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Completed cohomology and a two-dimensional Fontaine-Mazur conjecture

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The common theme of the Fontaine-Mazur conjectures is the idea that irreducible p-adic representations $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{Q}_p)$ of the absolute Galois group, which *look like* they come from algebraic geometry actually do come from algebraic geometry. Since their formulation in 1993, they have spawned a large body of work. While the case of one-dimensional representations is essentially proved via class field theory, higher dimensional cases are much more difficult to tackle. To study representations of higher dimension, it is natural (and has proved successful) to use modularity lifting results: Show that ρ actually comes from an automorphic representation of some $\mathbf{G}(\mathbb{A}^{\mathrm{fin}})$, the finite adelic points of a reductive group $\mathbf{G}$.

Major progress has been made by Colmez, Emerton and Kisin. The final result of this seminar is the following:

Theorem 1 (Fontaine-Mazur conjecture 3c for $n = 2$). *Let E be a finite extension of $\mathbb{Q}_p$ and let $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathrm{GL}_2(E)$ be a continuous, odd, absolutely irreducible representation that only ramifies at finitely many places and of which the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ is de Rham[1] of distinct Hodge-Tate weights. Then a twist of ρ by a character is modular.*

We will use a number of results as black boxes and concentrate our efforts on understanding the role Emerton's completed cohomology plays in the proof. At first glance it might seem as if completed cohomology is a technical and auxiliary construction, but according to Emerton it should be thought of as a suitable replacement for a space of p-adic automorphic forms — which so far is either undefined or not accessible by representation-theoretic methods. One of its technical key features is that completed cohomology carries a Hecke, Galois and $\mathbf{G}(\mathbb{Q}_p)$-action. It has for example been used to construct eigenvarieties and, as we will see in the course of this seminar, to realize certain parts of the p-adic local Langlands correspondence.

In [Eme06] and [Eme11], he uses this completed cohomology to formulate and later prove a certain local-global compatibility in the p-adic Langlands programme. This is the main focus of our seminar; deducing the theorem above is then quite simple in comparison.

[1] In the original article, the property "potentially semi-stable" is used here instead. Today we know by Berger's proof of the p-adic monodromy theorem that the properties "de Rham" and "potentially semi-stable" are in fact equivalent, which at the time the Fontaine-Mazur conjecture was formulated was still a conjecture. We decided to just use the de Rham property, since we can then omit introducing the notion of potential semistability.

We will mostly follow [Bre12] for this seminar, although we would like to point out that many arguments are taken from [Eme11], which sometimes provides more (or at least different) details.

Talk 1: The classical Langlands correspondences. The classical local Langlands correspondence gives a "well-behaved" bijection between isomorphism classes of certain n-dimensional Weil-Deligne representations of a local field F and those of certain representations of $\mathrm{GL}_n(F)$. They are completely proven. We focus on the special case $n = 2$.

First we explain the equivalence of Galois representations of $\mathrm{Gal}(\overline{F}|F)$ with coefficients in a finite extension of $\mathbb{Q}_\ell$, where ℓ is different from the residue characteristic of F, and Weil-Deligne representations. Then we introduce the notions of smooth and admissible representations of $\mathrm{GL}_2(F)$ with complex coefficients and mention that smooth irreducible ones are automatically admissible. We mention that one can attach L- and ε-factors to all those representations. Then we can state the local Langlands correspondence: There is a bijection between isomorphism classes of semisimple two-dimensional Weil-Deligne representations of F and those of smooth admissible representations of $\mathrm{GL}_2(F)$ respecting L- and ε-factors and compatible with character twisting.

Next we look at the global picture, which is still conjectural. We define an automorphic representation of a global field K as a representation of $\mathrm{GL}_2(\mathbb{A}_K)$ (where $\mathbb{A}_K$ is the ring of adeles of K) occuring as a subquotient in the standard L^2-representation. Each such representation is built from local ones for each place of K by the Tensor Product Theorem. Conjecturally, there should be a correspondence between such automorphic representations and representations of $\mathrm{Gal}(\overline{K}|K)$. Moreover, this correspondence should be compatible with the local one in a precise sense: This is the local-global compatibility conjecture for the classical (i. e., *not* p-adic) Langlands programme and is very similar to the analogous compatibility in class field theory.

