PRESENTATION OF MY WORK

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My research interests lie in dynamical systems, particularly actions of subgroups of Lie groups acting on homogeneous spaces.

Let G be a real linear, connected, semisimple Lie group, for instance $\mathrm{SL}(n,\mathbb{R})$ with $n \geq 3$ or $\mathrm{SL}(2,\mathbb{R})^k$. Let Γ be a discrete subgroup of G and H be a closed subgroup of G. I would like to understand the chaotic properties of the action $\Gamma\backslash G \curvearrowleft H$ or its dual $\Gamma \curvearrowright G/H$, using notions and tools from dynamical systems, in particular ergodic theory and topological dynamics, from probability theory, in particular random walks in Lie groups and from symmetric spaces of non-positive curvature.

Representation theory of Lie groups very efficiently adresses these problems when $\Gamma\backslash G$ has finite volume. When G has real rank one and Γ is a discrete subgroup of G, then the actions $\Gamma\backslash G \curvearrowleft H$ or $\Gamma\backslash G/Z_G(H) \curvearrowleft H$ have ties to the geometry of negatively curved manifolds. In particular chaotic aspects (i.e. ergodicity, mixing, entropy...) of the geodesic flow and horocyclic flows have been studied successfully by many people among which Hopf, Hedlund, Furstenberg, Bowen, Anosov, Ratner, Dani, Margulis, Eberlein... In higher rank and infinite volume, less is known. However, we know much about the asymptotic behaviour of random walks in Lie groups (see [BL85] or [BQ16]). This knowledge on the left action of Γ on G/H or $G/HZ_G(H)$, can help to study the right action of H on $\Gamma\backslash G$.

More specifically, I am interested in the chaotic properties of one parameter subgroups $H = (\phi_t)$ acting on $\Gamma\backslash G$ or $\Gamma\backslash G/Z_G(H)$. My first work concerns the study of their nonwandering set, their topological transitivity and mixing properties.

Let now m be a H-invariant Radon measure on $\Gamma\backslash G$. Then $(\Gamma\backslash G, \phi_t, m)$ is a measurable dynamical system and I would try to study ergodicity, conservativity or dissipativity in the infinite volume case.

In the future, I would like to add and solve other questions that come from ergodic theory, geometry of non positively curved manifolds and random walks on Lie groups.

Past and current works

Consider a Cartan subgroup A of G, the associated maximal compact subgroup K and a unipotent maximal subgroup N for which we have the Iwasawa decomposition $G = KAN$. I am interested in the dynamical properties of the right actions of one parameter subgroups of A on the homogeneous space $\Gamma\backslash G$. We denote by $\mathfrak{a}^+$ a choice of closed positive Weyl chamber in the Cartan subspace tangent to A. *Weyl chamber flows* are one parameter subgroups defined for all $\theta \in \mathfrak{a}^+$ by $\phi_\theta^t : g \mapsto ge^{t\theta}$. My first results are restricted to *regular* Weyl chamber flows, i.e. the Weyl chamber flows ϕ_θ^t such that θ is in the interior of the positive Weyl chamber.

Let us first describe the asymptotic behavior of random walks in G in the Cartan decomposition. Consider the Cartan subspace $\mathfrak{a}$ associated to A and the positive Weyl chamber $\mathfrak{a}^+$. By Cartan decomposition, for any element $g \in G$, there is a unique element $\kappa(g) \in \mathfrak{a}^+$ so that the non-unique polar decomposition $g \in Ke^{\kappa(g)}K$ holds. Then the map $\kappa : G \to \mathfrak{a}^+$ is called the Cartan decomposition. The Jordan projection is a map $\lambda : G \to \mathfrak{a}^+$ which encodes, for any element of G, the logarithm of its spectrum. The relationship between the Cartan projection and the Jordan projection is given by the spectral radius formula [BQ16, Corollary 5.34]

$$\lambda(g) = \lim_{n\to\infty} \frac{1}{n}\kappa(g^n).$$

In [Ben97], Benoist studies for any semigroup $\Gamma \subset G$ the smallest closed cone of $\mathfrak{a}^+$ containing $\lambda(\Gamma)$, called the limit cone. Denote it by

$$\mathcal{C}(\Gamma) := \overline{\underset{\gamma\in\Gamma}{\cup} \mathbb{R}_+\lambda(\gamma)}.$$

This cone can also be understood as the smallest closed cone of $\mathfrak{a}^+$ containing all directions θ whose associated Weyl chamber flow ϕ_θ^t acting on $\Gamma\backslash G/M$ has a periodic orbit. He proved that when Γ is Zariski dense, then the limit cone is convex of nonempty interior. In my thesis, I obtained two results on the limit cone. The first one concerns random walks on Lie groups, the second topological mixing in the Weyl chamber flow.

