TOPOLOGICAL MIXING OF POSITIVE DIAGONAL FLOWS

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ABSTRACT. Let G be a connected, real linear, semi-simple Lie group without compact factors and $\Gamma < G$ a Zariski dense, discrete subgroup. We study the topological dynamics of positive diagonal flows on $\Gamma\backslash G$. We extend Hopf coordinates to Bruhat-Hopf coordinates of G, which gives the framework to estimate the elliptic part of products of large generic loxodromic elements. By rewriting results of Guivarc'h-Raugi into Bruhat-Hopf coordinates, we partition the preimage in $\Gamma\backslash G$ of the non-wandering set of mixing regular Weyl chamber flows, into finitely many dynamically conjugated subsets. We prove a necessary condition for topological mixing, and when the connected component of the identity of the centralizer of the Cartan subgroup is abelian, we prove it is sufficient.

1. INTRODUCTION

Let G be a connected, real linear, semi-simple Lie group without compact factors. Let A be a maximal $\mathbb{R}$-split torus i.e. a maximal abelian subgroup whose Lie algebra $\mathfrak{a}$ is a Cartan subspace, denote by $\mathfrak{a}^+ \subset \mathfrak{a}$ a choice of closed positive Weyl chamber and by $\mathfrak{a}^{++}$ its interior, by $A^+ = \exp(\mathfrak{a}^+)$ and $A^{++} := \exp \mathfrak{a}^{++}$. Let $\Gamma < G$ be a Zariski dense, discrete subgroup. We study topological mixing of the right action by translation on $\Gamma\backslash G$ of one parameter subgroups of A that are parametrized by non-trivial elements of $\mathfrak{a}^+$.

1.1. Previous results.

In the case of lattices[1] i.e. $\Gamma\backslash G$ has finite volume for the Haar measure, topological mixing is a consequence of Howe–Moore [HM79] Theorem. Moore [Moo87] even proved that it is exponentially mixing for the Haar measure.

For the isometry group $\mathrm{SO}(n,1)^0$ of $\mathbb{H}^n$, the Cartan subspace $\mathfrak{a}$ is isomorphic to $\mathbb{R}$. Assume that Γ is Zariski dense, discrete and torsion free. Such right action corresponds to the geodesic frame flow of the hyperbolic orbifold $\Gamma\backslash\mathbb{H}^n$. The geodesic frame flow factors the geodesic flow on the unit tangent bundle $T^1\Gamma\backslash\mathbb{H}^n$. The latter identifies with the right action of A on $\Gamma\backslash\mathrm{SO}(n,1)^0/\mathrm{SO}(n-1)$, where $\mathrm{SO}(n-1)$ is the stabilizer in $\mathrm{SO}(n)$ of a fixed unit vector in $T^1\mathbb{H}^n$. The geodesic flow is topologically mixing on its non-wandering set[2].

Denote by Ω_G the preimage in $\Gamma\backslash\mathrm{SO}(n,1)^0$ of the non-wandering set of the geodesic flow. For convex cocompact subgroups, Winter [Win16] and Sarkar–Winter [SW20] proved exponential mixing for the push forward of the Bowen-Margulis-Sullivan (BMS) measure on the frame bundle. Since this measure is supported in Ω_G, these results imply topological mixing of the frame flow.

Under no other assumption for Γ than Zariski dense, Maucourant–Schapira [MS19] proved that the frame flow is topological mixing on Ω_G.

For rank one (i.e. $\dim A = 1$) locally symmetric spaces and discrete Zariski dense subgroup admitting a finite BMS measure, Winter [Win15] showed mixing for the frame flow.

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[1]lattices are Zariski dense subgroups by Borel density Theorem

[2]For example, topological mixing is equivalent to non-arithmeticity of the length spectrum by [Dal00], which follow, for Zariski dense subgroup from [Ben00] or [Kim06].

1.2. **Main setting.** In this article, we focus on a higher rank semisimple Lie group without compact factors G, meaning that $\dim A \geq 2$ and on an infinite covolume, discrete, Zariski dense subgroup Γ of G.

Let K be a maximal compact subgroup of G for which the Cartan decomposition KA^+K of elements of G holds. Denote by $M := Z_K(A)$ the centralizer subgroup of A in K.

For any $\theta \in \mathfrak{a}^+$, the *nonnegative diagonal flow* ϕ_θ^t corresponds to the right action by translation on $\Gamma\backslash G$ of $\exp(t\theta)$. When $\theta \in \mathfrak{a}^{++} \setminus \{0\}$, the flow ϕ_θ^t is called *positive diagonal*. Nonnegative diagonal flows ϕ_θ^t, where $\theta \in \mathfrak{a}^+$, induce right actions on $\Gamma\backslash G/M$, so called *Weyl chamber flows*. They are called *regular* when they are induced by positive diagonal flows. The latter will play the same role in higher rank as the geodesic flow in the unit tangent bundle of the hyperbolic orbifold.

1.3. **Mixing of regular Weyl chamber flows.** Conze–Guivarc'h [CG00] defined for $\mathrm{SL}(n,\mathbb{R})$ and Zariski dense discrete subgroups a right A-invariant closed subset $\Omega \subset \Gamma\backslash G/M$ (cf. § 5.1 for a detailed construction). Their construction generalizes to all semisimple Lie groups without compact factors.

Definition 1.1. *We denote by Ω the smallest closed A-invariant subset of $\Gamma\backslash G/M$ containing all periodic orbits of regular Weyl chamber flows and by Ω_G its preimage in $\Gamma\backslash G$.*

The closed subset Ω is the analogue for Weyl chamber flows of the non-wandering set of the geodesic flow in the hyperbolic case. With Glorieux [DG20], we obtained the following necessary and sufficient mixing condition for regular Weyl chamber flows.

Theorem 1.2 ([DG20]). *Let G be a connected, real linear, semi-simple Lie group, without compact factor. Let Γ be a Zariski dense, discrete subgroup of G.*

A regular Weyl chamber flow ϕ_θ^t is topologically mixing on Ω if and only if $\theta \in \overset{\circ}{\mathcal{C}}(\Gamma)$.

The limit cone $\mathcal{C}(\Gamma)$ was introduced by Benoist [Ben97b]. For every Zariski dense Γ, he proves that the limit cone is a closed, convex cone of $\mathfrak{a}^+$ of non-empty interior.

Definition 1.3. *Denote by $\lambda : G \to \mathfrak{a}^+$ the Jordan projection. The* limit cone *of Γ which is also called* Benoist cone *$\mathcal{C}(\Gamma)$, is the smallest closed cone of $\mathfrak{a}^+$ containing $\lambda(\Gamma)$.*

Mixing ratio for regular Weyl chamber flow ϕ_θ^t, where θ lies in the interior of the limit cone, were obtained by Thirion [Thi09] for Ping-Pong groups, Sambarino [Sam15] for Hitchin representations and Edwards–Lee–Oh [ELO20] for Borel Anosov groups.

1.4. **Main result.** We study the topological dynamics of non-negative diagonal flows $(\Omega_G, \phi_\theta^t)$. We focus on its topological mixing properties. Note that Ω_G is a right AM-invariant closed subset of $\Gamma\backslash G$ and a principal M-bundle over Ω, where M is not necessarily connected.

Using a result of Guivarc'h–Raugi [GR07], we partition Ω_G into finitely many A-invariant subsets that are dynamically conjugated to each other for nonnegative diagonal flows.

Theorem 1.4. *Let G be a connected, real linear, semi-simple Lie group, without compact factors. Let Γ be a Zariski dense, discrete subgroup of G.*

Then there exists a normal subgroup of finite index $M_0 \lhd M_\Gamma \lhd M$ and a partition of Ω_G denoted by $(\Omega_{[m]})_{[m]\in M/M_\Gamma}$ such that

(a) *every $\Omega_{[m]}$ is right AM_Γ-invariant and a principal M_Γ-bundle over Ω;*
(b) *for all $\theta \in \mathfrak{a}^+$, the dynamical systems $\{(\Omega_{[m]}, \phi_\theta^t)\}_{[m]\in M/M_\Gamma}$ are conjugated to each other;*
(c) *if $\theta \in \mathfrak{a}^{++}$ and $(\Omega_{[e_M]}, \phi_\theta^t)$ is topologically mixing then $\theta \in \overset{\circ}{\mathcal{C}}(\Gamma)$.*

If furthermore M_0 is abelian *and $\theta \in \mathfrak{a}^{++}$, then the converse of (c) is true:*

(d) $(\Omega_{[e_M]}, \phi_\theta^t)$ *is topologically mixing if and only if* $\theta \in \overset{\circ}{\mathcal{C}}(\Gamma)$.

We expect that condition (d) holds in the general case, because Maucourant–Schapira [MS19] proved topological mixing of the geodesic frame flow for $\mathrm{SO}(n,1)^0$ where $M = M_0 = \mathrm{SO}(n-1)$. Condition (c) is a consequence of the joint work with Glorieux.

Observe that M_0 is abelian for example: split real semisimple Lie groups i.e. $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}_0(p,p)$, $\mathrm{SO}_0(p,p+1)$; and also for $\mathrm{SU}(p,p+1)$, $\mathrm{SU}(p,p)$, $\mathrm{SO}_0(p,p+2)$ and $\mathrm{SL}(n,\mathbb{C})$. The closed, normal subgroup of finite index M_Γ of M containing the connected component of the identity M_0 of M, is defined in Guivarc'h–Raugi [GR07] by using the elliptic part of loxodromic elements of Γ. It was also defined and studied in the appendix of [Ben05]. We call it the *sign group* of Γ.

Labourie [Lab06] proved that M_Γ is trivial if Γ is the image of a Hitchin representations. It thus follows from the above result that in this case, there are 2^{n-1} disjoint subsets in $\Gamma\backslash\mathrm{PSL}(n,\mathbb{R})$ that share the same dynamical behavior for non-negative diagonal flows. Consequently, positive diagonal flows are topologically mixing on any of these subsets if and only if they are parametrized by directions of the interior of the limit cone.

For Borel Anosov subgroups and independently, Lee–Oh [LO20] prove that there is an A-ergodic decomposition of every BMS measure into AM_Γ-semi-invariant and A-ergodic measures parametrized by M/M_Γ. Any pair of such measures is the same up to right multiplication by elements of M/M_Γ, which concur with our result.

1.5. **Key ideas.**

Bruhat-Hopf coordinates. Denote by $\mathcal{F}^{(2)}$ the subset of transverse pairs in the Furstenberg boundary (cf. § 2.2) which identifies with G/AM (cf. Proposition 2.6). Thirion [Thi07] generalized Hopf coordinates in higher rank by parametrizing points of G/M with elements of $\mathcal{F}^{(2)} \times \mathfrak{a}$. The left action of G on G/M reads using the Iwasawa cocycle σ (cf. Definition 2.3) as follows

$$g(\xi, \check{\xi}\ ;\ x) = (g\xi, g\check{\xi}\ ;\ \sigma(g,\xi) + x).$$

The Weyl chamber flow reads by translating only the $\mathfrak{a}$ coordinate without changing the first two.

Consider the set $\{G_s\}_{s\in\mathcal{S}}$ of maximal Bruhat cells of G. For every $s \in \mathcal{S}$, we denote by $\mathcal{F}_s$ (resp. $\mathcal{F}_s^{(2)}$) the projection of G_s in $\mathcal{F}$ (resp. $\mathcal{F}^{(2)}$).

In Section 3, we construct *Bruhat-Hopf* coordinates $\mathcal{H}_s : G_s \to \mathcal{F}_s^{(2)} \times AM$ that extend Hopf coordinates (cf. Definition 3.2, 3.12, Proposition 3.10). Note that they differ from coordinates coming from the unique Bruhat decomposition of N^-MAN or their translate of the form hN^-MAN, where $h \in G$. The projection $G \to G/M$ reads for all $s \in \mathcal{S}$ by preserving the coordinates in $\mathcal{F}^{(2)}$ and projecting the AM-coordinates to $\mathfrak{a}$. The right translation by AM on G reads for all $(\xi, \check{\xi}; u)_s \in \mathcal{F}_s^{(2)} \times AM$ and $x \in AM$ as $(\xi, \check{\xi}; ux)_s$.

The left action of G on itself reads in this family of Bruhat-Hopf coordinates $(\mathcal{H}_s)_{s\in\mathcal{S}}$ equivariantly in the coordinates in $\mathcal{F}^{(2)}$ and via left multiplication by the *signed Iwasawa cocycles* $(\beta_{s',s})_{s,s'\in\mathcal{S}}$ (cf. Definition 3.7) of domains in $G \times \mathcal{F}$ and codomains in AM. They extend (cf. Proposition 3.10) the Iwasawa cocycle in the sense that for all $\xi \in \mathcal{F}_s$ and $g \in G$ such that $g\xi \in \mathcal{F}_{s'}$, then $\beta_{s',s}(g,\xi) \in \exp(\sigma(g,\xi))M$. We prove that the signed cocycles $(\beta_{s,s})_{s\in\mathcal{S}}$ are all cohomologous (cf. Fact 3.9) for the *transition maps* $\mathscr{T}_{s,s'} : \mathcal{F}_s \cap \mathcal{F}_{s'} \to AM$ of Definition 3.5.

Furthermore, Bruhat-Hopf coordinates induce local coordinates of K in $\mathcal{F} \times M$ by removing the second coordinate and projecting in M the third one.

Likewise, the reader can check that Bruhat-Hopf coordinates induce local coordinates on G/N, G/A and G/MN.

The elliptic part of loxodromic elements. Elements of G whose Jordan projection is in the positive Weyl chamber are called *loxodromic*. Denote by G^{lox} and Γ^{lox} the subset of loxodromic elements of the respective groups. Loxodromic elements (see §4) have trivial unipotent parts and are conjugated to elements in MA^{++}. The part in A^{++}, corresponding to the hyperbolic part, is given by the Jordan projection. In [Ben96], [Ben97b] and [Ben00], Benoist defines (r,ε)-loxodromic elements (see Definition 4.5) and obtains estimates for the Jordan projection of generic products of (r,ε)-loxodromic elements. We show that their elliptic part satisfy similar estimates.

The elliptic part of a loxodromic element is conjugated to an element of M which is defined up to conjugacy by M. Therefore, the latter is only well defined when M is abelian, in which case one can extend the Jordan projection from G^{lox} to $\mathfrak{a}^{++}\times M$. Bruhat-Hopf coordinates gives a framework to solve this technical difficulty in the general case.

Fix a loxodromic element g and denote by g^+ (resp. g^-) its attracting (resp. repelling) fixed point in $\mathcal{F}$ and by $\mathsf{b}(g^-)$ the basin of attraction of g^+ (cf. Proposition 4.4). Starting from the formula $\sigma(g,g^+)=\lambda(g)$ satisfied by loxodromic elements, we define a multiplicative and signed Jordan projection for g. For every $s\in\mathcal{S}$ such that $g^+\in\mathcal{F}_s$, we set $\mathscr{L}_s(g):=\beta_{s,s}(g,g^+)$. It is the unique element in $\exp(\lambda(g))M$ such that there is an element $h_s\in G_s$ unique up to right multiplication by A such that $h_s^{-1}gh_s=\mathscr{L}_s(g)$.

Using the continuous maps $\mathscr{R}_{s',s}$ given in Definition 4.8, we obtain an exact formula.

Proposition 1.5 (4.9 below). *Let G be a connected, real linear, semi-simple Lie group, without compact factor.*

Then for all loxodromic element $g\in G^{lox}$, all integer $n\geq 1$ and $\xi\in\mathsf{b}(g^-)$, for any suitable $s_0,s_1,s_2\in\mathcal{S}$ such that $(\xi,g^+,g^n\xi)\in\mathcal{F}_{s_0}\times\mathcal{F}_{s_1}\times\mathcal{F}_{s_2}$

$$\beta_{s_2,s_0}(g^n,\xi)=\mathscr{R}_{s_1,s_2}(g;g^n\xi)^{-1}\mathscr{L}_{s_1}(g)^n\mathscr{R}_{s_1,s_0}(g;\xi).$$

We estimate the elliptic part of generic products of (r,ε)-loxodromic elements. In order to do that, we introduce a family of constants $\{\delta_{r,\varepsilon}\mid 0<\varepsilon\leq r\}$ (cf. Definition 4.11) such that for all $r>0$, they satisfy $\lim_{\epsilon\to 0}\delta_{r,\epsilon}=0$ (cf. Proposition 4.10).

Proposition 1.6 (4.12 below). *Let G be a connected, real linear, semi-simple Lie group, without compact factor. For all $r>0$ and $\varepsilon\in(0,r]$ and every family $g_1,...,g_l\in G$ of (r,ε)-loxodromic elements such that*

$\star$ $r\leq\frac{1}{6}\mathrm{d}\big(\{g_{i-1}^+,g_i^+\},\partial\mathsf{b}(g_i^-)\big)$ *for all $1\leq i\leq l$ with the convention $g_0=g_l$.*

For all family $(s_i)_{0\leq i\leq l}\subset\mathcal{S}$ such that

$\star\star$ $\mathcal{F}_{s_i}\supset\mathcal{V}_r(\partial\mathsf{b}(g_i^-))^{\complement}$ *for every $1\leq i\leq l$ and $\mathcal{F}_{s_0}\supset\mathcal{V}_\varepsilon(\partial\mathsf{b}(g_1^-))^{\complement}$.*

Then for all integers $n_1,...,n_l\geq 1$, the element $g_l^{n_l}...g_1^{n_1}$ is $(2r,2\varepsilon)$-loxodromic with attracting (resp. repelling) point in $B(g_l^+,\varepsilon)$ (resp. $B(g_1^-,\varepsilon)$) and its extended Jordan projection satisfies

$$\mathscr{L}_{s_l}(g_l^{n_l}...g_1^{n_1})\in\mathscr{L}_{s_l}(g_l^{n_l})\mathscr{R}_{s_l,s_{l-1}}(g_l,g_{l-1}^+)...\mathscr{L}_{s_1}(g_1^{n_1})\mathscr{R}_{s_1,s_l}(g_1,g_l^+)B(e_{AM},2l\delta_{r,\varepsilon}).$$

Decorrelation. Denote by M^{ab} the abelianization of M. We define an abelianized Jordan projection for loxodromic elements $\mathscr{L}^{ab}:G^{lox}\to A^{++}M^{ab}$ using the previous local Jordan projections $\mathscr{L}_s$. The number of connected components of M^{ab} reached by the subset $\mathscr{L}^{ab}(\Gamma^{lox})$ suffices to understand M_Γ. Indeed, its abelianized M_Γ^{ab} is the subgroup of M^{ab} generated by the projection to M^{ab} of $\mathscr{L}^{ab}(\Gamma^{lox})$. Thanks to Guivarc'h–Raugi [GR07, Theorem 6.4] we deduce that the subgroup generated by $\mathscr{L}^{ab}(\Gamma^{lox})$ is dense in AM_Γ^{ab}. They also give a classification of

Γ-invariant minimal subsets of K. We rewrite their result using Bruhat-Hopf coordinates of K in Theorem 5.9 and define the invariant subsets $\Omega_{[m]}$ through their universal cover $\widetilde{\Omega}_{[m]}$ in G.

Denote by $L(\Gamma) \subset \mathcal{F}$ the limit set of Γ and by $L^{(2)}(\Gamma) := L(\Gamma) \times L(\Gamma) \cap \mathcal{F}^{(2)}$. The universal cover $\widetilde{\Omega}_G$ has Bruhat-Hopf coordinates $L^{(2)}(\Gamma) \times AM$.

Without loss of generality, by using the joint work with Glorieux [DG20], it suffices to prove the *decorrelation Proposition* 6.1 i.e. that there exists $(\xi_1, \check{\xi}_1) \in L^{(2)}(\Gamma)$ such that for every $x \in AM$ and small $\delta > 0$, the orbit $\Gamma(\xi_1, \check{\xi}_1 \; ; \; x)_{\check{c}_1}$ is δ-dense in an M_Γ-orbit of the form $(\xi_1, \check{\xi}_1 \; ; \; y_\delta x M_\Gamma)_{c_1}$ (for suitable $\check{c}_1, c_1 \in \mathcal{S}$).

The first step (Lemma 6.2) is to reach all connected components of M_Γ by the left action of finitely many (r,ε)-loxodromic elements of Γ of attracting point close to ξ_1. It does not use that M abelian.

In the second step (Lemma 6.4) we construct (r,ε)-loxodromic elements $\gamma_1, ..., \gamma_l \in \Gamma$ that satisfy the hypothesis of Proposition 1.6 and such that $\mathscr{L}^{ab}(\{\gamma_l^{n_l} ... \gamma_1^{n_1} \mid n_1, .., n_l \geq 1\})$ is δ-dense in an M_0-invariant set that projects to $\log \pi_A(y_\delta x) + \mathcal{C}_0$, where $\mathcal{C}_0 \subset \mathfrak{a}^{++}$ is a closed convex cone of non-empty interior. We rely on density of squares in M_0, as well as density lemmata deduced from the assumption that M_0 is abelian.

Finally, we use an overlapping cone argument to deduce the decorrelation.

1.6. **Organization of the paper.** In Section 2 we recall the classical Iwasawa, Bruhat decompositions of Lie groups and characterize the transverse points in the Furstenberg boundary. Section 3 is dedicated to the construction of Bruhat-Hopf coordinates. In Section 4, using Bruhat-Hopf coordinates, we estimate the elliptic part of products of generic loxodromic elements. In Section 5, we define the subgroup M_Γ, the Γ-invariant subsets of G and prove Theorem 1.4 (a)(b). Section 6 is dedicated to the proof of decorrelation. In Section 7 we prove the necessary and sufficient condition for topological mixing when M_0 abelian. In the appendix, we prove the density lemmata.

Relation to other works. Sections 2, 5, 6, 7 and the Appendix can be found in french in the author's PhD thesis [Dan19]. Sections 3 and 4 improve the thesis's construction of Bruhat-Hopf coordinates and its estimates of the elliptic and hyperbolic parts of products of loxodromic elements.

Bruhat-Hopf coordinates were independently studied by Lee–Oh [LO20].

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2. Background

> In the whole article, G is a semisimple, connected, real linear Lie group, without compact factor.

A classical reference for this section is [Hel01]. Let K be a maximal compact subgroup of G. Denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) the Lie algebra of G (resp. K). Consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a *Cartan subspace* i.e. a maximal abelian subspace of $\mathfrak{p}$ for which the adjoint endomorphism of every element is semisimple. Denote by $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

For every linear form $\alpha \in \mathfrak{a}^*$, set $\mathfrak{g}_\alpha := \{v \in \mathfrak{g} \mid \forall u \in \mathfrak{a},\ [u,v] = \alpha(u)v\}$. Note that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$. The set of *restricted roots* is given by $\Sigma := \{\alpha \in \mathfrak{a}^* \setminus 0 \mid \mathfrak{g}_\alpha \neq 0\}$. By simultaneous diagonalisation over the abelian family of endomorphisms $ad(\mathfrak{a})$, we deduce the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. Note that Σ is a finite set. Let us now choose a *positive Weyl chamber* of $\mathfrak{a}$ i.e. a connected component of $\mathfrak{a} \setminus \cup_{\alpha\in\Sigma} \ker(\alpha)$. Denote the closed positive Weyl chamber by $\mathfrak{a}^+$ and $\mathfrak{a}^{++}$ its interior. The set of *positive roots*, denoted by Σ^+, is the subset of restricted roots which take positive values in the positive Weyl chamber. This choice allows to define two particular nilpotent subalgebras $\mathfrak{n} = \oplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \oplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha}$.

Finally, denote by $A := \exp(\mathfrak{a})$ the maximal $\mathbb{R}$-split torus, $A^+ := \exp(\mathfrak{a}^+)$ the closed positive Weyl chamber, $A^{++} := \exp(\mathfrak{a}^{++})$ its interior, $N := \exp(\mathfrak{n})$ (resp. $N^- := \exp(\mathfrak{n}_-)$) the positive (resp. negative) maximal unipotent subgroups and M the centralizer of A in K, of Lie algebra $\mathfrak{m}$. By definition, A normalizes N and N^-. Furthermore, for all $a \in A^{++}$ and $h_\pm \in N^\pm$ the following convergences hold

$$a^{-n} h_\pm a^n \underset{\pm\infty}{\longrightarrow} e_G. \tag{1}$$

2.1. Furstenberg boundary.

By Iwasawa decomposition (cf. [Hel01, Chapter IX, Thm 1.3]) $G = KAN$ and $G = KAN^-$ and the maps (with the convention that $N^+ = N$)

$$\begin{aligned} K \times A \times N^\pm &\longrightarrow G \\ (k,a,n) &\longmapsto kan \end{aligned}$$

are diffeomorphisms. Denote by $g \mapsto \big(k_{\mathcal{I}^\pm}(g), a_{\mathcal{I}^\pm}(g), u_{\mathcal{I}^\pm}(g)\big) \in K \times A \times N^\pm$ the respective inverse diffeomorphisms. Note[3] that $[\mathfrak{g}_0, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$ for all $\alpha \in \Sigma_+$. Hence $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_-$ are Lie subalgebras of $\mathfrak{g}$. Consequently MAN and MAN^- are closed subgroups of G.

Definition 2.1. *The* Furstenberg boundary *is defined by $\mathcal{F} := G/MAN$. Denote by $k_\iota \in K$ a representative of the element in the Weyl group such that* $\mathrm{Ad}(k_\iota)\mathfrak{a}^+ = -\mathfrak{a}^+$. *Set $\eta_0 := MAN$ and $\check{\eta}_0 := k_\iota \eta_0$.*

The map $k \in K \mapsto k\eta_0 \in \mathcal{F}$ is surjective and equivariant for the left action of K. Furthermore, the stabilizer of η_0 is the closed subgroup M. Therefore, we deduce an identification of K/M with the Furstenberg boundary.

Let us sketch the construction of a K-invariant Riemannian distance on K. Start from a scalar product on $\mathfrak{k}$. Since K is a compact subgroup, its Haar measure is finite. By averaging the scalar product on $\mathfrak{k}$ along the Haar measure on K for the adjoint action, we obtain an $Ad(K)$-invariant scalar product and norm on $\mathfrak{k}$. Using the left action of K, we transport them on every tangent space and obtain a left K-invariant metric which is also invariant by conjugation. Hence K is endowed with an invariant Riemannian metric. Its induced Riemannian distance is thus K-invariant.

Definition 2.2. *Let d_K be a K-invariant Riemannian distance on K. For every $\xi, \eta \in \mathcal{F}$ for any choice of representatives $k_\xi, k_\eta \in K$ such that $k_\xi\eta_0 = \xi$ and $k_\eta\eta_0 = \eta$, we consider the induced left K-invariant distance in $\mathcal{F}$*

$$\mathrm{d}(\xi,\eta) := d_K(k_\xi M, k_\eta M).$$

Let us define the Iwasawa cocycle.