Talk 2: (φ, Γ)-modules and p-adic Hodge theory. In this talk we introduce (φ, Γ)-modules following [Col10]. The category of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ on p-adic vector spaces is a equivalent to the full subcategory of étale (φ, Γ)-modules.

The next citations all refer to [Col10]. Begin by introducing the field $\mathcal{E}$ and the rings $\mathcal{O}_\mathcal{E}$ and $\mathcal{O}_\mathcal{E}^+$ from §I.1.1 (see also §1 in the introduction). Then cover the material from §I.2.1, §I.2.2 and §I.3.1.

After that give a short overview on some notions of p-adic Hodge theory, where we refer to [BC09]. We need the period rings B_{HT}, B_{dR} and $\mathrm{B}_{\mathrm{cris}}$. We suggest to give the definition [BC09, Def. 2.4.7] of B_{HT} following the motivation in [BC09, §2.1–3], but for the other period rings to only explain how they relate to each other: B_{dR} is a filtered ring whose associated graded is B_{HT} and which contains $\mathrm{B}_{\mathrm{cris}}$ as a subring. Then you can define Hodge-Tate, de Rham and crystalline Galois representations and Hodge-Tate weights as in [BC09, §5]. We will not need any further details except these basic definitions.

Without any examples, it is hard to get a feeling for the meaning of these properties. We recommend you mention [BC09, Ex. 6.3.9, Cor. 9.3.2] and [Dal05, §1].

Other useful references for p-adic Hodge theory might be [Ber04, chap. I, II] or [FO08].

Talk 3: The p-adic local Langlands correspondence via the Colmez functor. In the p-adic Langlands programme we look on the one side at two-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ with coefficients in a finite extension of $\mathbb{Q}_p$, the category of which is fully faithfully embedded into the category of (φ, Γ)-modules. On the other side we look at representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ now on p-adic Banach spaces. There is a functor from the category of the latter ones to the category of étale (φ, Γ)-modules constructed by Colmez (sometimes called the "Montréal functor").

We start with some background on p-adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Give the definitions at the beginning of [Col10, §III.1.3], skip remark III.1.3 and lemmas III.1.4/5, then explain the content of §III.1.4. For this you need the tree introduced in §III.1.2 (middle of p. 72) and some of the notation from §III.1.1. Explain the notation $\mathcal{W}^{(0)}(\Pi)$ which is introduced at the beginning of §III.1.6 and state proposition III.1.16. We probably won't have time for its proof, but you should mention remark III.1.17. Finally introduce the notion of duals from the beginning of §III.2.1.

After these preliminaries we can start with the construction of Colmez's functor. Explain the content of §IV.1.1–3 and lemma IV.1.8, proposition IV.1.9, then continue with §IV.2.2. As a motivation for this construction, you might also want to read §6 in the introduction and [Bre12, §3.1]. Some technical results from earlier sections that we did not cover might be quoted here (such as lemma III.1.10); we won't have time to go into detail. At some point theorem IV.2.1 is quoted, which rests on a classification of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that we unfortunately cannot cover. Instead of stating this theorem, just explain that we rest on such a classification at this point. Finally explain §IV.2.3.

After these efforts we know how to associate a Galois reprenstation to a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. For the Langlands correspondence we need to be able to go in the other direction as well. Although Colmez also constructs such a functor, we follow [Bre12, §3.2] instead, where deformation theory is used to associate a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ to certain Galois representations. This is less general, but more suited to our application. We cannot say anything about the proof of theorem 3.3, but apart from that you should explain the material from §3.2.

Talk 4: Modular curves and modular Galois representations. This talk introduces modular curves and Hecke operators and sketches the construction of the Galois representation attached to a normalized eigenform. In this whole talk we will always work with $\mathbb{Q}$-schemes, so we don't have to bother with non-invertible integers.

Introduce elliptic curves and level structures as in [DR73, §IV.2.3, §IV.3, p. 68ff.] and [Eme06, §2.4] (note that "level H-structures" and "level K_f-structures" are the same). State that the moduli functors that associate to a $\mathbb{Q}$-scheme the set of isomorphism classes of such level structures are representable by affine curves over $\mathbb{Q}$ whenever n is large resp. K_f is small enough, and that in the remaining cases we still have coarse moduli spaces.[2]

The complex points of these curves can be identified with appropriate quotients of the upper half plane, see [Con09, §2.3.1, theorem 4.2.6.2], [DS05, §1.5]. Using the strong approximation theorem for GL_2 [Bum97, p. 293/294] they can alternatively be described as quotients of the adelic points of GL_2.

