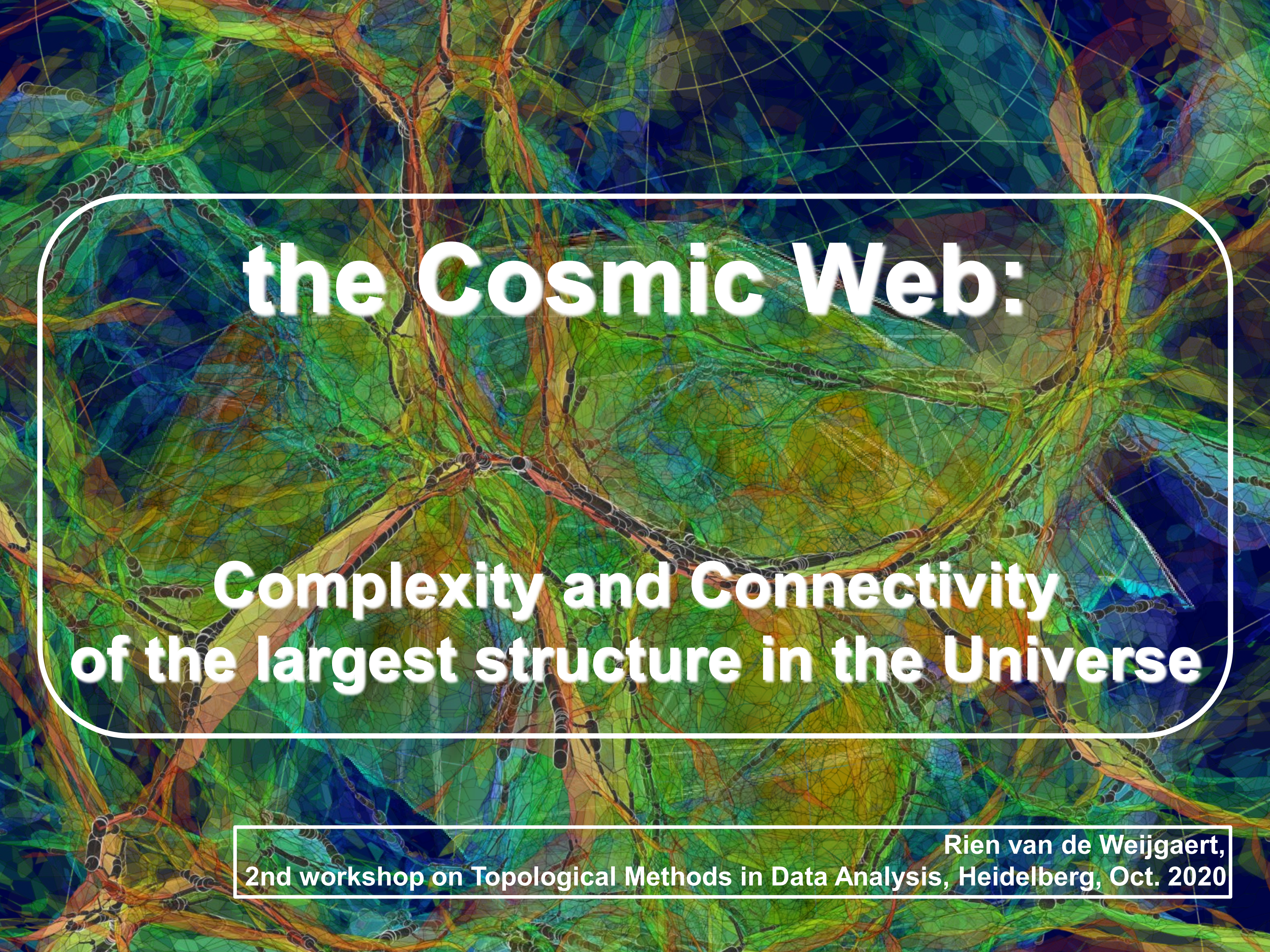
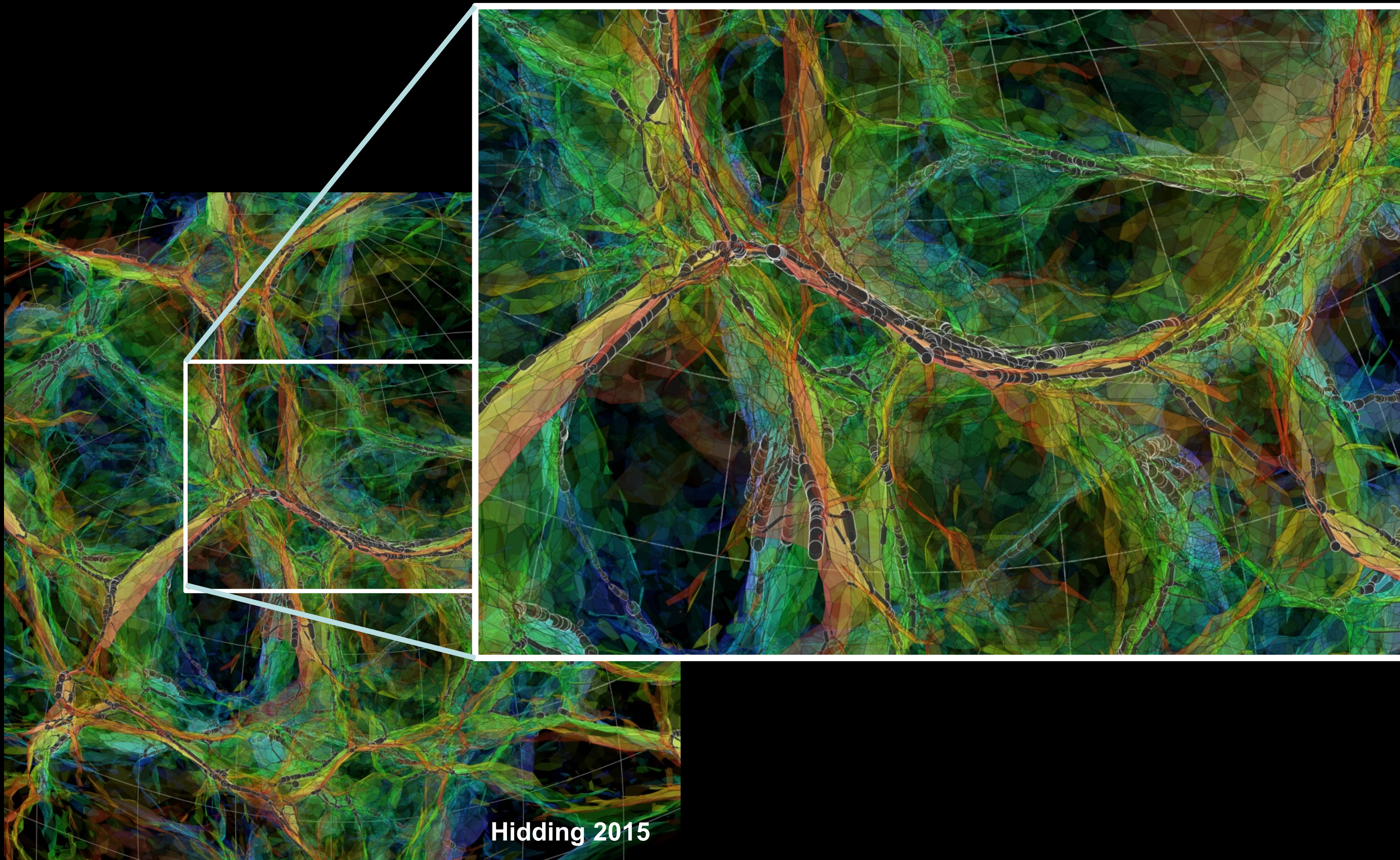


the Cosmic Web:

Complexity and Connectivity of the largest structure in the Universe

Rien van de Weijgaert,
2nd workshop on Topological Methods in Data Analysis, Heidelberg, Oct. 2020

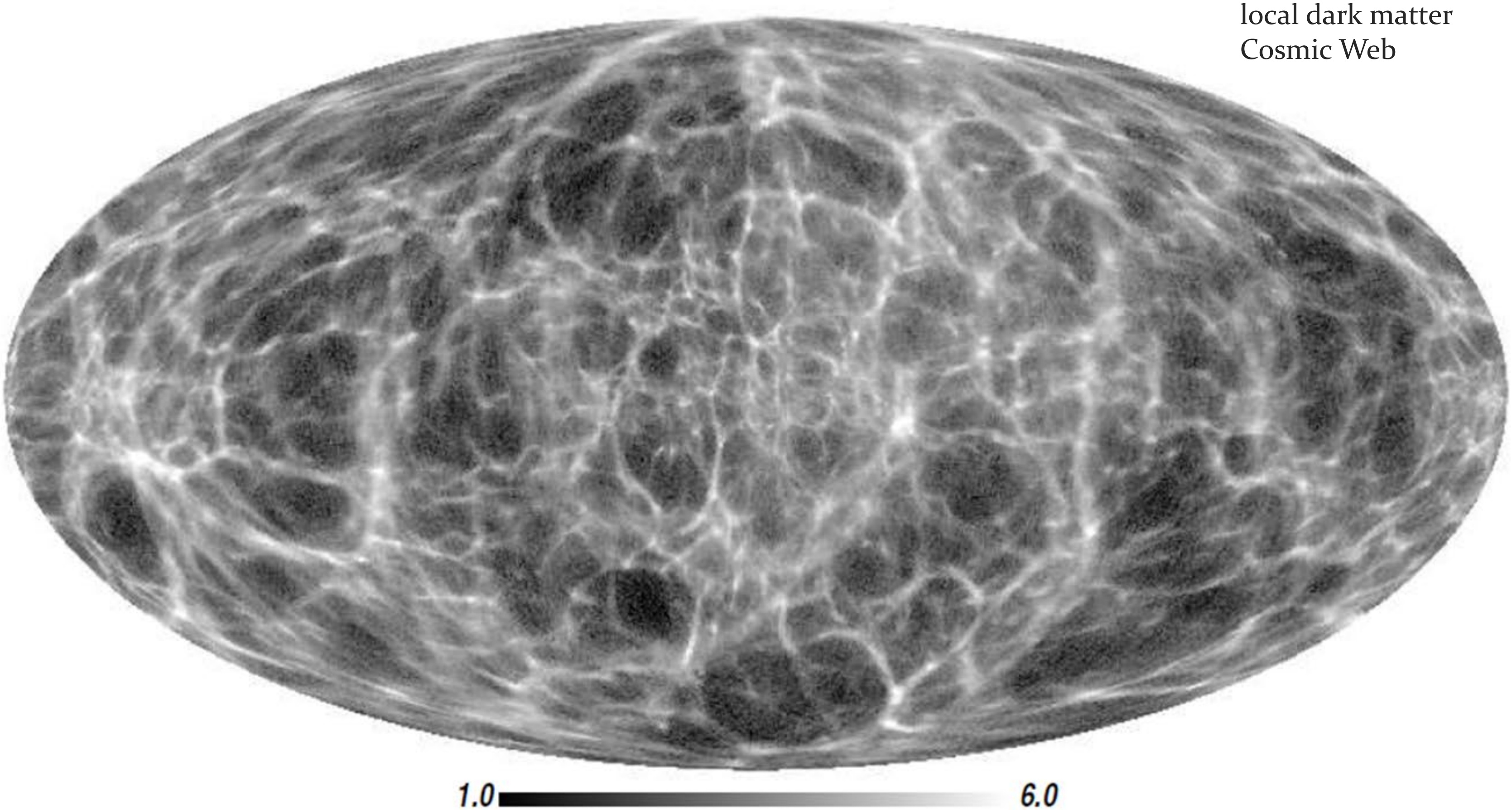
Pisces-Perseus Supercluster



local Cosmic Web: 2MRS

most detailed reconstruction
of the

local dark matter
Cosmic Web



Courtesy: Francisco Kitaura

S. Rieder 2014

on scales of ~5-100s Mpc

complex weblike pattern

in which

matter, gas & galaxies

aggregate in

- compact clusters,
- elongated filaments
- flattened sheets

around

- cosmic voids

Cosmic Web

The Cosmic Web

Physical Significance:

- **Manifestation mildly nonlinear clustering:**
  Transition stage between linear phase
  and fully collapsed/virialized objects

- **Weblike configurations contain**
  cosmological information:
  eg. *Void shapes & Alignments*

- **Cosmic environment within which to understand**
  the formation of galaxies.

Cosmic Web:

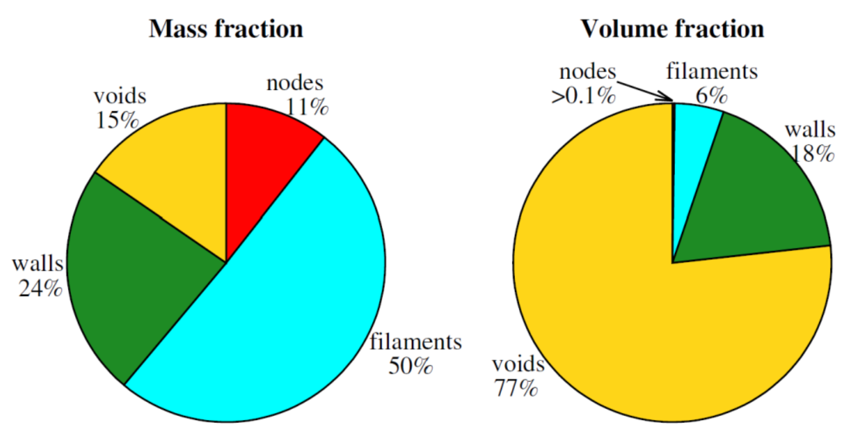
Density-Morphology Connection

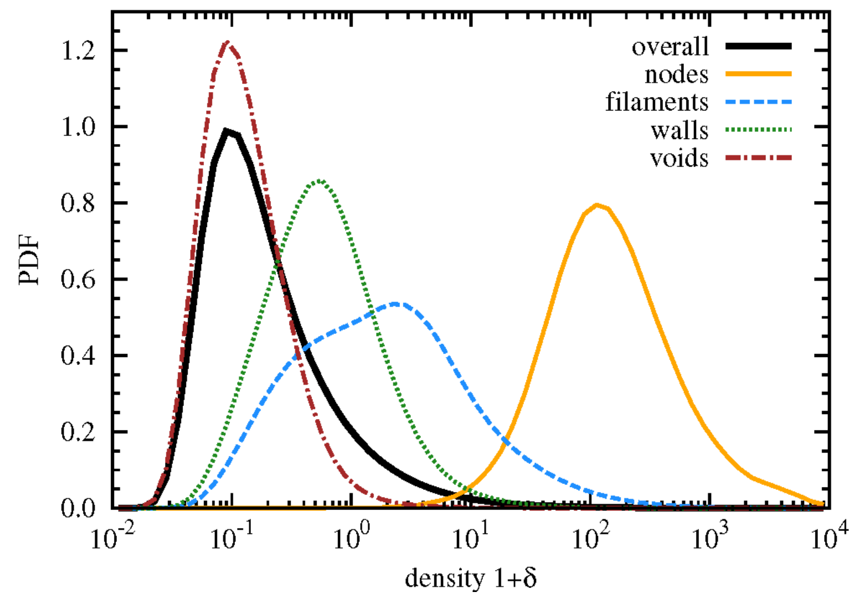
Mass & Volume content

Web morphologies

Density distribution

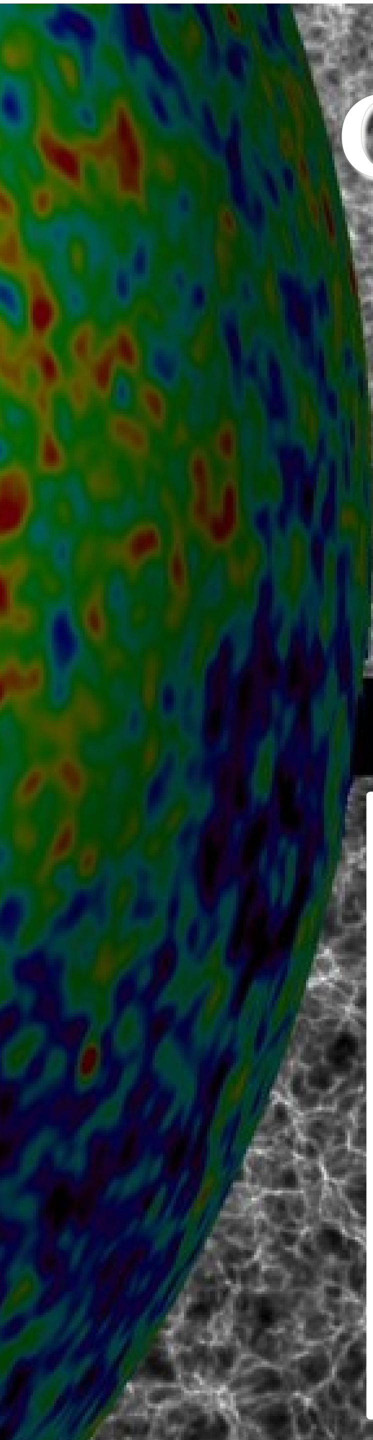
Individual morphologies





Cautun et al. 2014

Cosmic
Structure Formation



Cosmic Structure Formation

M. Aragon-Calvo

- Gravitationally driven
  growth & amplification

- Primordial Gaussian density & velocity fluctuations

- first nonlinear stage formation process:

  - contraction & collapse into walls and filaments

  - connecting at compact high-density nodes

  - expansion & evacuation low-density regions

- Cosmic Web: first cosmic structure
  emerging from primordial universe

Cosmic Web

Setting the Scene

Cosmic Web Characteristics

- **anisotropic structure:**

  - filaments dominant structural feature - elongated
  - sheets/walls - flattened

- **multiscale nature**

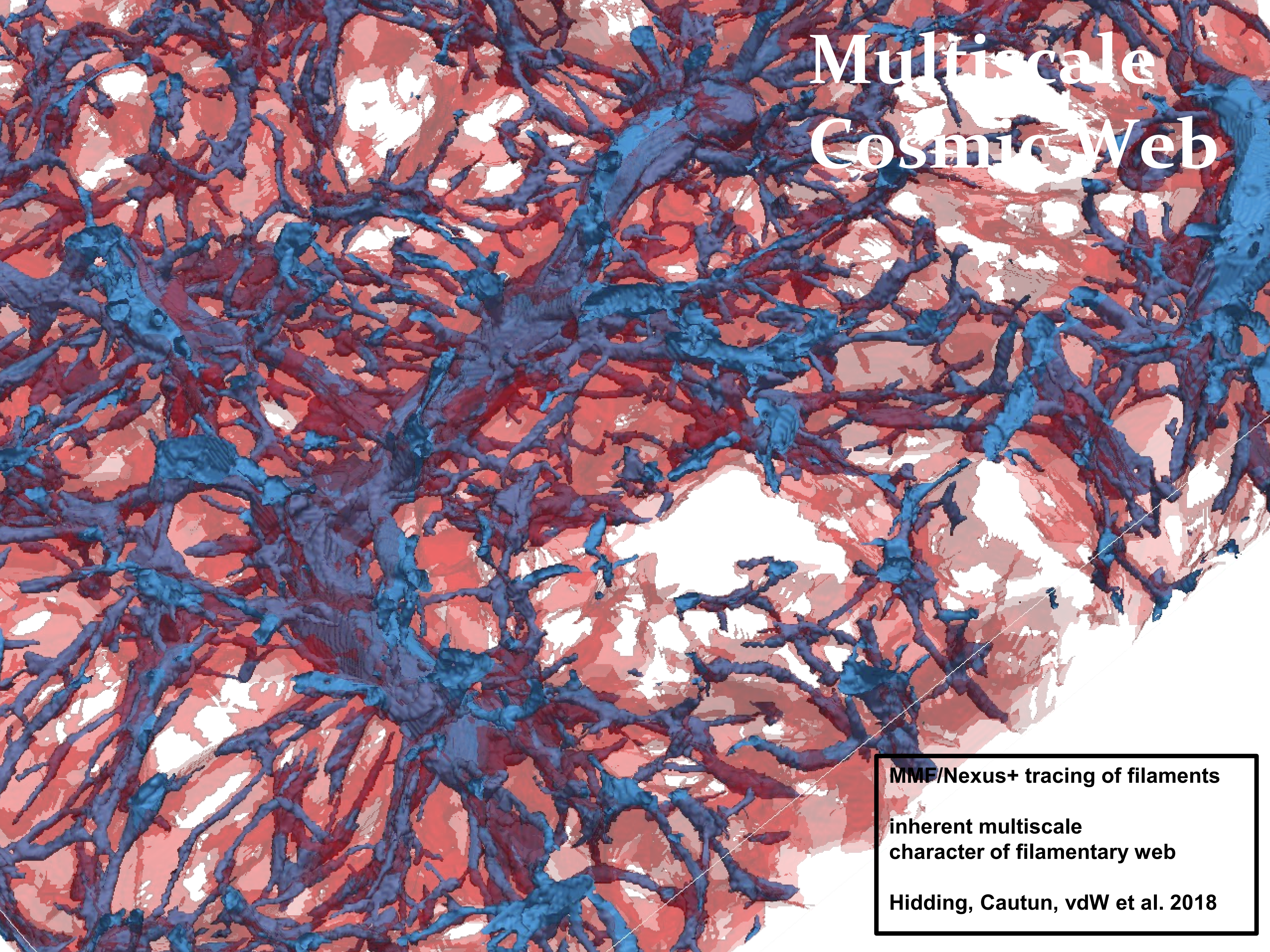
  - structure on wide range of scales
  - structures have wide range of densities

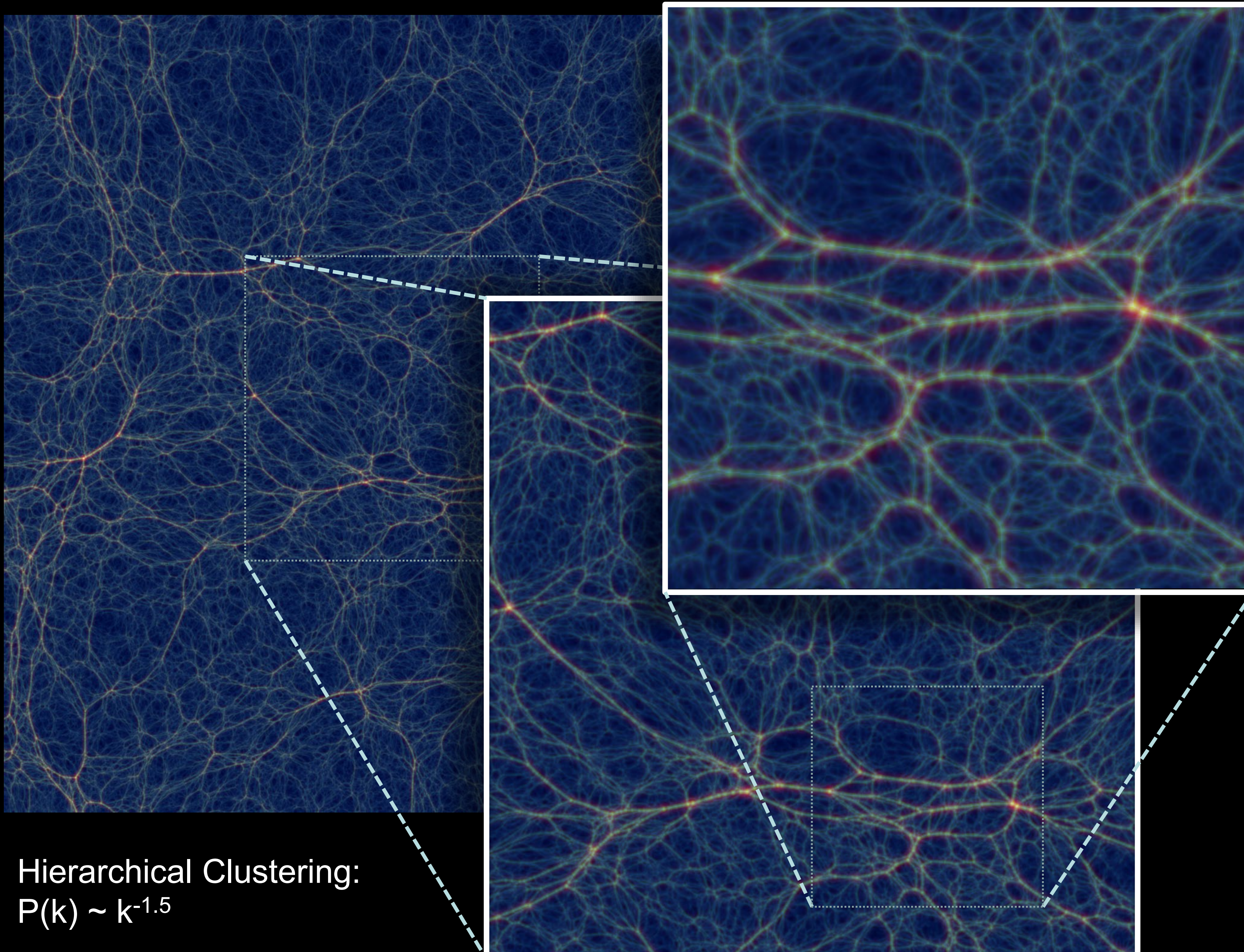
- **overdense-underdense asymmetry**

  - voids: underdense, large & roundish
  - filaments & walls: overdense, flattened/elongated
  - clusters: dense, massive & compact nodes

- **complex spatial connectivity**

  - all structural features connected in a complex, multiscale weblike network


Multiscale
Cosmic Web
MMF/Nexus+ tracing of filaments
inherent multiscale
character of filamentary web
Hidding, Cautun, vdW et al. 2018

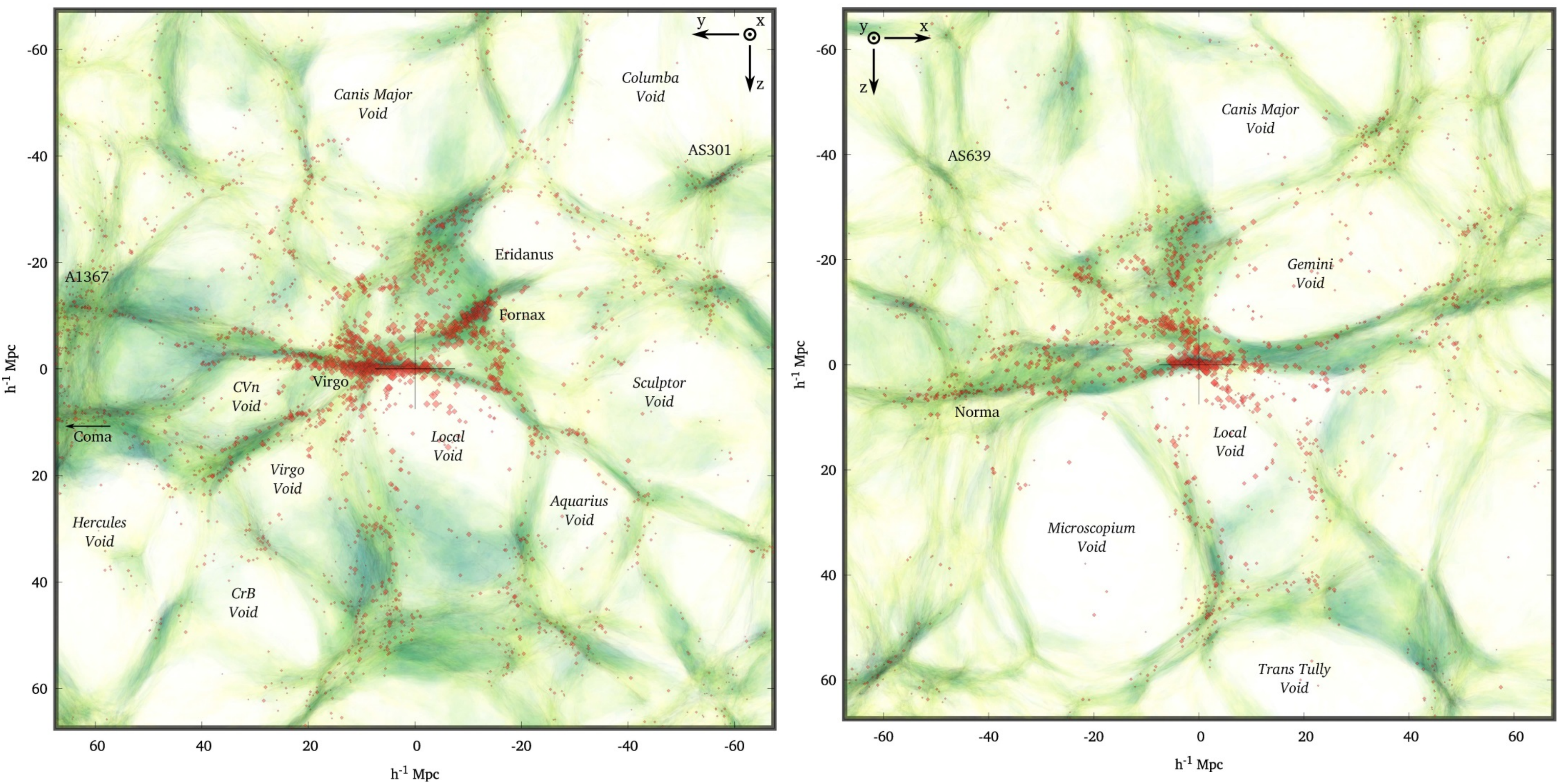


Hierarchical Clustering:
$P(k) \sim k^{-1.5}$

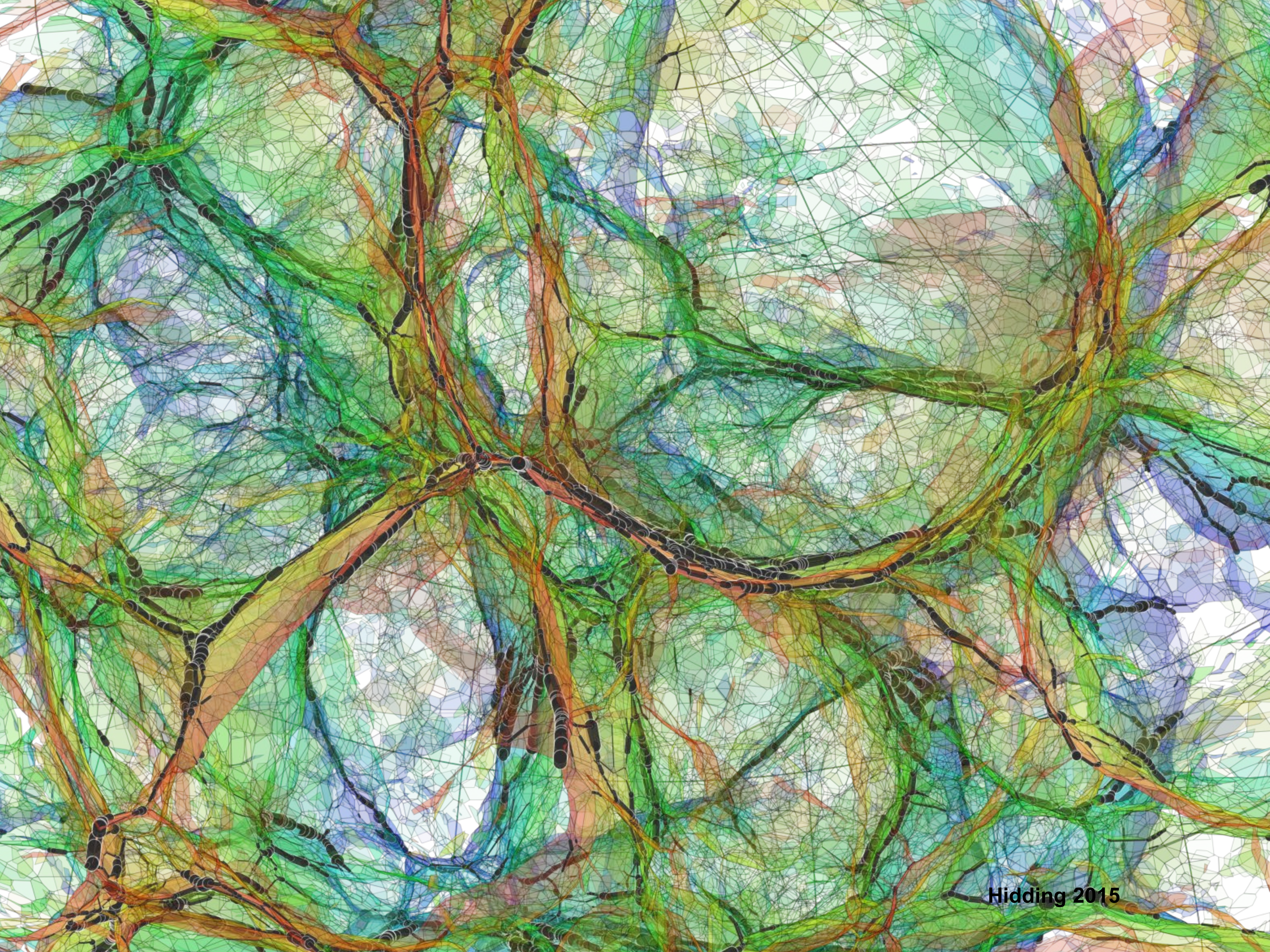
Void Population Local Universe

mean KIGEN-adhesion reconstruction (2MRS)

