we get the minimal polynomial $P_n(T) = (((1+T)^{p^n} - 1)/((1+T)^{p^{n-1}} - 1))$. Note that for $q' = \varphi(T)/T$ we get that $\varphi^{n-1}(q') = \varphi^n(T)/(\varphi^{n-1}(T)) = P_n(T)$ for $n \geq 1$. So we get that

$$f(T)u(T) = (\prod_{n=0}^{\infty} (1/p) P_i(T)^{j_n}) = T^{j_0} (\prod_{n=1}^{\infty} (\varphi^{n-1}(q')/p)^{j_n})$$

for some $u(T) \in (\mathcal{R}^L_{\mathbb{Q}_p})^\times$. This product converges since we have $\frac{1}{p} P_n(T) = \frac{1}{p}(1 + (1+T)^{p^n} + (1+T)^{2p^n} + \dots (1+T)^{(p-1)p^n})$ and we have for $k \in \{0, \dots, p-1\}$ that $|(1+T)^{kp^n}|$ tends to 1 for large n and hence so does $|(1/p)P_n(T)|$. Note that the right hand side of the equation has only coefficients in $\mathbb{Q}_p$, hence $(\pi_{\mathbb{Q}_p})^{j_0} (\prod_{n=1}^{\infty} (\varphi^{n-1}(q)/p)^{j_n}) \otimes 1$ is a generator for I, where the j_n's denote the multiplicity of the root $\varepsilon^{(n)} - 1$ in $f(T)$.

The fact that the j_n's are decreasing in case $\varphi(I) \subseteq I$, follows immediately from the fact that $\varphi(T) = Tq'$ and $\varphi(\varphi^{n-1}(q')) = \varphi^n(q')$.

$\square$

For the next lemma we first need to discuss some special properties of the Robba ring $\mathcal{R}^L_{\mathbb{Q}_p}$. We know that the Robba ring can be viewed as a module over itself via φ, more specifically $\mathcal{R}^L_{\mathbb{Q}_p} \cong \oplus_{i=0}^{p-1} b_i \varphi(\mathcal{R}^L_{\mathbb{Q}_p})$ for some $b_i \in \mathcal{R}^L_{\mathbb{Q}_p}$.

Lemma 3.3. *Let S be a torsion (φ, Γ)-module over $\mathcal{R}^L_{\mathbb{Q}_p}$. Then S is a successive extension of pure t-torsion (φ, Γ)-modules.*

Proof. By a result of Lazard in [9] we know that $\mathcal{R}^L_{\mathbb{Q}_p}$ is an adequate ring and hence allows a theory of elementary divisors (see [7, Theorem 3]), so there exists a set $\{e_1, \dots, e_d\} \subseteq S$ and chain of unique principal ideals $(r_1) \subset (r_2) \subset \cdots \subset (r_d)$ in $\mathcal{R}^L_{\mathbb{Q}_p}$, such that $S = \oplus_{i=1}^d \mathcal{R}^L_{\mathbb{Q}_p} e_i$, and $\mathrm{Ann}(e_i) = (r_i)$ for all i. Then for any $\gamma \in \Gamma_{\mathbb{Q}_p}$ we have $\oplus_{i=1}^d \mathcal{R}^L_{\mathbb{Q}_p} e_i = S = \gamma(S) = \oplus_{i=1}^d \mathcal{R}^L_{\mathbb{Q}_p} \gamma(e_i)$ and by the uniqueness of the r_i's we get that $(\gamma(r_i)) = (r_i)$, so (r_i) is $\Gamma_{\mathbb{Q}_p}$-invariant for all i.

Next we claim that the ideals (r_i) are stable under φ. We know that

$$\begin{aligned} S \cong \varphi^* S &= \mathcal{R}^L_{\mathbb{Q}_p} \otimes_{\varphi, \mathcal{R}^L_{\mathbb{Q}_p}} S \\ &= (\oplus_{j=0}^{p-1} \varphi(\mathcal{R}^L_{\mathbb{Q}_p}) b_j) \otimes_{\varphi, \mathcal{R}^L_{\mathbb{Q}_p}} S \\ &= \oplus_j (\varphi(\mathcal{R}^L_{\mathbb{Q}_p}) b_j \otimes_{\varphi, \mathcal{R}^L_{\mathbb{Q}_p}} S) \\ &= \oplus_j b_j \otimes_{\varphi, \mathcal{R}^L_{\mathbb{Q}_p}} S \\ &= \oplus_j b_j \otimes_{\varphi, \mathcal{R}^L_{\mathbb{Q}_p}} \oplus_{i=1}^d \mathcal{R}^L_{\mathbb{Q}_p} e_i \\ &= \oplus_{i,j} b_j \varphi(\mathcal{R}^L_{\mathbb{Q}_p}) \varphi(e_i) \\ &= \oplus_i \mathcal{R}^L_{\mathbb{Q}_p} \varphi(e_i). \end{aligned}$$

Now by the uniqueness property of the (r_i)'s we get that $\varphi((r_i)) \subseteq \mathrm{Ann}(\varphi(e_i)) = (r_i)$ for all i and so the ideals (r_i) are stable under φ. By the previous lemma we know that the ideals (r_i) are of the form $(r_i) = (\pi_{\mathbb{Q}_p}^{j_{0,i}} \prod_{n=1}^{\infty} (\varphi^{n-1}(q)/p)^{j_{n,i}} \otimes 1)$ and $j_{n,i} \in \mathbb{N}$ for all n, where $(j_{n,i})_n$ is a decreasing sequence. So the sequence will eventually become constant, let k_i denote this constant. We have that $t = \log(\pi_{\mathbb{Q}_p} + 1) = \pi_{\mathbb{Q}_p} \prod_{n=1}^{\infty} (\varphi^{n-1}(q)/p)$ and so we have that $(t^{k_i} \otimes 1) u = (\pi_{\mathbb{Q}_p}^{j_{0,i}} \prod_{n=1}^{\infty} (\varphi^{n-1}(q)/p)^{j_{n,i}} \otimes 1)$, where u is a finite product of $\varphi^{n-1}(q)/p$ and hence is a unit in $\mathcal{R}^L_{\mathbb{Q}_p}$. Therefore $(t^{k_i} \otimes 1) = (r_i)$.

By the chain property of the (r_i) we get that the k_i are decreasing, hence we then have $(t^{k_1} \otimes 1) S = 0$ and hence $0 = (t^{k_1} \otimes 1) S \subseteq (t^{k_1 - 1} \otimes 1) S \subseteq \cdots \subseteq S$ is a filtration of S, where all the subquotients are t-torsion.