After that define Hecke operators acting on the étale cohomology $H^1_{\text{ét}}(Y \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$ of modular curves. Here you could follow [Hid86, p. 565/566]. There, Φ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and Y is the quotient of the upper half plane by Φ and M is an abelian group with a certain additional action of a semigroup, which we may take as the trivial one here.[3] For certain matrices α an operator $[\Phi\alpha\Phi]$ on the (singular) cohomology with coefficients in the constant[4] sheaf M of Y is defined. The very same construction still works when we replace Φ by a subgroup $K_f \subseteq \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ as before and singular by étale cohomology. That the occurring maps between modular curves are then defined over $\mathbb{Q}$ is not obvious, but can be shown using the methods from [DR73]. Alternatively, one can use the comparison isomorphism between singular and étale cohomology to obtain the operators. The operators T_ℓ and S_ℓ are then obtained by setting α to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)$, respectively. An alternative way to introduce Hecke operators is shown in [Con09, §2.3.1].

For the topics mentioned so far, the overview given in [Füt17, §II.1.1, §II.1.2, §II.3.1–3] might be helpful (as well as a chat with its author).

Having defined Hecke operators, we can now view the étale cohomology of a modular curve as a module over the corresponding Hecke algebra, which is defined to be the $\mathbb{Q}_p$-subalgebra of the endomorphisms of this cohomology generated by the operators T_ℓ and S_ℓ. On these cohomology groups we also have an action of the absolute Galois group of $\mathbb{Q}$ which obviously commutes with the Hecke action.

The crucial step in the construction of the Galois representation is then the *Eichler-Shimura relation*, which says the following: if $\ell \neq p$ is a prime that does not divide the level of a modular curve Y, the characteristic polynomial of the geometric Frobenius Frob_ℓ acting on $H^1_{\text{ét}}(Y \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$ is $X^2 - T_\ell X + \ell S_\ell$. Here we view the étale cohomology as a module over the Hecke algebra, hence the polynomial has coefficients in this algebra. The proof of this theorem relies on an analysis of the reduction of the modular curve at ℓ; it is very deep and we probably cannot say much about it. See [Con09, esp. Thm. 4.1.2.1] or, for a short overview, [Pot].

After this insight, the construction of the Galois representation attached to a single normalized modular eigenform f of weight 2 is easy: f corresponds to a morphism λ from the Hecke

[2] More precisely: If $n \geq 3$ or K_f is torsion free, the geometric points of the representing stack do not have any nontrivial automorphisms, hence the stack is in fact an algebraic space [DR73, §IV.2.6]. By [DR73, Thm. IV.3.4 and p. 69] they are in fact schemes.

[3] This corresponds to restricting to weight 2, which is a good idea anyway, because many important phenomena can already be seen in this special case, while some technical complications can be avoided.

[4] In the higher weight case, the sheaf is no longer constant.

algebra to $\overline{\mathbb{Q}}_p$ sending T_ℓ and S_ℓ to the respective Hecke eigenvalues, and we just have to tensor the $H^1_{\text{ét}}(Y \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$ with $\overline{\mathbb{Q}}_p$ over the Hecke algebra along this morphism.

In the higher weight k case, one has to replace the constant sheaf $\mathbb{Q}_p$ in étale cohomology by $\mathrm{Sym}^{k-2} R^1\pi_*\mathbb{Q}_p$, where π denotes the map from the universal elliptic curve to the modular curve.

Talk 5: Completed cohomology. In this talk we introduce Emerton's completed cohomology and study some of its basic properties.

You should cover the following parts of [Eme11]: Start from §5 until the bottom of p. 45, then skip Lemma 5.1.3–4, continue with §5.2 until Rem. 5.2.3, skip the deformation theory stuff, state Lemma 5.2.4, skip the remainder of §5.2 and continue with §5.3, skipping Remarks 5.3.10 and 5.3.12, otherwise everything until Prop. 5.3.15.

