The affine Chabauty method

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Introduction and results

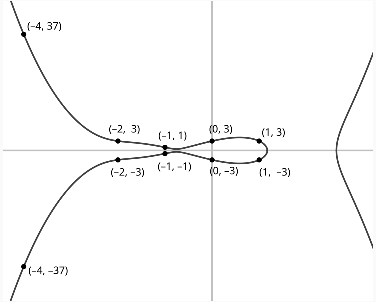
Motivation

What are the integer solutions of

$$y^2 = x^6 + 2x^5 - 7x^4 - 18x^3 + 2x^2 + 20x + 9?$$

What about solutions with $x, y \in \mathbb{Z}_S = \mathbb{Z}[\frac{1}{\ell} | \ell \in S]$?

Geometric formulation: What are the $\mathbb{Z}_S$-points of an affine curve $\mathcal{Y} \subseteq \mathbb{A}^2_{\mathbb{Z}}$?

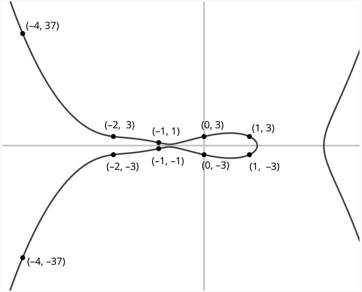


Geometric setup

- $X/\mathbb{Q}$ smooth proj. curve, genus g
- $D \subset X$ ("cusps"), $n = \#D(\overline{\mathbb{Q}}) > 0$
- $Y = X \setminus D$ affine curve
- S finite set of primes
- $\mathcal{X}/\mathbb{Z}$ regular model of X
- $\mathcal{D}$ the closure of D in $\mathcal{X}$
- $\mathcal{Y} = \mathcal{X} \setminus \mathcal{D}$ model of Y

Main goal: Compute $\mathcal{Y}(\mathbb{Z}_S)$ or bound $\#\mathcal{Y}(\mathbb{Z}_S)$.

- $g = 2$
- $D = \{\infty_\pm\}$, $n = 2$
- $y^2 = x^6 + 2x^5 + \cdots + 9$
- $S = \emptyset$



Main result

Main goal: Compute $\mathcal{Y}(\mathbb{Z}_S)$ or bound $\#\mathcal{Y}(\mathbb{Z}_S)$.

Theorem (Siegel, Mahler)

If $2 - 2g - n < 0$ then $\#\mathcal{Y}(\mathbb{Z}_S) < \infty$.

Theorem (L.-Lüdtke, 2025+)

Let $p \notin S$ be a prime, let $P_0 \in Y(\mathbb{Q})$ and let $r := \mathrm{rk}_{\mathbb{Z}} J(\mathbb{Q})$. If

$$\boxed{r + \#S + n_1(D) + n_2(D) - \#|D| < g + n - 1},$$

then there exist finitely many computable log differentials η_Σ and $c_\Sigma \in \mathbb{Q}_p$ s.t.

$$\mathcal{Y}(\mathbb{Z}_S) \subset \bigcup_\Sigma \left\{ P \in \mathcal{Y}(\mathbb{Z}_p) \;\middle|\; \int_{P_0}^{P} \eta_\Sigma = c_\Sigma \right\} =: \bigcup_\Sigma \mathcal{Y}(\mathbb{Z}_p)_{\eta_\Sigma, c_\Sigma}.$$

Corollaries and variants

Corollary (LL)

If moreover $p > 2g + n$, then we have the bound

$$\#\mathcal{Y}(\mathbb{Z}_p)_{\eta,c} \leq \#\mathcal{Y}(\mathbb{F}_p) + 2g - 2 + n.$$

Theorem (LL)

Let $\eta_1, \ldots, \eta_{g+n-1}$ be a basis of $\mathrm{H}^0(X, \Omega^1(D))$. Let $P_0, \ldots, P_{g+n-2}$ be known points in $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$. Then every point $P_{g+n-1} \in \mathcal{Y}(\mathbb{Z}_S)_\Sigma$ satisfies the equation

$$\det\left(\int_{P_0}^{P_i} \eta_j\right)_{\substack{1\leq i\leq g+n-1 \\ 1\leq j\leq g+n-1}} = 0.$$

Application to $y^2 = x^6 + 2x^5 - 7x^4 - 18x^3 + 2x^2 + 20x + 9$

Its $\mathbb{Z}$-points are $(-1, \pm1)$, $(0, \pm3)$, $(1, \pm3)$, $(-2, \pm3)$, $(-4, \pm37)$.

Affine Chabauty over number fields

We have a "Restriction of Scalars" variant to compute $\mathcal{Y}(\mathcal{O}_{K,S})_\Sigma$.

Details of the proof

3 ingredients

- log differential forms $\eta \in \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))$, i.e. only simple poles at D allowed
- partition S-integral points by (finitely many) reduction types

$$\mathcal{Y}(\mathbb{Z}_S) = \coprod_{\Sigma} \mathcal{Y}(\mathbb{Z}_S)_\Sigma$$

  $\Sigma = (\Sigma_\ell)_\ell$ prescribes for each ℓ the component of $\mathcal{X}_{\mathbb{F}_\ell}$ or (if $\ell \in S$) the cusp onto which the point reduces.

