Universidad de los Andes
Facultad de Ciencias
Departamento de Matemáticas

Teoría de Iwasawa

Ejercicios

Miércoles, 15 de agosto

Part A
Introduction to algebraic number theory and class field theory

Exercise A.10. For the following fields K, determine which primes are ramified and how the Frobenius automorphism at unramified primes looks like:
(a) $K = \mathbb{Q}(\sqrt{d})$ for $d \in \mathbb{Z}$ not a square;
(b) $K = \mathbb{Q}(\xi_n)$, where ξ_n is a primitive n-th root of unity, for any $n \in \mathbb{N}$.
Use the results to derive the quadratic reciprocity law

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{2}}$$

for odd primes p, q.

Exercise A.11. Let L/K be an extension of number fields of degree n. Then we have the norm map of ideals

$$\mathrm{N}_{L/K} \colon \operatorname{Div}(\mathcal{O}_L) \longrightarrow \operatorname{Div}(\mathcal{O}_K),$$

where $\operatorname{Div}(-)$ means the group a fractional ideals.
Show that for any ideals $I \subseteq \mathcal{O}_K$ we have

$$\mathrm{N}_{L/K}(I\mathcal{O}_L) = I^n.$$

Hint: Use that equality $\sum_i e_i f_i = n$, where e_i resp. f_i are the ramification resp. inertial degree for prime ideals as introduced in the lecture.

Exercise A.12. For $n \in \mathbb{N}$ let $\mathcal{F}_n = \mathbb{Q}(\mu_{p^{n+1}})$. Let A_n be the p-Sylow group of the ideal class group $\mathrm{Cl}(\mathcal{F}_n)$ of $\mathcal{F}_n$, L_n the p-Hilbert class field of $\mathcal{F}_n$ (i. e. the maximal unramified abelian p-extension of $\mathcal{F}_n$) and $X_n = \mathrm{Gal}(L_n/K_n)$. Then we have the Artin map

$$A_n \xrightarrow{\sim} X_n, \quad [\mathfrak{p}] \longmapsto \mathrm{Frob}_{\mathfrak{p}}. \tag{1}$$

We have an exact sequence

$$1 \longrightarrow \mathrm{Gal}(L_n/\mathcal{F}_n) \longrightarrow \mathrm{Gal}(L_n/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathcal{F}_n/\mathbb{Q}) \longrightarrow 1.$$

Find a canonical action of $\mathrm{Gal}(\mathcal{F}_n/\mathbb{Q})$ on $X_n = \mathrm{Gal}(L_n/\mathcal{F}_n)$ and note that we also have a natural action of$\mathrm{Gal}(\mathcal{F}_n/\mathbb{Q})$ on $A_n = \mathrm{Cl}(\mathcal{F}_n)_p$. Show that the Artin map is equivariant for these actions.

Part C

Local Units, Coleman's construction, higher logarithmic derivatives

Let $R = \mathbb{Z}_p[\![T]\!]$.

Exercise C.7. Check the relations between operators on R mentioned in the lecture:

(a) $D \circ \varphi = p\,\varphi \circ D$;

(b) $D \circ \psi = \frac{1}{p}\psi \circ D$;

(c) $D \circ \sigma = \chi(\sigma)\sigma \circ D$ for $\sigma \in \mathcal{G}$.

Exercise C.8. Check that the map

$$\Delta\colon R^{\times} \longrightarrow R, \quad f \longmapsto (1+T)\frac{f'(T)}{f(T)}$$

is a group homomorphism. Show that it maps $W = (R^{\times})^{\mathcal{N}=1}$ into $R^{\psi=1}$.

Exercise C.9. Later in the lectures, we want to construct a p-adic ζ-function, which will be given by a measure on $\mathcal{G}$, i. e. an element in the Iwasawa algebra $\Lambda(\mathcal{G})$. This measure will arise from the norm-compatible system $\mathbf{u}$ of cyclotomic units from exercise C.5.

Have a look at the various maps we studied and find a way to obtain such a measure from $\mathbf{u}$, i. e. find a map

$$\mathcal{U}_\infty \longrightarrow \Lambda(\mathcal{G}).$$

What do we know about this map?