Hidding, Kitaura, vdW & Hess 2016/2017



Cosmic Web Connectivity


Hidding 2015

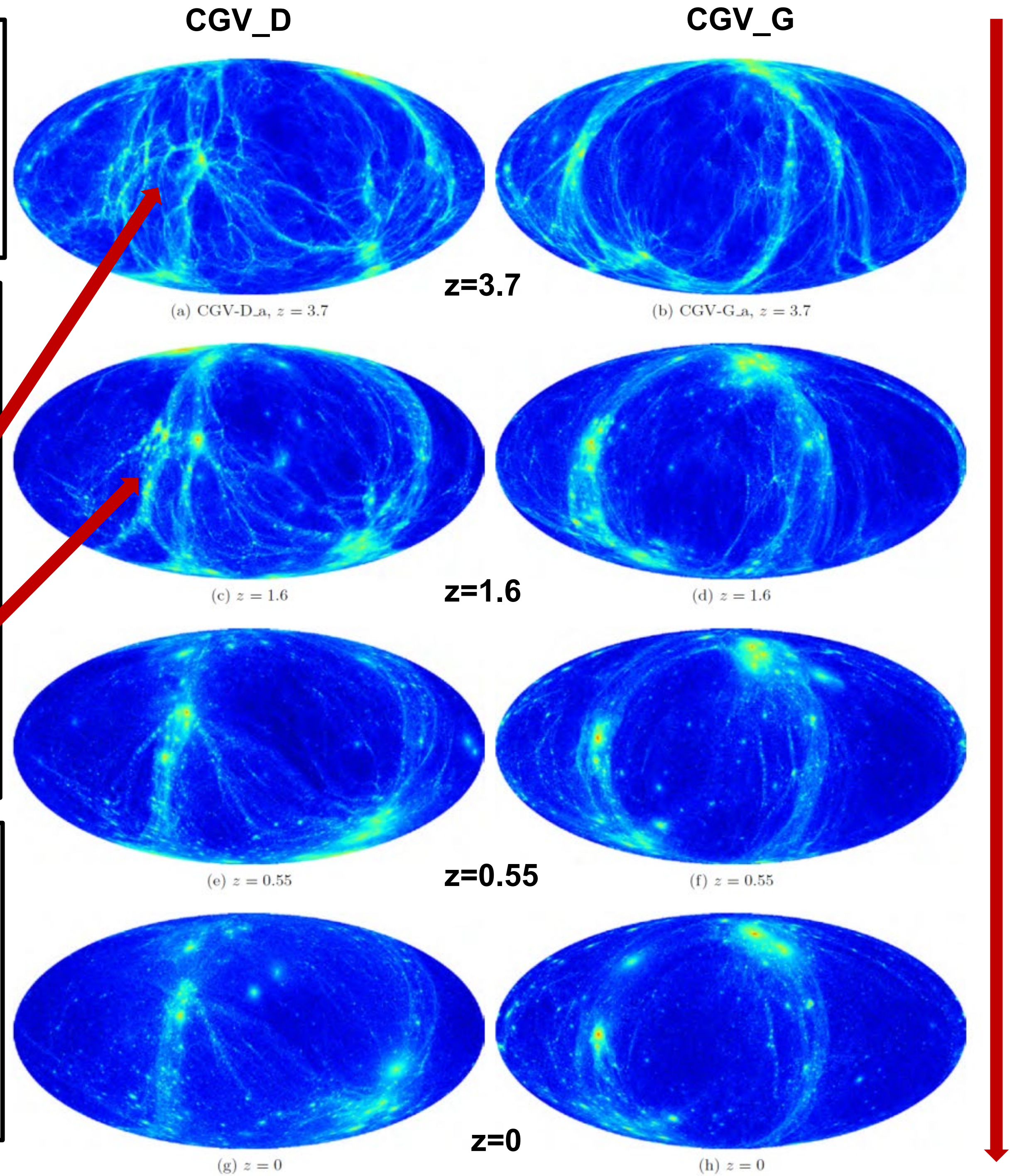
CGV:

on walls & filaments

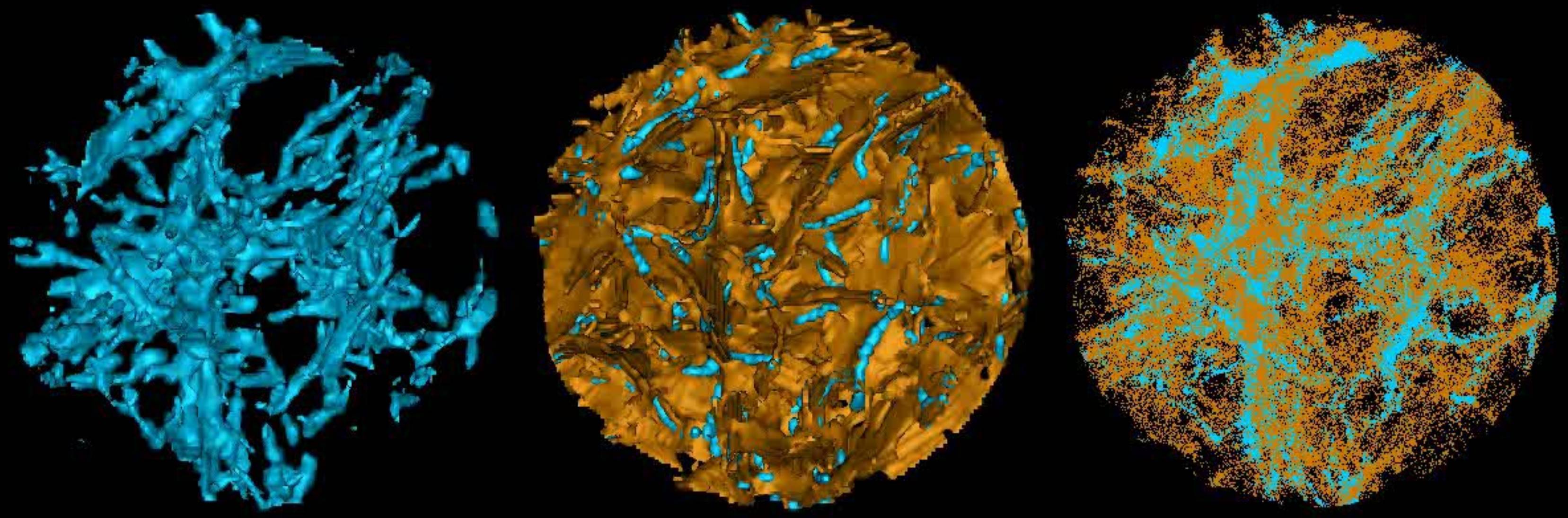
- Mollweide sky projection
  matter distribution around CGV halos
- CGV halos embedded in walls
- Walls dominate void infrastructure
- substantial fraction in filaments
  (embedded in walls)
- active dynamical evolution of
  wall-filament goes along with
  active void galaxy halo evolution

merging system of
Intravoid walls

Rieder et al. 2013



The Cosmic Web



MMF/Nexus
Cautun et al. 2013, 2014

Stochastic Spatial Pattern

- Clusters,
- Filaments &
- Walls

around

- Voids

in which matter & galaxies

have agglomerated

through gravity

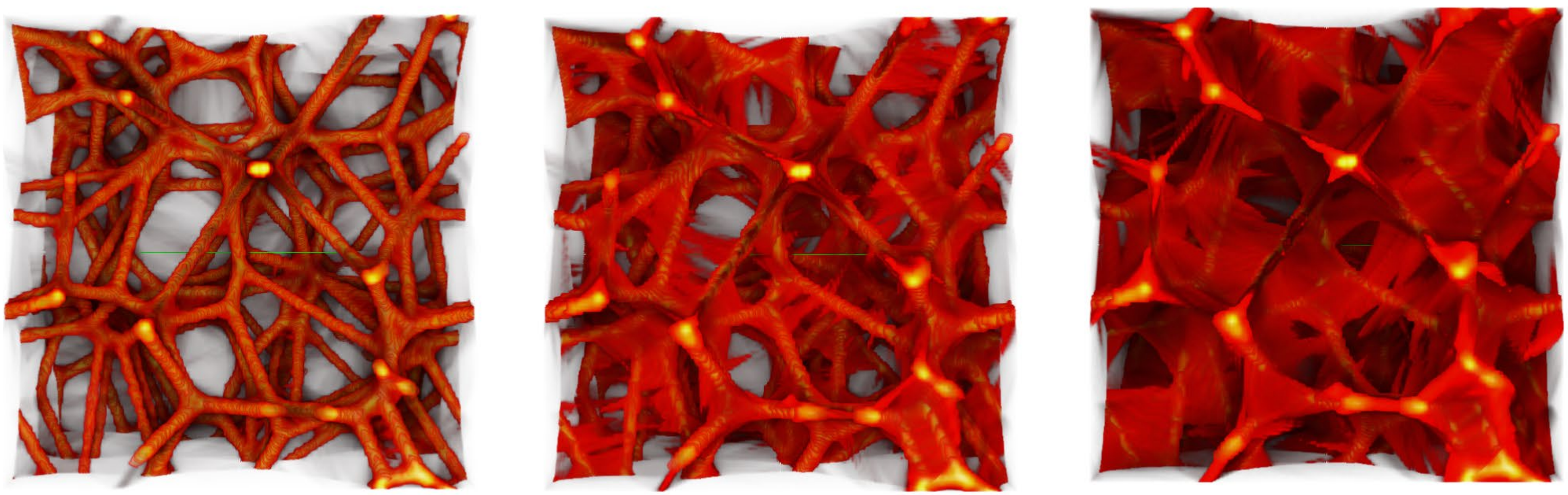
Multiscale Topology
&
Filtrations

Topological Hierarchy: Excursion Sets & Filtrations

Superlevel Sets

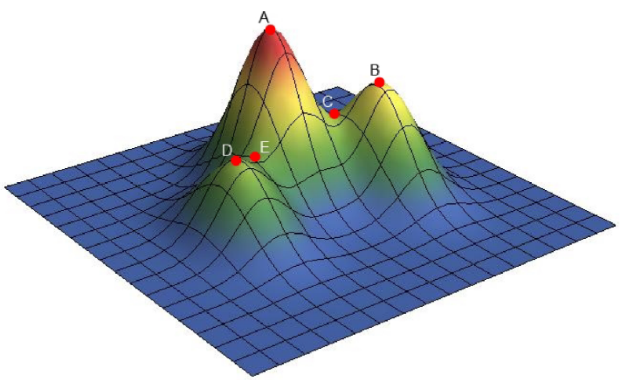
$$\mathfrak{M}_\nu = \left\{ \vec{x} \in \mathfrak{M} \mid f_s\left(\vec{x}\right) \in [f_\nu, \infty) \right\}$$

$$= f_s^{-1}\left[f_\nu, \infty\right)$$

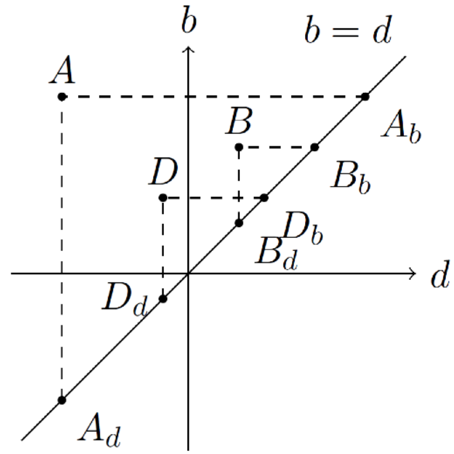


Pranav et al. 2013a

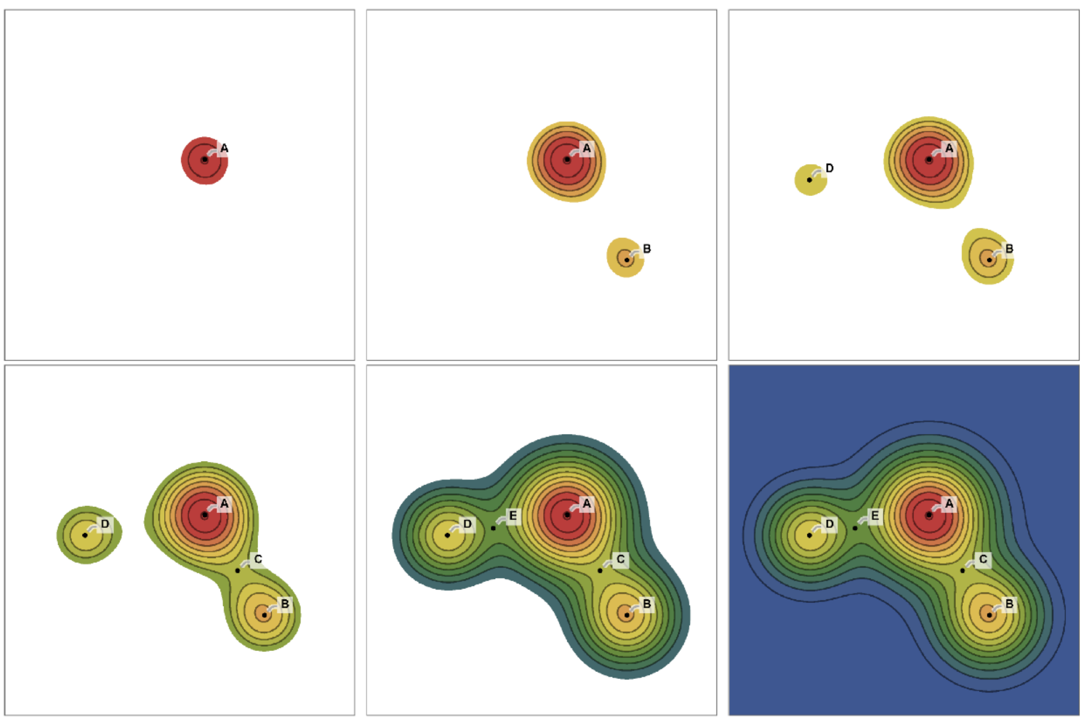
Topological Hierarchy: Persistence



(a) Realization of random field.



(b) Sketch of persistence diagram.



Persistence Diagram

Hierarchy
Field Singularities

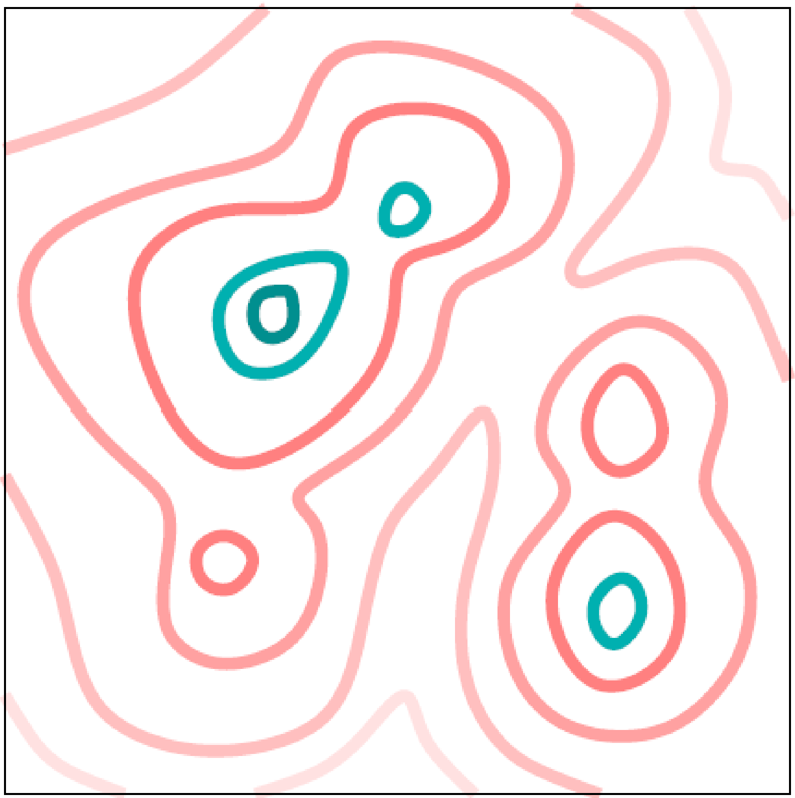
Persistent Pairs:

- **Birth level singularity**
- **Death level singularity**

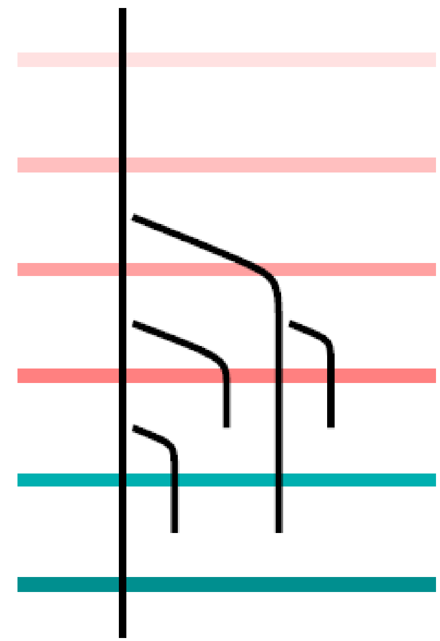
Feldbrugge, van Engelen, vdW et al. 2020

Topological Hierarchy

Persistent Homology:
“Cycling” over field filtration



Topology Tree



Edelsbrunner & Harer 2010

field value filtration
tree hierarchy

Topology
&
Geometry

Geometry & Topology

- Conventional Cosmological Topology Measure:

(Reduced) Genus

- # holes - # connected regions
- (Gott et al. 1986; Hamilton et al. 1986; Choi et al. 2010)

- Complete quantitative characterization of local geometry in terms of

Minkowski Functionals

- Minkowski Functionals:
  - Volume
  - Surface area
  - Integrated mean curvature
  - Genus/Euler Characteristic
- (Mecke, Buchert & Wagner 1994)

Genus, Euler & Betti

- For a surface with c components, the genus G specifies # handles on surface, and is related to the Euler characteristic χ via:

$$G = c - \frac{1}{2}\chi(\partial M)$$

where, according to the Gauss-Bonnet theorem

$$\chi(\partial M) = \frac{1}{2\pi}\oint\left(\frac{1}{R_1 R_2}\right) dS$$

- **Euler characteristic 3-D manifold & 2-D boundary manifold:**

$$\chi(M) = \frac{1}{2}\chi(\partial M)$$

$$\Downarrow$$

$$\chi(\partial M) = 2(\beta_0 - \beta_1 + \beta_2)$$

the Rule of Euler

from: Robert Adler

SIMPLICIAL TOPOLOGY

Simplices, complexes,
cycles, numbers of simplices,
Betti numbers

$$\sum_k (-1)^k \#\{k\text{-dimensional simplices}\}$$

INTEGRAL GEOMETRY

Convexity, convex ring
kinematic formulae
Minkowski functionals

$$\mathcal{M}_k(M) = c_{dk} \int_{\mathrm{Graff}(d,d-k)} \chi(M \cap V)\, d\mu^d_{d-k}(V)$$

$$\chi$$

ALGEBRAIC TOPOLOGY

$$\sum_k (-1)^k \beta_k$$

Homology, homotopy,
dimensions of groups,
Betti numbers, persistence

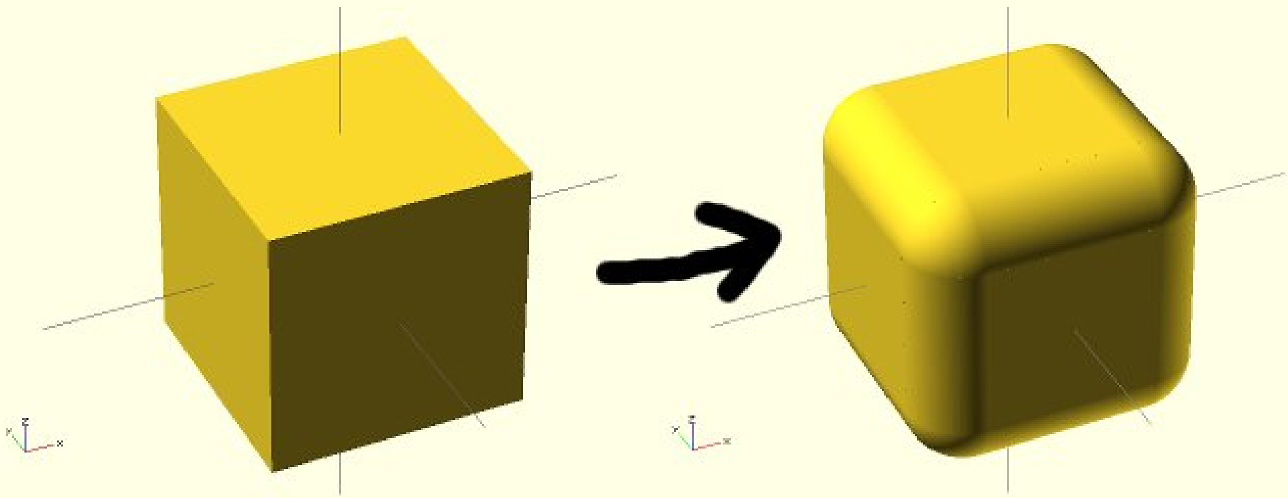
DIFFERENTIAL TOPOLOGY

$$\sum_k (-1)^k \#\{\text{critical points of index } k\}$$

$$\int_M \mathrm{Tr}(R^{m/2}) \mathrm{Vol}_g$$

Curvature, forms, Betti numbers,
Morse theory, integration,
Lipschitz-Killing curvatures

Minkowski functionals



- **Weyl's Tube formula:**

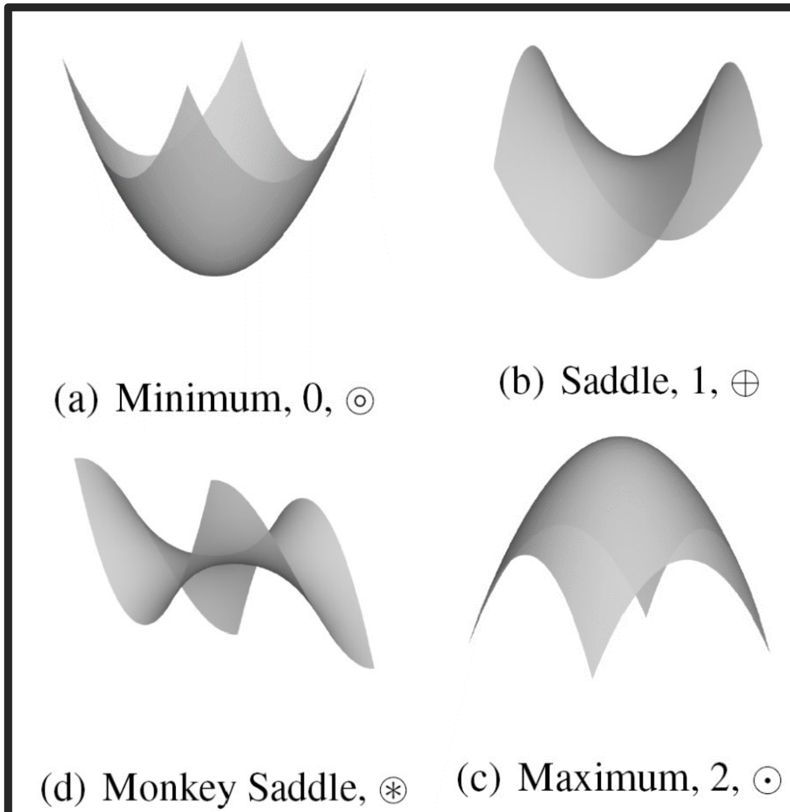
Minkowski functionals Q_k are the parameters specifying the contribution of volumes r^k to the volume of a cube M^r with rounded edges of radius r:

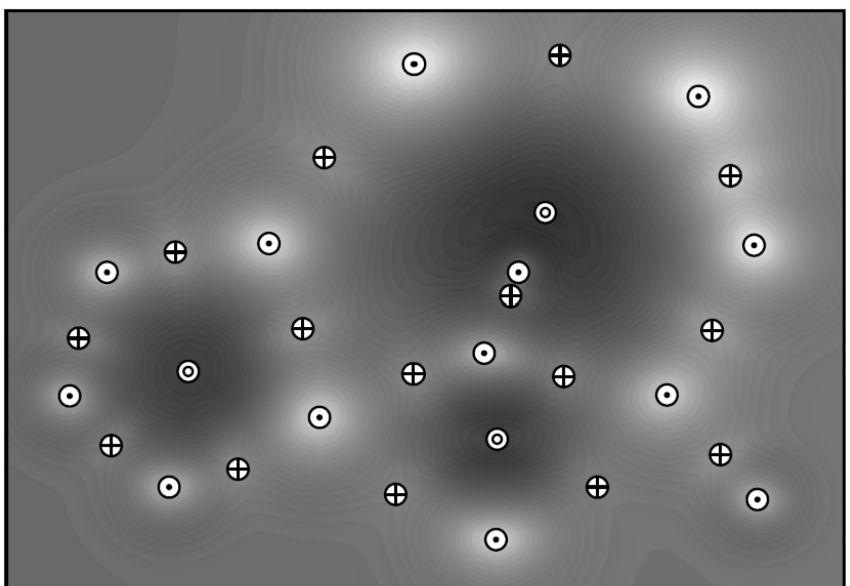
$$\mathrm{Vol}\left(M^r\right) = Q_0 + Q_1 r + Q_2 r^2 + Q_3 r^3$$

Euler & Morse

Number of singularities in field determines Euler characteristic:

- ζ_0 : **minima**
- ζ_1 : **saddle 1**
- ζ_2 : **saddle 2**
- ζ_3 : **maxima**





$$\chi = \sum_{k=0}^{d} (-1)^k \zeta_k$$

Euler-Poincare

Euler Characteristic χ is alternating sum of Betti Numbers

3-manifold □:

$$\chi(M) = \beta_0 - \beta_1 + \beta_2 + \beta_3$$

$$\approx \beta_0 - \beta_1 + \beta_2$$

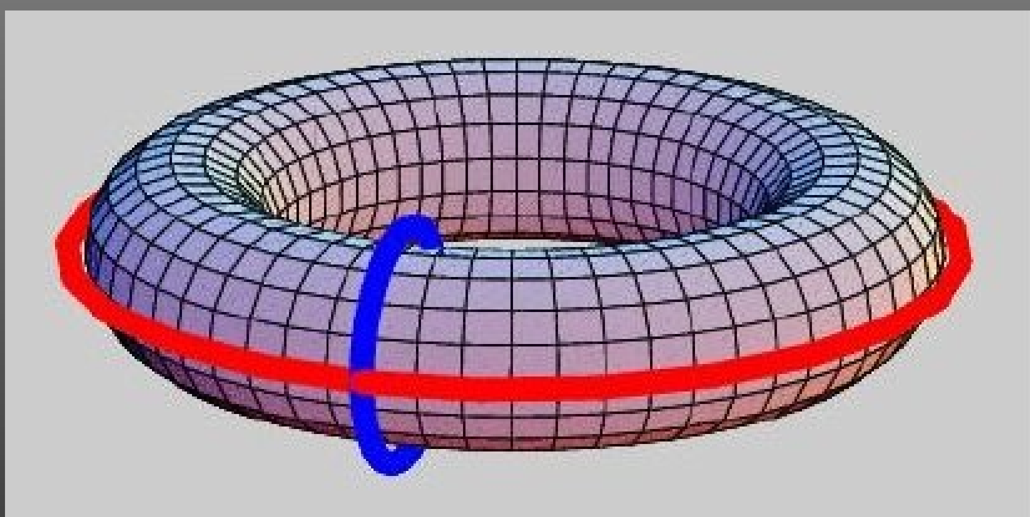
boundary 2-manifold □□:

$$\chi(\partial M) = \beta_{0b} - \beta_{1b} + \beta_{2b}$$

Betti Numbers - Homology Groups

Topology:
Study of connectivity and spatial relations that remain invariant under homeomorphisms (= continuous mapping between two topological objects)

Homology:
Description of topology of a space in terms of the relationship between cycles and boundaries.



Torus:	one	0-cycle:	rank group H_0:	1
	two	1-cycles:	rank group H_1:	2
	one	2-cycle	rank group H_2:	1

p-chain: sum of p-simplices
p-cycle: boundary of (p+1) chain

0-cycle: closed component
1-cycle: closed loop of edges, or finite union
2-cycle: closed surface, or finite union

adding two p-cycles $\longrightarrow$ p-cycle

Group of p-cycles:

Homology group H

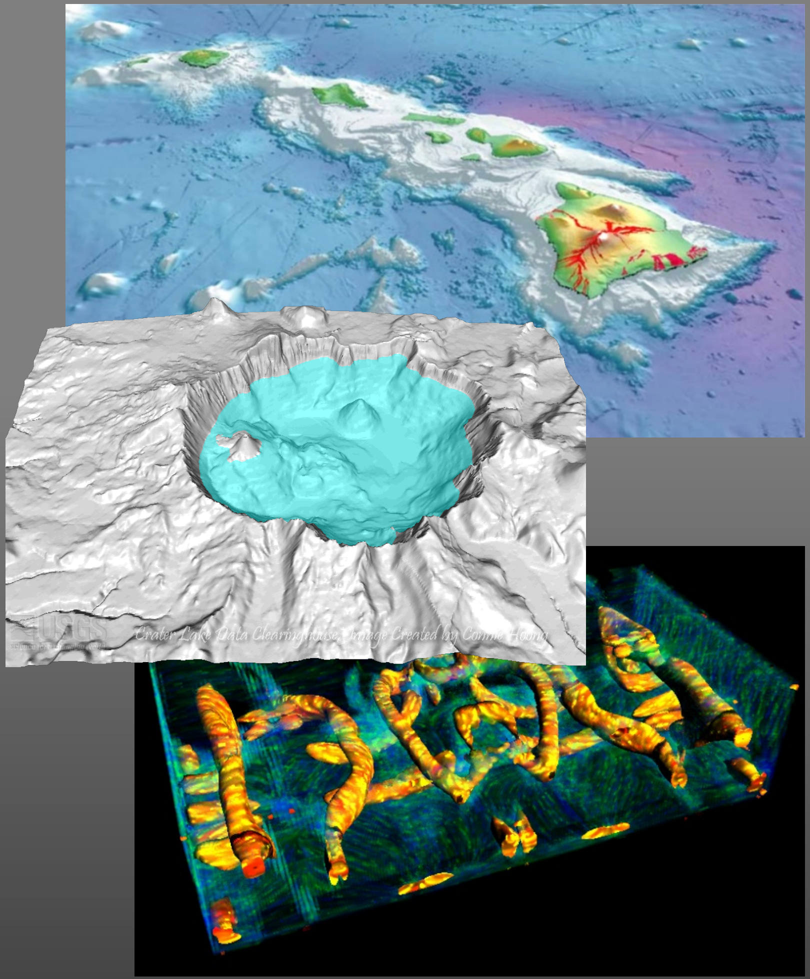
Betti Numbers - Cosmological View

Homology:

focus on structural features & holes

⇩

relating to surrounding spaces/structures



Holes different Dimension

0-dim: connected components
"islands" connected mass clumps

Superclusters

1-dim: Loops

Filaments & Tunnels

2-dim: Closed surface,

Voids

Betti Numbers:

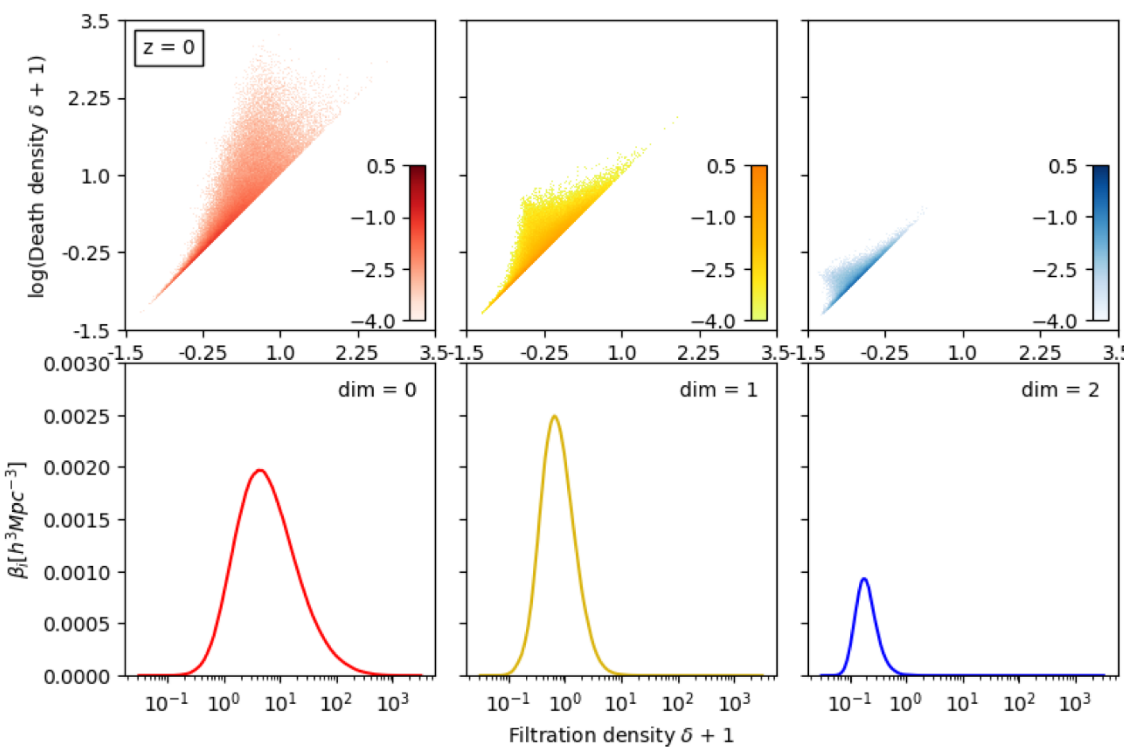
d-dimensional features

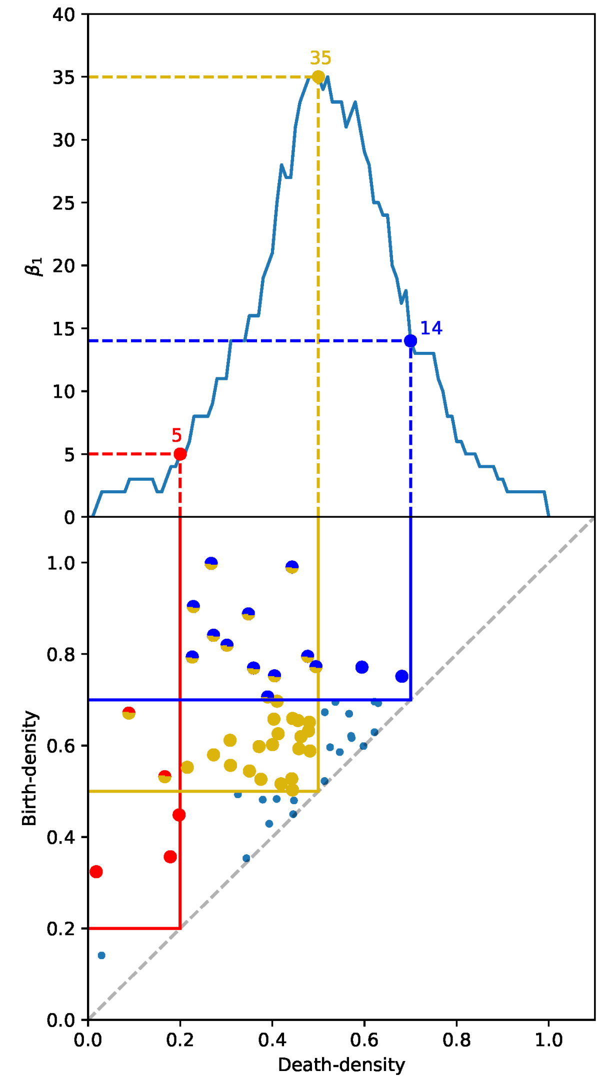
Betti Numbers - Persistence

Betti Number $\beta_i(\nu)$:

Integral over persistence pairs:

- **birth level** $\nu > \nu_{birth}$
- **death level** $\nu < \nu_{death}$





Reionization Bubble Network:

Multiscale Topology

Elbers & vdW 2019, 2021

100 seconds

380 000 years

300–500 million years

Light and matter are coupled

Dark matter evolves independently: it starts clumping and forming a web of structures

