Spectral Sequences

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We explain the notion of a spectral sequence and various ways of constructing them, before illustrating the concept by some examples. We do not treat spectral sequences in the greatest possible generality but tried to keep the text accessible.

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Parts of this text follow [Wei94] rather closely.

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1 Spectral Sequences and Convergence

Let C be an abelian category.

Definition 1.1 (Spectral sequences): A (homologically graded) *spectral sequence* is a family of objects $\{E^r_{pq}\}$ of C, for all $p, q \in \mathbb{Z}$ and $r \geq a$ (with a fixed $a \in \mathbb{Z}$), together with differentials $d^r_{pq} \colon E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$ which satisfy $d^r \circ d^r = 0$. Furthermore we require that there are isomorphisms

$$E^{r+1}_{pq} \cong H(E^r_{pq}) = \ker(d^r_{pq}) \Big/ \operatorname{im}(d^r_{p+r,q-r+1}) \ .$$

By $n := p + q$ we denote the *total degree* of E^r_{pq}.

The collections $(E^r_{pq})_{p,q\in\mathbb{Z}}$ for fixed r are called the *sheets* or *pages* of the spectral sequence. By the isomorphisms required above, we imagine that we get from one sheet to the next one ("turing a page around") by taking homology.

To get an imagination of how the differentials look like, we first regard the sheet of the spectral sequence where $r = 0$.

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,3} & E^0_{1,3} & E^0_{2,3} & E^0_{3,3} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

On the first sheet, the differentials go from the right to the left, as is shown here:

$$\begin{array}{c}
\ldots \longleftarrow E^1_{0,3} \longleftarrow E^1_{1,3} \longleftarrow E^1_{2,3} \longleftarrow E^1_{3,3} \longleftarrow \ldots \\[1em]
\ldots \longleftarrow E^1_{0,2} \longleftarrow E^1_{1,2} \longleftarrow E^1_{2,2} \longleftarrow E^1_{3,2} \longleftarrow \ldots \\[1em]
\ldots \longleftarrow E^1_{0,1} \longleftarrow E^1_{1,1} \longleftarrow E^1_{2,1} \longleftarrow E^1_{3,1} \longleftarrow \ldots \\[1em]
\ldots \longleftarrow E^1_{0,0} \longleftarrow E^1_{1,0} \longleftarrow E^1_{2,0} \longleftarrow E^1_{3,0} \longleftarrow \ldots
\end{array}$$

In the following picture we see as an example the 2-sheet with differentials $d^2_{pq} \colon E^2_{p,q} \longrightarrow E^2_{p-2,q+1}$.

$$\begin{tikzcd}
E^2_{0,3} & E^2_{1,3} & E^2_{2,3} & E^2_{3,3} \\
E^2_{0,2} & E^2_{1,2} & E^2_{2,2} \arrow[ull] & E^2_{3,2} \arrow[ull] \\
E^2_{0,1} & E^2_{1,1} & E^2_{2,1} \arrow[ull] & E^2_{3,1} \arrow[ull] \\
E^2_{0,0} & E^2_{1,0} & E^2_{2,0} \arrow[ull] & E^2_{3,0} \arrow[ull]
\end{tikzcd}$$

Example 1.2 (First quadrant spectral sequence): A spectral sequence with $E^r_{pq} = 0$ for $p < 0$ or $q < 0$ is called a *first quadrant spectral sequence*. We observe that for r large enough the differential with codomain E^r_{pq} has domain 0 and the differential with domain E^r_{pq} has codomain 0. We get

$$E^{r+1}_{pq} = H(E^r_{pq}) = {}^{\ker(d)}\big/_{\operatorname{im}(d)} = {}^{E^r_{pq}}\big/_{0} = E^r_{pq}.$$

The stable value $E^r_{pq} = E^k_{pq}$ for $k \geq$ such r is named E^∞_{pq}.

We restrict ourselves to first quadrant spectral sequences, which makes our life much easier.

Definition 1.3 (Convergence): Let $\{H_n\}$ be a family of objects of C.
We say a spectral sequence ...

(a) ... *weakly converges* to H_* if there exists a filtration

$$\ldots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \ldots \subseteq H_n$$

for each $n \in \mathbb{Z}$ and furthermore isomorphisms

$$\beta_{pq} \colon E^\infty_{pq} \cong {}^{F_pH_{p+q}}\big/_{F_{p-1}H_{p+q}} \, .$$

(b) ... *approaches* H_* if it weakly converges to H_* and[1]

$$H_n = \bigcup F_pH_n \quad \text{and} \quad \bigcap F_pH_n = 0.$$

(c) ... *converges* to H_* if it approaches H_* and $H_n = \varprojlim \left(H_n/F_pH_n\right)$.[2]

We denote convergence by $E^r_{pq} \Rightarrow H_{p+q}$.

Remark 1.4: Let a spectral sequence converging to H_* have $E^2_{pq} = 0$ unless $p = 0, 1$. Then for each n there is a short exact sequence

$$0 \longrightarrow E^2_{0n} \longrightarrow H_n \longrightarrow E^2_{1,n-1} \longrightarrow 0.$$

[1] The first of the two conditions is called *exhaustive* and the second one *Hausdorff*.
[2] A filtration which satisfies $\forall n \; \exists t : \; F_tH_n = H_n$ fulfills this additional condition.

Proof: First we draw a picture of the 2-sheet of the spectral sequence:

$$\begin{tikzcd}
0 & 0 & E^2_{0,3} & E^2_{1,3} & 0 & 0 \\
0 & 0 & E^2_{0,2} \arrow[ull] & E^2_{1,2} \arrow[ull] & 0 \arrow[ull] & 0 \arrow[ull] \\
0 & 0 & E^2_{0,1} \arrow[ull] & E^2_{1,1} \arrow[ull] & 0 \arrow[ull] & 0 \arrow[ull] \\
0 & 0 & E^2_{0,0} \arrow[ull] & E^2_{1,0} \arrow[ull] & 0 \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}$$

