Topological mixing of the Weyl chamber flow

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Plan of the talk

(1) Weyl chamber flows
(2) Mixing condition
(3) Benoist cone
(4) Non-diverging orbits
(5) Necessary condition for mixing
(6) Sufficient condition for mixing
(6.1) A dense orbit in the space of oriented flats
(6.2) Generic product of loxodromic elements

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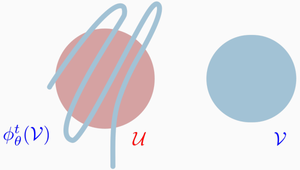
Topological mixing:

$\Omega \curvearrowleft \phi^t$ is *topologically mixing* if for all (non-empty) $\mathcal{U}, \mathcal{V} \subset \Omega$, there exists $T > 0$ such that $\phi^t(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ for all $t \geq T$.



Weyl chamber flows

G semisimple Lie group of non-compact type,

A maximal torus, $\mathfrak{a} := Lie(A)$ the Cartan subspace,

$\mathfrak{a}^+$ closed positive Weyl chamber, $\mathfrak{a}^{++}$ its interior.

M the compact subgroup such that $AM = Z_G(A)$.

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A maximal torus, $\mathfrak{a} := Lie(A)$ the Cartan subspace,	$diag(e^{t_1}, ..., e^{t_n})$ $\sum_{i=1}^{n} t_i = 0$
$\mathfrak{a}^+$ closed positive Weyl chamber, $\mathfrak{a}^{++}$ its interior.	$diag(t_1, ..., t_n)$ $t_1 \geq ... \geq t_n$
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For all non zero $\theta \in \mathfrak{a}^+$, denote by $\phi_\theta^t(\Gamma g M) := \Gamma g e^{t\theta} M$ the Weyl chamber flow $\phi_\theta^t : \Gamma\backslash G/M \curvearrowleft$. The flow ϕ_θ^t is a *regular Weyl chamber flow* when $\theta \in \mathfrak{a}^{++}$.

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Γ lattice: Howe-Moore(79) mixing, Moore (87) exponential rate

$G = \mathrm{SO}(n,1)_0$ (... Dal'bo 2000, Kim 2006)

Geodesic flow acting on $T^1\Gamma\backslash\mathbb{H}^n$ is topologically mixing on its non-wandering set.

Mixing condition

Theorem (D-Glorieux 20)

Let G be a connected, real linear, semisimple Lie group of non-compact type and $\Gamma < G$ discrete, Zariski dense. The following holds.

(a) If ϕ_θ^t has a non-diverging orbit, then $\theta \in \mathcal{B}(\Gamma)$.

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Definition

Ω is the smallest closed, A-invariant subset of $\Gamma \backslash G/M$ containing the periodic orbits of regular Weyl chamber flows.

Conze-Guivarc'h (2000) gave a construction for $\mathrm{SL}(n, \mathbb{R})$, they proved that there are dense A-orbits in Ω.

Benoist cone

Jordan decomposition

g	$=$	g_e	g_h	g_u
commuting parts		*elliptic* $\subset$ max. compact subgp	*hyperbolic* $\subset$ conjugate of A	*unipotent* $\subset$ max unip subgp

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Jordan projection, $g \in G \mapsto \lambda(g) \in \mathfrak{a}^+$ unique element such that $g_h \sim e^{\lambda(g)}$.

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$\gamma = h_\gamma m_\gamma e^{\lambda(\gamma)} h_\gamma^{-1}$

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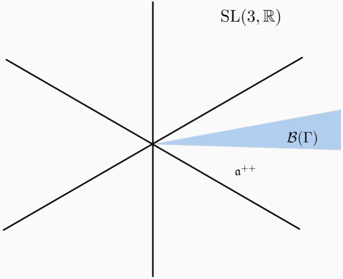
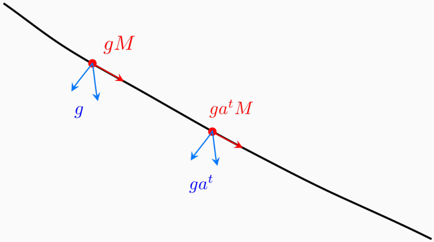
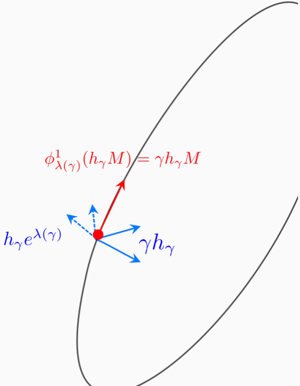
Non-wandering set