□

For $K = \mathbb{Q}_p$, take a (φ, Γ)-module S, which is of pure t^k-torsion, and let $d = \mathrm{rank}_{\mathcal{R}^L_{\mathbb{Q}_p}/(t^k \otimes 1)} S$ and let $\{e_1, \dots e_d\}$ be a basis of S. Let A be the transformation matrix for φ in this basis. Since we have $\varphi^* S \cong S$, there is a matrix B such that $AB = BA = 1_d$. And since Γ is topologically finitely generated we can find an r_0, such that the entries of A,B and all the transformation matrices of the Γ-operation have entries in $B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1)$. For any $r \geq r_0$ we set S_r to be the $B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1)$-submodule of S, which is spanned by $\{e_1 \dots e_d\}$. We get a Γ-action on S_r and the map $\varphi : S_r \to S_{pr}$ induces an isomorphism $1 \otimes \varphi : B^{\dagger,pr,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1) \otimes_{B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k\otimes 1)} S_r \to S_{pr}$.

Lemma 3.4. *For $r \geq p-1$ and $n(r)$ the smallest $N \in \mathbb{N}$ such that satifies $(p-1)p^{N-1}r \geq 1$ the following holds*

(i) For $n \geq n(r)$ the maps $B^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1) \to B^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k) \otimes 1)$ induce an isomorphism

$$B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1) \to \prod_{n \geq n(r)}^{\infty} B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k) \otimes 1).$$

(ii) For $n \geq n(r)$ and $l \in L$ mapping $\pi \otimes l$ to $(\varepsilon^{(n)} e^{t/p^n} - 1) \otimes l$ induces a $\Gamma_{\mathbb{Q}_p}$-equivariant isomorphism

$$B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k) \otimes 1) \to (L(\varepsilon^{(n)})[t])/(t^k).$$

(iii) For $r' \geq r$ and $n \geq n(r')$ using the isomorphism (ii), we get that the natural inclusion $B^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p} \to B^{\dagger,r'}_{\mathrm{rig},\mathbb{Q}_p}$ induces the following commutative diagramm

$$\begin{array}{ccc} B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k) \otimes 1) & \xrightarrow{\mathrm{id}} & B^{\dagger,r',L}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k) \otimes 1) \\ \downarrow & & \downarrow \\ L(\varepsilon^{(n)})[t]/(t^k) & \xrightarrow{\mathrm{id}} & L(\varepsilon^{(n)})[t]/(t^k) \end{array}$$

(iv) We can describe $\varphi : B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1) \to B^{\dagger,pr,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1)$ via the isomorphism (i) by setting $\varphi((x_n)_{n \geq n(r)}) = (y_n)_{n \geq n(r)+1}$ and $y_{n+1} = x_n$ for $n \geq n(r)$ for any $(x_n)_{n \geq n(r)} \in \prod_{n \geq n(r)} B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k) \otimes 1)$.

Proof. Since $(t^k \otimes 1) = (t^k) \otimes L$ we get that $B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1) \cong B^{\dagger,r,}_{\mathrm{rig},\mathbb{Q}_p}/(t^k) \otimes_{\mathbb{Q}_p} L$ and since $(\varphi^n(q^k) \otimes 1) = (\varphi^n(q^k)) \otimes L$ we get that $B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k) \otimes 1) \cong B^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k)) \otimes_{\mathbb{Q}_p} L$ and hence the proof is the same as in [4, Prop. 3.15]. □

Let S be a pure t^k-torsion module over $\mathcal{R}^L_{\mathbb{Q}_p}$, such that $d = \mathrm{rank}_{\mathcal{R}^L_{\mathbb{Q}_p}/(t^k \otimes 1)} S$, then by the previous lemma we can embed $L(\varepsilon^{(n)})[t]/(t^k)$ in $B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}/(t^k \otimes 1)$ and hence we can set $S^n = S_r \otimes_{L(\varepsilon^{(n)})[t]} L(\varepsilon^{(n)})[t]/(t^k)$, which is a free $L(\varepsilon^{(n)})[t]/(t^k)$-module of rank d, with a $\Gamma_{\mathbb{Q}_p}$-action. By the previous

lemma we get that the image φS^n lies in S^{n+1} and furthermore we get that φ is injective on S^n. This however implies that
$1 \otimes \varphi : L(\varepsilon^{(n)})[t]/(t^k) \otimes_{L(\varepsilon^{(n)})[t]/(t^k)} S^n \cong S^{n+1}$. Hence we can view S^n as a submodule of S^{n+1}. This gives us the following theorem.

Theorem 3.5. *(i) With the above notation we get that for $n \geq n(r)$ the natural map $S_r \to S^n$ induces an isomorphism*

$$S_r \cong \prod_{n \geq n(r)}^{\infty} S^n$$

as $\Gamma_{\mathbb{Q}_p}$-modules over $L(\varepsilon^{(n(r))})[t]/(t^k)$.

(ii) With the isomorphism of (i) and $r \geq p-1$, we can deduce that for $r' \geq r$ there exists a natural map

$$\begin{aligned} S_r &\longrightarrow S_{r'} \\ (x_n)_{n \geq n(r)} &\longmapsto (x_n)_{n \geq n(r')}. \end{aligned}$$

(iii) The map φ on S_r is given by

$$\begin{aligned} S_r &\longrightarrow S_{pr} \\ (x_n)_{n \geq n(r)} &\longmapsto (y_n)_{n \geq n(r)+1}, \end{aligned}$$

where $y_{n+1} = x_n$ for $n \geq n(r)$.

Proof. As in Lemma 3.4, this theorem can be proven in the same way as in [10, Thm. 3.3]. □

Let E/K be a finite extension of fields. Note that we can extend Shapiro's lemma to generalised (φ, Γ)-modules. This can be done by defining the induced $\mathcal{R}_K^L$-module $\mathrm{Ind}_E^K D$ of a generalised (φ, Γ)-module D over $\mathcal{R}_E^L$ in the same way we did in Section 2.4. The lemma for generalised (φ, Γ)-modules can be proven in the same way, so we get:

Theorem 3.6 (Shapiro's lemma for generalised (φ, Γ)-modules)**.** *Let D be a generalised (φ, Γ)-module over $\mathcal{R}_E^L$, then there exist isomorphisms*

$$H^i(D) \cong H^i(\mathrm{Ind}_E^K D)$$

for $i = 0, 1, 2$, which are functorial and comaptible with cup products.

