Affine (Quadratic) Chabauty

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Rational points on curves

Projective curves

Let $X/\mathbb{Q}$ be a smooth projective geometrically integral curve of genus g (i.e. Euler characteristic $\chi = 2 - 2g$).

Theorem (Mordell, Weil, Faltings, . . .)

If $X(\mathbb{Q}) \neq \emptyset$, then

- $g = 0$ *($\chi = 2$): $X(\mathbb{Q})$ is infinite.*
- $g = 1$ *($\chi = 0$): $X(\mathbb{Q})$ is a finitely generated abelian group.*
- $g \geq 2$ *($\chi < 0$): $X(\mathbb{Q})$ is finite.*

Open question: Compute $X(\mathbb{Q})$ or a bound on $\#X(\mathbb{Q})$.

Let $Y = X \setminus D$. Say Y is affine if $D \neq \emptyset$. Its Euler characteristic is $\chi = 2 - 2g - n$, where $n = \#D(\overline{\mathbb{Q}})$.

Theorem (Mordell, Weil, Faltings, Siegel, ...)

Let S be a finite set of primes. If $\chi < 0$ (i.e. Y is hyperbolic), then $Y(\mathbb{Z})$ is finite.

Open question: Compute $Y(\mathbb{Z})$ or a bound on $\#Y(\mathbb{Z})$.

Results

Theorem (L.–Lütdke–Müller)

If $\alpha_1 := g - r + b > 0$, *then* $Y(\mathbb{Z})$ *is finite. Moreover*

$$\#Y(\mathbb{Z}) \leq \kappa_p \cdot \prod_{\ell} n_\ell \cdot \#Y(\mathbb{F}_p) \cdot (4g + 2n - 2)^2 (g + 1)\,.$$

Theorem (LLM)

If $\alpha_2 := \alpha_1 + \rho_f > 0$, *then* $Y(\mathbb{Z})$ *is finite.*

Theorem (LLM)

If $\beta := \frac{1}{2}g(g + 3) - \frac{1}{2}r(r + 3) + \rho_f + b > 0$, *then*

$$\#Y(\mathbb{Z}) \leq \kappa_p \cdot \prod_{\ell} n_\ell \cdot \#Y(\mathbb{F}_p) \cdot (4g + 2n - 2)^2 (g + 1)\,.$$

Results

Theorem (LLM)

If $\alpha_1 := g - r + b > 0$*, then* $Y(\mathbb{Z}_p)_1$ *is finite.*

Theorem (LLM)

If $\alpha_2 := \alpha_1 + \rho_f > 0$*, then* $Y(\mathbb{Z}_p)_2$ *is finite.*

Theorem (LLM)

If $\beta := \frac{1}{2}g(g+3) - \frac{1}{2}r(r+3) + \rho_f + b - s > 0$*, then*

$$\#Y(\mathbb{Z}_p)_2 \leq \kappa_p \cdot \prod_{\ell} n_\ell \cdot \#Y(\mathbb{F}_p) \cdot (4g + 2n - 2)^2 (g+1)\,.$$

Chabauty–Kim method

Idea of Chabauty

Let J be the Jacobian of X: an Abelian variety of dimension g.

$$\begin{array}{ccccc} X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) & & \\ \downarrow & & \downarrow & \searrow & \\ J(\mathbb{Q}) & \hookrightarrow & J(\mathbb{Q}_p) & \xrightarrow[\log]{\sim} & \mathbb{Z}_p^g \end{array}$$

If $r < g$, then the set

$$X(\mathbb{Q}_p)_1 := \{z \in X(\mathbb{Q}_p) \mid \int_b^z \omega = 0\} \subset X(\mathbb{Q}_p)$$

is finite and contains $X(\mathbb{Q})$.

Theorem (Coleman)

If $p > 2g$ and $r < g$, then $\#X(\mathbb{Q}_p)_1 \leq \#X(\mathbb{F}_p) + 2g - 2$.

