Universidad de los Andes
Facultad de Ciencias
Departamento de Matemáticas

Teoría de Iwasawa

Ejercicios

Martes, 14 de agosto

Part A
Introduction to algebraic number theory and class field theory

Exercise A.6. Let L/K be a finite extension of fields. For $a \in L$ the multiplication

$$L \longrightarrow L, \quad x \longmapsto ax$$

is K-linear and hence has a well-defined trace and determinant. The determinant is called *norm* in this case, and we denote trace and norm by

$$\mathrm{Tr}_{L/K}(a), \quad \mathrm{N}_{L/K}(a).$$

Fix an algebraic closure $\overline{K}$ of K and let $\sigma_1, \dots, \sigma_n$ be the K-linear embeddings of L into $\overline{K}$. Show that

$$\mathrm{Tr}_{L/K}(a) = \sum_{i=1}^{n} \sigma(a), \quad \mathrm{N}_{L/K}(a) = \prod_{i=1}^{n} \sigma(a).$$

In particular, $\mathrm{N}_{L/K}(x) = x^n$ if $x \in K$.

Exercise A.7. Let K be a number field with ring of integers $\mathcal{O}_K$ and $I \subseteq \mathcal{O}_K$ an ideal. In the lecture the norm of I was defined as $\mathrm{N}(I) = \#(\mathcal{O}_K/I)$.

(a) Show that if I is a principal ideal, $I = a\mathcal{O}_K$, then $\mathrm{N}(I) = \left|\mathrm{N}_{K/\mathbb{Q}}(a)\right|$.

(b) Show that the norm is multiplicative, i. e. $\mathrm{N}(IJ) = \mathrm{N}(I)\,\mathrm{N}(J)$ for any ideals $I, J \subset \mathcal{O}_K$.

Exercise A.8. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$.

(a) Let K be a finite extension of $\mathbb{Q}_p$, $\alpha, \beta \in \overline{\mathbb{Q}}_p$ and let $\alpha_2, \dots, \alpha_n$ be the other zeroes (apart from α) of the minimal polynomial of α over K. Assume that

$$\forall i \in \{2, \dots, n\}\colon \ |\beta - \alpha| < |\alpha_i - \alpha| \, .$$

Show that then $K(\alpha) \subseteq K(\beta)$.

This statement is known as *Krasner's lemma.*

(b) Show that $\overline{\mathbb{Q}}_p$. is not topologically complete.

(c) Show that the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ is algebraically closed.

Exercise A.9. Show that for two different primes $p \neq \ell$ the fields $\mathbb{Q}_p$ and $\mathbb{Q}_\ell$ are not isomorphic to each other (as abstract fields).

Part B
Structure theory of Iwasawa modules and Iwasawa's asymptotic formula

Exercise B.5. (a) Show that a pseudo-null module is torsion.
(b) Show that over a Dedekind ring a module is pseudo-null if and only if it is trivial.

Part C
Local Units, Coleman's construction, higher logarithmic derivatives

As in the lecture we use the following notation:

- $R = \mathbb{Z}_p[\![T]\!]$;
- $(\zeta_n)_{n\geq 0}$ is a compatible system of p-power roots of unity, i. e. ζ_n is a primitive p^{n+1}-th root of unity and $\zeta_{n+1}^p = \zeta_n$ for all $n \geq 0$;
- $\pi_n = \zeta_n - 1$ for $n \geq 0$;
- $\mathcal{K}_n = \mathbb{Q}_p(\zeta_n)$, $\mathcal{O}_n$ is the ring of integers in $\mathcal{K}_n$ and $\mathcal{U}_n = \mathcal{O}_n^\times$ for all $n \geq 0$;
- $\mathcal{U}_\infty = \varprojlim_n \mathcal{U}_n$;
- $W = \{f \in R^\times : \mathcal{N}(f) = f\}$.

Exercise C.4. In the lecture we constructed for each $\mathbf{u} \in \mathcal{U}_\infty$ an interpolating power series $f_{\mathbf{u}} \in R$. Show that this defines an isomorphism of $\mathcal{G}$-modules

$$\mathcal{U}_\infty \xrightarrow{\sim} W, \quad \mathbf{u} \longmapsto f_{\mathbf{u}}.$$

Exercise C.5. The aim of this exercise is to give an example for a Coleman power series. Let $a, b \in \mathbb{Z}$ be nonzero and prime to p. For $n \geq 0$ define

$$u_n = \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}}.$$

Show that $u_n \in \mathcal{U}_n$ and that $\mathbf{u} \coloneqq (u_n)_{n\geq 0} \in \mathcal{U}_\infty$. Find an explicit power series $f_{\mathbf{u}} \in R$ such that $f_{\mathbf{u}}(\pi_n) = u_n$ for all $n \geq 0$. (*Hint:* Consider $((1+T)^{-k/2} - (1+T)^{k/2})/T$ for $k \in \mathbb{Z}$).