Persistence decomposition algorithms

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Joint work with
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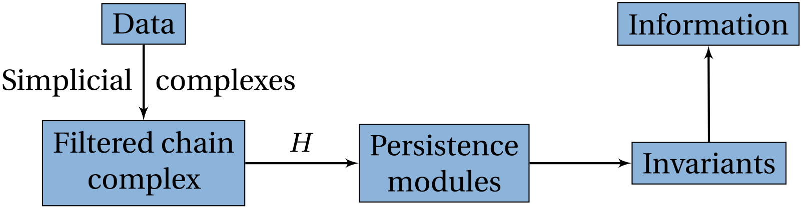
Graz University of Technology

Persistence pipeline

Data

Information

Persistence pipeline



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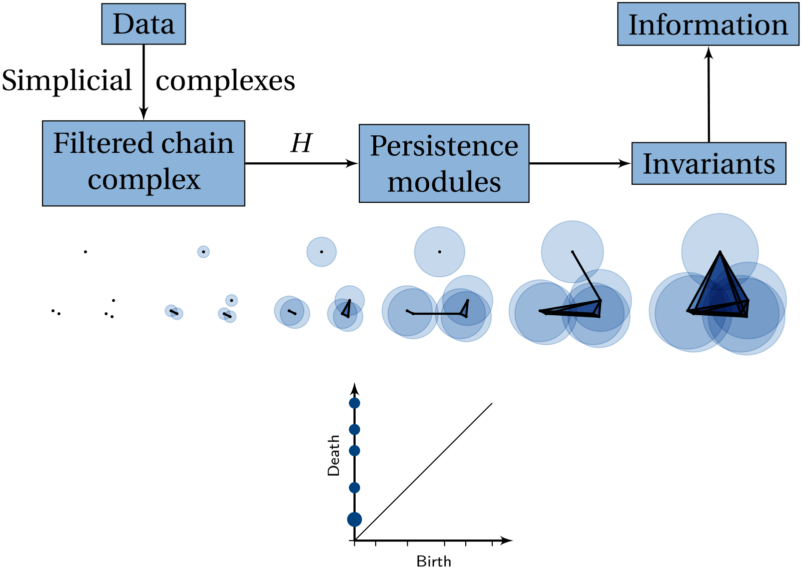

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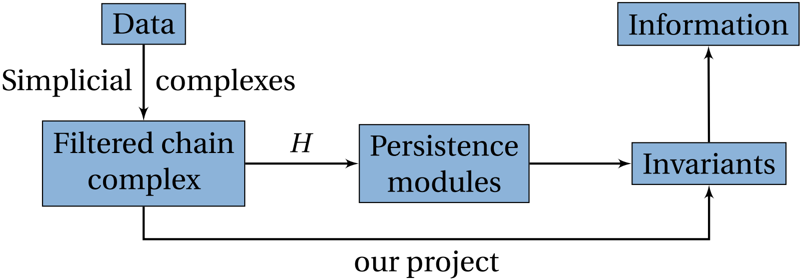

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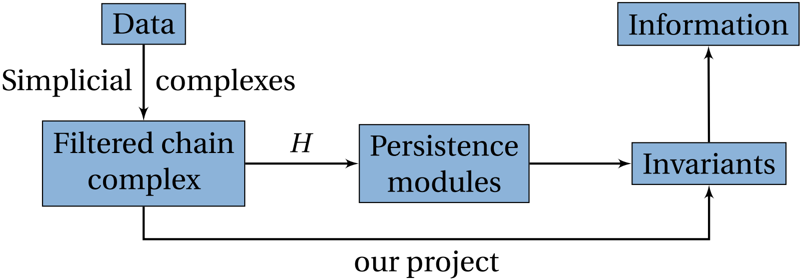

Studying persistence at the filtered chain complex level give use insights (and possibly new information).

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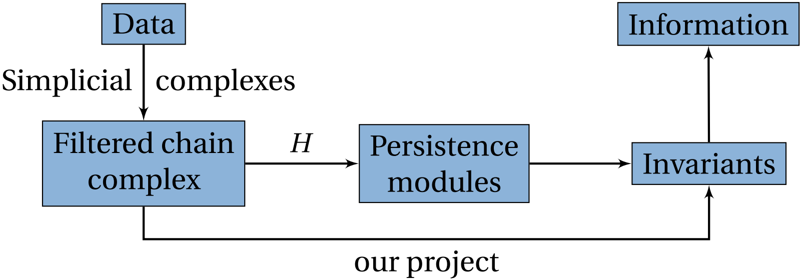

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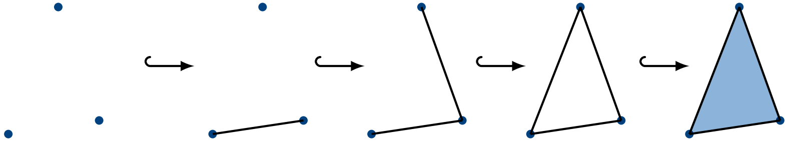
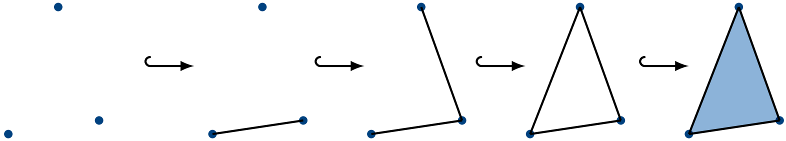
http://www.zotero.org/groups/TDA-Applications

Filtered chain complexes



Filtered chain complexes



Definition

A **filtered chain complex** is a finite sequence of chain complexes and inclusions between them.