Light and matter separate

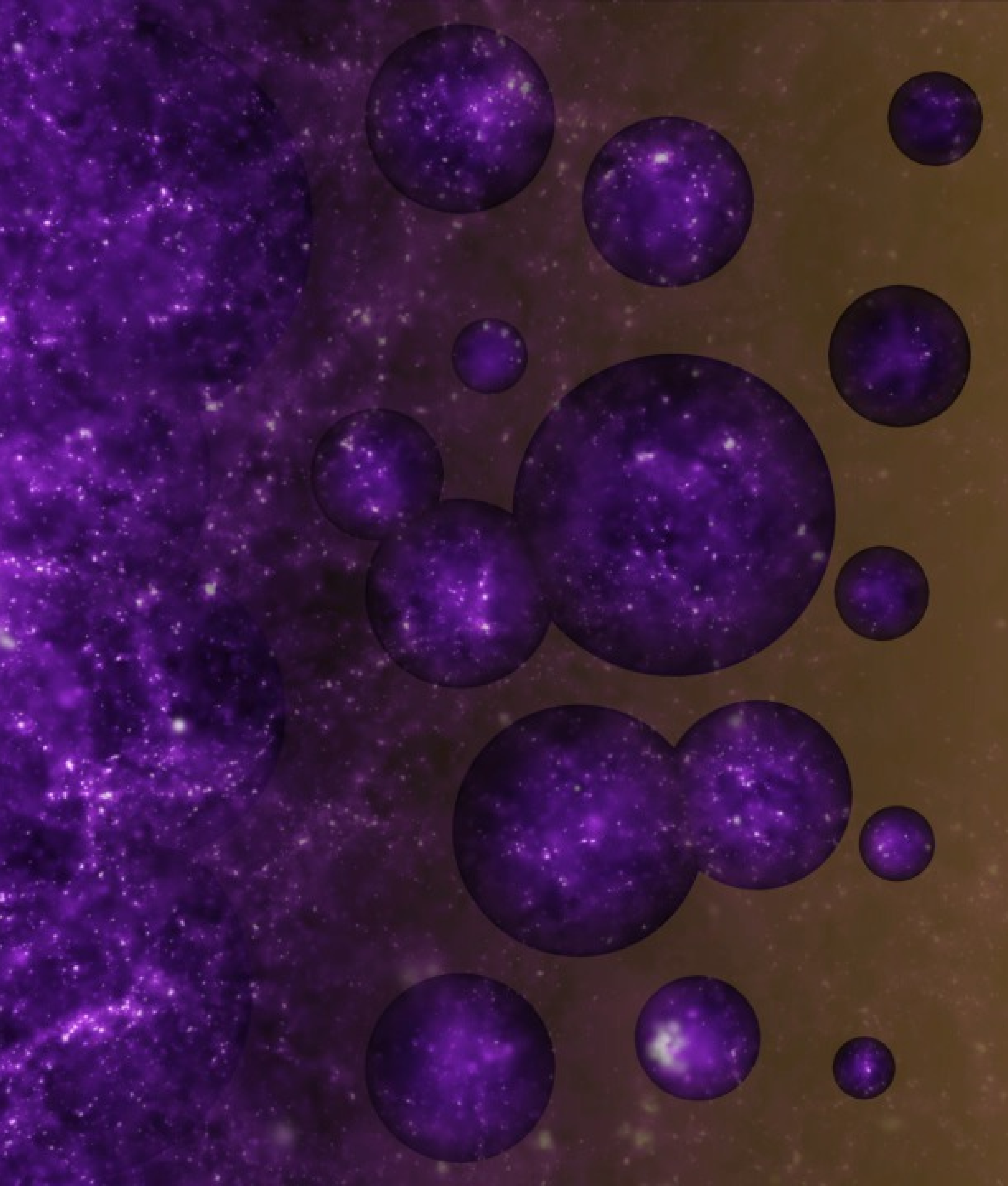
- Protons and electrons form atoms
- Light starts travelling freely: it will become the Cosmic Microwave Background (CMB)

Dark ages

Atoms start feeling the gravity of the cosmic web of dark matter

First stars

The first stars and galaxies form in the densest knots of the cosmic web



the intense UV radiation
of the first stars and galaxies

and possibly
the hard Xray radiation
of the first
Supermassive Black Holes

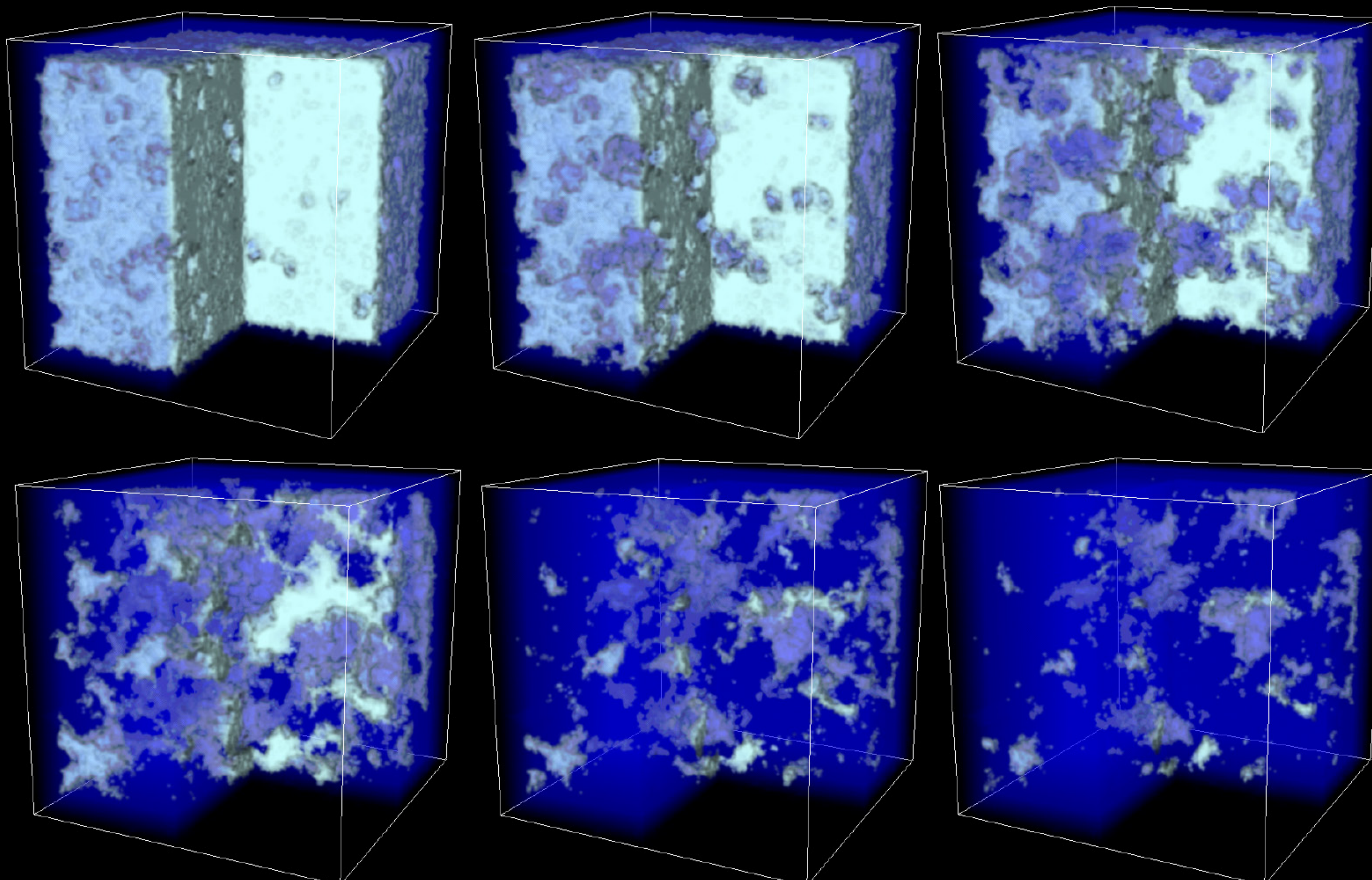
start to ionize the
neutral hydrogen in the
intergalactic medium

the IGM is mostly concentrated
in the filaments of the early Cosmic Web,
with still substantial amounts in the voids

the ionization bubbles around
these early objects expand and
gradually merge and occupy
major fractions of the Universe:

EPOCH OF REIONIZATION (EOR)

Reionization Process



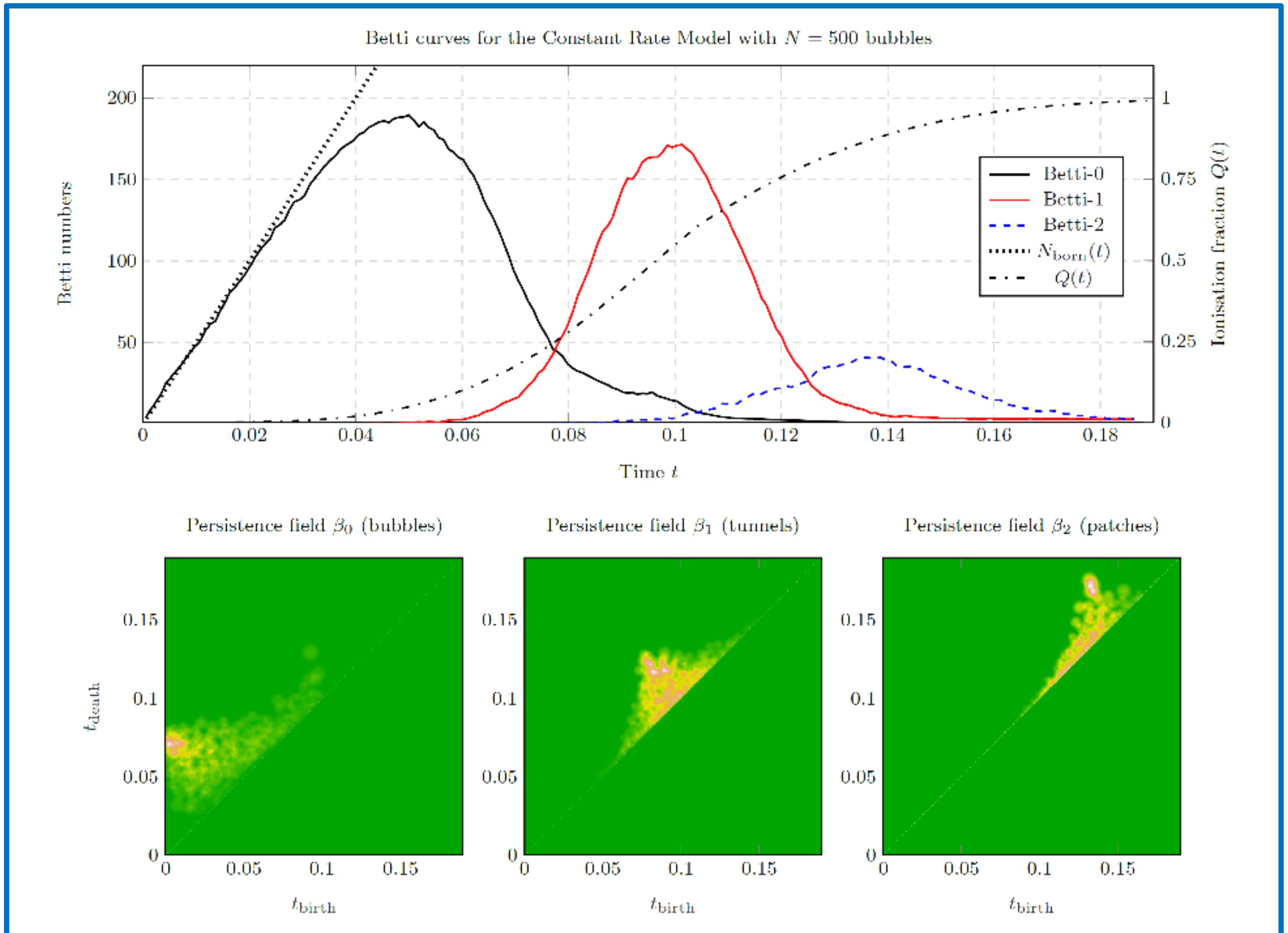
J. Ritzerveld, V. Icke

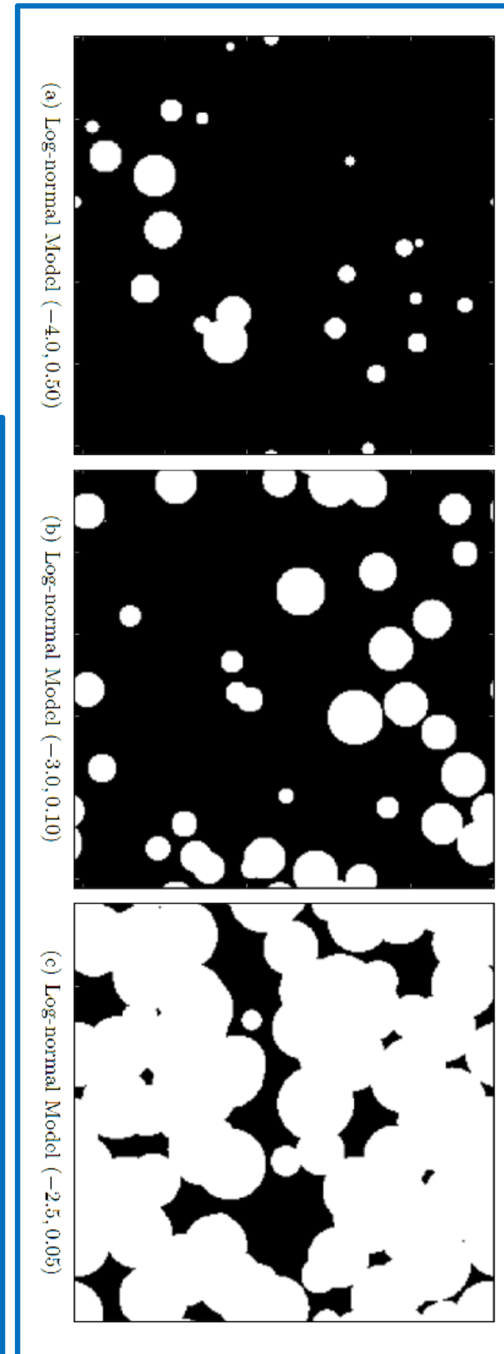
Cosmic Bubble Bath

M. Alvarez, T Abel.

Reionization
Bubble Network
Persistent Topology

Elbers & vdW 2019, 2021

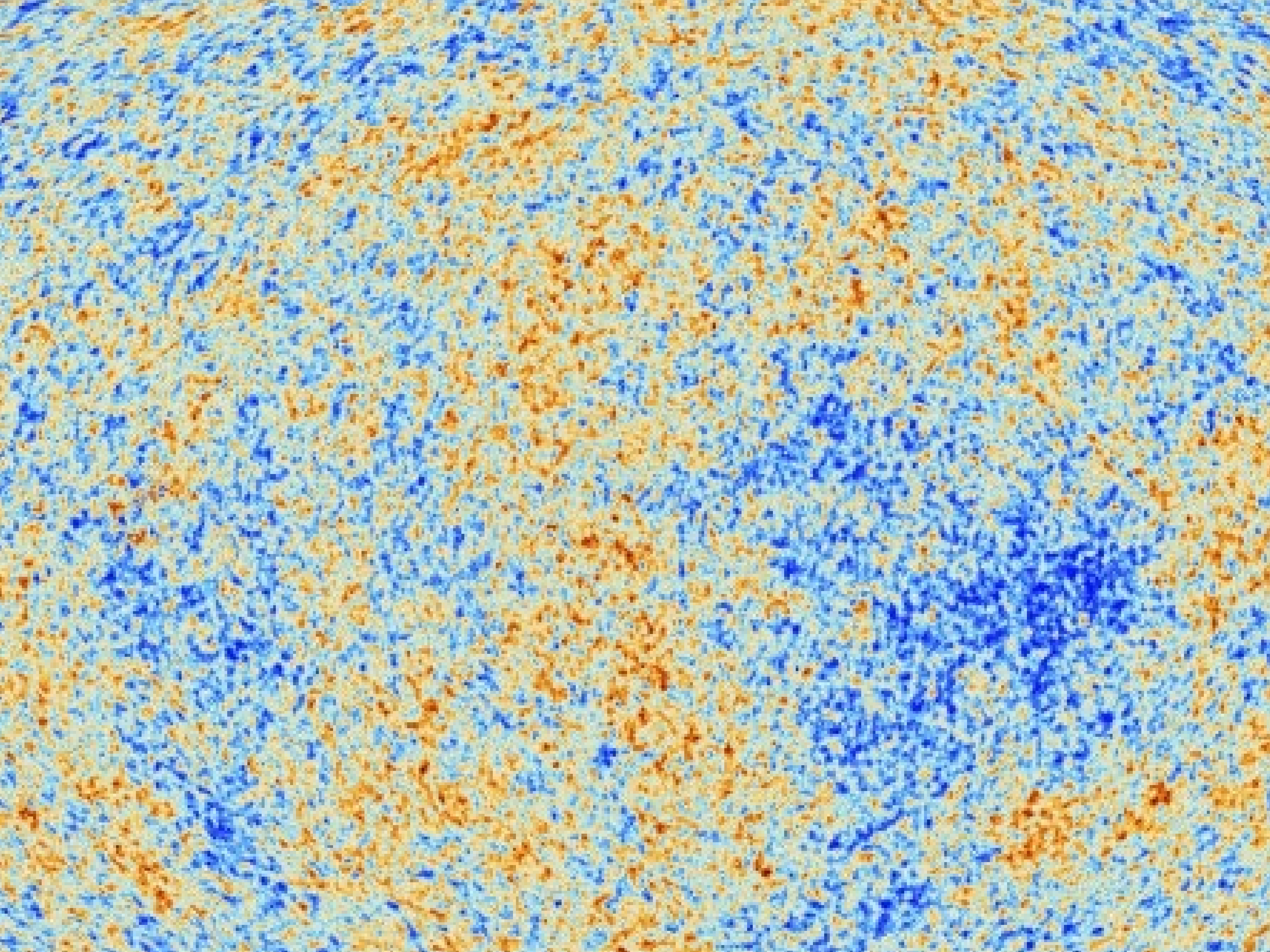




Topology

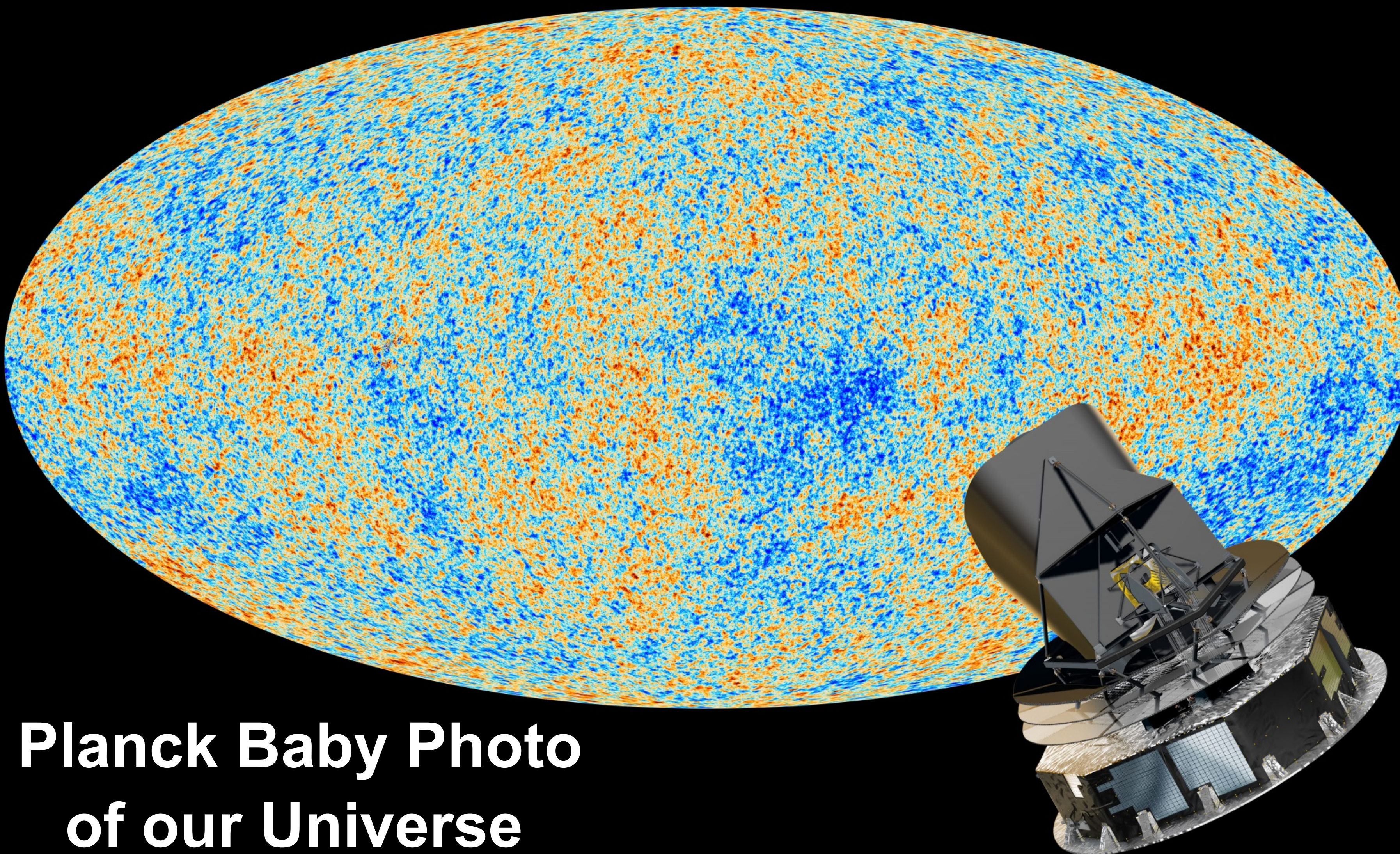
of the

Primordial Gaussian Field



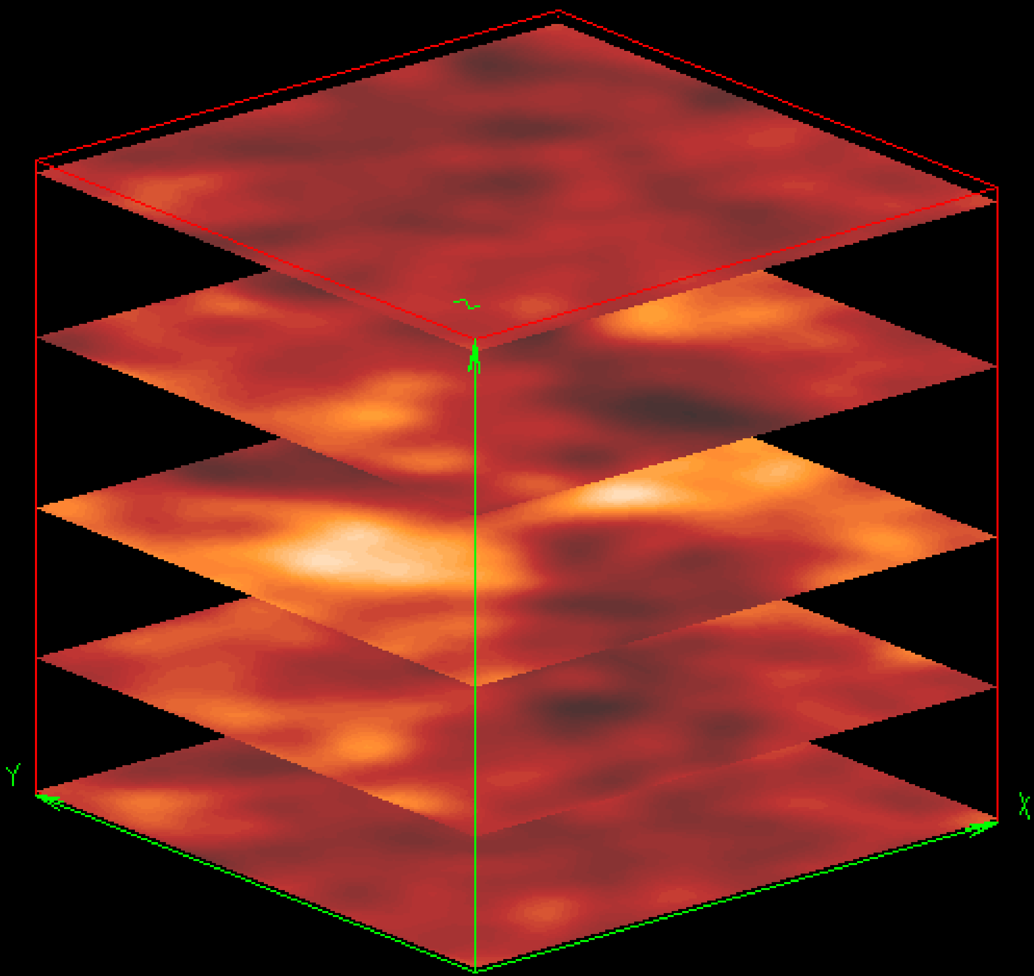
Cosmic Origins

- Universe 380.000 yrs after Big Bang
- 13.8 Gyrs ago (13.798±0.037 Gyrs)
- Temperature T = 2.72548±0.00057 K
- temperature/density fluctuations ($\Delta T/T<10^{-5}$)



Planck Baby Photo of our Universe

Primordial Gaussian Field



$$f\left(\vec{x}\right) = \int \frac{d\vec{k}}{\left(2\pi\right)^3} \hat{f}\left(\vec{k}\right) e^{-i\vec{k}\cdot\vec{x}}$$

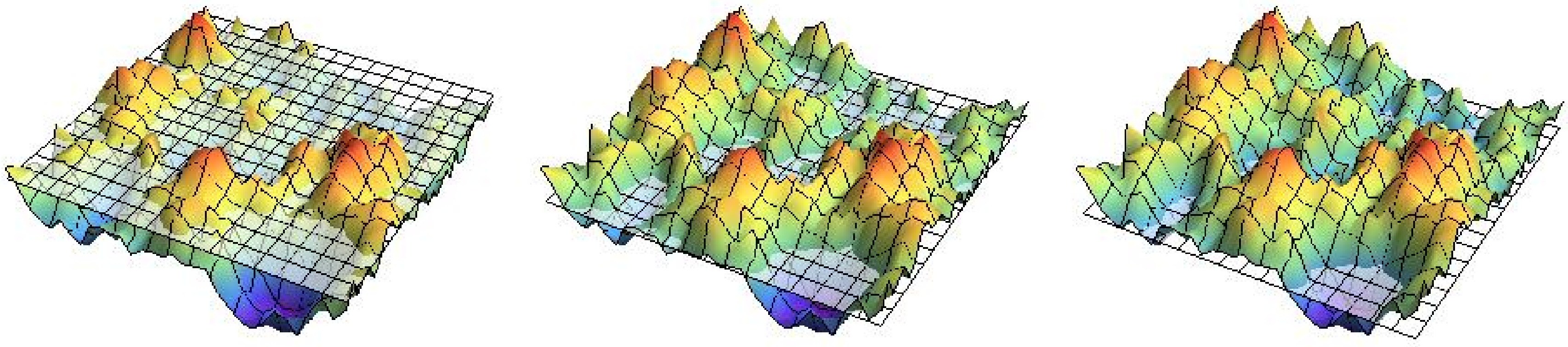
$$\hat{f}\left(\vec{k}\right) = \hat{f}_r\left(\vec{k}\right) + i\,\hat{f}_i\left(\vec{k}\right) = \left|\hat{f}\left(\vec{k}\right)\right| e^{i\theta\left(\vec{k}\right)}$$

$$P_N = \frac{\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} f_i \left(M^{-1}\right)_{ij} f_j\right]}{\left[\left(2\pi\right)^N \left(\det M\right)\right]^{1/2}} \prod_{k=1}^{N} df_k$$

$$P_N \propto \exp\left(-\sum_i \frac{\left|\hat{f}\left(\vec{k}_i\right)\right|^2}{2P\left(k_i\right)}\right) \propto \prod_i \exp\left(-\frac{\left|\hat{f}\left(\vec{k}_i\right)\right|^2}{2P\left(k_i\right)}\right)$$

$$P_1\left(\left|\hat{f}\left(\vec{k}\right)\right|\right) d\left|\hat{f}\left(\vec{k}\right)\right| = \exp\left(-\frac{\left|\hat{f}\left(\vec{k}\right)\right|^2}{2P(k)}\right) \frac{\left|\hat{f}\left(\vec{k}\right)\right| d\left|\hat{f}\left(\vec{k}\right)\right|}{P(k)}$$

Gaussian Random Fields: Excursion Sets



Superlevel Sets

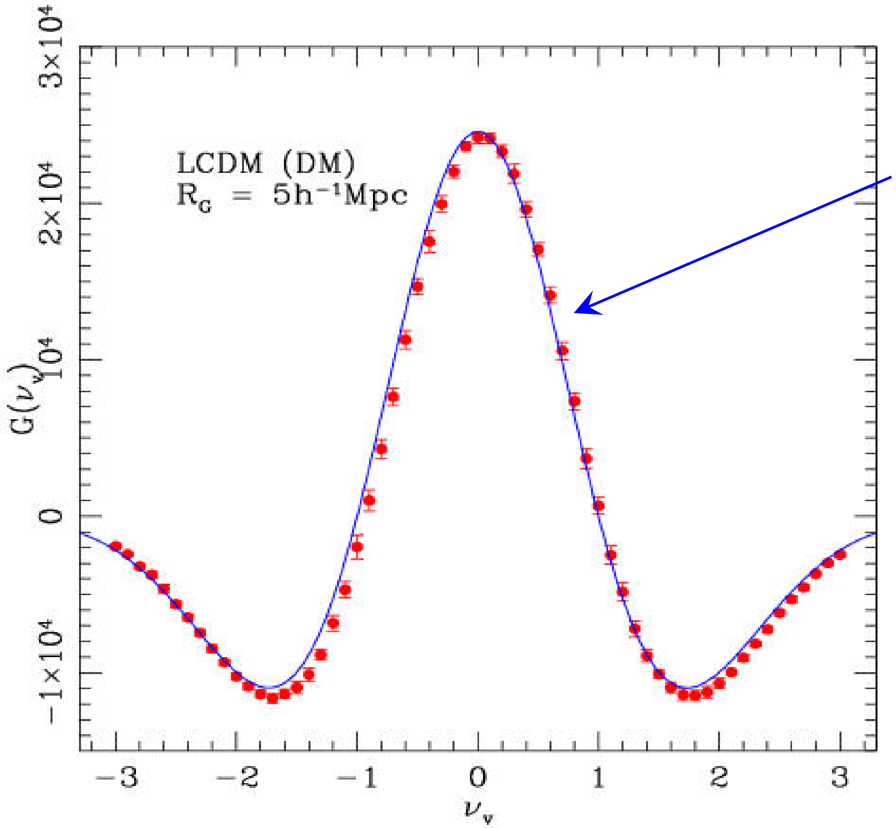
$$\mathfrak{M}_\nu = \left\{ \vec{x} \in \mathfrak{M} \mid f_s\left(\vec{x}\right) \in [f_\nu, \infty) \right\}$$

$$= f_s^{-1}\left[f_\nu, \infty \right)$$

Gaussian Random Fields: Genus

Genus Gaussian Field, the "cosmological" way :

$$g = G - c$$



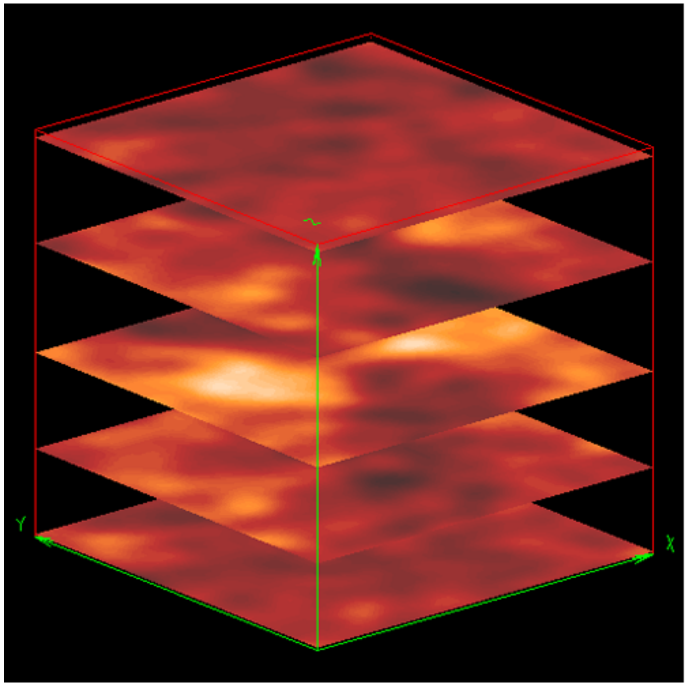
$$g(\nu) = -\frac{1}{8\pi^2}\left(\frac{\langle k^2\rangle}{3}\right)^{3/2}\left(1-\nu^2\right)e^{-\nu^2/2}$$

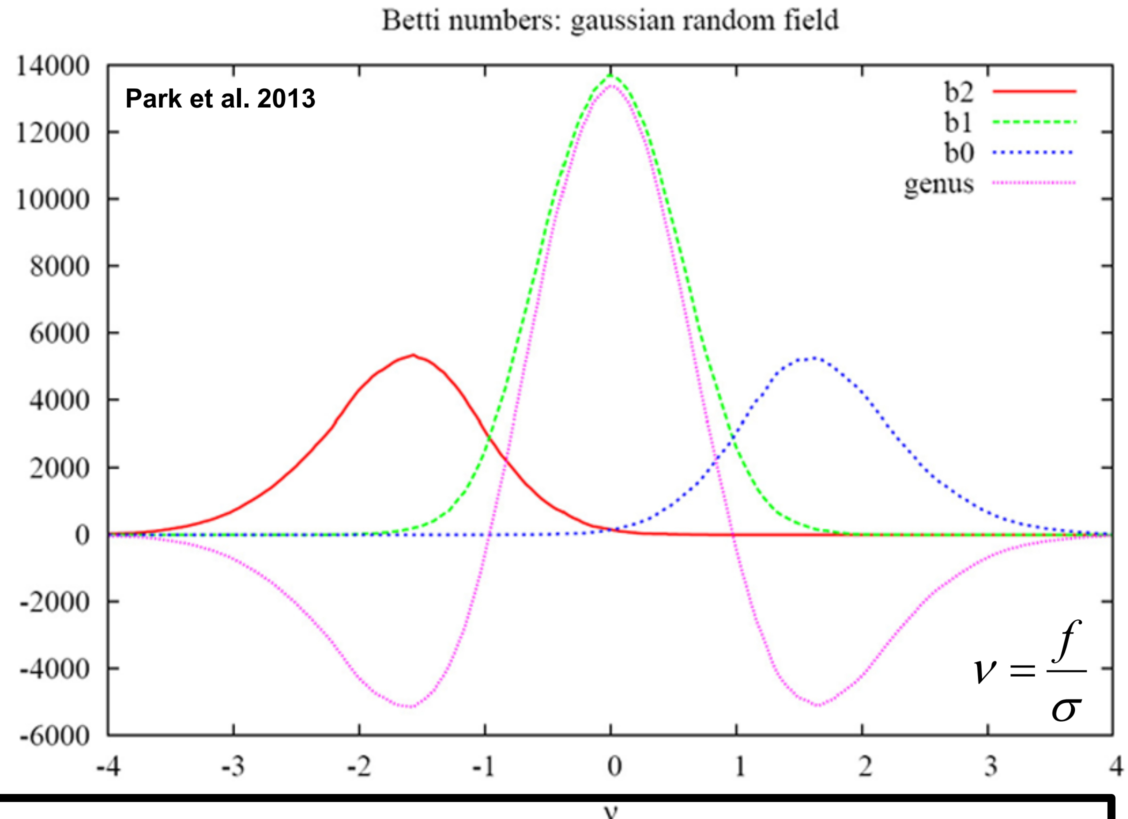
$$g(\nu) = -\beta_0(\nu) + \beta_1(\nu) - \beta_2(\nu)$$

