Universität Heidelberg
Mathematisches Institut
Arbeitsgruppe Arithmetische Geometrie

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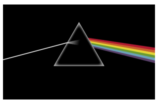
Prismatic Cohomology

Oberseminar Arithmetische Geometrie (AG Venjakob) im Sommersemester 2022



1. Introduction

Prismatic cohomology is a recently discovered cohomology theory for algebraic varieties over p-adically complete rings that unifies various (integral) cohomology theories of interest, in the sense that they can be recovered as special cases of prismatic cohomology. These specialisations are analogous to how a prism splits white light into various individual colours, giving rise to the terminology "prismatic". In this seminar, we aim for an introduction to the notion of prismatic cohomology, culminating in a few applications in the end, e.g. the following Theorem by Bhatt-Morrow-Scholze:

Theorem. *Let X be a smooth projective variety over $\mathbb{Q}$ with good reduction outside some integer $N > 0$. Take homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{Z}[1/N][x_1, \ldots, x_n] \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ defining $X \subseteq \mathbb{P}^n$. For any prime p that does not divide N, write X_p for the smooth projective variety over $\mathbb{F}_p$ obtained by reducing the equations defining X modulo p, and write X^{an} for the projective complex manifold defined by X.*
Then we have an inequality

$$\dim H^i(X^{an}, \mathbb{F}_p) \leq \dim H^i_{\mathrm{dR}}(X_p).$$

We will see how this theorem is proved, roughly, by "deforming $H^*_{\mathrm{dR}}(X_p)$ to $H^*(X^{an}, \mathbb{F}_p)$", with prismatic cohomology being the passageway for this.
Another application we will study is the proof of an improved version of the almost purity theorem allowing ramification along arbitrary closed subsets.

2. Time and Place

We meet on Thursdays at 11:15 o'clock in Heidelberg, INF 205 (Mathematikon) Seminarraum 8. The seminar will take place in hybrid form (with the possibility to participate online -- please contact us if you would like to receive the link to the Zoom online meeting). The first talk will be on the 28th of April.

3. Contact

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Please contact us if you are interested in giving a talk.

4. Talks

We will follow the lecture notes [Bha18] and [Ked21], where the latter are based on the former but are more extensive since they fill in some details and also cover more background material. We encourage the speaker to compare the material of their talk to the relevant section of the paper [BS19] as well.

The duration of each talk should be 90 minutes, perhaps with some time for discussion included.

Talk 1. Delta rings (28.04., Sriram Chinthalagiri Ventaka)
References: [Ked21, Sec. 2 and 3.3], [Bha18, Lec. II].
Discuss the definition and examples of δ-rings, p-derivations and Frobenius lifts, the category of δ-rings and its properties (adjunctions, free objects). Explain how Witt vectors induce an equivalence of categories of perfect rings of characteristic p and p-adically complete perfect δ-rings.

Talk 2. Derived completeness (05.05., Rustam Steingart)
References: [Ked21, Sec. 6], [Bha18, Lec. III §2].
This talk is a digression into background material on derived completions. To work effectively with prisms, it will be useful to have a good theory of completions along an ideal. Unfortunately, the rings we shall encounter (such as perfect or perfectoid rings) are often non-noetherian, and the modules we shall encounter are often not finitely generated. In this setting, the classical theory of completions does not behave so well; for example, the cokernel of a map of complete modules can fail to be separated (and is thus not complete). This defect is rectified by passage to the derived variant of the notion of completeness. This talk will cover all the definitions, facts and properties we will need.

Talk 3. Distinguished elements and prisms (12.05., Max Witzelsperger)
References: [Ked21, Sec. 5 and 7], [Bha18, Lec. III §§1 and 3, Lec. IV §1].
Discuss the definition, examples and properties of distinguished elements, prisms and perfect prisms. Cover tilting and slicing of prisms.

Talk 4. Perfectoid rings ("lenses") and perfect prisms (19.05., Milan Malcic)
References: [Ked21, Sec. 8], [Bha18, Lec. IV §§2 and 3].
A perfectoid ring in the sense of Bhatt-Morrow-Scholze is called a *lens* in [Ked21]. Discuss the results on the structure of perfectoid rings, their properties and characterizations. Explain the equivalence of categories of perfect prisms and perfectoid rings. The results of this talk will enable us to produce many perfectoid rings in cases where it is not so obvious how to give a direct construction of the corresponding perfect prism. In particular, we can make some prisms without specifying either a δ-ring structure or a Frobenius lift.