0.1. **On random walks.** Take μ a probability measure on the Lie group G and consider $(X_n)_{n\geq 1}$ a sequence of i.i.d random variables of law μ. The behavior of random products $X_n...X_1$ and $X_1...X_n$ were studied by many mathematicians, among which Furstenberg, Kesten, Kifer, Goldsheid, Guivarc'h, Le Page, Raugi, Conze, Bougerol, Oseledets. Furstenberg-Kesten Theorem [FK60] is an extension to Lie groups of the Law of Large Number. It states that for any probability measure μ on G of finite first moment (i.e. so that $\int_G \|\kappa(g)\|d\mu(g) < +\infty$), then

$$\frac{1}{n}\kappa(X_n...X_1) \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \overrightarrow{\lambda}_\mu \in \mathfrak{a}^+$$

where $\overrightarrow{\lambda}_\mu$ can be defined by this and is called the Lyapunov vector of μ. It follows from Guivarc'h-Raugi [GR85] and Goldsheid-Margulis [GM89] that for any probability measure μ on G with finite first moment and such that the support generates a Zariski dense subsemigroup of G, then $\overrightarrow{\lambda}_\mu \in \mathfrak{a}^{++}$.

More recently, Sert proved in [Ser16] (see also [BS18]) that for any probability measure μ of finite second moment (i.e. $\int_G \|\kappa(g)\|^2 d\mu(g) < +\infty$) and such that the subsemigroup Γ_μ generated by $supp(\mu)$ is Zariski dense in G, then

$$\overrightarrow{\lambda}_\mu \in \overset{\circ}{\mathcal{C}}(\Gamma_\mu).$$

Denote by $\mathcal{M}_Z^2(\Gamma)$ the set of probability measures on G of finite second moment and such that the support generates a Zariski dense subsemigroup of G in Γ. In my thesis, I proved the following Theorem.

Theorem 0.1. *Let G be a semisimple, real linear Lie group. Let Γ be a Zariski dense subgroup of G. Then the map*

$$\begin{aligned}\overrightarrow{\lambda} : \mathcal{M}_Z^2(\Gamma) &\longrightarrow \overset{\circ}{\mathcal{C}}(\Gamma)\\ \mu &\longmapsto \overrightarrow{\lambda}_\mu\end{aligned}$$

is continuous and surjective.

0.2. **Topological mixing.** In the finite volume case, the mixing property of $\Gamma\backslash G \curvearrowleft H$ when H is a closed non compact subgroup of G is given by Howe-Moore Theorem. In particular, for any $\theta \in \mathfrak{a}^+$, it implies that $\Gamma\backslash G \curvearrowleft \exp(\mathbb{R}\theta)$ is mixing .

In rank one, $\Gamma\backslash G/M$ is the unit tangent bundle of the manifold $\Gamma\backslash G/K$ and the right A-action on $\Gamma\backslash G/M$ is the geodesic flow. This particular situation has been widely studied by many people, among which for example Bowen, Hopf, Margulis, Sullivan... Dal'bo [Dal00] proved that the geodesic flow is mixing (on its nonwandering set) if and only if the length spectrum is non arithmetic. The latter holds when Γ is a Zariski dense subgroup, see Benoist [Ben00], Kim [Kim06].

In higher rank when $\Gamma\backslash G/M$ has infinite volume, less is known.

Consider the smallest closed A-invariant subset Ω of $\Gamma\backslash G/M$ containing every periodic orbit of all Weyl chamber flows. In [CG02] Conze and Guivarc'h proved that the right action of A on Ω is topologically transitive i.e. there is a dense orbit. In the particular case of so-called Ping-Pong subgroups of $\mathrm{PSL}(k+1,\mathbb{R})$, Thirion [Thi07], [Thi09] proved mixing with respect to a natural measure on Ω for a one parameter flow associated to the "maximal growth vector" introduced by Quint in [Qui02]. Sambarino [Sam15] generalized this result for hyperconvex representations.

With Glorieux in [DG19], we studied the topological properties regular Weyl chamber flows for any Zariski dense discrete subgroup of G. Denote by Ω_G the preimage in $\Gamma\backslash G$ of Ω by the projection $\Gamma\backslash G \to \Gamma\backslash G/M$.

Theorem 0.2 ([DG19]). *Let G be a semisimple, connected, real linear Lie group, of non-compact type. Let Γ be a Zariski dense, discrete subgroup of G. Let $\theta \in \mathfrak{a}^+$. If there is a non diverging orbit in Ω_G, then θ is in the limit cone. If there is a dense orbit and $\theta \in \mathfrak{a}^{++}$, then θ is in the interior of the limit cone.*

Actually, we even obtained a sufficient and necessary condition for topological mixing of regular Weyl chamber flows.