$$
\begin{tikzcd}
\vdots \arrow[d, "\partial_n^0"] & \vdots \arrow[d, "\partial_n^{\tau_1}"] & \cdots & \vdots \arrow[d, "\partial_n^{\tau_i}"] & \vdots \arrow[d, "\partial_n^{\tau_{i+1}}"] & \\
X_n^0 \arrow[r, hook] \arrow[d, "\partial_{n-1}^0"] & X_n^{\tau_1} \arrow[r, hook] \arrow[d, "\partial_{n-1}^{\tau_1}"] & \cdots \arrow[r, hook] & X_n^{\tau_i} \arrow[r, hook] \arrow[d, "\partial_{n-1}^{\tau_i}"] & X_n^{\tau_{i+1}} \arrow[r, hook] \arrow[d, "\partial_{n-1}^{\tau_{i+1}}"] & \cdots \\
X_{n-1}^0 \arrow[r, hook] \arrow[d, "\partial_{n-2}^0"] & X_{n-1}^{\tau_1} \arrow[r, hook] \arrow[d, "\partial_{n-2}^{\tau_1}"] & \cdots \arrow[r, hook] & X_{n-1}^{\tau_i} \arrow[r, hook] \arrow[d, "\partial_{n-2}^{\tau_i}"] & X_{n-1}^{\tau_{i+1}} \arrow[r, hook] \arrow[d, "\partial_{n-2}^{\tau_{i+1}}"] & \cdots \\
\vdots & \vdots & \cdots & \vdots & \vdots &
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Chain Complex

Parameter

Decomposition of filtered chain complexes

Let $n \in \mathbf{N}$, $s \in [0, \infty)$ and $e \in [s, \infty]$.

Decomposition of filtered chain complexes

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The **interval sphere** $I^n\,[s, e)$ is the filtered chain complex given by:

$$
\begin{array}{ccccc}
t<s & & s \leq t<e & & e \leq t \\[1em]
0 & \longrightarrow & S^n & \hookrightarrow & D^{n+1}
\end{array}
$$

$$
\begin{array}{ccc}
0 & n+1 & \mathbb{F} \\
\big\downarrow & & \big\downarrow{\scriptstyle 1} \\
\mathbb{F} & n & \mathbb{F}
\end{array}
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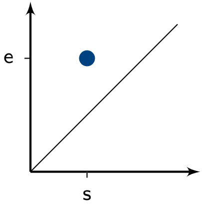

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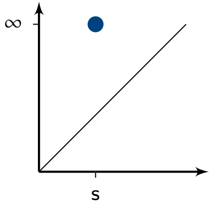

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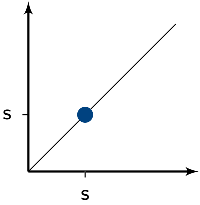

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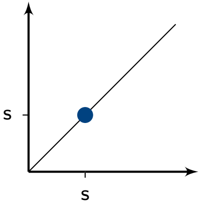

Theorem

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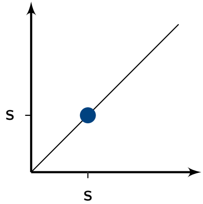

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$\Longrightarrow$ applying homology, we get the persistence diagram of the associated persistence module

Generators

Let X be a filtered chain complex. There is a finite collection of interval spheres $\{I^{n_i}[s_i, s_i)\}_{i=1,\dots,m}$ and an epimorphism

$$\varphi\colon \oplus_{i=1}^{m} I^{n_i}[s_i, s_i) \to X$$

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$$\deg(x_i) = n_i+1 \qquad \mathrm{ent}(x_i) = s_i$$

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Definition

The set $\mathcal{G} = \{x_1, \ldots, x_m\}$ is called a **set of generators of** X. If $\mathcal{G}$ is minimal degreewise, then it is called a **quasi-minimal** set of generators.

Generators

The simplices of a filtered simplicial complexes form a quasi-minimal set of generators.

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Consider the interval sphere $I^n\,[s,e)$. A quasi-minimal set of generators of it is $\mathcal{G} = \{x_s, x_e\}$, with $\deg(x_s) = n$, $\deg(x_e) = n{+}1$.

$$\begin{tikzcd}
\cdots & i{=}s{-}\varepsilon & i{=}s & \cdots & i{=}e{-}\varepsilon & i{=}e & \cdots & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots & n{+}1 \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots & n
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Standard persistence algorithm

Let X be a filtered chain complex with quasi-minimal set of generators $\{x_1, \ldots, x_m\}$, totally ordered first by entrance time. Let D be its boundary matrix.

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A matrix C is called **column reduced** if $low\,(j) \neq low\,(j')$ for all non-zero columns $j \neq j'$. The element C_j^i is called a **column pivot** (of C) if $i = low\,(j)$.

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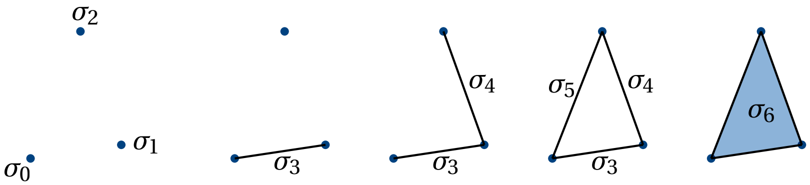
Theorem

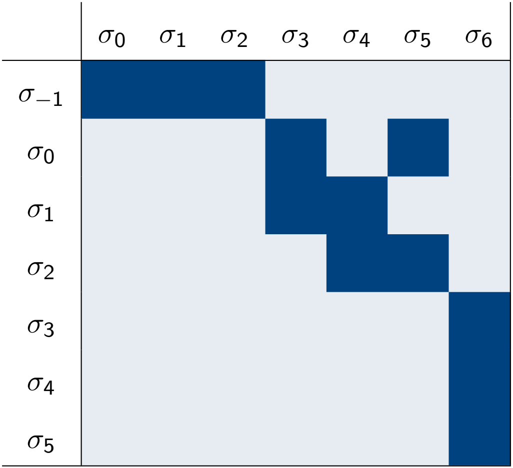
The column pivot pairing of a reduced boundary matrix provides the barcode decomposition of the associated persistent module.