We see that $E^2_{pq} = E^\infty_{pq}$ and since we know that the spectral sequence converges, we know that there exists a filtration F_pH_n which fulfills $E^\infty_{pq} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$.
If $p \neq 0, 1$ we get $0 = E^2_{pq} = F_pH_{p+q}/F_{p-q}H_{p+q}$ which tells us $F_pH_{p+q} = F_{p-1}H_{p+q}$ and therefore the filtration looks like

$$\ldots F_{-2}H_n = F_{-1}H_n \subseteq F_0H_n \subseteq F_1H_n = F_2H_n = \ldots$$

Moreover we know that $\bigcap F_pH_n = 0$, so $F_{-1}H_n = F_{-2}H_n = \ldots = 0$ and since $\bigcup F_pH_n = H_n$ we get $F_1H_n = F_2H_n = \ldots H_n$.
Now let $p = 0$. We notice that

$$E^2_{0,n} = E^\infty_{0,n} \cong F_0H_n \big/ F_{-1}H_n = F_0H_n.$$

For $p = 1$ we get

$$E^2_{1,n-1} = E^\infty_{1,n-1} \cong F_1H_n \big/ F_0H_n = H_n \big/ F_0H_n$$

and obviously the short exact sequence

$$0 \longrightarrow F_0H_n \hookrightarrow H_n \twoheadrightarrow H_n \big/ F_0H_n \longrightarrow 0$$

turns into the short exact sequence

$$0 \longrightarrow E^2_{0,n} \longrightarrow H_n \longrightarrow E^2_{1,n-1} \longrightarrow 0. \qquad \square$$

Remark 1.5: Let a spectral sequence converging to H_* have $E^2_{pq} = 0$ unless $q = 0, 1$. Then there is a long exact sequence

$$\ldots H_{p+1} \longrightarrow E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \longrightarrow H_p \longrightarrow E^2_{p0} \xrightarrow{d} E^2_{p-2,1} \longrightarrow H_{p-1} \ldots.$$

Proof: Again we first draw a picture

$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
E^2_{-1,1} & E^2_{0,1} & E^2_{1,1} \arrow[ull] & E^2_{2,1} \arrow[ull] \\
E^2_{-1,0} & E^2_{0,0} & E^2_{1,0} \arrow[ull] & E^2_{2,0} \arrow[ull] \\
0 & 0 & 0 \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}$$

and we see that the objects at the 3-sheet are the objects E^∞_{pq}. For $q \neq 1, 0$ we get

$$F_pH_{p+q} \Big/ F_{p-1}H_{p+q} \cong E^\infty_{pq} = E^3_{pq} = H(E^2_{pq}) = \ker(d^2_{pq}) \Big/ \mathrm{im}(d^2_{p+2,q-1}) = 0$$

and therefore $F_pH_{p+q} = F_{p-1}H_{p+q}$.
Let's have a look at the filtration of H_n:

$$\ldots = F_{n-3}H_n = F_{n-2}H_n \subseteq F_{n-1}H_n \subseteq F_nH_n = F_{n+1}H_n = \ldots$$

$\bigcap F_pH_n = 0$ implies $F_rH_n = 0$ for $r \leq p + q - 2$ and $\bigcup F_pH_n = H_n$ implies $F_rH_n = H_n$ for $r \geq n = p + q$.
Let $q = 0$. We get

$$H_p \Big/ F_{p-1}H_p = F_pH_p \Big/ F_{p-1}H_p \cong E^\infty_{p,0} = E^3_{p,0} = \ker(d^2_{p,0}) \Big/ \mathrm{im}(d^2_{p+2,-1}) = \ker(d^2_{p,0}).$$

And $q = 1$ leads to

$$\begin{aligned} F_pH_{p+1} = F_pH_{p+1} \Big/ F_{p-1}H_{p+1} &\cong E^\infty_{p,1} = E^3_{p,1} \\ &= H(E^2_{p,1}) = ker(d^2_{p,1}) \Big/ \mathrm{im}(d^2_{p+2,0}) = E^2_{p,1} \Big/ \mathrm{im}(d^2_{p+2,0}) \,. \end{aligned}$$

We can now put all this together to a long exact sequence:

$$
\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & & \\
 & & & & E^2_{p+1,0}/\ker d^2_{p+1,0} \arrow[ur] \arrow[dr, hook] & & & & H_p/F_{p-1}H_p & \\
\cdots \arrow[r, dashed] & H_{p+1} \arrow[rr, dashed] \arrow[dr] & & E^2_{p+1,0} \arrow[rr, dashed] \arrow[ur] & & E^2_{p-1,1} \arrow[rr, dashed] \arrow[dr] & & H_p \arrow[r, dashed] \arrow[ur] & \cdots . & \\
 & & \ker d^2_{p+1,0} \arrow[ur, hook] \arrow[dr] & & & & F_{p-1}H_p \arrow[ur, hook] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

□

2 Construction of Spectral Sequences

In this section, we introduce the important notions of exact couples and filtered complexes. Every exact couple yields a spectral sequence, and every filtered complex yields an exact couple. Finally, we prove a convergence theorem for spectral sequences obtained in this way.

Construction 2.1 (Exact couples and their derivations): An *exact couple* is a pair of objects D and E of $\mathcal{C}$ and morphisms i, j and k such that the diagram

$$
\begin{tikzcd}
D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\
 & E \arrow[ul, "k"] &
\end{tikzcd}
$$

is exact at each vertex. Since $d := jk$ complies with $d^2 = (jk)^2 = j(kj)k = j0k = 0$ we can apply homology by setting $H(E) = \ker(d)/\mathrm{im}(d)$ and get the *derived exact couple*

$$
\begin{tikzcd}
i(D) \arrow[rr, "i|_{i(D)}"] & & i(D) \arrow[dl, "j^{(2)}"] \\
 & H(E) \arrow[ul, "k^{(2)}"] &
\end{tikzcd}
$$

with $j^{(2)}(i(x)) = [j(x)]$ and $k^{(2)}([e]) = k(e)$, $x \in D$ and $[e] \in H(E)$. It is an easy computation that $j^{(2)}$ and $k^{(2)}$ are well-defined and that the derived couple is exact. It suggests itself to iterate this process and set $E^1 := E$ and $E^r = H(E^{r-1})$, $d^1 := d = jk$ and $d^r := j^{(r)}k^{(r)}$.

