Affine Chabauty II

Marius Leonhardt, Universität Heidelberg
joint work in progress with Martin Lüdtke

30th June 2025

33èmes Journées Arithmétiques, Luxembourg

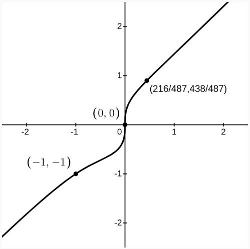
Recap Affine Chabauty I

Setup

- $X/\mathbb{Q}$ projective curve
- $D \subset X$ nonempty ("cusps")
- $Y = X \setminus D$ affine curve
- S finite set of primes
- $\mathcal{X}/\mathbb{Z}$ regular model of X
- $\mathcal{D}$ the closure of D in $\mathcal{X}$
- $\mathcal{Y} = \mathcal{X} \setminus \mathcal{D}$ model of Y

Main goal: Compute $\mathcal{Y}(\mathbb{Z}_S)$ or bound $\#\mathcal{Y}(\mathbb{Z}_S)$.

- $X\colon y^2 + y = x^3 - 1$
- $D = \{(\zeta_3^i, 0) \mid i = 0, 1, 2\}$
- $Y\colon v^3 = u(u^2 + u + 1)$
- $S = \{487\}$



$\mathcal{Y}(\mathbb{Z}_S)$ contains $\{(0,0), (-1,-1), (\frac{216}{487}, \frac{438}{487})\}$.

Affine Chabauty diagram

Let $\Sigma = (\Sigma_\ell)_\ell$ be a reduction type.

$$
\begin{array}{ccccccc}
\mathcal{Y}(\mathbb{Z}_S)_\Sigma & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) & \overset{\int_{P_0}}{\searrow} & \\
\downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \\
\mathrm{Sel}(P_0, \Sigma) & \hookrightarrow & J_Y(\mathbb{Q}) & \hookrightarrow & J_Y(\mathbb{Q}_p) & \xrightarrow{\log_{J_Y}} & \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))^\vee
\end{array}
$$

Theorem (L.-Lüdtke, 2025+)

If $\boxed{r + \#S + n_1(D) + n_2(D) - \#|D| < g + n - 1}$*, then there exist a computable log differential* η *and* $c \in \mathbb{Q}_p$ *s.t.*

$$
\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subset \left\{ P \in \mathcal{Y}(\mathbb{Z}_p) \;\middle|\; \int_{P_0}^{P} \eta = c \right\} =: \mathcal{Y}(\mathbb{Z}_p)_{\eta,c}.
$$

Corollaries and variants

Corollary (LL)

If moreover $p > 2g + n$, then we have the bound

$$\#\mathcal{Y}(\mathbb{Z}_p)_{\eta,c} \leq \#\mathcal{Y}(\mathbb{F}_p) + 2g - 2 + n.$$

Theorem (LL)

Let $\eta_1, \ldots, \eta_{g+n-1}$ be a basis of $\mathrm{H}^0(X, \Omega^1(D))$. Let $P_0, \ldots, P_{g+n-2}$ be known points in $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$. Then every point $P_{g+n-1} \in \mathcal{Y}(\mathbb{Z}_S)_\Sigma$ satisfies the equation

$$\det\left(\int_{P_0}^{P_i} \eta_j\right)_{\substack{1\leq i\leq g+n-1\\ 1\leq j\leq g+n-1}} = 0.$$

Affine Chabauty over number fields

We have a “Restriction of Scalars” variant to compute

$$\mathcal{Y}(\mathcal{O}_{K,S})_\Sigma.$$

The affine Chabauty method

Strategy

$$\begin{array}{ccccccc}
\mathcal{Y}(\mathbb{Z}_S)_\Sigma & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) & \overset{\int_{P_0}}{\searrow} & \\
\downarrow \mathsf{AJ}_{P_0} & & \downarrow \mathsf{AJ}_{P_0} & & \downarrow \mathsf{AJ}_{P_0} & & \\
\mathrm{Sel}(P_0,\Sigma) & \hookrightarrow & J_Y(\mathbb{Q}) & \hookrightarrow & J_Y(\mathbb{Q}_p) & \xrightarrow{\log_{J_Y}} & \mathrm{H}^0(X_{\mathbb{Q}_p},\Omega^1(D))^\vee
\end{array}$$