Example



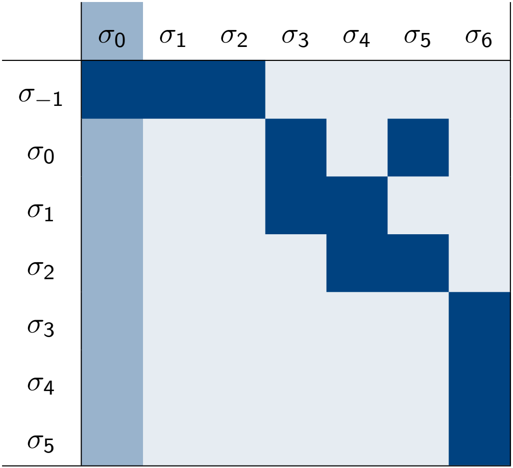
σ_2
σ_1
σ_0
σ_3
σ_4
σ_3
σ_5
σ_4
σ_3
σ_6

Example



σ_0 σ_1 σ_2 σ_3 σ_4 σ_5 σ_6
σ_{-1}
σ_0
σ_1
σ_2
σ_3
σ_4
σ_5

Example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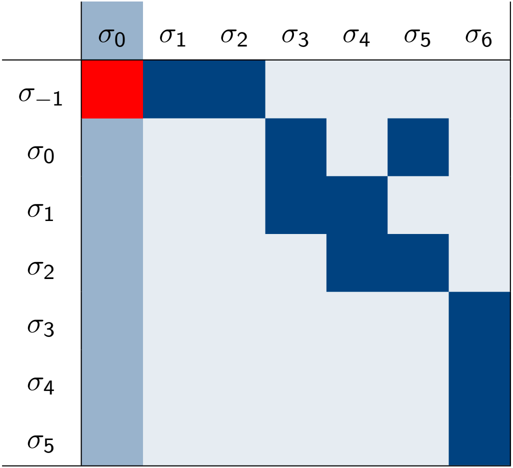
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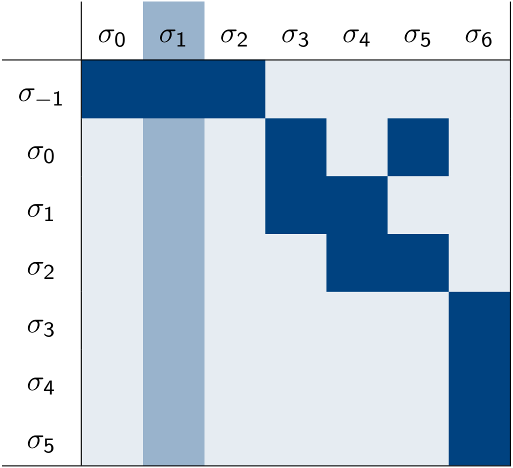

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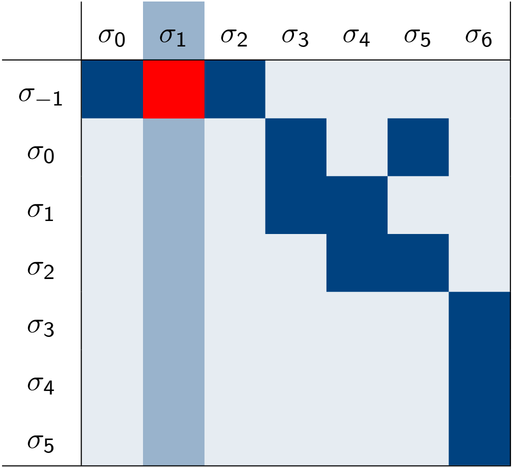

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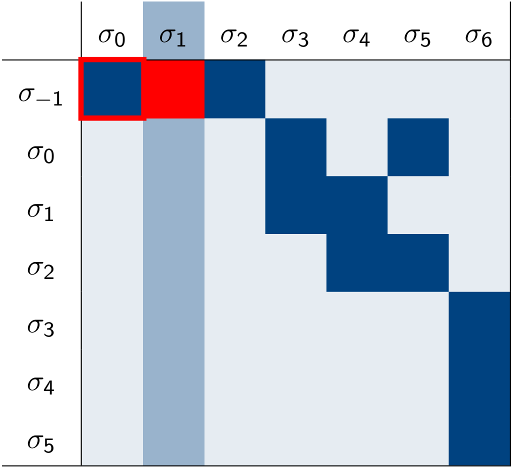

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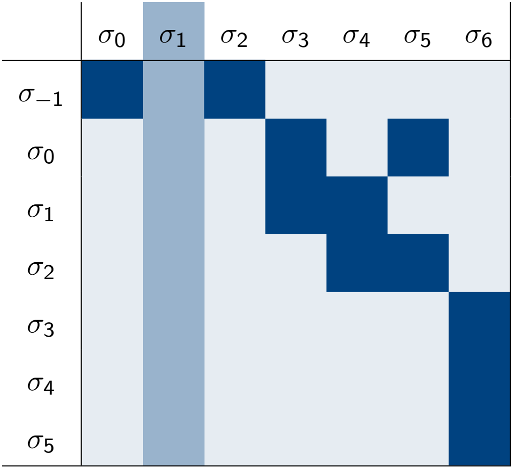

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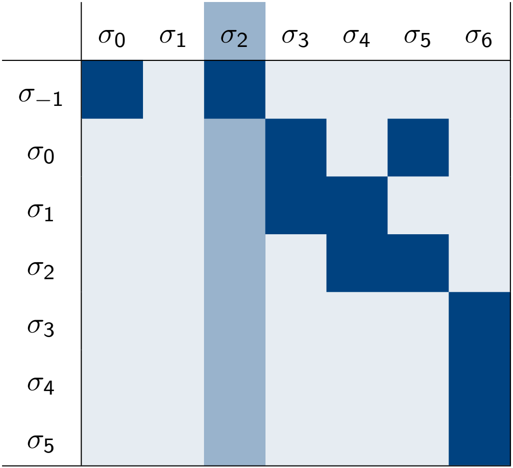
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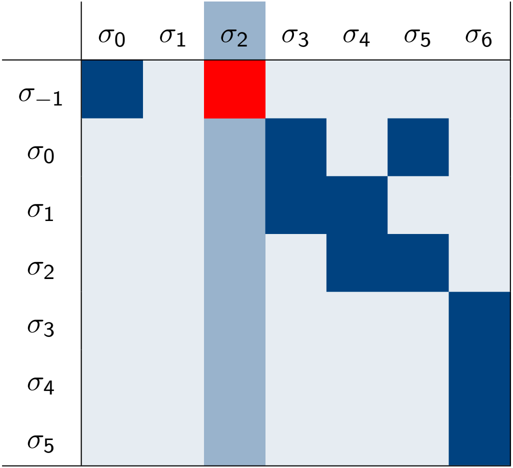