[3] using Jacobi identity

Definition 2.3. *For all $g \in G$ and $\xi \in \mathcal{F}$, we denote by $\sigma(g,\xi)$ the unique element*[4] *in $\mathfrak{a}$ such that for all $k_\xi \in K$ such that $k_\xi \eta_0 = \xi$,*

$$gk_\xi \in K \exp(\sigma(g,\xi))N.$$

The map $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is the Iwasawa cocycle.

2.2. **Transverse pairs in the Furstenberg boundary.** The following subset of $\mathcal{F} \times \mathcal{F}$ is a higher rank analogue to the set of pair of points in the geometric boundary of the hyperbolic plane $\mathbb{H}^2$ that parametrize oriented geodesics. It also identifies for $\mathrm{SL}(n,\mathbb{R})$ with the space of transverse complete flags of $\mathbb{R}^n$.

Definition 2.4. *The subset of* ordered transverse pairs *of $\mathcal{F} \times \mathcal{F}$ is defined by*

$$\mathcal{F}^{(2)} := \{(g\eta_0, g\check{\eta}_0) \mid g \in G\}.$$

Since k_ι is an involution, $(\check{\eta}_0, \eta_0)$ is also an ordered transverse pair. Consequently, we say that $\xi, \eta \in \mathcal{F}$ are transverse *if any of the ordered pairs (ξ,η) of (η,ξ) are transverse.*

Denote by $W := N_K(A)/Z_K(A)$ the Weyl group of G. We choose for every $w \in W$ a representative $k_w \in N_K(A)$. Then by Bruhat decomposition [Hel01, Chapter IX, Thm 1.4],

$$G = \sqcup_{w \in W} B k_w B$$

where $B = MAN$. Note that $N^- = k_\iota N k_\iota^{-1}$ and that $G = \sqcup_{w\in W} k_\iota B k_w B$, meaning that N^-MAN is a cell in the Bruhat decomposition of G.

Corollary 2.5 (Chapter IX, Cor. 1.9 [Hel01])**.** *The map*

$$\begin{aligned} N^- &\longrightarrow N^-\eta_0 \\ n_- &\longmapsto n_-\eta_0 \end{aligned}$$

is a diffeomorphism, its image is an open submanifold of $\mathcal{F}$ and its complement is a finite union of disjoint submanifolds of stricly smaller dimensions.

Thus N^-MAN is a maximal cell for the Bruhat decomposition. We will call sets of the form hN^-MAN as well as their projection to $\mathcal{F}$, where $h \in G$, *maximal Bruhat cells*. We describe below the subset of transverse pairs in the Furstenberg boundary and include a proof for completeness.

Proposition 2.6. *The following holds,*

(i) *the set of transverse points to $\check{\eta}_0$ is $N^-\eta_0$,*
(ii) *for all $\eta, \xi \in \mathcal{F}$ and $k_\eta, \check{k}_\xi \in K$ such that $k_\eta \eta_0 = \eta$ and $\check{k}_\xi \check{\eta}_0 = \xi$,*
$$(\eta,\xi) \in \mathcal{F}^{(2)} \iff \check{k}_\xi^{-1} k_\eta \in N^-MAN,$$
(iii) *for all $\xi \in \mathcal{F}$ and $\check{k}_\xi \in K$ such that $\check{k}_\xi \check{\eta}_0 = \xi$, the set of transverse points to ξ is $\check{k}_\xi N^- \eta_0$.*
(iii') *for all $\xi \in \mathcal{F}$ and $k_\xi \in K$ such that $k_\xi \eta_0 = \xi$, the set of transverse points to ξ is $k_\xi N \check{\eta}_0$.*

Furthermore, the $G-$equivariant map

$$\begin{aligned} G/AM &\longrightarrow \mathcal{F}^{(2)} \\ gAM &\longmapsto (g\eta_0, g\check{\eta}_0) \end{aligned}$$

is a diffeomorphism.

[4]because M normalises N, this element does not depend on the choice of the representative in K of ξ.

Proof. (i) First remark that $N^-(\eta_0, \check{\eta}_0) = (N^-\eta_0, \check{\eta}_0)$. Let us now prove the converse i.e. that any point transverse to $\check{\eta}_0$ must be in $N^-\eta_0$. Let $g \in G$ such that $(g\eta_0, \check{\eta}_0) \in \mathcal{F}^{(2)}$. Then by definition, there exists $h \in G$ such that

$$(g\eta_0, \check{\eta}_0) = h(\eta_0, \check{\eta}_0).$$

On one hand $h\check{\eta}_0 = \check{\eta}_0$, hence $h \in \mathrm{Stab}(\check{\eta}_0) = k_\iota MANk_\iota^{-1}$. Since $N^- = k_\iota N k_\iota^{-1}$ and MA is invariant by conjugation by k_ι, we deduce that

$$h \in MAN^-.$$

On the other hand $g\eta_0 = h\eta_0$, hence $h^{-1}g \in \mathrm{Stab}(\eta_0) = MAN$. Thus

$$g \in hMAN \subset MAN^-MAN.$$

Since MA normalizes N^-, we deduce that $g \in N^-MAN$. Hence $g\eta_0 \in N^-\eta_0$.

(ii) It follows from (i) and by noticing that the pair $(k_\eta\eta_0, \check{k}_\xi\check{\eta}_0) \in \mathcal{F}^{(2)}$ if and only if $(\check{k}_\xi^{-1}k_\eta\eta_0, \check{\eta}_0) \in \mathcal{F}^{(2)}$.

(iii) It follows from (ii) since $\check{k}_\eta(N^-\eta_0, \check{\eta}_0) \in \mathcal{F}^{(2)}$.

For the last statement, remark first that G acts transitively on $\mathcal{F}^{(2)}$. Furthermore

$$\mathrm{Stab}_G(\eta_0, \check{\eta}_0) = MAN \cap MAN^- = AM.$$

We thus deduce the G-equivariance and bijectivity of the map

$$\begin{aligned} G/AM &\longrightarrow \mathcal{F}^{(2)} \\ gAM &\longmapsto (g\eta_0, g\check{\eta}_0). \end{aligned}$$

The left action of G on the Furstenberg boundary $\mathcal{F} = G/MAN$ is differentiable and so is its action on $\mathcal{F} \times \mathcal{F}$. Thus, the map $g \mapsto (g\eta_0, g\check{\eta}_0)$ is differentiable. The kernel of the differential in e_G of the map $g \mapsto (g\eta_0, g\check{\eta}_0)$ contains $\mathfrak{m} \oplus \mathfrak{a}$. Since the maps $N^- \to N^-\eta_0$ and $N \to N\check{\eta}_0$ are diffeomorphisms, the differential in e_G of $g \mapsto (g\eta_0, g\check{\eta}_0)$ is surjective from $\mathfrak{g}$ to $\mathfrak{n}_- \oplus \mathfrak{n}_+$. By Bruhat decomposition in the Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, we deduce that the kernel of the differential in e_G of $g \mapsto (g\eta_0, g\check{\eta}_0)$ is equal to $\mathfrak{a} \oplus \mathfrak{m}$. Thus, the map $G/AM \to \mathcal{F}^{(2)}$ is a local diffeomorphism in AM. Finally, by transitivity of the left G action on G/AM, we deduce that it is a diffeomorphism. □

We parametrize the maximal Bruhat cells of the Furstenberg boundary.

Definition 2.7. *Let $\check{\eta} \in \mathcal{F}$, then for any representative $h(\check{\eta}) \in G$ such that $\check{\eta} = h(\check{\eta})\check{\eta}_0$, we denote by $\mathsf{b}(\check{\eta}) := h(\check{\eta})N^-\eta_0$ the maximal Bruhat cell opposite to $\check{\eta}$.*

Thanks to the previous Proposition, the representative $h(\check{\eta}) \in G$ is chosen up to right multiplication by MAN^-. Remark that $\mathsf{b}(\eta_0) = N\check{\eta}_0$ and $\mathsf{b}(\check{\eta}_0) = N^-\eta_0$. Using this notation, the set of Bruhat cells of $\mathcal{F}$ is naturally endowed with a left action of G which satisfies $h\mathsf{b}(\check{\eta}_0) := \mathsf{b}(h\check{\eta}_0)$ for all $h \in G$.

3. Bruhat-Hopf coordinates

In his thesis, Thirion [Thi07, Chapter 8 §8.G.2] introduced Hopf coordinates for $\mathrm{SL}(n,\mathbb{R})/M$. His construction generalizes to every semisimple Lie group without compact factors. It is defined by the following map

$$\begin{aligned} \mathcal{H} : G/M &\longrightarrow \mathcal{F}^{(2)} \times \mathfrak{a} \\ hM &\longmapsto (h\eta_0, h\check{\eta}_0 \; ; \; \sigma(h, \eta_0)). \end{aligned}$$

The left action of G and right action of A on G/M read in those coordinates as follows. For all $(g,\theta,t)\in G\times\mathfrak{a}\times\mathbb{R}$ and $(\xi,\check{\xi}\ ;\ x)\in\mathcal{F}^{(2)}\times\mathfrak{a}$,

$$\tilde{\phi}^t_\theta\big(g(\xi,\check{\xi}\ ;\ x)\big)=(g\xi,g\check{\xi}\ ;\ \sigma(g,\xi)+x+t\theta).$$

The projection $G/M\to\mathcal{F}$ (resp. $G/M\to G/AM$) reads as the projection to the first coordinate in $\mathcal{F}$ (resp. by removing the coordinate in $\mathfrak{a}$).

In this section, we extend locally and equivariantly (for the left action of G and right action of A) Hopf coordinates to G.

Any local trivialisation of $G\to G/AM$ provides local coordinates in $\mathcal{F}^{(2)}\times AM$ that are equivariant for the right action of A. The restricted left G-action provides a local AM-cocycle. In general, neither these cocycles extend the Iwasawa cocycle nor will those coordinates locally extend Hopf coordinates. We give a general method to extend locally Hopf coordinates to $\mathcal{F}^{(2)}\times AM$, while the local AM-cocycle extends the Iwasawa cocycle.

In §3.1, using Bruhat decomposition, we construct from any cross-sections s of $G\to\mathcal{F}$ of domain $\mathcal{F}_s$, open sets $G_s:=s(\mathcal{F}_s)MAN$ of G and the local coordinate map $\mathcal{B}_s: G_s\hookrightarrow\mathcal{F}^{(2)}\times AM$ (cf. Definition 3.2). We set notations for the rest of the article and in Definition 3.3, define covering families of cross-sections and of the same type (i.e. that are translates of one another by G-action).

In §3.2, we set notations for the transition functions between different sets of coordinates in Definition 3.5. In Proposition 3.6 we compute these functions in some cases and prove that the cross-section parameters satisfy chain rule relations.

In §3.3, for every family of differentiable cross-sections $(s_i)_{i\in I}$ of $G\to\mathcal{F}$ whose domain cover $\mathcal{F}$, we read the left action of G on itself in the $\mathcal{B}_{s_i}$ coordinates. The behavior is the same as for Hopf coordinates for the coordinates in $\mathcal{F}^{(2)}$. We define AM-valued functions in Definition 3.7 of domain in $G\times\mathcal{F}$. We prove in Proposition 3.8 that those functions are cocycles that encode the information in AM for the left action of G on itself. This implies in particular that the information contained in the second and third coordinate in $\mathcal{F}^{(2)}\times AM$ are not needed when one reads the left action of G. In Fact 3.9, we obtain for the cocyle a chain rule formula compatible with the one we had for the transition functions.

In §3.4, we prove that when the cross-section s takes value in K, then the coordinate map $\mathcal{B}_s$ extends the Hopf coordinates. Indeed, in Proposition 3.10, we prove that when the cross-sections $(s_i)_{i\in I}$ take value in K, the signed multiplicative Iwasawa cocycles $(\beta_{s_i,s_j})_{i,j\in I}$ defined in the third paragraph generalize the Iwasawa cocycle. We obtain an equivariant and commutative diagram with Hopf coordinates.

In §3.5, we prove in Proposition 3.11 that local coordinates of G that extends Hopf coordinates provide local coordinates of K that take value in $\mathcal{F}\times M$. Furthermore, the map $k_{\mathcal{I}}: G\to K$ reads in those coordinates by keeping the first coordinate in $\mathcal{F}$ and projecting the last one in M.

In the last paragraph, using Bruhat decomposition and Iwasawa decomposition, we construct two families of cross-sections of $G\to\mathcal{F}$ defined on Bruhat cells of $\mathcal{F}$: *unipotent* and *compact Bruhat sections* in Definition 3.12 We define *Bruhat-Hopf coordinates* as the local extensions of Hopf coordinates given by Proposition 3.10 with respect to the compact Bruhat sections. In Proposition 3.14 we parametrize these cross-sections.

3.1. **Local trivialisations.** Let s be a non-trivial cross-section of the MAN-bundle $G\to\mathcal{F}$, we denote by $\mathcal{F}_s$ its domain. Denote by $G_s:=s(\mathcal{F}_s)MAN$ the preimage of $\mathcal{F}_s$ by the projection $G\to\mathcal{F}$, by $\mathcal{F}_s^{(2)}:=G_s(\eta_0,\check{\eta}_0)=(\mathcal{F}_s\times\mathcal{F})\cap\mathcal{F}^{(2)}$ the image of G_s by the projection $G\to\mathcal{F}^{(2)}$. The following Fact will allow us to define the coordinate map $G_s\to\mathcal{F}_s^{(2)}\times AM$.

Fact 3.1. *Let s be a differentiable cross-section of $G\to\mathcal{F}$.*

Then the two maps below are diffeomorphisms.

$$\begin{aligned}\mathcal{F}_s \times N \times AM &\longrightarrow s(\mathcal{F}_s)NAM \subset G\\ (\xi, u, x) &\longmapsto s(\xi)ux.\end{aligned}$$

$$\begin{aligned}\mathcal{F}_s \times N \times AM &\longrightarrow \mathcal{F}_s^{(2)} \times AM\\ (\xi, u, x) &\longmapsto (\xi, s(\xi)u\check{\eta}_0 \ ; \ x)_s.\end{aligned}$$

Proof. By hypothesis, the map $s : \mathcal{F}_s \to G$ is a cross-section and by Iwasawa decomposition in NAM, we deduce that the first map is a diffeomorphism.

By Proposition 2.6 (iii') for every $\xi \in \mathcal{F}_s$, the set of transverse points to ξ is $s(\xi)N\check{\eta}_0$. Hence the map

$$\begin{aligned}\mathcal{F}_s \times N &\longrightarrow \mathcal{F}_s^{(2)}\\ (\xi, u) &\longmapsto (\xi, s(\xi)u\check{\eta}_0)\end{aligned}$$

is a diffeomorphism. Consequently, the second map is a diffeomorphism. □

Definition 3.2. *For every differentiable cross-section s of $G \to \mathcal{F}$, we denote by $\mathcal{B}_s$ the following differentiable coordinate map*

$$\begin{aligned}\mathcal{B}_s : G_s \subset G &\longrightarrow \mathcal{F}_s^{(2)} \times AM\\ g = s(\xi)ux &\longmapsto (g\eta_0, g\check{\eta}_0 \ ; \ x)_s.\end{aligned}$$

When s is compact valued i.e. a cross-section of $K \to \mathcal{F}$, the same map is denoted by $\mathcal{H}_s$.

In order to write in such coordinates every element of G, we construct families of differentiable cross-sections whose domain cover $\mathcal{F}$. For all $g \in G$ and any cross-section $s : \mathcal{F}_s \to G$, we define the left translate by

$$\begin{aligned}g \cdot s : g\mathcal{F}_s &\longrightarrow G\\ \xi &\longmapsto gs(g^{-1}\xi).\end{aligned}$$

This provides a left G action on the space of cross-sections of $G \to \mathcal{F}$. For any $b \in MAN$, we define the cross-section

$$\begin{aligned}s.b : \mathcal{F}_s &\longrightarrow G\\ \xi &\longmapsto s(\xi)b.\end{aligned}$$

Definition 3.3. *A family of cross-section $(s_i)_{i\in I}$ of the bundle $G \to \mathcal{F}$ is* covering *when the family of domains $\{\mathcal{F}_{s_i}\}_{i\in I}$ covers $\mathcal{F}$ i.e.*

$$\mathcal{F} \subset \cup_{i\in I}\mathcal{F}_{s_i}.$$

The family $(s_i)_{i\in I}$ is of the same type if for any $i, j \in I$ there exists $g_{ij} \in K$ such that

$$s_i = g_{ij} \cdot s_j.$$

Since $\mathcal{F}$ is compact and the action $K \curvearrowright \mathcal{F}$ is transitive, one can construct covering families of differentiable cross-sections of the same type. We provide two such families in Definition 3.12.

3.2. **Transition functions.** Given two differentiable cross-sections s, s' of $G \to \mathcal{F}$, we compute the change of coordinates. This information is contained in the so-called transition map $\mathscr{T}_{s,s'}$ below. We prove in Proposition 3.6 that the cross-section subscripts in the notation $\mathscr{T}_{s,s'}$ follow a chain rule relation and compute for all $b \in MAN$ the transition function between s and $s.b$.

Fact 3.4. *Let s and s' be two differentiable cross-sections of $G \to \mathcal{F}$ such that $\mathcal{F}_s \cap \mathcal{F}_{s'} \neq \emptyset$. Then for all $\xi \in \mathcal{F}_s \cap \mathcal{F}_{s'}$,*

$$a_{\mathcal{I}}\big(s(\xi)^{-1}s'(\xi)\big)\ \ k_{\mathcal{I}}\big(s(\xi)^{-1}s'(\xi)\big) \in AM.$$

Proof. For every $\xi \in \mathcal{F}_s \cap \mathcal{F}_{s'}$, we denote by $\mathscr{T}_{s,s'}(\xi) := a_{\mathcal{I}}\big(s(\xi)^{-1}s'(\xi)\big)\ \ k_{\mathcal{I}}\big(s(\xi)^{-1}s'(\xi)\big)$. Due to the hypothesis that s and s' are both cross-sections of $G \to \mathcal{F}$, we deduce that $s'(\xi) \in s(\xi)MAN$. Hence the compact part $k_{\mathcal{I}}\big(s(\xi)^{-1}s'(\xi)\big)$ is in M and $\mathscr{T}_{s,s'}(\xi)$ is in AM. □

Definition 3.5. *Let s and s' be two differentiable cross-sections of $G \to \mathcal{F}$ such that $\mathcal{F}_s \cap \mathcal{F}_{s'} \neq \emptyset$. We define the* transition map

$$\begin{aligned}\mathscr{T}_{s,s'} : \mathcal{F}_s \cap \mathcal{F}_{s'} &\longrightarrow AM\\ \xi &\longmapsto a_{\mathcal{I}}\big(s(\xi)^{-1}s'(\xi)\big)\ \ k_{\mathcal{I}}\big(s(\xi)^{-1}s'(\xi)\big),\end{aligned}$$

which associate to every $\xi \in \mathcal{F}_s \cap \mathcal{F}_{s'}$, the unique element in AM such that

$$s'(\xi) \in s(\xi)N\mathscr{T}_{s,s'}(\xi).$$

Remark that if both s and s' take value in K, then the transition functions takes value in M. Let us compute the change of coordinates between $\mathcal{B}_s$ and $\mathcal{B}_{s'}$.

Proposition 3.6. *Let s and s' be differentiable cross-sections of $G \to \mathcal{F}$ such that $\mathcal{F}_s \cap \mathcal{F}_{s'} \neq \emptyset$. Then the following holds.*

(i) *The map $\mathscr{T}_{s,s'}$ is differentiable and the identity map of $s(\mathcal{F}_s \cap \mathcal{F}_{s'})NAM$ reads in $\mathcal{B}_{s'}$ and $\mathcal{B}_s$ coordinates as follows:*

$$\begin{aligned}\big(\mathcal{F}^{(2)}_{s'} \cap \mathcal{F}^{(2)}_{s}\big) \times AM &\longrightarrow \big(\mathcal{F}^{(2)}_{s} \cap \mathcal{F}^{(2)}_{s'}\big) \times AM\\ (\xi, \check{\xi}\ ;\ x)_{s'} &\longmapsto (\xi, \check{\xi}\ ;\ \mathscr{T}_{s,s'}(\xi)x)_s.\end{aligned}$$

(ii) *For all differentiable cross-section s'' such that $\mathcal{F}_{s''} \cap \mathcal{F}_{s'} \cap \mathcal{F}_s \neq \emptyset$, and all ξ in the triple intersection,*

$$\mathscr{T}_{s'',s}(\xi) = \mathscr{T}_{s'',s'}(\xi)\mathscr{T}_{s',s}(\xi).$$

(iii) *For all $\xi \in \mathcal{F}_{s'} \cap \mathcal{F}_s$,*

$$\mathscr{T}_{s',s}(\xi) = \mathscr{T}_{s,s'}(\xi)^{-1}.$$

(iv) *For all $x \in AM$ and $u \in N$,*

$$\mathscr{T}_{s,s.xu} = x = \mathscr{T}_{s,s.ux}.$$

The first three points enforce the computational 'chain rule' that double cross-sections subscript cancel.

Proof. (i) note that $s'(\mathcal{F}_s \cap \mathcal{F}_{s'})NAM = s(\mathcal{F}_s \cap \mathcal{F}_{s'})NAM = G_s \cap G_{s'}$ since s and s' are both cross-sections of $G \to \mathcal{F}$. We want to write every element in $G_s \cap G_{s'}$ in $\mathcal{B}_s$ and $\mathcal{B}_{s'}$ coordinates. By Definition 3.2, the first two coordinates in $\mathcal{F}^{(2)}$ do not depend on s and s'. We only need to compute the change in the last coordinate. Fix an element $g \in s'(\mathcal{F}_s \cap \mathcal{F}_{s'})NAM$ and denote by $(\xi, \check{\xi}\ ;\ x)_{s'} \in \mathcal{F}^{(2)}_{s'} \times AM$ its coordinates with respect to the section s'. Using Fact 3.1 on g and s', there exists a unique element $u_{\check{\xi}} \in N$ such that g admits the following decomposition

$$g = s'(\xi)u_{\check{\xi}}x.$$

Let us deduce the last $\mathcal{B}_s$ coordinate of g by finding its decomposition in $s(\mathcal{F}_s \cap \mathcal{F}_{s'})NAM$. Since $\xi \in \mathcal{F}_{s'} \cap \mathcal{F}_s$, by Definition 3.5, there exists a unique element $u_{s',s}(\xi) \in N$ such that

$$s'(\xi) = s(\xi) u_{s',s}(\xi) \mathscr{T}_{s,s'}(\xi).$$

Then we replace it in $s'(\xi)u_{\check{\xi}}x$,

$$s'(\xi)u_{\check{\xi}}x = s(\xi)u_{s',s}(\xi)\mathscr{T}_{s,s'}(\xi)u_{\check{\xi}}x.$$

Since AM normalizes N and $\mathscr{T}_{s,s'}(\xi) \in AM$, we deduce the following $s(\mathcal{F}_s \cap \mathcal{F}_{s'})NAM$ decomposition of g,

$$g = s'(\xi)u_{\check{\xi}}x = s(\xi)\ \left(u_{s',s}(\xi)\mathscr{T}_{s,s'}(\xi)u_{\check{\xi}}\mathscr{T}_{s,s'}(\xi)^{-1}\right)\ \mathscr{T}_{s,s'}(\xi)x.$$

Hence, the $\mathcal{B}_s$-coordinates of g is $\left(\xi, \check{\xi}\ ;\ \mathscr{T}_{s,s'}(\xi)x\right)_s$.

(ii) is a direct consequence of the relation $\mathcal{B}_s\mathcal{B}_{s''}^{-1} = \mathcal{B}_s\mathcal{B}_{s'}^{-1}\ \mathcal{B}_{s'}\mathcal{B}_{s''}^{-1}$ where each map is restricted to $s\big(\mathcal{F}_{s''} \cap \mathcal{F}_{s'} \cap \mathcal{F}_s\big)NAM$.

(iii) follows from (ii) since $e_{AM} = \mathscr{T}_{s,s} = \mathscr{T}_{s,s'}\mathscr{T}_{s',s}$.

(iv) we recall that for all $x \in AM$ and $u \in N$ the section $s.xu$ (resp. $s.ux$) is defined for every $\xi \in \mathcal{F}_s$ by $s.xu(\xi) = s(\xi)xu$ (resp. $s.ux(\xi) = s(\xi)ux$). Using that AM normalises N, we deduce the unique decomposition in $s(\mathcal{F}_s)NAM$,

$$s.xu(\xi) = s(\xi)\ (xux^{-1})\ x.$$

Hence the maps $\mathscr{T}_{s,s.xu}$ and $\mathscr{T}_{s,s.ux}$ are constant equal to x. □

3.3. **Cocycle.** Fix a covering family of differentiable cross-sections $(s_i)_{i\in I}$ of $G \to \mathcal{F}$ and let us read in $(\mathcal{B}_{s_i})_{i\in I}$ coordinates the left action of G on itself. The left action of G on the first two coordinates in $\mathcal{F}^{(2)}$ is given by $g(\xi, \check{\xi}) = (g\xi, g\check{\xi})$.

In Proposition 3.8, we prove that the AM-valued function defined below, called *signed Iwasawa cocycle*, contains the remaining information on the third coordinate. Its domain is in $G \times \mathcal{F}$, meaning that the information contained in the second and third coordinate in $\mathcal{F}^{(2)} \times AM$ are not needed when one reads the left action of G.

In Fact 3.9, we prove a chain rule relation for the cross-section parameter subscripts of the cocyle. Such a relation is compatible with the one we had for the transition functions.

Definition 3.7. *Let s_0, s_1 be differentiable cross-sections of $G \to \mathcal{F}$.*

For every $g \in G$ and $\xi \in \mathcal{F}_{s_0}$ such that $g\xi \in \mathcal{F}_{s_1}$, we denote by $\beta_{s_1,s_0}(g,\xi)$ the unique element in AM such that

$$gs_0(\xi) \in s_1(g\xi)\beta_{s_1,s_0}(g,\xi)N.$$

When $s_1 = s_0$, we set $\beta_{s_0} := \beta_{s_0,s_0}$.

Whenever s_0 and s_1 take value in K, the cocycle β_{s_1,s_0} is called signed (multiplicative) Iwasawa cocycle *or in a shorter way,* signed cocycle.