Let $\eta : \mathbb{Z}_p^\times \to \mathcal{O}_E^\times$ be a character of finite order. The conductor of η is given by $p^{N(\eta)}$, where $N(\eta) = 0$ if $\eta = 1$, else $N(\eta)$ is the smallest $n \in \mathbb{N}$ such that η is trivial on $1 + p^n\mathbb{Z}_p$. Next for any $N \in \mathbb{N}$ take ζ_p^N to be a primitive

p^N-th root of unity, such that $(\zeta_{p^{N+1}})^p = \zeta_{p^N}$ for all $N \in \mathbb{N}$. We can then define the Gauss sum $G(\eta)$ by setting $G(\eta) = 1$, if $\eta = 1$ and else

$$G(\eta) = \sum_{x \in (\mathbb{Z}/p^{n(\eta)}\mathbb{Z})^\times} \eta(x)\zeta_{p^{N(\eta)}}^x \in (E(\mu_{p^{N(\eta)}}))^\times$$

Lemma 3.7. *For any $k \in \mathbb{N}$, we have*

(i) For any $\eta : \mathbb{Z}_p^\times \to \mathcal{O}_E$ a character of finite order and $0 \leq i \leq k-1$ we have that

$$g(G(\eta)t^i) = (\eta^{-1}\chi^i)(g) \cdot (G(\eta)t^i)$$

for any $g \in \Gamma$.

(ii) For any $n \in \mathbb{N}$ we have that

$$E(\varepsilon^{(n)})[t]/(t^k) = \oplus_{\eta, N(\eta) \leq n} \oplus_{0 \leq i \leq k-1} E \cdot G(\eta)t^i.$$

Proof. See [4, Prop. 3.13] for the proof. □

With all the statements from above, we can proof the next theorem, which states how the cohomology of a torsion (φ, Γ)-module over R behaves and is crucial for the proof of Euler-Poincaré formula and Tate duality.

Theorem 3.8 (Euler-Poincaré formula for torsion (φ, Γ)-modules)**.** *Let S be a torsion (φ, Γ)-module over $\mathcal{R}_K^L$. Then we have*

(i) $\dim_L H^0(S) = \dim_L H^1(S) < \infty$

(ii) $\varphi - 1$ *is surjective on S and hence $H^2(S) = 0$.*

Proof. We can use Shapiro's lemma to reduce to the case that $K = \mathbb{Q}_p$.

Next we claim that if we have an exact sequence of torsion modules over $\mathcal{R}_{\mathbb{Q}_p}^L$, say

$$0 \to S' \to S \to S'' \to 0$$

and the theorem holds for both S' and S'', it must also hold for S.

We consider the long exact sequence of cohomology and from that we immediately conclude that (ii) holds for S. Again from the long exact sequence of cohomology we get that $\dim_L H^i(S) < \infty$ and we get that
$\dim_L H^0(S') + \dim_L H^0(S'') + \dim_L H^1(S) = \dim_L H^0(S) + \dim_L H^1(S')$
$+ \dim_L H^1(S'')$. Then since (i) holds for S' and S'' we get that
$\dim_L H^0(S) = \dim_L H^1(S)$, so (i) holds for S. Then by Lemma 3.3 we can assume that S is of pure t^k-torsion. We now show that (ii) holds for S.

We claim that the map $\varphi - 1 : S_r \to S_{pr}$ is surjective for $r \geq p-1$. From this we can conclude that S satisfies (ii), since $S = \bigcup_r S_r$. Recall that $S_r = \prod_{n \geq n(r)}^\infty S^n$. Take $(y_n)_{n \geq n(r)+1} \in S_{pr}$ and set $x_n = -\sum_{i=n(r)}^n y_i$

(set $y_{n(r)} = 0$). Then $(x_n)_{n \geq n(r)} \in S_r$ and $(\varphi - 1)((x_n)_{n \geq n(r)}) = (y_n)_{n \geq n(r)+1}$, which proves the surjectivity of $\varphi - 1$ on S.

We still need to show that (i) holds. We start by computing the group $H^0(S)$. For this set $(S_r)' = (S_r)^{\Delta_{\mathbb{Q}_p}}$ and $(S^n)' = (S^n)^{\Delta_{\mathbb{Q}_p}}$, where $\Delta_{\mathbb{Q}_p}$ is a torsion subgroup of $\Gamma_{\mathbb{Q}_p}$. Then we have $(S_r)' = \prod_{n \geq n(r)}^{\infty} (S^n)'$, since the $\Gamma_{\mathbb{Q}_p}$-action acts componentwise on the right hand side. Let $a \in S$ then there is a representation $a = (a_n)_{n \geq n(r)} \in (S_r)'$ for some r. We then have $a = 0$ if and only if $a_n = 0$ for almost all n by Thm. 3.5. Now take $a \in H^0(S)$ with $a = (a_n)_{n \geq n(r)} \in (S_r)'$ for some r, such that $(\varphi - 1)a = 0$, then by Thm. 3.4, this implies that a_n becomes constant for n large enough.

Therefore, since elements of $H^0(S)$ are $\Gamma_{\mathbb{Q}_p}$-invariant, we can write

$$H^0(S) = \varinjlim_{n \to \infty} ((S^n)')^{\Gamma_{\mathbb{Q}_p}/\Delta_{\mathbb{Q}_p}} = \varinjlim_{n \to \infty} (S^n)^{\Gamma_{\mathbb{Q}_p}}.$$

Next we will compute $H^1(S)$. For this let $(a, b) \in Z^1(S)$. We now claim that we can assume $b = 0$. Since we know $(\varphi - 1)$ is surjective on S, there exists a $c \in S$ such that $(\varphi - 1)c = b$, but then we have that $(a, b) = (a - (\gamma_{\mathbb{Q}_p} - 1)c, b - (\varphi - 1)c) = (a - (\gamma_{\mathbb{Q}_p} - 1)c, 0)$, hence we can assume that $b = 0$.

We then take for some r a representative $(a_n)_{n \geq n(r)} \in (S_r)'$ of a. Then we have $(\varphi - 1)a = 0$ and this means that the a_n become constant for n large enough. We also have that $(a, 0) \in B^1(S)$ if and only if $a_n \in (\gamma_K - 1)(S^n)'$ for $n \geq n_0$. This implies that

$$H^1(S) = \varinjlim_{n \to \infty} (S^n)'/(\gamma_{\mathbb{Q}_p} - 1).$$

Now since $(S^n)'$ is a finitedimensional L-vector space, we get that $\dim_L (S^n)^{\Gamma_{\mathbb{Q}_p}} = \dim_L (S^n)'/(\gamma_{\mathbb{Q}_p} - 1)$. Recall that we can view S^n as a submodule of S^{n+1} and hence we also have an injection $(S^n)^{\Gamma_{\mathbb{Q}_p}} \hookrightarrow (S^{n+1})^{\Gamma_{\mathbb{Q}_p}}$.