The $(r+1)$-th exact couple looks like

$$\begin{tikzcd} i^r(D) \arrow[rr, "i|_{i^r(D)}"] & & i^r(D) \arrow[dl, "j^{(r+1)}"] \\ & E^{r+1} \arrow[ul, "k^{(r+1)}"] & \end{tikzcd}.$$

Remark: Notice that the cycles can be described as $Z^r = k^{-1}(i^r(D))$ and the boundaries as $B^r = j(\ker(i^r))$.

Proof: One just calculates

$$\begin{aligned} Z^r &= \ker(d^r) = \ker(j^{(r)}k^{(r)}) = \ker(ji^{-r+1}k) \\ &= k^{-1}(\ker(ji^{-r+1})) = k^{-1}(i^{r-1}(\ker j)) \\ &= k^{-1}(i^{r-1}(\operatorname{im} i)) = k^{-1}(i^r(D)), \end{aligned}$$

$$\begin{aligned} B^r &= \operatorname{im}(d^r) = \operatorname{im}(j^{(r)}k^{(r)}) = \operatorname{im}(ji^{-r+1}k) \\ &= j(i^{-r+1}k(E_{pq})) = j(i^{-r+1}\operatorname{im}(k)) \\ &= j(i^{-r+1}(\ker(i)) = j(i^{-r}(0)) = j(\ker(i^r)). \end{aligned}$$ □

We now regard $D = \bigoplus D_{pq}$ and $E = \bigoplus E_{pq}$ with the morphisms i, j and k having bidegree $(1,-1)$, $(0,0)$[3] and $(-1,0)$ respectively (i.e. $i(D_{pq}) \subseteq D_{p+1,q-1}$ and so on). The bidegrees of i and k do not change by passage to the derived couple while the bidegree of $j^{(r)}$ in the r-th couple is $\deg(j^{(r)}) = \deg(j) - (r-1)\deg(i) = (0,0) - (r-1)(1,-1) = (-(r-1), r-1)$ since $j^{(r)}(i^{r-1}(x)) = [j(x)]$ and therefore the bidegree of the differential $d^r = j^{(r)}k^{(r)}$ is $\deg(d^r) = (-r, r-1)$ just as claimed in the definition of spectral sequences.
In this way the exact couple yields a spectral sequence with objects E^r_{pq}.

Example 2.2 (Exact couple of a filtered complex): Let C_* be a complex with a filtration

$$\ldots F_pC_* \subseteq F_{p+1}C_* \subseteq \ldots \subseteq C_*$$

such that there are integers $s < t$ for each n with $F_sC_n = 0$ and $F_tC_n = C_n$. (Such a filtration is called *bounded*. Particularly this means $F_kC_n = 0$ for all $k \leq s$ and $F_kC_n = C_n$ for all $k \geq t$. We will always consider *canonically bounded* filtrations, i.e. $s = -1$ and $F_nC_n = C_n$ for all n – what leads to a first quadrant spectral sequence.)
We get short exact sequences

$$0 \longrightarrow F_{p-1}C_* \xrightarrow{i} F_pC_* \xrightarrow{\pi_p} F_pC_*\big/F_{p-1}C_* \longrightarrow 0$$

and, by applying homology, long exact secquences

$$\begin{aligned} \ldots \longrightarrow H_{p-1,q+1}(F_{p-1}C_*) &\xrightarrow{i} H_{p+q}(F_pC_*) \\ &\xrightarrow{j} H_{p+q}\left(F_pC_*\big/F_{p-1}C_*\right) \xrightarrow{\delta} H_{p-1+q}(F_pC_*) \longrightarrow \ldots \end{aligned}$$

[3] To avoid confusion because of all the indices and variables we assume $a = 0$ – but j could also have bidegree $(-a, a)$. $a \neq 0$ would not be more difficult, but we would have to take care of it.

where i and j are the maps induces by i and π_p above and δ is the map delivered by the snake lemma.

Now we can roll up those long exact sequences into an exact triangle

$$\begin{tikzcd} \bigoplus H_{p+q}(F_pC_*) \arrow[rr, "i"] & & \bigoplus H_{p+q}(F_pC_*) \arrow[dl, "j"] \\ & \bigoplus H_{p+q}(F_pC_*/F_{p-1}C_*) \arrow[ul, "k"] & \end{tikzcd}$$

which gives us a spectral sequence E^r_{pq}.

From now on we concentrate on spectral sequences arising from an exact couple which complies the following two conditions: we require $D_{pq} = 0$ if $p < 0$ and for $q < 0$ we want $i\colon D_{p-1,q+1} \longrightarrow D_{pq}$ to be an isomorphism.

Remark: Those two facts guarantee that the spectral sequence lives in the first quadrant.

Proof: First let $p < 0$. Since $k\colon E_{pq} \longrightarrow D_{p-1,q} = 0$ has $\operatorname{im}(k) = 0$ we get $d = jk = 0$. Furthermore $0 = D_{pq} \xrightarrow{j} E_{pq}$ says that

$$E_{pq} = \ker(d)\big/\operatorname{im}(d) = \ker(d) = \operatorname{im}(j) = 0.$$

Now we look at the case $q < 0$. We have $d\colon E_{pq} \longrightarrow D_{p-1,q} \xrightarrow{j} D_{p-1,q}$ and we know that $i\colon D_{p-1,q} \xrightarrow{\sim} D_{p,q-1}$ is an isomorphism. But since $\operatorname{im}(k) = \ker(i) = 0$ this leads to $d = jk = 0$. Moreover we have $D_{p-1,q+1} \overset{i}{\xrightarrow{\sim}} D_{pq} \xrightarrow{j} E_{pq}$ and since $\ker(j) = \operatorname{im}(i) = D_{p-1,q+1}$ we get

$$E_{pq} = \ker(d) = \ker(k) = \operatorname{im}(j) = 0. \qquad \square$$

Let $Z^r_{pq} = \ker(d^r_{pq})$ and $B^r_{pq} = \operatorname{im}(d^r_{p+r,q-r+1})$ be the cycles and boundaries given by the fact that the r-sheet of the spectral sequence contains chain complexes.

