Universidad de los Andes
Facultad de Ciencias
Departamento de Matemáticas

Teoría de Iwasawa

Ejercicios

Lunes, 13 de agosto

Part A
Introduction to algebraic number theory and class field theory

Exercise A.1. (a) Show that $\mathbb{Z}_p$ is a principal ideal domain and that each ideal is generated by a power of p.
(b) Show that $\mathbb{Q}_p$ is the quotient field of $\mathbb{Z}_p$.

Exercise A.2. (a) Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{Q}_p$. Show that the series $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{Q}_p$ if and only if $\lim_{n\to\infty} a_n = 0$.
(b) Show that any $x \in \mathbb{Z}_p$ can be written uniquely as

$$x = \sum_{n=0}^{\infty} b_n p^n$$

with $b_n \in \{0, \ldots, p-1\}$. What about $\mathbb{Q}_p$?
(c) Show that $\mathbb{Z}_p$ is uncountable.

Exercise A.3. (a) Show that

$$\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n \cong \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n \mid \forall i \geq j \colon x_i \bmod p^j = x_j \right\}.$$

(b) Show that there is a canonical isomorphism

$$\mathbb{Z}_p \cong \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n .$$

Exercise A.4. Show that $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$.

Exercise A.5. Let $K = \mathbb{Q}(\zeta_p)$, where ζ_p is a primitive p-th root of unity.
(a) Show that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$.
(b) Describe the isomorphism class of the unit group $\mathcal{O}_K^\times$.

Part B
Structure theory of Iwasawa modules and Iwasawa's asymptotic formula

As in the lectures, let K be a finite extension of $\mathbb{Q}_p$ with ring of integers R and maximal ideal $\mathfrak{m}$.

Exercise B.1. Show that the minimal prime ideals in $R[\![T]\!]$ are $\mathfrak{m}R[\![T]\!]$ and $F \cdot R[\![T]\!]$ for irreducible distinguished polynomials F.

Exercise B.2. Let G be a profinite group and let $\Lambda(G)$ be the corresponding Iwasawa algebra. Let S be any profinite[1] R-algebra and let $\varphi\colon G \longrightarrow S^\times$ be a continuous group homomorphism. Show that φ extends uniquely to a continuous homomorphism of R-algebras

$$\varphi^*\colon \Lambda(G) \longrightarrow S.$$

Exercise B.3. Fix a profinite group G. Let M be a profinite R-module on which G acts continuously and R-linearly. Show that M is in a unique way a compact $\Lambda(G)$-module.

Exercise B.4. Let A be a commutative ring and G a finite group such that $\#G \in A^\times$. For any homomorphism $\chi\colon G \longrightarrow A^\times$ define

$$e_\chi = \frac{1}{\#G}\sum_{g\in G} \chi(g)^{-1} g \in A[G].$$

Show the following equalities:

(a) $e_\chi^2 = e_\chi$ for any χ;
(b) $e_\chi g = \chi(g) e_\chi$ for any χ and $g \in G$;
(c) $e_\chi e_\psi = 0$ if $\chi \neq \psi$;
(d) $\sum_\chi e_\chi = 1$, where the sum runs over all possible χ.

Let now M be an $A[G]$-module. For any $\chi\colon G \longrightarrow A^\times$ we write

$$M[\chi] = \{m \in M | \forall g \in G\colon gm = \chi(g)m\}.$$

(e) Show that $M[\chi] = e_\chi M$.
(f) Conclude that $M = \bigoplus_\chi M[\chi]$, where again the sum runs over all possible χ.

Part C
Local Units, Coleman's construction, higher logarithmic derivatives

As in the lecture we write $R = \mathbb{Z}_p[\![T]\!]$.

Exercise C.1. Let $a \in \mathbb{Z}_p^\times$ and $n \in \mathbb{N}$. Show that $\binom{a}{n} \in \mathbb{Z}_p$.

Exercise C.2. Check that the endomorphisms of R defined by φ, ψ, $\mathcal{N}$ and $\sigma \in \mathcal{G}$ are continuous. Show that φ, ψ and $\mathcal{N}$ commute with the action of $\mathcal{G}$.

Exercise C.3. Show that $\sum_{\xi\in\mu_n} \xi f$ is divisible by p for any $f \in R$.

[1] A profinite R-algebra is a topological R-algebra which can be written as a limit of finite R-algebras.

Part D
Measures and Iwasawa-algebras

Let G be a profinite group and K a complete non-archimedean field with ring of integers R. Denote by $D(G, R)$ the R-module of integral measures.

Exercise D.1. In the lecture a canonical isomorphism $\mathrm{D}(G, R) \cong \Lambda(G)$ of R-modules was constructed. Show that it is in fact an isomorphism of R-*algebras* if we define the multiplication on $\mathrm{D}(G, R)$ by convolution, i. e.

$$\int_G f \,\mathrm{d}(\mu_1 * \mu_2) = \int_G \left(\int_G f(gh) \mathrm{d}\,\mu_1(g) \right) \mathrm{d}\,\mu_2(h) \text{ for } f \in \mathrm{C}^\infty(G, R),\ \mu_1,\ \mu_2 \in D(G, R).$$

Exercise D.2. Let $\varphi\colon G \longrightarrow R^\times$ be a continuous homomorphism. Then by Exercise B.2 we have an induced homomorphism

$$\varphi^* : \Lambda(G) \longrightarrow R.$$

On the other hand, we have an isomorphism $\mathrm{D}(G, R) \cong \Lambda(G)$. Show that the composition

$$\mathrm{D}(G, R) \xrightarrow{\sim} \Lambda(G) \xrightarrow{\varphi^*} R$$

is given by

$$\mu \longmapsto \int_G \varphi \,\mathrm{d}\mu.$$

Michael Fütterer, mfuetterer@mathi.uni-heidelberg.de