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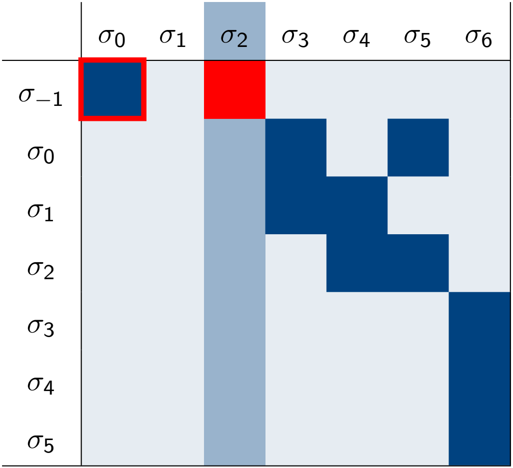
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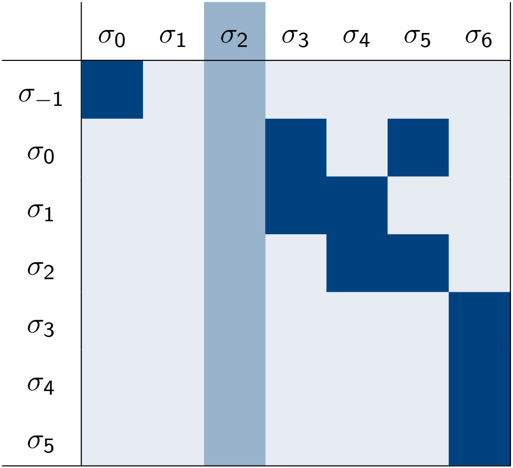
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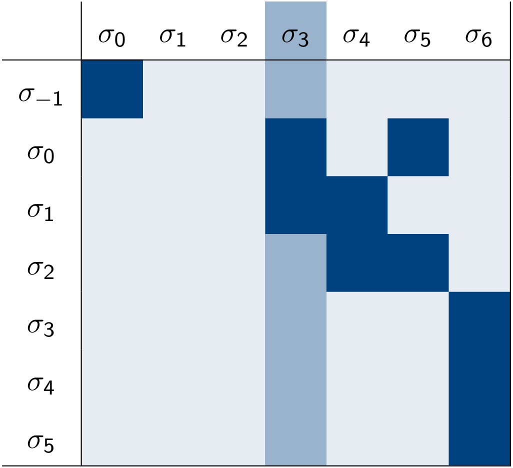
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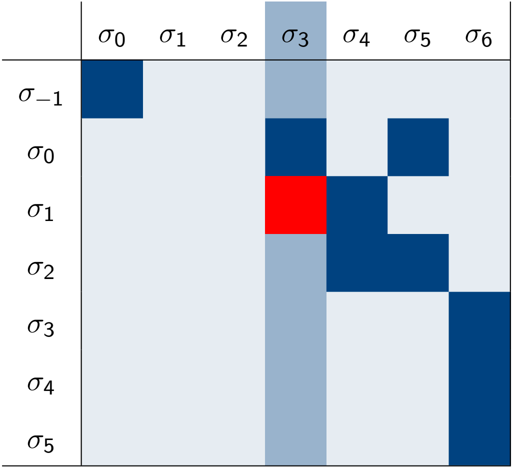

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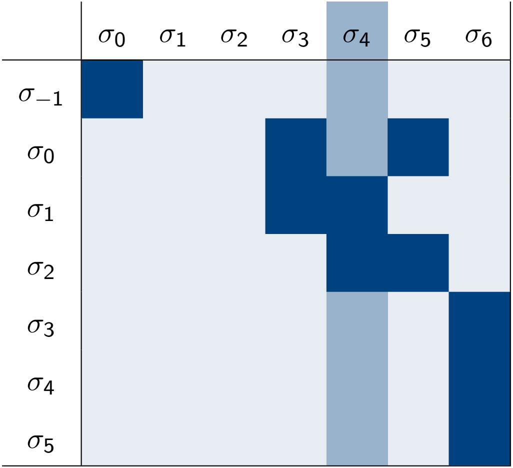
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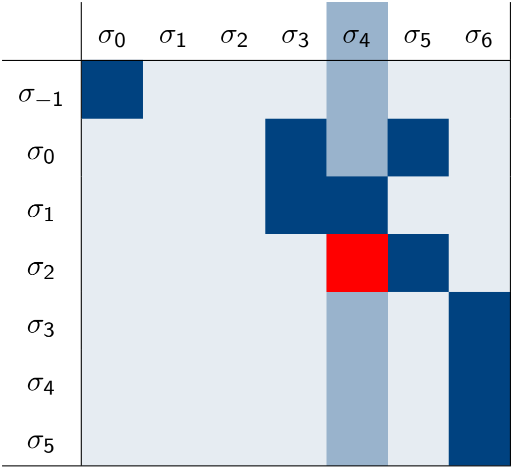