To conclude that $\dim_L H^0(S) = \dim_L H^1(S)$ holds, it now suffices to show that $(S^n)'/(\gamma_{\mathbb{Q}_p} - 1) \to (S^{n+1})'/(\gamma_{\mathbb{Q}_p} - 1)$ is an injection for all n.

From Lemma 3.7 we know that $L(\varepsilon^{(n)})[t]/(t^k)$ is a direct summand of $L(\varepsilon^{(n+1)})[t]/(t^k)$ as $\Gamma_{\mathbb{Q}_p}$-modules. This implies that also S^n is a direct summand of S^{n+1} and hence $(S^n)'$ is a direct summand of $(S^{n+1})'$, so the natural injection induces an injection $(S^n)'/(\gamma_{\mathbb{Q}_p} - 1) \hookrightarrow (S^{n+1})'/(\gamma_{\mathbb{Q}_p} - 1)$.

We now still need to show that $\dim_L H^0(S) < \infty$. For this we will show that $\dim_L (S^n)^{\Gamma_{\mathbb{Q}_p}}$ has an upper bound which does not depend on n. Take

$s \in \mathbb{N}$, then by Lemma 3.7, we get that

$$\begin{aligned}(S^{n+s})^{\Gamma_{\mathbb{Q}_p}} &= \Big(\bigoplus_{\eta, n < N(\eta) \le n+s} \bigoplus_{i=0}^{k-1} (L \cdot G(\eta) t^i) \otimes S^n \Big)^{\Gamma_{\mathbb{Q}_p}} \\ &= \bigoplus_{\eta, n < N(\eta) \le n+s} \bigoplus_{i=0}^{k-1} \Big((L \cdot G(\eta) t^i) \otimes S^n \Big)^{\Gamma_{\mathbb{Q}_p}}.\end{aligned}$$

Therefore we get that

$$\dim_L (S^{n+s})^{\Gamma_{\mathbb{Q}_p}} \le \sum_{\eta, n < N(\eta) \le n+s} \sum_{i=0}^{k-1} \dim_L ((L \cdot G(\eta) t^i) \otimes S^n)^{\Gamma_{\mathbb{Q}_p}} \le \dim_L S^n.$$

Here the last inequality follows from the fact that $((L \cdot G(\eta)t^i) \otimes S^n)^{\Gamma_{\mathbb{Q}_p}} = S^n(\eta^{-1}\chi^i)^{\Gamma_{\mathbb{Q}_p}} \subset S^n$. And since all the characters are distinct, we have that all $S^n(\eta^{-1}\chi^i)$ are disjoint subspaces of S^n.

Therefore $\dim_L (S^n)^{\Gamma_{\mathbb{Q}_p}}$ has an upper bound independant of n and therefore $\dim_L H^0(S)$ is finite.

$\square$

Remark. For any torsion (φ, Γ)-module S over $\mathcal{R}_K^L$ Thm 3.8 proves that $\chi(S) = 0$ and $\dim_{\mathbb{Q}_p} H^i(S) < \infty$ for $i = 0, 1, 2$. Since we also have rank $S = 0$, we in fact know that the Euler-Poincaré formula holds for torsion (φ, Γ)-module over $\mathcal{R}_K^L$.

4 Euler-Poincaré formula

In this section we will prove the Euler-Poincaré formula for generalised (φ, Γ)-modules over the ring $\mathcal{R}_K^L$.

In case D is a generalised (φ, Γ)-module over $\mathcal{R}_K^L$ the Euler-Poincaré formula states that:

(i) $\dim_L H^i(D)$ is finite for all $0 \leq i \leq 2$

(ii) $\chi(D) = \sum_{i=0}^{2} \dim_L H^i(D) = -[K : \mathbb{Q}_p]\text{rank } D$

Lemma 4.1. *Let D be a (φ, Γ)-module over $\mathcal{R}_K^L$. Then for $i = 0, 1, 2$, we have $\dim_L H^i(D) < \infty$ if and only if $\dim_L H^i(D(x)) < \infty$. And also if $H^i(D)$ is finite dimensional for all i, then $\chi(D) = \chi(D(x))$.*

Proof. If we compare the φ- and Γ-operation of the module tD and $D(x)$, we see that they coincide. To see this, we can for any $d \in D$ calculate $\varphi(td) = p\varphi(d)$ and $\varphi(d \otimes v) = p\varphi(d) \otimes v$ as well as $\gamma(td) = \chi(\gamma)\gamma(d)$ and $\gamma(d \otimes v) = \chi(\gamma)\gamma(d) \otimes v$ for any $\gamma \in \Gamma$. Hence there exists a natural isomorphism between tD and $D(x)$ by mapping td to $d \otimes v$ for any $d \in D$. Since D/tD is a torsion module we get by Thm. 3.8 that $\dim_L H^i(D/D(x))$ is finite for all i and for $i = 2$ one has $H^2(D/D(x)) = 0$. We consider the following short exact sequence

$$0 \longrightarrow D(x) \longrightarrow D \longrightarrow D/D(x) \longrightarrow 0$$

and with it the long exact sequence of cohomology

$$0 \longrightarrow H^0(D(x)) \longrightarrow H^0(D) \longrightarrow H^0(D/D(x)) \longrightarrow \cdots \longrightarrow H^2(D) \longrightarrow 0$$

We can now see for $0 \leq i \leq 2$ that if either $H^i(D)$ or $H^i(D(x))$ are finitedimensional over L, then the other one also has to be finitedimensional. From the long exact sequence of cohomology we get that

$$\dim_L H^0(D(x)) + \dim_L H^0(D/D(x)) + \dim_L H^1(D) + \dim_L H^2(D(x))$$
$$= \dim_L H^0(D) + \dim_L H^1(D(x)) + \dim_L H^1(D/D(x)) + \dim_L H^2(D).$$

Since by Thm. 3.8 we have that $\dim_L H^0(D/D(x)) = \dim_L H^1(D/D(x))$, we conclude that $\chi(D) = \chi(D(x))$.