Park et al. 2013

Gaussian Random Fields: Betti Numbers

$$\chi = \beta_0 - \beta_1 + \beta_2$$

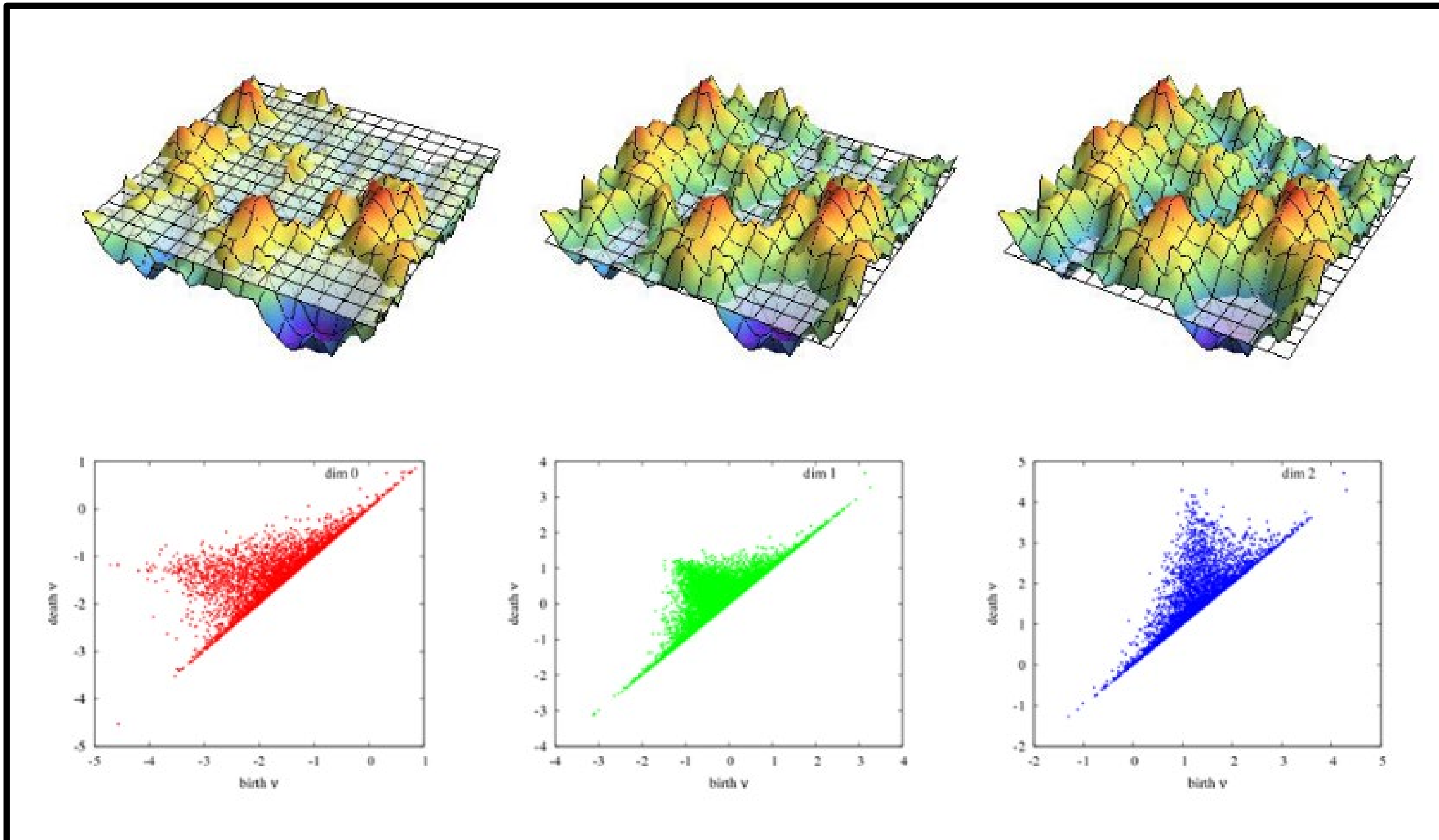




In a Gaussian field:

- # tunnels dominant at intermediate density levels, when superlevel domain spongelike
- overlap between β_0 and β_2 at v=0, domain punctured by clumps with cavities
- # clumps/islands reaches maximum at $\nu = \sqrt{3}$, # cavities/voids at $\nu = -\sqrt{3}$

Gaussian Random Fields: Persistence



2D Gaussian Field

Graph
towards Betti Numbers

Feldbrugge, van Engelen, vdW, Vegter et al. 2014

Gaussian Field Singularities

Number density of singularities in Gaussian random field
(BBKS 1986, Pogosyan et al. 2011, Wintraecken)

$$N_i(\nu) = \iint \left|\det \mathrm{H}\right| \, P(f, f_i = 0, J_i)\left(J_1{}^2 - J_2\right) dJ_1 \, dJ_2$$

N_0(v) : number density minima
N_1(v) : number density saddles
N_2 (v) : number density maxima

P(f,f_i,J_i): Gaussian probability

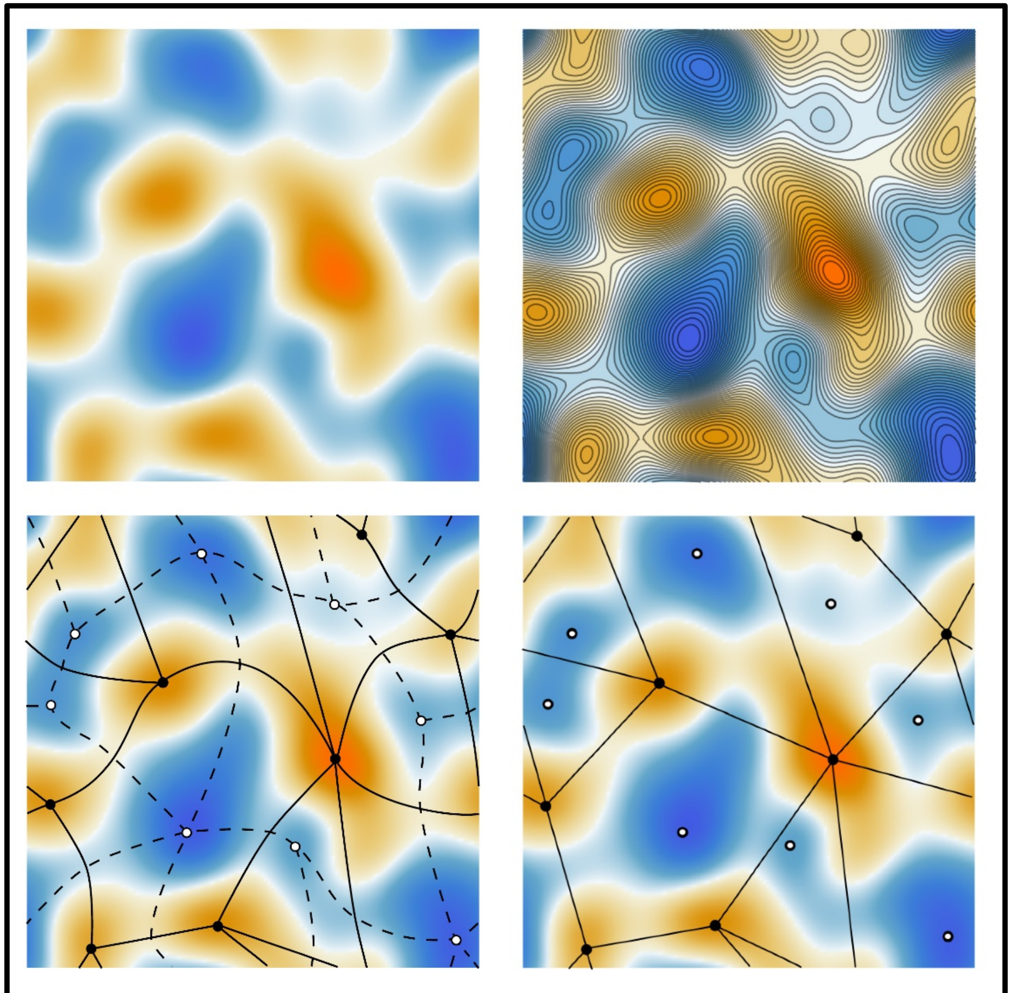
$$f_i = \frac{\partial f}{\partial x_i}; \qquad f_1 = f_2 = 0;$$

$$H: \qquad f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j};$$

$$J_1 = \lambda_1 + \lambda_2 = f_{11} + f_{22}$$

$$J_2 = \left(\lambda_1 - \lambda_2\right)^2 = \left(f_{11} - f_{22}\right)^2 + 4 f_{12}{}^2$$

Gaussian Random Fields: Morse-Smale complex



density field: isolevel set contours

$N_0(\nu)$, $N_1(\nu)$, $N_2(\nu)$:
maxima, saddles, minima in field

Morse-Smale complex

Tessellation of Space

Simplified Simplicial Representation:

Morse Graph

Incremental Algorithm

Computing Betti Numbers:
using ***adapted Morse-Smale complex***, in combination with ***incremental algorithm.***

Critical point	index	β_0	β_1	β_2	addition of
Minimum	0		$\downarrow$	$\uparrow$	face
Saddle point	1	$\downarrow$	$\uparrow$		edge
Maximum	2	$\uparrow$			vertex

Influence of Critical Points on the Betti Numbers of a simplicial complex

Betti Number Formulae

Using the incremental algorithm,

$$\beta_0(\nu_0) = \int_{\nu_0}^{\infty} \left[N_0(\nu) - \left(1 - g(\nu)\right) N_1(\nu) \right] d\nu$$

$$\beta_1(\nu_0) = \int_{\nu_0}^{\infty} \left[g(\nu) N_1(\nu) - N_2(\nu) \right] d\nu$$

where $g(\nu)$ is the probability that an added saddle point at height ν increases β_1 by 1

Note:
because of symmetry maxima and minima in Gaussian field,

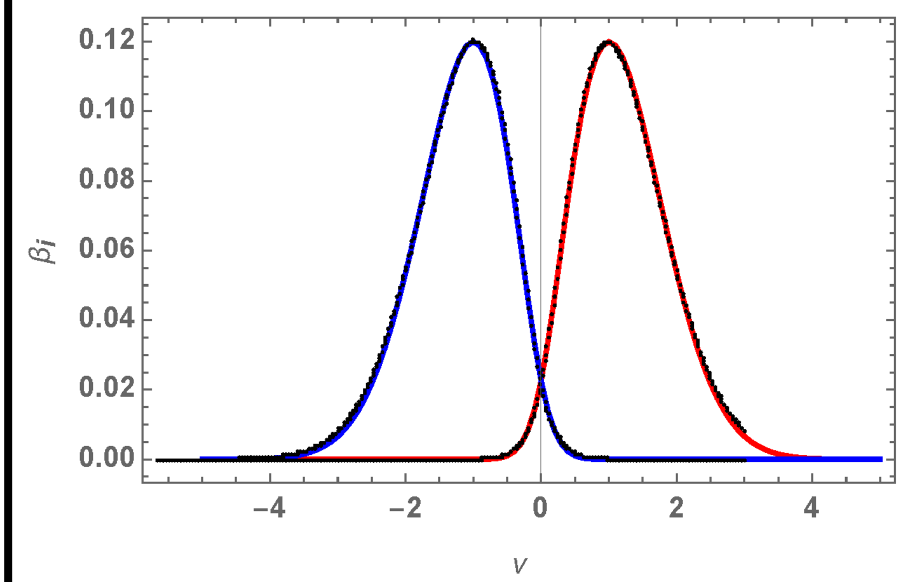
$$\beta_0(\nu) = \beta_1(-\nu)$$

Betti Number Prediction

g(ν) can be accurately fitted by the function,

$$g(\nu) \approx \frac{e^{-\alpha\nu}\left(1-\frac{N_0(\nu)}{N_1(\nu)}\right)+e^{\alpha\nu}\frac{N_2(\nu)}{N_1(\nu)}}{e^{-\alpha\nu}+e^{\alpha\nu}}$$

$$\alpha \sim 2.7$$



Primordial non-Gaussianities

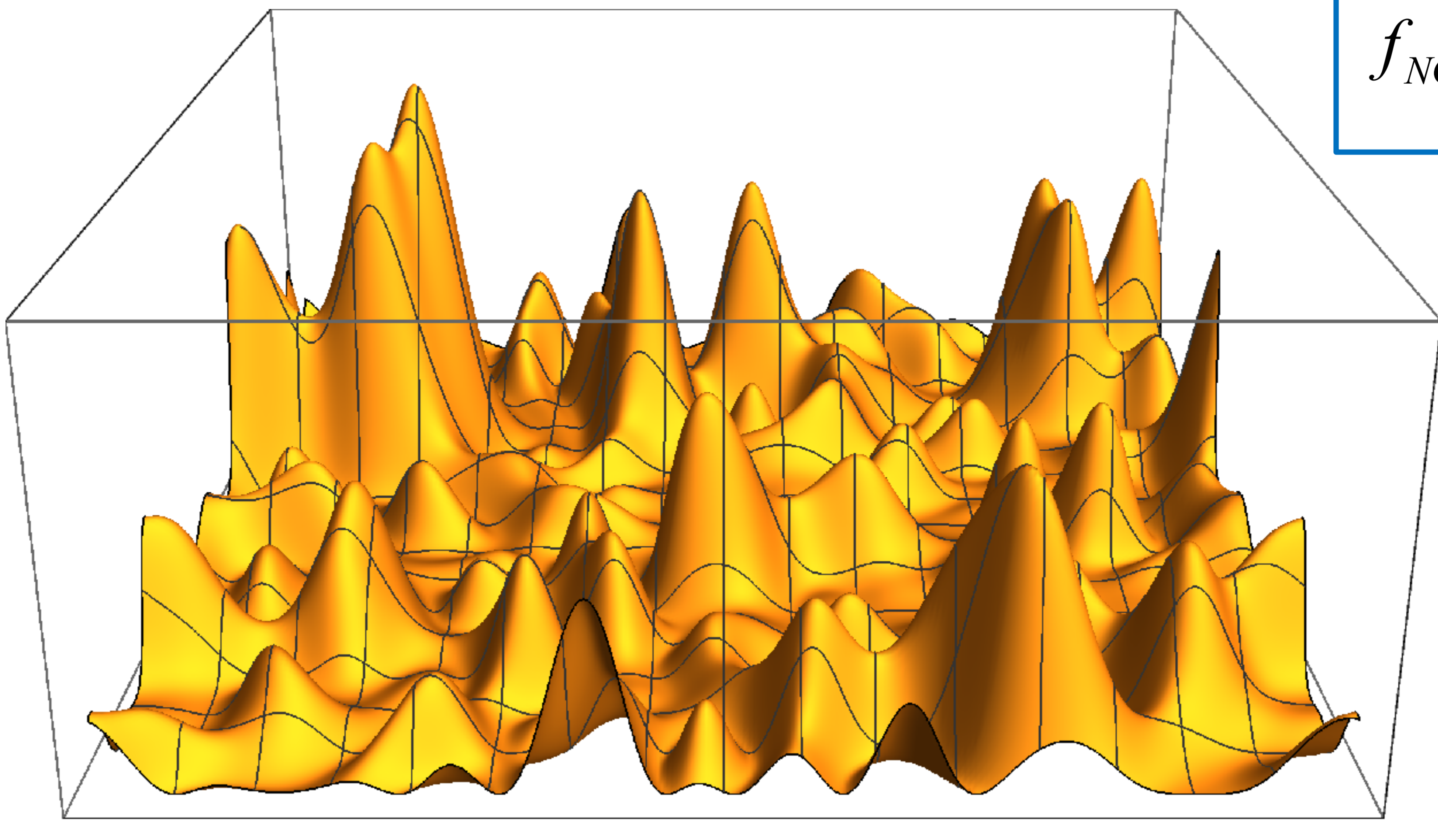
Topological Signatures

Feldbrugge, van Engelen, vdW, Vegter et al. 2014

Primordial non-Gaussianities: f_{NL} parameter

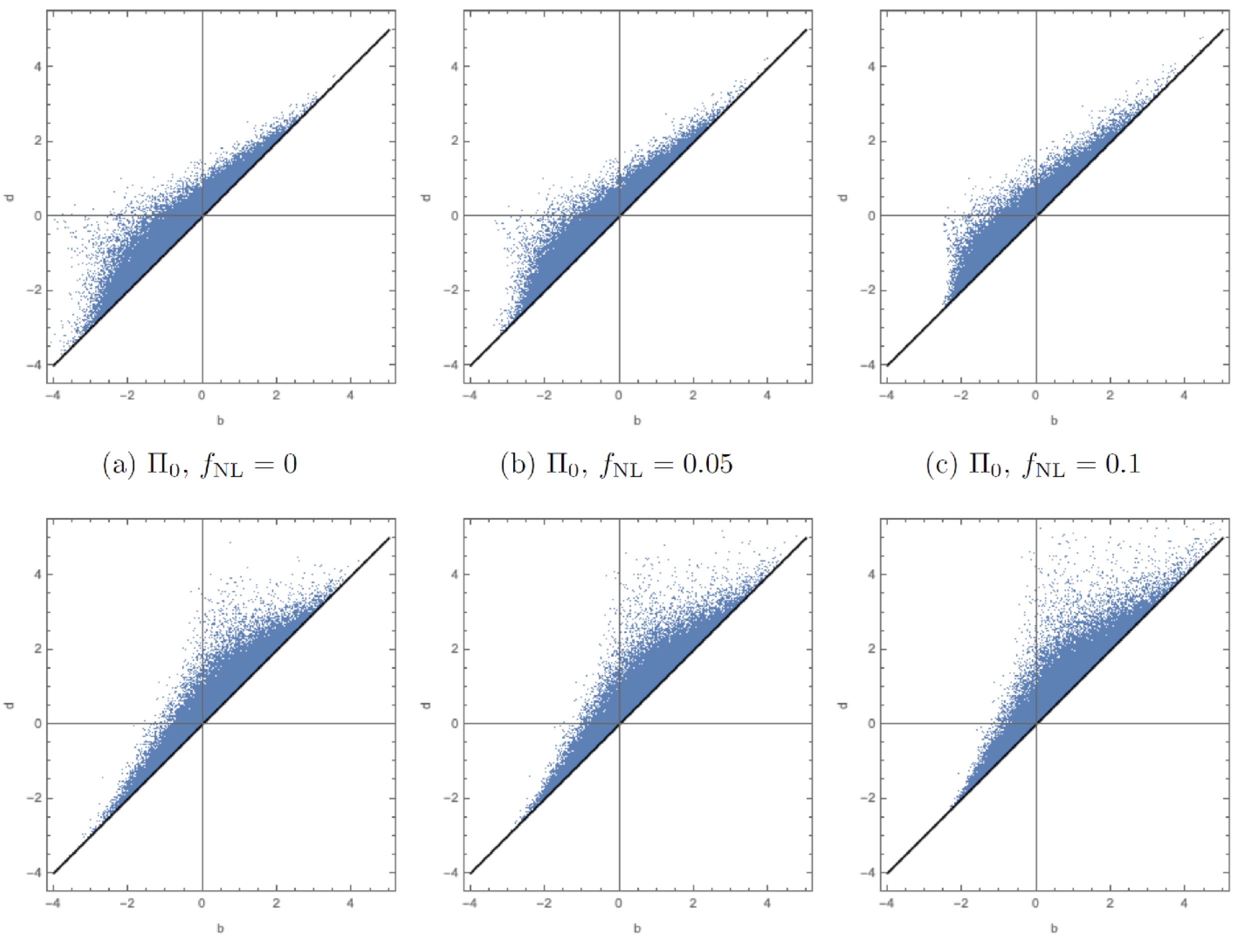
Feldbrugge, van Engelen, vdW, Vegter et al. 2014

also see Cole, Chiu et al. 2019, 2020



$$f_{NG} = f_G + f_{NL} f_G{}^2$$

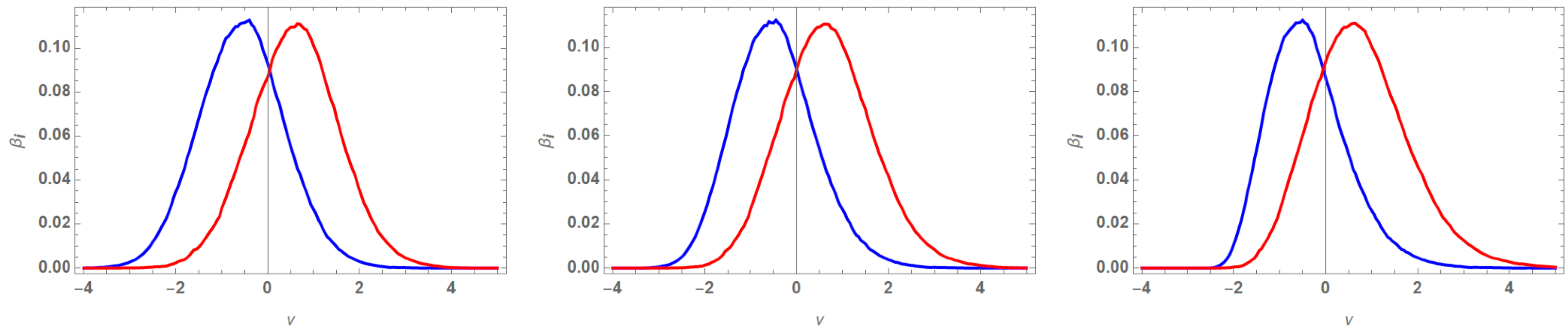
Primordial non-Gaussianities: Persistence



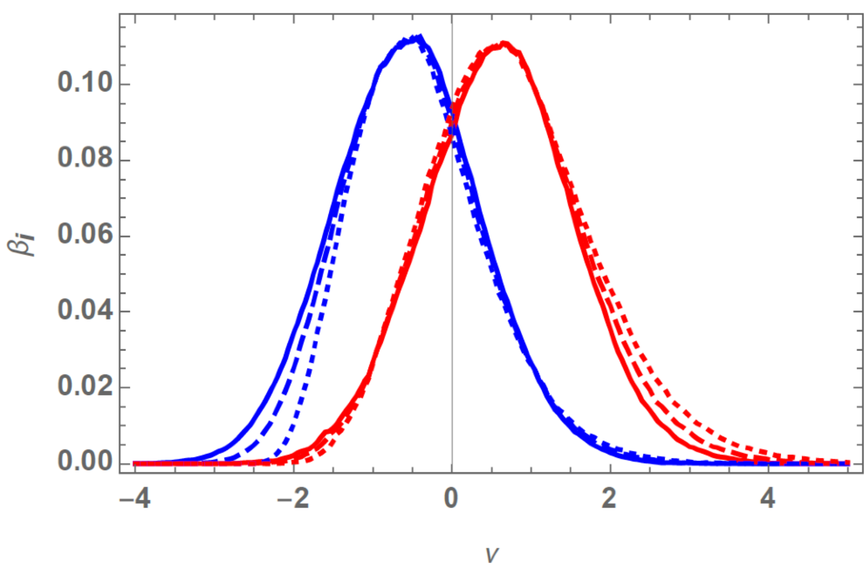
(a) Π_0, $f_{\mathrm{NL}} = 0$ (b) Π_0, $f_{\mathrm{NL}} = 0.05$ (c) Π_0, $f_{\mathrm{NL}} = 0.1$

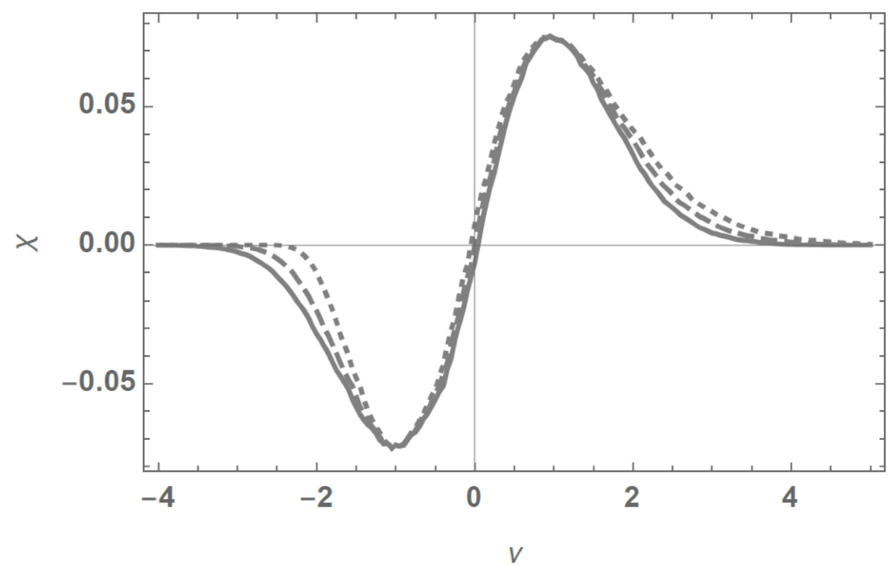
(d) Π_1, $f_{\mathrm{NL}} = 0$ (e) Π_1, $f_{\mathrm{NL}} = 0.05$ (f) Π_1, $f_{\mathrm{NL}} = 0.1$

Primordial non-Gaussianities: Betti Numbers



(a) $f_{\mathrm{NL}} = 0$ (b) $f_{\mathrm{NL}} = 0.05$ (c) $f_{\mathrm{NL}} = 0.1$

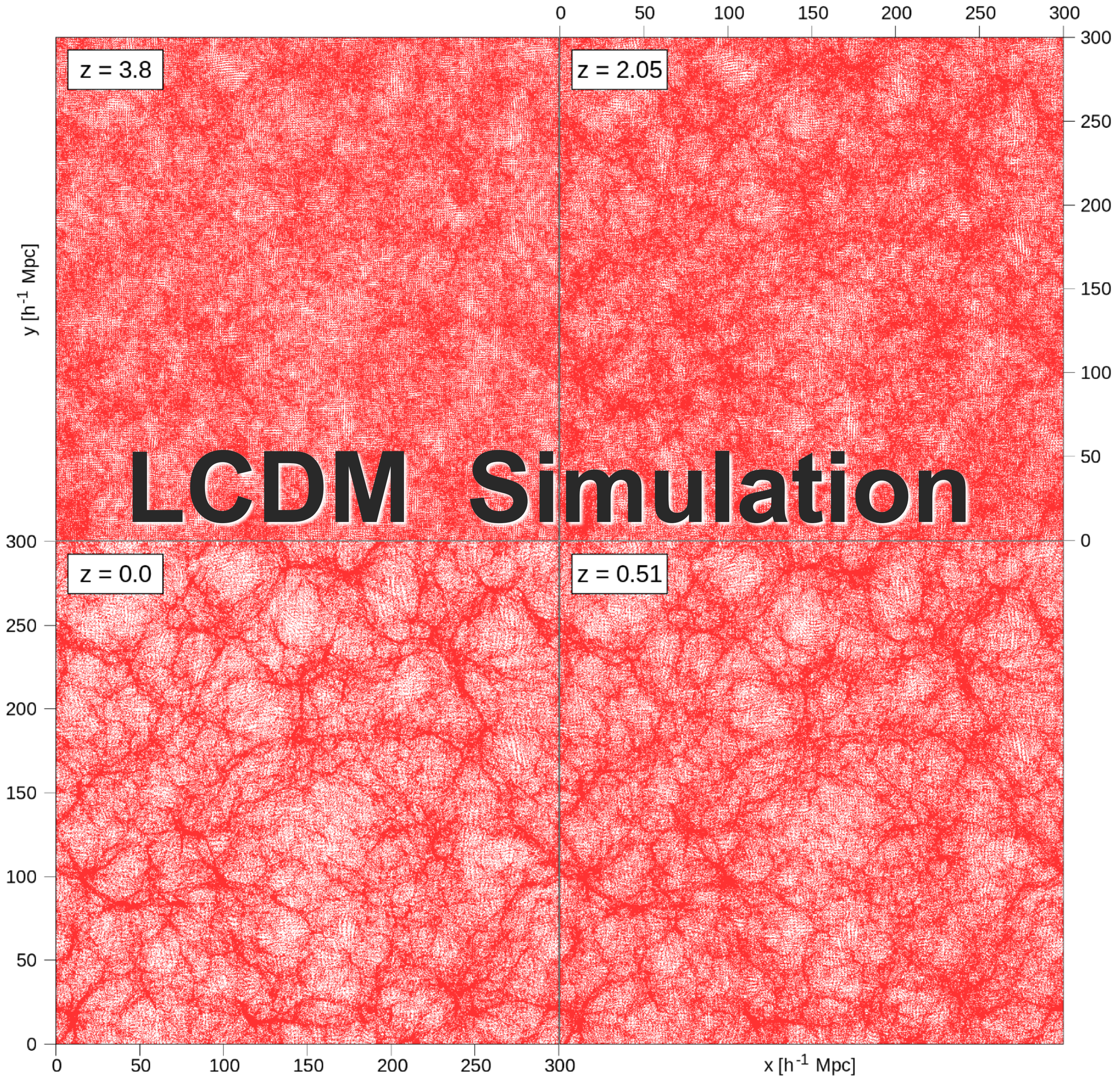




Cosmic Web Topology:

LCDM cosmology

Wilding Nevenzeel vdW et al. 2021
Bermejo-Sanchez, Wilding, vdW, Jones 2021

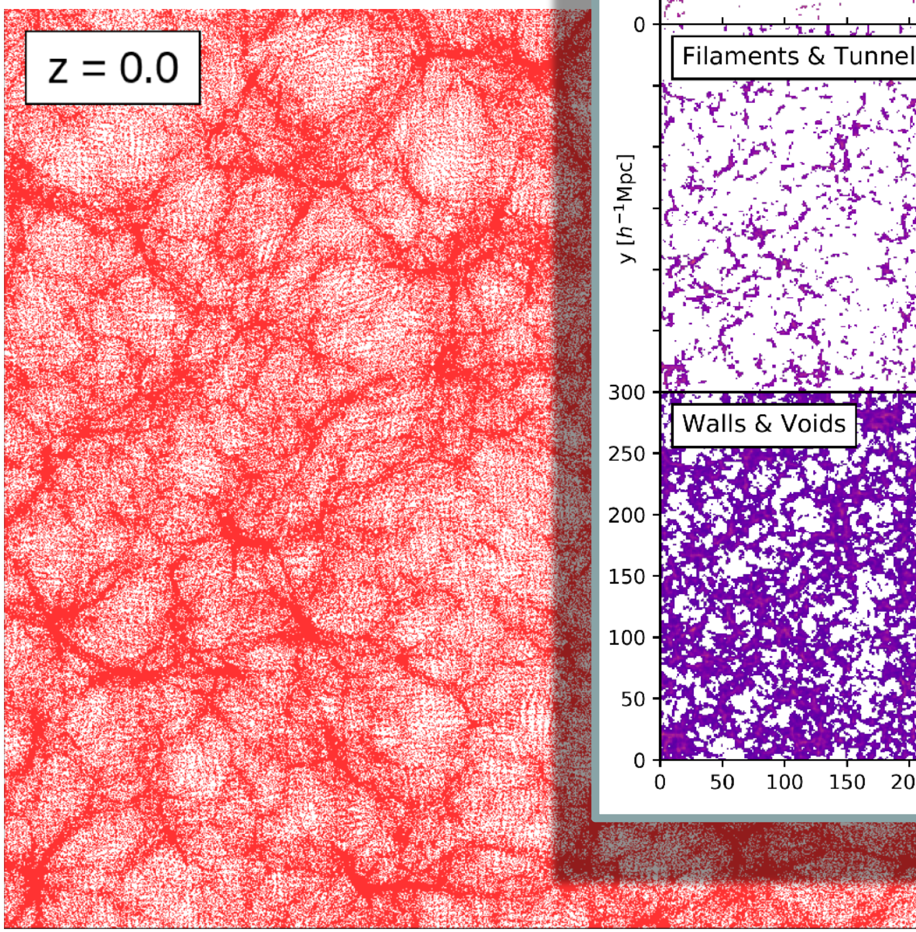


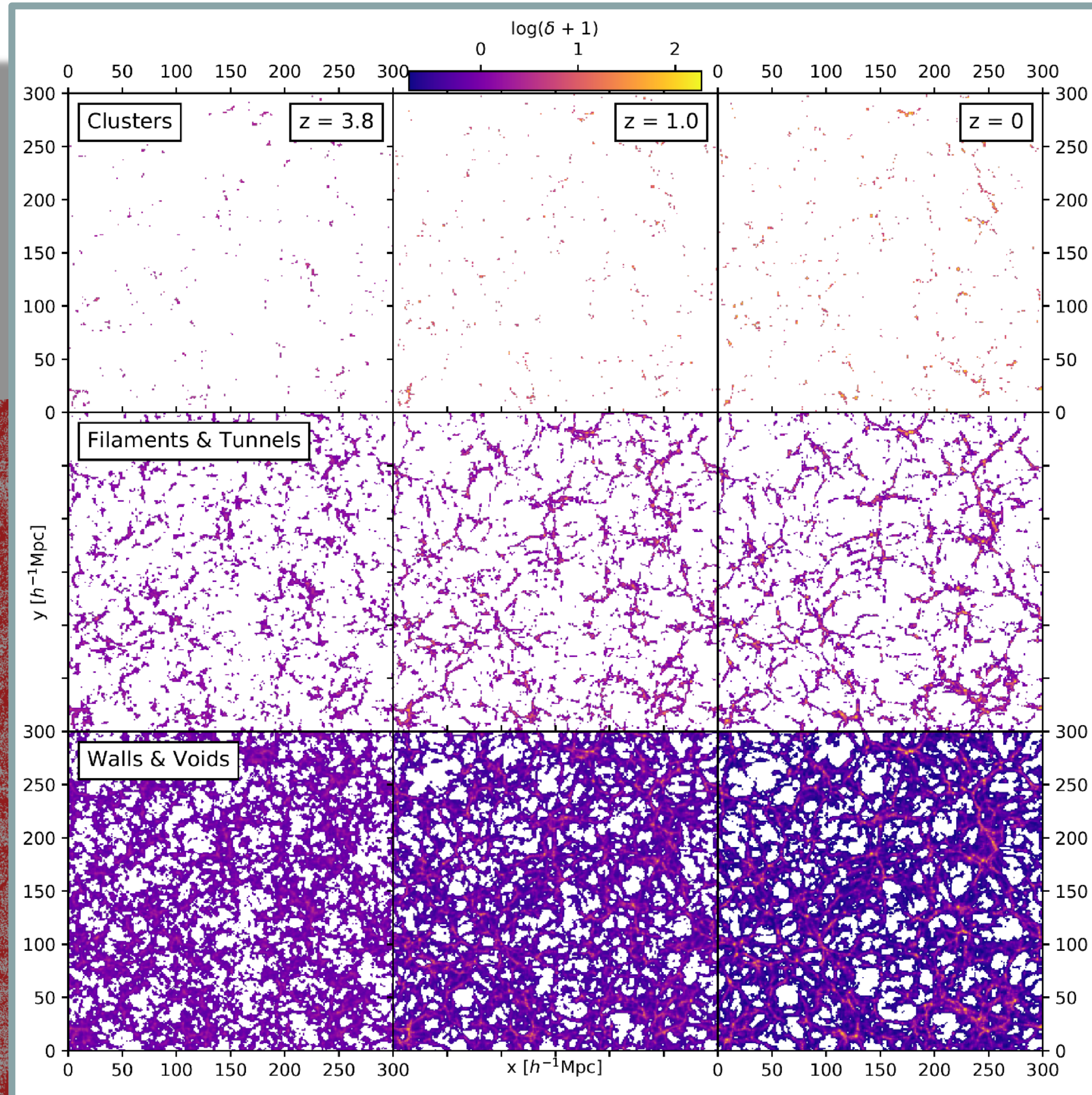
LCDM Simulation
z = 3.8
z = 2.05
z = 0.0
z = 0.51
y [h^{-1} Mpc]
x [h^{-1} Mpc]
0
50
100
150
200
250
300