Talk 5. The prismatic site (02.06., Rızacan Çiloğlu)

References: [Ked21, Sec. 11.3-4, 11.1, 11.6, 12.3], [Bha18, Lec. V].

Define the prismatic site and the perfect prismatic site in the affine situation. Define prismatic and Hodge-Tate cohomology. Recall Čech-Alexander resolutions as the general tool for computing cohomology in this context. Construct the Hodge-Tate comparison map, which will be proven to be an isomorphism in the next talk.

Talk 6. A special case of the Hodge-Tate comparison Theorem (09.06., Otmar Venjakob)

References: [Ked21, Sec. 14.2-4]. See also [Bha18, Lec. VI].

The goal of this talk is to prove the Hodge-Tate comparison Theorem for the prism $(\mathbb{Z}_p, (p))$. Introduce divided powers and relate this notion to δ-rings. Use this to compute the prismatic cohomology over a crystalline prism in terms of crystalline cohomology. With these preparations, prove the above-mentioned special case of the theorem.

Talk 7. Extending prismatic cohomology to the singular case (23.06., Sriram Chinthalagiri Ventaka)

References: [Bha18, Lec. VII], [Ked21, Sec. 16.4 and 18].

Introduce non-abelian derived functors and derived prismatic cohomology over a fixed base prism (A, I). This extends the prismatic cohomology functor $R \longrightarrow {\Delta}_{R/A}$ from formally smooth A/I-algebras to arbitrary p-complete A/I-algebras. This will later allow us to state the étale comparison Theorem in great generality.

Talk 8. Perfectoidization (30.06., Jakob Burgi)

References: [Ked21, Sec. 19.2-4], [Bha18, Lec. VIII §§2 and 3] (possibly include 19.1 resp. VIII §1 for motivation).

Define the *perfection* and *perfectoidization* of a p-complete A/I-algebra R (Kedlaya calls those *prismatic coperfection* and *lens coperfection* respectively). Prove their basic properties, in particular their independence of the base.

Sketch the proof of André's flatness Lemma and conclude that for a semiperfectoid ring S, the canonical map from S to its perfectoidization is surjective. This last result will be crucial for the prismatic treatment of the almost purity Theorem later.

Talk 9. The étale comparison Theorem and applications (07.07., Amine Koubaa)

References: [Bha18, Lec. IX §§1-4] or [Ked21, Sec. 20-22] for the proof; [Bha18, Lec. IX §5] and [Ked21, Sec. 23] for applications.

State the étale comparison Theorem and sketch a proof. Here one can either follow Bhatt and cite the necessary tools from the étale cohomology of adic spaces, or adopt Kedlaya's approach using the arc topology and Artin-Schreier-Witt exact sequences to avoid any use of p-adic geometry.

Also cover some applications of the theorem, in particular the dimension inequality between étale and de Rham cohomology.

It may take two sessions to complete this talk.

Talk 10. The almost purity theorem (14.07., Max + Milan)

References: [Ked21, Sec. 24.2-4, 25.1-3].

Give a very brief introduction to almost commutative algebra. State the necessary definitions and results for almost mathematics of perfectoid rings.

Go on to cover the various versions of the almost purity Theorem and sketch their proofs. It may take two sessions to complete this talk.

References

[Bha18] B. Bhatt. *Geometric aspects of prismatic cohomology*. Eilenberg Lectures at Columbia University, http://www-personal.umich.edu/~bhattb/teaching/prismatic-columbia/. Fall of 2018.

[BS19] B. Bhatt and P. Scholze. *Prisms and Prismatic Cohomology*. 2019. DOI: 10.48550/ARXIV.1905.08229. URL: https://arxiv.org/abs/1905.08229.

[Ked21] K. Kedlaya. *Notes on prismatic cohomology*. Graduate topics course at UC San Diego, https://kskedlaya.org/papers/prismatic-ptx.pdf. Spring of 2021.