Furthermore we set $H_n := \varinjlim D_{p,n-p}$ (the injective limit along the maps $i\colon D_{pq} \longrightarrow D_{p+1,q-1}$, i.e. the disjoint union of the $D_{p,n-p}$ where two elements are said to be equal if they are equal under compositions of i) and $F_pH_n := \operatorname{im}(D_{pq} \longrightarrow H_{p+q})$ by what we get a filtration

$$\ldots F_{p-1}H_n \subseteq F_pH_n \subseteq \ldots$$

since $i(D_{p-1,n-p-1}) \subseteq D_{p,n-p}$. This filtration is even canonically bounded: $D_{p,n-p} = 0$ for $p < 0$ causes $F_pH_n = 0$. If $p > n$ (which is equal to $q = n - p < 0$) we know that $D_p, n-p \cong D_{p+1,n-(p+1)} \cong \ldots$ and therefore $H_{p+q} = \operatorname{im}(D_{p,n-p} \longrightarrow H_{p+q}) = F_pH_n$. In particular the filtration satisfies $\bigcup_p F_pH_n = H_n = \varprojlim (H_n/F_pH_n)$ and $\bigcap_p F_pH_n = 0$. So, if a spectral sequence weakly converges to H_* it even converges to H_*.

Proposition 2.3: *There is a natural inclusion of $F_pH_n/F_{p-1}H_n$ in $E^\infty_{p,n-p}$. The spectral sequence E^r_{pq} weakly converges to H_* if and only if*

$$Z^\infty = \bigcap_r k^{-1}(i^r D) \text{ equals } k^{-1}(0) = j(D).$$

Proof: First we define K_{pq} to be the kernel of $D_{pq} \longrightarrow H_{p+q}$. Beeing an element of K_{pq} is equivalent to lying in the kernel of i^r for an integer r, so $K_{pq} = \bigcup_r \ker(i^r)$. It follows that $j(K_{pq}) = \bigcup_r j(\ker(i^r)) = \bigcup_r B^r_{pq} = B^\infty_{pq}$, as one can easily see that the infinite boundaries are the union of the B^r_{pq} (and similarly $Z^\infty_{pq} = \bigcap_r Z^r_{pq}$). We look at the following diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & K_{p-1,q+1} \arrow[r] \arrow[d, "i|_{K_{p-1,q+1}}"] & D_{p-q,q+1} \arrow[r] \arrow[d, "i"] & F_{p-1}H_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{pq} \arrow[r] \arrow[d] & D_{pq} \arrow[r] \arrow[d] & F_pH_n \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker}(i|_{K_{p-1,q+1}}) & \operatorname{coker}(i) & F_pH_n/F_{p-1}H_n &
\end{tikzcd}$$

with short exact sequences as rows.

Before applying the snake lemma we regard the cokernels.

$$D_{pq}\big/ i(D_{p-1,q+1}) = D_{pq}\big/\operatorname{im}(i) = D_{pq}\big/\ker(j)$$

tells us that $\operatorname{coker}(i) = j(D_{pq})$. Furthermore we have $\operatorname{im}(i|_{K_{p-1,q+1}}) = \ker(j|_{K_{pq}})$ (!) and therefore $\operatorname{coker}(i|_{K_{p-1,q+1}}) \cong j(K_{pq}) = B^\infty_{pq}$. The snake lemma now says that

$$0 \longrightarrow B^\infty_{pq} \longrightarrow j(D_{pq}) \longrightarrow F_pH_n\big/F_{p-1}H_n \longrightarrow 0$$

is a short exact sequence.

For all $r \in \mathbb{Z}$ we have

$$j(D_{pq}) = \operatorname{im}(j) = \ker(k) = k^{-1}(0) \subseteq k^{-1}(i^r D_{p-r-1,q+r}) = Z^r_{pq}$$

what means $j(D_{pq}) \subseteq \bigcap_r Z^r_{pq} = Z^\infty_{pq}$. We get a natural inclusion

$$F_pH_n\big/F_{p-q}H_n \cong j(D_{pq})\big/B^\infty_{pq} \subseteq Z^\infty_{pq}\big/B^\infty_{pq} = E^\infty_{pq}$$

and in particular we know now that the spectral sequences weakly converges to H_*, i.e. $F_pH_n/F_{p-1}H_n = E^\infty_{pq}$ if and only if $j(D) = Z^\infty$. □

Corollary 2.4 (Convergence Theorem): *A spectral sequence with the requirements mentioned above converges to $H_* = \varinjlim D$:*

$$E^r_{pq} \Rightarrow H_{p+q}.$$

If a spectral sequence arises from an exact couple of a complex C_* with canonically bounded filtration it converges to $\varinjlim D = H_*$ which is the homology of the filtered complex C_*.

Proof: A remark above tells us, that the spectral sequence converges to H_* if it weakly converges to H_*. All we have to show is that $Z^\infty = k^{-1}(0)$, if the spectral sequence complies the two facts that $D_{pq} = 0$ for $p < 0$ and $i\colon D_{p-1,q+1} \xrightarrow{\sim} D_{pq}$ for $q < 0$.

Let p and q be arbitrary integers. There exists an integer r such that $p - r - 1 < 0$ which implies $i^r(D_{p-r-1,q+r}) = i^r(0) = 0$. We now see that $Z^r_{pq} = k^{-1}(i^r(D_{p-r-1,q+r}) = k^{-1}(0)$. Then $Z^\infty_{pq} = \bigcap_{r'} Z^{r'}_{pq} = k^{-1}(0)$ and therefore $Z^\infty = k^{-1}(0)$.
That means $E^r_{pq} \Rightarrow H_{p+q}$.
Let $\dots F_{p-1}C_* \subseteq F_pC_* \subseteq \dots \subseteq C_*$ be a canonically bounded filtration of a complex C_*.
Since $F_pC_n = 0$ for all $p < 0$ we get $D_{pq} = H_{p+q}(F_pC_*) = 0$ for those p. For $q < 0$ we have $p = n - q > n$ and $F_pC_n = C_n$. This leads to $D_{pq} = H_n(F_pC_*) = H_n(C_*) = H_n(F_{p+1}C_*) = D_{p+1,q-1} = D_{p+1,q-2} \dots$. Alltogether it follows that a spectral sequence arising from a canonically bounded complex converges to H_*.
It remains to show hat H_* is the homology of C_*.
Let $p < n$; we then know that i is an isomorphism and therefore we get

$$H_{p+q} \cong \operatorname{im}(D_{p,n-p} \longrightarrow H_{p+q}) \cong D_{p,n-p} = H_{p+q}(F_pC_*) = H_{p+q}(C_*). \qquad \square$$

3 Double Complexes

In Example 2.2, we explained that a filtered complex yields an exact couple, which in turn yields a spectral sequence. We will now go one step further and introduce double complexes, which yield filtered complexes and finally spectral sequences.