LCDM simulation

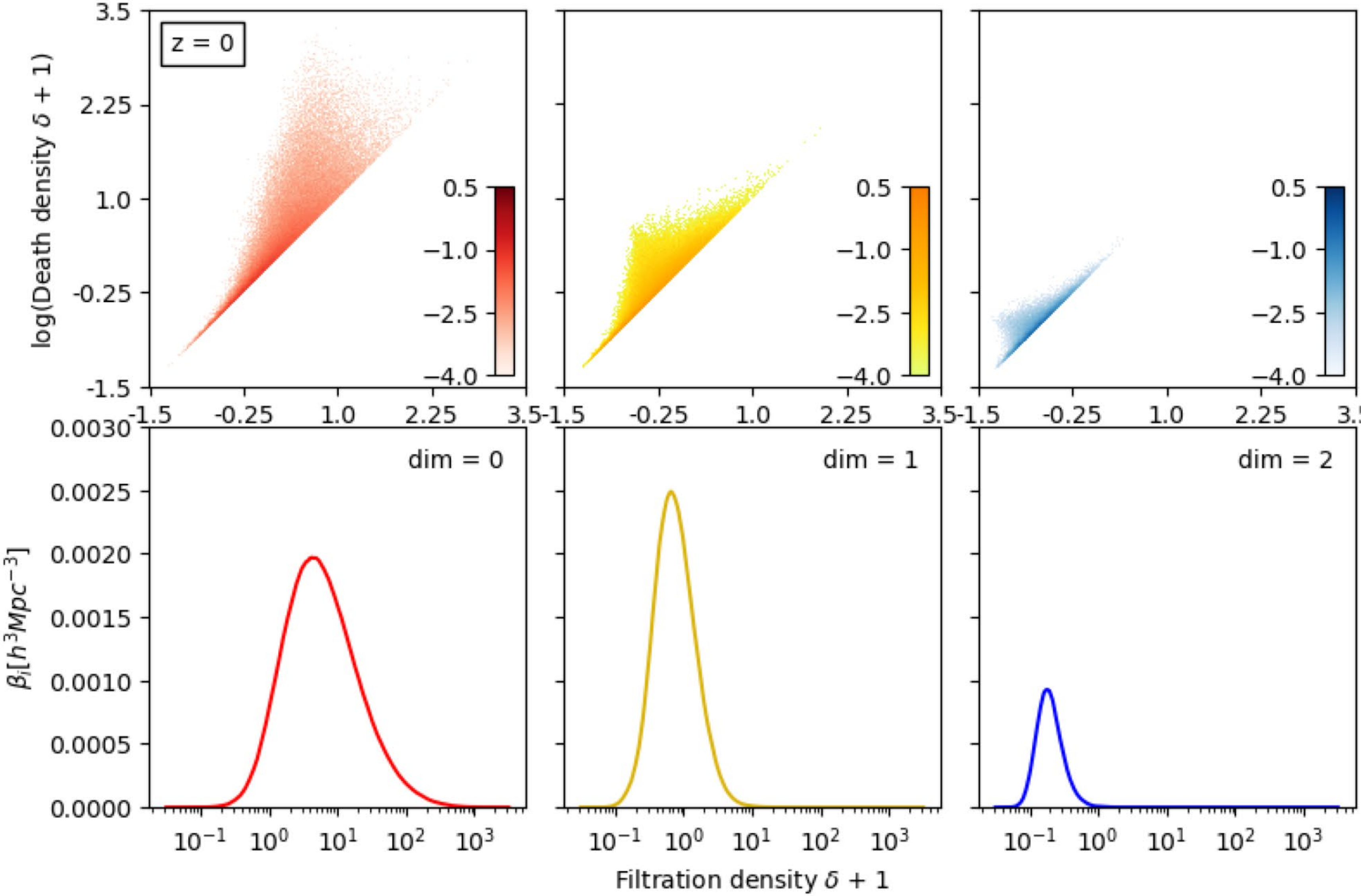
Density field

Superlevel sets

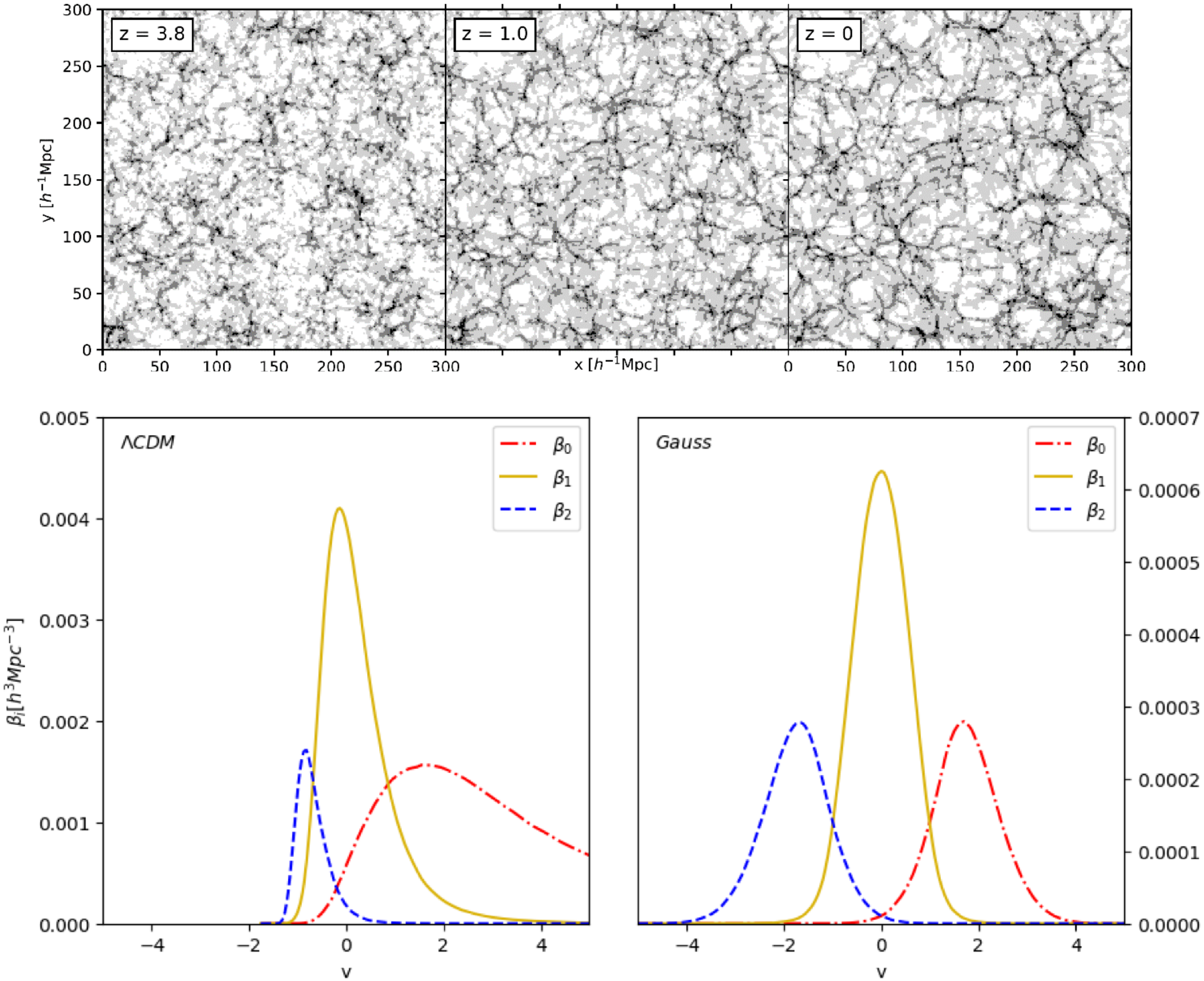




LCDM Betti- Persistence



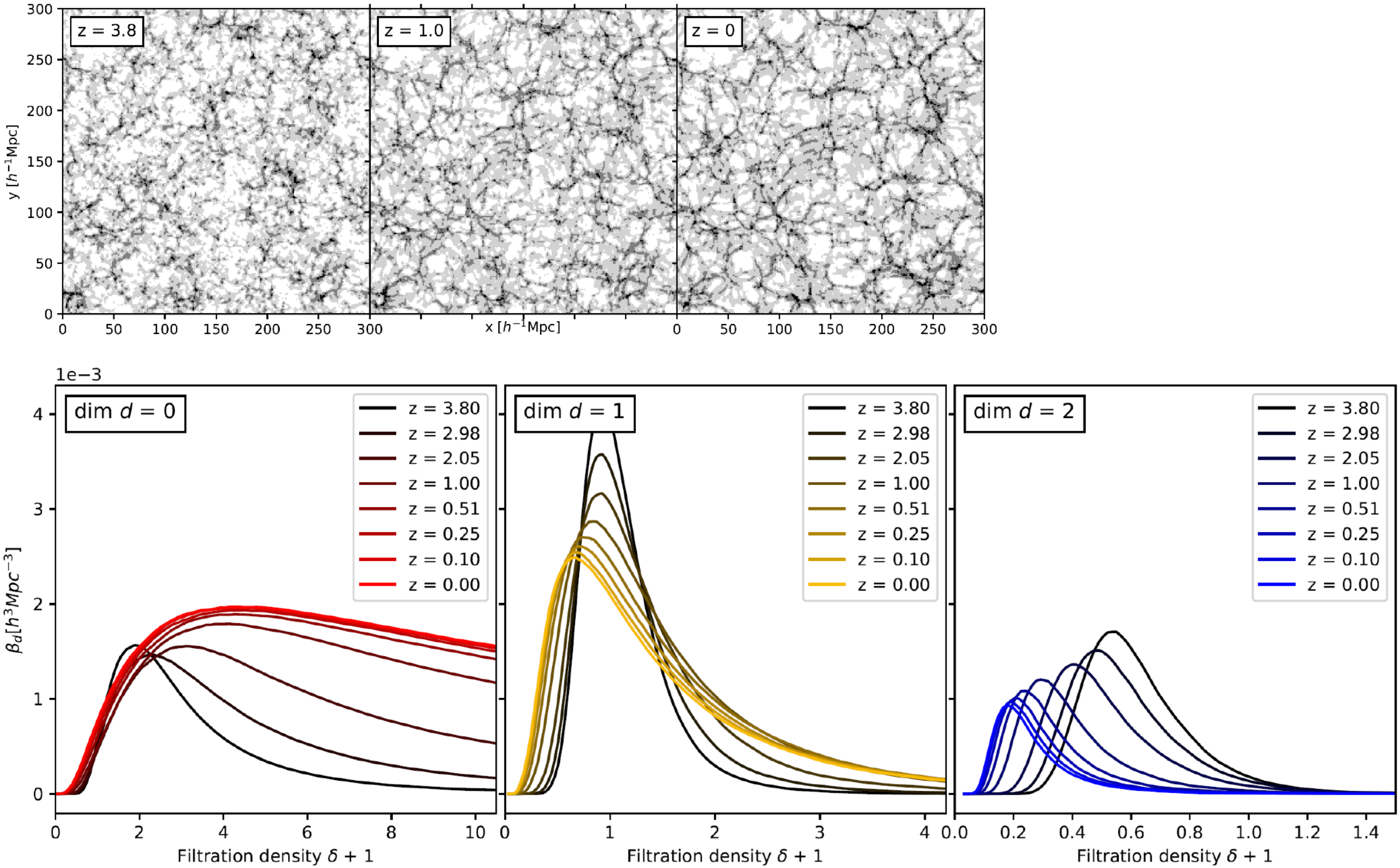
LCDM:
Betti Number Evolution



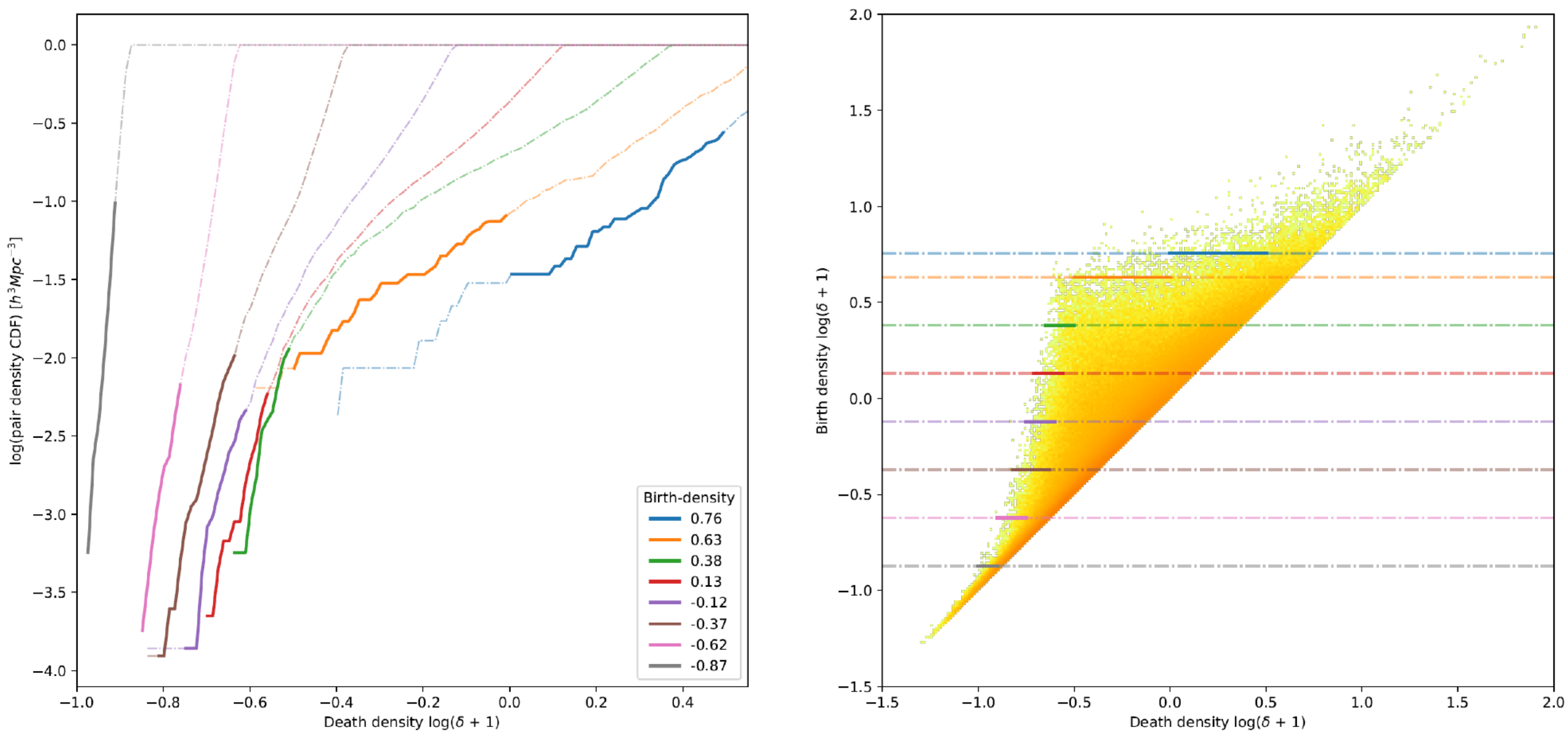
Betti Curve Evolution:

- **Shift from symmetric Gaussian curves**
- **Asymmetry Overdense-Underdense**
- **High number of overdense clumps/islands**
- **Lower number of large deep voids**

LCDM:
Betti Number Evolution



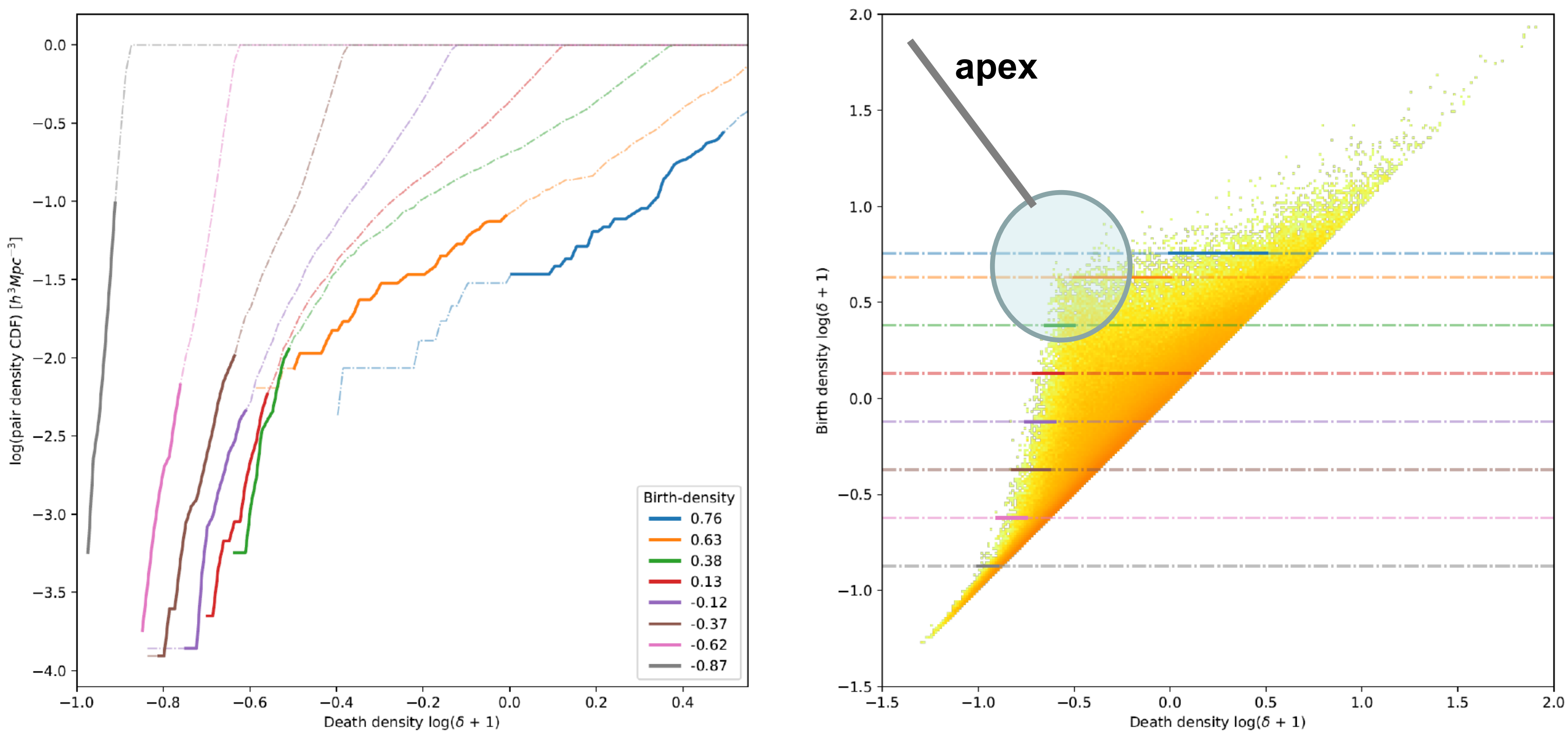
Singularity Persistence



LCDM

Internal structure - dim=1 persistence diagram

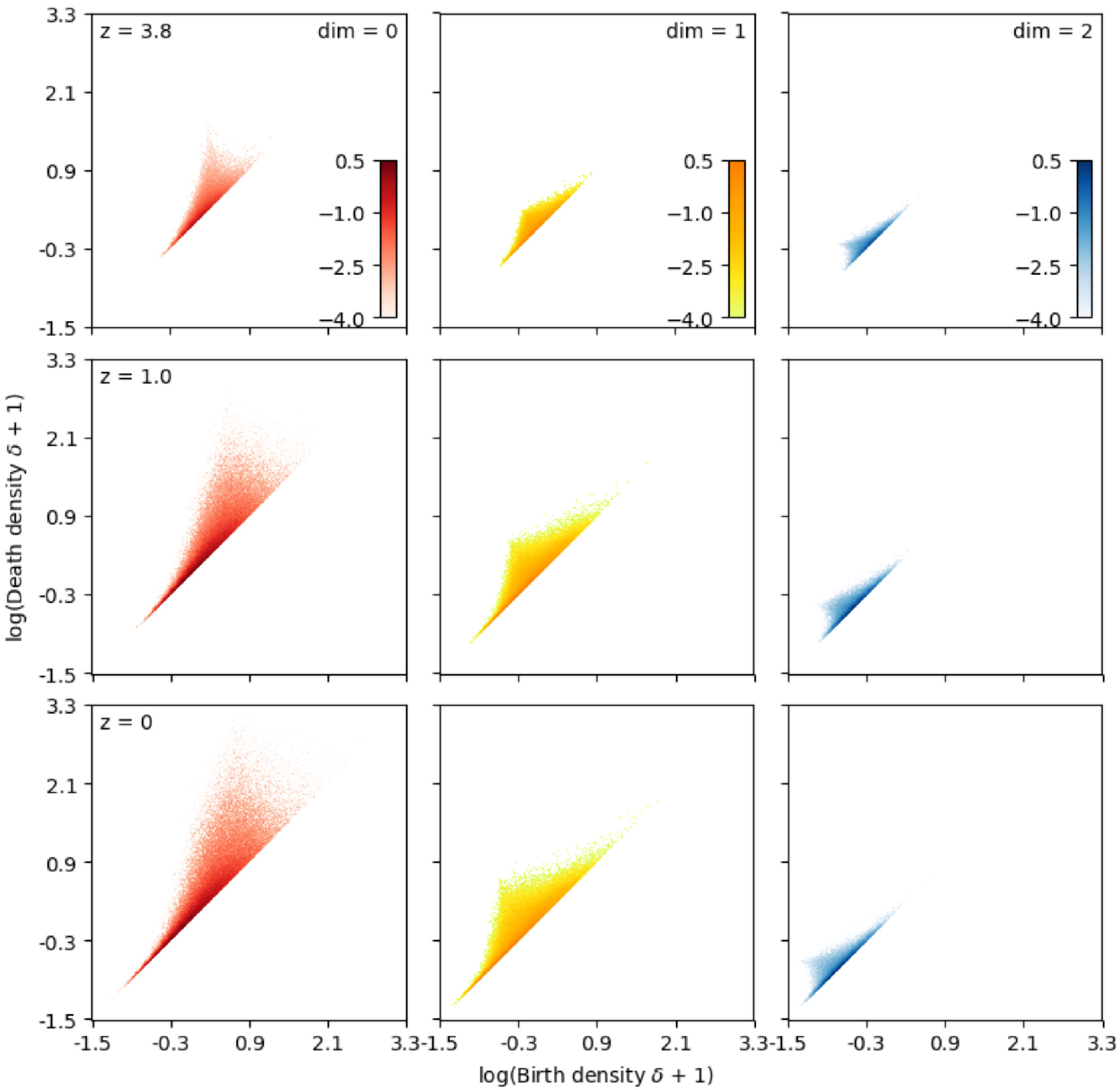
Singularity Persistence



Characteristic features correspond with dynamics structure formation process:

- **Apex corresponds to characteristic densities major transition in population objects**
- **Dim=0 - Dim=1: disappearance supercluster islands - appearance weblike network by filaments**
- **Dim=1 - Dim=2: filaments disappear as walls close - birth of void population**

LCDM: Persistent Evolution

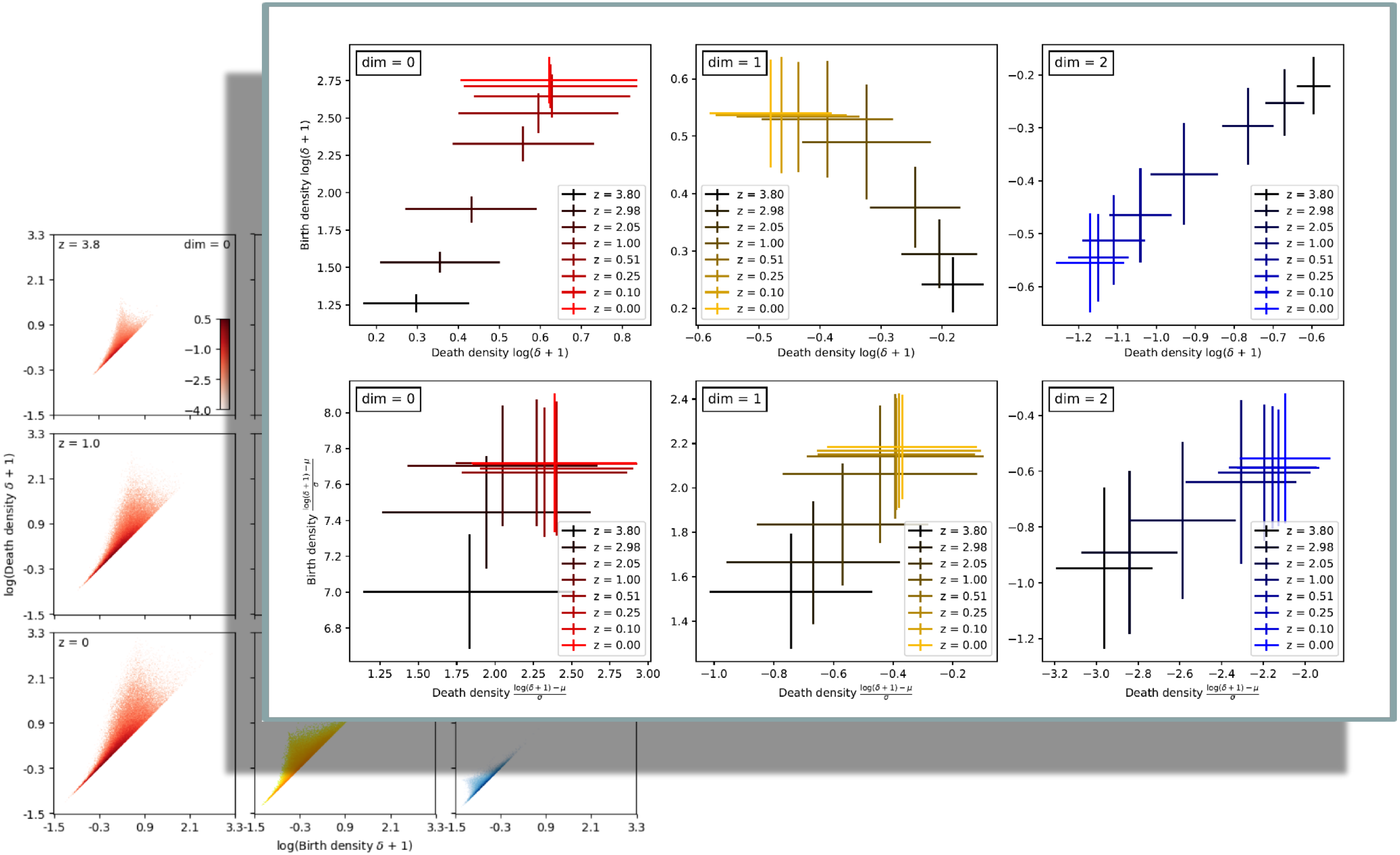


Shifting Persistence Diagrams:

- **Evolving Cosmic Web**
- **Traces characteristic density levels of**
  - **island complexes**
  - **filaments/tunnels**
  - **voids**
- **Dynamical transitions:**
  - **apex shift**

Persistence Apex Shift:

Self-similar (hierarchical) evolution



Topological Bias:

Halo Population

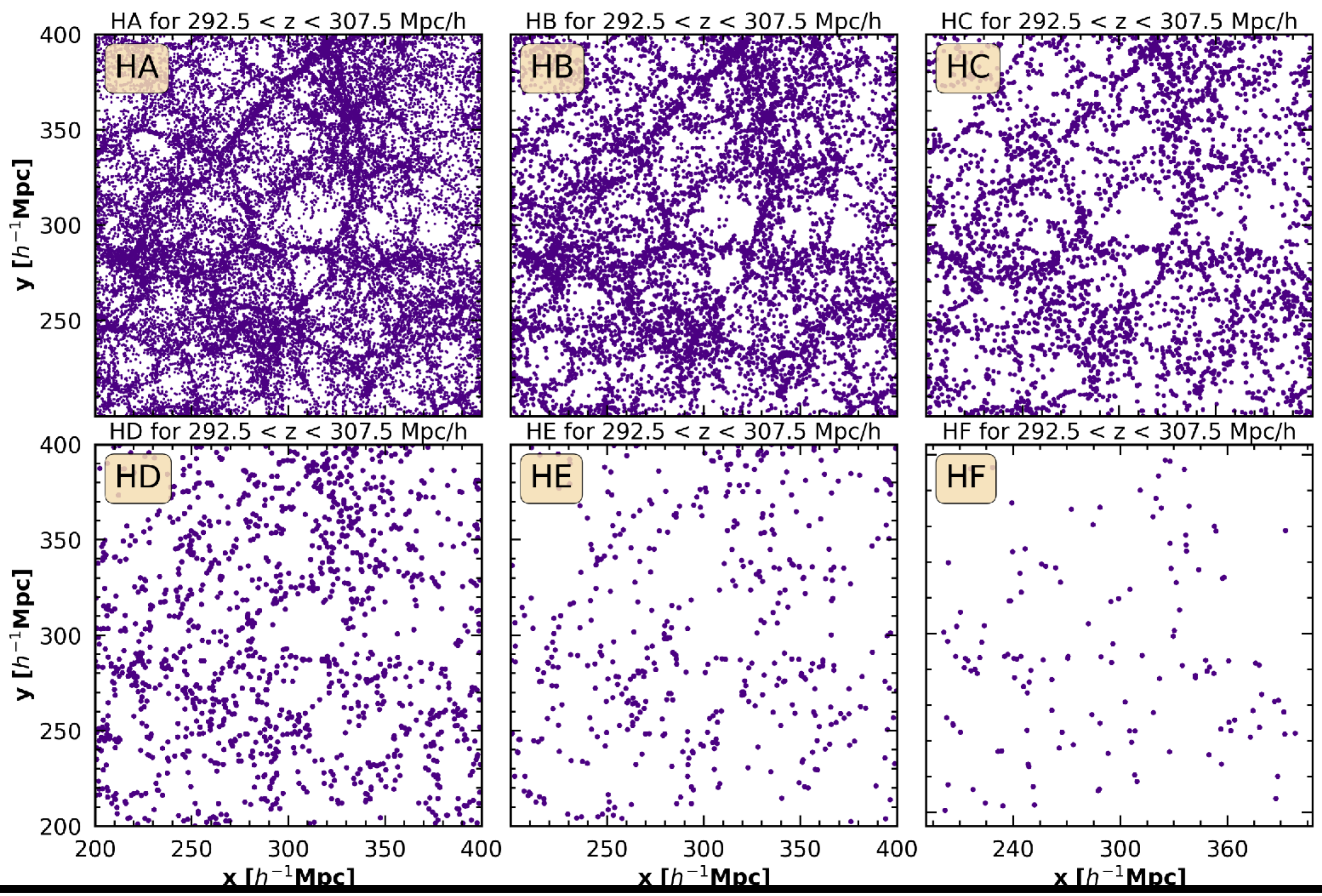
Bermejo-Sanchez, Wilding, vdW, Jones 2021


Millennium
Springel 2005

Halo Tracing – Cosmic Web

Cosmic Web Observed:

- traced by galaxies – galaxy halos
- Dependence observed structure cosmic web on halo/galaxy population
- Dependence topological characteristics on halo/galaxy population

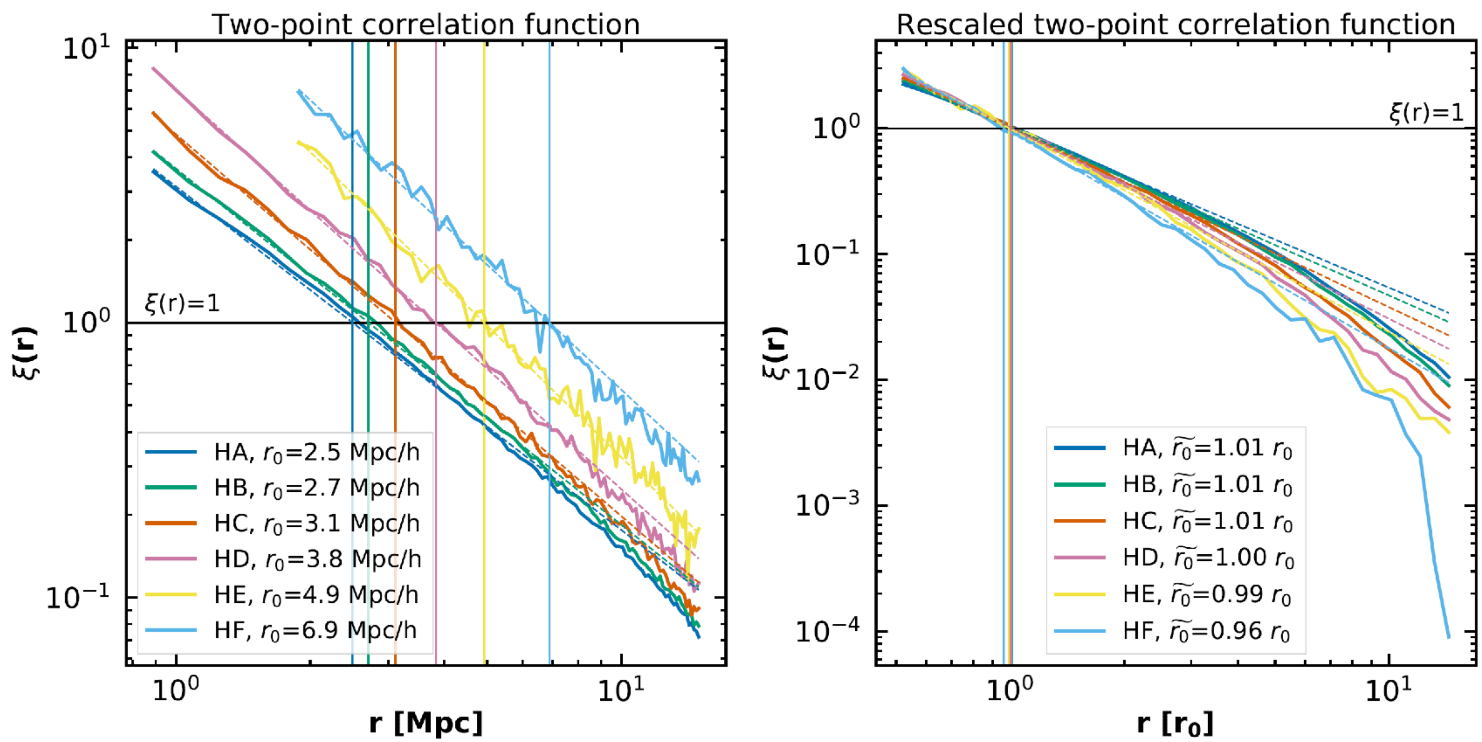


Halo Tracing – Cosmic Web

Cosmic Web Observed:

- **P-Millennium simulation (Cole et al.)**
- **Halo population in 6 mass ranges**
- **Clustering:**
- **2nd order correlation function ξ(r)**
- **Rescaling by correlation length**