Proposition 3.8. *Let s_0, s_1 be differentiable cross-sections of $G \to \mathcal{F}$.*

For all $g \in G$ and every element in G_{s_0} of coordinates $(\xi, \check{\xi}\ ;\ x)_{s_0} \in \mathcal{F}_{s_0}^{(2)} \times AM$ such that $g\xi \in \mathcal{F}_{s_1}$, we denote by $g\big(\xi, \check{\xi}\ ;\ x\big)_{s_0}$ its left multiplication by g. Then the latter's coordinates with respect to s_1 are

$$g\big(\xi, \check{\xi}\ ;\ x\big)_{s_0} = \big(g\xi, g\check{\xi}\ ;\ \beta_{s_1,s_0}(g,\xi)x\big)_{s_1}. \tag{2}$$

For every covering family of smooth cross-sections $(s_i)_{i\in I}$ of $G \to \mathcal{F}$, for every $i,j,k \in I$, all $\xi_i \in \mathcal{F}_{s_i}$ and $g_j, g_k \in G$ such that $g_j\xi_i \in \mathcal{F}_{s_j}$ and $g_kg_j\xi_i \in \mathcal{F}_{s_k}$ then we have the cocycle relation

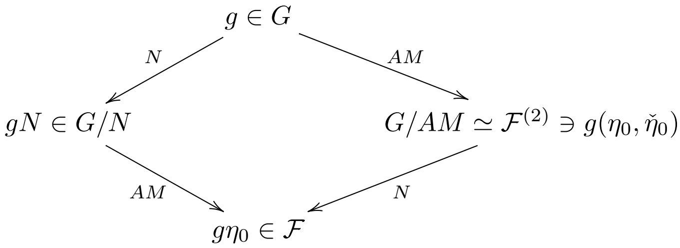
$$\beta_{s_k,s_i}(g_kg_j,\xi_i) = \beta_{s_k,s_j}(g_k, g_j\xi_i)\ \beta_{s_j,s_i}(g_j,\xi_i). \tag{3}$$

For all $y \in AM$, for every element of coordinates $(\xi, \check{\xi}\ ;\ x)_{s_0}$, denote by $(\xi, \check{\xi}\ ;\ x)_{s_0} y$ its right multiplication by y, then

$$(\xi, \check{\xi}\ ;\ x)_{s_0} y = (\xi, \check{\xi}\ ;\ xy)_{s_0}. \tag{4}$$

As in Proposition 3.6 concerning the transition functions, (2) and (3) enforce the 'chain rule' that double cross-sections subsequent subscript cancel. Equation (4) provides a key argument for the properties given in §5 Proposition 5.12 of the invariant subsets of G for the dynamics of nonnegative diagonal flows.

Proof. Because AM normalizes N, the following diagram is G-equivariant for the left action of G and commutative.



Thanks to the lower left side $G/N \to \mathcal{F}$ of the diagram we deduce that local trivializations of $G \to \mathcal{F}$ induces local trivializations of $G/N \to \mathcal{F}$, of fiber AM. Indeed, for every differentiable cross-section $s : \mathcal{F}_s \to G$, the map $\mathcal{F}_s \times AM \to G/N$ that associates to $(\xi\ ;\ x)_s \in \mathcal{F}_s \times AM$ the element $s(\xi)xN \in G/N$ is the inverse of a local coordinate system.
Let $(s_i)_{i\in I}$ be a covering family of cross-sections of $G \to \mathcal{F}$. Then the cocycles $(\beta_{s_i,s_j})_{i,j\in I}$ of Definition 3.7 and the left action of G on $\mathcal{F}$ encode the left action of G on G/N. Indeed, let $hN \in G/N$ be an element of coordinates $(\xi\ ;\ x)_{s_i} \in \mathcal{F}_{s_i} \times AM$ and $g \in G$ such that $g\xi \in \mathcal{F}_{s_j}$. By the restricted coordinates map, we write $hN = s_i(\xi)xN$. Hence

$$ghN = gs_i(\xi)xN.$$

By Definition 3.7 of β_{s_j,s_i}, there exists a unique $u \in N$ such that $gs_i(\xi) = s_j(\xi)\beta_{s_j,s_i}(g,\xi)u$. Replacing it in the expression of ghN and using that AM normalizes N, we get

$$ghN = s_j(g\xi)\beta_{s_j,s_i}(g,\xi)uxN = s_j(g\xi)\beta_{s_j,s_i}(g,\xi)x\left(x^{-1}uxN\right).$$

Hence ghN has coordinates $(g\xi\ ;\ \beta_{s_j,s_i}(g,\xi)x)_{s_j}$.
Thanks to the higher right hand side of the diagram, the same cocycles $(\beta_{s_i,s_j})_{i,j\in I}$ combined with the left action of G on $\mathcal{F}^{(2)}$ allow us to write in local trivialisations the left action of G on itself. Hence, equation (2) holds.

The cocycle relation given by equation (3) follows from the equivariance of the diagram for the left action of G.

For equation (4), note first that for every cross-section s of $G \to \mathcal{F}$, the subset $s(\mathcal{F}_s)NAM$ is invariant by right AM-translation. Furthermore, right translating by AM preserve the parts of the decomposition in $s(\mathcal{F}_s)N$. Finally, this translates in $\mathcal{F}_s^{(2)} \times AM$ to a trivial action in the $\mathcal{F}^{(2)}$ coordinates and a translation in the third AM coordinate. □

Lastly, let us combine the relations between the transition maps and the cocycles for the coordinate system $(\mathcal{B}_{s_i})_{i\in I}$.

Fact 3.9. *Let s_0, s_0', s_1, s_1' be differentiable cross-sections of $G \to \mathcal{F}$. Then for all $g \in G$ and $\xi \in \mathcal{F}_{s_0} \cap \mathcal{F}_{s_0'}$ such that $g\xi \in \mathcal{F}_{s_1} \cap \mathcal{F}_{s_1'}$,*

$$\beta_{s_1',s_0'}(g,\xi) = \mathscr{T}_{s_1',s_1}(g\xi)\ \beta_{s_1,s_0}(g,\xi)\ \mathscr{T}_{s_0,s_0'}(\xi).$$

Note that cross-section subscript that are doubled, cancel out with our notations.

Proof. Let $(\xi,\check\xi\ ;\ x)_{s_0'} \in \left(\mathcal{F}_{s_0'} \cap \mathcal{F}_{s_0}\right)^{(2)} \times AM$ and $g \in G$ such that $g\xi \in \mathcal{F}_{s_1'} \cap \mathcal{F}_{s_1}$. By equation (2) of the previous Proposition 3.8 for the local coordinates given by s_1' and s_0',

$$g(\xi,\check\xi\ ;\ x)_{s_0'} = (g\xi, g\check\xi\ ;\ \beta_{s_1',s_0'}(g,\xi)x)_{s_1'}.$$

Then by the transition identity of Proposition 3.10 (i) between s_0' and s_0 on the left side of the previous equation,

$$g(\xi,\check\xi\ ;\ x)_{s_0'} = g(\xi,\check\xi\ ;\ \mathscr{T}_{s_0,s_0'}(\xi)x)_{s_0}.$$

Again by the cocycle identity on the right hand side between s_0 and s_1,

$$g(\xi,\check\xi\ ;\ x)_{s_0'} = (g\xi, g\check\xi\ ;\ \beta_{s_1,s_0}(g,\xi)\mathscr{T}_{s_0,s_0'}(\xi)x)_{s_1}.$$

Lastly, the transition identity between s_1 and s_1' on the right side of the equation yields

$$(g\xi, g\check\xi\ ;\ \beta_{s_1',s_0'}(g,\xi)x)_{s_1'} = (g\xi, g\check\xi\ ;\ \mathscr{T}_{s_1',s_1}(g\xi)\beta_{s_1,s_0}(g,\xi)\mathscr{T}_{s_0,s_0'}(\xi)x)_{s_1'}.$$

□

3.4. **Local extensions of Hopf coordinates.** Given a family of covering differentiable cross-sections $(s_i)_{i\in I}$ of $G \to \mathcal{F}$, the associated cocycles do not extend the Iwasawa cocycle. Hence, in a general setting, the maps $(\mathcal{B}_{s_i})_{i\in I}$ do not extend Hopf coordinates of G/M.

We prove that when the cross-sections $(s_i)_{i\in I}$ take value in K, the signed multiplicative cocycles $(\beta_{s_i,s_j})_{i,j\in I}$ generalize the Iwasawa cocycle. We obtain an equivariant and commutative diagram with the Hopf coordinates.

Proposition 3.10. *Let s be a* compact *valued, differentiable cross-section of $G \to \mathcal{F}$, then $\mathcal{H}_s$ extends the Hopf coordinates restricted to G_sM i.e. the following diagram is commutative.*

$$\begin{array}{ccc} G_s & \longrightarrow & \mathcal{F}_s^{(2)} \times AM \ni (\xi,\check\xi\ ;\ x)_s \\ \big\downarrow{\scriptstyle \pi_M} & & \big\downarrow \\ G/M & \longrightarrow & \mathcal{F}^{(2)} \times \mathfrak{a} \ni (\xi,\check\xi\ ;\ \log x_A) \end{array}$$

Moreover, it is equivariant with the left action of G, i.e. for all $\xi \in \mathcal{F}_s$, for all $g \in G$ and all compact valued section s' such that $g\xi \in \mathcal{F}_{s'}$, the element

$$g(\xi,\check\xi\ ;\ x)_s = (g\xi, g\check\xi\ ;\ \beta_{s',s}(g,\xi)x)_{s'}$$

projects in G/M to

$$g(\xi,\check\xi\ ;\ \log x_A) = (g\xi, g\check\xi\ ;\ \sigma(g,\xi) + \log x_A).$$

Similarly, it is equivariant with the right action of A i.e. for all $(\xi,\check\xi\ ;\ x) \in \mathcal{F}_s^{(2)} \times AM$ and all $\theta \in \mathfrak{a} \setminus \{0\}$, the element

$$\tilde\phi_\theta^t(\xi,\check\xi\ ;\ x)_s = (\xi,\check\xi\ ;\ xe^{t\theta})_s$$

projects to

$$\tilde\phi_\theta^t(\xi,\check\xi\ ;\ \log x_A) = (\xi,\check\xi\ ;\ \log x_A + t\theta).$$

The proof in §7 of the main mixing Theorem 7.1 of this paper, relies on key results (Cf. Proposition 7.2, 5.2 below) of the joint work [DG20] on mixing of regular Weyl chamber flow on $\Gamma\backslash G/M$. These results provide the arguments in $\mathcal{F}^{(2)} \times A$. By constructing local extensions of Hopf coordinates, we provide a first technical background step in the construction of the invariant sets and in the proof of mixing.

Proof. Recall that the lower part of the diagram reads as $gM \mapsto (g\eta_0, g\check{\eta}_0\ ;\ \sigma(g,\eta_0))$. The upper part reads as $g \mapsto (g\eta_0, g\check{\eta}_0\ ;\ x)_s$ where x is the component in MA given by the Iwasawa decomposition of $s(g\eta_0)^{-1}g$. Commutativity of the diagram then follows from the hypothesis $s(\mathcal{F}_s) \subset K$ and the Definition 2.1 of the Iwasawa cocycle $\exp(\sigma(g,\eta_0)) = a_{\mathcal{I}}(g) = a_{\mathcal{I}}(s(g\eta_0)^{-1}g)$.

Let us check the left G-equivariance. Let s and s' be compact valued differentiable cross-sections. By Proposition 3.8 (2) the left G-action that sends elements of G_s to $G_{s'}$ is given in the AM-coordinate by the cocycle $\beta_{s',s}$. Since s and s' take value in K, the maps $\mathcal{H}_s$ and $\mathcal{H}_{s'}$ are local extensions of the Hopf coordinates where the left G-action is given in the $\mathfrak{a}$-coordinate by the Iwasawa cocycle. Hence the equivariance.

The last part follows from the commutativity of the diagram and Proposition 3.8 (4) that describes how to read in coordinates the right multiplication by elements of AM. □

3.5. **Local coordinates of K.** We prove that every differentiable compact valued cross-section of $G \to \mathcal{F}$ also induces local coordinates of K that take value in $\mathcal{F} \times M$. We read the map $k_{\mathcal{I}}$ in those coordinates.

By endowing K with the left G-action defined for every $g \in G$ and $k \in K$ by $g.k = k_{\mathcal{I}}(gk)$, we make the projection G-equivariant.

Proposition 3.11. *Let s be a differentiable* compact *valued cross-section of $G \to \mathcal{F}$. Then the restriction to the first and last coordinates of $\mathcal{H}_s$ provide local coordinates of K as follows.*

$$\begin{aligned} \mathcal{F}_s \times M &\longrightarrow s(\mathcal{F}_s)M \subset K \\ (\xi\ ;\ c)_s &\longmapsto s(\xi)c. \end{aligned}$$

Furthermore, the map $k_{\mathcal{I}} : G \to K$ reads in coordinates as

$$\begin{aligned} \mathcal{F}_s^{(2)} \times AM &\longrightarrow \mathcal{F}_s \times M \\ (\xi, \check{\xi}\ ;\ x)_s &\longmapsto (\xi\ ;\ x_M)_s \end{aligned}$$

and for every covering family of compact valued cross-sections $(s_i)_{i\in I}$ of $G \to \mathcal{F}$, the M-coordinate of the cocycles $(\beta_{s_i,s_j})_{i,j\in I}$ parametrize the left G action on K.

In §5, we use the Proposition above to rewrite in local coordinates the results of Guivarc'h–Raugi [GR07] on the action of Γ on K. The relations of these coordinates with the extended Hopf coordinates of G allow us to construct the invariant sets in $\Gamma\backslash G$ for the dynamics of the nonnegative diagonal flows. Proposition 5.12 is a direct consequence of the properties of the extended Hopf coordinates and of the results of Guivarc'h–Raugi.

Proof. The map $k \mapsto k\eta_0$ allows to identify $\mathcal{F}$ with K/M. Consequently, every compact valued differential cross-section induces a local trivialization.

Let s be a compact valued differential cross-section of $G \to \mathcal{F}$. Then $s(\mathcal{F}_s)M \subset K$. Coordinates in $\mathcal{F}_s^{(2)} \times AM$ are the same as unique decompositions in $s(\mathcal{F}_s)NAM$ where the N part is associated to the second coordinate in $\mathcal{F}$ and the AM part the last coordinate. Since AM normalises N and M commutes with A, the compact part of the Iwasawa decomposition KAN of every element in $s(\mathcal{F}_s)NAM$ is given by the product of its elements in $s(\mathcal{F}_s)$ and M. Hence, the following diagram

$$\begin{array}{ccccc} G & \overset{AN}{\longrightarrow} & K & \overset{M}{\longrightarrow} & \mathcal{F} \\ g & \longmapsto & k_{\mathcal{I}}(g) & \longmapsto & g\eta_0 \end{array}$$

reads in local $\mathcal{B}_s$ coordinates as

$$\begin{array}{ccccc} \mathcal{F}_s^{(2)} \times AM & \longrightarrow & \mathcal{F}_s \times M & \longrightarrow & \mathcal{F}_s \\ (\xi, \check{\xi}\ ;\ x)_s & \longmapsto & (\xi\ ;\ x_M)_s & \longmapsto & \xi. \end{array}$$

Let $(s_i)_{i\in I}$ be a family of covering differentiable cross-sections of $K \to \mathcal{F}$. That the left G action in K reads as the projection in M of the cocycles $\left(\beta_{s_i,s_j}\right)_{i,j\in I}$ now follows from the equivariance of the second diagram in local coordinates. □

3.6. **Bruhat-Hopf coordinates.** We define two (covering) families of cross-sections of $G \to \mathcal{F}$ defined on maximal Bruhat cells of $\mathcal{F}$: *unipotent* and *compact Bruhat sections.* We define *Bruhat-Hopf coordinates* as the extensions of Hopf coordinates on maximal Bruhat cells of G (Cf. Proposition 3.10). In Proposition 3.14 we prove that every unipotent (resp. compact) Bruhat section is parametrized by a point of $\mathcal{F}$ and an element in AM (resp. M).

By Corollary 2.5 of Bruhat decomposition, the map

$$\begin{aligned} N^- &\longrightarrow N^-\eta_0 = \mathsf{b}(\check{\eta}_0) \\ u &\longmapsto u\eta_0 \end{aligned}$$

is a diffeomorphism. Denote by $[e]$ its inverse.

Definition 3.12. *A* unipotent Bruhat section *is a left translate by G of the map $[e]$. We denote them by $[h] := h \cdot [e]$ where $h \in G$. For every $h \in G$, the unipotent Bruhat section $[h]$ has domain $hN^-\eta_0 = \mathsf{b}(h\check{\eta}_0)$, codomain hN^- and is defined for all $\xi \in \mathsf{b}(h\check{\eta}_0)$ by*

$$[h](\xi) = h[e](h^{-1}\xi).$$

A compact Bruhat section *is the compact component in the KAN decomposition of a unipotent Bruhat section, meaning that for every $h \in G$, the associated compact Bruhat section is defined by $k_{\mathcal{I}} \circ [h]$.*

Bruhat-Hopf coordinates *(resp.* Bruhat coordinates*) are the families of coordinates of G given by covering families of compact (resp. unipotent) Bruhat sections.*

The relations between Bruhat coordinates and Bruhat-Hopf coordinates play an important role in the estimates of the elliptic part of products of loxodromic elements of §4 as well as in the proofs of decorrelation in §6. In the rest of the section, we lighten the notations for unipotent and compact Bruhat sections.

For every $h \in G$, the unipotent Bruhat section $[h]$ and the compact Bruhat section $k_{\mathcal{I}} \circ [h]$ share the same domain: the maximal Bruhat cell $\mathsf{b}(h\check{\eta}_0)$ opposite to $h\check{\eta}_0$. Using compactness of $\mathcal{F}$, one can choose finite families of covering Bruhat sections of any type.

For every $\check{\xi} \in \mathcal{F}$, we pick a compact element $h_{\check{\xi}} \in K$ such that $h_{\check{\xi}}\check{\eta}_0 = \check{\xi}$. The choice of this compact family $(h_{\check{\xi}})_{\check{\xi}\in\mathcal{F}} \subset K$ determines a covering family of unipotent Bruhat sections. Abusing notation, we denote each of them by

$$[\check{\xi}] := [h_{\check{\xi}}].$$

Likewise, we determine a choice of compact Bruhat section for every domain $\mathsf{b}(\check{\xi})$ where $\check{\xi} \in \mathcal{F}$. We denote them by

$$k(\check{\xi}) := k_{\mathcal{I}} \circ [\check{\xi}].$$

Remark 3.13. *The Proposition below implies that for any $h \in G$ such that $h\check{\eta}_0 = \check{\xi}$, there is a unique element $x_* \in AM$ such that $[h] = [\check{\xi}].x_*$.*

Similarly, any compact Bruhat section s is determined by its domain $\mathsf{b}(\check{\xi})$ with $\check{\xi} \in \mathcal{F}$ and an element $c \in M$ such that $s = k(\check{\xi}).c$.

Recall that $k_{\mathcal{I}^-}$ (resp. $a_{\mathcal{I}^-}$) denotes the coordinate in K (resp. A) in the Iwasawa decomposition $G = KAN^-$ and that for every cross-sections s, s' of $G \to \mathcal{F}$, for all $\xi \in \mathcal{F}_s \cap \mathcal{F}_{s'}$, we defined $\mathscr{T}_{s,s'}(\xi)$ as the unique element in AM such that

$$s'(\xi) \in s(\xi)N\mathscr{T}_{s,s'}(\xi).$$

We compute the transition functions for particular cases of unipotent Bruhat sections.

Proposition 3.14. *The following holds.*

(1) *For every $u_* \in N^-$, then $[u_*] = [e]$ i.e.*

$$\mathscr{T}_{[e],[u_*]} = e_{AM}.$$

(2) *For every $x_* \in AM$ and $u_* \in N^-$, then $[x_*u_*] = [e].x_* = [u_*x_*]$ i.e.*

$$\mathscr{T}_{[e],[x_*u_*]} = x_* = \mathscr{T}_{[e],[u_*x_*]}.$$

(3) *For every $h \in G$, then $[h] = [k_{\mathcal{I}^-}(h)].a_{\mathcal{I}^-}(h)$ i.e.*

$$\mathscr{T}_{[k_{\mathcal{I}^-}(h)],[h]} = a_{\mathcal{I}^-}(h).$$

Proof. Note that for every $h \in N^-AM$, because $h\check{\eta}_0 = \check{\eta}_0$, then $\mathcal{F}_{[h]} = \mathcal{F}_{[e]} = \mathsf{b}(\check{\eta}_0)$.

Let us prove (1) i.e. that for all $u_* \in N^-$ and $\xi \in \mathcal{F}_{[u_*]}$ then $[u_*](\xi) \in [e](\xi)N$. Since $[u_*](\xi) = u_*[e](u_*^{-1}\xi)$ for every $\xi \in \mathsf{b}(\check{\eta}_0)$, then by Definition 3.12 of $[e]$, we deduce that $[u_*]$ takes value in N^-. Hence $[e](\xi)^{-1}[u_*](\xi) \in N^-$. Furthermore, using that $[e]$ and $[u_*]$ are cross-sections of $G \to \mathcal{F}$, we deduce $[e](\xi)^{-1}[u_*](\xi) \in N^- \cap MAN$. Therefore, by uniqueness of the Bruhat decomposition $[e](\xi)^{-1}[u_*](\xi) = e_G$ and $\mathscr{T}_{[e],[u_*]} = e_{AM}$.

For statement (2), for all $(u_*, x_*) \in N^- \times AM$ and $\xi \in \mathcal{F}_{[e]}$, then $[x_*u_*](\xi) = x_*u_*[e](u_*^{-1}x_*^{-1}\xi)$. Using that AM normalizes N^-, we deduce that the map $\xi \mapsto [x_*u_*](\xi)x_*^{-1}$ is a differentiable cross-section of $G \mapsto \mathcal{F}$ taking value in N^- and of domain $\mathsf{b}(\check{\eta}_0)$. Hence, by uniqueness of the Bruhat decomposition in N^-NAM, we deduce that $\mathscr{T}_{[e],[x_*u_*].x_*^{-1}} = e_{AM}$. Now we apply Proposition 3.6 (ii) on transition functions to deduce that

$$e_{AM} = \mathscr{T}_{[e],[x_*u_*].x_*^{-1}} = \mathscr{T}_{[e],[x_*u_*]}\mathscr{T}_{[x_*u_*],[x_*u_*].x_*^{-1}}.$$

Then point (iv) of the same Proposition yields $\mathscr{T}_{[x_*u_*],[x_*u_*].x_*^{-1}} = x_*^{-1}$, hence

$$\mathscr{T}_{[e],[x_*u_*]}x_*^{-1} = e_{AM}.$$

For the second part of the equality, note that $[u_*x_*] = [x_*(x_*^{-1}u_*x_*)]$. Since AM normalizes N^-, the conjugated term is in N^- and the rest follows from the previous point.

For statement (3), we write the KAN^- decomposition $h = k_{\mathcal{I}^-}(h)a_{\mathcal{I}^-}(h)u_{\mathcal{I}^-}(h)$. Then by properties of the left action of G on $[e]$, we deduce that

$$h \cdot [e] = k_{\mathcal{I}^-}(h) \cdot [a_{\mathcal{I}^-}(h)u_{\mathcal{I}^-}(h)].$$

Hence by statement (2), we deduce $[h] = [k_{\mathcal{I}^-}(h)].a_{\mathcal{I}^-}(h)$. □

4. Products of loxodromic elements

Recall that an element of G is *unipotent* (resp. *elliptic*, *hyperbolic*) if it is conjugated to an element in N (resp. K, A). By semisimplicity of the Lie group, every element $g \in G$ admits a unique decomposition $g = g_e g_h g_u$, called the *Jordan decomposition*, where g_e, g_h and g_u commute and g_e (resp. g_h, g_u) is called the *elliptic part* (resp. *hyperbolic part*, *unipotent part*) of g.

Definition 4.1. *For any element $g \in G$, there is a unique element $\lambda(g) \in \mathfrak{a}^+$ such that the hyperbolic part of g is conjugated to $\exp(\lambda(g)) \in A^+$. The map $\lambda : G \to \mathfrak{a}^+$ is called the* Jordan projection.

An element $g \in G$ is *loxodromic* if $\lambda(g) \in \mathfrak{a}^{++}$. Denote by G^{lox} (resp. Γ^{lox}) the subset of loxodromic elements of G (resp. Γ). Since any element of N that commutes with A^{++} is trivial, the unipotent part of loxodromic elements is trivial. Furthermore, the only elements of K that commute with A^{++} are in M. We deduce that the elliptic part of loxodromic elements is conjugated to elements in M. Therefore, g is loxodromic if and only if there exists $h \in G$ such that $h^{-1}gh \in MA^{++}$.

Hence, for every loxodromic element $g \in G$, there exists $h_g \in G$ and $m(g) \in M$ so that we can write $g = h_g m(g) e^{\lambda(g)} h_g^{-1}$. However, for every $m \in M$ we can also write

$$g = (h_g m)(m^{-1}m(g)m)e^{\lambda(g)}(h_g m)^{-1}.$$

Which means that the *angular* part $m(g)$ is only well defined up to conjugacy by M. We thus use specific cross-sections of $G \to G/AM$, to study the elliptic part of products of loxodromic elements.

For every loxodromic element $g \in G$, denote by $g^+ := h\eta_0$ and $g^- := h\eta_0$. The Iwasawa cocycle of g on g^+ is equal to its Jordan projection (see for instance [DG20, Fact 2.6])

$$\sigma(g, g^+) = \lambda(g).$$

In §4.1, by using differential cross-sections of $G \to \mathcal{F}$ that factor the projection $G \to G/AM$, we extend locally and to loxodromic elements the previous formula.

In §4.2, we recall the dynamical properties of the left action of loxodromic elements on the Furstenberg boundary. This leads us to another definition of (r, ε)-loxodromic elements, where r is a positive number that measures the distance between the attracting point of the loxodromic element and the boundary of its basin of attraction and ε measures how contracting it is. Using the Bruhat sections of $G \to \mathcal{F}$, we give another proof that every loxodromic element, iterated enough times, will become (r, ε)-loxodromic.

In §4.3, we compute for every loxodromic element, the signed cocycle given by the unipotent Bruhat section supported on the basin of attraction and on each point of the basin. Benoist in [Ben00] gave estimates for the Jordan projection of products of loxodromic elements involving the Jordan projection of each term and some explicit error term maps. We improve those error term maps into so-called *Ratio maps* that take value in AM and obtain an exact formula in Proposition 4.9.

We define in §4.4 a family of equicontinuity constants $\delta_{r,\varepsilon}$ for compact Bruhat sections. We claim the construction can be adapted for any family of covering K-valued cross-sections of $G \to \mathcal{F}$ of the same type.

In the last paragraph, we estimate simultaneously the elliptic and hyperbolic part of products of generic loxodromic elements in Proposition 4.12, extending the estimates of Benoist to the elliptic part. The proof is based on a Ping-Pong argument.