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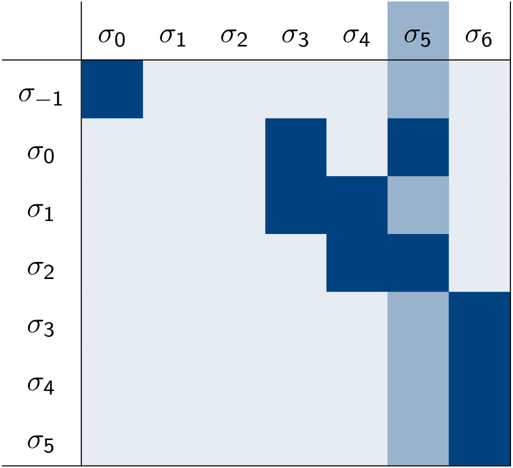
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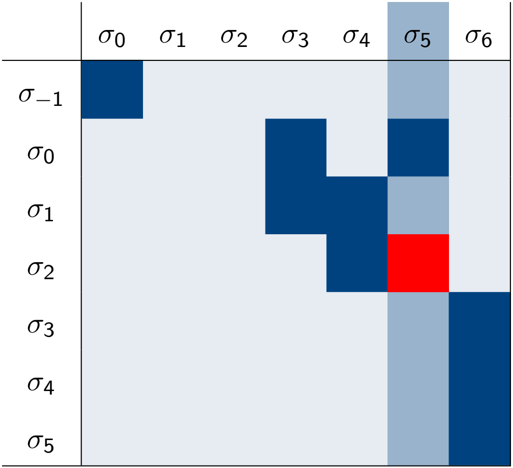
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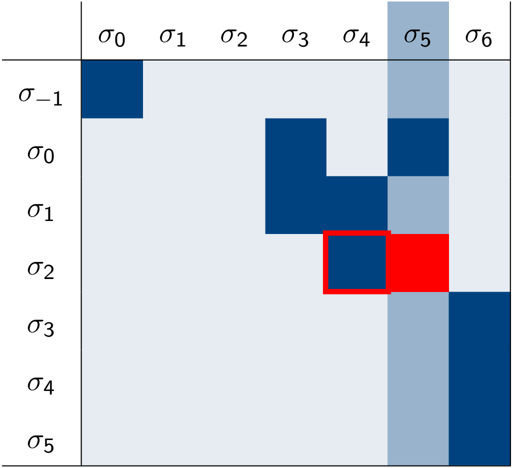
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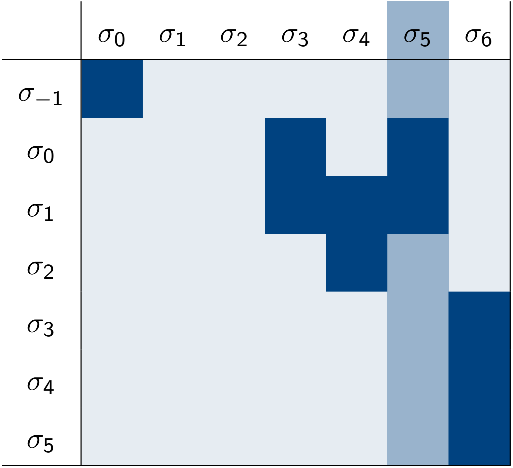
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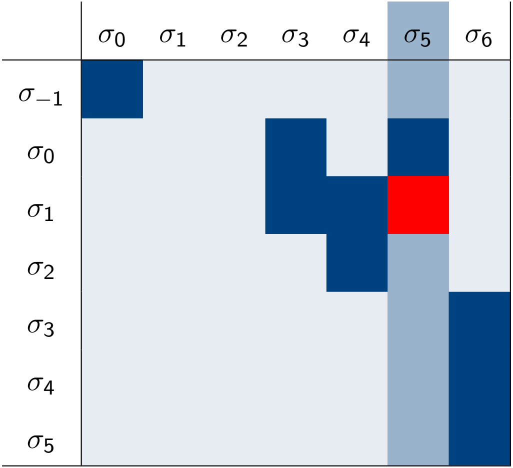
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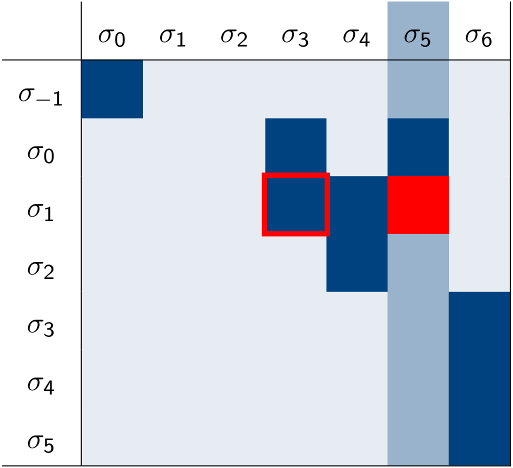
σ_0 σ_1 σ_2 σ_3 σ_4 σ_5 σ_6
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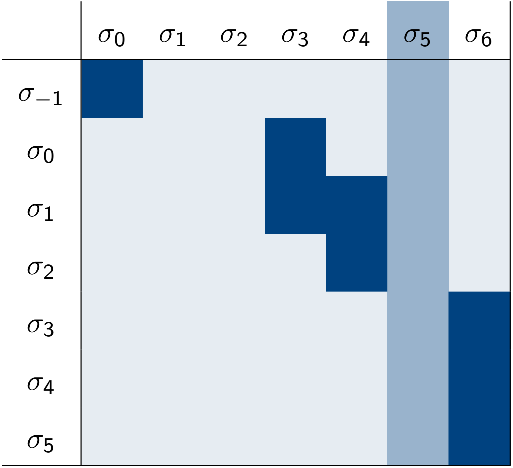

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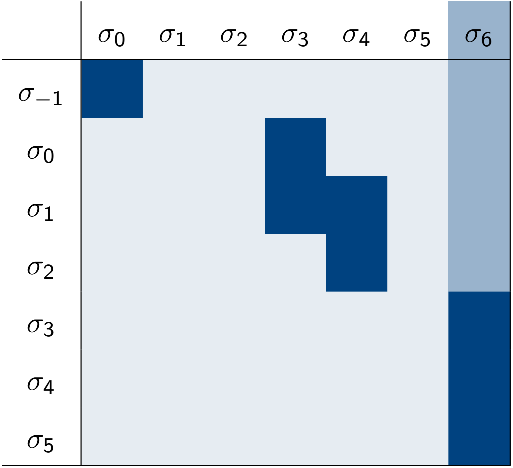
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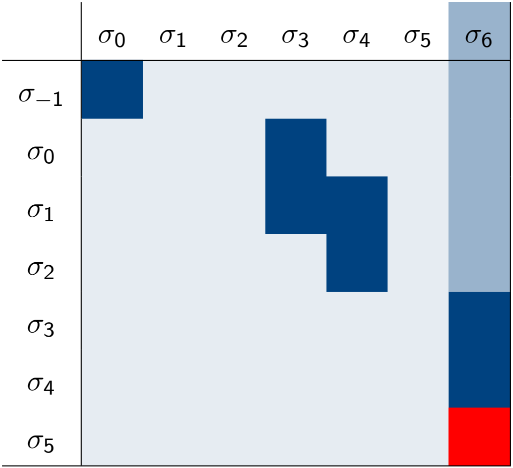
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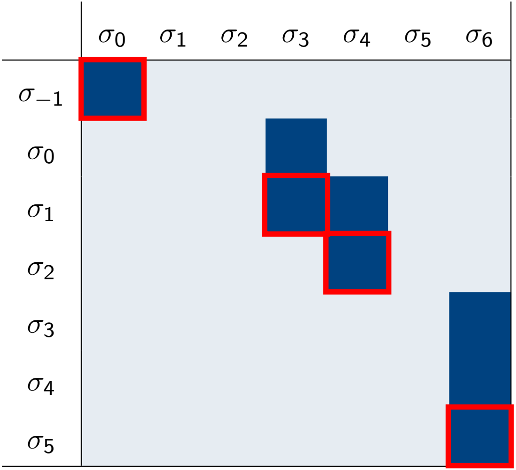
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Clear and compress