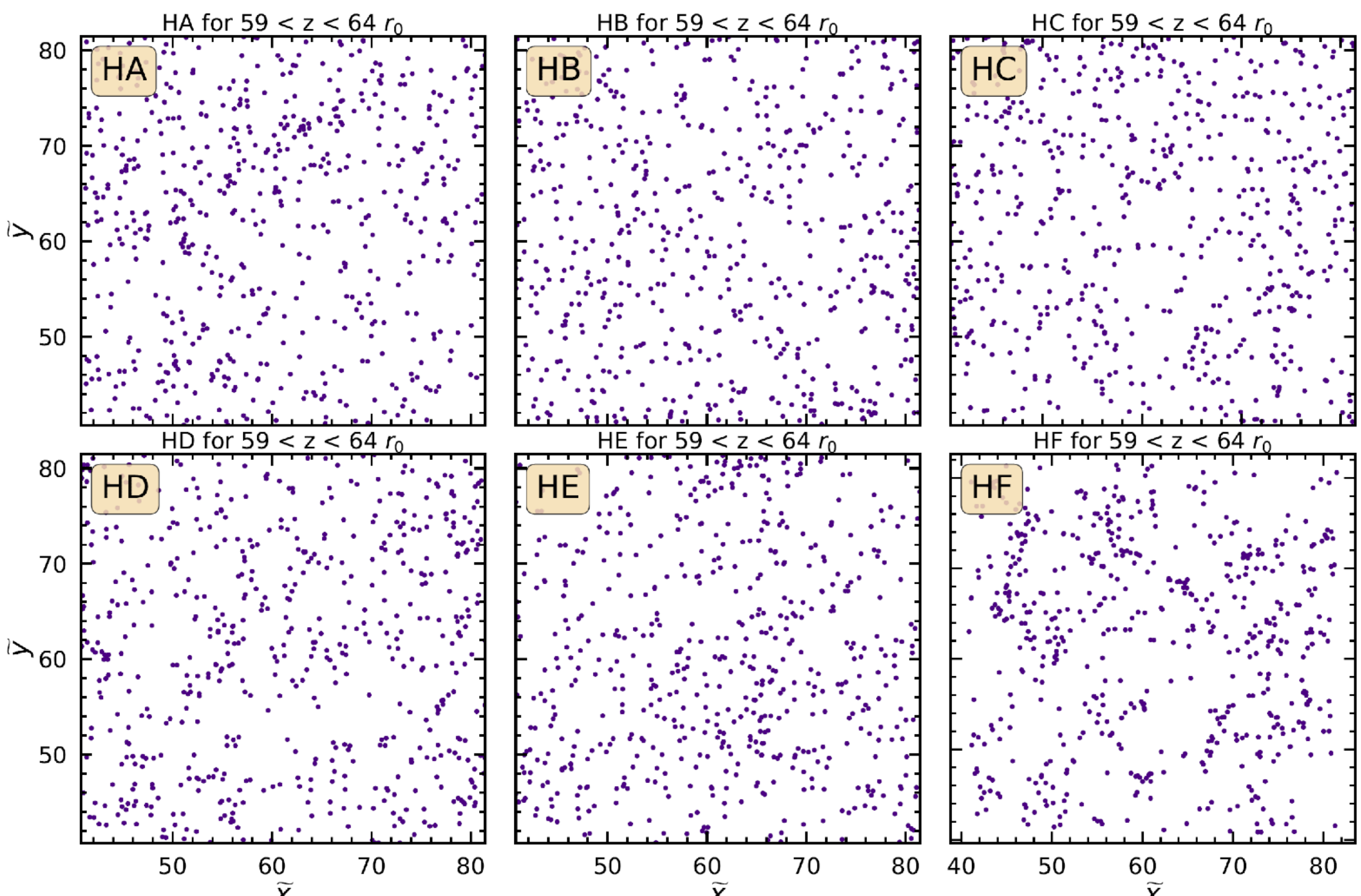
Halo Population	$\log_{10}$(M) $[h^{-1}M_\odot]$	r_0 $[h^{-1}Mpc]$	γ	$\widetilde{r_0}$ $[r_0]$	$\widetilde{\gamma}$
HA	(10.5,11]	2.49±0.01	-1.251±0.008	1.011±0.005	-1.27±0.01
HB	(11,11.5]	2.70±0.01	-1.297±0.006	1.009±0.004	-1.34±0.01
HC	(11.5,12]	3.11±0.02	-1.390±0.008	1.008±0.004	-1.43±0.01
HD	(12,12.5]	3.83±0.02	-1.454±0.006	1.005±0.005	-1.52±0.01
HE	(12.5,13]	4.92±0.04	-1.59±0.02	0.99±0.01	-1.61±0.03
HF	(13,13.5]	6.89±0.09	-1.51±0.02	0.961±0.007	-1.72±0.03



Halo Tracing – Cosmic Web

Cosmic Web Observed:

- **traced by galaxies – galaxy halos**
- **Incorporating 2nd order clustering – rescaling by correlation length**
- **Still dependence observed structure cosmic web on halo/galaxy population: higher order structural pattern**

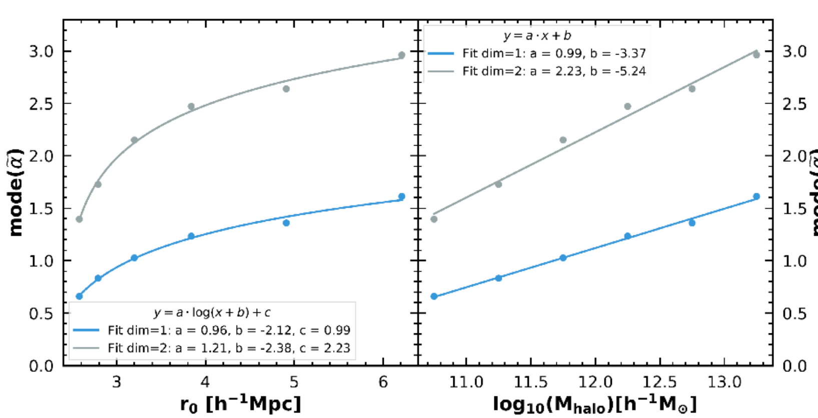


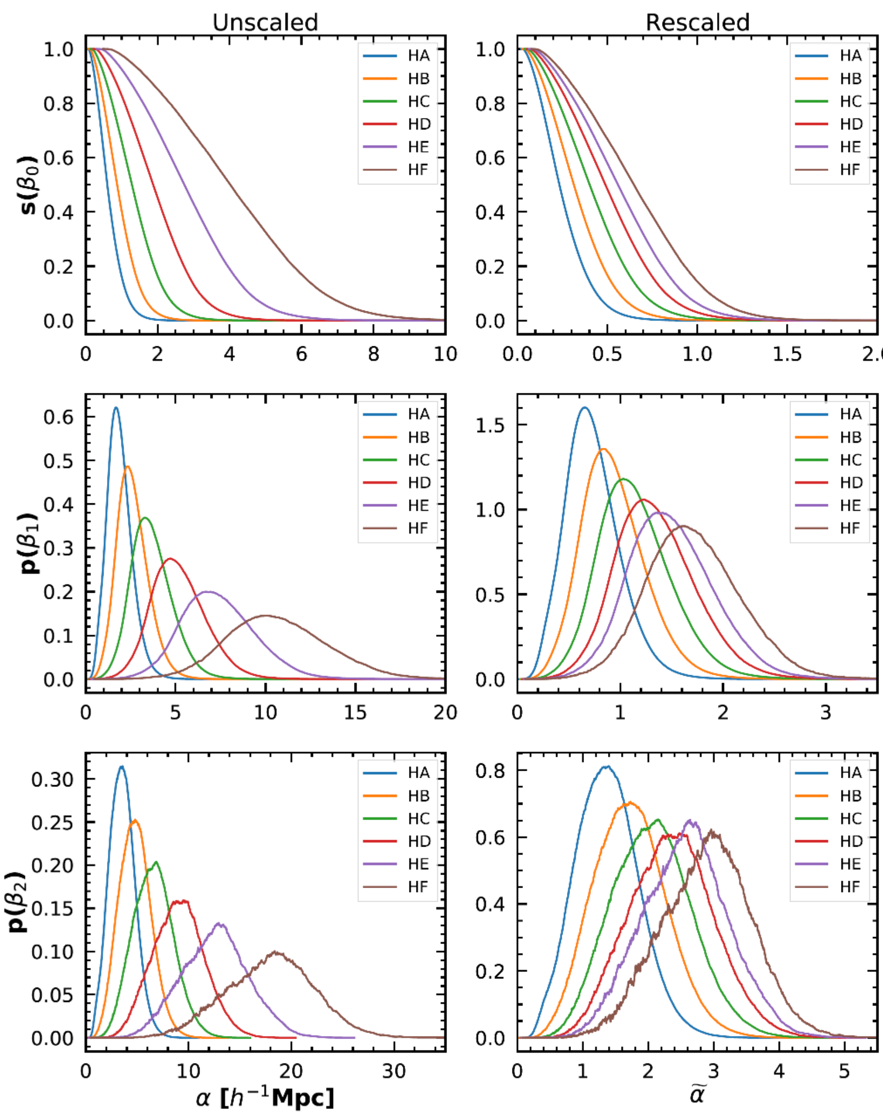
Betti Number - Halo Population

Cosmic Web Observed:

- Expected scaling with increasing mass range: shift to larger scales
- After rescaling by correlation length:
  - still shift to larger (scaled) scales
  - beautiful self-similar increase characteristic scale

- different topology cosmic web traced by different populations
- eg. Voids different halo tracers significantly different features !





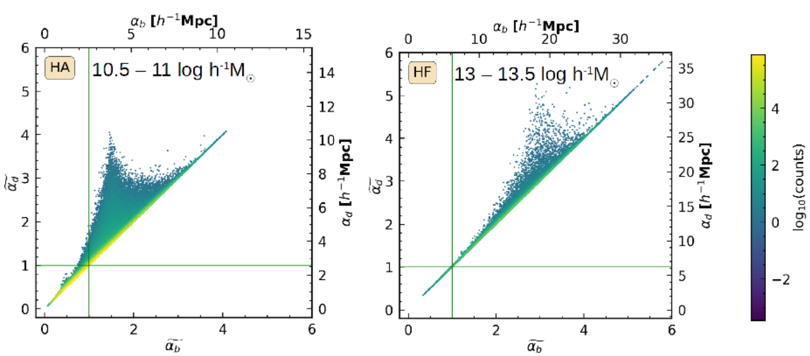
Persistence - Halo Population

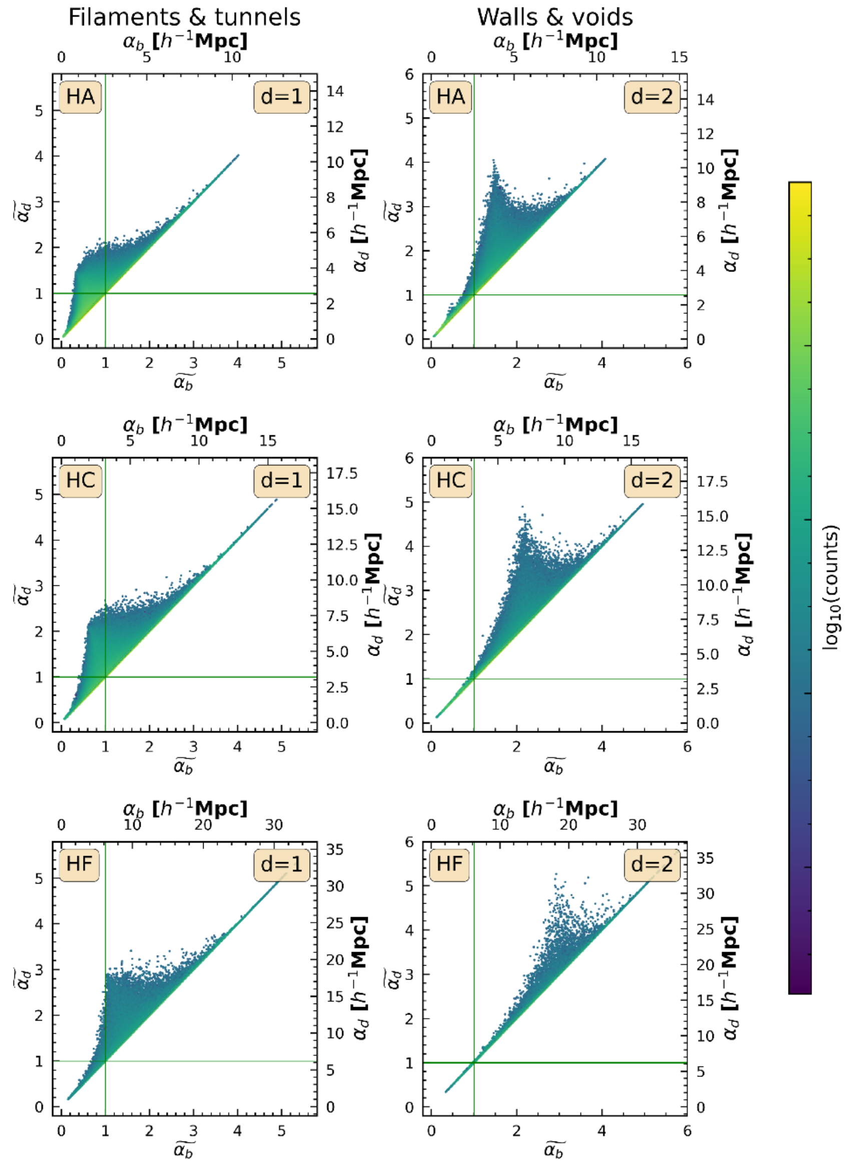
Cosmic Web Observed:

- Expected scaling with increasing mass range: shift to larger scales

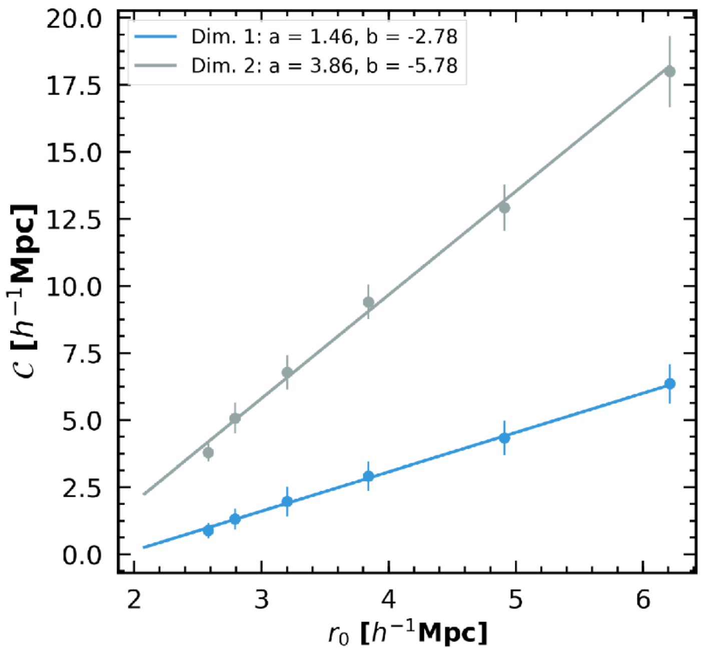
- After rescaling by correlation length:
  - still shift to larger (scaled) scales
  - beautiful self-similar increase characteristic scale

  - different topology cosmic web traced by different populations
  - eg. Voids different halo tracers significantly different features !

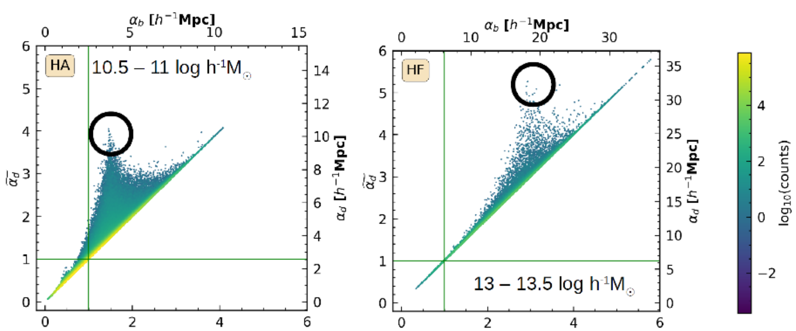


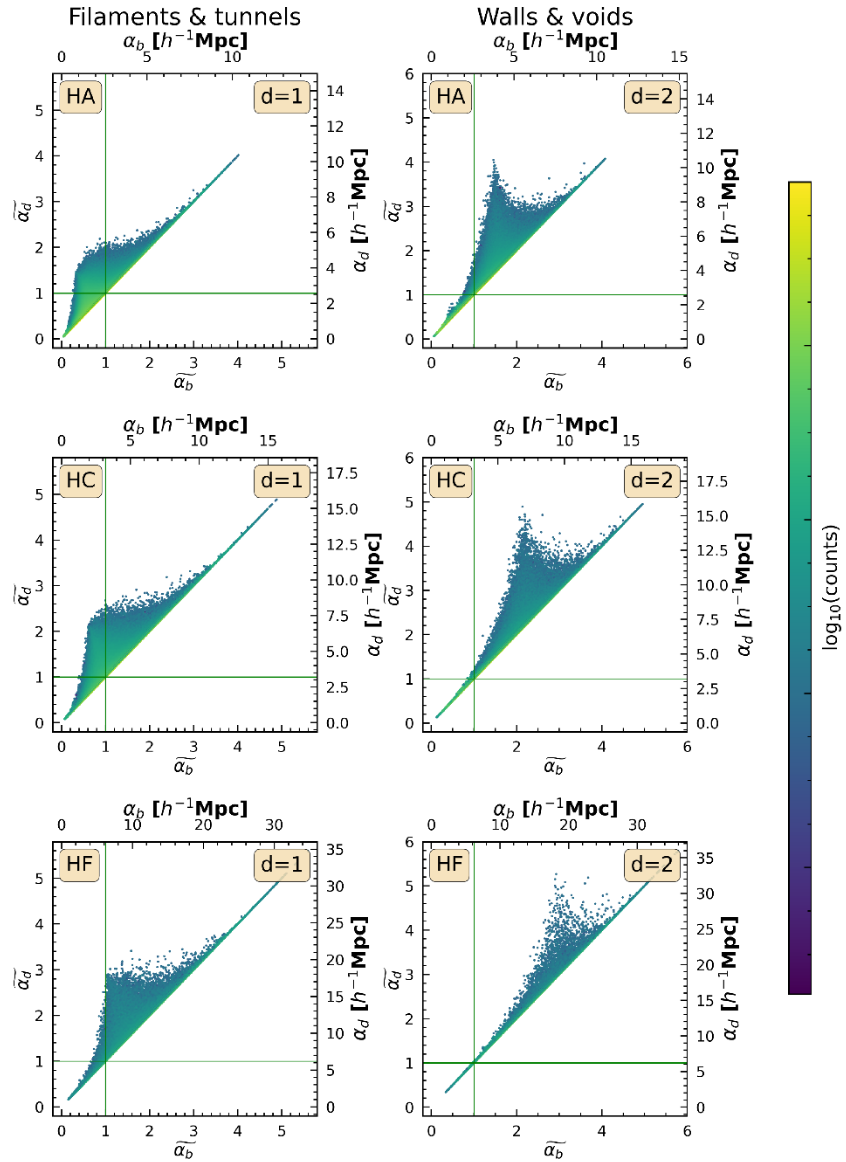


Persistence - Halo Population



- • **beautiful self-similar increase characteristic scale apex**





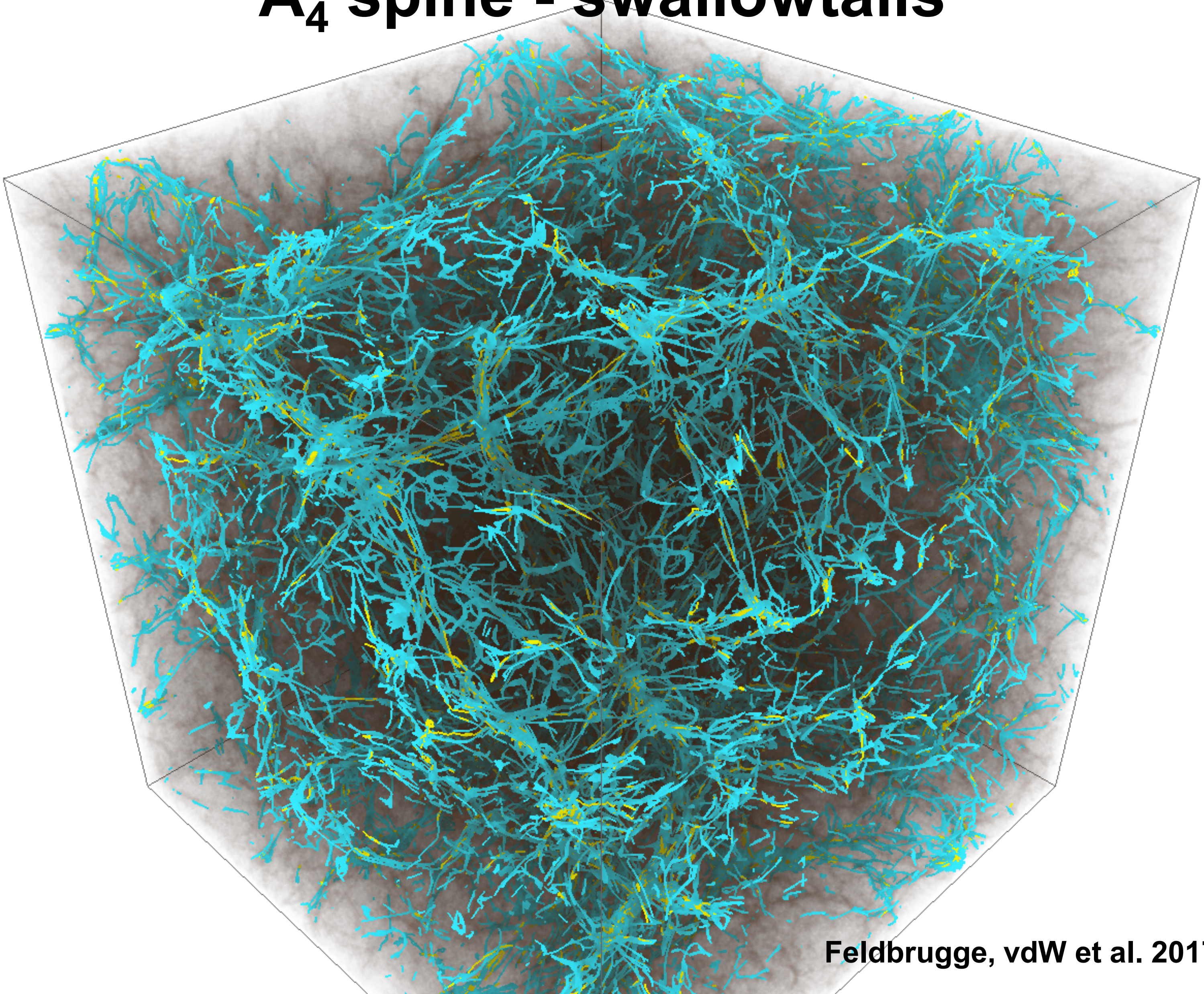
Cosmic Web Connectivity:

Caustic Skeleton

Feldbrugge, vdW et al. 2018, 2019
Hidding, Shandarin & vdW 2014
Wilding, Feldbrugge, vdW 2021

Skeleton (3D) Cosmic Web:

A_4 spine - swallowtails

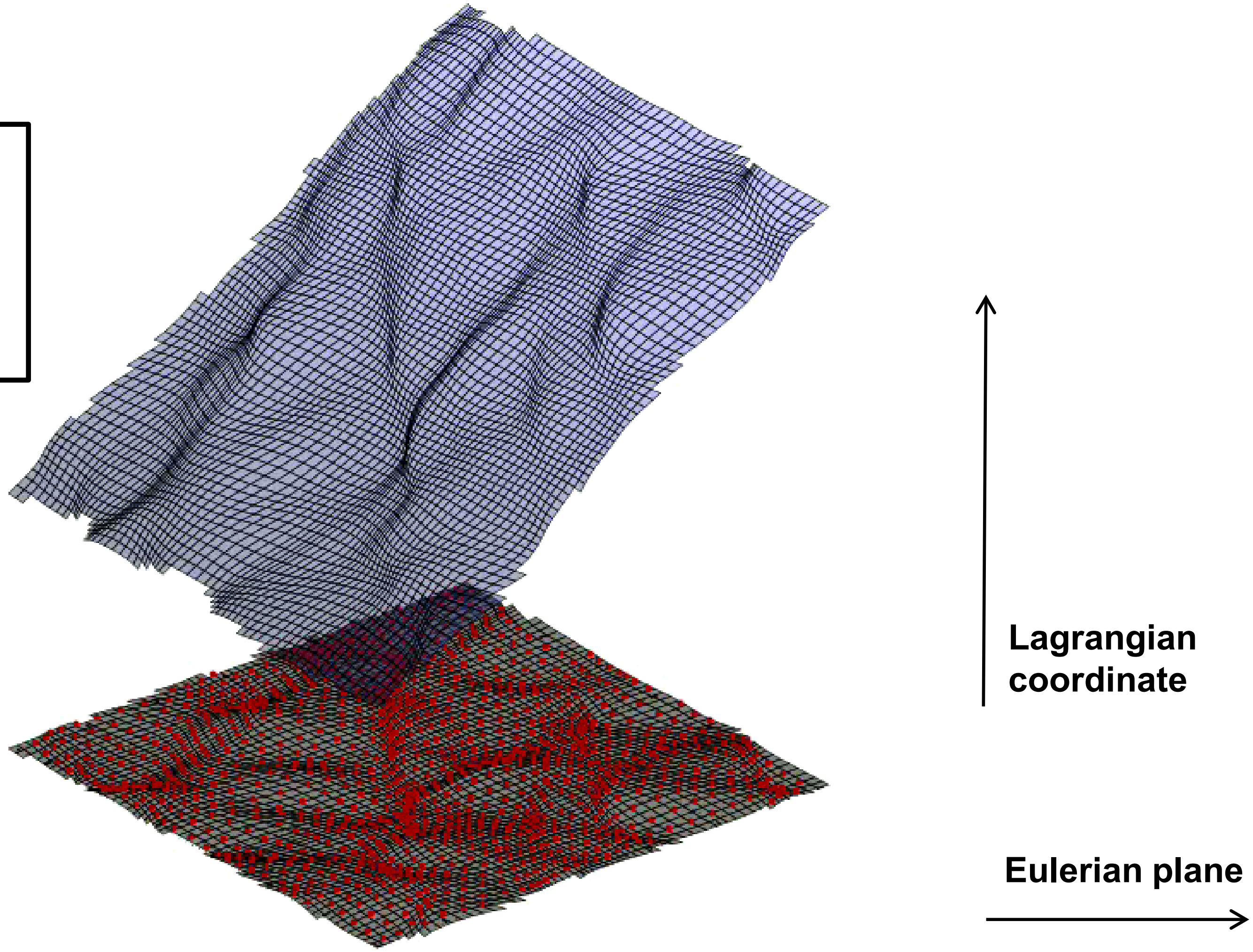


Feldbrugge, vdW et al. 2017b

Phase-Space Dynamics

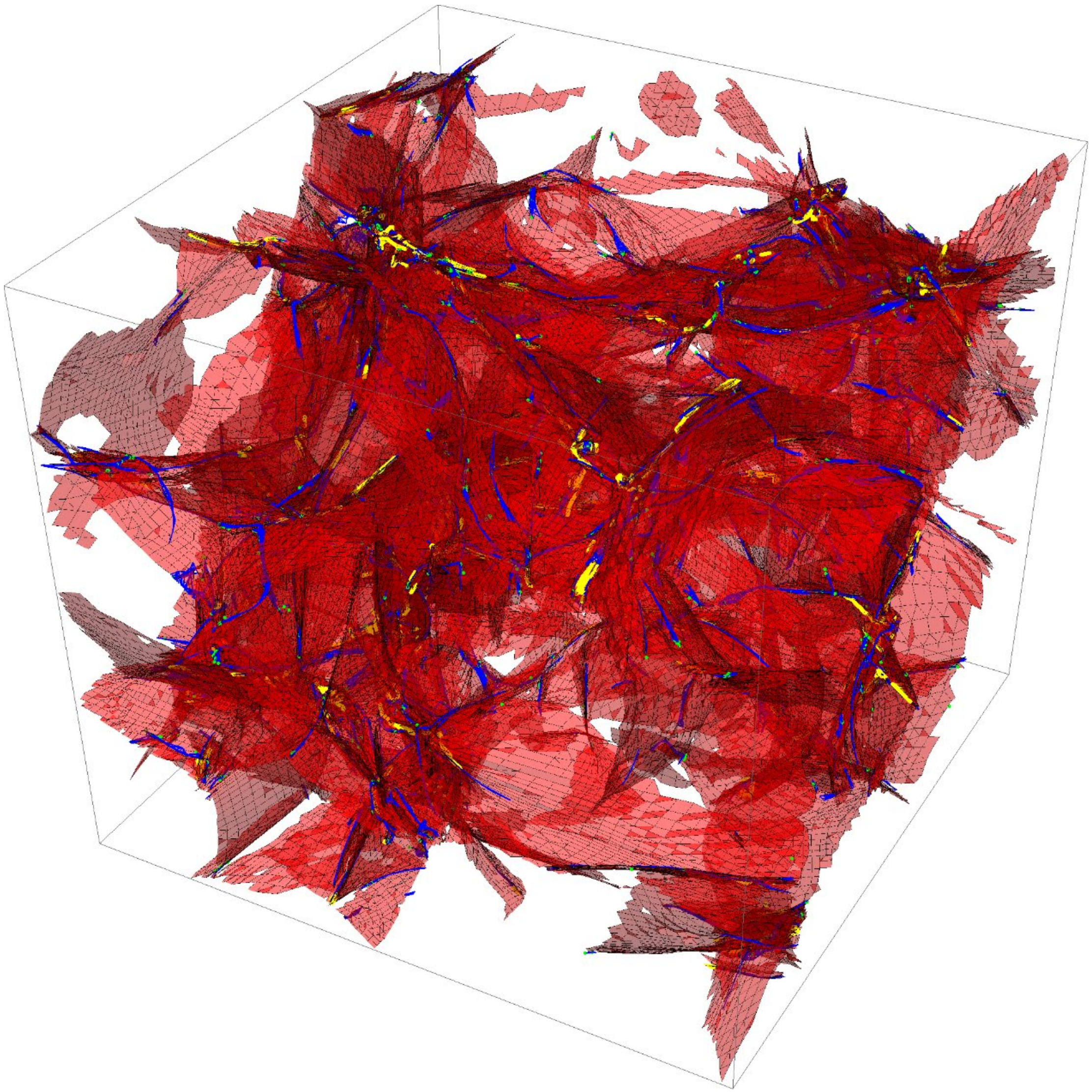
Dynamical Evolution:

folding the phase-space sheet {q,x}



Hidding 2014

Cosmic Web - Caustic Skeleton



Cosmic Web: Caustic features

⇩

Deformation field (Gaussian initial density field)

Caustic structure

Red sheets - cusps A3 singularities - walls

Blue lines - swallowtail A4 sing. - filaments

Green dots - butterfly A5 sing. - nodes

Yellow lines - D4 umbilics - filaments

Caustic connectivity:

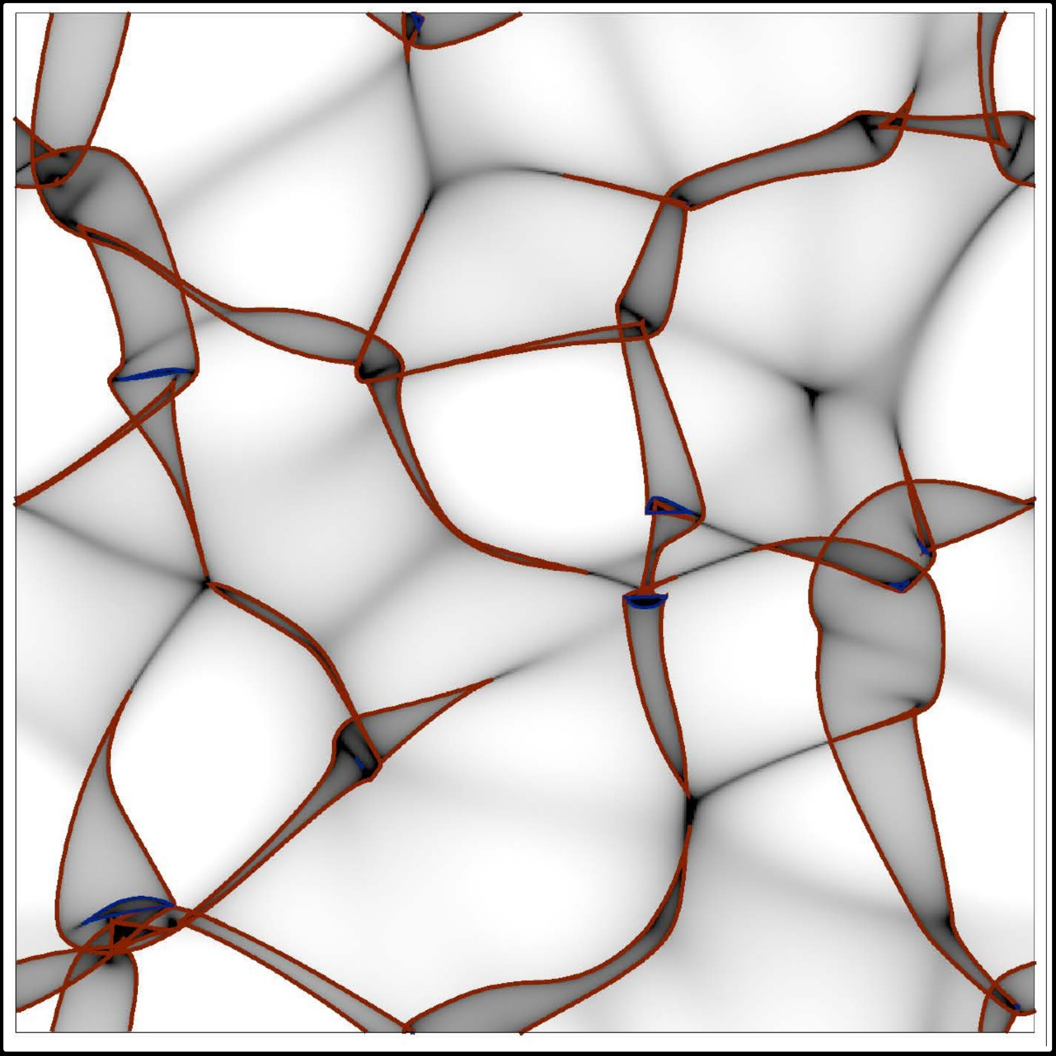
Topology of the deformation field

Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$\Phi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\phi_{lin}(\vec{q})$$



Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$d_{ij} = -\frac{\partial u_i}{\partial q_j}$$

$$\rho\left(\vec{q},t\right) = \frac{\rho_u\left(t\right)}{\left(1 - D(t)\lambda_1\left(\vec{q}\right)\right)\left(1 - D(t)\lambda_2\left(\vec{q}\right)\right)\left(1 - D(t)\lambda_3\left(\vec{q}\right)\right)}$$

structure of the cosmic web determined by the spatial field of eigenvalues

$$\lambda_1, \lambda_2, \lambda_3$$

Deformation, Streaming & Caustics

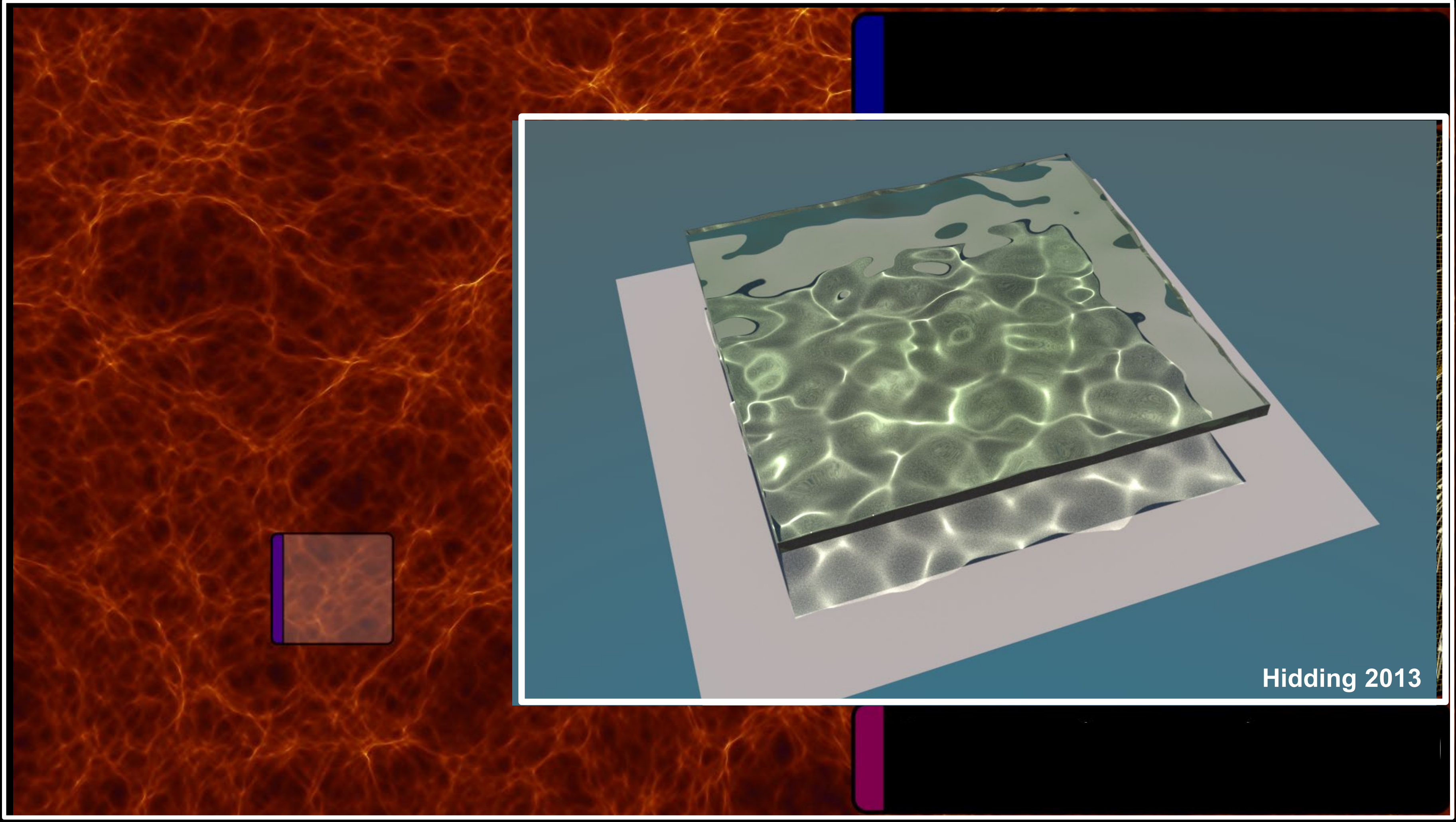
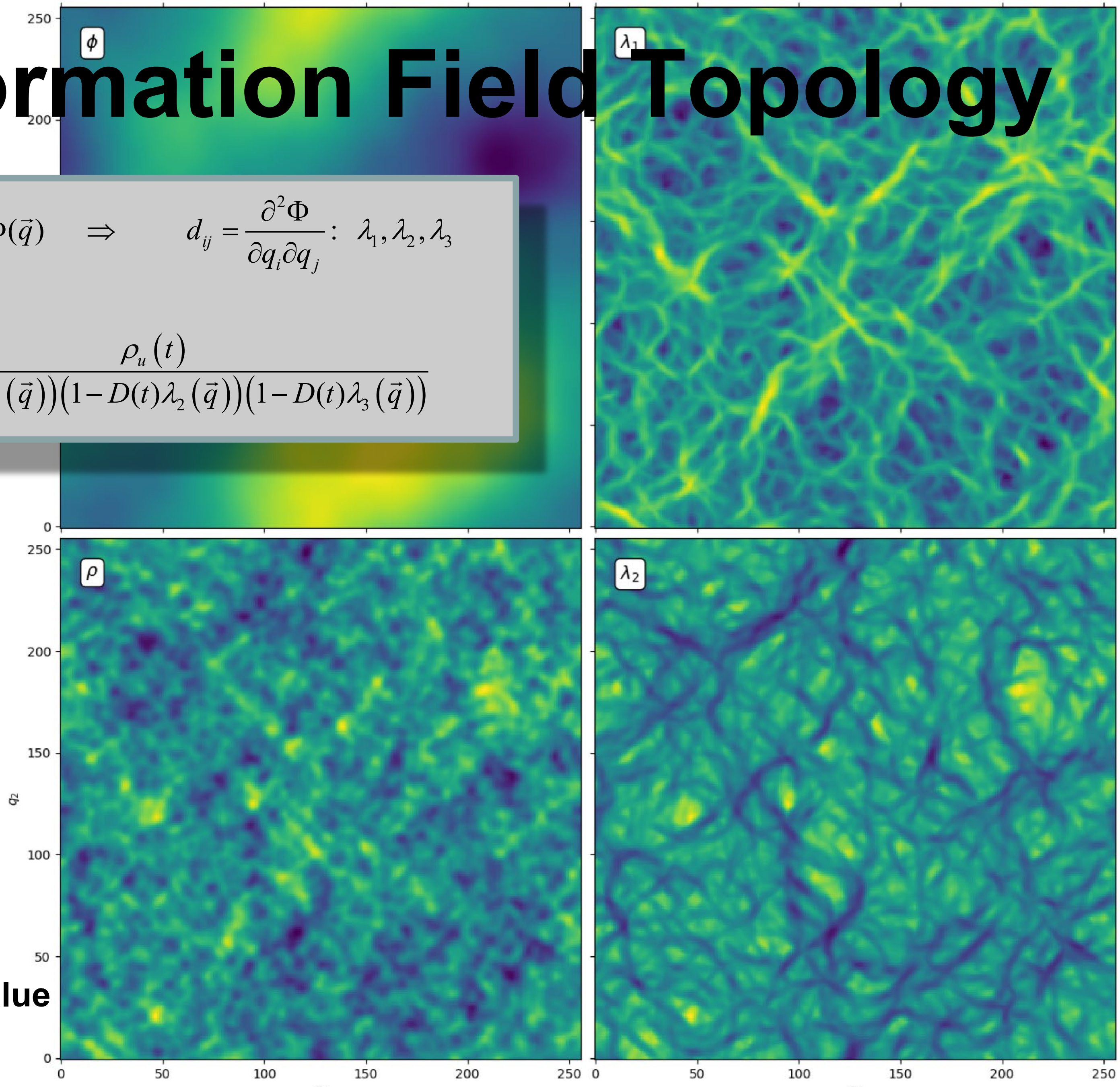


Illustration of the formation of caustics due to
streaming paths of light through deforming medium

Deformation Field Topology

$$\vec{x}(\vec{q},t) = \vec{q} - D(t)\vec{\nabla}\Phi(\vec{q}) \quad \Rightarrow \quad d_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j} : \; \lambda_1, \lambda_2, \lambda_3$$

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1 - D(t)\lambda_1(\vec{q})\right)\left(1 - D(t)\lambda_2(\vec{q})\right)\left(1 - D(t)\lambda_3(\vec{q})\right)}$$



Zeldovich deformation eigenvalue landscape

Caustic Conditions & Structure Formation: Eigenvalues & Eigenvectors Deformation Field

In Lagrangian space L (coordinates q):

A singularity forms in a manifold $\mathrm{M} \subset L$ at location q_s when at q_s,

- the deformation tensor eigenvalue $\mu_i(q_s)$
- the corresponding eigenvector $\vec{v}_i\,(q_s)$

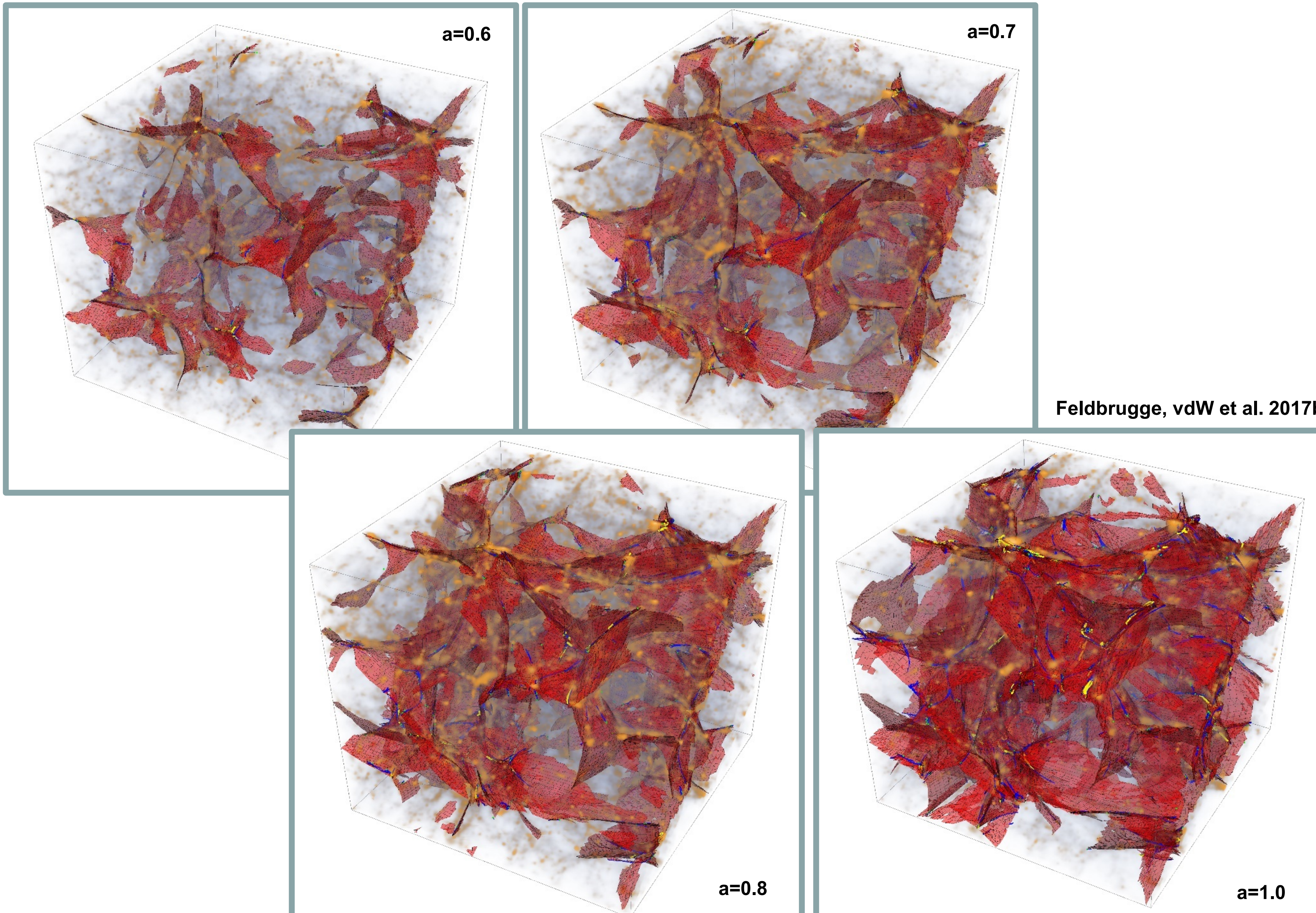
when at least one nonzero tangent vector $\vec{T}$ such that

$$\{1+\mu_i(q_s)\}\;\; \vec{v}_i^{\,*}(q_s)\cdot\vec{T} = 0$$

NOTE: Nature of singularity not only dependent on EIGENVALUES $\mu_i(q_s)$, *but also EIGENVECTORS* $\vec{v}_i\,(q_s)$

Feldbrugge, vdW et al. 2018

Cosmic Web - Evolving Skeleton

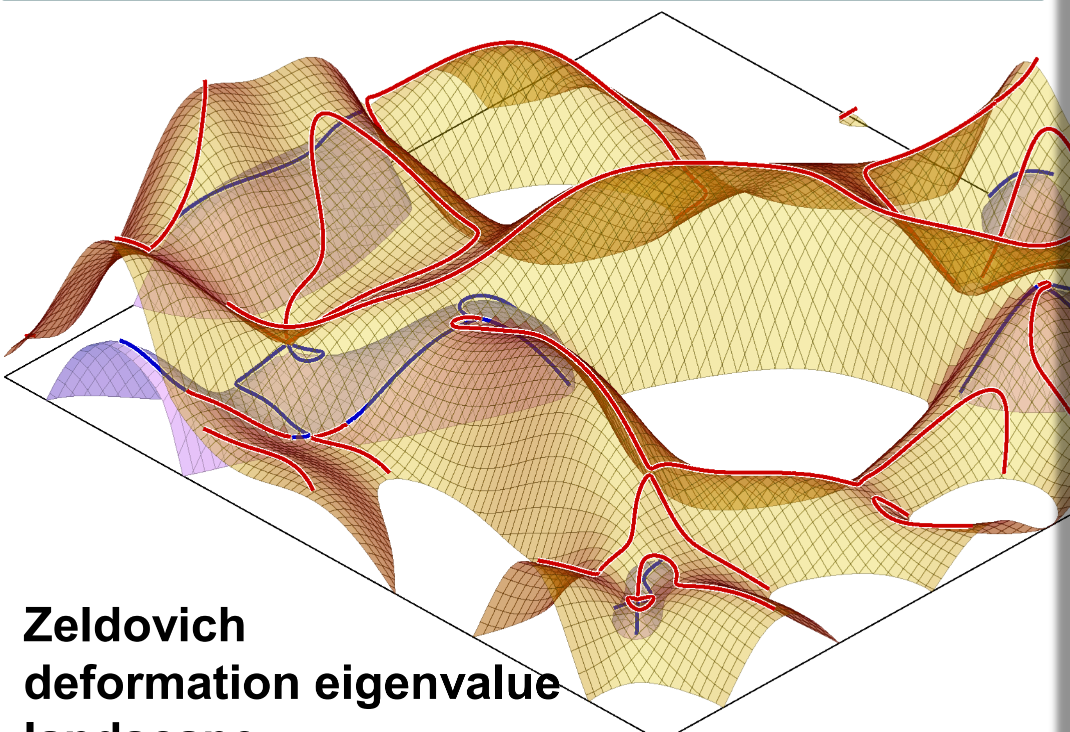


Feldbrugge, vdW et al. 2017b

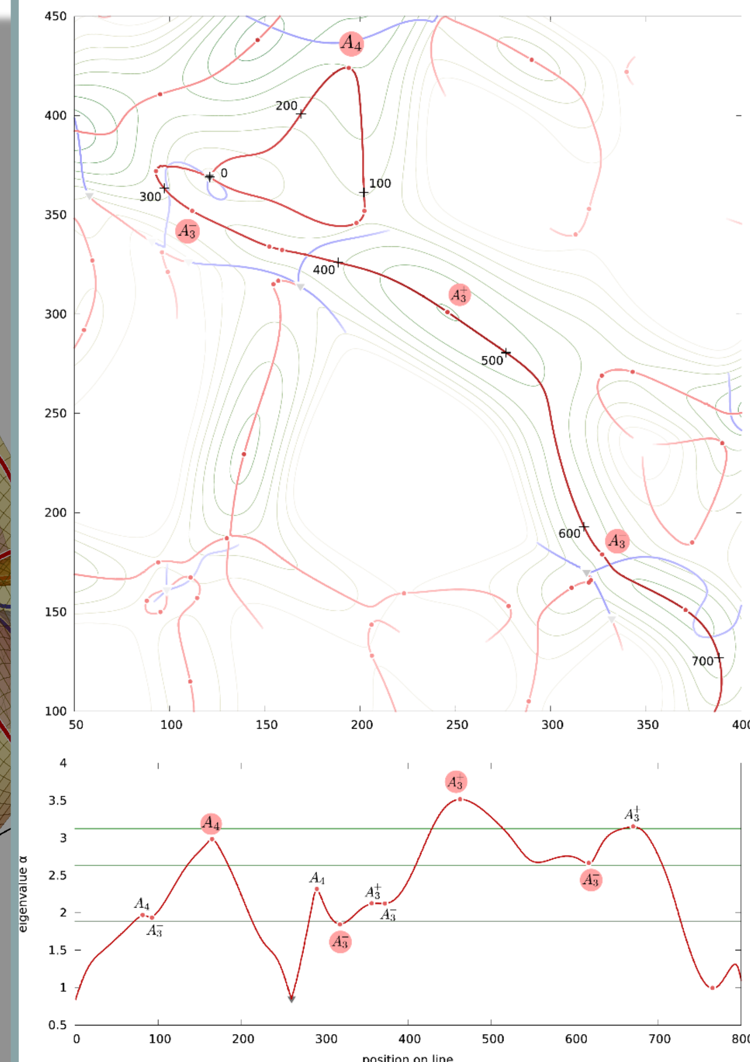
Singularities & Catastrophes:
Deformation Field

$$\vec{x}(\vec{q},t)=\vec{q}-D(t)\vec{\nabla}\Phi(\vec{q}) \quad \Rightarrow \quad d_{ij}=\frac{\partial^2\Phi}{\partial q_i \partial q_j}:\ \lambda_1,\lambda_2,\lambda_3$$

$$\rho(\vec{q},t)=\frac{\rho_u(t)}{\left(1-D(t)\lambda_1(\vec{q})\right)\left(1-D(t)\lambda_2(\vec{q})\right)\left(1-D(t)\lambda_3(\vec{q})\right)}$$



Zeldovich deformation eigenvalue landscape

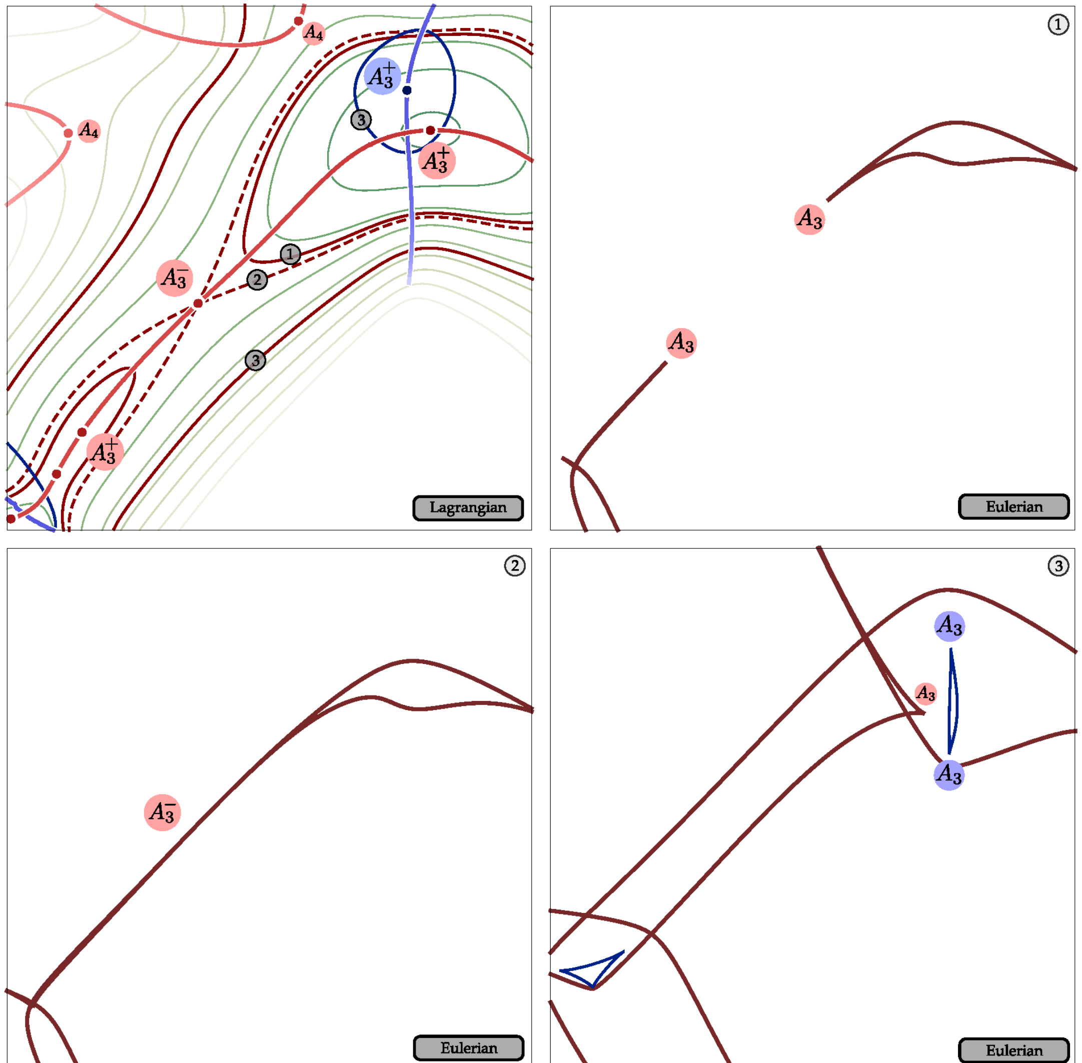


Hidding, Shandarin & vdW 2014

Caustic Merging & Annihilation

Caustic Hierarchy

Merging & Annihilation of Structures



Evolution & Merging Caustic Features:

Topological transformations in the deformation field:

- Growth of islands ("pancakes") starts at **maxima (A_3^+)** in deformation field λ_i
- Merging of islands at **saddle points (A_3^-)** in deformation field λ_i

The merger of two pancakes.

- Top-left: Lagrangian space.

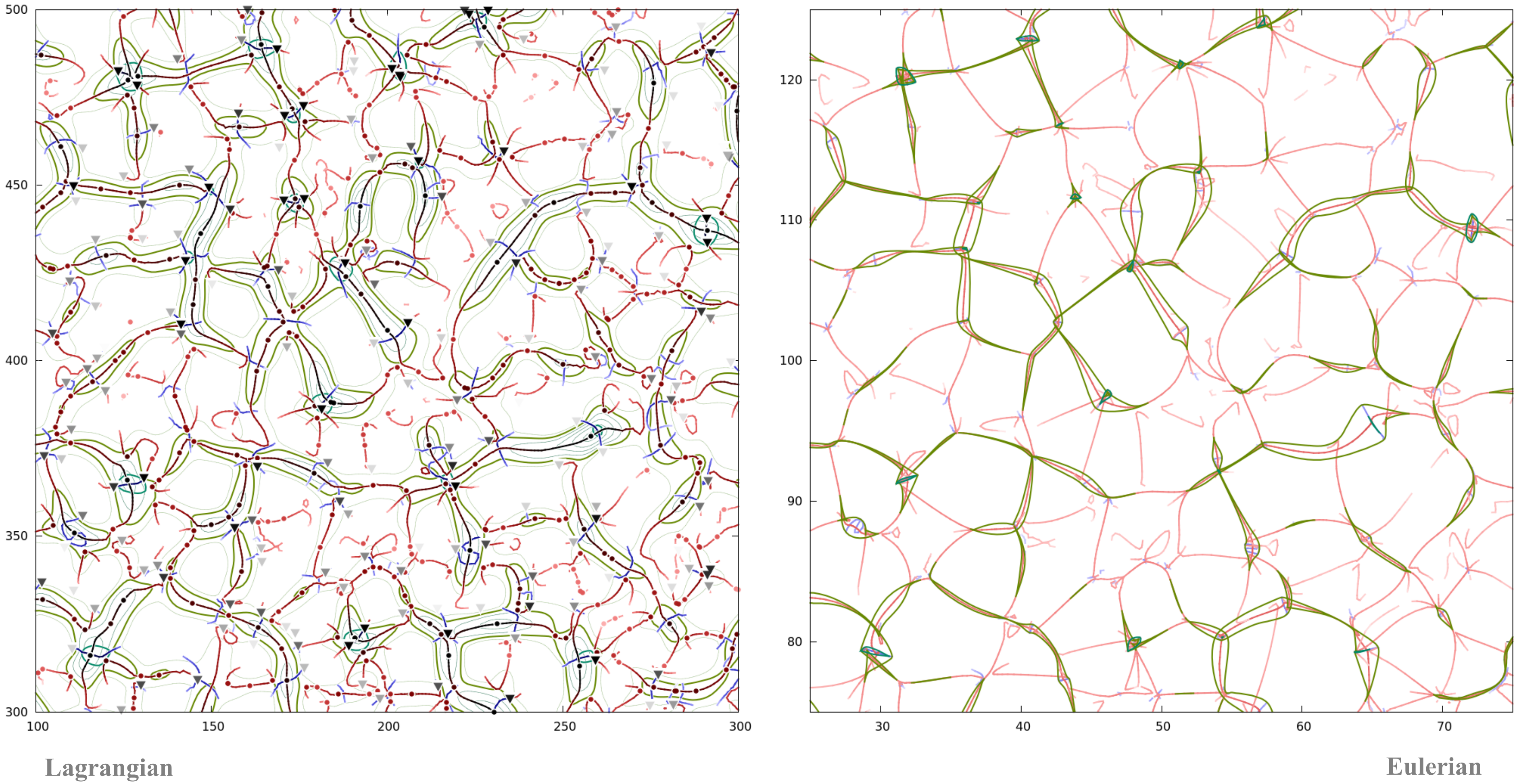
- Other panels: the evolution of caustics in Eulerian space.

Hierarchical Buildup of Cosmic Skeleton:

- Topological transformations at maxima and saddles
- Persistence diagrams (birth-death)