$$\Omega := \overline{\{wA \quad | \quad \theta \in \lambda(\Gamma) \cap \mathfrak{a}^{++} \text{ and } \phi_\theta^t(w) \text{ is periodic}\}}.$$

$\gamma \in \Gamma^{lox}$ then $w = h_\gamma M$ is $\phi^t_{\lambda(\gamma)}$ periodic in $\Gamma \backslash G/M$, where $\gamma = h_\gamma m_\gamma e^{\lambda(\gamma)} h_\gamma^{-1}$.







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When Γ is Zariski dense, $\mathcal{B}(\Gamma)$ is the cone asymptotic to $\mu(\Gamma)$ i.e.

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Pick $h \in G$ such that $\phi_\theta^t(\Gamma hM)$ is non-diverging,
pick $C \subset G/M$ compact, $t_n \to \infty$ and γ_n such that

$$\phi_\theta^{t_n}(hM) = he^{t_n \theta} M \in \gamma_n C \subset \Gamma.C.$$

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The Cartan projection is a proper map,

$$\mu(he^{t_n\theta}) = \mu(\gamma_n) + O(\mu(C))$$

$$t_n\theta = \mu(\gamma_n) + O(\mu(C), \mu(h)).$$

$$\theta = \frac{1}{t_n}\mu(\gamma_n) + O(\frac{1}{t_n}).$$

Hence $\theta \in \mathcal{B}(\Gamma)$.

Necessary condition for mixing

(a) ϕ_θ^t has a non-diverging orbit $\Rightarrow \theta \in \mathcal{B}(\Gamma)$.
$\Omega := \overline{\{wA \quad | \quad \theta \in \lambda(\Gamma) \cap \mathfrak{a}^{++} \text{ and } \phi_\theta^t(w) \text{ is periodic}\}}$.

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Proof: Take $h \in G$ such that

$$\Gamma.hAM \subset \overline{\phi_\theta^t(\Gamma.hM)}.$$

For all $v \in \mathfrak{a}$, there exists $\gamma_n \in \Gamma$, $t_n \to \infty$, $m_n \in M$, $\delta_n \to e_G$ such that

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Idea: $h e^{-v+t_n\theta} m_n h^{-1}$ is (r, ε_n)-loxodromic when n large. $\delta_n' \to e_G$.
γ_n is $(2r, 2\varepsilon_n)$-loxodromic when n large enough and

$$\lambda(\gamma_n) = \lambda(h e^{-v+t_n\theta} m_n h^{-1}) + o(1)$$

$$\lambda(\gamma_n) = -v + t_n \theta + o(1).$$

Generic product of loxodromic elements

$g \in G^{lox}$ loxodromic, pick $(h_g, m_g) \in G \times M$ such that $g = h_g e^{\lambda(g)} m_g h_g^{-1}$.
$N = \{u \in G \mid a^{-n} u a^n \to e_G \text{ for all } a \in A^{++}\}$ maximal unipotent subgroup.
$P := MAN$ and $\mathcal{F} = G/P$ the Furstenberg boundary, P^- s.t. $\mathrm{Stab}_G(P^-) = MAN^-$.

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Let $r > 0$ and $\varepsilon \in (0, r]$, we say g is (r, ε)-loxodromic if

(i) $r \leq \frac{1}{2}\mathrm{d}(g^+, \partial \mathrm{b}(g^-))$;

(ii) $g\mathcal{V}_\varepsilon(\partial \mathrm{b}(g^-))^{\complement} \subset B(g^+, \varepsilon)$ and the restriction is ε-Lipschitz.

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Proposition (Benoist 97)

There exists $\delta_{r,\varepsilon} \to 0$ such that the following holds.