Two time optimisations of the standard persistence algorithm are the clear and compress.

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Dualisation of the standard persistence algorithm

Let X be a filtered chain complex with quasi-minimal set of generators $\{x_1, \ldots, x_m\}$, totally ordered first by entrance time. Let D be its boundary matrix.

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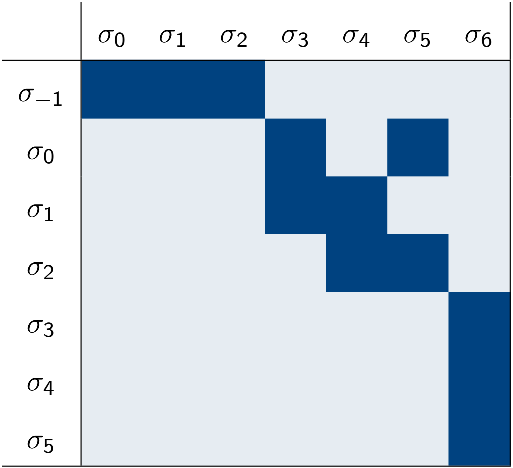
Theorem

The row pivot pairing of a reduced boundary matrix provides the barcode decomposition of the associated persistent module.

Example


σ_0 σ_1 σ_2 σ_3 σ_4 σ_5 σ_6
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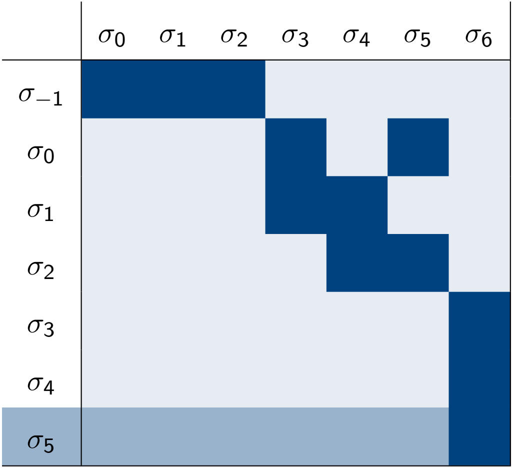
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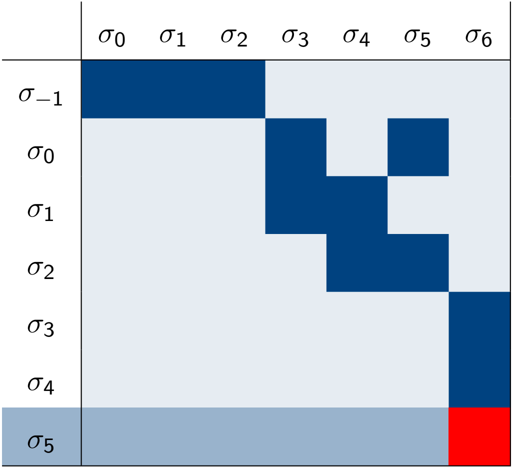
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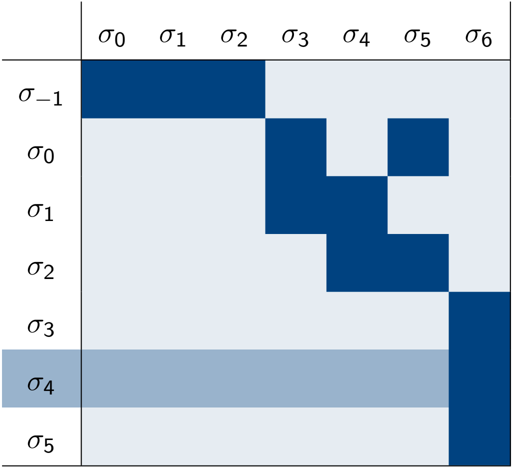