Definition 3.1 (Double complex): A *double complex* C_{**} is a family $(C_{pq})_{p,q\in\mathbb{Z}}$ of objects of an abelian category together with maps

$$d^{\mathrm{h}}_{pq} \colon C_{pq} \longrightarrow C_{p-1,q}, \quad d^{\mathrm{v}}_{pq} \colon C_{pq} \longrightarrow C_{p,q-1}$$

such that for each q resp. p, the families $(C_{*q}, d^{\mathrm{h}}_{*q})$ resp. $(C_{p*}, d^{\mathrm{v}}_{p*})$ are chain complexes and additionally, in every square the relation

$$d^{\mathrm{v}}_{p-1,q} \circ d^{\mathrm{h}}_{pq} = \pm\, d^{\mathrm{h}}_{p,q-1} \circ d^{\mathrm{v}}_{pq}$$

is satisfied. In case of a "$+$" in this relation, we call the double complex *commutative* and in case of a "$-$" we call it *anticommutative*. We say that a double complex lives in the *first quadrant* if $C_{pq} = 0$ whenever $p < 0$ or $q < 0$.

We can picture a double complex like this (here we have a first quadrant double complex):

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
C_{02} \arrow[d] & C_{12} \arrow[l] \arrow[d] & C_{22} \arrow[l] \arrow[d] & \dots \arrow[l] \\
C_{01} \arrow[d] & C_{11} \arrow[l] \arrow[d] & C_{21} \arrow[l] \arrow[d] & \dots \arrow[l] \\
C_{00} & C_{10} \arrow[l] & C_{20} \arrow[l] & \dots . \arrow[l]
\end{tikzcd}$$

Definition 3.2: Let C_{**} be an anticommutative double complex. Then we define the *total complex* attached to C_{**} as the chain complex $\mathrm{Tot}_*(C_{**})$ with entries[4]

$$\mathrm{Tot}_n(C_{**}) = \bigoplus_{p+q=n} C_{pq}$$

and differentials $d = d^{\mathrm{v}} + d^{\mathrm{h}}$.

A straightforward calculation using anticommutativity shows $d^2 = 0$, so that the total complex is indeed a chain complex.

We can turn the total complex of a given double complex into a filtered chain complex by considering the following two obvious filtrations:

$$\text{“Filtration by rows”} \quad \begin{matrix} 0 & 0 & 0 & \dots \\ * & * & * & \dots \\ * & * & * & \dots \end{matrix} \qquad \text{“Filtration by columns”} \quad \begin{matrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \\ \vdots & \vdots & \vdots \end{matrix}$$

More precisely, we set

$$F_p^{(\mathrm{R})} \mathrm{Tot}_n(C_{**}) = \bigoplus_{i \leq p} C_{i,n-i}, \quad F_q^{(\mathrm{C})} \mathrm{Tot}_n(C_{**}) = \bigoplus_{j \leq q} C_{n-j,j}.$$

In both cases, we get

$$\left(F_p \mathrm{Tot}_*(C_{**}) \big/ F_{p-1} \mathrm{Tot}_*(C_{**}) \right)_n = C_{pq}$$

for the n-th entry of the filtration quotients (where $p + q = n$). Note that when the double complex lives in the first quadrant, then both filtrations are canonically bounded. Therefore we will assume from now on that every double complex lives in the first quadrant.

The machinery developped in the previous sections now yields two spectral sequences which we call ${}^{(\mathrm{R})}E^r_{**}$ and ${}^{(\mathrm{C})}E^r_{**}$, respectively. By Corollary 2.4 *both* spectral sequences converge to the homology of the total complex $\mathrm{Tot}_*(C_{**})$. Often one is not really interested in this homology, but one can play these sequences off against one another to obtain information about the entries of the spectral sequence.

Can we write down more explicitly how these spectral sequences look like? We have to apply the technique of exact couples to our situation. Doing so, we obtain:

Proposition 3.3: *The E^1 page of ${}^{(\mathrm{C})}E^r_{**}$ is obtained from the initial double complex C_{**} by taking homology in vertical direction, and the differentials d^1 on that page are the maps induced by the horizontal differentials d^{h} of C_{**} on this homology. Dually, the E^1 page of ${}^{(\mathrm{R})}E^r_{**}$ is obtained by “transposing” this procedure.*

[4] It is also common to consider a slightly differently defined total complex in which the direct sum is replaced by a direct product. As we will mostly consider double complexes living in the first quadrant, this does not make a difference for us.

Idea of proof: To prove this, we would have to go into the details of how the long homology sequence coming from the short exact sequence of chain complexes

$$0 \longrightarrow F^?_{p-1} \operatorname{Tot}_*(C_{**}) \longrightarrow F^?_p \operatorname{Tot}_*(C_{**}) \longrightarrow F^?_p \operatorname{Tot}_*(C_{**}) \Big/ F^?_{p-1} \operatorname{Tot}_*(C_{**}) \longrightarrow 0$$

is constructed, which in the end boils down to investigating the definition of the connecting homomorphism coming from the snake lemma. As this is a rather tedious calculation, we will omit this here. □

This proposition suggests to view the initial double complex C_{**} as the E^0 page of either spectral sequence (possibly "transposed").

Before giving some examples of applications of this construction, we record the following useful result.

Lemma 3.4: *If we have a converging spectral sequence*

$$E^r_{pq} \Rightarrow H_{p+q}$$

and the E^2 page consists of 0 everywhere except in the bottom row, then we have

$$E^2_{p0} = H_p.$$

Proof: This follows directly from Remark 1.4. □

Finally, we note that we can also apply this whole technique to a *commutative* double complex. We can turn any such double complex into an anticommutative one by inserting some -1's, for example at every second horizontal or vertical arrow: in this way in every square exactly one map is multiplied by -1. This does not affect the remaining properties of being a double complex and does also not change homology.

4 Examples

We conclude this note with a few examples that demonstrate the power of our techniques. The first examples are very simple, whereas the later ones are a bit more interesting.