4.1. **Extended Jordan projections for loxodromic elements.** For every loxodromic element $g \in G$, denote by $g^+ := h\eta_0$ and $g^- := h\eta_0$. Let us define a multiplicative and local extension to MA^{++} of the Jordan projection of loxodromic elements.

Definition 4.2. *Let s be a differentiable cross-section of $G \to \mathcal{F}$. For every loxodromic element $g \in G$ such that $g^+ \in \mathcal{F}_s$, we denote by*

$$\mathscr{L}_s(g) := \beta_s(g, g^+).$$

For compact or unipotent Bruhat sections, such a map is called a signed Jordan projection *(for loxodromic elements).*

Fact 4.3. *Fix a family of unipotent Bruhat sections denoted by $([\xi])_{\xi \in \mathcal{F}}$ of respective domains $\mathsf{b}(\xi)$. Let $g \in G$ be a loxodromic element. The following holds.*

(1) *For every $h \in G$ such that $h^{-1}gh \in MA^{++}$, we have $\mathscr{L}_{[h]}(g) = h^{-1}gh$.*
(2) *Denote by h_g the element of Bruhat coordinates $(g^+, g^- \,;\, e_{AM})_{[g^-]}$.*

$$\mathscr{L}_{[g^-]}(g) = \mathscr{L}_{[h_g]}(g) = h_g^{-1} g h_g.$$

Let s be a cross-section of $G \to \mathcal{F}$ such that $g^+ \in \mathcal{F}_s$. Then
(3) $\mathscr{L}_s(g) = \mathscr{T}_{[g^-],s}(g^+)^{-1}\, \mathscr{L}_{[g^-]}(g)\, \mathscr{T}_{[g^-],s}(g^+)$,
(4) $\mathscr{L}_s(g) \in M e^{\lambda(g)}$.

Proof. (1) By Definition 3.7 of the cocycle, $\beta_{[h]}(g, g^+)$ is the unique element in AM such that

$$g[h](g^+) \in [h](g^+)\beta_{[h]}(g, g^+)N.$$

By Definition 3.12 of the unipotent Bruhat section, $[h](g^+) = h[e](h^{-1}g^+)$. Since $g^+ = h\eta_0$, we deduce that $[h](g^+) = h[e](\eta_0) = h$. We rewrite the inclusion, with the definition of the extended Jordan projection $\beta_{[h]}(g, g^+) = \mathscr{L}_{[h]}(g)$

$$gh \in h\mathscr{L}_{[h]}(g)N.$$

Since $h^{-1}gh \in MA^{++}$, we deduce that $\mathscr{L}_{[h]}(g) \in MA^{++}$ and the N-coordinate is trivial, i.e.

$$gh = h\mathscr{L}_{[h]}(g).$$

(2) The unipotent Bruhat section $[g^-]$ shares the same domain as $[h]$. By Remark 3.13, these cross-sections are defined only up to their domain and by right multiplication by an element in AM. Since h_g the unique element in hMA of Bruhat coordinates $(g^+, g^- \,;\, e_{AM})_{[g^-]}$, then

$$g(g^+, g^- \,;\, e_{AM})_{[g^-]} = (g^+, g^- \,;\, \beta_{[g^-]}(g, g^+))_{[g^-]}.$$

Using properties of the right translation by AM in Bruhat coordinates, we deduce that

$$gh_g = h_g \mathscr{L}_{[g^-]}(g).$$

(3) Follows first from the identity of Fact 3.9 between transition functions and cocycle. Then using that g^+ is a fixed point for the action of g on $\mathcal{F}$, we apply Proposition 3.6 (iii) on $\mathscr{T}_{s,[g^-]}(g^+)$.
(4) Follows from (3) because we are conjugating by an element in AM. □

4.2. **Dynamical action on the Furstenberg boundary.** We study the left action of loxodromic elements on the Furstenberg boundary. We give an alternative proof that the basin of attraction is the Bruhat cell opposite to the repelling point. This leads to a Definition 4.5 of (r, ε)-loxodromic elements using the K-invariant distance on $\mathcal{F}$. We give another proof that large iterates of loxodromic element are (r, ε)-loxodromic.

Proposition 4.4. *Let $g \in G$ be a loxodromic element.*
Then g^+ is an attracting point for the action of g on the Furstenberg boundary. Furthermore, the basin of attraction *of g^+ is $\mathsf{b}(g^-)$, the Bruhat cell opposite to its repelling point.*

The classical proof uses the fundamental representations of G introduced Tits (cf. [Sam14, Corollary 3.12]) and involves the notion of simultaneaous proximality in those representations (cf. [Ben97b]). We only rely here on Bruhat decomposition and the convergence (1).

Proof. Let us first assume that $g \in MA^{++}$. Then $g^+ = \eta_0$ and $g^- = \check{\eta}_0$ and we are going to prove that its basin of attraction is $\mathsf{b}(\check{\eta}_0)$.

Since MA normalizes N^-, we deduce that g stabilizes the Bruhat cell $\mathsf{b}(\check{\eta}_0)$. Indeed, for every $u_* \in N^-$, then $gu_*\eta_0 = gu_*g^{-1}\eta_0 \in N^-\eta_0$. Furthermore for any $u_* \in N^-$, then $g^n u_* g^{-n} \to e_G$ when $n \to +\infty$. This implies that $\mathsf{b}(\check{\eta}_0)$ is in the bassin of attraction of η_0.

Conversely, let $\xi \in \mathcal{F}$ be in the basin of attraction of η_0. Choose for every element in the Weyl group $w \in W$ a representative $k_w \in N_K(A)$ and recall that $k_\iota \in N_K(A)$ denotes an element such that $N^- = k_\iota N k_\iota^{-1}$. Apply Bruhat decomposition $G = k_\iota \sqcup_{w\in W} Bk_wB$ where $B = MAN$. Then there exists $u_* \in N^-$ and $k_w \in N_K(A)$ such that $\xi = u_* k_w \eta_0$. Now $g^n\xi = g^n u_* g^{-n}\ (g^n k_w \eta_0)$. Since ξ is in the basin of attraction and $g^n u_* g^{-n}$ converges to e_G, we deduce that $g^n k_w \eta_0 \to \eta_0$. Using that k_w normalizes MA, we deduce that $g^n k_w \eta_0 = k_w(k_w^{-1} g^n k_w)\eta_0 = k_w\eta_0$. The sequence is stationary at $k_w\eta_0$, by uniqueness of the limit $k_w\eta_0 = \eta_0$. Hence $k_w \in M$ and $\xi \in N^-\eta_0$.

In the general case, let $g \in G$ be a loxodromic element. Consider the unique element $h_g \in G$ given by Fact 4.3 such that

$$h_g^{-1} g h_g = \mathscr{L}_{[g^-]}(g) \in MA^{++}.$$

Hence, the attracting point of g is $h_g\eta_0 = g^+$ and its basin of attraction is $h_g\mathsf{b}(\check\eta_0) = \mathsf{b}(g^-)$. □

We give a definition of (r,ε)-loxodromic elements which is slightly different from what the reader may find in [Ben97b] or [Ben00] because it does not use the notion of simultaneous proximality in the fundamental representations of G given by Tits. However, using our choice of distance on $\mathcal{F}$ and the intrinsic characterization of the basin of attraction of loxodromic elements, one can check that both definitions are equivalent.

For all $\varepsilon > 0$, all $\check\xi \in \mathcal{F}$, we set the following notation.

$$\mathcal{V}_\varepsilon(\partial \mathsf{b}(\check\xi))^{\complement} := \{\xi \in \mathcal{F} \mid \mathrm{d}(\xi, \partial\mathsf{b}(\check\xi)) \geq \varepsilon\}.$$

Definition 4.5. *Let $r > 0$ be a positive number and $\varepsilon \in (0, r]$. An element $g \in G$ is (r,ε)-loxodromic if it satisfies the following conditions.*

(i) *The element g is loxodromic and $r \leq \frac{1}{2}\mathrm{d}(g^+, \partial\mathsf{b}(g^-))$.*
(ii) *It maps the compact set $\mathcal{V}_\varepsilon(\partial\mathsf{b}(g^-))^{\complement}$ into the ball $B(g^+,\varepsilon)$.*
(iii) *The restriction of g to $\mathcal{V}_\varepsilon(\partial\mathsf{b}(g^-))^{\complement}$ is an ε-Lipschitz map.*

These remarks follow from the previous definition.

1) If an element is (r,ε)-loxodromic, then it is (r',ε)-loxodromic for every $\varepsilon \leq r' \leq r$.
2) If an element is (r,ε)-loxodromic, then it is (r,ε')-loxodromic for every $r \geq \epsilon' \geq \varepsilon$.
3) If g is is (r,ε)-loxodromic, then g^n is also is (r,ε)-loxodromic for every $n \geq 1$.

Note that loxodromic elements that are not sufficiently contracting, for instance those too close to e_G, will never satisfy the second condition for being (r,ε)-loxodromic. However, we give below another proof that every loxodromic element, iterated a large enough amount of times will be (r,ε)-loxodromic.

Proposition 4.6. *Let $g \in G$ be a loxodromic element.*

Then for all positive number $r \leq \frac{1}{2}\mathrm{d}(g^+, \partial\mathsf{b}(g^-))$ and all $\varepsilon \in (0, r]$, there exists an integer $N_{r,\varepsilon} \geq 1$ such that for all $n \geq N_{r,\varepsilon}$, the element g^n is (r,ε)-loxodromic.

The Proposition above is used in the Ping-Pong arguments of §6.3 : in the proofs of decorrelation.

Proof. Let $g \in G$ be a loxodromic element and fix $r \leq \frac{1}{2}\mathrm{d}(g^+, \partial\mathsf{b}(g^-))$ and $\varepsilon \in (0, r]$. By choice of these parameters, condition (i) holds.

Note that $\mathcal{V}_\varepsilon(\partial\mathsf{b}(g^-))^{\complement}$ is a compact subset of $\mathsf{b}(g^-)$, which by Proposition 4.4 is the basin of attraction of g^+. Hence $\{g^n\mathcal{V}_\varepsilon(\partial\mathsf{b}(g^-))^{\complement}\}_{n\geq 1}$ is a sequence of compact sets in the basin of attraction shrinking towards g^+. Consequently, condition (ii) holds for every $n \geq N_2$ sufficiently large.

Let us now prove that there exists an integer $N_{r,\varepsilon}$ such that for every $n \geq N_{r,\varepsilon}$ the restriction of g^n to $\mathcal{V}_\varepsilon(\partial\mathsf{b}(g^-))^{\complement}$ is an ε-Lipschitz map. By Fact 4.3, we consider the element $h_g \in G$ such that $h_g^{-1} g h_g = \mathscr{L}_{[g^-]}(g)$. Using that AM normalises N^-, we express the action of g on $\mathsf{b}(g^-)$ in the unipotent charts $[g^-](\mathsf{b}(g^-)) = h_g N^-$ by

$$\begin{aligned} c(g) : h_g N^- &\longrightarrow h_g N^- \\ h_g u_* &\longmapsto h_g\big(\mathscr{L}_{[g^-]}(g) u_* \mathscr{L}_{[g^-]}(g)^{-1}\big). \end{aligned}$$

The chosen metric on $\mathcal{F}$ is induced by the identification $K/M \simeq \mathcal{F}$. Furthermore, the compact Bruhat section $k(g^-) : \mathsf{b}(g^-) \to K$ defined by $k_{\mathcal{I}} \circ [g^-]$ is a differentiable chart of $\mathsf{b}(g^-)$. Therefore, any upper bound of the differential of the map

$$\begin{aligned} k(g^-)(\mathsf{b}(g^-)) &\longrightarrow k(g^-)(\mathsf{b}(g^-)) \\ k_{\mathcal{I}}(h_g u_*) &\longmapsto k_{\mathcal{I}} \circ c(g)(h_g u_*). \end{aligned}$$

restricted to $k(g^-)(\mathcal{V}_\varepsilon(\partial \mathsf{b}(g^-))^{\complement})$ provides a Lipschitz constant for the map

$$\begin{aligned} \mathcal{V}_\varepsilon(\partial \mathsf{b}(g^-)^{\complement} &\longrightarrow \mathsf{b}(g^-) \\ \xi &\longmapsto g\xi. \end{aligned}$$

Set $C_{r,\varepsilon} := \sup_{u_* \in [g^-]B(g^+,\varepsilon)} \|D_{u_*} k_{\mathcal{I}}\| \ \sup_{u_* \in [g^-]\mathcal{V}_\varepsilon(\partial \mathsf{b}(g^-))^{\complement}} \|D_{u_*} k_{\mathcal{I}}\|^{-1}$.
At every point, the eigenvalues of the differential of $c(g)$ are $\{e^{-\alpha(\lambda(g))}\}_{\alpha\in\Sigma_+}$ where Σ_+ is the set of positive roots. Denote by $\ell_g := \min_{\alpha\in\Sigma^+} \alpha(\lambda(g))$. Since g is loxodromic, ℓ_g is a positive number and we obtain the uniform exponential decay of the differential of $c(g)$ i.e. for every $n \geq 1$,

$$\sup_{u_* \in h_g N^-} \|D_{u_*} c(g^n)\| \leq e^{-n\ell_g}.$$

By hypothesis on r and ε, we deduce that $B(g^+,\varepsilon) \subset \mathcal{V}_\varepsilon(\partial \mathsf{b}(g^-))^{\complement}$. Let $n \geq N_2$. Then by choice of ℓ_g and $C_{r,\varepsilon}$, we deduce that $C_{r,\varepsilon} e^{-n\ell_g}$ is a Lipschitz constant for the action of g^n restricted to this compact subset of the basin of attraction. Since this sequence decays exponentially fast, there exists $N_{r,\varepsilon} \geq N_2$ such that for every $n \geq N_{r,\varepsilon}$, then $C_{r,\varepsilon} e^{-n\ell_g} \leq \varepsilon$ and condition (iii) is satisfied. ☐

4.3. **Cocycle on the basin of attraction.** Let g be a loxodromic element. We prove in Lemma 4.7 that the cocycle in Bruhat coordinates $\beta_{[g^-]}$, applied to g and every point ξ in $\mathsf{b}(g^-)$ is everywhere equal to the signed Jordan projection $\mathscr{L}_{[g^-]}(g)$. Then we define the so-called ratio maps which allow us to write in Proposition 4.9 the relation between the signed cocycle of g on every point ξ in its basin of attraction and any local signed Jordan projection of g, provided that it is well defined.

Lemma 4.7. *For all loxodromic element $g \in G$, all $n \geq 1$ and every $\xi \in \mathsf{b}(g^-)$,*

$$\beta_{[g^-]}(g^n, \xi) = \mathscr{L}_{[g^-]}(g)^n.$$

We give a different proof from Lee-Oh [LO20], using Bruhat coordinates.

Proof. Denote by h_g the element of G of Bruhat coordinates $(g^+, g^- \,;\, e_{AM})_{[g^-]}$. By Fact 4.3 we deduce that $g h_g = h_g \mathscr{L}_{[g^-]}(g)$.

By property of the unipotent Bruhat section, for every $\xi \in \mathsf{b}(g^-) = h_g N^- \eta_0$, there exists a unique $u_\xi \in N^-$ such that $\xi = h_g u_\xi \eta_0$ and $h_g u_\xi$ reads in Bruhat coordinates as $(\xi, g^- \,;\, e_{AM})_{[g^-]}$.

On one hand, by definition of the cocycle and because $\mathsf{b}(g^-)$ is the basin of attraction of g^+, for all $n \geq 1$, the element $g^n h_g u_\xi$ reads as

$$g^n(\xi, g^- \,;\, e_{AM})_{[g^-]} = (g^n\xi, g^- \,;\, \beta_{[g^-]}(g^n,\xi)\,)_{[g^-]}.$$

Note that $\beta_{[g^-]}(g^n,\xi)$ is the unique element in AM such that $g^n h_g u_\xi \in h_g N^- \beta_{[g^-]}(g^n,\xi)N$. That the second coordinate remains equal to g^- means that the part in N is trivial.

On the other hand, using the definition of the signed Jordan projection,

$$g^n h_g u_\xi = h_g \mathscr{L}_{[g^-]}(g)^n u_\xi = h_g\big(\mathscr{L}_{[g^-]}(g)^n u_\xi \mathscr{L}_{[g^-]}(g)^{-n}\big)\mathscr{L}_{[g^-]}(g)^n.$$

Since AM normalizes N^-, we deduce that $\mathscr{L}_{[g^-]}(g)^n u_\xi \mathscr{L}_{[g^-]}(g)^{-n} \in N^-$, hence

$$g^n h_g u_\xi \in h_g N^- \mathscr{L}_{[g^-]}(g)^n.$$

This allows us to deduce by uniqueness of the Bruhat decomposition in $h_g N^- MAN$ that $\beta_{[g^-]}(g^n,\xi) = \mathscr{L}_{[g^-]}(g)^n$. □

Definition 4.8. *Given two cross-sections s_1, s_2 and a Bruhat cell $\mathsf{b}(\check\xi)$, then for all $\xi_1 \in \mathcal{F}_{s_1}\cap \mathsf{b}(\check\xi)$ and $\xi_2 \in \mathcal{F}_{s_2}\cap \mathsf{b}(\check\xi)$ we define the* ratio

$$\mathscr{R}_{s_1,s_2}(\check\xi;\xi_1,\xi_2) := \mathscr{T}_{s_1,[\check\xi]}(\xi_1)\mathscr{T}_{[\check\xi],s_2}(\xi_2).$$

When $s_1 = s_2$, we shorten the notation $\mathscr{R}_{s_1} := \mathscr{R}_{s_1,s_1}$. For every loxodromic element $g\in G$ such that $g^+ \in \mathcal{F}_{s_1}\cap \mathsf{b}(g^-)$, for all $\xi \in \mathcal{F}_{s_2}\cap \mathsf{b}(g^-)$, set

$$\mathscr{R}_{s_1,s_2}(g,\xi) := \mathscr{R}_{s_1,s_2}(g^-;g^+,\xi).$$

The regularity of the ratio map depends on the regulatity of the transfer maps which in turn depend on that of the cross-sections. Because the transition functions between the unipotent and compact Bruhat sections take value in AM, for any compact Bruhat sections s_1, s_2, the *ratio map* $\mathscr{R}_{s_1,s_2}$ is continuous on its domain and takes value in AM.

Using the ratio map, the following statement follows from Lemma 4.7.

Proposition 4.9. *For all loxodromic element $g \in G^{lox}$, all integer $n \geq 1$ and $\xi \in \mathsf{b}(g^-)$, for any choice of compact (Bruhat) sections s_0, s_1, s_2 such that $(\xi, g^+, g^n\xi) \in \mathcal{F}_{s_0}\times\mathcal{F}_{s_1}\times\mathcal{F}_{s_2}$,*

$$\beta_{s_2,s_0}(g^n,\xi) = \mathscr{R}_{s_1,s_2}(g;g^n\xi)^{-1}\mathscr{L}_{s_1}(g)^n\mathscr{R}_{s_1,s_0}(g;\xi). \tag{5}$$

Proof. Using first the transition functions between s_2, s_0 and $[g^-]$, then applying Lemma 4.7 on the middle term and finally using the transition function between $[g^-]$ and s_1 in the middle term, we get

$$\begin{aligned}\beta_{s_2,s_0}(g^n,\xi) &= \mathscr{T}_{s_2,[g^-]}(g^n\xi)\quad \beta_{[g^-]}(g^n,\xi)\quad \mathscr{T}_{[g^-],s_0}(\xi)\\ &= \mathscr{T}_{s_2,[g^-]}(g^n\xi)\quad \mathscr{L}_{[g^-]}(g^n)\quad \mathscr{T}_{[g^-],s_0}(\xi)\\ &= \mathscr{T}_{s_2,[g^-]}(g^n\xi)\mathscr{T}_{[g^-],s_1}(g^+)\ \mathscr{L}_{s_1}(g^n)\ \mathscr{T}_{s_1,[g^-]}(g^+)\mathscr{T}_{[g^-],s_0}(\xi).\end{aligned}$$

Finally, using Definition 4.8 and the properties of the transition functions, we check that $\mathscr{T}_{s_2,[g^-]}(g^n\xi)\mathscr{T}_{[g^-],s_1}(g^+) = \mathscr{R}_{s_1,s_2}(g;g^n\xi)^{-1}$ and $\mathscr{T}_{s_1,[g^-]}(g^+)\mathscr{T}_{[g^-],s_0}(\xi) = \mathscr{R}_{s_1,s_0}(g;\xi)$. □

4.4. **Equicontinuity constants.** The first term on the left side of the equation (5), will converge towards $\mathscr{R}_{s_1,s_2}(g;g^+)^{-1} = e_{AM}$ as n goes to infinity. Therefore, when n is large enough, the signed cocycle $\beta_{s_2,s_0}(g^n,\xi)$ is well approximated by $\mathscr{L}_{s_1}(g)^n\mathscr{R}_{s_1,s_0}(g;\xi)$. In the next paragraph §4.4, we define a suitable distance on AM and the so-called equicontinuity constants $\delta_{r,\varepsilon}$ that control this approximation for (r,ε)-loxodromic elements.

After constructing a distance of AM_0 that is symmetric and left and right invariant, we introduce for every $r>0$ and $\check\xi \in \mathcal{F}$, a family $(\delta_{r,\varepsilon}(\check\xi))_{\varepsilon\in(0,r]}$ of equicontinuity constants of a continuous fonction defined over a compact set. These constants are thus positive and converge to zero when ε goes to zero. Furthermore, using the K-invariance of the distance on $\mathcal{F}$ and the action of K on the compact and unipotent Bruhat sections, we show that these constants do not depend on the choice of $\check{\xi} \in \mathcal{F}$.

Let us now choose a distance on AM which is left and right AM invariant i.e. for every $x, y, z, w \in AM$, then $\mathrm{d}_{AM}(wxz, wyz) = \mathrm{d}_{AM}(x, y)$. The maximal $\mathbb{R}$-split torus A is an abelian group of finite dimension so any norm on its Lie algebra $\mathfrak{a}$ will induce by the exponential map a suitable distance on A. Since M is the centralizer of A in K, it is sufficient to construct an M invariant distance between arc connected points and setting an infinite distance otherwise. We will then endow AM with the distance d_{AM} induced by the product group structure $A \times M$.

Now we construct an M-invariant norm on the Lie algebra of M. Starting from an euclidean norm on $\mathfrak{m}$, we make it $Ad(M)$-invariant by taking its average with respect to the Haar measure on M. Since M is compact, its Haar measure is finite. Therefore, the average is an $Ad(M)$-invariant norm on $\mathfrak{m}$. It induces an $Ad(M)$-invariant scalar product on $T_{e_M}M$. By transporting it on the tangent space over every point by left multiplication by M we obtain a left invariant metric. The induced riemannian distance on M is only defined between arc connected points and is, by construction, left M-invariant and invariant by conjugation. This suffices to deduce the M-invariance of such a distance.

Recall that for every $\check{\xi} \in \mathcal{F}$, then $[\check{\xi}]$ denotes a choice of unipotent Bruhat section of domain $\mathsf{b}(\check{\xi})$ and $k(\check{\xi}) := k_{\mathcal{I}} \circ [\check{\xi}]$ is the associated compact Bruhat section. Therefore $k(\mathcal{F})$ denotes the family of such compact Bruhat sections.

Proposition 4.10. *Let $r > 0$. Consider the compact, symmetric and invariant by conjugacy by K subset*

$$K_r := \{h \in K \mid h\mathcal{V}_r(\partial \mathsf{b}(\check{\eta}_0)) \subset \mathcal{V}_{2r}(\partial \mathsf{b}(\check{\eta}_0))\}.$$

For every $\check{\xi} \in \mathcal{F}$, denote by

$$\delta_{r,\varepsilon}(\check{\xi}) := \sup_{s \in K_r \cdot k(\check{\xi})} \Big\{ \mathrm{d}_{AM}(\mathscr{R}_s(\check{\xi}\ ;\ \xi_1, \xi_2), e_{AM}) \ \Big|\ \xi_1 \in \mathcal{V}_{3r}(\partial \mathsf{b}(\check{\xi}))^{\complement} \text{ and } \xi_2 \in B(\xi_1, \varepsilon) \Big\}.$$

Then the following holds.

(a) *For every $r > 0$ and every $\varepsilon \in (0, r]$, the constant $\delta_{r,\varepsilon}(\check{\eta}_0)$ is non-zero and*

$$\delta_{r,\varepsilon}(\check{\eta}_0) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

(b) *For every $\check{\xi} \in \mathcal{F}$, the equality holds $\delta_{r,\varepsilon}(\check{\eta}_0) = \delta_{r,\varepsilon}(\check{\xi})$.*

Proof. First, by M-invariance of the distance we deduce that

$$\mathrm{d}_{AM}(\mathscr{R}_s(\check{\xi}\ ;\ \xi_1, \xi_2), e_{AM}) = \mathrm{d}_{AM}(\mathscr{T}_{s,[\check{\xi}]}(\xi_1)\ ,\ \mathscr{T}_{s,[\check{\xi}]}(\xi_2)).$$

(a) Because the Bruhat sections and the Iwasawa decomposition are differentiable, by Definition 3.5 of the transition maps, the map

$$\begin{aligned} K_r \times \mathcal{V}_{2r}(\partial \mathsf{b}(\check{\eta}_0))^{\complement} &\longrightarrow AM \\ (c, \xi) &\longmapsto \mathscr{T}_{c \cdot k(\check{\eta}_0), [\check{\eta}_0]}(\xi) \end{aligned}$$

is continuous. It is defined over a compact set and using the Definition 4.8 of the ratio map, we notice that $\delta_{r,\varepsilon}(\check{\eta}_0)$ is for every $\varepsilon > 0$ bounded above by the equicontinuity constant of this map and bounded below by the equicontinuity constants for the restriction to $K_r \times \mathcal{V}_{3r}(\partial \mathsf{b}(\check{\eta}_0))^{\complement}$. Hence the positivity and convergence to zero.