For all $0 < \varepsilon \leq r$ and $g_k, ..., g_1 \in G$ be (r, ε)-loxodromic and r-generically ordered, there exists $\nu(g_k, ..., g_1) \in \mathfrak{a}$ such that for all $n_k, ..., n_1 \geq 1$

$$\lambda(g_k^{n_k} ... g_1^{n_1}) \overset{l\delta_{r,\varepsilon}}{\simeq} n_k \lambda(g_k) + ... + n_1 \lambda(g_1) + \nu.$$

and $g = g_k^{n_k} ... g_1^{n_1}$ is $(2r, 2\varepsilon)$-loxodromic with $(g^+, g^-) \in B(g_k^+, \varepsilon) \times B(g_1^-, \varepsilon)$.

Sufficient condition for mixing

Step 1

There exists $h \in G$ with $\Gamma hM \in \Omega$ such that for all $\delta > 0$, for all $v \in \mathfrak{a}$, there exists $T > 0$ such that for all $t \geq T$

$$\Gamma.B(hM, \delta) \cap B(he^{v+t\theta}M, \delta) \neq \emptyset.$$

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Proposition (Benoist 1997)

Γ Zariski dense, $\theta \in \overset{\circ}{\mathcal{B}}(\Gamma)$. Then there is an r-generic family $S \subset \Gamma$ of dim A elements and $\varepsilon_n \to 0$ such that

- θ is in the interior of the polygonal cone spanned by $\lambda(S)$.
- S_n is an r-generic family of (r, ε_n)-loxodromic elements and spans a Zariski dense semigroup of G, for all $n \geq 1$.

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$$\Gamma.B(hM, \delta) \cap B(he^{v+t\theta}M, \delta) \neq \emptyset.$$

Proposition (Benoist 1997)

Γ Zariski dense, $\theta \in \overset{\circ}{\mathcal{B}(\Gamma)}$. Then there is an r-generic family $S \subset \Gamma$ of dim A elements and $\varepsilon_n \to 0$ such that

- θ is in the interior of the polygonal cone spanned by $\lambda(S)$.
- S_n is an r-generic family of (r, ε_n)-loxodromic elements and spans a Zariski dense semigroup of G, for all $n \geq 1$.

Proposition (Benoist 2000)

Γ Zariski dense. Then $\overline{\langle \lambda(\Gamma) \rangle} = \mathfrak{a}$.

Sufficient condition for mixing

Step 1

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Proposition (Benoist 2000)

Γ Zariski dense. Then $\overline{\langle \lambda(\Gamma) \rangle} = \mathfrak{a}$.

Choose $g_k, \ldots, g_1$ in the semigroup spanned by S_n so that $\langle \lambda(g_j) \rangle$ is δ_{r,ε_n} dense in $\mathfrak{a}$, where $k \leq 3 \dim A$. Pick $h \in G$ such that $(hP, hP^-) = (g_k^+, g_1^-)$.

$$g_k^{n_k} \ldots g_1^{n_1} hM \overset{l\delta_{r,\varepsilon_n}}{\simeq} h \exp(\nu + \sum n_j \lambda(g_j))M.$$

A dense orbit in the space of oriented flats

Hopf coordinates

$$G/MA \xrightarrow{\sim} \mathcal{F} \overset{\triangle}{\times} \mathcal{F}$$

$$G/M \xrightarrow{\sim} \mathcal{F} \overset{\triangle}{\times} \mathcal{F} \times \mathfrak{a}.$$

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The limit set $L_\Gamma \subset \mathcal{F}$ is closed, Γ-invariant minimal.

$$\widetilde{\Omega} \overset{\sim}{\to} L_\Gamma \overset{\triangle}{\times} L_\Gamma \times \mathfrak{a}.$$

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Theorem (Conze-Guivarc'h (2000), D-Glorieux)

The action of Γ on $L_\Gamma \overset{\triangle}{\times} L_\Gamma$ has a dense orbit.

Rmk: $h \in G$ such that $(hP, hP^-) = (g_k^+, g_1^-)$ where $g_k, g_1 \in \Gamma^{lox}$, then $\Gamma hM \in \Omega$!
$\Rightarrow$ topological mixing!

Thank you for your attention!