4.1 The five lemma

As a first example we want to prove the five lemma. Given have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r, "g"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r, "f'"'] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r, "g'"'] & E' \end{tikzcd}$$

with exact rows, we want to prove:

(a) When β, δ are monomorphisms and α is an epimorphism, then γ is a monomorphism.

(b) When β, δ are epimorphisms and ε is a monomorphism, then γ is an epimorphism.

To get a double complex, we flip this diagram, insert kernels and cokernels on the left and right and fill everything else with 0 to obtain

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \ldots \arrow[l] \\
\operatorname{coker} g \arrow[d] & E \arrow[l] \arrow[d] & D \arrow[l] \arrow[d] & C \arrow[l] \arrow[d] & B \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & \ker f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \ldots \arrow[l] \\
\operatorname{coker} g' & E' \arrow[l] & D' \arrow[l] & C' \arrow[l] & B' \arrow[l] & A' \arrow[l] & \ker f' \arrow[l] & 0 \arrow[l] & \ldots . \arrow[l]
\end{tikzcd}$$

In the following, we will omit the 0's again and draw only the non-zero part.

From this, we see immediately that the E^1 page of ${}^{(R)}E^r_{**}$ consists only of 0: because to get this E^1 page, we have to take homology in horizontal direction, but as the rows are exact, this vanishes. So we conclude that ${}^{(R)}E^r_{**}$ converges to 0, and our theory then tells us that also ${}^{(C)}E^r_{**}$ must converge to 0. What does this imply?

Taking homology in vertical direction, we obtain the E^1 page of ${}^{(C)}E^r_{**}$:

$$\begin{tikzcd}
* & \ker \varepsilon \arrow[l] & \ker \delta \arrow[l] & \ker \gamma \arrow[l] & \ker \beta \arrow[l] & \ker \alpha \arrow[l] & * \arrow[l] \\
* & \operatorname{coker} \varepsilon \arrow[l] & \operatorname{coker} \delta \arrow[l] & \operatorname{coker} \gamma \arrow[l] & \operatorname{coker} \beta \arrow[l] & \operatorname{coker} \alpha \arrow[l] & *. \arrow[l]
\end{tikzcd}$$

Here we have written "$*$" to indicate any entry we are not interested in.

We now use the assumption from (a) to get $\ker \delta = \ker \beta = \operatorname{coker} \alpha = 0$. By taking homology again, we finally look at the E^2 page:

$$\begin{tikzcd}
* & * & * & \ker \gamma & * & * & * \\
* & * & * \arrow[ull] & * \arrow[ull] & * \arrow[ull] & 0 \arrow[ull] & * \arrow[ull]
\end{tikzcd}$$

As the spectral sequence converges to 0, we know that on the E^∞ page no entry can be left. But this means that $\ker \gamma$ has to vanish since otherwise it could never disappear on the following pages. This is just what we wanted to show in (a). The claim in (b) is shown similarly.

4.2 The snake lemma

The next application of our theory will be a proof of the snake lemma, which proceeds quite similar to the one of the five lemma. We start again with a commutative diagram

$$
\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}
$$

with exact rows, and as before we insert kernels, cokernels and zeros to obtain the E^0 page of our two spectral sequences:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l, leftrightarrow] \arrow[d] & \dots \arrow[l] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & C \arrow[l] \arrow[d] & B \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & \ker f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\
0 & \operatorname{coker} g' \arrow[l] & C' \arrow[l] & B' \arrow[l] & A' \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \dots . \arrow[l]
\end{tikzcd}
$$

As the rows are exact, horizontal homology vanishes and again, both spectral sequences converge to 0. Now taking homology in vertical direction yields the E^1 page of ${}^{(C)}E^r_{**}$:

$$
\begin{tikzcd}
0 & \ker \gamma \arrow[l] & \ker \beta \arrow[l, "\varphi"] & \ker \alpha \arrow[l] & \ker f \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 & \operatorname{coker} g' \arrow[l] & \operatorname{coker} \gamma \arrow[l] & \operatorname{coker} \beta \arrow[l] & \operatorname{coker} \alpha \arrow[l, "\psi"] & 0. \arrow[l]
\end{tikzcd}
$$

Here we see part of the snake lemma sequence in the two rows, and standard arguments show that these are exact. Finally we take homology again to look at the E^2 page:

$$
\begin{tikzcd}
0 & \operatorname{coker} \varphi & 0 & 0 & 0 \\
0 & 0 & 0 \arrow[ull] & \ker \psi \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}
$$

As on the E^∞ page everything must have vanished, this one remaining map must be an isomorphism. By inverting this isomorphism we can get a connecting homomorphism and obtain the snake lemma sequence:

$$
\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] & \ker \alpha \arrow[r] & \ker \beta \arrow[r, "\varphi"] & \ker \gamma \arrow[r, dashed] \arrow[d, two heads] & \operatorname{coker} \alpha \arrow[r, "\psi"] & \operatorname{coker} \beta \arrow[r] & \operatorname{coker} \gamma \arrow[r] & \operatorname{coker} g' \arrow[r] & 0. \\
& & & & \operatorname{coker} \varphi & \ker \psi \arrow[l, "\sim"'] \arrow[u, hook] & & & &
\end{tikzcd}
$$

By simple arguments one can prove that this is indeed exact.

4.3 Balancing Tor

The next application we want to consider is balancing Tor. So let R be a ring, A a right R-module and B a left R-module. Denoting by $\mathrm{Tor}_*(A,-)$ the derived functors of $A \otimes_R -$ and by $\mathrm{Tor}_*(-,B)$ the derived functors of $- \otimes_R B$, we want to show

$$\mathrm{Tor}_*(A,-)(B) = \mathrm{Tor}_*(-,B)(A)$$

using spectral sequences.

To get our machinery started, we need a double complex. We choose projective resolutions

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$
$$\dots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0$$

of A and B, respectively, and define the double complex C_{**} by

$$C_{pq} = P_p \otimes Q_q,$$

where the maps are the induced ones coming from the maps in the projective resolutions. Since projective modules are flat, the rows and columns of this double complex are indeed complexes, and the squares in the double complex are commutative.