(b) Let $l \in K$ such that $\check{\xi} = l\check{\eta}_0$ and $k(\check{\xi}) = l \cdot k(\check{\eta}_0)$. Since $l\mathcal{V}_{2r}(\partial \mathsf{b}(\check{\eta}_0))^{\complement} = \mathcal{V}_{2r}(\partial \mathsf{b}(\check{\xi}))^{\complement}$, note that $\delta_{r,\varepsilon}(\check{\xi})$ is associated to the continuous map

$$K_r \times \mathcal{V}_{2r}(\partial \mathsf{b}(\check{\eta}_0))^{\complement} \longrightarrow AM$$
$$(c, \xi) \longmapsto \mathscr{T}_{cl\cdot k(\check{\eta}_0), l\cdot[\check{\eta}_0]}(l\xi).$$

Recall that the transition function is the unique element in AM such that

$$\big(cl \cdot k(\check{\eta}_0)\big)(l\xi) \in \big(l \cdot [\check{\eta}_0]\big)(l\xi) N \mathscr{T}_{cl\cdot k(\check{\eta}_0), l\cdot[\check{\eta}_0]}(l\xi).$$

By first applying the definition of the translation $l \cdot s(l\xi) = ls(\xi)$, then multiplying by l^{-1} on the left, we obtain

$$\big(l^{-1}cl \cdot k(\check{\eta}_0)\big)(\xi) \in [\check{\eta}_0](\xi) N \mathscr{T}_{cl\cdot k(\check{\eta}_0), l\cdot[\check{\eta}_0]}(l\xi).$$

Therefore $\mathscr{T}_{cl\cdot k(\check{\eta}_0), l\cdot[\check{\eta}_0]}(l\xi) = \mathscr{T}_{l^{-1}cl\cdot k(\check{\eta}_0), [\check{\eta}_0]}(\xi)$. Since K_r is invariant by conjugation, in particular $l^{-1}K_r l = K_r$. The continuous maps associated to $\delta_{r,\varepsilon}(\check{\xi})$ and $\delta_{r,\varepsilon}(\eta_0)$ coincide, hence the constants are equal. □

Definition 4.11. *Let $r > 0$. We define the family of* equicontinuity constants

$$\delta_{r,\varepsilon} := \sup_{\check{\xi} \in \mathcal{F}} \sup_{s \in K_r \cdot k(\check{\xi})} \Big\{ \mathrm{d}_{AM}(\mathscr{R}_s(\check{\xi}\ ;\ \xi_1, \xi_2), e_{AM}) \ \Big|\ \xi_1 \in \mathcal{V}_{3r}(\partial \mathsf{b}(\check{\xi}))^{\complement} \text{ and } \xi_2 \in B(\xi_1, \varepsilon) \Big\}.$$

4.5. **Estimates for products of generic loxodromic elements.** Let $g_1, ..., g_l \in G^{lox}$ be loxodromic elements. Taking the convention that $g_0 = g_l$, we say the (ordered) family is *generic* if g_{i-1}^+, g_i^- are transverse for every $1 \leq i \leq l$ or in other words $g_{i-1}^+ \in \mathsf{b}(g_i^-)$. The statement below gives new information on the elliptic part of a product of generic very contracting loxodromic elements. However, because the elliptic part of loxodromic elements is well defined only up to conjugation and M is not abelian in general, we need to specify the choice of diagonalisation of each loxodromic element via a family of cross-sections s_i.

Proposition 4.12. *Let $r > 0$ and $\varepsilon \in (0, r]$, let $g_1, ..., g_l \in G$ be a generic family of (r, ε)-loxodromic elements such that*

$\star$ $r \leq \frac{1}{6}\mathrm{d}(\{g_{i-1}^+, g_i^+\}, \partial \mathsf{b}(g_i^-))$ for all $1 \leq i \leq l$ with the convention $g_0 = g_l$.

Fix a choice of compact Bruhat sections $(s_i)_{0 \leq i \leq l}$ such that

$\star\star$ $\mathcal{F}_{s_i} \supset \mathcal{V}_r(\partial \mathsf{b}(g_i^-))^{\complement}$ for every $1 \leq i \leq l$ and $\mathcal{F}_{s_0} \supset \mathcal{V}_\varepsilon(\partial \mathsf{b}(g_1^-))^{\complement}$.

Then for all $\xi_0 \in \mathcal{V}_\varepsilon(\partial \mathsf{b}(g_1^-))^{\complement}$ and for all integers $n_1, ..., n_l \geq 1$,

$$\beta_{s_l, s_0}(g_l^{n_l}...g_1^{n_1}, \xi_0) \in \mathscr{L}_{s_l}(g_l^{n_l})\mathscr{R}_{s_l, s_{l-1}}(g_l, g_{l-1}^+)...\mathscr{L}_{s_1}(g_1^{n_1})\mathscr{R}_{s_1, s_0}(g_1, \xi_0) B(e_{AM}, (2l-1)\delta_{r,\varepsilon}).$$

Furthermore, $g_l^{n_l}...g_1^{n_1}$ is $(2r, 2\varepsilon)$-loxodromic with attracting (resp. repelling) point in $B(g_l^+, \varepsilon)$ (resp. $B(g_1^-, \varepsilon)$) and its extended Jordan projection satisfies

$$\mathscr{L}_{s_l}(g_l^{n_l}...g_1^{n_1}) \in \mathscr{L}_{s_l}(g_l^{n_l})\mathscr{R}_{s_l, s_{l-1}}(g_l, g_{l-1}^+)...\mathscr{L}_{s_1}(g_1^{n_1})\mathscr{R}_{s_1, s_l}(g_1, g_l^+) B(e_{AM}, 2l\delta_{r,\varepsilon}).$$

Benoist in [Ben00, Lemma 3.6] gave a proof that $g_l^{n_l}...g_1^{n_1}$ is $(2r, 2\varepsilon)$-loxodromic with attracting (resp. repelling) point in $B(g_l^+, \varepsilon)$ (resp. $B(g_1^-, \varepsilon)$). Using the fundamental representations of G given by Tits and simultaneous proximality of loxodromic elements, he defined some error terms $\nu(g_i, \xi) \in \mathfrak{a}$, equicontinuity constants $\delta_{r,\varepsilon}$ and proved the following estimate

$$\left\| \lambda(g_l^{n_l}...g_1^{n_1}) - \sum_{1 \leq i \leq l} (n_i \lambda(g_i) + \nu(g_i, g_{i-1}^+)) \right\| \leq 2l\delta_{r,\varepsilon}.$$

The statement in the above Proposition is multiplicative, taking value in AM, replacing λ with the signed Jordan projections $\mathscr{L}_{s_i}$ and ν with the ratio maps $\mathscr{R}_{s_i,s_{i-1}}$. It gives information on both the elliptic and hyperbolic part of product of generic loxodromic elements.

Finally, this Proposition is an important component in the proofs of §6 for decorrelation.

Proof. Let us prove the estimate for the signed cocycle, the extended Jordan projection's estimate will follow from the Definition 4.2 that for every loxodromic element g and a suitable cross-section s such that $g^+ \in \mathcal{F}_s$, then $\mathscr{L}_s(g) = \beta_s(g, g^+)$.

For all $1 \le j \le l$, we set $\xi_j := g_j^{n_j} ... g_1^{n_1}\xi_0$. At each step starting from $j = 1$, the element g_j is (r,ε)-loxodromic and $\xi_{j-1} \in \mathcal{V}_\varepsilon(\partial\mathsf{b}(g_j^-))^\complement \cap \mathcal{F}_{s_{j-1}}$. Hence $\xi_j := g_j^{n_j}\xi_{j-1} \in B(g_j^+,\varepsilon)$, which by choice of s_j and $r \le \frac{1}{6}\mathrm{d}(g_j^+, \partial\mathsf{b}(g_j^-) \cup \partial\mathsf{b}(g_{j+1}^-))$ is inside $\mathcal{V}_r(\partial\mathsf{b}(g_{j+1}^-))^\complement \cap \mathcal{V}_r(\partial\mathsf{b}(g_j^-))^\complement \subset \mathcal{F}_{s_j}$. We deduce, by induction, that $\xi_j \in B(g_j^+,\varepsilon) \subset \mathcal{F}_{s_j}$ for every $1 \le j \le l$.

By the cocycle relation and recognizing ξ_j for every $1 \le j \le l-1$,

$$\begin{aligned}\beta_{s_l,s_0}(g_l^{n_l}...g_1^{n_1},\xi_0) &= \beta_{s_l,s_{l-1}}(g_l^{n_l}, g_{l-1}^{n_{l-1}}...g_1^{n_1}\xi_0)\ \beta_{s_{l-1},s_0}(g_{l-1}^{n_{l-1}}...g_1^{n_1},\xi_0)\\ &= \beta_{s_l,s_{l-1}}(g_l^{n_l},\xi_{l-1}) \qquad \beta_{s_{l-1},s_0}(g_{l-1}^{n_{l-1}}...g_1^{n_1},\xi_0)\\ &= \beta_{s_l,s_{l-1}}(g_l^{n_l},\xi_{l-1}) \cdots \beta_{s_j,s_{j-1}}(g_j^{n_j},\xi_{j-1}) \cdots \beta_{s_1,s_0}(g_1^{n_1},\xi_0).\end{aligned}$$

We will first prove that the first term on the right hand side is in a $2\delta_{r,\varepsilon}$ neighbourhood of $\mathscr{L}_{s_l}(g_l^{n_l})\mathscr{R}_{s_l,s_{l-1}}(g_l\ ;\ g_{l-1}^+)$. It will then follow by induction that every term of the form $\beta_{s_j,s_{j-1}}(g_j^{n_j},\xi_{j-1})$ where $2 \le j \le l$ is $2\delta_{r,\varepsilon}$ close to $\mathscr{L}_{s_j}(g_j^{n_j})\mathscr{R}_{s_j,s_{j-1}}(g_j\ ;\ g_{j-1}^+)$. Finally, we prove that the last term is in a $\delta_{r,\varepsilon}$ neighbourhood of $\mathscr{L}_{s_1}(g_1^{n_1})\mathscr{R}_{s_1,s_0}(g_1;\xi_0)$.

Let us apply Proposition 4.9, then replace $g_l^{n_l}\xi_{l-1}$ with ξ_l

$$\begin{aligned}\beta_{s_l,s_{l-1}}(g_l^{n_l},\xi_{l-1}) &= \mathscr{R}_{s_l}(g_l\ ;\ g_l^{n_l}\xi_{l-1})^{-1}\ \mathscr{L}_{s_l}(g_l)^{n_l}\ \mathscr{R}_{s_l,s_{l-1}}(g_l\ ;\ \xi_{l-1})\\ &= \mathscr{R}_{s_l}(g_l\ ;\ \xi_l)^{-1}\ \mathscr{L}_{s_l}(g_l)^{n_l}\ \mathscr{R}_{s_l,s_{l-1}}(g_l\ ;\ \xi_{l-1}).\end{aligned}$$

By Definition 4.8 of the ratio $\mathscr{R}_{s_l}(g_l\ ;\ \xi_l)^{-1} = \mathscr{R}_{s_l}(g_l^-\ ;\ g_l^+,\xi_l)^{-1}$. Since $\xi_l \in B(g_l^+,\varepsilon)$ and by choice of $r \le \frac{1}{6}\mathrm{d}(g_l^+, \partial\mathsf{b}(g_l^-))$, we deduce by Definition 4.11 of $\delta_{r\varepsilon}$ that $\mathscr{R}_{s_l}(g_l\ ;\ \xi_l) \in B(e_{AM},\delta_{r,\varepsilon})$. The first term is small, it remains to show that the third term is close to $\mathscr{R}_{s_l,s_{l-1}}(g_l^-; g_l^+, g_{l-1}^+)$. By definition of the ratio map,

$$\begin{aligned}\mathscr{R}_{s_l,s_{l-1}}(g_l^-; g_l^+,\xi_{l-1}) &= \mathscr{T}_{s_l,[h_l]}(g_l^+) \qquad\qquad \mathscr{T}_{[h_l],s_{l-1}}(\xi_{l-1})\\ &= \mathscr{T}_{s_l,[h_l]}(g_l^+)\ \mathscr{T}_{[h_l],s_{l-1}}(g_{l-1}^+)\\ &\qquad \mathscr{T}_{s_{l-1},[h_l]}(g_{l-1}^+)\mathscr{T}_{[h_l],s_{l-1}}(\xi_{l-1})\\ &= \mathscr{R}_{s_l,s_{l-1}}(g_l^-; g_l^+, g_{l-1}^+) \qquad \mathscr{R}_{s_{l-1}}(g_l^-; g_{l-1}^+,\xi_{l-1}).\end{aligned}$$

Hence, the cocycle can be written as follows,

$$\beta_{s_l,s_{l-1}}(g_l^{n_l},\xi_{l-1}) = \mathscr{R}_{s_l}(g_l\ ;\ \xi_l)^{-1} \quad \mathscr{L}_{s_l}(g_l)^{n_l}\ \mathscr{R}_{s_l,s_{l-1}}(g_l\ ;\ g_{l-1}^+) \quad \mathscr{R}_{s_{l-1}}(g_l^-; g_{l-1}^+,\xi_{l-1}).$$

Finally, by choice of $r \le \frac{1}{6}\mathrm{d}(g_{l-1}^+, \partial\mathsf{b}(g_l^-))$ and by Definition 4.11 of $\delta_{r,\varepsilon}$, the third term is small i.e. $\mathscr{R}_{s_{l-1}}(g_l^-; g_{l-1}^+,\xi_{l-1}) \in B(e_{AM},\delta_{r,\varepsilon})$. Given that the distance in AM is symmetric and invariant by conjugation, we deduce that

$$\beta_{s_l,s_{l-1}}(g_l^{n_l},\xi_{l-1}) \in \mathscr{L}_{s_l}(g_l)^{n_l}\ \mathscr{R}_{s_l,s_{l-1}}(g_l\ ;\ g_{l-1}^+)\ B(e_{AM},2\delta_{r,\varepsilon}).$$

By induction, for every $2 \le j \le l$

$$\beta_{s_j,s_{j-1}}(g_j^{n_j},\xi_{j-1}) \in \mathscr{L}_{s_j}(g_j)^{n_j}\ \mathscr{R}_{s_j,s_{j-1}}(g_j\ ;\ g_{j-1}^+)\ B(e_{AM},2\delta_{r,\varepsilon}).$$

Now for $\beta_{s_1,s_0}(g_1^{n_1},\xi_0)$, by Proposition 4.9 and by replacing $g_1^{n_1}\xi_0$ with ξ_1

$$\beta_{s_1,s_0}(g_1^{n_1},\xi_0) = \mathscr{R}_{s_1}(g_1\ ;\ g_1^{n_1}\xi_0)^{-1}\ \mathscr{L}_{s_1}(g_1)^{n_1}\ \mathscr{R}_{s_1,s_1}(g_1\ ;\ \xi_0).$$

Similarly, by choice of r and by definition of $\delta_{r,\varepsilon}$, we deduce that

$$\beta_{s_1,s_0}(g_1^{n_1},\xi_0) \in \mathscr{L}_{s_1}(g_1)^{n_1}\ \mathscr{R}_{s_1,s_1}(g_1\ ;\ \xi_0)\ B(e_{AM},\delta_{r,\varepsilon}).$$

Hence,

$$\beta_{s_l,s_0}(g_l^{n_l}...g_1^{n_1},\xi_0) \in \mathscr{L}_{s_l}(g_l^{n_l})\mathscr{R}_{s_l,s_{l-1}}(g_l,g_{l-1}^+)...\mathscr{L}_{s_1}(g_1^{n_1})\mathscr{R}_{s_1,s_0}(g_1,\xi_0)B(e_{AM},(2l-1)\delta_{r,\varepsilon}).$$

Finally, for the extended Jordan projection, we apply the cocycle estimate for the attracting point g^+ of $g_l^{n_l}...g_1^{n_1}$ with cross-section $s_0 = s_l$. By [Ben00, Lemma 3.6], it is in $B(g_l^{n_l},\varepsilon)$, therefore by choice of $r \leq \frac{1}{6}\mathsf{d}(g_l^+,\partial\mathsf{b}(g_1^-))$, we deduce that $\mathscr{R}_{s_1,s_l}(g_1,g^+) \in \mathscr{R}_{s_1,s_l}(g_1,g_l^+)B(e_{AM},\delta_{r,\varepsilon})$.

Hence $\mathscr{L}_{s_l}(g_l^{n_l}...g_1^{n_1}) \in \mathscr{L}_{s_l}(g_l^{n_l})\mathscr{R}_{s_l,s_{l-1}}(g_l,g_{l-1}^+)...\mathscr{L}_{s_1}(g_1^{n_1})\mathscr{R}_{s_1,s_l}(g_1,g_l^+)B(e_{AM},2l\delta_{r,\varepsilon})$. □

Definition 4.13. *Let $0 < \varepsilon \leq r$. A semigroup $\Gamma \subset G$ is* strongly (r,ε)-Schottky *if*

(i) *every element is (r,ε)-loxodromic,*
(ii) *$d(h^+,\partial\mathsf{b}(h'^-)) \geq 6r$ for all $h,h' \in \Gamma$.*

We also write that Γ is a strong (r,ε)-Schottky semigroup.

5. Invariant sets

In this section, $\Gamma < G$ is a Zariski dense subsemigroup of G.

In the first paragraph, following [DG20], we construct the non-wandering set $\Omega \subset \Gamma\backslash G/M$ for regular Weyl chamber flows. We notice that it is the smallest closed A-invariant subset of $\Gamma\backslash G/M$ containing all the periodic orbits of the flows $\phi^t_{\lambda(\Gamma^{lox})}$.

Denote by $\widetilde{\Omega}_G$ the preimage of Ω via the projection $G \to \Gamma\backslash G/M$. Such a subset is closed, left Γ-invariant and right AM-invariant. Denote by M_0 the connected component of the identity of M. In the second paragraph, following Guivarc'h–Raugi [GR07], we introduce the sign group M_Γ, a normal subgroup of finite index of M containing M_0. One can find another construction of the sign group in [Ben05].

Finally, using Guivarc'h–Raugi's classification of Γ-invariant subsets of K (cf. Theorem 5.9) we construct a partition of left Γ-invariant right AM_Γ-invariant subsets of $\widetilde{\Omega}_G$. We prove in Proposition 5.12 that the topological dynamics of diagonal flows on these subsets are all conjugated.

5.1. In the space of Weyl chamber.

Definition 5.1. *A point $\eta \in \mathcal{F}$ is a* limit point *if there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that $\big((\gamma_n)_*\mathrm{Haar}_{\mathcal{F}}\big)_{n\geq 1}$ converges weakly towards the Dirac measure in η.*

The limit set *of Γ, denoted by $L_+(\Gamma)$, is the set of limit points of Γ. It is a closed, Γ-invariant subset of $\mathcal{F}$.*

Denote by $L_-(\Gamma)$ the limit set of Γ^{-1} and finally let $L^{(2)}(\Gamma) = \Big(L_+(\Gamma) \times L_-(\Gamma)\Big) \cap \mathcal{F}^{(2)}$.

Note that when Γ is a subgroup, then $L_+(\Gamma) = L_-(\Gamma)$ and $L^{(2)}(\Gamma)$ is the subset of pair of points of $L_+(\Gamma)$ in general position. For the hyperbolic plane, we get the product of the usual limit set minus the diagonal.

By [Ben97b] Lemma 3.6, the set of pairs of attracting and repelling points of loxodromic elements of Γ is dense in $L_+(\Gamma)\times L_-(\Gamma)$. Therefore, using Hopf coordinates and the construction of attracting and repelling points of loxodromic elements, $L^{(2)}(\Gamma)$ identifies with smallest closed Γ-invariant subset of G/AM containing

$$\Big\{h_\gamma AM \mid \exists\gamma \in \Gamma^{lox} \text{ such that } h_\gamma^{-1}\gamma h_\gamma \in MA^{++}\Big\}.$$

Theorem 5.2 (Theorem 4.5 [DG20]). *The (diagonal) action of Γ on $L^{(2)}(\Gamma)$ is topologically transitive, i.e. there are dense Γ-orbits.*

The transitivity, along with the background work on Bruhat-Hopf coordinates, is one of the main arguments in the proof of the main mixing Theorem 7.1 in §7.

Definition 5.3. *We denote by* $\widetilde{\Omega}$ *the subset of* non-wandering Weyl chambers, *defined through the Hopf parametrization by:*

$$\widetilde{\Omega} := \mathcal{H}^{-1}(L^{(2)}(\Gamma) \times \mathfrak{a}).$$

This is a Γ*-invariant and right* A*-invariant subset of* G/M. *When* Γ *is a subgroup, we denote by* $\Omega := \Gamma\backslash\widetilde{\Omega}$ *the quotient space.*

By Theorem 5.2 the quotient Ω is the smallest closed A-invariant subset of $\Gamma\backslash G/M$ containing the following subset

$$\left\{\phi^{\mathbb{R}}_{\lambda(\gamma)}(h_\gamma M) \mid \gamma \in \Gamma^{lox} \text{ and } h_\gamma^{-1}\gamma h_\gamma \in MA^{++}\right\}.$$

Note that in rank one, the above set is the reunion of all periodic orbits for the geodesic flow.

5.2. **The sign subgroup.** Denote by $M^{ab} := M/[M,M]$ the abelianisation of the compact group M and by $\pi_{ab} : M \to M^{ab}$ the projection. Abusing notations, π_{ab} also denotes the projection $AM \to AM^{ab}$.

Fact 5.4. *For all cross-sections* s *and* s' *the projection into* AM^{ab} *of the signed translation maps* $\mathscr{L}_s$ *and* $\mathscr{L}_{s'}$ *coincide on the intersection of their domains i.e. for every loxodromic element* $g \in G$ *such that* $g^+ \in \mathcal{F}_s \cap \mathcal{F}_{s'}$, *then*

$$\pi_{ab}\big(\mathscr{L}_s(g)\big) = \pi_{ab}\big(\mathscr{L}_{s'}(g)\big).$$

Proof. By Definition 4.2 of the signed translation map, we write

$$\mathscr{L}_{s'}(g) = \beta_{s'}(g, g^+).$$

First apply the relations between the signed cocycles and transition functions of the Fact 3.9, then using that that g^+ is fixed by g, we deduce that

$$\mathscr{L}_{s'}(g) = \mathscr{T}_{s',s}(g^+)\beta_s(g, g^+)\mathscr{T}_{s,s'}(g^+).$$

The middle term is an extended Jordan projection and the first and last term are inverse (see Proposition 3.6 (iii) on transition functions). Hence

$$\mathscr{L}_{s'}(g) = \mathscr{T}_{s',s}(g^+)\mathscr{L}_s(g)\mathscr{T}_{s',s}(g^+)^{-1}.$$

The claim then follows by projecting the relation into the abelian group AM^{ab}. □

Denote by $\mathscr{L}^{ab}$ the map that associates to every loxodromic element $g \in G$ the projection into AM^{ab} of any signed Jordan projection. We call this map the *abelian signed Jordan projection.* Denote by $\pi_{M^{ab}}$ the projection $AM^{ab} \to M^{ab}$.

Definition 5.5. *Denote by* Γ^{lox} *the subset of loxodromic elements of* Γ. *We define the* abelian sign group *of* Γ *by*

$$M^{ab}_\Gamma := \pi_{M^{ab}}\left(\overline{\langle \mathscr{L}^{ab}(\Gamma^{lox})\rangle}\right).$$

The sign group *of* Γ, *denoted by* M_Γ *is given by* $M_\Gamma := \pi_{ab}^{-1}(M^{ab}_\Gamma)$.

The following Theorem will imply non-arithmeticity in AM^{ab}_Γ of the abelian signed Jordan projections of Γ.

Theorem 5.6 (Theorem 6.4 [GR07]). *The closed subgroup spanned by* $\mathscr{L}^{ab}(\Gamma^{lox})$ *is of finite index in* AM^{ab}.

Corollary 5.7. *The closed subgroup spanned by $\mathscr{L}^{ab}(\Gamma^{lox})$ and the sign group M_Γ are related as follows.*

$$\overline{\langle \mathscr{L}^{ab}(\Gamma^{lox}) \rangle} = AM_\Gamma^{ab}.$$

Proof. Denote by $H := \overline{\langle \mathscr{L}^{ab}(\Gamma^{lox}) \rangle}$. By definition, it is a closed subgroup of AM^{ab}. In particular, AM^{ab}/H is Hausdorff. According to the previous Theorem 5.6, it is a finite group. By endowing it with the discrete topology, we deduce that the morphism

$$\varphi : A \longrightarrow AM^{ab}/H$$
$$a \longmapsto aH$$

is a continuous map that takes value in a finite group. Since A is connected, φ is constant to $e_A H$, hence $\varphi(A) = AH = H$ and $A \subset H$.

By Definition 5.5 of the sign group, $H \subset AM_\Gamma^{ab}$. Conversely, for every $x \in A$ and $m \in M_\Gamma^{ab}$, there exists $y \in A$ such that $ym \in H$. We now write xm as a product $xm = (xy^{-1})ym$. In the right hand side, the first term is in A hence in H and the second term is in H, hence $xm \in H$. We thus conclude that $H = AM_\Gamma^{ab}$. □

Theorem 5.8 (Theorem 8.2 [Ben05],Theorem 1.9 [GR07])**.** *The following holds.*

(a) *M_Γ is a closed normal subgroup of finite index of M and contains the connected component of the identity M_0.*
(b) *There exists an integer $p_\Gamma \in [0, \dim \mathfrak{a}]$ such that M_Γ/M_0 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{p_\Gamma}$.*
(c) *$M_{\Gamma^{-1}} = k_\iota M_\Gamma k_\iota^{-1}$ where $k_\iota \in N_K(A)$ is an element such that $Ad(k_\iota)\mathfrak{a}^+ = -\mathfrak{a}^+$.*
(d) *For all $g \in G$, the groups satisfy $M_{g\Gamma g^{-1}} = M_\Gamma$.*

When G is a split, real linear, algebraic group, Y. Benoist in [Ben97a] studies the following conditions:

(C1) There exists a Zariski dense subgroup $\Gamma \subset G$ such that $M_\Gamma = M_0$.
(C2) There exists a Zariski dense subgroup $\Gamma \subset G$ with $M_\Gamma \supsetneq M_0$ such that the sign group of every Zariski dense subgroup of Γ strictly contains M_0.

In particular, he proves for $\mathrm{SL}(m, \mathbb{R})$ that both conditions hold when m is a multiple of 4, in fact (C2) is true for all m. . However, when m is even but not divisible by 4, condition (C1) is false i.e. the sign group of every Zariski dense subgroup of $\mathrm{SL}(m, \mathbb{R})$ is non trivial.

5.3. **Γ-invariant subsets of G.** The G-equivariant projection $K \to \mathcal{F}$ endows K with a fiber bundle structure of fiber M over the Furstenberg boundary. We apply a result of Guivarc'h–Raugi [GR07] to the left action of G on K. Denote by $L_G(\Gamma)$ the preimage in K of the limit set $L(\Gamma) \subset \mathcal{F}$. Then the closed right M-invariant and left Γ-invariant subset $L_G(\Gamma) \subset K$ partitions into $|M/M_\Gamma|$ closed, Γ-invariant, minimal subsets. Furthermore, these invariant subsets are right M_Γ-invariant. Lastly, using Iwasawa decomposition, we partition $\widetilde{\Omega}_G$ into left Γ-invariant and right AM_Γ-invariant subsets of G.

