Universidad de los Andes
Facultad de Ciencias
Departamento de Matemáticas

Teoría de Iwasawa

Ejercicios

Jueves, 16 de agosto

Part B
Structure theory of Iwasawa modules and Iwasawa's asymptotic formula

Exercise B.6. Let M be a finitely generated torsion module over $R = \mathbb{Z}_p[\![T]\!]$. In the lecture we showed that there is a pseudo-isomorphism

$$M \longrightarrow \bigoplus_{j=1}^{s} R\Big/p^{m_j} \oplus \bigoplus_{i=1}^{t} R\Big/F_i^{n_i}$$

where the F_j are irreduble distinguished polynomials, and we defined the characteristic polynomial of M as

$$F_M := p^{\mu(M)} \prod_{i=1}^{n_i} F_i^{n_i} \in \mathbb{Z}_p[T].$$

Let $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$, which is a finite-dimensional $\mathbb{Q}_p$-vector space, and let F_T be the characteristic polynomial of the multiplication by T map $v \longmapsto T \cdot v$ on V.
Show that $F_M = p^{\mu(M)} F_T$. (This explains the name "characteristic polynomial" for F_M.)

Part E
Cyclotomic units and Iwasawa's theorem

Exercise E.1. We constructed the p-adic zeta function $\boldsymbol{\zeta}_p$, which is a pseudo-measure on the Galois group $\mathcal{G}$ whose integrals are related to special values of the Riemann zeta function. Use this to show the following congruences for these special values:
Let $r \in \mathbb{N}$. Then there exists a $t \in \mathbb{N}$ (depending on r) such that for integers $m, n \geq 1$ that satisfy $m \equiv n \mod p^t(p-1)$, we have

$$(1 - p^{n-1})\zeta(1-n) \equiv (1 - p^{m-1})\zeta(1-m) \mod p^r.$$

These congruences are known as the *Kummer congruences.*
Hint: Show that for any $i \in \mathbb{Z}$ the function

$$\mathbb{Z}_p \longrightarrow \mathbb{Z}_p, \quad s \longmapsto \int_{\mathcal{G}} \langle \chi \rangle_1^i \, \langle \chi \rangle_2^s \, \mathrm{d}\boldsymbol{\zeta}_p$$

is well-defined and uniformly continuous. Here $\langle \cdot \rangle_1$ and $\langle \cdot \rangle_2$ denote the projections onto the first resp. second factor in

$$\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p).$$

Exercise E.2. Why is the group C_∞^1 a module over $\Lambda(\mathcal{G})$?

Exercise E.3. For this exercise use the following fact about complex functions. Let

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be a series with coefficients $a_n \in \mathbb{C}$ that converges for at least one $s \in \mathbb{C}$. For real numbers $t > 0$ set

$$f(t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-nt}$$

(this converges!) and assume that this function can be written as

$$f(t) = \sum_{n=-1}^{\infty} b_n \frac{t^n}{n!}, \quad t > 0$$

with coefficients $b_n \in \mathbb{C}$. Then the function L has a meromorphic continuation to all of $\mathbb{C}$ and

$$L(-n) = (-1)^n b_n$$

for all $n \in \mathbb{Z}$, $n \geq 0$.
Apply this to the Riemann zeta function to deduce that

$$\zeta(1-n) = -\frac{B_n}{n} \text{ for all } n \in \mathbb{N},$$

where B_n denotes the n-th Bernoulli number as in the lecture.
Why is $B_n \in \mathbb{Q}$ for all $n \geq 1$? Why is $B_n = 0$ for all odd $n \geq 3$?

Part F
Survey on the proof of the main conjecture and Euler systems

Exercise F.1. Consider the $\Lambda(\mathcal{G})$-module $M := \Lambda(\mathcal{G})/I(\mathcal{G})\zeta_p$, where ζ_p is the p-adic zeta function and $I(\mathcal{G})$ is the augmentation ideal. Show the equivalences

$$\begin{aligned} M \neq 0 &\iff \text{one of the numbers } \zeta(-n) \text{ for } n \geq 1 \text{ odd is divisible by } p \\ &\iff \text{one of the numbers } \zeta(-1), \zeta(-3), \ldots, \zeta(4-p) \text{ is divisible by } p. \end{aligned}$$

Hint: Use the idempotents for powers of the Teichmüller character.

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