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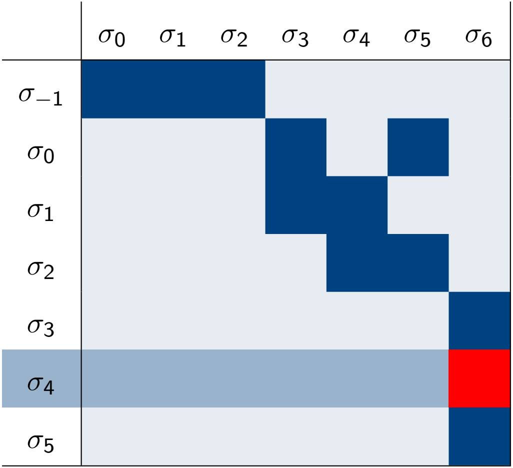
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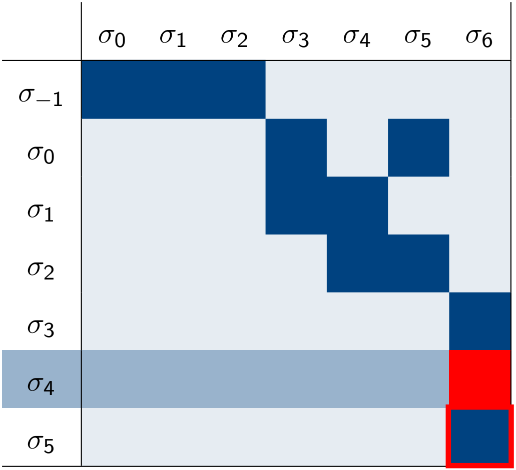
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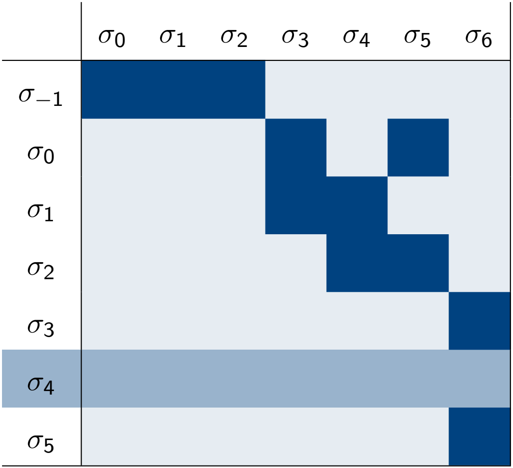
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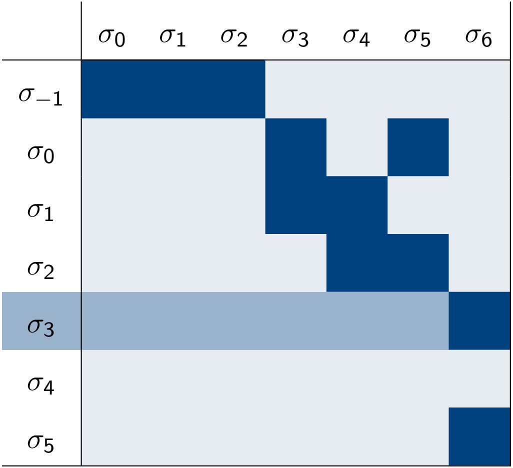
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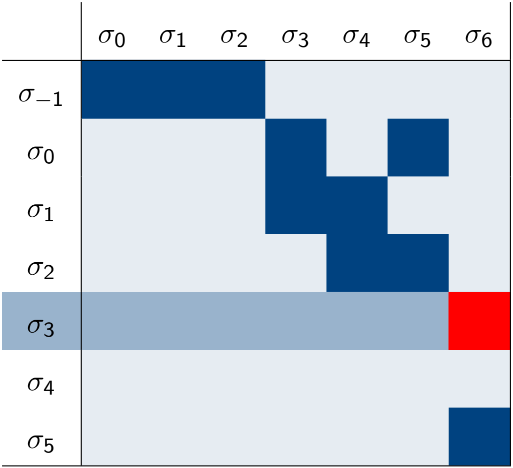
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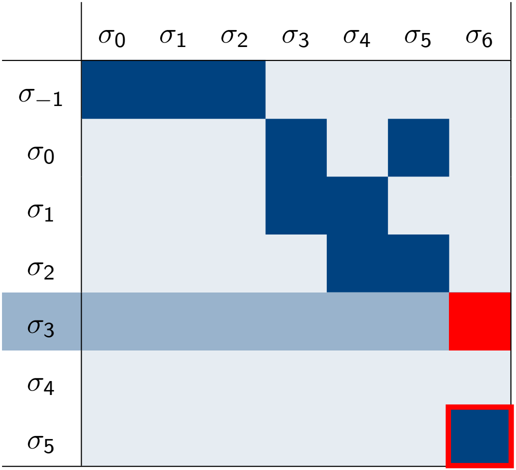
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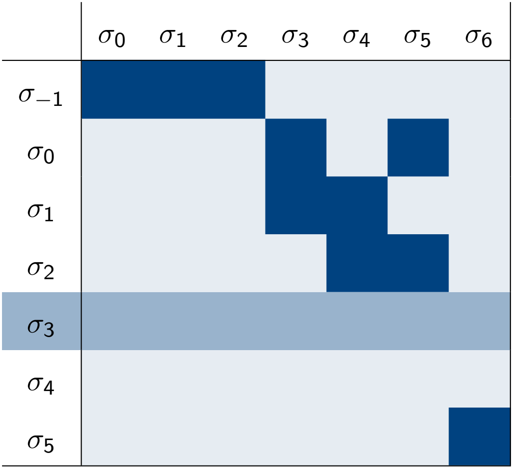

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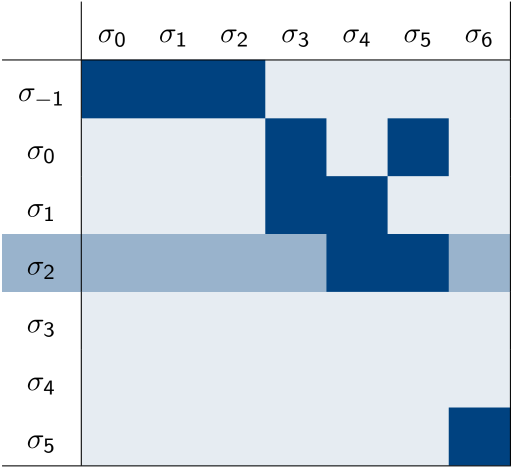
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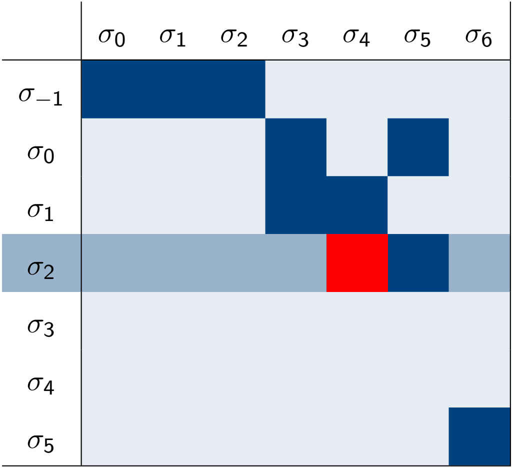