Theorem 5.9 (Theorem 2 [GR07])**.** *The following holds.*

1) *$L_G(\Gamma) \subset K$ partitions into $|M/M_\Gamma|$ closed, minimal Γ-invariant subsets i.e. in each partition, every Γ-orbit is dense.*
2) *There is an indexation of this partition by M/M_Γ i.e. $L_G(\Gamma) = \sqcup_{[m] \in M/M_\Gamma} L_{[m]}(\Gamma)$ such that for every $m \in M$,*

$$L_{[m]}(\Gamma) = L_{[e_M]}(\Gamma)m.$$

3) *Every element of the partition turns out to be right M_Γ-invariant.*

Recall that for every compact Bruhat section s, the map $g \in G_s \mapsto k_{\mathcal{I}}(g) \in K$ reads in $\mathcal{B}_s$ coordinates for the source and target as

$$\begin{aligned} \mathcal{F}_s^{(2)} \times AM &\longrightarrow \mathcal{F}_s \times M \\ (\xi, \eta \,;\, x)_s &\longmapsto (\xi \,;\, x_M)_s. \end{aligned}$$

Bruhat-Hopf coordinates make the following diagram commutative and equivariant for every compact Bruhat section s.

$$\begin{array}{ccc} \mathcal{F}_s^{(2)} \times AM \simeq G_s \subset G & \longrightarrow & \mathcal{F}_s \times M \simeq s(\mathcal{F}_s)M \subset K \\ {\scriptstyle M}\big\downarrow & & \big\downarrow{\scriptstyle M} \\ \mathcal{H}^{-1}\big(\mathcal{F}_s^{(2)} \times \mathfrak{a}\big) \subset G/M & \longrightarrow & \mathcal{F}_s \end{array}$$

Let us now translate Theorem 5.9 using the right side of the diagram.

Corollary 5.10. *Let s be a compact Bruhat section of $K \to \mathcal{F}$. Then the following holds.*

1) *For every element in $L_G(\Gamma)$ of coordinates $(\xi \,;\, x)_s \in \big(\mathcal{F}_s \cap L(\Gamma)\big) \times M$, there exists a unique element $[m] \in M/M_\Gamma$ such that $(\xi \,;\, x)_s$ is in $L_{[m]}(\Gamma)$. Furthermore, the Γ-orbit $\Gamma(\xi \,;\, x)_s$ is dense in $L_{[m]}(\Gamma)$.*
2) *For every element in $L_{[e_M]}(\Gamma)$ of coordinate $(\xi \,;\, x)_s$ and for all $m \in M$, the translate of coordinate $(\xi \,;\, xm)_s$ is in $L_{[m]}(\Gamma)$.*
3) *For every element in $L_{[m]}(\Gamma)$ of coordinate $(\xi \,;\, x)_s$ and for all $c \in M_\Gamma$ then the element of coordinate $(\xi \,;\, xc)_s$ remains in $L_{[m]}(\Gamma)$.*

Denote by $\widetilde{\Omega}_G$ the preimage in G of $\widetilde{\Omega} \subset G/M$ by the projection $G \to G/M$. It is a closed, left Γ-invariant and right AM-invariant subset of G. For every compact Bruhat section $s \in k(\mathcal{F})$, the intersection $\widetilde{\Omega}_G \cap s(\mathcal{F}_s)NAM$ reads in Bruhat-Hopf coordinates as

$$\mathcal{B}_s\big(\widetilde{\Omega}_G \cap s(\mathcal{F}_s)NAM\big) = \big(L^{(2)}(\Gamma) \cap \mathcal{F}_s^{(2)}\big) \times AM.$$

In other words, every element of coordinate $(\xi, \eta \,;\, x)_s \in L^{(2)}(\Gamma) \times AM$ with $\xi \in \mathcal{F}_s$ is in $\widetilde{\Omega}_G$. The previous Theorem and left side of the diagram allow us to partition it into closed left Γ-invariant and right AM_Γ-invariant subsets. To simplify notations, for every $x \in AM$, we denote by x_M its projection in M.

Definition 5.11. *For every $m \in M$, we denote by $\widetilde{\Omega}_{[m]} := L_{[m]}(\Gamma)AN \cap \widetilde{\Omega}_G$ and $\Omega_{[m]} := \Gamma \backslash \widetilde{\Omega}_{[m]}$. In other words, $\widetilde{\Omega}_{[m]}$ is the subset of elements of coordinate $(\xi, \eta \,;\, x)_s \in L^{(2)}(\Gamma) \times AM$ whose compact Iwasawa projection of coordinate $(\xi \,;\, x_M)_s$ is in $L_{[m]}(\Gamma)$, for every suitable compact Bruhat section s.*

Proposition 5.12. *The sets $\widetilde{\Omega}_{[m]}$, with $[m] \in M/M_\Gamma$ satisfy the following properties.*

(a) *The left Γ-invariant and right AM_Γ-invariant subsets $\widetilde{\Omega}_{[m]}$ form a partition of $\widetilde{\Omega}_G$, i.e.*

$$\widetilde{\Omega}_G = \bigsqcup_{[m] \in M/M_\Gamma} \widetilde{\Omega}_{[m]}.$$

(b) *For every $m \in M$, then $\widetilde{\Omega}_{[m]} = \widetilde{\Omega}_{[e_M]} m$.*

(c) *For all $[m] \in M/M_\Gamma$, the dynamical systems $(\Omega_{[m]}, \phi_\theta^t)$ and $(\Omega_{[e_M]}, \phi_\theta^t)$ are conjugated.*

Proof. The left Γ-invariance in (a) is a consequence of the first point of Theorem 5.9 and of the left Γ-invariance of $\widetilde{\Omega}_G$. It also follows from the same point that the subsets $\big(\widetilde{\Omega}_{[m]}\big)_{[m] \in M/M_\Gamma}$

form a partition of $\widetilde{\Omega}_G$. The right AM_Γ-invariance is due to the right M_Γ-invariance of $L_{[m]}(\Gamma)$ and the properties of the Bruhat-Hopf coordinates given by Proposition 3.8 and Proposition 3.11.

Point (b) is a direct consequence of the second point of Theorem 5.9 and the compatibility of the Bruhat-Hopf coordinates with the compact Iwasawa projection.

Point (c) follows from the commutativity of the right action by multiplication by M with that of A, because every element of M commute with every element of A. □

6. Decorrelation

In the remaining parts of this paper, unless it is specified otherwise in the statements, $\Gamma < G$ is a Zariski dense subgroup of G.

We construct a pair of points $(\xi_1, \check{\xi}_1) \in L^{(2)}(\Gamma)$ and show that there exists (r,ε)-loxodromic elements in Γ of attracting and repelling points in an ε-neighbourhood of these points and whose signed cocycle are dense in an M_Γ-invariant set.

Consider the family of equicontinuity constants $\delta_{r,\varepsilon}$ of Definition 4.11. To simplify notations, we introduce the family of constants

$$\tilde{\delta}_{r,\varepsilon} := (8\dim\mathfrak{a} + 4\dim M_0 + 5)\delta_{r,\varepsilon}.$$

In this section, we prove the following Proposition.

Proposition 6.1. Assume that M_0 is abelian. *Then there exists*

1) $(\xi_1, \check{\xi}_1) \in L^{(2)}(\Gamma)$,
2) *a real positive number*

$$0 < r_1 \leq \frac{1}{6}\mathrm{d}(\xi_1, \partial\mathsf{b}(\check{\xi}_1)),$$

such that for all $r \in (0, r_1]$ and $\varepsilon \in (0, r]$, for any choice of compact Bruhat sections $c_1, \check{c}_1$ with

$$B(\xi_1, r) \subset \mathcal{F}_{c_1} \text{ and } \mathcal{V}_{6r}\big(\partial\mathsf{b}(\check{\xi}_1)\big)^{\complement} \subset \mathcal{F}_{\check{c}_1}$$

there exists a finite family $(g_i)_{i\in I} \subset \Gamma$ and a point $a_{r,\varepsilon} \in A$ that satisfy the following conditions.

† *For all $i \in I$, the element g_i is $(2r, 2\varepsilon)$-loxodromic with*

$$(g_i^+, g_i^-) \in B(\xi_1, \varepsilon) \times B(\check{\xi}_1, \varepsilon).$$

‡ *For all $\eta \in \mathcal{V}_{6r}\big(\partial\mathsf{b}(\check{\xi}_1)\big)^{\complement}$ and $(\eta_i)_{i\in I} \subset B(\eta, \varepsilon)$, the family $\{\beta_{c_1,\check{c}_1}(g_i, \eta_i)\}_{i\in I}$ is $\tilde{\delta}_{r,\varepsilon}$-dense in $a_{r,\varepsilon}\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1; \xi_1, \eta)M_\Gamma$ i.e.*

$$a_{r,\varepsilon}\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1; \xi_1, \eta)M_\Gamma \subset \cup_{i\in I} B(\beta_{c_1,\check{c}_1}(g_i, \eta_i), \tilde{\delta}_{r,\varepsilon}).$$

In the first paragraph, we construct (Cf. Lemma 6.2) families of finite products of loxodromic elements whose signed cocycle reach all the connected components of AM_Γ.

In the second paragraph, we prove the density, in an M_0 orbit of AM_Γ that projects into a convex cone of non-empty interior of $\mathfrak{a}^{++}$, of the signed Jordan projection of a family of products of loxodromic elements. More specifically, we construct in Lemma 6.4:

(a) a convex cone of non-empty interior $\mathcal{C}_0 \subset \mathfrak{a}^{++}$,
(b) a pair of transverse points $(\xi_0, \check{\xi}_0) \in L^{(2)}(\Gamma)$,
(c) a real positive number $r_0 > 0$.

for which there exists, for all $0 < \varepsilon \leq r \leq r_0$ an (r,ε)-Schottky generating family $F_{r,\varepsilon} = (\gamma_1, ..., \gamma_l)$, of at most $4\dim\mathfrak{a} + 2\dim M_0$ elements, such that the signed Jordan projection of the elements of the form

$$\{\gamma_1^{n_1}...\gamma_l^{n_l} \mid n_1, ...n_l \geq 1\}$$

are $2l\delta_{r,\varepsilon}$-dense in a translate in AM_Γ of $\exp(\mathcal{C}_0)M_0$. Note that the constants $\delta_{r,\varepsilon}$ converge to 0 as ε goes to 0.

In the third paragraph, we prove Proposition 6.1 by combining the previous Lemmata with an overlapping cone argument.

6.1. **The connected components of AM_Γ.** Since M/M_0 is abelian, the projection in M/M_0 of every signed Jordan projection does not depend on the choice of the cross-section. The following Lemma does not require that M_0 is abelian.

Lemma 6.2. *Denote by p the integer such that M_Γ/M_0 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^p$ and by $\pi_{M/M_0} : AM \to M/M_0$ the projection.*

Then for all $\xi_0 \in L_+(\Gamma)$, there exists $h_1, ..., h_p \in \Gamma^{lox}$ such that taking the notation $h_0^+ := \xi_0$, the following holds.

(i) *For every choice of cross-sections $s_1, ..., s_p$ such that $h_i^+ \in \mathcal{F}_{s_i}$ for all $1 \le i \le p$, the set $\{\pi_{M/M_0}(\mathscr{L}_{s_i}(h_i))\}_{1\le i\le p}$ forms a basis of the vector space M_Γ/M_0.*

(ii) *For all $1 \le i \le p$, the pair $(h_{i-1}^+, h_i^-) \in L^{(2)}(\Gamma)$ is transverse.*

(iii) *Assume now that s_0, s_p are compact Bruhat sections of respective domains $\mathsf{b}(h_1^-)$ and $\mathsf{b}(h_p^-)$, then there exists $m_p \in M$ and a large integer $N \in \mathbb{N}$ such that for all $\nu \in \{0,1\}^p$, for all $n \ge N$,*

$$\pi_{M/M_0}\Big(\beta_{s_p m_p, s_0}(h_p^{2n+\nu_p}...h_1^{2n+\nu_1}, \xi_0)\Big) = \nu.$$

For the first step of the proof of this Lemma we use the non-arithmeticity Corollary 5.7 to choose p loxodromic elements in Γ. We order them. For the second step, since the repelling point of the ith element is not necessarily transverse to the attracting point of the $i-1$th term, we conjugate inductively these elements. Thanks to the Fact below, the abelianised Jordan projection of the conjugated element will remain in the same connected component of AM_Γ. To obtain the third point, we use the explicit formula of the cocycle given by Proposition 4.9 and the cocycle relation and combine it with a Ping-Pong argument. Finally, the corrective term $m_p \in M$ of the cross-section is chosen using the Definition 4.8 of the ratio maps.

Fact 6.3. *For all $u \in G$ and all loxodromic element $g \in G$, the conjugate ugu^{-1} is loxodromic of attracting point ug^+ and basin of attraction $u\mathsf{b}(g^-) = \mathsf{b}(ug^-)$. Furthermore,*

$$\mathscr{L}^{ab}(ugu^{-1}) = \mathscr{L}^{ab}(g).$$

Proof. By Proposition 4.4, a loxodromic element g has attracting point g^+ in $\mathcal{F}$ and its basin of attraction is the Bruhat cell opposite to its repelling point $\mathsf{b}(g^-)$. By Fact 4.3, consider $h_g \in G$ such that $\mathscr{L}_{[g^-]}(g) = h_g^{-1} g h_g$. Since the Jordan projection is invariant by conjugation, ugu^{-1} is also loxodromic and diagonalised by uh_g. Therefore, its attracting point is ug^+ of basin of attraction $\mathsf{b}(ug^-)$. The abelian signed Jordan projection relation comes from Fact 4.3 by choosing to compute $\mathscr{L}_{u\cdot[g^-]}(ugu^{-1}) = \mathscr{L}_{[g^-]}(g)$ and then using Fact 6.3 to argue that the abelian signed Jordan projection does not depend on the choice of the cross-sections. □

Proof of Lemma 6.2. Since $\mathbb{Z}/2\mathbb{Z}$ is a field, M_Γ/M_0 is a vector field over it. By Corollary 5.7, the abelian signed Jordan projection of Γ^{lox} spans AM_Γ^{ab} i.e. $AM_\Gamma^{ab} = \overline{\langle \mathscr{L}^{ab}(\Gamma^{lox})\rangle}$. Because M_0 is a closed normal subgroup of M and $M_0 \supset [M,M]$, we deduce that $M_\Gamma^{ab}/M_0^{ab} = M_\Gamma/M_0$. Using that this is a discrete vector space and projecting the abelian signed Jordan projection to M^{ab}, we get

$$M_\Gamma/M_0 = M_\Gamma^{ab}/M_0^{ab} = \Big\langle \pi_{M^{ab}/M_0^{ab}}\big(\mathscr{L}^{ab}(\Gamma^{lox})\big)\Big\rangle.$$

The left and middle sides are $\mathbb{Z}/2\mathbb{Z}$ vector space of dimension p. The right hand side provides us with a generating set of the vector space, we extract a basis from it. Hence there exists $g_1, ..., g_p \in \Gamma^{lox}$ such that

$$M_\Gamma/M_0 = \Big\langle \pi_{M^{ab}/M_0^{ab}}\big(\mathscr{L}^{ab}(g_p)\big), ..., \pi_{M^{ab}/M_0^{ab}}\big(\mathscr{L}^{ab}(g_1)\big)\Big\rangle.$$

Now using that $M_\Gamma^{ab}/M_0^{ab} = M_\Gamma/M_0$, we deduce for every suitable choice of compact Bruhat sections $b_1, ..., b_p$, that

$$M_\Gamma/M_0 = \Big\langle \pi_{M/M_0}\big(\mathscr{L}_{b_p}(g_p)\big), ..., \pi_{M/M_0}\big(\mathscr{L}_{b_1}(g_1)\big)\Big\rangle.$$

By the above Fact 6.3, condition (i) holds for every family $h_1, ..., h_p$ such that for every $i = 1, ..., p$ the element h_i is a conjugate of g_i.

Let us now construct $h_1, ..., h_p$. Set $u_0 := e_G$ and $g_0^+ := \xi_0$. We are going to choose by induction $u_1, ..., u_p \in \Gamma$ such that for every $i = 1, ..., p$,

$$\big(u_{i-1}^{-1} g_{i-1}^+, u_i^{-1} g_i^-\big) \in \Big(L_+(\Gamma) \times L_-(\Gamma)\Big) \cap \mathcal{F}^{(2)}.$$

Repelling points of loxodromic elements lie in $L_-(\Gamma)$ i.e. for every $i = 1, ..., p$,

$$g_i^- \in L_-(\Gamma).$$

By minimality of the action of Γ^{-1} on $L_-(\Gamma)$ and because there are no isolated points in this subset, we choose $u_1 \in \Gamma$ such that $u_1^{-1} g_1^-$ also lies in the Bruhat cell opposite to ξ_0, meaning that $u_1^{-1} g_1^- \in L_-(\Gamma) \cap \mathsf{b}(\xi_0)$. By Proposition 2.6, we deduce the first step $(\xi_0, u_1^{-1} g_1^-) \in L^{(2)}(\Gamma)$. Using the same minimality arguments on the action of Γ^{-1} on $L_-(\Gamma)$, we proceed as such to construct u_i given $u_1, ..., u_{i-1}$ such that $\big(u_{i-1}^{-1} g_{i-1}^+, u_i^{-1} g_i^-\big) \in L^{(2)}(\Gamma)$. Now that $u_1, ..., u_p \in \Gamma$ are chosen, we set for every $i = 1, ..., p$

$$h_i := u_i^{-1} g_i u_i.$$

By the above Fact 6.3, condition (i) holds. Furthermore, because Γ is a subgroup, every h_i is a loxodromic element of Γ with

$$(h_i^+, h_i^-) = (u_i^{-1} g_i^+, u_i^{-1} g_i^-).$$

The family $h_1, ..., h_p$ verifies condition (ii) by construction of the u_i.

Let us now check condition (iii). Choose $s_1, ..., s_p$ compact Bruhat sections of respective domains $\mathsf{b}(h_1^-), ..., \mathsf{b}(h_p^-)$ and set $s_1 = s_0$. For all $n_1, ..., n_p \geq 1$, denote by $\underline{n} := (n_1, ..., n_p)$ and for all $i = 1, ..., p$ we set

$$\xi_{i,\underline{n}} := h_i^{n_i} ... h_1^{n_1} \xi_0.$$

Let us compute the signed cocycle $\beta_{s_p,s_0}(h_p^{n_p} ... h_1^{n_1}, \xi_0)$. We want to understand which connected component of AM these cocycles can reach. Condition (ii) ensures that $\beta_{s_i,s_{i-1}}(h_i^{n_i}, \xi_{i-1,\underline{n}})$ is well-defined for every $1 \leq i \leq p$. Hence we start by applying the cocycle relation, then we apply the exact formula of Proposition 4.9 using that the domain of s_i is $\mathsf{b}(h_i^-)$ for every $1 \leq i \leq p$.

$$\begin{aligned}
\beta_{s_p,s_0}(h_p^{n_p} ... h_1^{n_1}, \xi_0) &= \beta_{s_p,s_{p-1}}(h_p^{n_p}, h_{p-1}^{n_{p-1}} ... h_1^{n_1} \xi_0) \ ... \ \beta_{s_1,s_0}(h_1^{n_1}, \xi_0) \\
&= \beta_{s_p,s_{p-1}}(h_p^{n_p}, \xi_{p-1,\underline{n}}) \ ... \ \beta_{s_2,s_1}(h_2^{n_2}, \xi_{1,\underline{n}}) \beta_{s_1,s_0}(h_1^{n_1}, \xi_0) \\
&= \qquad \mathscr{R}_{s_p}(h_p; \xi_{p,\underline{n}})^{-1} \quad \mathscr{L}_{s_p}(h_p)^{n_p} \quad \mathscr{R}_{s_p,s_{p-1}}(h_p; \xi_{p-1,\underline{n}}) \ ... \\
&\qquad ... \quad \mathscr{R}_{s_2}(h_2; \xi_{2,\underline{n}})^{-1} \quad \mathscr{L}_{s_2}(h_2)^{n_2} \quad \mathscr{R}_{s_2,s_1}(h_2; \xi_{1,\underline{n}}) \\
&\qquad\qquad \mathscr{R}_{s_1}(h_1; \xi_{1,\underline{n}})^{-1} \quad \mathscr{L}_{s_1}(h_1)^{n_1} \quad \mathscr{R}_{s_1,s_0}(h_1; \xi_0).
\end{aligned}$$

Condition (i) allow us to deduce that the products of the middle terms $\mathscr{L}_{s_p}(h_p)^{n_p}...\mathscr{L}_{s_1}(h_1)^{n_1}$ take value in the connected component of AM_Γ corresponding to the projection of $\underline{n}$ in $\big(\mathbb{Z}/2\mathbb{Z}\big)^p$ that we denote by ν. Then

$$\pi_{M/M_0}\big(\mathscr{L}_{s_p}(h_p)^{n_p}...\mathscr{L}_{s_1}(h_1)^{n_1}\big) = \nu. \tag{6}$$

Note that this equation does not depend on the choice of compact Bruhat section $s_1,...,s_p$ of same domains.

It remains to control the connected components of AM in which the ratio terms take value. First, by a Ping-Pong argument, we choose a large integer N which will allows us to control the sequence $(\xi_{i,\underline{n}})_{1\leq i\leq p}$. Then we slightly modify the choice of $s_1,...,s_p$ while preserving their domains. Lastly, we check that under these modifications the ratio terms are AM_0 valued.

Let us start by the Ping-Pong argument. For all $i = 2,...,p$ denote by $\mathsf{b}^0(h_i^-, h_{i-1}^-)$ the connected component of $\mathsf{b}(h_i^-)\cap\mathsf{b}(h_{i-1}^-)$ containing h_{i-1}^+. By condition (ii) then $\xi_0 \in \mathsf{b}(h_1^-)$ and $h_1^+ \in \mathsf{b}(h_2^-)$. By Proposition 4.4 applied on the loxodromic element h_1, there exists a large integer $N_1 \geq 1$ such that for every $n_1 \geq N_1$, the element $h_1^{n_1}\xi_0$ is sufficiently close to h_1^+ to satisfy

$$h_1^{n_1}\xi_0 = \xi_{1,n_1} \in \mathsf{b}^0(h_2^-, h_1^-).$$

Assume for any $i = 1,...,p$ the following induction hypothesis, that there exists a large integer N_{i-1} such that for every $\underline{n} \in \big([N_{i-1},+\infty)\cap\mathbb{N}\big)^p$ and every $j = 1,...,i-1$

$$\xi_{j,\underline{n}} \in \mathsf{b}^0(h_{j+1}^-, h_j^-).$$

In particular $\xi_{i-1,\underline{n}} \in \mathsf{b}(h_i^-)$. Also, by condition (ii) then $h_i^+ \in \mathsf{b}(h_{i+1}^-)$. As before, we apply Proposition 4.4 on h_i to choose a large integer $N_i \geq N_{i-1}$ such that for all $\underline{n} \in \big([N_i,+\infty)\cap\mathbb{N}\big)^p$,

$$h_i^{n_i}\xi_{i-1,\underline{n}} = \xi_{i,\underline{n}} \in \mathsf{b}^0(h_{i+1}^-, h_i^-).$$

Since N_i is larger that N_{i-1}, the induction hypothesis is inherited for every $j = 1,...,i$ i.e. $\xi_{j,\underline{n}} \in \mathsf{b}^0(h_{j+1}^-, h_j^-)$. Hence, by induction, there exists a large integer $N \geq 1$ such that for all $\underline{n} \in \big([N,+\infty)\cap\mathbb{N}\big)^p$ and all $i = 1,...,p$

$$\xi_{i,\underline{n}} \in \mathsf{b}^0(h_{i+1}^-, h_i^-). \tag{7}$$

Now that the large integer N is chosen, assume that $\underline{n} \in \big([2N,+\infty)\cap\mathbb{N}\big)^p$. Let us now modify the sections by right multiplication by elements of M and prove that the ratio terms for the new family of compact Bruhat section take value in AM_0. Recall the Definition 4.8 of the ratio map.

$$\mathscr{R}_{s_i,s_{i-1}}(h_i;\xi_{i-1,\underline{n}}) = \mathscr{T}_{s_i,[h_i^-]}(h_i^+)\mathscr{T}_{[h_i^-],s_{i-1}}(\xi_{i-1,\underline{n}}).$$

By Definition 3.5 of the transition functions, the domain of $\mathscr{R}_{s_i,s_{i-1}}(h_i;.)$ is $\mathsf{b}(h_i^-)\cap\mathsf{b}(h_{i-1}^-)$. Set $m_0 = e_M$. By induction, we multiply $s_1,...,s_p$ on the right by elements $m_1,...,m_p \in M$ such that for every $i = 1,...,p$ the restriction to the connected component containing h_{i-1}^+ of the map

$$\begin{aligned} \mathsf{b}^0(h_i^-, h_{i-1}^-) &\longrightarrow AM \\ \xi_{i-1} &\longmapsto \mathscr{R}_{s_i.m_i,s_{i-1}.m_{i-1}}(h_i;\xi_{i-1}) \end{aligned}$$

takes value in AM_0. In particular, by choice of N such that condition (7) holds, we deduce that all $\mathscr{R}_{s_i.m_i,s_{i-1}.m_{i-1}}(h_i;\xi_{i-1,\underline{n}})$ term take value in AM_0. Replacing them in the cocycle expression, we write

$$\begin{aligned} \beta_{s_p.m_p,s_0}(h_p^{n_p}...h_1^{n_1},\xi_0) = \mathscr{R}_{s_p.m_p}(h_p;\xi_{p,\underline{n}})^{-1}\,\mathscr{L}_{s_p.m_p}(h_p)^{n_p}\,\mathscr{R}_{s_p.m_p,s_{p-1}.m_{p-1}}(h_p;\xi_{p-1,\underline{n}})\,... \\ ...\,\mathscr{R}_{s_1.m_1}(h_1;\xi_{1,\underline{n}})^{-1}\,\mathscr{L}_{s_1.m_1}(h_1)^{n_1}\,\mathscr{R}_{s_1.m_1,s_0}(h_1;\xi_0). \end{aligned}$$

Let us now prove that the left hand terms of the form $\mathscr{R}_{s_i.m_i}(h_i;\xi_{i,\underline{n}})^{-1}$ take value in AM_0. Recall that,

$$\mathscr{R}_{s_i.m_i}(h_i;\xi_{i,\underline{n}}) = \mathscr{T}_{s_i.m_i,[h_i^-]}(h_i^+)\mathscr{T}_{[h_i^-],s_i.m_i}(\xi_{i,\underline{n}}).$$

Using that the domain of $s_i.m_i$ is $\mathsf{b}(h_i^-)$, we deduce that $\mathscr{R}_{s_i.m_i}(h_i;.)$ is well defined on it. Furthermore, by Proposition 3.6, (iii)

$$\mathscr{T}^{-1}_{s_i.m_i,[h_i^-]} = \mathscr{T}_{[h_i^-],s_i.m_i}.$$

Hence by continuity of the transition functions defined in a connected set, we deduce that the continuous maps $\xi_i \in \mathsf{b}(h_i^-) \mapsto \mathscr{R}_{s_i.m_i}(h_i;\xi_i)^{-1}$ take value in AM_0.