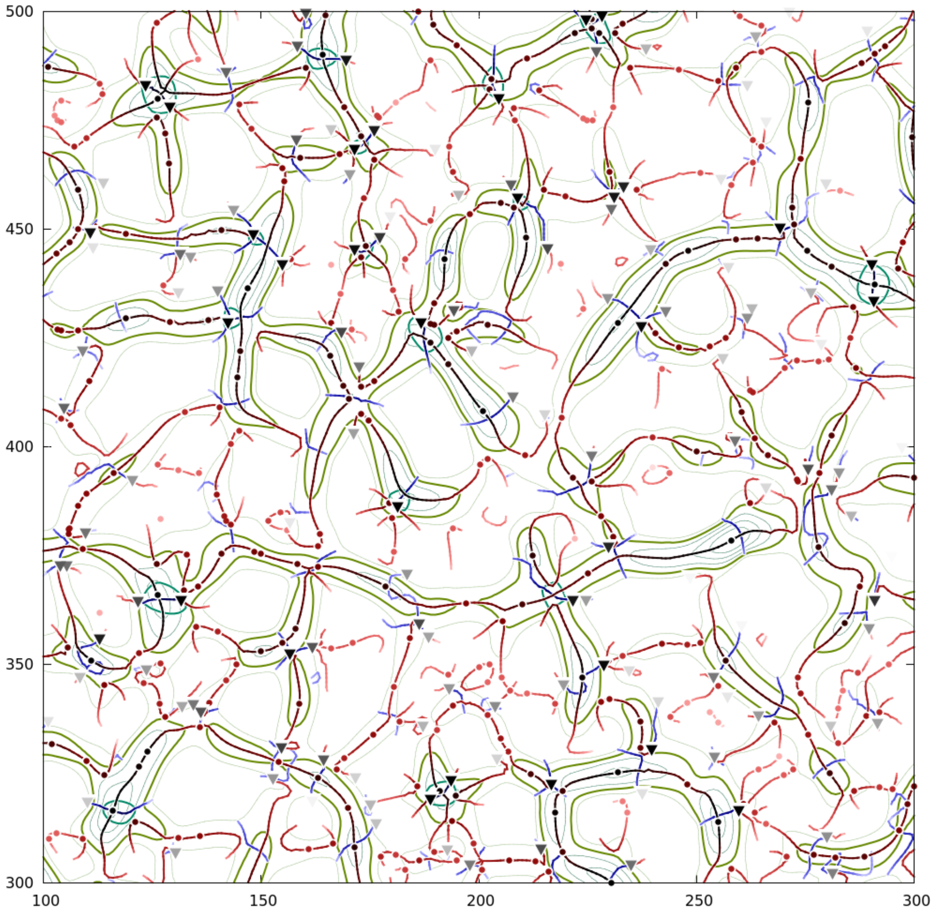
Caustic Hierarchy

Merging & Annihilation of Structures

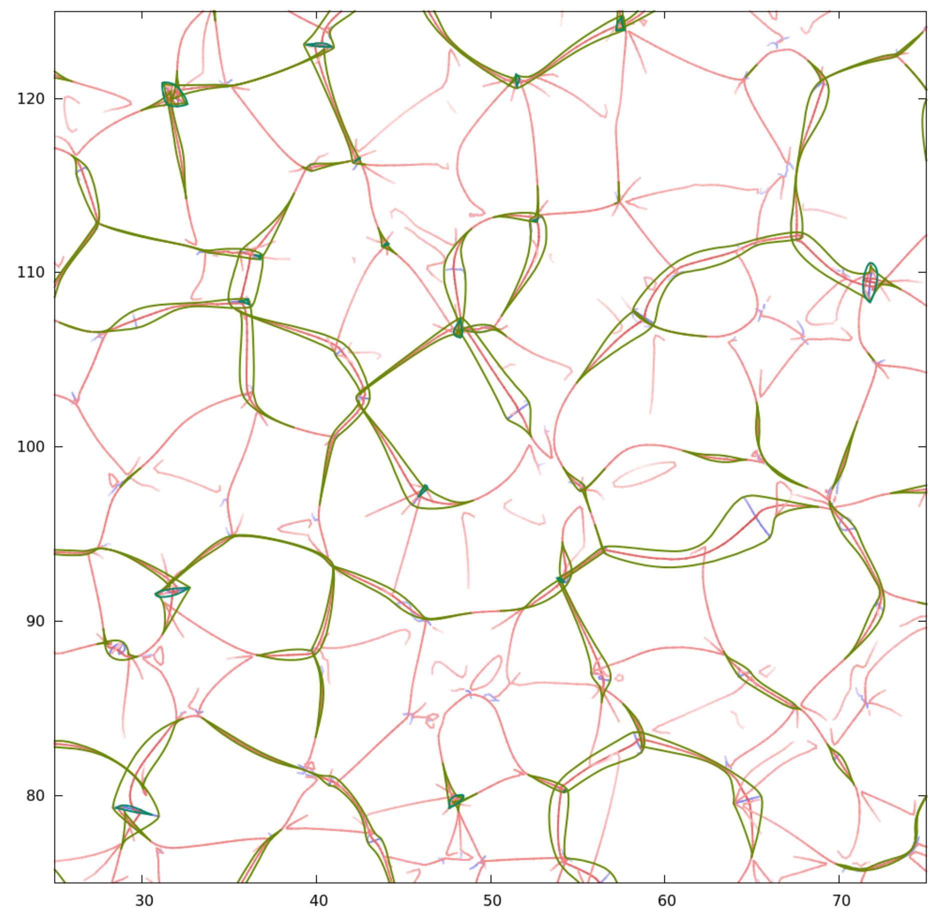


Caustic Hierarchy

Merging & Annihilation of Structures



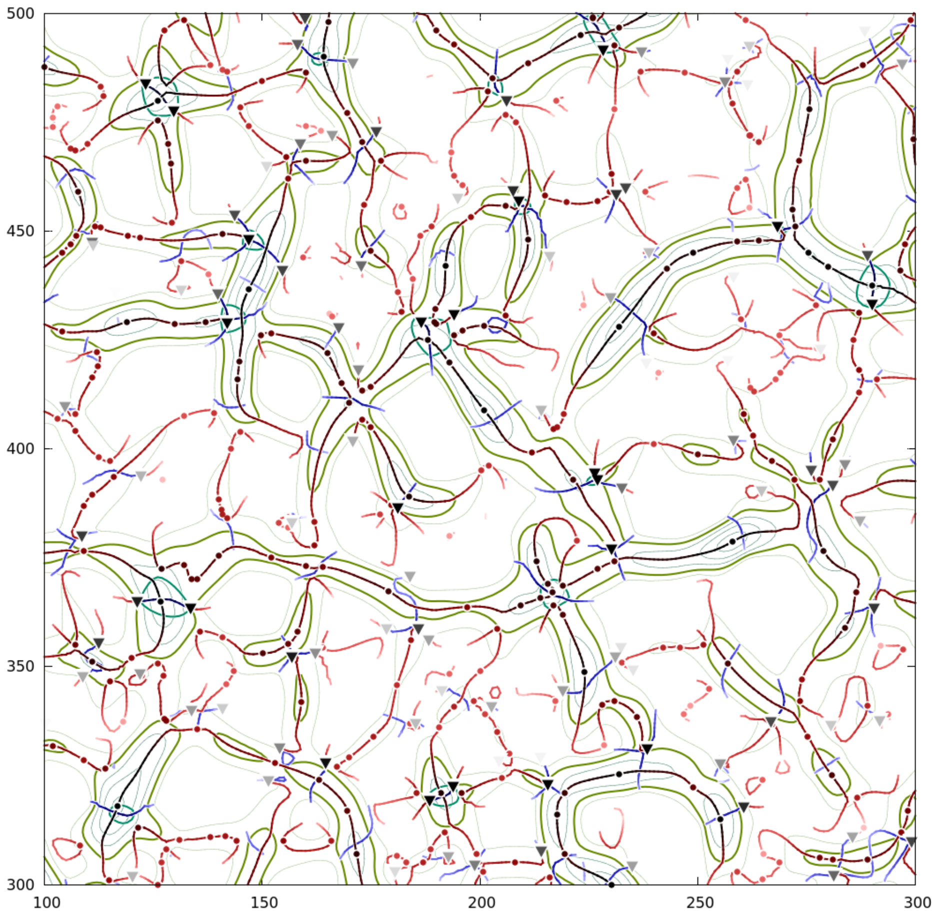
Lagrangian

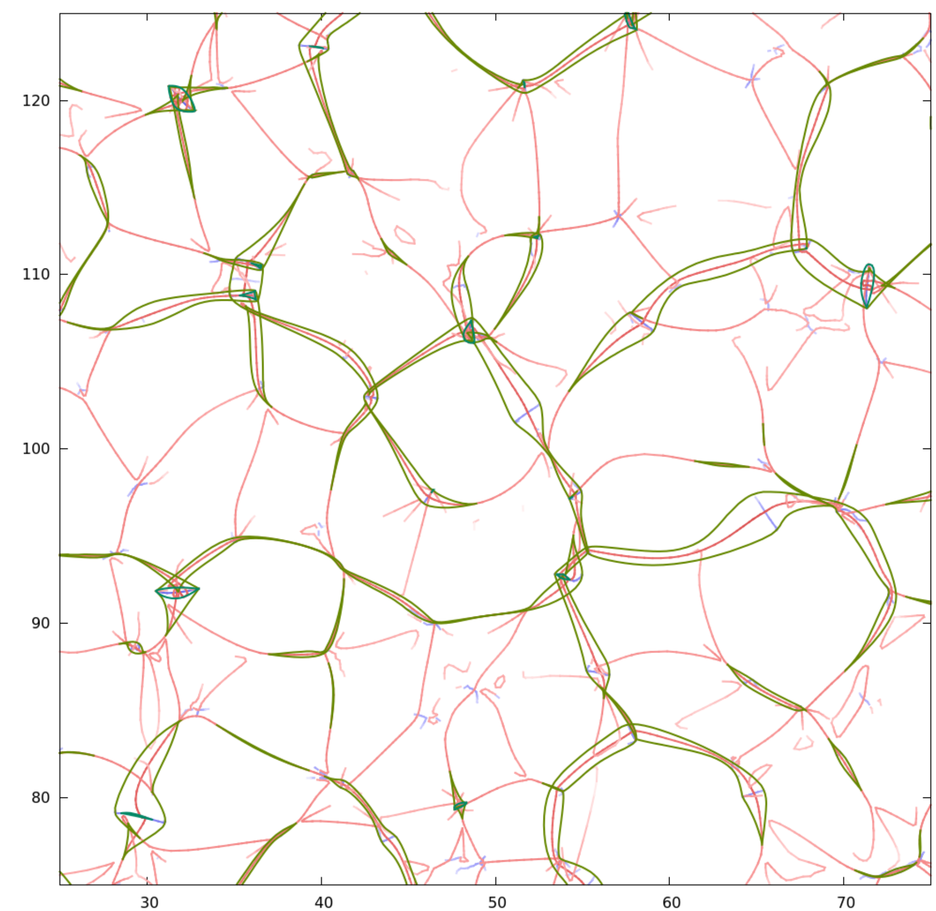


Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



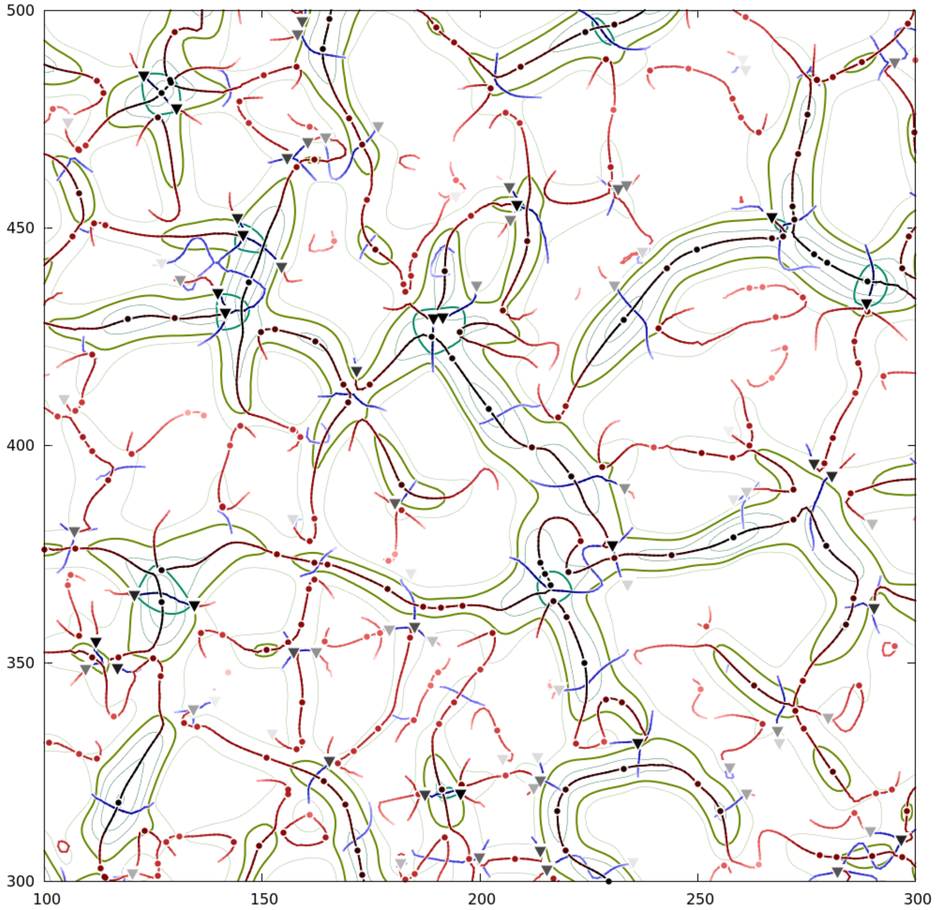


Lagrangian

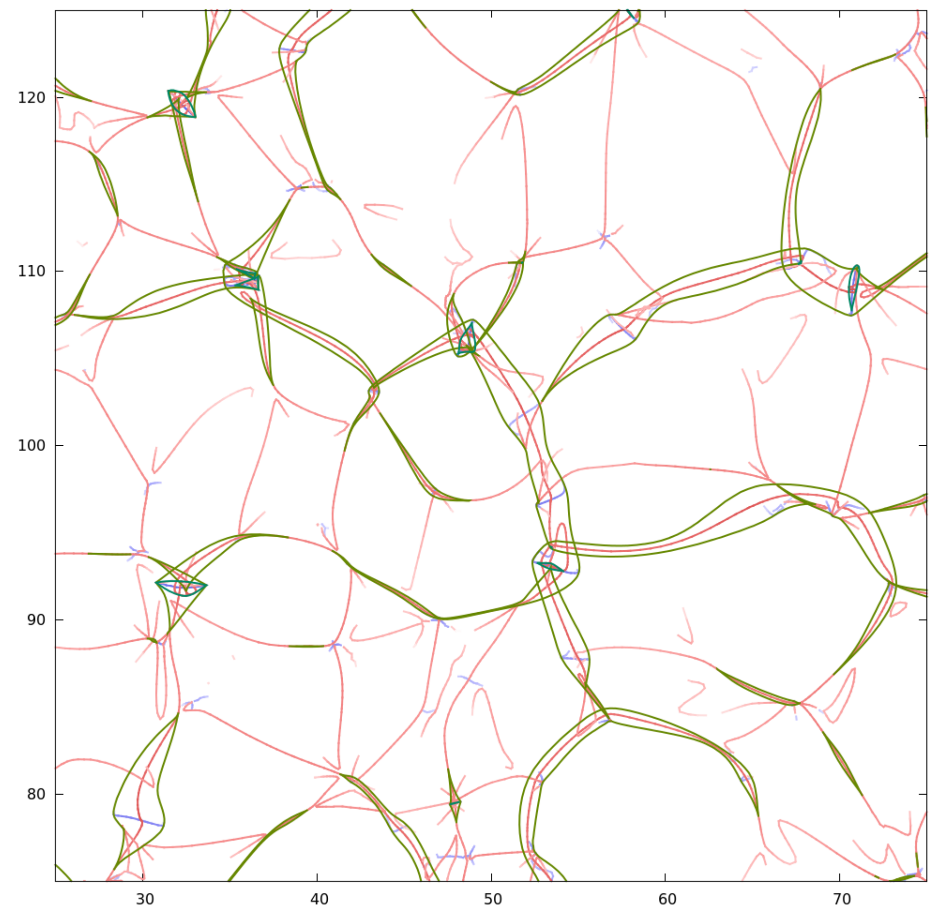
Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



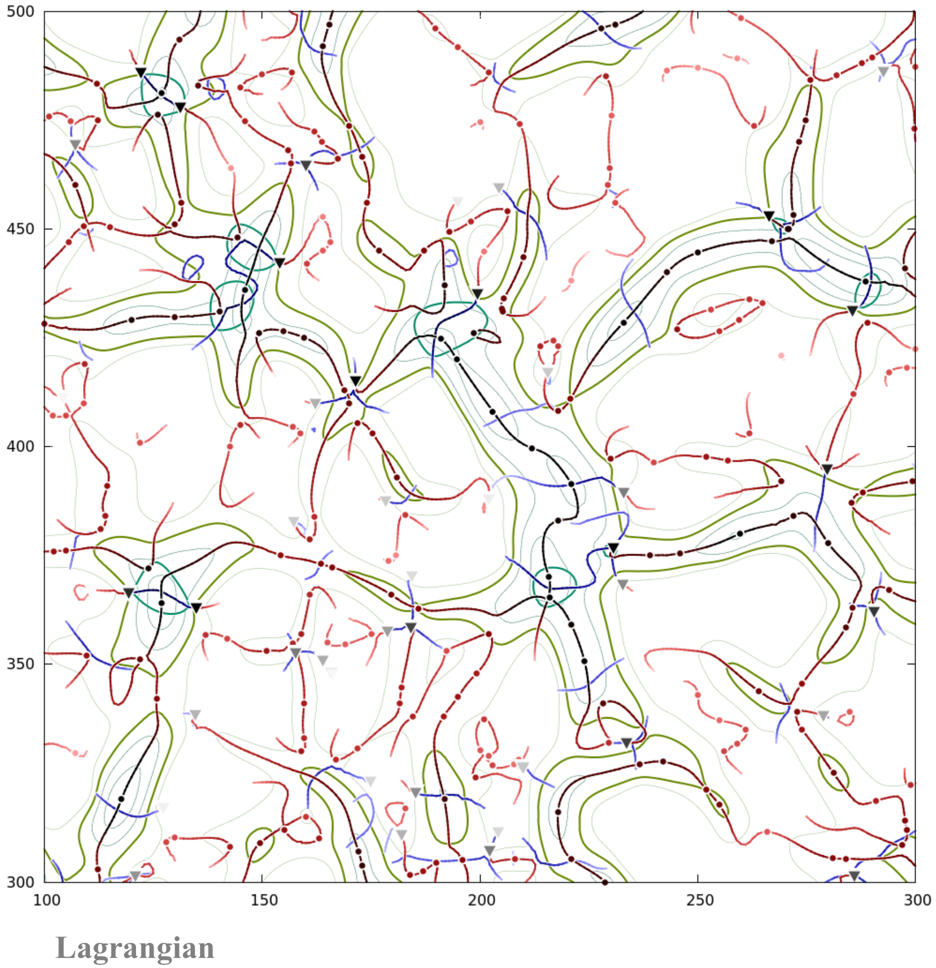
Lagrangian

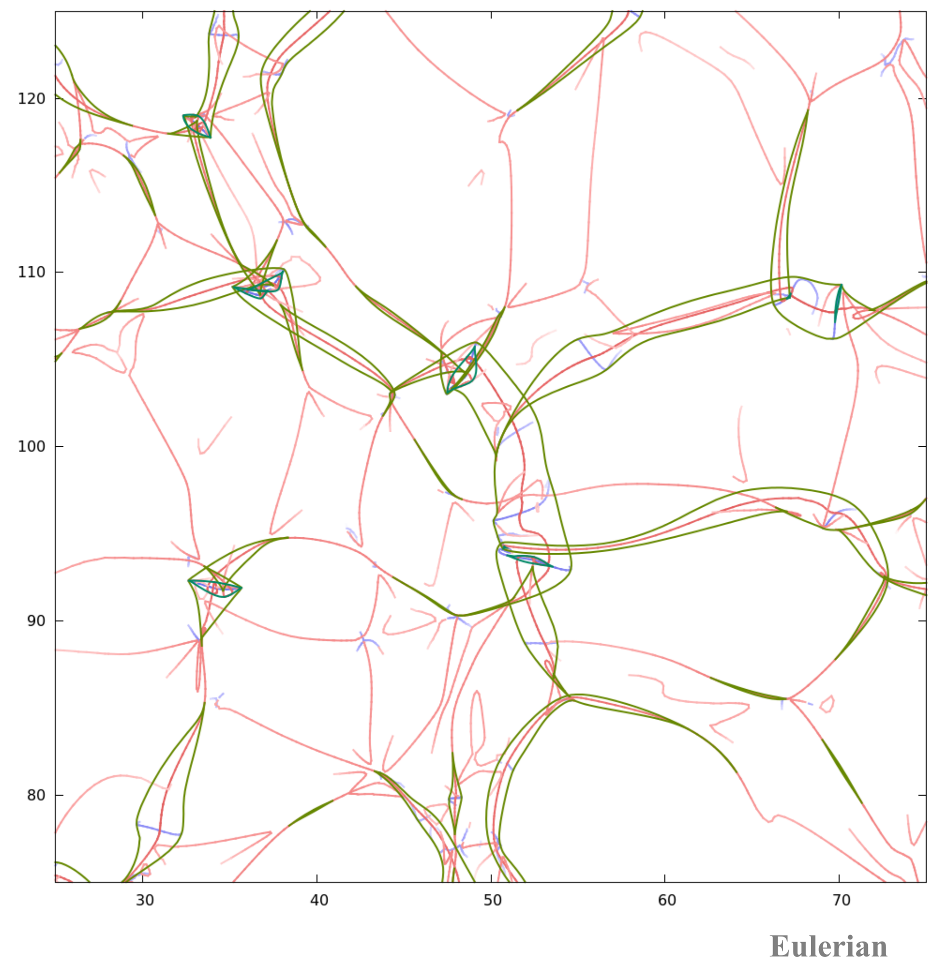


Eulerian

Caustic Hierarchy

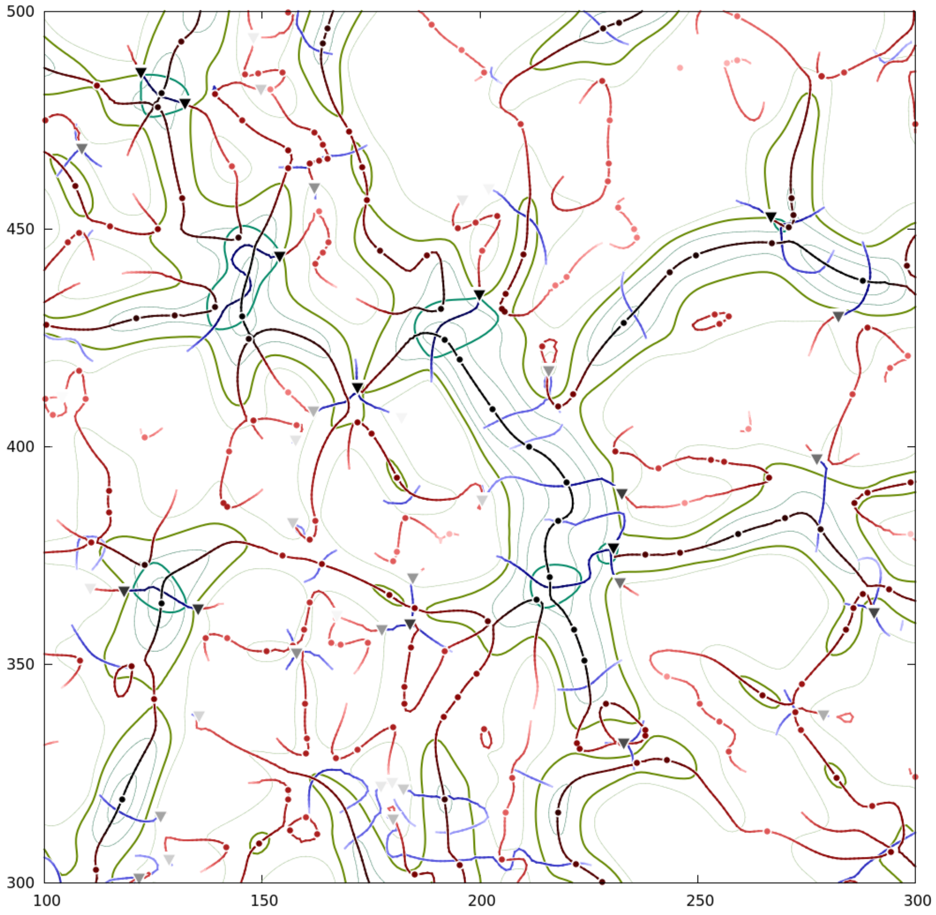
Merging & Annihilation of Structures



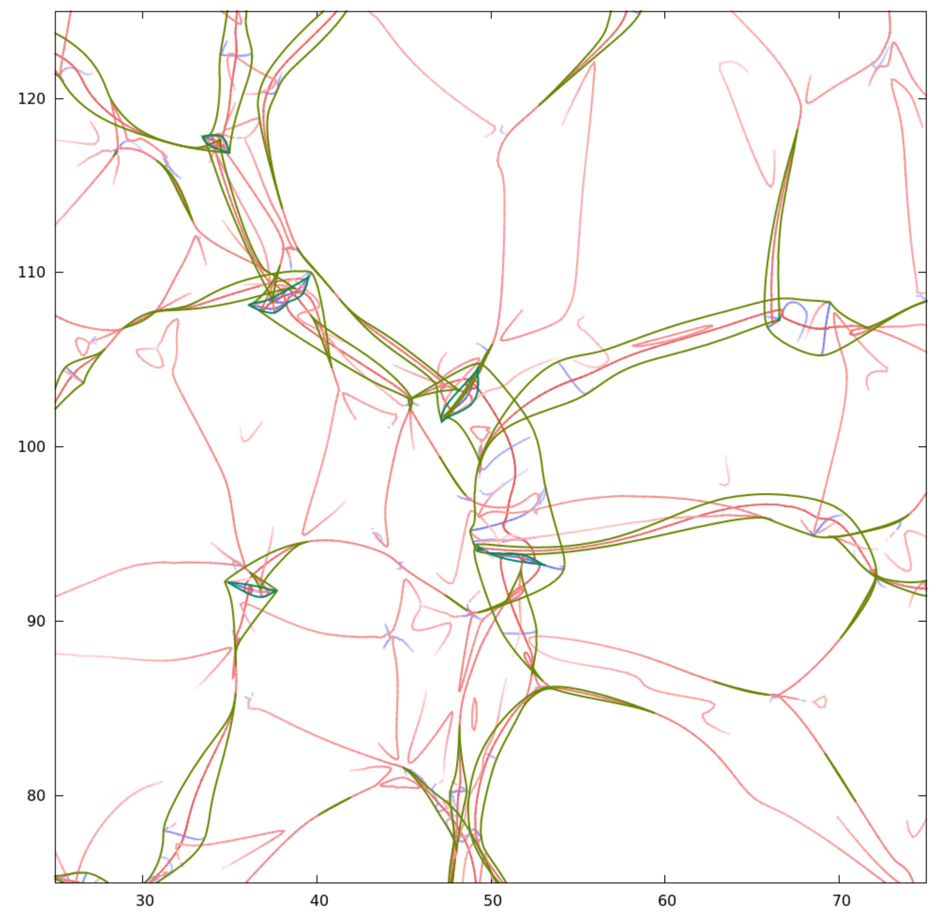


Caustic Hierarchy

Merging & Annihilation of Structures



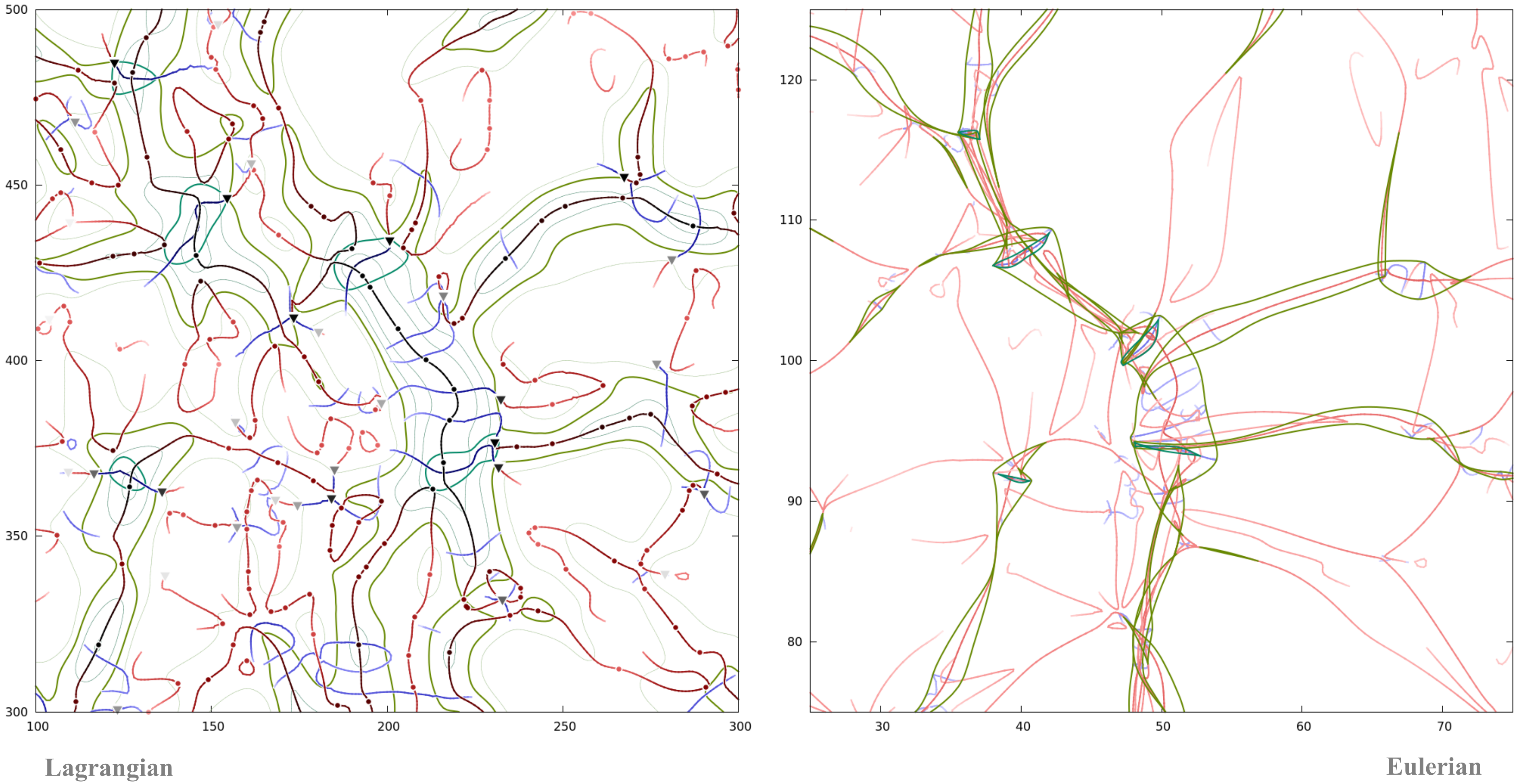
Lagrangian



Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



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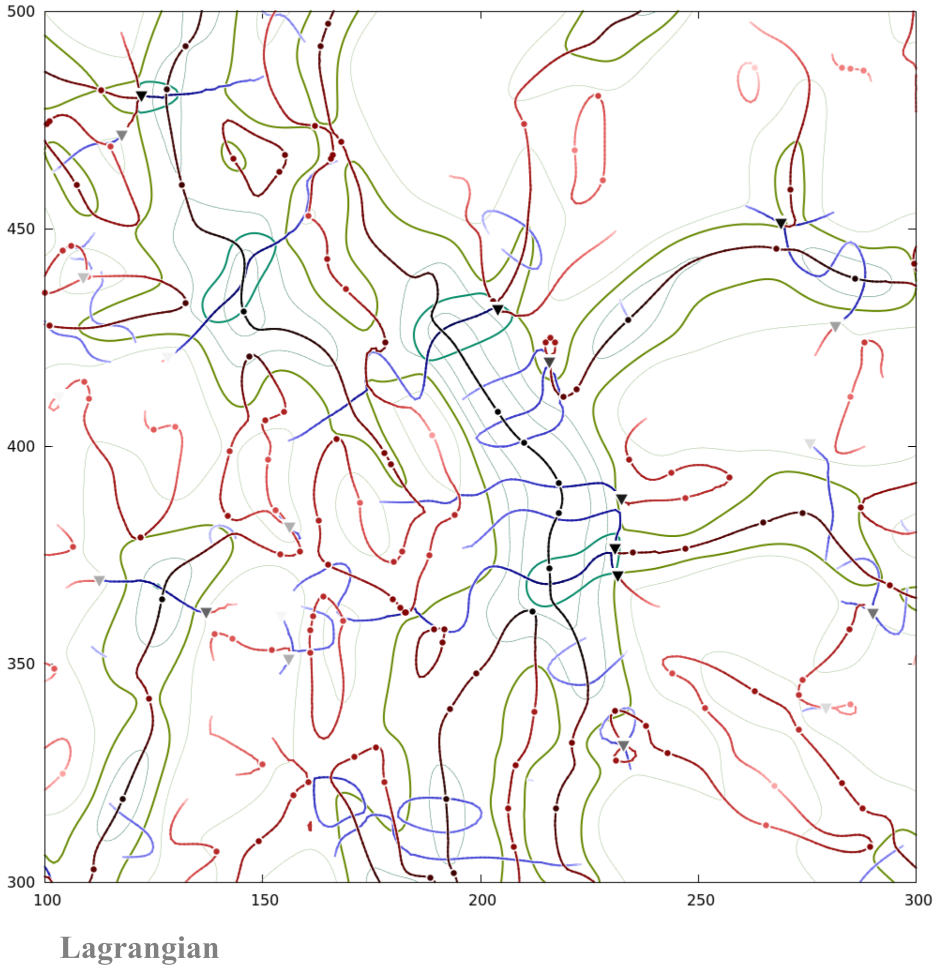

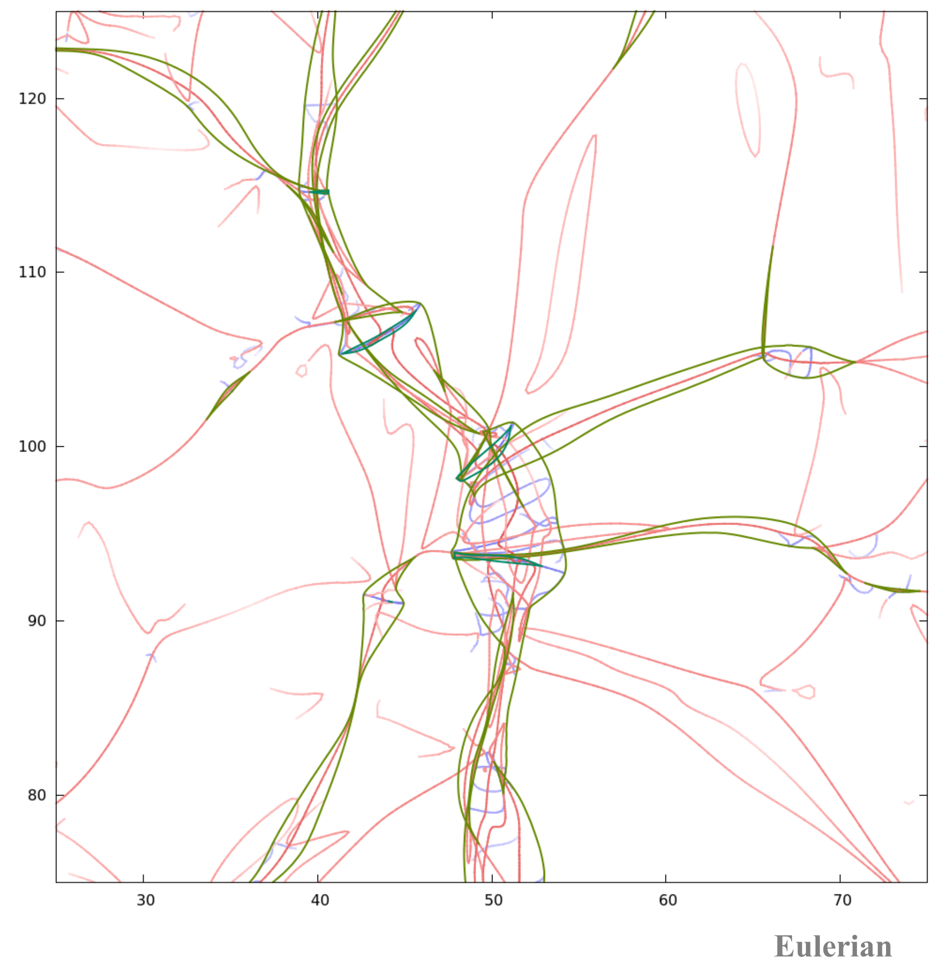

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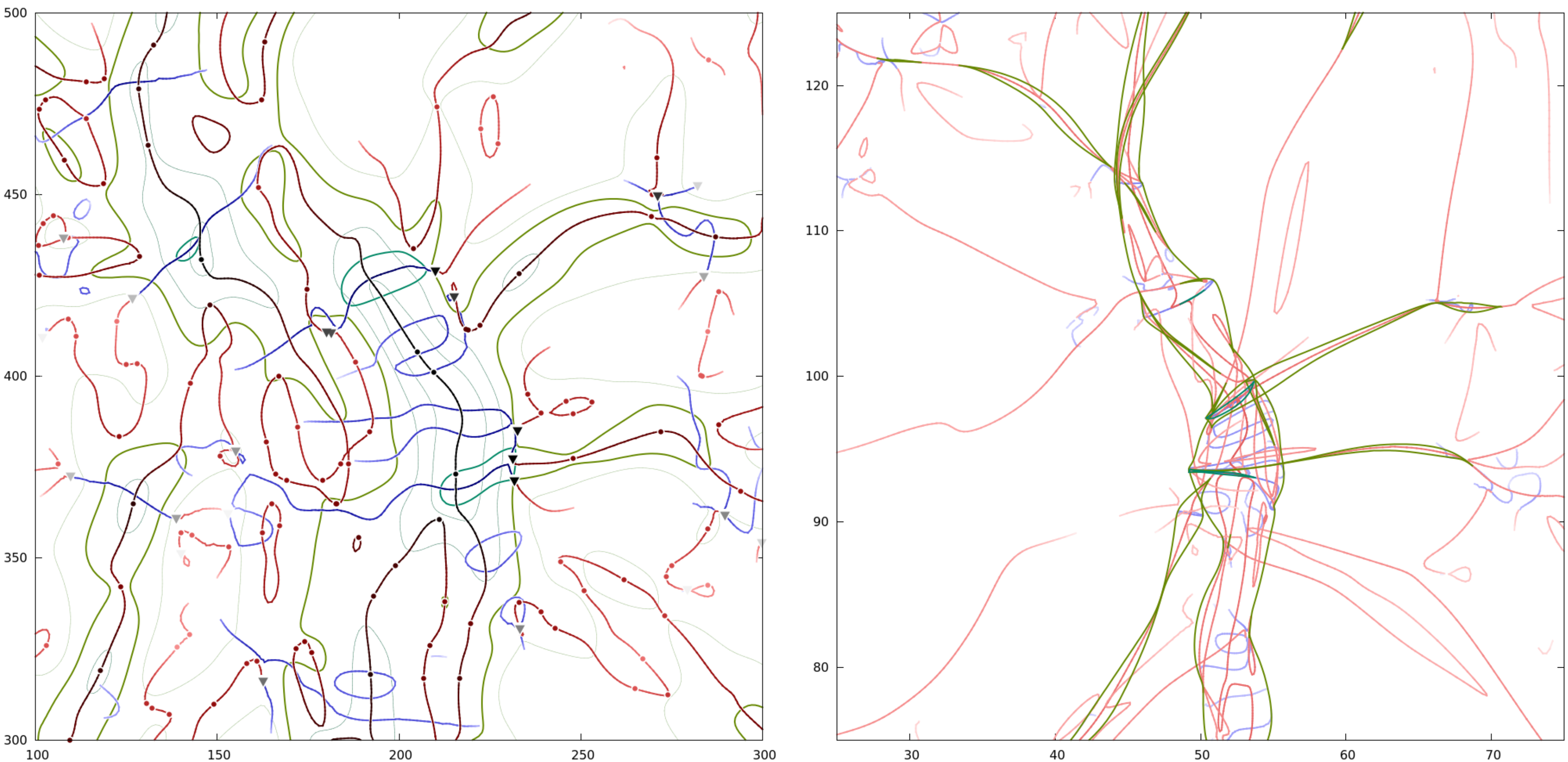
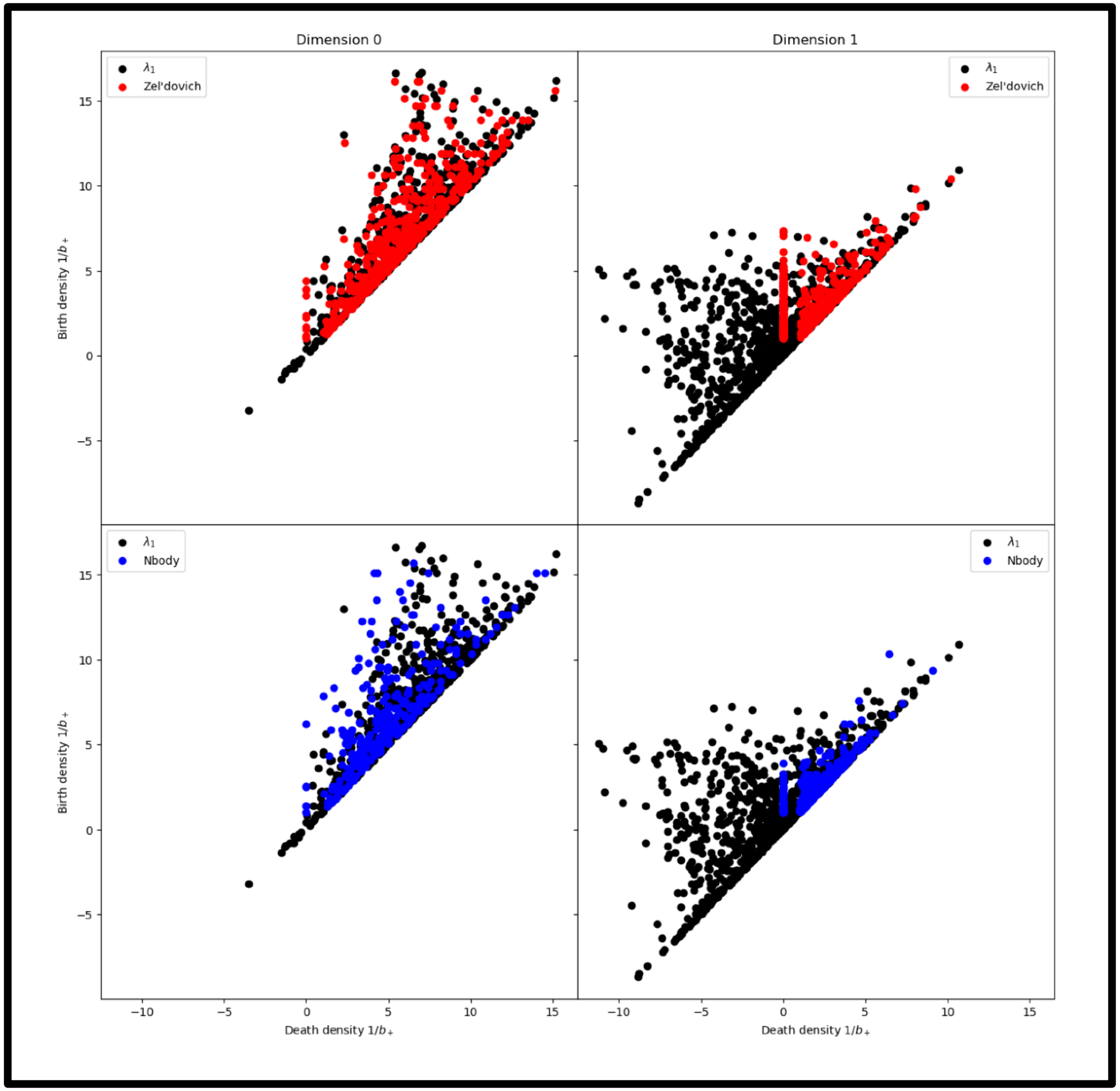


Lagrangian

Eulerian

Caustic Hierarchy & Connectivity

Deformation Persistence



Cosmic Web Persistence