Writing down the E^0 page of the column-filtered spectral sequence, we have

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
P_0 \otimes Q_2 & P_1 \otimes Q_2 & P_2 \otimes Q_2 & \dots \\
\downarrow & \downarrow & \downarrow & \\
P_0 \otimes Q_1 & P_1 \otimes Q_1 & P_2 \otimes Q_1 & \dots \\
\downarrow & \downarrow & \downarrow & \\
P_0 \otimes Q_0 & P_1 \otimes Q_0 & P_2 \otimes Q_0 & \dots.
\end{array}$$

Now since every P_p is projective (hence flat), the sequence

$$\dots \longrightarrow P_p \otimes Q_1 \longrightarrow P_p \otimes Q_0 \longrightarrow P_p \otimes B \longrightarrow 0$$

is exact, so we have

$$\mathrm{coker}(P_p \otimes Q_1 \longrightarrow P_p \otimes Q_0) = P_p \otimes B.$$

Therefore the E^1 page looks like this:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dots \\
P_0 \otimes B & \longleftarrow & P_1 \otimes B & \longleftarrow & P_2 \otimes B & \longleftarrow & \dots.
\end{array}$$

What we have here in the bottom row is exactly the complex used to calculate the derived functors of $- \otimes B$. So taking homology of this yields as E^2 page of ${}^{(C)}E^r_{**}$

$$\begin{tikzcd}[column sep=small]
0 & 0 & 0 & 0 & 0 \\
\mathrm{Tor}_0(-,B)(A) & \mathrm{Tor}_1(-,B)(A) & \mathrm{Tor}_2(-,B)(A) \arrow[ull] & \mathrm{Tor}_3(-,B)(A) \arrow[ull] & \mathrm{Tor}_4(-,B)(A). \arrow[ull]
\end{tikzcd}$$

In a totally analogous way one can see that the E^2 page of ${}^{(R)}E^r_{**}$ is

$$\begin{tikzcd}[column sep=small]
0 & 0 & 0 & 0 & 0 \\
\mathrm{Tor}_0(A,-)(B) & \mathrm{Tor}_1(A,-)(B) & \mathrm{Tor}_2(A,-)(B) \arrow[ull] & \mathrm{Tor}_3(A,-)(B) \arrow[ull] & \mathrm{Tor}_4(A,-)(B). \arrow[ull]
\end{tikzcd}$$

Now in both spectral sequences, we have an E^2 page whose only nonzero entries are in the bottom row. In this situation we can apply Lemma 3.4, which tells us that

$${}^{(C)}E^r_{pq} \Rightarrow \mathrm{Tor}_{p+q}(-,B)(A) \quad \text{and} \quad {}^{(R)}E^r_{pq} \Rightarrow \mathrm{Tor}_{p+q}(A,-)(B).$$

But as both spectral sequences converge to the same thing, we are done now.

4.4 Base change for Tor

We now consider the following situation: let R, S be rings, $f\colon R \longrightarrow S$ a ring homomorphism, A an R-module and B an S-module. By f we can also consider B as an R-module. As in the previous example, we consider again projective resolutions

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$
$$\ldots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0$$

of A as an R-module and B as an S-module, respectively, and form the double complex C_{**} given by

$$C_{pq} = P_p \otimes_R Q_q.$$

Here Q_q might not be projective as an R-module! But P_p is, so $P_p \otimes_R -$ is exact, and when we consider the spectral sequence ${}^{(C)}E^r_{**}$ filtered by columns, we see by the same argument as an the previous example that it converges to $\mathrm{Tor}^R_{p+q}(A,B)$. So we know that also ${}^{(C)}E^r_{**}$ converges to the same thing, but how does this spectral sequence look like explicitly?
The E^0 page looks like this:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_2 \otimes_R Q_0 \arrow[d] & P_2 \otimes_R Q_1 \arrow[d] & P_2 \otimes_R Q_2 \arrow[d] & \ldots \\
P_1 \otimes_R Q_0 \arrow[d] & P_1 \otimes_R Q_1 \arrow[d] & P_1 \otimes_R Q_2 \arrow[d] & \ldots \\
P_0 \otimes_R Q_0 & P_0 \otimes_R Q_1 & P_0 \otimes_R Q_2 & \ldots.
\end{tikzcd}$$

We now use

$$P_p \otimes_R Q_q = (P_p \otimes_R S) \otimes_S Q_q.$$

As $(P_p \otimes_R S)_p$ is just the complex used to calculate the derived functors of $- \otimes_R S$ and every Q_q is projective, hence $- \otimes_S Q_q$ is exact, we obtain the following E^1 page:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathrm{Tor}_2^R(A,S) \otimes_S Q_0 & \longleftarrow & \mathrm{Tor}_2^R(A,S) \otimes_S Q_1 & \longleftarrow & \mathrm{Tor}_2^R(A,S) \otimes_S Q_2 & \longleftarrow & \dots \\
\mathrm{Tor}_1^R(A,S) \otimes_S Q_0 & \longleftarrow & \mathrm{Tor}_1^R(A,S) \otimes_S Q_1 & \longleftarrow & \mathrm{Tor}_1^R(A,S) \otimes_S Q_2 & \longleftarrow & \dots \\
\mathrm{Tor}_0^R(A,S) \otimes_S Q_0 & \longleftarrow & \mathrm{Tor}_0^R(A,S) \otimes_S Q_1 & \longleftarrow & \mathrm{Tor}_0^R(A,S) \otimes_S Q_2 & \longleftarrow & \dots .
\end{array}$$

Now the complexes in these rows are the ones used to calculate the derived functors of $- \otimes_S B$, and this means that the pq-th entry on the E^2 page is $\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A,S), B)$. So we have proved:

Proposition 4.1: *There exists a spectral sequence*

$$E^2_{pq} = \mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A,S), B) \Rightarrow \mathrm{Tor}_{p+q}^R(A,B).$$

Why is this useful? Let us consider the special case that S is flat as an R-module. Then $\mathrm{Tor}_i^R(A,S)$ vanishes for $i > 0$, hence on the E^2 page of the spectral sequence we constructed, only the bottom row contains nonzero entries. Therefore we can again use Lemma 3.4, which gives us

$$\mathrm{Tor}_i^S(A \otimes_R S, B) = \mathrm{Tor}_i^R(A,B).$$

The requirement that S is a flat R-module is specifically fulfilled when G is a group, H is a subgroup, and we take $R = \mathbb{Z}[H]$ and $S = \mathbb{Z}[G]$. In this case, S is even R-free: one can use any system of coset representatives as a basis. In terms of group homology (setting $B = \mathbb{Z}$), the statement then reads

$$H_i(G, \mathrm{ind}_H^G A) = H_i(H, A).$$

This is just Shapiro's lemma.