Finally, since all ratio terms take value in AM_0 and by equation (6), we deduce condition (iii) that for all $\nu \in \{0,1\}^p$, all $\underline{n}$ of the form $(2n+\nu_i)_{1\le i\le p}$ such that $n \ge N$,

$$\pi_{M/M_0}\big(\beta_{s_p.m_p,s_0}(h_p^{n_p}...h_1^{n_1},\xi_0)\big) = \pi_{M/M_0}\big(\mathscr{L}_{s_p}(h_p)^{n_p}...\mathscr{L}_{s_1}(h_1)^{n_1}\big) = \nu.$$

□

6.2. The connected component AM_0.

Lemma 6.4. Assume that M_0 is abelian *and that Γ is a Zariski dense subsemigroup. Then there exists*

(a) *a convex cone of non empty interior $\mathcal{C}_0$,*
(b) *a pair of transverse points $(\xi_0,\check{\xi}_0) \in L^{(2)}(\Gamma)$,*
(c) *a real positive number $r_0 > 0$,*

such that for all $r \in (0,r_0]$ and $\varepsilon \in (0,r]$ and any Bruhat section s_0 of domain $\mathsf{b}(\check{\xi}_0)$, there exists $F_{r,\varepsilon} \subset \Gamma$ and $x_{r,\varepsilon} \in AM_\Gamma$ such that the following holds.

- ♡ *$F_{r,\varepsilon}$ is a finite subset of at most $4\dim\mathfrak{a} + 2\dim M_0$ elements.*
- ♣ *$F_{r,\varepsilon}$ is a subset of a strong (r,ε)-Schottky Zariski dense subsemigroup.*
- ♢ *There exists an ordering of $F_{r,\varepsilon} = (\gamma_1,...,\gamma_l)$ such that $\gamma_1^- = \check{\xi}_0$ and $\gamma_l^+ = \xi_0$, for which every element of the form $w = \gamma_l^{n_l}...\gamma_1^{n_1}$ with $n_1,...,n_l \ge 1$, satisfies*

$$(w^+,w^-) \in B(\xi_0,\varepsilon)\times B(\check{\xi}_0,\varepsilon).$$

- ♠ *For such an ordering, the set*

$$\mathscr{L}_{s_0}\big(\{\gamma_l^{n_l}...\gamma_1^{n_1} \mid n_1,...,n_l \ge 1\}\big)$$

is $l_{AM_0}\delta_{r,\varepsilon}$-dense in $\exp(\mathcal{C}_0)x_{r,\varepsilon}M_0$, where $l_{AM_0} := 8\dim\mathfrak{a} + 4\dim M_0 + 1$.

The family of constants $\delta_{r,\varepsilon}$ is given in Definition 4.11, for every $r>0$, they converge to 0 when ε goes to 0.

The first step of the proof is given by the following Lemma, which is a consequence of [Ben97b, Proposition 4.3]. We give a reference for a proof. The last steps involve the non-arithmeticity of Corollary 5.7 and density Lemmata 7.3, 7.5 of the appendix. These statements require that M_0 is abelian.

Lemma 6.5 (Lemme 5.6 [DG20]). *Let $\Gamma \subset G$ be Zariski dense subsemigroup. For all θ in the interior of the limit cone $\mathcal{C}(\Gamma)$, there exists a finite set $S \subset \Gamma$, a positive number $r_0 > 0$ such that*

(i) *θ is in the interior of the convex cone $C(\lambda(S)) := \sum_{g\in S}\mathbb{R}_+\lambda(g)$,*
(ii) *the elements of $\lambda(S)$ form a basis of $\mathfrak{a}$,*
(iii) *for all $r \in (0,r_0]$ and $\varepsilon \in (0,r]$, there exists an integer $N>0$ such that for all $n \ge N$, the family $S_n := (g^n)_{g\in S}$ spans a Zariski-dense strong (r,ε)-Schottky semigroup of Γ.*

Proof of Lemma 6.4 . First fix θ in the interior of the limit cone. Apply Lemma 6.5. Set $\mathcal{C}_0 := \sum_{g\in S}\mathbb{R}_+\lambda(g)$. By (i), it is indeed a convex cone of non-empty interior. Let us now order the elements $S := (g_1, ..., g_{r_G})$ where $r_G = \dim\mathfrak{a}$ by (ii). By (iii), for any integer n sufficiently large, S_n spans a strong (r,ε)-Schottky Zariski dense subsemigroup. We deduce that $g_{r_G}^+$ is in the basin of attraction of g_1, meaning that $g_{r_G}^+ \in \mathsf{b}(g_1^-)$, which by Proposition 2.6 is the same as $(g_{r_G}^+, g_1^-) \in L^{(2)}(\Gamma)$.

Let $r \in (0, r_0]$ and $\varepsilon \in (0, r]$, fix a compact Bruhat section s_0 of domain $\mathsf{b}(\check{\xi}_0)$. Let us choose $F_{r,\varepsilon} \subset \Gamma$. Consider a large integer N such that for every $n \geq N$, the subset S_n spans a Zariski dense, strong (r,ε)-Schottky subsemigroup.

By Theorem 5.8, the group M_Γ/M_0 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^p$. Consequently, for every element $m \in M_\Gamma$, its square m^2 is in M_0. In particular, for every loxodromic element $\gamma \in \Gamma$ and any suitable compact Bruhat section s such that $\gamma^+ \in \mathcal{F}_s$,

$$\mathscr{L}_s(\gamma^2) = \big(\mathscr{L}_s(\gamma)\big)^2 \in AM_0.$$

Since M_0 is abelian and a normal subgroup of M_Γ, we deduce that the multiplicative Jordan projection of squares does not depend on the choice of s and coincides with $\mathscr{L}^{ab}$. We therefore remove the subscript. Denote by Γ_n the Zariski dense subsemigroup generated by S_{2n}. By Corollary 5.7 and using that M_0 is abelian,

$$\overline{\langle \mathscr{L}^{ab}(\Gamma_n)\rangle} = AM_{\Gamma_n}^{ab} \supset AM_0.$$

Let us prove that the subset of squares $\mathscr{L}(\Gamma_n)^2$ spans a dense subgroup of AM_0. Every element $x \in AM_0$ admits a square root that we denote by $\sqrt{x} \in AM_0$. Now we approximate it in $\langle \mathscr{L}^{ab}(\Gamma_n)\rangle$. For all $\delta > 0$, there exists a finite number of integers $(k_j)_{j\in J} \subset \mathbb{Z}$ and a finite number of elements $(\gamma_j)_{j\in J}$ such that

$$\sqrt{x} \in B\Big(\prod_{j\in J}\mathscr{L}^{ab}(\gamma_j)^{k_j}, \sqrt{\delta}\Big).$$

Taking the squares, we obtain the approximation by squares,

$$x \in B\Big(\prod_{j\in J}\mathscr{L}(\gamma_j)^{2k_j}, \delta\Big).$$

Hence

$$\overline{\langle \mathscr{L}(\Gamma_n)^2\rangle} = AM_0.$$

Apply density Lemma 7.3 in AM_0 for the family of squares $\mathscr{L}(\Gamma_n)^2$. Consider $F'_{r,\varepsilon}$ of at most $3\dim\mathfrak{a} + 2\dim M_0$ elements such that the subgroup spanned by squares $\mathscr{L}(F'_{r,\varepsilon})^2$ is $\delta_{r,\varepsilon}$-dense in AM_0. Denote by

$$F_{r,\varepsilon} := S_{2n} \cup \{\gamma^2 \mid \gamma \in F'_{r,\varepsilon}\}.$$

The subgroup spanned by $\mathscr{L}(F_{r,\varepsilon})$ is $\delta_{r,\varepsilon}$-dense in AM_0. Apply now density Lemma 7.5 to such a family. There exists $v_{r,\varepsilon} \in \mathfrak{a}$ such that the subsemigroup generated by $\mathscr{L}(F_{r,\varepsilon})$ is $\delta_{r,\varepsilon}$-dense in

$$\exp\left(v_{r,\varepsilon} + \sum_{\gamma\in F_{r,\varepsilon}}\mathbb{R}_+\lambda(\gamma)\right)M_0.$$

Now since $F_{r,\varepsilon}$ contains S_{2n} and by choice of $\mathcal{C}_0$,

$$\mathcal{C}_0 \subset \sum_{\gamma\in F_{r,\varepsilon}}\mathbb{R}_+\lambda(\gamma),$$

we deduce $\delta_{r,\varepsilon}$-density of the subsemigroup generated by $\mathscr{L}(F_{r,\varepsilon})$ in $\exp(v_{r,\varepsilon} + \mathcal{C}_0)M_0$.

Let us now compute $x_{r,\varepsilon} \in AM$. We order $F_{r,\varepsilon} = (\gamma_1, ..., \gamma_l)$ such that $\gamma_1 := g_1^{2n}$ and $\gamma_l := g_{rG}^{2n}$. Fix compact Bruhat sections $s_1, ..., s_l$ such that for every $i = 1, ..., l$

$$B(\gamma_i^+, \varepsilon) \subset \mathcal{F}_{s_i}.$$

We assume that for every $i = 2, ..., l$ then $\mathscr{R}_{s_i,s_{i-1}}(\gamma_i, \gamma_{i-1}^+) \in AM_0$. Since $\mathscr{R}_{s_i,s_{i-1}}(\gamma_i, .)$ restricted to the connected component of $\mathsf{b}(\gamma_i^-) \cap \mathcal{F}_{s_{i-1}}$ containing γ_{i-1}^+ takes value in a connected component of AM, by multiplying s_i on the right by an element of M one can always assume that this restricted map takes value in AM_0. Set $\mathscr{R}_i := \mathscr{R}_{s_i,s_{i-1}}(\gamma_i, \gamma_{i-1}^+)$ with convention that $s_0 = s_l$ and $\gamma_0 = \gamma_l$ and

$$x_{r,\varepsilon} := \exp(v_{r,\varepsilon})\mathscr{R}_l...\mathscr{R}_2\mathscr{R}_1.$$

Let us now check $\heartsuit, \clubsuit, \diamondsuit, \spadesuit$. Since S_{2n} contains $\dim \mathfrak{a}$ elements and generates the strong (r, ε)-Schottky Γ_n and by choice of $F'_{r,\varepsilon}$, the subset $F_{r,\varepsilon}$ satisfies both $\heartsuit, \clubsuit$.

By choice of ordering $\gamma_1^- = g_1^- = \check{\xi}_0$ and $\gamma_l^+ = g_{rG}^+ = \xi_0$. Apply Proposition 4.12, for all $n_1, ..., n_l \geq 1$ the element $w = \gamma_l^{n_l}...\gamma_1^{n_1}$ is loxodromic and satisfies $(w^+, w^-) \in B(\xi_0, \varepsilon) \times B(\check{\xi}_0, \varepsilon)$. Hence $\diamondsuit$ is satisfied.

Furthermore, by Proposition 4.12 we estimate the Jordan projection

$$\mathscr{L}_{s_l}(w) \in \mathscr{L}_{s_l}(\gamma_l)^{n_l}\mathscr{R}_l...\mathscr{L}_{s_1}(\gamma_1)^{n_1}\mathscr{R}_1\ B\big(e_{AM}, 2l\delta_{r,\varepsilon}\big).$$

Note that $\mathscr{R}_l, ..., \mathscr{R}_2$ take value in the abelian group AM_0 by choice of $s_2, .., s_l$. Furthermore, the γ_i are squares, hence integer powers of $\mathscr{L}_{s_i}(\gamma_i)$ take value in AM_0 and we can remove the subscript. Hence by reordering the terms in AM_0,

$$\mathscr{L}_{s_l}(w) \in \mathscr{L}(\gamma_l)^{n_l}...\mathscr{L}(\gamma_1)^{n_1}\mathscr{R}_l...\mathscr{R}_1\ B\big(e_{AM}, 2l\delta_{r,\varepsilon}\big).$$

The first part of the left hand side

$$\{\mathscr{L}(\gamma_l)^{n_l}...\mathscr{L}(\gamma_1)^{n_1} \mid n_1, ..., n_l \geq 1\}$$

coincides with the subsemigroup of AM_0 generated by $\mathscr{L}(F_{r,\varepsilon})$ which is $\delta_{r,\varepsilon}$-dense in

$$\exp(\mathcal{C}_0)\exp(v_{r,\varepsilon})M_0.$$

We deduce that

$$\{\mathscr{L}_{s_l}(\gamma_l^{n_l}...\gamma_1^{n_1}) \mid n_1, ..., n_l \geq 1\}$$

is $(2l + 1)\delta_{r,\varepsilon}$-dense in

$$\exp(\mathcal{C}_0)\exp(v_{r,\varepsilon})M_0\ \mathscr{R}_l...\mathscr{R}_1 = \exp(\mathcal{C}_0)M_0x_{r,\varepsilon},$$

using that M_0 centralises A. Since M_0 is a normal subgroup, we deduce $(2l + 1)\delta_{r,\varepsilon}$-density in $\exp(\mathcal{C}_0)M_0x_{r,\varepsilon} = \exp(\mathcal{C}_0)x_{r,\varepsilon}M_0$. By $\heartsuit$, then $l \leq 4\dim \mathfrak{a} + 2\dim M_0$, hence

$$(2l + 1)\delta_{r,\varepsilon} \leq (8\dim \mathfrak{a} + 4\dim M_0 + 1)\delta_{r,\varepsilon}$$

and condition $\spadesuit$ is satisfied. □

6.3. **Proof of Proposition 6.1.** Let us first find the pair $(\xi_1, \check{\xi}_1) \in L^{(2)}(\Gamma)$ using the previous Lemmas of this section. Consider the pair of transverse points $(\xi_0, \check{\xi}_0) \in L^{(2)}(\Gamma)$ given by the decorrelation in AM_0 Lemma 6.4 (b). Apply Lemma 6.2 to ξ_0 to reach every connected component of AM_Γ. There exists loxodromic elements $h_1, ..., h_p \in \Gamma^{lox}$ such that taking the notation $h_0^+ := \xi_0$, the following holds.

(i) For every choice of sections $s_1, ..., s_p$ such that $h_i^+ \in \mathcal{F}_{s_i}$ for all $1 \leq i \leq p$, the set $\{\pi_{M/M_0}(\mathscr{L}_{s_i}(h_i))\}_{1\leq i\leq p}$ forms a basis of the vector space M_Γ/M_0.

(ii) For all $1 \leq i \leq p$, the pair $(h_{i-1}^+, h_i^-) \in L^{(2)}(\Gamma)$ is transverse.

(iii) Assume now that s_0, s_p are compact Bruhat sections of respective domains $\mathsf{b}(h_1^-)$ and $\mathsf{b}(h_p^-)$, then there exists $m_p \in M$ and a large integer $N \in \mathbb{N}$ such that for all $\nu \in \{0,1\}^p$, for all $n \geq N$,

$$\pi_{M/M_0}\Big(\beta_{s_p m_p, s_0}(h_p^{2n+\nu_p}...h_1^{2n+\nu_1}, \xi_0)\Big) = \nu.$$

Since h_p^+ has no reason to be transverse to $\check{\xi}_0$, we need the following choice. By density of attracting and repelling points of loxodromic elements in $L^{(2)}(\Gamma)$, there exists a loxodromic element $h_{p+1} \in \Gamma^{lox}$ such that

$$\begin{cases} (h_{p+1}^+, \check{\xi}_0) \in L^{(2)}(\Gamma) \\ (h_p^+, h_{p+1}^-) \in L^{(2)}(\Gamma) \end{cases}$$

Such a choice is always possible because there are no isolated points in the limit sets $L_{\pm}(\Gamma)$. Set now

$$(\xi_1, \check{\xi}_1) := (h_{p+1}^+, \check{\xi}_0). \tag{8}$$

Let us now find the positive number r_1. Consider the real number r_0 given by Lemma 6.4 (c). We set

$$r_0' := \inf_{1\leq i\leq p+1}\left\{\frac{1}{6}\mathrm{d}(h_{i-1}^+, \partial\mathsf{b}(h_i^-)), \frac{1}{2}\mathrm{d}(h_i^+, \partial\mathsf{b}(h_i^-))\right\}.$$

By (ii), choice of h_{p+1} and using that $h_1, ..., h_{p+1}$ are loxodromic, we deduce that both r_0 and r_0' are positive real numbers. This leads us to define the positive real number

$$r_1 := \inf(r_0, r_0'). \tag{9}$$

Let $r \in (0, r_1]$ and $\varepsilon \in (0, r]$. Fix a choice of compact Bruhat sections $c_1, \check{c}_1$ such that

$$B(\xi_1, r) \subset \mathcal{F}_{c_1} \text{ and } \mathcal{V}_{6r}(\partial\mathsf{b}(\check{\xi}_1))^{\complement} \subset \mathcal{F}_{\check{c}_1}.$$

Reaching every connected component of AM_Γ

By Proposition 4.6 on loxodromic elements $h_1, ..., h_{p+1}$, there exists a large integer $N_{r,\varepsilon} \geq 1$ such that for every $n \geq N_{r,\varepsilon}$, each h_i^n are (r,ε)-loxodromic.

Since ξ_0 is in the basin of attraction of h_1, then by Proposition 4.4, we choose another integer $N_1 \geq 1$ such that for all $n \geq N_1$ large enough, $h_1^n\xi_0 \in B(h_1^+, \varepsilon)$. Set $N_2 := \sup(N_1, N_{r_\varepsilon})$. By a Ping-Pong argument using the dynamical properties of (r,ε)-loxodromic elements, this implies that for all $n_1, ..., n_p \geq N_2$, then $h_p^{n_p}...h_1^{n_1}\xi_0 \in B(h_p^+, \varepsilon)$. For all $\underline{n} := (n_p, ..., n_1)$ family of positive integers, denote by $\xi_{p,\underline{n}} := h_p^{n_p}...h_1^{n_1}\xi_0$. By Proposition 4.9, for all $\xi_p \in B(h_p^+, \varepsilon)$, then

$$\beta_{c_1, s_p m_p}(h_{p+1}^{2n}, \xi_p) = \mathscr{R}_{c_1}(h_{p+1}; h_{p+1}^{2n}\xi_p)^{-1}\mathscr{L}(h_{p+1})^{2n}\mathscr{R}_{c_1, s_p m_p}(h_{p+1}; \xi_p).$$

Note that by choice of $\varepsilon \leq r_1$ the balls $B(h_p^+, \varepsilon)$ resp. $B(h_{p+1}^+, \varepsilon)$ are included in connected components of $\mathsf{b}(h_{p+1}^-) \cap \mathcal{F}_{s_p}$ resp. $\mathcal{F}_{c_1} \cap \mathsf{b}(h_{p+1}^-)$. Therefore, using that h_{p+1}^{2n} is (r,ε)-loxodromic when $n \geq N_2$, the restriction to $B(h_p^+, \varepsilon)$ of $\beta_{c_1, s_p m_p}(h_{p+1}^{2n}, .)$ to $B(h_p^+, \varepsilon)$ is constant mod AM_0. For all $n \geq N_2$, the map

$$\begin{aligned} \big([N_2, \infty) \cap \mathbb{N}\big)^p &\longrightarrow AM \\ \underline{n} &\longmapsto \beta_{c_1, s_0}(h_{p+1}^{2n} h_p^{n_p}...h_1^{n_1}, \xi_0) \end{aligned}$$

reaches every connected components of AM_Γ. Indeed, by the cocycle relation

$$\beta_{c_1, s_0}(h_{p+1}^{2n} h_p^{n_p}...h_1^{n_1}, \xi_0) = \beta_{c_1, s_p m_p}(h_{p+1}^{2n}, \xi_{p,\underline{n}})\beta_{s_p m_p, s_0}(h_p^{n_p}...h_1^{n_1}, \xi_0),$$

and by (iii), we control which connected component of AM_Γ the right term hits, the left term being constant mod AM_0 as discussed above. Thus, for all $\nu \in \{0,1\}^p \simeq M_\Gamma/M_0$ there exists and we choose $n_p(\nu), ..., n_1(\nu) \geq N_2$ such that if we denote by

$$\begin{cases} h_{[\nu]} := h_{p+1}^{2n} h_p^{n_p(\nu)} ... h_1^{n_1(\nu)} \\ x_{[\nu]} := \beta_{c_1,s_0}(h_{[\nu]}, \xi_0) \end{cases}$$

then $\pi_{AM/AM_0}(x_{[\nu]}) = \nu$.

A particular subset of loxodromic elements of Γ

Consider now the subset $F_{r,\varepsilon}$, the point $x_{r,\varepsilon} \in AM_\Gamma$ and the convex cone of non-empty interior $\mathcal{C}_0$ given by Lemma 6.4. They satisfy

♡ $F_{r,\varepsilon}$ is a finite subset of at most $4 \dim \mathfrak{a} + 2 \dim M_0$ elements.

♣ $F_{r,\varepsilon}$ is a subset of a strong (r, ε)-Schottky Zariski dense subsemigroup.

♢ There exists an ordering of $F_{r,\varepsilon} = (\gamma_1, ..., \gamma_l)$ such that $\gamma_1^- = \check{\xi}_0$ and $\gamma_l^+ = \xi_0$, for which every element of the form $w = \gamma_l^{n_l} ... \gamma_1^{n_1}$ with $n_1, ..., n_l \geq 1$, satisfies

$$(w^+, w^-) \in B(\xi_0, \varepsilon) \times B(\check{\xi}_0, \varepsilon).$$

♠ For such an ordering, the set

$$\mathscr{L}_{s_0}\big(\{\gamma_l^{n_l} ... \gamma_1^{n_1} \mid n_1, ..., n_l \geq 1\}\big)$$

is $l_{AM_0}\delta_{r,\varepsilon}$-dense in $\exp(\mathcal{C}_0)x_{r,\varepsilon}M_0$ where $l_{AM_0} := 8 \dim \mathfrak{a} + 4 \dim M_0 + 1$.

We are going to choose $(g_i)_{i \in I}$ among elements of the form $h_{[\nu]}\gamma_l^{n_l} ... \gamma_1^{n_1}$, where $n_l, ..., n_1 \geq 1$ are integers and $\nu \in \{0,1\}^p$.

By choice of r_1, we deduce † for all elements of

$$\{h_{[\nu]}\gamma_l^{n_l} ... \gamma_1^{n_l} \mid n_1, ..., n_l \geq 1 \text{ and } \nu \in (\mathbb{Z}/2\mathbb{Z})^p\}.$$

Meaning that all elements of the set above are $(2r, 2\varepsilon)$-loxodromic with attracting and repelling points in $B(\xi_1, \varepsilon) \times B(\check{\xi}_1, \varepsilon)$.

