Plectic phenomena on Hilbert modular varieties

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Plectic group

The *plectic Galois group* is

$$\Gamma_{\mathrm{pl}} := S_X \ltimes \Gamma_F^X.$$

We have the embedding

$$\rho_s \colon \Gamma_{\mathbb{Q}} \hookrightarrow S_X \ltimes \Gamma_F^X, \quad \gamma \mapsto ([x \mapsto \gamma x], (h_x)_{x \in X}).$$

Recall the transfer map

$$V \colon \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \to \Gamma_F^{\mathrm{ab}}, \quad V(\gamma) = \prod_{x \in X} h_x|_{F^{\mathrm{ab}}}.$$

Applications of Shimura varieties

Modular Curves

- Moduli of elliptic curves
- Applications:
  - Mazur: Torsion of elliptic curves
  - Wiles: Modularity
  - Gross-Zagier-Kolyvagin: BSD cases

General Shimura varieties

- E.g. moduli of AVs
- Similar Applications?
  - seems hard.
  - Maybe using plectic cohomology?!

Plectic Conjecture

- Dream (Nekovář-Scholl): If $G = R_{F/\mathbb{Q}}H$, then the associated Shimura variety Y can be "plectified", i. e.

$$\begin{array}{ccc} Y & \dashrightarrow & Y_{\mathrm{pl}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Q}) & \dashrightarrow & \mathrm{Spec}(\mathbb{Q})_{\mathrm{pl}}. \end{array}$$

- Facts:
  - Plectic reflex Galois group.
  - Plectic Hodge theory.
- Application: Special values of L-functions

Goal

Define an action of Γ_{pl} on CM points of the Hilbert modular variety.

Hilbert modular variety

Let $G = R_{F/\mathbb{Q}} \operatorname{GL}_2$. The Hilbert modular variety is the Shimura variety associated to $(G, (\mathbb{C} \setminus \mathbb{R})^X)$.

Fact

There is a bijection

$$\operatorname{Sh}(G, (\mathbb{C} \setminus \mathbb{R})^X) \cong \{(A, i, \eta)\}/\text{isom}.$$

where

- A is a complex abelian variety of dimension $[F : \mathbb{Q}]$,
- $i \colon F \hookrightarrow \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ (say A has real multiplication by F),
- $\eta \colon \mathbb{A}_{F,f}^2 \xrightarrow{\sim} \widehat{V}(A)$ is an $\mathbb{A}_{F,f}$-module-isomorphism.

CM points of the Hilbert modular variety

Call $[A, i, \eta]$ a *CM point* if A has complex multiplication, i. e. there exist K/F totally imaginary quadratic and an extension of i to K.
Then:

- (A, i) determines a CM type Φ of K.
- There exists a lattice $\mathfrak{a}$ in K such that

$$[A, i, \eta] = [\mathbb{C}^{\Phi}/\Phi(\mathfrak{a}), i_{\Phi}, \eta'].$$

Intermediate goal

Describe Galois action on CM points.

Tate's half transfer

For $\varphi \in \mathrm{Hom}(K, \overline{\mathbb{Q}}) = \Gamma_{\mathbb{Q}}/\Gamma_K$, choose coset representatives $w_\varphi \in \Gamma_{\mathbb{Q}}$ such that $w_{c\varphi} = cw_\varphi$. We define *Tate's half transfer* by

$$F_\Phi : \Gamma_{\mathbb{Q}} \to \Gamma_K^{\mathrm{ab}}, \quad F_\Phi(\gamma) := \prod_{\varphi \in \Phi} \left(w_{\gamma\varphi}^{-1} \gamma w_\varphi \right) \Big|_{K^{\mathrm{ab}}}.$$

The *Taniyama element* is the unique $f_\Phi(\gamma) \in \mathbb{A}_{K,f}^\times / K^\times$ such that

- $r_K(f_\Phi(\gamma)) = F_\Phi(\gamma)$,
- ${}^{1+c} f_\Phi(\gamma) = \chi_{\mathrm{cyc}}(\gamma) K^\times$.