□

Remark. Note that if $0 \to D_1 \to D \to D_2 \to 0$ is an exact sequence of free (φ, Γ)-modules over R. Then we have $\text{rank}D_1 + \text{rank}D_2 = \text{rank}(D)$ and the long exact sequence of cohomology shows that also $\chi(D_1) + \chi(D_2) = \chi(D)$, if the cohomolgy of these modules is finitedimensional.

Next we will construct pure (φ, Γ)-modules of arbitrary slopes, which can be obtained through succesive extensions of $\mathcal{R}^L_{\mathbb{Q}_p}(x^i)$.

Lemma 4.2. *There is a (φ, Γ)-module E over $\mathcal{R}^L_{\mathbb{Q}_p}$ of rank d such that E is pure of slope $1/d$ and which is a successive extension of $\mathcal{R}^L_{\mathbb{Q}_p}(x^i)$ with $i = 0, 1$.*

Proof. We will proof this by induction on the rank d. For $d = 1$ set $E = \mathcal{R}^L_{\mathbb{Q}_p}(x)$. The module E is pure, since all rank 1 modules are pure by Lemma 1.4(iv) and we have $\mu(E) = 1$. Now for $d \geq 2$ assume we can find a module E_0 of rank $d-1$, which is pure of slope $1/(d-1)$ and is a successive extension of $\mathcal{R}^L_{\mathbb{Q}_p}(x^i)$. We claim there is a nontrivial extension of $\mathcal{R}^L_{\mathbb{Q}_p}$ by E_0. To see this we first note that $\mathcal{R}^L_{\mathbb{Q}_p}$ is étale and hence by Thm. 2.9 $\chi(\mathcal{R}^L_{\mathbb{Q}_p}) = -1$. By Lemma 4.1 we also have that $\chi(\mathcal{R}^L_{\mathbb{Q}_p}) = \chi(\mathcal{R}^L_{\mathbb{Q}_p}(x))$ and since E_0 is a successive extension of $\mathcal{R}^L_{\mathbb{Q}_p}(x^i)$ we get that $\chi(E_0) = (\text{rank} E_0)(\chi(\mathcal{R}^L_{\mathbb{Q}_p}))$.

Therefore we have

$$\dim_L H^1(E_0) \geq -\chi(E_0) = (\text{rank} E_0)(\chi(\mathcal{R}^L_{\mathbb{Q}_p})) = (d-1) \geq 1.$$

And hence there exists a nontrivial extension E of $\mathcal{R}^L_{\mathbb{Q}_p}$ by E_0, i.e.

$$0 \longrightarrow E_0 \longrightarrow E \xrightarrow{\alpha} \mathcal{R}^L_{\mathbb{Q}_p} \longrightarrow 0$$

We now claim that E is the module we want. By Lemma 1.4 (i) we have that $\deg(E) = \deg(E_0) + \deg(\mathcal{R}^L_{\mathbb{Q}_p}) = \deg(E_0)$ and hence $\mu(E) = 1/d$. So it suffices to show that E is pure. For this we now assume that E is not semistable, that means there is a submodule $P \subset E$ such that $\mu(P) < 1/d$. Then since $\text{rank}(P) \leq d$ we have $\deg(P) \leq 0$, since the deg is $\mathbb{Z}$-valued. Therefore $\mu(P) \leq 0$. Since E_0 is pure of positive slope we have that $P \cap E_0 = \{0\}$. And therefore $P \subset E \cap \ker(\alpha) = \{0\}$ and hence $\alpha|_P$ is injective. So we can view P as a submodule of $\mathcal{R}^L_{\mathbb{Q}_p}$. Then by Lemma 1.4 (iv) we get that $\mu(P) \geq 0$ and $\mu(P) = 0$ if and only if $P \cong \mathcal{R}^L_{\mathbb{Q}_p}$. But we have seen earlier $\mu(P) \leq 0$, so this implies $\mu(P) = 0$ and hence $P \cong \mathcal{R}^L_{\mathbb{Q}_p}$. But this implies that the extension E is trivial, which is a contradiction. So we can assume that E is semistable and hence also pure. Clearly E is a successive extension of $\mathcal{R}^L_{\mathbb{Q}_p}(x^i)$ and this concludes the induction step. □

Remark. For any $q = c/d \in \mathbb{Q}$ this theorem allows us to find a pure (φ, Γ)-module of slope $\mu(D) = c/d$, which is a successive extension of $\mathcal{R}^L_{\mathbb{Q}_p}(x^i)$ where the i's can be taken from $\mathbb{Z}$. This can be done by taking $D = (\otimes^s_{j=1} E)(x^k)$, where $c/d = k + s/d$ and $k \in \mathbb{Z}$ and $0 \leq s < d$. Here E is the module constructed in Lemma 4.2 with slope $1/d$. Then D is in fact pure since the tensor product of pure modules is again pure.

Before we can prove the Euler-Poinare formula, we need to define the localization at $\varepsilon^{(n)} - 1$ of a generalized (φ, Γ)-module over $\mathcal{R}^L_{\mathbb{Q}_p}$. Note that there is an injective morphism $\iota_n : B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p} \to L(\varepsilon^{(n)})[[t]]$ for $n \geq n(r)$, which is constructed in [1, Chapter 2]. The Γ- and φ-operation of $L(\varepsilon^{(n)})[[t]] = \mathbb{Q}_p(\varepsilon^{(n)})[[t]] \otimes_{\mathbb{Q}_p} L$ act trivially on L. This map can be viewed as evaluating the function at $\varepsilon^{(n)} - 1$. We can then define $D^{+,n}_{\mathrm{dif}}(D) = D_r \otimes_{B^{\dagger,r,L}_{\mathrm{rig},\mathbb{Q}_p}} L(\varepsilon^{(n)}[[t]]$, the localization at $\varepsilon^{(n)} - 1$ of D.