Cocycle estimates

By equation (8) recall that $\check{\xi}_0 = \check{\xi}_1 = \gamma_1^-$. Let $n_l, ..., n_1 \geq 1$ be integers. Then by choice of $r \leq r_1$ and $\varepsilon \leq r$, the element $\gamma = \gamma_l^{n_l} ... \gamma_1^{n_1}$ is (r, ε)-loxodromic, of attracting point in $B(\gamma_l^+, \varepsilon)$ and repelling point in $B(\gamma_1^-, \varepsilon)$. By Proposition 4.9 on loxodromic element γ and $\eta \in \mathcal{V}_{6r}(\partial \mathsf{b}(\check{\xi}_1))^{\complement}$, by Definition 4.11 of the equicontinuity constant $\delta_{r,\varepsilon}$, we deduce

$$\beta_{s_0,\check{c}_1}(\gamma, \eta) \in \mathscr{L}_{s_0}(\gamma)\mathscr{R}_{s_0,\check{c}_1}(\gamma; \eta)B(e_{AM}, \delta_{r,\varepsilon}).$$

Now, $\mathscr{R}_{s_0,\check{c}_1}(\gamma; \eta)$ is $\delta_{r,\varepsilon}$ close to $\mathscr{R}_{s_0,\check{c}_1}(\gamma_1^-; \gamma_l^+, \eta)$. Hence using $\xi_0 = \gamma_l^+$ and $\check{\xi}_1 = \gamma_1^-$, we deduce

$$\beta_{s_0,\check{c}_1}(\gamma, \eta) \in \mathscr{L}_{s_0}(\gamma)\mathscr{R}_{s_0,\check{c}_1}(\check{\xi}_1; \xi_0, \eta)B(e_{AM}, 2\delta_{r,\varepsilon}).$$

For all $\nu \in \{0,1\}^p$, by the cocycle relation,

$$\beta_{c_1,\check{c}_1}(h_{[\nu]}\gamma, \eta) = \beta_{c_1,s_0}(h_{[\nu]}, \gamma\eta)\beta_{s_0,\check{c}_1}(\gamma, \eta).$$

By a Ping-Pong argument on $\gamma_1, ..., \gamma_l$ we deduce that $\gamma\eta \in B(\gamma_l^+, \varepsilon)$. Similarly, the same type of argument on the (r, ε)-loxodromic elements $h_1^{n_1(\nu)}, ..., h_p^{n_p(\nu)}, h_{p+1}^{2n}$ yields that

$$\beta_{c_1,s_0}(h_{[\nu]}, \gamma\eta) \in \beta_{c_1,s_0}(h_{[\nu]}, \gamma_l^+)B(e_{AM}, \delta_{r,\varepsilon}).$$

Using $\gamma_l^+ = \xi_0$ and the definition of $x_{[\nu]}$, we deduce the following estimate

$$\beta_{c_1,\check{c}_1}(h_{[\nu]}\gamma, \eta) \in x_{[\nu]}\mathscr{L}_{s_0}(\gamma)\mathscr{R}_{s_0,\check{c}_1}(\check{\xi}_1; \xi_0, \eta)B(e_{AM}, 3\delta_{r,\varepsilon}).$$

To recover the term $\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1; \xi_1, \eta)$ as in ‡, one can check using the definition of the Ratio maps that $\mathscr{R}_{s_0,\check{c}_1}(\check{\xi}_1; \xi_0, \eta) = \mathscr{R}_{c_1,s_0}(\check{\xi}_1; \xi_1, \xi_0)^{-1}\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1, \xi_1, \eta)$. Denote by $y_0 := \mathscr{R}_{c_1,s_0}(\check{\xi}_1; \xi_1, \xi_0)^{-1}$. Then for all $\nu \in (\mathbb{Z}/2\mathbb{Z})^p$, all $\gamma \in \{\gamma_l^{n_l} ... \gamma_1^{n_1} \mid n_1, ..., n_l \geq 1\}$ and all $\eta \in \mathcal{V}_{6r}(\partial \mathsf{b}(\check{\xi}_1))^{\complement}$,

$$\beta_{c_1,\check{c}_1}(h_{[\nu]}\gamma, \eta) \in x_{[\nu]}\mathscr{L}_{s_0}(\gamma)y_0\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1; \xi_1, \eta)B(e_{AM}, 3\delta_{r,\varepsilon}). \tag{10}$$

Overlapping cone argument

Using ♠ on the Jordan term, we deduce that for every $\eta \in \mathcal{V}_{6r}(\partial \mathsf{b}(\check{\xi}_1))^{\complement}$, the subset of cocycles

$$\{\beta_{c_1,\check{c}_1}(h_{[\nu]}\gamma_l^{n_l}...\gamma_1^{n_1},\eta) \mid n_1,...,n_l \geq 1\}$$

is $(3+l_{AM_0})\delta_{r,\varepsilon}$-dense in the translated cone $x_{[\nu]}\exp(\mathcal{C}_0)x_{r,\varepsilon}M_0\, y_0\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta)$. The left terms $x_{[\nu]}$ ensures that when ν varies in $(\mathbb{Z}/2\mathbb{Z})^p$, all connected components of AM_Γ are reached. Denote by $\pi_A : AM \to A$ the projection. Using that $\mathcal{C}_0$ is convex of non-empty interior, we deduce that there exists $a_{r,\varepsilon} \in A$ such that the intersection, over the number of connected components of AM_Γ, of the projection in A of these translated cones, contains $a_{r,\varepsilon}\exp(\mathcal{C}_0)$, i.e.

$$a_{r,\varepsilon}\exp(\mathcal{C}_0) \subset \bigcap_{\nu\in(\mathbb{Z}/2\mathbb{Z})^p} \pi_A\big(x_{[\nu]}\exp(\mathcal{C}_0)x_{r,\varepsilon}M_0y_0\big).$$

Hence the disjoint union of translated cones contains $a_{r,\varepsilon}\exp(\mathcal{C}_0)M_\Gamma$ i.e.

$$a_{r,\varepsilon}\exp(\mathcal{C}_0)M_\Gamma \subset \bigsqcup_{\nu\in(\mathbb{Z}/2\mathbb{Z})^p} x_{[\nu]}\exp(\mathcal{C}_0)x_{r,\varepsilon}M_0\, y_0.$$

Hence by right multiplication by $\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta)$, we deduce that

$$a_{r,\varepsilon}\exp(\mathcal{C}_0)M_\Gamma\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta) \subset \bigsqcup_{\nu\in(\mathbb{Z}/2\mathbb{Z})^p} x_{[\nu]}\exp(\mathcal{C}_0)x_{r,\varepsilon}M_0\, y_0\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta).$$

Using the $(3+l_{AM_0})\delta_{r,\varepsilon}$ density of cocycles in the disjoint union on the right yields

$$a_{r,\varepsilon}\exp(\mathcal{C}_0)M_\Gamma\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta) \subset \bigcup_{\substack{\nu\in(\mathbb{Z}/2\mathbb{Z})^p\\ n_1,...,n_l\geq 1}} \beta_{c_1,\check{c}_1}(h_{[\nu]}\gamma_l^{n_l}...\gamma_1^{n_1},\eta)B(e_{AM},(3+l_{AM_0})\delta_{r,\varepsilon}).$$

By compacity, we choose a finite family

$$(g_i)_{i\in I} \subset \{h_{[\nu]}\gamma_l^{n_l}...\gamma_1^{n_l} \mid n_1,...,n_l \geq 1 \text{ and } \nu \in (\mathbb{Z}/2\mathbb{Z})^p\}$$

such that for all $\eta \in \mathcal{V}_{6r}(\partial \mathsf{b}(\check{\xi}_1))^{\complement}$,

$$a_{r,\varepsilon}M_\Gamma\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta) \subset \bigcup_{i\in I} B\big(\beta_{c_1,\check{c}_1}(g_i,\eta),(3+l_{AM_0})\delta_{r,\varepsilon}\big).$$

Set $\tilde{\delta}_{r,\varepsilon} := (8\dim\mathfrak{a} + 4\dim M_0 + 5)\delta_{r,\varepsilon} = (l_{AM_0}+4)\delta_{r,\varepsilon}$. Finally, we apply for every family $(\eta_i)_{i\in I} \subset B(\eta,\varepsilon)$ the Proposition 4.9 on $\beta_{c_1,\check{c}_1}(g_i,\eta_i)$ and by Definition 4.11 the ratio maps $\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta_i)$ are $\delta_{r,\varepsilon}$ close to $\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta)$, hence ‡

$$a_{r,\varepsilon}M_\Gamma\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\eta) \subset \bigcup_{i\in I} B\big(\beta_{c_1,\check{c}_1}(g_i,\eta_i),\tilde{\delta}_{r,\varepsilon}\big).$$

7. Conditions for topological mixing

We prove the following necessary and sufficient conditions.

Theorem 7.1. *Let G be a real linear, connected, semisimple Lie group of non-compact type (i.e. without compact factors) and Γ be a Zariski dense subgroup of G. For all $\theta \in \mathfrak{a}^{++}$, the following topological mixing conditions occur.*

- (NC) *If the dynamical system $(\Omega_{[e_M]},\phi_\theta^t)$ is topologically mixing then $\theta \in \mathring{\mathcal{C}}(\Gamma)$.*
- (SC) *Assume that the connected component of the identity M_0 of M is* abelian. *Then the converse is true i.e. if θ is in the interior of the Benoist cone, then the dynamical system $(\Omega_{[e_M]},\phi_\theta^t)$ is topologically mixing.*

7.1. **Necessary condition.** Let $\theta \in \mathfrak{a}^{++}$. We prove that if the dynamical system $(\Omega_{[e_M]}, \phi_\theta^t)$ is topologically mixing, then θ is in the interior of the limit cone $\mathcal{C}(\Gamma)$.

Since this dynamical system factors (Ω, ϕ_θ^t), we deduce topological mixing of the regular Weyl chamber flow. Using now $\theta \in \mathfrak{a}^{++}$ and the necessary and sufficient condition for mixing [DG20], we deduce that

$$\theta \in \overset{\circ}{\mathcal{C}}(\Gamma).$$

7.2. **Sufficient condition.** The key arguments are given by Theorem 5.2, decorrelation Proposition 6.1 and the Proposition 7.2 below.

Let $\theta \in \mathfrak{a}^{++}$ be in the interior of the limit cone. We want to prove that for all non-empty open sets $U, V \subset \Omega_{[e_M]}$, there exists $T > 0$ such that for every $t \geq T$,

$$\phi_\theta^t(U) \cap V \neq \emptyset.$$

It is equivalent to prove that for all non-empty open sets $\mathcal{U}, \mathcal{V} \subset \widetilde{\Omega}_{[e_M]}$, there exists $T > 0$ such that for every $t \geq T$,

$$\mathcal{U}e^{t\theta} \cap \Gamma\mathcal{V} \neq \emptyset.$$

By Theorem 5.2, the action of Γ on $L^{(2)}(\Gamma)$ has dense orbits. The latter are the first and second Bruhat-Hopf coordinates of $\widetilde{\Omega}_{[e_M]}$. Using that left and right actions commute, we align $\mathcal{U}$ and $\mathcal{V}$ in the same AM orbit as a right AM-invariant subsets given by Proposition 6.1: of first and second Bruhat-Hopf coordinates in a neighbourhood of $(\xi_1, \check{\xi}_1)$.

We apply the Proposition 7.2 to θ and the neighbourhood of $(\xi_1, \check{\xi}_1)$: the Jordan projection of the elements in $\gamma \in \Gamma^{lox}$ such that (γ^+, γ^-) is in that neighbourhood of $(\xi_1, \check{\xi}_1)$ is dense in affine half-lines of direction θ. We thus construct elements in Γ that will satisfy the mixing statement up to right multiplication by M_Γ. Finally decorrelation Proposition 6.1 allows to choose very contracting loxodromic elements in Γ whose attracting and repelling points are in a neighbourhood of $(\xi_1, \check{\xi}_1)$ and whose signed Jordan projection are dense in an M_Γ-invariant set of AM.

Proposition 7.2 (Proposition 5.4 [DG20])**.** *Fix $\theta \in \mathfrak{a}^{++}$ of norm 1 in the interior of the limit cone $\mathcal{C}(\Gamma)$.*

Then for every nonempty open subset $\mathcal{O}^{(2)} \subset L^{(2)}(\Gamma)$, for all $x_0 \in A$ and $\delta_0 > 0$ there exists $T_0 > 0$ such that for all $t \geq T_0$ there exists a loxodromic element $\gamma_t \in \Gamma$ with

$$\begin{cases} (\gamma_t^+, \gamma_t^-) \in \mathcal{O}^{(2)} \\ \exp\big(\lambda(\gamma_t)\big) \in B\big(x_0 e^{t\theta}, \delta_0\big) \end{cases} \tag{11}$$

Recall that for every compact Bruhat section s of the M-bundle $K \to \mathcal{F}$, we denote by $\mathcal{F}_s$ its domain, by $G_s := s(\mathcal{F}_s)MAN$ the domain of the Bruhat-Hopf coordinates map $\mathcal{H}_s$ that takes value in $\mathcal{F}_s^{(2)} \times AM$. Denote by π_A the projection $AM \to A$.

Proof of Theorem 7.1 (SC). Let $\theta \in \mathfrak{a}^{++}$ be in the interior of the limit cone.

We want to prove the following statement given in Bruhat-Hopf coordinates: for all non-empty open sets $\mathcal{U}^{(2)} \subset L^{(2)}(\Gamma)$ and $\mathcal{V}^{(2)} \subset L^{(2)}(\Gamma)$, for all $u, v \in AM_\Gamma$ and $\delta > 0$, there exists $T_1 > 0$ such that for every $t \geq T_1$, for all compact Bruhat sections c_U, c_V such that $\mathcal{U}^{(2)} \subset \mathcal{F}_{c_U}^{(2)}$ and $\mathcal{V}^{(2)} \subset \mathcal{F}_{c_V}^{(2)}$,

$$\phi_\theta^t\big(\mathcal{U}^{(2)} \times B(u, \delta)\big)_{c_U} \cap \Gamma\big(\mathcal{V}^{(2)} \times B(v, \delta)\big)_{c_V} \neq \emptyset,$$

meaning that there exists $h_t \in \Gamma$ such that

$$\big(\mathcal{U}^{(2)} \times B(ue^{t\theta}, \delta)\big)_{c_U} \cap h_t\big(\mathcal{V}^{(2)} \times B(v, \delta)\big)_{c_V} \neq \emptyset.$$

Consider the pair $(\xi_1, \check{\xi}_1) \in L^{(2)}(\Gamma)$, the real positive number $r_1 > 0$ given by Proposition 6.1 and the associated compact Bruhat sections $c_1, \check{c}_1$.

Step 1: Apply topological transitivity of the action of Γ on $L^{(2)}(\Gamma)$ given by Theorem 5.2. Then there exists $h_U, h_V \in \Gamma$ such that

$$\begin{cases} h_U \mathcal{U}^{(2)} \ni (\xi_1, \check{\xi}_1) \\ h_V \mathcal{V}^{(2)} \ni (\xi_1, \check{\xi}_1). \end{cases}$$

By left Γ invariance and right AM_Γ invariance of $\widetilde{\Omega}_{[e_M]}$, there exists $u_1, v_1 \in AM_\Gamma$ such that in Bruhat-Hopf coordinates,

$$\begin{cases} h_U \big(\mathcal{U}^{(2)} \times B(u,\delta)\big)_{c_U} \ni (\xi_1, \check{\xi}_1 \;;\; u_1)_{c_1} \\ h_V \big(\mathcal{V}^{(2)} \times B(v,\delta)\big)_{c_V} \ni (\xi_1, \check{\xi}_1 \;;\; v_1)_{\check{c}_1}. \end{cases}$$

Choose $r \in (0, r_1]$ and $\delta_1 > 0$ small enough such that in Bruhat-Hopf coordinates

$$\begin{cases} h_U \big(\mathcal{U}^{(2)} \times B(u,\delta)\big)_{c_U} \supset \big(B(\xi_1, r) \times B(\check{\xi}_1, r) \times B(u_1, \delta_1)\big)_{c_1} \\ h_V \big(\mathcal{V}^{(2)} \times B(v,\delta)\big)_{c_V} \supset \big(B(\xi_1, r) \times B(\check{\xi}_1, r) \times B(v_1, \delta_1)\big)_{\check{c}_1}. \end{cases}$$

By Proposition 6.1, for all $\varepsilon \in (0, r]$, there exists a finite family $(g_i)_{i\in I} \subset \Gamma$ and a point $a_{r,\varepsilon} \in A$ satisfying the following conditions.

† For all $i \in I$, the element g_i is $(2r, 2\varepsilon)$-loxodromic with

$$(g_i^+, g_i^-) \in B(\xi_1, \varepsilon) \times B(\check{\xi}_1, \varepsilon).$$

‡ For all $\eta \in \mathcal{V}_{6r}\big(\partial \mathsf{b}(\check{\xi}_1)\big)^{\complement}$ and $(\eta_i)_{i\in I} \subset B(\eta, \varepsilon)$, the family $\{\beta_{c_1,\check{c}_1}(g_i, \eta_i)\}_{i\in I}$ is $\tilde{\delta}_{r,\varepsilon}$-dense in $a_{r,\varepsilon}\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1; \xi_1, \eta) M_\Gamma$ i.e.

$$a_{r,\varepsilon}\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1; \xi_1, \eta) M_\Gamma \subset \cup_{i\in I} B(\beta_{c_1,\check{c}_1}(g_i, \eta_i), \tilde{\delta}_{r,\varepsilon}).$$

Step 2: Choose $\varepsilon \in (0, r]$ such that $\tilde{\delta}_{r,\varepsilon} \leq \delta_1/2$. Denote by $\mathcal{O}^{(2)} := B(\xi_1, \varepsilon) \times B(\check{\xi}_1, \varepsilon)$. We are going to prove the topological mixing statement for $u_1, v_1 \in AM_\Gamma$, small $\delta_1 > 0$, when $\mathcal{U}^{(2)} = \mathcal{V}^{(2)} = \mathcal{O}^{(2)}$.

Let us apply Proposition 7.2 to θ which is in the interior of the limit cone, to the above open subset $\mathcal{O}^{(2)} \subset L^{(2)}(\Gamma)$, to $x_0 := \pi_A\big(a_{r,\varepsilon}^{-1} u_1 v_1^{-1}\big)$ and to $\delta_1/2$. We thus consider $T_0 > 0$ and a family of loxodromic elements $(\gamma_t)_{t\geq T_0}$ satisfying the system (11). Apply †, since g_i^- is the attracting point of g_i^{-1}, we deduce for all $i \in I$

$$\gamma_t^{-1} g_i^{-1} B(\check{\xi}_1, \varepsilon) \subset B(\check{\xi}_1, \varepsilon).$$

Hence for all $i \in I$ and every $\check{\xi} \in \gamma_t^{-1} g_i^{-1} B(\check{\xi}_1, \varepsilon)$,

$$\begin{aligned} g_i\gamma_t\big(\gamma_t^+, \check{\xi} \;;\; v_1\big)_{\check{c}_1} &= \big(g_i\gamma_t^+, g_i\gamma_t\check{\xi} \;;\; \beta_{c_1,\check{c}_1}(g_i\gamma_t, \gamma_t^+)v_1\big)_{c_1} \\ &= \big(g_i\gamma_t^+, g_i\gamma_t\check{\xi} \;;\; \beta_{c_1,\check{c}_1}(g_i, \gamma_t^+)\mathscr{L}_{\check{c}_1}(\gamma_t)v_1\big)_{c_1} \\ &\in \mathcal{O}^{(2)} \times \{\beta_{c_1,\check{c}_1}(g_i, \gamma_t^+)\mathscr{L}_{\check{c}_1}(\gamma_t)v_1\}. \end{aligned}$$

We discuss the cocycle terms using the decorrelation. By ‡, the set

$$\{\beta_{c_1,\check{c}_1}(g_i, \gamma_t^+)\mathscr{L}_{\check{c}_1}(\gamma_t)v_1 \mid i \in I\} \tag{12}$$

is $\delta_1/2$-dense in

$$\Big(a_{r,\varepsilon}\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1; \xi_1, \xi_1) M_\Gamma\Big)\mathscr{L}_{\check{c}_1}(\gamma_t)v_1.$$

Since the ratio $\mathscr{R}_{c_1,\check{c}_1}(\check{\xi}_1;\xi_1,\xi_1)$ is trivial and M_Γ is a normal subgroup of M, we deduce that the above subset of cocycles (12) is $\delta_1/2$-dense in $a_{r,\varepsilon}\mathscr{L}_{\check{c}_1}(\gamma_t)u_1M_\Gamma$. Furthermore, by equation (11) it is δ_1- dense in $a_{r,\varepsilon}x_0e^{t\theta}u_1M_\Gamma$. By choice of x_0, remark $\pi_A(a_{r,\varepsilon}x_0e^{t\theta}v_1)=\pi_A(u_1e^{t\theta})$. Hence

$$\pi_A(u_1e^{t\theta})M_\Gamma \subset \bigcup_{i\in I} B(\beta_{c_1,\check{c}_1}(g_i\gamma_t,\gamma_t^+)v_1,\delta_1).$$

Since $u_1e^{t\theta}\in\pi_A(u_1e^{t\theta})M_\Gamma$, we choose for all $t\geq T_0$ an element $h_t\in\{g_i\gamma_t\}_{i\in I}$ such that

$$u_1e^{t\theta}\in B(\beta_{c_1,\check{c}_1}(h_t,\gamma_t^+)v_1,\delta_1).$$

Consider $w\in B(v_1,\delta_1)$ such that $\beta_{c_1,\check{c}_1}(h_t,\gamma_t^+)w=u_1e^{t\theta}$. Then for all $\check{\xi}\in h_t^{-1}B(\check{\xi}_1,\varepsilon)$,

$$\begin{aligned} h_t\big(\gamma_t^+,\check{\xi}\,;\,w\big)_{\check{c}_1} &= \big(h_t\gamma_t^+,h_t\check{\xi}\,;\,\beta_{c_1,\check{c}_1}(h_t,\gamma_t^+)w\big)_{c_1}\\ &= \big(h_t\gamma_t^+,h_t\check{\xi}\,;\,u_1e^{t\theta}\big)_{c_1}\in\phi_\theta^t\big(\mathcal{O}^{(2)}\times B(u_1,\delta_1)\big)_{c_1}. \end{aligned}$$

Therefore, all points of such coordinates are in $\phi_\theta^t\big(\mathcal{O}^{(2)}\times B(u_1,\delta_1)\big)_{c_1}\cap h_t\big(\mathcal{O}^{(2)}\times B(v_1,\delta_1)\big)_{\check{c}_1}$. Hence for all $t\geq T_0$, there exists $h_t\in\Gamma$ such that

$$\phi_\theta^t\big(\mathcal{O}^{(2)}\times B(u_1,\delta_1)\big)_{c_1}\cap h_t\big(\mathcal{O}^{(2)}\times B(v_1,\delta_1)\big)_{\check{c}_1}\neq\emptyset. \tag{13}$$

By choice of $\varepsilon>0$, remark that

$$\begin{cases} h_U\big(\mathcal{U}^{(2)}\times B(u,\delta)\big)_{c_U}\supset\big(\mathcal{O}^{(2)}\times B(u_1,\delta_1)\big)_{c_1}\\ h_V\big(\mathcal{V}^{(2)}\times B(v,\delta)\big)_{c_V}\supset\big(\mathcal{O}^{(2)}\times B(v_1,\delta_1)\big)_{\check{c}_1}. \end{cases}$$

Note that relation (13) ensures that $\phi_\theta^t\big(h_U\big(\mathcal{U}^{(2)}\times B(u,\delta)\big)_{c_U}\big)\cap h_th_V\big(\mathcal{V}^{(2)}\times B(v,\delta)\big)_{\check{c}_V}\neq\emptyset$. Since the flow commutes with left multiplication by Γ, we deduce that for all $t\geq T_0$,

$$\phi_\theta^t\big(\mathcal{U}^{(2)}\times B(u,\delta)\big)_{c_U}\cap h_U^{-1}h_th_V\big(\mathcal{V}^{(2)}\times B(v,\delta)\big)_{\check{c}_V}\neq\emptyset.$$

□

Appendix : density Lemmata

Lemma 7.3. *Let C be a compact connected abelian real linear Lie group and V be a finite dimensionnal real vector space.*

Then for all subset $E\subset V\times C$ that span a dense subgroup in $V\times C$, for all small real number $\delta>0$, there exists a finite subset $F_\delta\subset E$ of at most $3\dim V+2\dim C$ elements such that the subgroup generated by F_δ is δ-dense in $V\times C$.

It a consequence of the following Lemma.

Lemma 7.4 (Lemma 6.1 [DG20]). *Let V be a finite dimensionnal real vector space.*

Then for all subset $E\subset V$ that span a dense subgroup in V, for all small real number $\delta>0$ and all basis $B\subset E$ of V, there exists a finite subset $F_\delta\subset E$ of at most $2\dim V$ elements such that the subgroup generated by $B\cup F_\delta$ is δ-dense in V.

Proof of Lemma 7.3 . By Corollary 3.7 of [BtD85], the group C is isomorphic to a torus. Consequently, its universal cover $\tilde{C}$ is a real vector space of dimension $\dim(C)$.

Fix a small real number $\delta>0$. Denote by $\widetilde{V}=V\times\tilde{C}$ the universal cover of $V\times C$. Then $\widetilde{V}$ is a real vector space of dimension $\tilde{d}=d+\dim C$.

We want to apply Lemma 7.4 on this vector space. Let us first construct out of E a subset that spans a dense additive subgroup. Denote by $p:\widetilde{V}\to V\times C$ the covering map. Fix a basis $(b_1,...,b_d,b_{d+1},...,b_{\tilde{d}})$ of $\widetilde{V}$ such that $(p(b_1),...,p(b_d))$ is a basis of $V\times\{0\}$ and the additive subgroup generated by $(b_{d+1},...,b_{\tilde{d}})$ is the kernel of the covering map $\ker(p)$. With such a basis, we explicit the isomorphism between $\widetilde{V}/\langle b_{d+1},...,b_{\tilde{d}}\rangle$ and $V\times C$. Then the following subset

$$\widetilde{\mathcal{D}} := Vect(b_1,...,b_d)\times\left(\prod_{j=1}^{\dim C}[0,1[b_{d+j}\right),$$

is a fundamental domain of the covering. Consider now the subset of elements of this fundamental domain that project into elements of E,

$$\widetilde{E} := p^{-1}(E)\cap\widetilde{\mathcal{D}}.$$

We deduce that $\widetilde{E}\cup\{b_{d+1},...,b_{\tilde{d}}\}$ spans a dense additive subgroup of $\widetilde{V}$. Fix now a subset $B'\subset\widetilde{E}$ such that $\pi_V(p(B'))$ is a basis of V.

Apply now density Lemma 7.4 on $\widetilde{V}$, for the subset $\widetilde{E}\cup\{b_{d+1},...,b_{\tilde{d}}\}$ and choice of basis $B'\cup\{b_{d+1},...,b_{\tilde{d}}\}$. There exists and we choose a finite subset $\widetilde{F}\subset\widetilde{E}$ of at most $2\tilde{d}$ elements, such that $\widetilde{F}\cup B'\cup\{b_{d+1},...,b_{\tilde{d}}\}$ spans a δ-dense additive subgroup of $\widetilde{V}$.

Finally, we project $\widetilde{F}\cup B'\cup\{b_{d+1},...,b_{\tilde{d}}\}$ to $V\times C$ using the covering map. Then $p(\widetilde{F}\cup B')\subset E$ is a finite subset of at most $3d+2\dim C$ elements that spans a δ-dense additive subgroup of $V\times C$. □

Lemma 7.5. *Let C be a compact connected abelian real linear Lie group and V be a finite dimensionnal real vector space. Fix $\delta>0$ a small real number.*

Then for all finite subset $F\subset V\times C$ that spans a δ-dense subset of $V\times C$, there exist an element $v_F\in V$ such that the semigroup *genenerated by F is $\delta-$dense in*

$$\left(v_F+\sum_{f\in F}\mathbb{R}_+\pi_V(f)\right)\times C.$$

Proof. We adapt a proof of Y. Benoist [Ben00, Lemma 6.2].

Consider the compact subset of V

$$\widetilde{D} := \left\{\sum_{f\in F}t_f\pi_V(f)\;\middle|\;0\le t_f\le 1\right\}.$$

Then $\widetilde{D}\times C$ is a compact subset of $V\times C$. By hypothesis, the additive subgroup generated by F is $\delta-$dense in $V\times C$. Then, applying compacity, we choose a finite subset $X\subset\langle F\rangle$ that is δ-dense in $\widetilde{D}\times C$, i.e. such that

$$\widetilde{D}\times C\subset\bigcup_{x\in X}B(x,\delta).$$

Denote by $\langle F\rangle_+$ the subsemigroup generated by F. Choose an element of the additive subsemigroup $h\in\langle F\rangle_+$ such that $hX\subset\langle F\rangle_+$. Such a choice is possible because $V\times C$ is *abelian.*

Then the translate $h(\widetilde{D}\times C)$ is δ-covered by $hX\subset\langle F\rangle_+$, i.e.

$$h(\widetilde{D}\times C)\;\subset\;\bigcup_{x\in X}B(hx,\delta)\;\subset\;\bigcup_{x\in\langle F\rangle_+}B(x,\delta). \tag{14}$$

Remark now that

$$h(\widetilde{D}\times C)=(\pi_V(h)+\widetilde{D})\times\pi_C(h)C=(\pi_V(h)+\widetilde{D})\times C.$$

Denote now by L the close convex cone generated by $\pi_V(F)$, i.e. $L:=\sum_{f\in F}\mathbb{R}_+\pi_V(f)$. Then, by translating on the left by $\langle F\rangle_+$ in the previous equality, a translate of L appears on the right hand side i.e.

$$\langle F\rangle_+\big((\pi_V(h)+\widetilde{D})\times C\big)=\big((\pi_V(h)+L)\times C\big).$$

Finally, combining with (14), we deduce that $\langle F\rangle_+$ is δ-dense in $\big((\pi_V(h)+L)\times C\big)$ i.e.

$$\big((\pi_V(h)+L)\times C\big)\subset\bigcup_{x\in\langle F\rangle_+}B(x,\delta).$$

$\square$

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