We need to determine the kernel of the map $\log^*_{J_Y}$ between affine linear functions on $\mathrm{H}^0(X_{\mathbb{Q}_p},\Omega^1(D))^\vee$ and $\mathrm{Sel}(P_0,\Sigma)$.

For $F \in \mathrm{Div}^0(Y)$ and η a log differential we have

$$\log^*_{J_Y}(\eta)(F) = \int_F \eta.$$

Fundamental calculation

From now on: Simplifying assumption $D(\overline{\mathbb{Q}}) = D(\mathbb{Q})$.

Let $G_1, \ldots, G_r$ be a basis of $J(\mathbb{Q})$. Then

$$F = \sum_i x_i(F) G_i + \mathrm{div}(f) \qquad \text{and hence}$$

$$\begin{aligned}\int_F \eta &= \sum_{i=1}^{r} \left(\int_{G_i} \eta \right) x_i(F) + \int_{\mathrm{div}(f)} \eta \\ &= \sum_{i=1}^{r} \left(\int_{G_i} \eta \right) x_i(F) + \sum_{Q \in D(\mathbb{Q})} \mathrm{Res}_Q(\eta) \log f(Q)\end{aligned}$$

by the p-adic residue theorem (= Coleman reciprocity).

If $F \in \mathrm{Sel}(P_0, \Sigma)$, get formula for prime factorisation of $f(Q)$.

Linear algebra

Describe $\log^*_{J_Y} : (\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))^\vee)^{\text{aff.lin.}} \to \mathrm{Sel}(P_0, \Sigma)^{\text{aff.lin.}}$.

- basis for LHS: $\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_{n-1}, 1$
- basis for RHS: $x_1, \ldots, x_r, (t_\ell)_{\ell \in S}, 1$
- matrix of $\log^*_{J_Y}$ w.r.t. these bases is

$$M(\Sigma) := \begin{pmatrix} A & B & 0 \\ 0 & D(\Sigma) & 0 \\ 0 & e(\Sigma) & 1 \end{pmatrix}$$

and its entries look like

$$B_{i,j} = \int_{G_i} \eta_j - \sum_{Q \in D(\mathbb{Q})} \mathrm{Res}_Q\, \eta_j \sum_\ell i_{\mathcal{Q} \bmod \ell}(\Psi_\ell(G_i), \mathcal{Q}) \log \ell.$$

Algorithm

Input

Curve Y, base point $P_0 \in Y(\mathbb{Q})$, prime p.

Precalculations

- Divisors $G_1, \ldots, G_r \in \mathrm{Div}^0(Y)$ generating $J(\mathbb{Q})$.
- Vertical $\mathbb{Q}$-divisor $\Phi_\ell(G_i) \rightsquigarrow \Psi_\ell(G_i) := \mathcal{G}_i + \Phi_\ell(G_i)$.
- Basis $\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_{n-1}$ of log differentials and their residues $\mathrm{Res}_Q(\eta_j)$.

Finding η

- Compute the matrix $M(\Sigma)$, i.e. compute $\int_{G_i} \omega_j$, $\int_{G_i} \eta_j$, and correction terms.
- Find $(a_1, \ldots, a_g, b_1, \ldots, b_{n-1}, -c)$ in the kernel of $M(\Sigma)$.
- Output: $\eta = \sum_i a_i\omega_i + \sum_j b_j\eta_j$ and c

Example $Y\colon v^3 = u(u^2+u+1)$

Input

$X\colon y^2 + y = x^3 - 1$ and $D = \{Q_1, Q_2\}$ with $Q_1 = (1,0)$ and $Q_2 = (\zeta_3, 0)$, base point $P_0 = \infty$, prime $p = 7$.