Theorem 4.3 (Euler-Poincaré formula)**.** *Let D be a generalized (φ, Γ)-module over R, then the Euler-Poincaré formula holds for D.*

Proof. We can use Shapiro's lemma to reduce to the case $K = \mathbb{Q}_p$ since $\mathrm{rank}_{\mathcal{R}^L_K} D = [K : \mathbb{Q}_p] \mathrm{rank}_{\mathcal{R}^L_{\mathbb{Q}_p}} \mathrm{Ind}^{\mathbb{Q}_p}_K D$. Next we can reduce to the case that D is a (φ, Γ)-module. For this suppose that S is the torsion submodule of D. We then have that D/S is a (φ, Γ)-module. We assume now that the Euler-Poincaré fomula holds for D/S. Consider the short exact sequence

$$0 \longrightarrow S \longrightarrow D \longrightarrow D/S \longrightarrow 0.$$

Since $H^i(S)$ is finitedimensional for $0 \leq i \leq 2$ by Thm. 3.8 and $H^i(D/S)$ are finitedimensional by our assumption above, we get from the long exact sequence of cohomology that $\dim_L H^i(D)$ is also finite for all $0 \leq i \leq 2$. So (i) holds. For (ii) note that by Thm. 3.8, we have that $H^2(S) = 0$, hence by the long exact sequence of cohomology we immediately get that $\dim_L H^2(D) = \dim_L H^2(D/S)$ and so the remaining sequence is as follows

$$0 \longrightarrow H^0(S) \longrightarrow H^0(D) \longrightarrow H^0(D/S) \longrightarrow \cdots \longrightarrow H^1(D/S) \longrightarrow 0$$

Now using the fact that from Thm. 3.8 we get $\dim_L H^0(S) = \dim_L H^1(S)$ and hence by the long exact sequence of cohomology we have $\dim_L H^0(D) - \dim_L H^1(D) = \dim_L H^0(D/S) - \dim_L H^1(D/S)$. And therefore we get that $\chi(D) = \chi(D/S)$. And since D and D/S have the same rank (ii) also holds for D. Therefore we can now assume that D is a (φ, Γ)-module.

Let $d = \mathrm{rank}\ D$. We will now proof that $\dim_L H^0(D) \leq d$.

Now for an r large enough we know that D_r exists and we have a natural injection $\alpha : D_r \hookrightarrow D^{+,n}_{\mathrm{dif}}(D)[1/t]$, where $n \geq n(r)$. We first claim that $\dim_L (D^{+,n}_{\mathrm{dif}}(D)[1/t])^\Gamma \leq d$. We will prove this by contradiction, so we assume that there are $e_1, \dots e_{d+1} \in (D^{+,n}_{\mathrm{dif}}(D)[1/t])^\Gamma$, which are linearly independent over L.

Since we have $\mathrm{rank}\ D_r = d$, we get that $D^{+,n}_{\mathrm{dif}}(D)[1/t]$ is a d-dimensional $L(\varepsilon^{(n)})((t))$-vectorspace. And therefore $e_1, \dots e_{d+1}$ are linearly dependent in

$D_{\mathrm{dif}}^{+,n}(D)[1/t]$ over $L(\varepsilon^{(n)})((t))$. We now choose a minimal linear dependent subset of $\{e_1, \ldots, e_{d+1}\}$. We may assume $\{e_1, \ldots, e_k\}$ is such a subset. We then find $a_i \in L(\varepsilon^{(n)})((t))$ such that $\sum_{i=1}^{k} a_i e_i = 0$. By minimality we may assume that $a_1 \neq 0$ and hence we get

$$e_1 + \sum_{i=1}^{k} (a_i/a_1) e_i = 0.$$

Because the e_i are Γ-invariant applying any $\gamma \in \Gamma$ to the above equation gives us

$$e_1 + \sum_{i=1}^{k} \gamma(a_i/a_1) e_i = 0.$$

By minimality again we get that $\gamma(a_i/a_1) = a_i/a_1$ for all i and all $\gamma \in \Gamma$. And therefore $a_i/a_1 \in (L(\varepsilon^{(n)})((t)))^{\Gamma} = L$, since both t and $\varepsilon^{(n)}$ are not Γ-invariant. This however would imply that $e_1, \ldots e_k$ are linearly dependent over L, which is a contradiction, so $\dim_L (D_{\mathrm{dif}}^{+,n}(D)[1/t])^{\Gamma} \leq d$. Since α is an injection we get that $\dim_L (D_r)^{\Gamma} \leq d$ for any r, this means we also get $\dim_L H^0(D) \leq \dim_L D^{\Gamma} \leq d < \infty$.

We will now proceed the proof by induction on $\mathrm{rank}(D)$. For $d \geq 1$, assume now that the theorem holds for all (φ, Γ)-modules of rank less than d and let $d = \mathrm{rank}(D)$. For the rest we need to note that by the long exact sequence of cohomology we get that the Euler-Poincaré formula is preserved by extensions. Then if the slope filtration for D of Thm 1.5 is non-trivial, we are done by our induction hyptothesis. Note that since the slope filtration is unique, we get that the slope filtration of D is trivial if and only if D is pure. Hence we can now assume D is pure. Assume that $\mu(D) = c/d$ then by the remark after Lemma 4.2 we can find a pure (φ, Γ)-module F with slope $\mu(F) = -c/d$, which is a successive extension of $\mathcal{R}_{\mathbb{Q}_p}^{L}(x^i)$. Then the module $D \otimes F$ is étale and by usual Euler-Poincaré formula, we get that $\chi(D \otimes F) = -\mathrm{rank}(D \otimes F) = -(\mathrm{rank}\ D)\ (\mathrm{rank}\ F)$.

Also by the construction of F we get that $D \otimes F$ is a successive extension of $D(x^i)$ and hence there is a $j \in \mathbb{Z}$ such that

$$0 \longrightarrow D(x^j) \longrightarrow D \otimes F \longrightarrow G \longrightarrow 0$$

for some quotient G, which is itself obtained through successive extensions of $D(x^i)$. Note that both $D(x^j)$ and G are free (φ, Γ)-modules, so by the above we have that both $H^0(G)$ and $H^0(D(x^j))$ are finite dimensional over

L. Now we get the following long exact sequence of cohomology

$$\cdots \longrightarrow H^0(G) \longrightarrow H^1(D(x^j)) \longrightarrow H^1(D \otimes F) \longrightarrow H^1(G)$$

$$\longrightarrow H^2(D(x^j)) \longrightarrow H^2(D \otimes F) \longrightarrow \cdots$$

By usual Euler-Poincaré forumla we have $\dim_L H^1(D \otimes F) < \infty$ and since $H^0(G)$ is also finitedimensional we get that $\dim_L H^1(D(x^j)) < \infty$. So by repeatedly applying Lemma 4.1 we get that $\dim_L H^1(D(x^i)) < \infty$ for all $i \in \mathbb{Z}$. So in particular we have that $H^1(D) = H^1(D(x^0))$ is finitedimensional over L, as well as $\dim_L H^1(G) < \infty$, since it is obtained through successive extensions by $D(x^i)$. Since by usual Euler-Poincaré formula $H^2(D \otimes F)$ is also finitedimensional over L, we get from the above sequence that $\dim_L H^2(D(x^j)) < \infty$. And again by repeatedly applying Lemma 4.1 we get that $\dim_L H^2(D) < \infty$. And hence (i) holds.