Chabauty–Kim

$J = \pi_1(X)^{ab}$. Idea: Use larger quotients U of π_1! Then

$$\begin{array}{ccccc} Y(\mathbb{Z}) & \hookrightarrow & Y(\mathbb{Z}_p) & & \\ \downarrow & & \downarrow j_p & \searrow & \\ Sel_U(\mathbb{Q}_p) & \xrightarrow[\mathrm{loc}_p]{} & H^1_f(G_p, U)(\mathbb{Q}_p) & \xrightarrow[\log]{\sim} & U_{dR}/Fil^0 \end{array}$$

Define $Y(\mathbb{Z}_p)_U := j_p^{-1}(im(\mathrm{loc}_p)) \subset Y(\mathbb{Z}_p)$.

Theorem

If $\dim Sel_U(\mathbb{Q}_p) < \dim H^1_f(G_p, U)$, *then* $Y(\mathbb{Z}_p)_U$ *is finite.*

Taking $U = \pi_1(Y)_n$, we get a sequence

$$Y(\mathbb{Z}_p) \supset Y(\mathbb{Z}_p)_1 \supset Y(\mathbb{Z}_p)_2 \supset \cdots \supset Y(\mathbb{Z}).$$

Quotients of $\pi_1(Y)$

Depth 1

$$1 \longrightarrow \underbrace{\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1)}_{\mathrm{gr}^W_{-2} U_Y^{\mathrm{ab}}} \longrightarrow U_Y^{\mathrm{ab}} \longrightarrow \underbrace{V_pJ}_{\mathrm{gr}^W_{-1} U_Y^{\mathrm{ab}}} \longrightarrow 1$$

Lemma

- $\dim H^1_f(G_Q, V_pJ) = r_p$.
- $\dim H^1_f(G_p, V_pJ) = g$.
- $\dim H^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-2} U_Y^{\mathrm{ab}}) = n_1 + n_2 - \#|D|$.
- $\dim H^1_f(G_p, \mathrm{gr}^W_{-2} U_Y^{\mathrm{ab}}) = n - 1$.

Depth 2

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{gr}^W_{-2} U_Y & \longrightarrow & U_{Y,2} & \longrightarrow & \mathrm{gr}^W_{-1} U_Y & \longrightarrow & 1 \\
 & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \| & & \\
1 & \longrightarrow & W \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1) & \longrightarrow & U & \longrightarrow & V_pJ & \longrightarrow & 1
\end{array}$$

with $W := (\mathbb{Q}_p \otimes \mathrm{NS}(J_{\overline{\mathbb{Q}}}))^\vee(1)$.

Lemma

- $\dim H^1_f(G_Q, V_pJ) = r_p$.
- $\dim H^1_f(G_p, V_pJ) = g$.
- $\dim H^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-2} U) = n_1 + n_2 - \#|D| + \dim W - \rho_f$.
- $\dim H^1_f(G_p, \mathrm{gr}^W_{-2} U) = n - 1 + \dim W$.

Weight filtration and bounds

$$
\begin{array}{ccccc}
Y(\mathbb{Z}) & \hookrightarrow & Y(\mathbb{Z}_p) & & \\
\downarrow & & \downarrow j_p & \searrow & \\
Sel_U(\mathbb{Q}_p) & \xrightarrow[\mathrm{loc}_p]{} & H^1_f(G_p, U)(\mathbb{Q}_p) & \xrightarrow[\log]{\sim} & U_{dR}/Fil^0
\end{array}
$$

Can put "weight" filtration $W_\bullet$ on the Selmer schemes.

Theorem

If $\dim W_m\mathcal{O}(Sel_{S,U}(\mathbb{Q}_p)) < \dim W_m\mathcal{O}(H^1_f(G_p, U))$, *then there exists a function of weight* $\leq m$ *vanishing on* $Y(\mathbb{Z}_p)_U$.

Take away

- If $\chi < 0$, then $Y(\mathbb{Z})$ is finite.
- Chabauty–Kim is a p-adic method to compute $Y(\mathbb{Z})$, using cohomology of the (unipotent) fundamental group.
- If certain inequalities (involving g, r, $\#|D|, \dots$) hold, we get a bound on $\#Y(\mathbb{Z})$.

Thank you for your attention!

Questions?