Precalculations

- $r = 1$ and $G_1 = (7,18) - \infty$ generates $J(\mathbb{Q})$
- log differentials $\omega_1 = \frac{dx}{2y+1}, \eta_1 = \frac{1}{y}\frac{dx}{2y+1}, \eta_2 = \frac{x-1}{y}\frac{dx}{2y+1}$.
- $\mathrm{Res}_{Q_1}(\eta_1) = \frac{1}{3}$, $\mathrm{Res}_{Q_2}(\eta_1) = \frac{\zeta_3}{3}$, $\mathrm{Res}_{Q_2}(\eta_2) = \frac{\sqrt{-3}}{3}$.

$M(\Sigma)$

$$\begin{pmatrix} \int_{G_1}\omega_1 & \int_{G_1}\eta_1 - (5\cdot 7 + 4\cdot 7^2 + \dots) & \int_{G_1}\eta_2 & 0 \\ 0 & 5\cdot 7^2 + 3\cdot 7^3 + \dots & 2\cdot 7 + 5\cdot 7^2 + \dots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example $Y\colon v^3 = u(u^2+u+1)$

$M(\Sigma)$

$$\begin{pmatrix} 4\cdot 7+2\cdot 7^3 & 5\cdot 7+7^2+2\cdot 7^3 & 6\cdot 7+6\cdot 7^3 & 0 \\ 0 & 5\cdot 7^2+3\cdot 7^3 & 2\cdot 7+5\cdot 7^2+2\cdot 7^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\eta = a_1\omega_1 + b_1\eta_1 + b_2\eta_2$ **and** c

$(a_1, b_1, b_2, c) = (1, 2+7+2\cdot 7^2+4\cdot 7^4, 2\cdot 7+6\cdot 7^2, 0)$

and indeed

$$\int_{\infty}^{(\frac{216}{487},\frac{438}{487})} \eta = 0.$$

Take away

- $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ is contained in the vanishing locus of the integral of a log differential η.
- We can bound the size of this vanishing locus.
- We can compute η (and its vanishing locus) by computing
  - Coleman integrals of basic log differentials η_j,
  - their residues,
  - and arithmetic intersection numbers with the cusps.

Examples

More examples

Punctured elliptic curves

- 128.a2, $D = \{\infty, (0,1)\}$: $\mathbb{Z}$-points
- 243.a1, $D = \{(\zeta_3^i, 0)\}$: $\mathbb{Z}$-points and $\mathbb{Z}_S$-points with $\#S = 1$.

Split even degree hyperelliptic curves of genus 2

- 6081.b.164187.1, $D = \{\infty_\pm\}$: $\mathbb{Z}$-points on $Y\colon y^2 = x^6 + 2x^5 - 7x^4 - 18x^3 + 2x^2 + 20x + 9$ are $\{(-1, \pm 1),\ (0, \pm 3),\ (1, \pm 3),\ (-2, \pm 3),\ (-4, \pm 37)\}$.
- 1549.a.1549.1, $D = \{\infty_\pm\}$: $\mathbb{Z}[\zeta_3]$-points on $Y\colon y^2 = x^6 - 4x^5 + 2x^4 + 6x^3 + x^2 - 10x + 1$ are $\left\{(0, \pm 1),\ (2, \pm 1),\ (1, \pm\sqrt{-3})\right\}$.

Take away

- $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ is contained in the vanishing locus of the integral of a log differential η.
- We can bound the size of this vanishing locus.
- We can compute η (and its vanishing locus) by computing
    - Coleman integrals of basic log differentials η_j,
    - their residues,
    - and arithmetic intersection numbers with the cusps.

Thank you for your attention!