To see (ii) note that by Lemma 4.1 $\chi(D(x^i)) = \chi(D)$ for all $i \in \mathbb{Z}$ and since $D \otimes F$ is obtained through successive extensions by $D(x^i)$, so by the the remark after Lemma 4.1 we get that $\chi(D \otimes F) = (\text{rank } F)\chi(D)$. Combining this with our earlier result, we get

$$-(\text{rank } F)(\text{rank } D) = \chi(D \otimes F) = (\text{rank } F)\chi(D)$$

and therefore

$$\chi(D) = -(\text{rank } D).$$

□

5 Tate duality

In this section we prove the Tate local duality theorem for (φ, Γ)-modules D over $\mathcal{R}_K^L$. Tate local dualtiy states that the cup product

$$H^i(D) \times H^{2-i}(D^\vee(\omega)) \longrightarrow H^2(D \otimes D^\vee(\omega)) \xrightarrow{\alpha} H^2(R(\omega)) \cong L$$

is a perfect pairing for any $0 \leq i \leq 2$. Here the map α is defined by $\alpha\big(x \otimes (f \otimes r)\big) = f(x) \otimes r$ for any $x \in D, f \in D^\vee, r \in \mathcal{R}_K^L(\omega)$.

This map is a perfect pairing if and only if the induced map $H^{2-i}(D^\vee(\omega)) \to H^i(D)^\vee$ is an isomorphism.

Lemma 5.1. *If for an exact sequence* $0 \to D' \to D \to D'' \to 0$ *of* (φ, Γ)*-modules over* $\mathcal{R}_K^L$ *the local Tate Duality is true for any two modules, it also is true for the third one.*

Proof. First we note that $0 \to D''^\vee(\omega) \to D^\vee(\omega) \to D'^\vee(\omega) \to 0$ is also an exact sequence. From the above exact sequences we can obtain the long exact sequence of cohomology and therefore the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & H^{2-i}(D''^\vee(\omega)) \arrow[r] \arrow[d] & H^{2-i}(D^\vee(\omega)) \arrow[r] \arrow[d] & H^{2-i}(D'^\vee(\omega)) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^i(D'')^\vee \arrow[r] & H^i(D)^\vee \arrow[r] & H^i(D')^\vee \arrow[r] & \dots
\end{tikzcd}$$

This diagram is commutative and hence by the Five lemma, we get that if Tate duality holds for two of these modules, it also holds for the third one. $\square$

Lemma 5.2. *Tate duality is true for* $\mathcal{R}_{\mathbb{Q}_p}^L(|x|)$

Proof. To see this we need to show that the maps $H^{2-i}(\mathcal{R}_{\mathbb{Q}_p}^L(|x|)^\vee(\omega)) \to H^i(\mathcal{R}_{\mathbb{Q}_p}^L(|x|))^\vee$ are isomorphisms for all i. Note that by Lemma 2.7 we have $\mathcal{R}_{\mathbb{Q}_p}^L(|x|)^\vee(\omega) = \mathcal{R}_{\mathbb{Q}_p}^L(x)$. We will show that for $i = 0, 2$ this map is trivial and show it for the case $i = 1$ by using the fact that Tate duality is known for étale (φ, Γ)-modules (see Thm 2.6) and using the Euler-Poincaré forumla.

From the Euler-Poincaré formula we get that $\dim_L H^1(\mathcal{R}_{\mathbb{Q}_p}^L(x|x|^{-1})) \geq \chi(\mathcal{R}_{\mathbb{Q}_p}^L(x|x|^{-1})) = 1$. Therefore there exists a nonsplit short exact sequence

$$0 \to \mathcal{R}_{\mathbb{Q}_p}^L(x) \to D \xrightarrow{\alpha} \mathcal{R}_{\mathbb{Q}_p}^L(|x|) \to 0.$$

One has $\deg(D) = \deg(\mathcal{R}_{\mathbb{Q}_p}^L(x)) + \deg(\mathcal{R}_{\mathbb{Q}_p}^L(|x|)) = 0$ by Lemma 1.4 (i). We will now show that D is in fact étale. For this we need to show that D

is semistable. Assume D is not semistable, then we can find a submodule $P \subset D$ with $\mu(P) < 0$. Then P must be of rank 1 by Lemma 1.4(iv) and therefore we get $\mu(P) \leq -1$, since the deg is $\mathbb{Z}$-valued. Since rank 1 modules are always semistable we get $P \cap \mathcal{R}^L_{\mathbb{Q}_p}(x) = \{0\}$. This however implies that the map α is injective when restricted to P, so we can view P as a submodule of $\mathcal{R}^L_{\mathbb{Q}_p}(|x|)$ and by Lemma 1.4 (iv) we have $\mu(P) \geq -1$. Combining these results we conclude $\mu(P) = -1 = \mu(\mathcal{R}^L_{\mathbb{Q}_p}(|x|))$. And hence $P \cong \mathcal{R}^L_{\mathbb{Q}_p}(|x|)$ again by Lemma 1.4 (iv). But this means $D = \mathcal{R}^L_{\mathbb{Q}_p}(x) \oplus P$, which is a contradiction since the sequence above is supposed to be nonsplit. So we can now assume that D is étale.

Recall from the remark preceding Lemma 2.7 that the cohomology groups $H^0(x)$ and $H^0(|x|)$ are trivial. Then by the long exact sequence of cohomology we get that $H^0(D)$ is trivial as well. We will now study the cohomolgy of the dual modules. Note that by Lemma 2.7 we have
$\mathcal{R}^L_{\mathbb{Q}_p}(x)^\vee \otimes \mathcal{R}^L_{\mathbb{Q}_p}(\omega) = \mathcal{R}^L_{\mathbb{Q}_p}(|x|)$ and $\mathcal{R}^L_{\mathbb{Q}_p}(|x|)^\vee \otimes \mathcal{R}^L_{\mathbb{Q}_p}(\omega) = \mathcal{R}^L_{\mathbb{Q}_p}(x)$. So taking the dual exact sequence of the sequence above and tensoring with $\mathcal{R}^L_{\mathbb{Q}_p}(\omega)$ gives us the following sequence:

$$0 \to \mathcal{R}^L_{\mathbb{Q}_p}(x) \to D^\vee(\omega) \to \mathcal{R}^L_{\mathbb{Q}_p}(|x|) \to 0.$$