Galois action on CM points

Theorem (Main theorem of complex multiplication)

Let

- K/F *be a CM field,* $[K:F]=2$*, and* Φ *be a CM type of* K*.*
- $[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi, \eta]$ *be a CM point of* $\mathrm{Sh}(G, (\mathbb{C}\setminus\mathbb{R})^X)$*.*
- $\gamma \in \Gamma_{\mathbb{Q}}$ *and* $f \in \mathbb{A}_{K,f}^\times$ *such that* $f_\Phi(\gamma) = fK^\times$*.*

Then:

$$\gamma[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi, \eta] = [\mathbb{C}^{\gamma\Phi}/\gamma\Phi(f\mathfrak{a}), i_{\gamma\Phi}, f\circ\eta].$$

Main ingredients:

- Action of $\Gamma_{\mathbb{Q}}$ on CM types, i. e. on $\mathrm{Hom}(K, \overline{\mathbb{Q}})$.
- Taniyama element $f_\Phi : \Gamma_{\mathbb{Q}} \to \mathbb{A}_{K,f}^\times/K^\times$.

Plectify

Want:

- Action of Γ_{pl} on $\mathrm{Hom}(K, \overline{\mathbb{Q}}) = \mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}\{1, c\}$, so OK.
- A *plectic half transfer*

$$\widetilde{F}_\Phi : \Gamma_{\mathrm{pl}} \to \Gamma_K^{\mathrm{ab}}.$$

  OK.
- A *plectic Taniyama element*

$$\widetilde{f}_\Phi : \Gamma_{\mathrm{pl}} \to \mathbb{A}_{K,f}^\times / K^\times.$$

  OK once we have a splitting χ_F.

Plectic Galois action on CM points

Let

- K/F be a CM field, $[K:F]=2$, and Φ be a CM type of K.
- $[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi, \eta]$ be a CM point of $\mathrm{Sh}(G, (\mathbb{C}\setminus\mathbb{R})^X)$.
- $\gamma \in \Gamma_{\mathrm{pl}}$ and $\widetilde{f} \in \mathbb{A}_{K,f}^\times$ such that $\widetilde{f_\Phi}(\gamma) = \widetilde{f}K^\times$.

Define

$$\gamma[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi, \eta] := [\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}, \widetilde{f}\circ\eta].$$

Lemma

This gives a well-defined group action of Γ_{pl} on the set of CM points of $\mathrm{Sh}(G, (\mathbb{C}\setminus\mathbb{R})^X)$.

Set of connected components

Fact

$$\pi_0(\mathrm{Sh}(G, (\mathbb{C} \setminus \mathbb{R})^X)) = \pi_0(C_F),$$

where $C_F = \mathbb{A}_F^\times / F^\times$ is the idele class group of F.

- Galois action given by

$$\Gamma_{\mathbb{Q}} \xrightarrow{V} \Gamma_F^{\mathrm{ab}} \xrightarrow{r_F^{-1}} \pi_0(C_F).$$

- Define a plectic action by

$$\Gamma_{\mathrm{pl}} \xrightarrow{(1,\mathrm{prod})} \Gamma_F^{\mathrm{ab}} \xrightarrow{r_F^{-1}} \pi_0(C_F).$$

Plectic equivariance of π_0

Theorem (L. 2019)

The map

$$\pi_0 : \mathrm{Sh}(G, (\mathbb{C} \setminus \mathbb{R})^X) \longrightarrow \pi_0(\mathrm{Sh}(G, (\mathbb{C} \setminus \mathbb{R})^X))$$

is Γ_{pl}-equivariant on CM points.

Summary

The plectic group

$$\Gamma_{\mathrm{pl}} = S_X \ltimes \Gamma_F^X$$

acts on

- CM points via plectic half transfer,
- the set of connected components via $(1, \mathrm{prod})$.

Thank you for your attention!

Questions?