4.5 The universal coefficients theorem

As a last application we want to prove a universal coefficients theorem. Let $(C_*, d_*) = (C_q, d_q)_q$ be a chain complex consisting of *free* abelian groups and A be any abelian group. In this situation, what can we say about the relation between $H_*(C_*)$ and $H_*(C_* \otimes A)$?

We choose a projective resolution

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

of A and consider the double complex $(P_p \otimes C_q)_{pq}$.

We first look at the spectral sequence coming from the filtration by columns. Its E^0 page is

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes C_2 \arrow[d] & P_1 \otimes C_2 \arrow[d] & P_2 \otimes C_2 \arrow[d] & \dots \\
P_0 \otimes C_1 \arrow[d] & P_1 \otimes C_1 \arrow[d] & P_2 \otimes C_1 \arrow[d] & \dots \\
P_0 \otimes C_0 & P_1 \otimes C_0 & P_2 \otimes C_0 & \dots.
\end{tikzcd}$$

Since $P_p \otimes -$ is exact, we get as E^1 page

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
P_0 \otimes H_2(C) & P_1 \otimes H_2(C) \arrow[l] & P_2 \otimes H_2(C) \arrow[l] & \dots \arrow[l] \\
P_0 \otimes H_1(C) & P_1 \otimes H_1(C) \arrow[l] & P_2 \otimes H_1(C) \arrow[l] & \dots \arrow[l] \\
P_0 \otimes H_0(C) & P_1 \otimes H_0(C) \arrow[l] & P_2 \otimes H_0(C) \arrow[l] & \dots. \arrow[l]
\end{tikzcd}$$

Now the rows here are the complexes used to calculate the derived functors of $A \otimes -$, so the pq-th entry on the E^2 page is $\mathrm{Tor}_q^{\mathbb{Z}}(A, H_p(C))$. We want to examine this more closely. Applying $A \otimes -$ to the short exact sequence of chain complexes

$$0 \longrightarrow \operatorname{im} d_* \longrightarrow \ker d_* \longrightarrow H_*(C_*) \longrightarrow 0$$

and deriving yields a long exact sequence

$$\begin{aligned}
\dots \longrightarrow \mathrm{Tor}_2(A, \operatorname{im} d_{n-1}) &\longrightarrow \mathrm{Tor}_2(A, \ker d_n) \longrightarrow \mathrm{Tor}_2(A, H_n(C)) \\
\xrightarrow{\partial} \mathrm{Tor}_1(A, \operatorname{im} d_{n-1}) &\longrightarrow \mathrm{Tor}_1(A, \ker d_n) \longrightarrow \mathrm{Tor}_1(A, H_n(C)) \\
\xrightarrow{\partial} \mathrm{Tor}_0(A, \operatorname{im} d_{n-1}) &\longrightarrow \dots
\end{aligned}$$

for every $n \in \mathbb{N}_0$. But as $\operatorname{im} d_{n-1}$ and $\ker d_n$ are subgroups of the *free* abelian group C_n, they are both free themselves, and therefore the higher Tor groups vanish. By the long exact sequence, this means that also $\mathrm{Tor}_i(A, H_n(C))$ vanishes for $i \geq 2$. Hence our E^2 page looks like this:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
\mathrm{Tor}_0(A, H_2(C)) & \mathrm{Tor}_1(A, H_2(C)) & 0 & 0 & \dots \\
\mathrm{Tor}_0(A, H_1(C)) & \mathrm{Tor}_1(A, H_1(C)) & 0 \arrow[ull] & 0 \arrow[ull] & \dots \\
\mathrm{Tor}_0(A, H_0(C)) & \mathrm{Tor}_1(A, H_0(C)) & 0 \arrow[ull] & 0 \arrow[ull] & \dots.
\end{tikzcd}$$

We now have a look at the other spectral sequence coming from the filtration by rows. From the E^0 page

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
P_2 \otimes C_0 & P_2 \otimes C_1 & P_2 \otimes C_2 & \dots \\
\downarrow & \downarrow & \downarrow & \\
P_1 \otimes C_0 & P_1 \otimes C_1 & P_1 \otimes C_2 & \dots \\
\downarrow & \downarrow & \downarrow & \\
P_0 \otimes C_0 & P_0 \otimes C_1 & P_0 \otimes C_2 & \dots
\end{array}$$

we as pq-th entry of the E^1 page $\mathrm{Tor}_q(A, C_p)$. For $q > 0$ this vanishes because every C_p is free, so the E^1 page looks like this:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dots \\
A \otimes C_0 & \longleftarrow & A \otimes C_1 & \longleftarrow & A \otimes C_2 & \longleftarrow & \dots .
\end{array}$$

Hence on the E^2 page, we have again that everything except the bottom row is 0, and in the bottom row we have

$$H_0(A \otimes C_*) \quad H_1(A \otimes C_*) \quad H_2(A \otimes C_*) \quad \dots .$$

So by Lemma 3.4, this (and hence both) spectral sequences converge to $H_*(A \otimes C_*)$. We now apply Remark 1.4 to the first spectral sequence. For every $n \in \mathbb{N}_0$ we have ${}^{(C)}E^2_{0n} = \mathrm{Tor}_0(A, H_n(C_*)) = H_n(C_*) \otimes A$ and ${}^{(C)}E^2_{1,n-1} = \mathrm{Tor}_1(H_{n-1}(C_*), A)$. So what we get is the following universal coefficients theorem that describes the relation between $H_*(C_*)$ and $H_*(C_* \otimes A)$ in terms of the latter Tor group:

Proposition 4.2: *There is a short exact sequence*

$$0 \longrightarrow H_n(C_*) \otimes A \longrightarrow H_n(C_* \otimes A) \longrightarrow \mathrm{Tor}_1(H_{n-1}(C), A) \longrightarrow 0.$$

References

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