Talk 6: Emerton's local-global compatibility conjecture in the p-adic Langlands programme. State Emerton's local-global compatibility conjecture, which is [Eme11, conjecture 1.1.1]. Then define what it means for a Galois representation to be promodular [Bre12, Def. 4.2] and state Breuil's special case of Emerton's conjecture [Bre12, Thm. 2.1,], which we are going to prove during the next talks. Then present the first reduction step in Breuil's proof following [Bre12, §4.2]. A comparison with [Eme11, proposition 6.1.12] might be useful. We also found Emerton's thoughts on Ihara's lemma on StackExchange rather illuminating: `https://math.stackexchange.com/questions/629707/how-should-i-think-about-iharas-lemma`

It will also be insightful to actually look at the explicit Langlands correspondence: In the situation we consider, the representation ρ is unramified, hence it factors as a representation of $\widehat{\mathbb{Z}}$, which is (after semisimplification) a sum of two unramified characters χ_1 and χ_2. The classical local Langlands correspondence is very explicit in this situation: It associates to ρ the principal series representation $B(\chi_1, \chi_2)$ (it has to because of its compatibility with class field theory, but see also (4.1.8) in `https://www2.math.uni-paderborn.de/fileadmin/Mathematik/People/wedhorn/publications/LocalLanglands.pdf`). Breuil then claims (implicitly) that the representation $\text{c-ind}_{\mathrm{GL}_2(\mathbb{Z}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1/(T_\ell - \lambda(T_\ell), S_\ell - \lambda(S_\ell))$ is isomorphic to $B(\chi_1, \chi_2)$. This follows from [Bum97, Thm. 4.6.4, Prop. 4.6.6].

Talk 7: Proof of the local-global compatibility I. The main result here is that the set of classical crystalline maximal ideals of the Hecke algebra is Zariski dense, which is a key ingredient later on.

Using deformation theory, this yields a certain *universal* representation π_Σ, which will play an important role later in the definition of $X_{\mathcal{O}_E}$.

First, state [Bre12, theorem 4.1] and especially equation (6) (for a definition of locally algebraic vectors and representations see [Col10, beginning of §VI.2.1,2]). Afterwards, follow §4.3 (loc. cit.).

Talk 8: Proof of the local-global compatibility II. The aim of this talk is to prove [Eme11, theorem 6.3.12], disguised as [Bre12, lemma 4.6], which studies the properties of a certain module over a Hecke algebra that Breuil denotes $X_{\mathcal{O}_E}$ while Emerton calls it $X(\pi_\Sigma^{\mathrm{m}})$. The non-vanishing of certain submodules of this module (which is subject of the following talk) will then imply the weak version of local-global compatibility.

As we don't have time to review the theory of the eigencurve, we can't prove [Eme11, lemma 5.4.9], but apart from that, everything in §6.3 up to 6.3.12 (loc. cit.) should be proved. (We can make an exception for details concerning locally convex vector spaces.) For the $\mathcal{O}$-torsion-freeness used in the proof of Thm. 6.3.12 see [Eme06, Lem. 7.2.1].

Talk 9: Proof of the local-global compatibility III. After having studied the structure of $X_{\mathcal{O}_E}$ in the previous talk, we will use this module to show [Bre12, theorem 2.1] as done in §4.4 (loc. cit.). We joyfully restrict ourselves to $p > 2$ and absolutely irreducible ρ_p.

The idea is basically the following: One can show that for classical crystalline maximal ideals $\mathfrak{p}$ not containing p in the Hecke algebra, $X_{\mathcal{O}_E}$ has non-vanishing $\mathfrak{p}$-torsion. A density argument then implies that $X_{\mathcal{O}_E}$ has non-vanishing $\mathfrak{p}$-torsion for every maximal ideal $\mathfrak{p}$ not containing p of the Hecke algebra. This in turn yields a nontrivial morphism $\rho \otimes_E B(\rho_p) \longrightarrow \hat{H}^1_{E,\Sigma}$, which finishes the proof of the aforementioned theorem.