Since we know D is étale we get from the ususal Tate Duality that
$H^0(D) \cong H^2(D^\vee(\omega)) = 0$. By the long exact sequence of cohomology of the dual sequence we get that also $H^2(|x|) = 0$. Since $H^0(|x|)$ is also trivial the Euler-Poincaré forumla gives us $\dim_L H^1(|x|) = 1$. The cup pairings now give us the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H^1(x) \arrow[r, "\delta_1"] \arrow[d, "\alpha"] & H^1(D^\vee(\omega)) \arrow[r, "\delta_2"] \arrow[d, "\beta"] & H^1(|x|) \arrow[r, "\delta_3"] \arrow[d] & H^2(x) \arrow[d] \\
H^2(x)^\vee \arrow[r] & H^1(|x|)^\vee \arrow[r] & H^1(D)^\vee \arrow[r] & H^1(x)^\vee \arrow[r] & 0
\end{tikzcd}$$

Since D is étale we get that the map $\beta : H^1(D^\vee(\omega)) \to H^1(D)^\vee$ is an isomorphism by Thm 2.9. Since both δ_1 and β are injective the map α is injective as well and therefore $\dim_L H^1(x) \leq 1$. By Euler-Poincaré formula we get $\chi(\mathcal{R}^L_{\mathbb{Q}_p}(x)) = -1$ and hence $\dim_L H^1(x) = 1$ and $H^2(x) = 0$. This gives us that α is in fact an isomorphism. Therefore the maps
$H^{2-i}(x) \to H^i(|x|)^\vee$ are isomorphisms for all i and the Tate duality holds for $\mathcal{R}^L_{\mathbb{Q}_p}(|x|)$; the map is trivial for $i = 0, 2$.

$\square$

Theorem 5.3 (Tate Duality). *For any (φ, Γ)-module D over $\mathcal{R}_K^L$ the cup product*

$$H^i(D) \times H^{2-i}(D^\vee(\omega)) \to H^2(R(\omega))$$

is a perfect pairing.

Proof. We can use Shapiro's lemma and Thm. 2.9 to reduce to the case $K = \mathbb{Q}_p$.

Applying the slope filtration theorem to D and using Lemma 5.1 it suffices to show the theorem for pure modules. So we conclude that D is pure of rank d. Since $\mu(D) = -\mu(D^\vee(\omega))$ (note that ω does not change the φ-action). We can now proof the theorem by induction on $s = \deg(D)$.

If $s = 0$, D is étale and the Tate duality follows from usual Tate duality (see Thm. 2.6).

Let $s > 0$ and assume the theorem holds for any pure (φ, Γ)-module $\widetilde{D}$ satisfying $0 \leq \deg(\widetilde{D}) < s$. Now by the Euler-Poincaré formula we have that $\dim_L H^1(D(|x|^{-1})) \geq -\chi(D(|x|^{-1}) \geq d \geq 1$. Therefore there exists a nontrivial extension $0 \to D(|x|^{-1}) \to \overline{E} \to \mathcal{R}_{\mathbb{Q}_p}^L \to 0$. Tensoring with $\mathcal{R}_{\mathbb{Q}_p}^L(|x|)$ gives us the exact sequence

$$0 \to D \to E \xrightarrow{\alpha} \mathcal{R}_{\mathbb{Q}_p}^L(|x|) \to 0.$$

Since $\deg(\mathcal{R}_{\mathbb{Q}_p}^L(|x|)) = -1$, we have that $\deg(E) = s - 1$ by Lemma 1.4 (i) and therefore $\mu(E) = \frac{s-1}{d+1} < \mu(D)$. Applying the slope filtration theorem to E gives us a chain $0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$ of pure, saturated (φ, Γ)-submodules, which satisfy $\mu(E_1) < \mu(E_2/E_1) < \ldots \mu(E/E_{l-1})$.
We claim that for all of these quotients we have that
$\deg(E_1), \deg(E_2/E_1), \ldots, \deg(E_l/E_{l-1}) < s$. We consider the exact sequence

$$0 \to E_1 \cap D \to E_1 \xrightarrow{\alpha} E_1/(E_1 \cap D) \to 0$$

and will show now that $\deg(E_1) \geq 0$.

Since D is pure of positive slope one has $\deg(E_1 \cap D) > 0$ or $E_1 \cap D = 0$. We can view $E_1/(E_1 \cap D)$ as a submodule of $\mathcal{R}_{\mathbb{Q}_p}^L(|x|)$ via the map α. Hence by Lemma 1.4 (iv) we have $\deg(E_1/(E_1 \cap D)) \geq -1 = \deg(\mathcal{R}_{\mathbb{Q}_p}^L(|x|))$, since $(\mathcal{R}_{\mathbb{Q}_p}^L(|x|))$ is pure. Then by Lemma 1.4 (i) and since the underlying valuation is discrete, we have that $\deg(E_1) = \deg(E_1 \cap D) + \deg(E_1/(E_1 \cap D)) \geq 0$, if $E_1 \cap D \neq 0$.

In case $E_1 \cap D = 0$ we get that $E_1 \cong E_1/(E_1 \cap D) \subseteq \mathcal{R}_{\mathbb{Q}_p}^L(|x|)$. But if $\deg(E_1) < 0$, we even get that $\deg(E_1) \leq -1$. But from the above we have that $\deg(E_1) \geq -1$, so $\deg(E_1) = \deg(\mathcal{R}_{\mathbb{Q}_p}^L(|x|))$ and hence by

Lemma 1.4 (iv) we get that $\mathcal{R}^L_{\mathbb{Q}_p}(|x|) \cong E_1$. But this implies $E = D \oplus E_1$ is nonsplit, which is a contradiction.

Hence we can assume $\deg(E_1) \geq 0$ and therefore $\mu(E_1) \geq 0$. By the slope filtration theorem we then also have $\mu(E_j/E_{j-1}) \geq 0$ for all j. So all summands in the sum $\sum_{j=1}^{l} \deg(E_j/E_{j-1}) = \deg(E) = s-1$ are nonnegative and therefore each summand satisfies $\deg(E_j/E_{j-1}) < s$. Hence the Tate dualtity is true for E_j/E_{j-1} for all j by the induction hypothesis. And then by Lemma 5.1 the Tate duality is true for E. Since by Lemma 5.2 Tate duality is true for $\mathcal{R}^L_{\mathbb{Q}_p}(|x|)$, again by theorem 5.1 we get that Tate duality holds for D, which finishes the induction.

□

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