Theorem 0.3 ([DG19]). *Let G be a semisimple, connected, real linear Lie group, of non-compact type. Let Γ be a Zariski dense, discrete subgroup of G. Let $\theta \in \mathfrak{a}^{++}$.*

Then the dynamical system (Ω, ϕ_t^θ) is topologically mixing if and only if θ is in the interior of the limit cone $\mathcal{C}(\Gamma)$.

Using a Theorem of Guivarc'h and Raugi [GR07], I improved in my thesis [Dan19] the mixing criteria to $\Gamma\backslash G$ when M is abelian. Guivarc'h-Raugi's theorem allows to partition Ω_G into a finite number of dynamically conjugated A-invariant subsets $(\Omega_{G,i})_{i\in I}$ that project onto $\Omega \subset \Gamma\backslash G/M$.

Theorem 0.4 (in preparation). *Let G be a semisimple, connected, real linear Lie group, of non-compact type such that M is abelian. Let Γ be a Zariski dense, discrete subgroup of G. Let $\theta \in \mathfrak{a}^{++}$.*

Then the dynamical system $(\Omega_{G,i}, \phi_t^\theta)$ is topologically mixing for any $i \in I$ if and only if θ is in the interior of the limit cone $\mathcal{C}(\Gamma)$.

References

[Ben97] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.

[Ben00] Y. Benoist. Propriétés asymptotiques des groupes linéaires. II. In *Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997)*, volume 26 of *Adv. Stud. Pure Math.*, pages 33–48. Math. Soc. Japan, Tokyo, 2000.

[BL85] P Bougerol and Lacroix. *Products of random matrices with applications to Schrödinger operators*, volume 8 of *Progress in Probability*. Birkhäuser Basel, 1985.

[BQ16] Y. Benoist and J-F. Quint. *Random walks on reductive groups*, volume 62 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2016.

[BS18] E. Breuillard and Ç. Sert. The joint spectrum. *ArXiv e-prints*, September 2018.

[CG02] J-P. Conze and Y. Guivarc'h. Densité d'orbites d'actions de groupes linéaires et propriétés d'équidistribution de marches aléatoires. In *Rigidity in dynamics and geometry (Cambridge, 2000)*, pages 39–76. Springer, Berlin, 2002.

[Dal00] F. Dal'bo. Topologie du feuilletage fortement stable. *Ann. Inst. Fourier (Grenoble)*, 50(3):981–993, 2000.

[Dan19] Nguyen-Thi Dang. *Dynamics of group action on homogeneous spaces of higher rank and infinite volume.* Theses, Université Rennes 1, September 2019.

[DG19] N.-T. Dang and O. Glorieux. Topological mixing of the Weyl chamber flow. *Ergodic theory and dynamical systems*, September 2019.

[FK60] H. Furstenberg and H. Kesten. Products of random matrices. *Ann. Math. Statist.*, 31:457–469, 1960.

[GM89] I. Ya. Goldsheid and G. A. Margulis. Lyapunov exponents of a product of random matrices. *Uspekhi Mat. Nauk*, 44(5(269)):13–60, 1989.

[GR85] Y. Guivarc'h and A. Raugi. Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence. *Z. Wahrsch. Verw. Gebiete*, 69(2):187–242, 1985.

[GR07] Y. Guivarc'h and A. Raugi. Actions of large semigroups and random walks on isometric extensions of boundaries. *Ann. Sci. École Norm. Sup. (4)*, 40(2):209–249, 2007.

[Kim06] I. Kim. Length spectrum in rank one symmetric space is not arithmetic. *Proc. Amer. Math. Soc.*, 134(12):3691–3696, 2006.

[Qui02] Jean-François Quint. Divergence exponentielle des sous-groupes discrets en rang supérieur. *Commentarii Mathematici Helvetici*, 77(3):563–608, 2002.

[Sam15] A. Sambarino. The orbital counting problem for hyperconvex representations. *Ann. Inst. Fourier (Grenoble)*, 65(4):1755–1797, 2015.

[Ser16] Ç. Sert. *Joint Spectrum and Large Deviation Principles for Random Matrix Products*. PhD thesis, Paris-Saclay, Université, 2016.

[Thi07] X. Thirion. *Sous-groupes discrets de* $\mathrm{SL}(d,\mathbb{R})$ *et équidistribution dans les espaces symétriques*. PhD thesis, Tours, 2007.

[Thi09] X. Thirion. Propriétés de mélange du flot des chambres de Weyl des groupes de ping-pong. *Bull. Soc. Math. France*, 137(3):387–421, 2009.

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