Talk 10: Deducing a special case of a Fontaine-Mazur conjecture. The Fontaine-Mazur conjectures are a number of conjectures that were published in [FM95]. In this talk, the relevant conjectures should be introduced and given a bit of context (the original source seems very adequate). The focus should of course be on conjecture 3c. Explain how conjecture 3c follows from conjecture 1 via Khare-Wintenberger. Our previous work implies $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})}(\rho, (\hat{H}^1_E)^{\mathrm{alg}}) \neq 0$, and the description of $(\hat{H}^1_E)^{\mathrm{alg}}$ together with the construction of the Galois representation attached to a modular form yields the result.

It might be interesting to see how other conjectures can also be derived with the same methods (cf. [Bre12, §2.2, remainder of §5.2]), but only if time allows.

References

[BC09] Olivier Brinon and Brian Conrad. *CMI Summer School notes on p-adic Hodge theory. Preliminary version.* 2009. URL: `http://math.stanford.edu/~conrad/papers/notes.pdf`.

[Ber04] Laurent Berger. "An introduction to the theory of p-adic representations". In: *Geometric aspects of Dwork theory. Vol. I, II.* Walter de Gruyter, Berlin, 2004, pp. 255–292.

[Bre12] Christophe Breuil. "Correspondance de Langlands p-adique, compatibilité local-global et applications [d'après Colmez, Emerton, Kisin, ...]". In: *Astérisque* 348 (2012). Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042, Exp. No. 1031, viii, 119–147. ISSN: 0303-1179.

[Bum97] Daniel Bump. *Automorphic forms and representations*. Vol. 55. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997, pp. xiv+574. ISBN: 0-521-55098-X. DOI: 10.1017/CBO9780511609572. URL: https://doi.org/10.1017/CBO9780511609572.

[Col10] Pierre Colmez. "Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ, Γ)-modules". In: *Astérisque* 330 (2010), pp. 281–509. ISSN: 0303-1179.

[Con09] Brian Conrad. *Modular Forms, Cohomology, and the Ramanujan Conjecture. Draft II.* 2009. eprint: http://www.math.leidenuniv.nl/~edix/public_html_rennes/brian.ps.

[Dal05] Chandan Singh Dalawat. *Good reduction, bad reduction. Notes for a lecture at the conference on Commutative Algbera and Algebraic Geometry, Madras, August 1–6, 2005.* 2005. URL: http://arxiv.org/pdf/math/0605326.pdf.

[DR73] P. Deligne and M. Rapoport. "Les schémas de modules de courbes elliptiques". In: *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*. Springer, Berlin, 1973, 143–316. Lecture Notes in Math., Vol. 349.

[DS05] Fred Diamond and Jerry Shurman. *A first course in modular forms*. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436. ISBN: 0-387-23229-X.

[Eme06] Matthew Emerton. "A local-global compatibility conjecture in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$". In: *Pure Appl. Math. Q.* 2.2, Special Issue: In honor of John H. Coates. Part 2 (2006), pp. 279–393. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2006.v2.n2.a1. URL: https://doi.org/10.4310/PAMQ.2006.v2.n2.a1.

[Eme11] Matthew Emerton. *Local-global compatibility in the p-adic Langlands programme for* $\mathrm{GL}_{2/\mathbb{Q}}$. 2011. eprint: http://www.math.uchicago.edu/~emerton/pdffiles/lg.pdf.

[FM95] Jean-Marc Fontaine and Barry Mazur. "Geometric Galois representations". In: *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*. Ser. Number Theory, I. Int. Press, Cambridge, MA, 1995, pp. 41–78.

[FO08] Jean-Marc Fontaine and Yi Ouyang. *Theory of p-adic Galois Representations*. Springer, 2008. URL: http://www.math.u-psud.fr/~fontaine/galoisrep.pdf. Not published yet.

[Füt17] Michael Fütterer. "A p-adic L-function with canonical motivic periods for families of modular forms". Dissertation. Ruprecht-Karls-Universität Heidelberg, 2017.

[Hid86] Haruzo Hida. "Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms". In: *Invent. Math.* 85.3 (1986), pp. 545–613. ISSN: 0020-9910. DOI: 10.1007/BF01390329. URL: https://doi.org/10.1007/BF01390329.

[Pot] Jonathan Pottharst. *Attaching ℓ-adic Representations to Elliptic Modular Forms.* URL: http://vbrt.org/writings/l-adic-talk.pdf.