- The generalised Jacobian J_Y of Y is a semiabelian variety with

$$J_Y(\mathbb{Q}) = \mathrm{Div}^0(Y) \,/\, \left\{\mathrm{div}(f) : f \in k(X)^\times, f|_D = 1\right\}$$

  an Abel-Jacobi map $\mathrm{AJ}_{P_0} \colon Y \to J_Y$, $P \mapsto [P] - [P_0]$

Affine Chabauty diagram

$$
\begin{array}{ccccccc}
\mathcal{Y}(\mathbb{Z}_S)_\Sigma & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) & \xrightarrow{\int_{P_0}} & \\
\downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \\
\mathrm{Sel}(P_0, \Sigma) & \hookrightarrow & J_Y(\mathbb{Q}) & \hookrightarrow & J_Y(\mathbb{Q}_p) & \xrightarrow{\log_{J_Y}} & \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))^\vee
\end{array}
$$

Key insight: $\mathrm{AJ}_{P_0}(\mathcal{Y}(\mathbb{Z}_S)_\Sigma)$ is contained in a translate of a f. g. subgroup of rank

$$r + \#S + n_1(D) + n_2(D) - \#|D| < g + n - 1 = \dim_{\mathbb{Q}_p} \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))$$

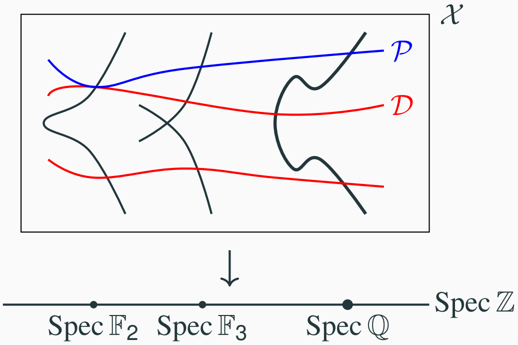
$\implies$ Main Theorem.

Arithmetic intersection theory

We construct the *D-intersection map*

$$\sigma_\ell \colon J_Y(\mathbb{Q}) \to Z_0(\mathcal{D}_{\mathbb{F}_\ell})/[\mathcal{D}_{\mathbb{F}_\ell}] \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$F \mapsto \sum_{x \in |\mathcal{D}_{\mathbb{F}_\ell}|} i_x(\Psi_\ell(F), \mathcal{D})[x]$$

where $\Psi_\ell(F) = \mathcal{F} + \Phi_\ell(F)$ with $\mathcal{F}$ horizontal and $\Phi_\ell(F)$ vertical $\mathbb{Q}$-divisor.



Fundamental calculation: For $P \in \mathcal{Y}(\mathbb{Z}_S)_\Sigma$ we have

$$\sigma_\ell(\mathrm{AJ}_{P_0}(P)) = \underbrace{\sigma_\ell(\mathcal{P})}_{=0 \text{ (if } \ell \notin S) \text{ or } \in \mathbb{Z}_{\geq 0} \cdot [\Sigma_\ell] \text{ (if } \ell \in S)} - \underbrace{\sigma_\ell(\mathcal{P}_0)}_{\text{constant}} + \underbrace{\sigma_\ell(\Phi_\ell(P - P_0))}_{\text{only depends on } \Sigma_\ell}$$

Selmer sets

Fundamental calculation: For $P \in \mathcal{Y}(\mathbb{Z}_S)_\Sigma$ we have

$$\sigma_\ell(\mathrm{AJ}_{P_0}(P)) = \underbrace{\sigma_\ell(\mathcal{P})}_{=0 \text{ (if } \ell \notin S\text{) or } \in \mathbb{Z}_{\geq 0} \cdot [\Sigma_\ell] \text{ (if } \ell \in S\text{)}} - \underbrace{\sigma_\ell(\mathcal{P}_0)}_{\text{constant}} + \underbrace{\sigma_\ell(\Phi_\ell(P - P_0))}_{\text{only depends on } \Sigma_\ell}$$

We get

$$\sigma_\ell(\mathrm{AJ}_{P_0}(\mathcal{Y}(\mathbb{Z}_S)_\Sigma)) \subseteq \mathfrak{S}_\ell(P_0, \Sigma) \text{ of rank 0 } (\ell \notin S) \text{ or 1 } (\ell \in S).$$

Define the Selmer set

$$\mathrm{Sel}(P_0, \Sigma) := \{F \in J_Y(\mathbb{Q}) \mid \forall \ell : \sigma_\ell(F) \in \mathfrak{S}_\ell(P_0, \Sigma)\}.$$

Lemma

$\mathrm{Sel}(P_0, \Sigma)$ *contains* $\mathrm{AJ}_{P_0}(\mathcal{Y}(\mathbb{Z}_S)_\Sigma)$ *and is a translate of a subgroup of small rank.*

Take away

$$
\begin{array}{ccccccc}
\mathcal{Y}(\mathbb{Z}_S)_\Sigma & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) & \xrightarrow{\int_{P_0}} & \\
\downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \downarrow{\scriptstyle \mathrm{AJ}_{P_0}} & & \\
\mathrm{Sel}(P_0, \Sigma) & \hookrightarrow & J_Y(\mathbb{Q}) & \hookrightarrow & J_Y(\mathbb{Q}_p) & \xrightarrow{\log_{J_Y}} & \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))^\vee
\end{array}
$$

- $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ is contained in the vanishing locus of the integral of a log differential η.
- We can bound the size of this vanishing locus.
- To compute η (and its vanishing locus), we embed Y into its generalised Jacobian J_Y and compute arithmetic intersection numbers with the cusps.

Thank you for your attention!