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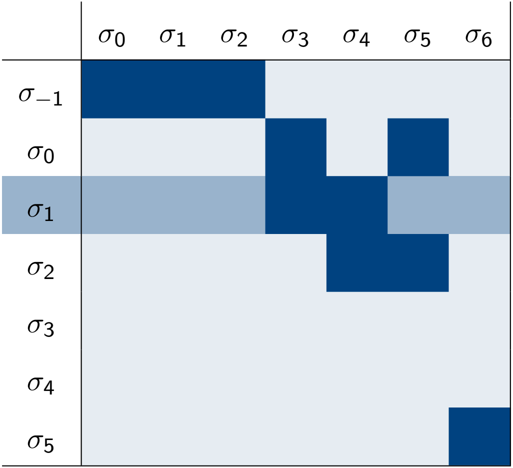

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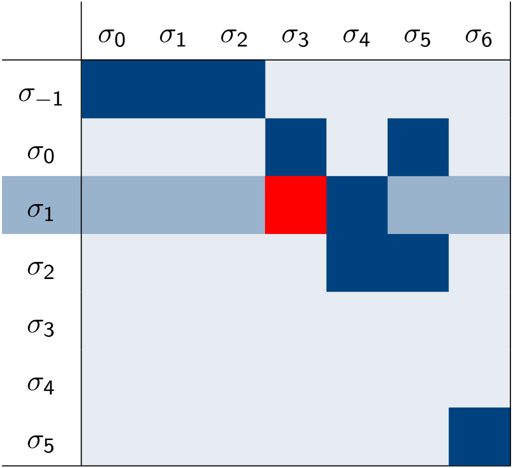
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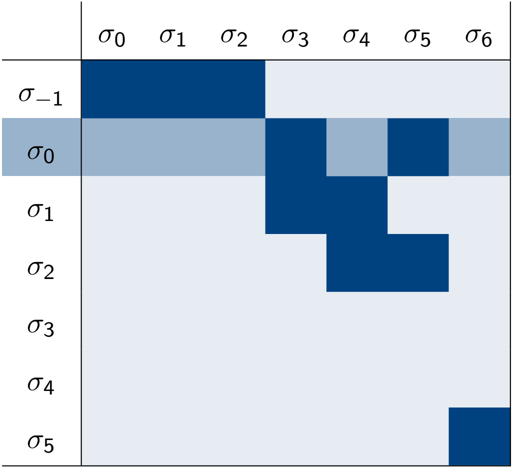
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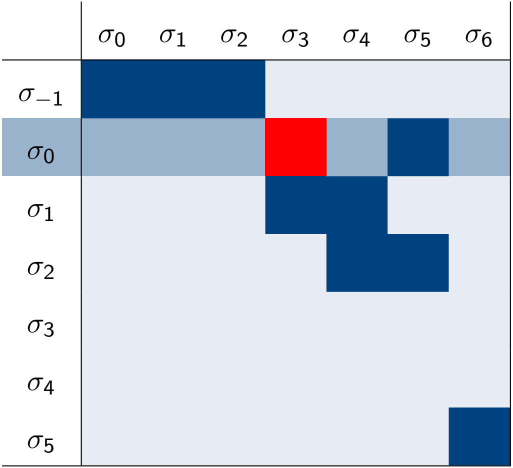
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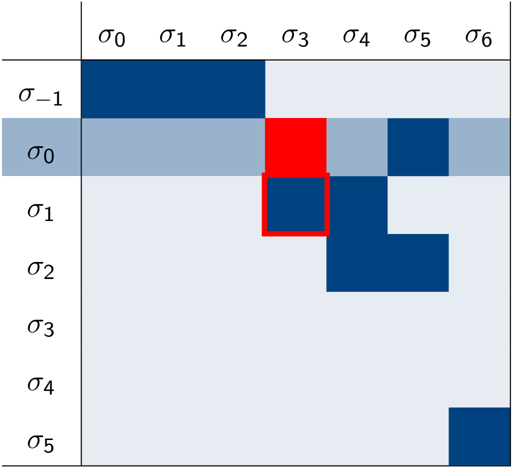

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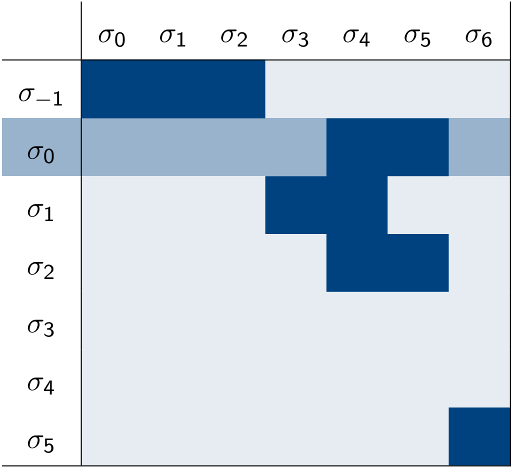
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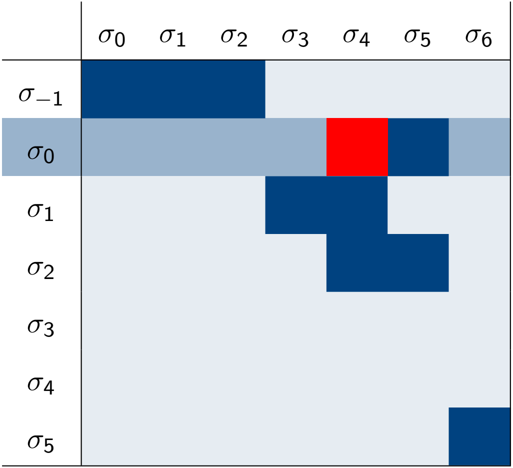
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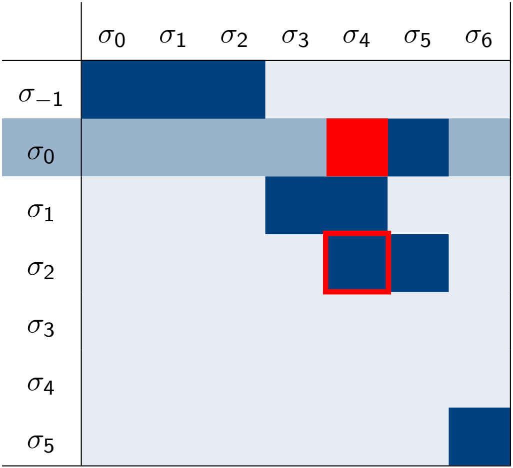
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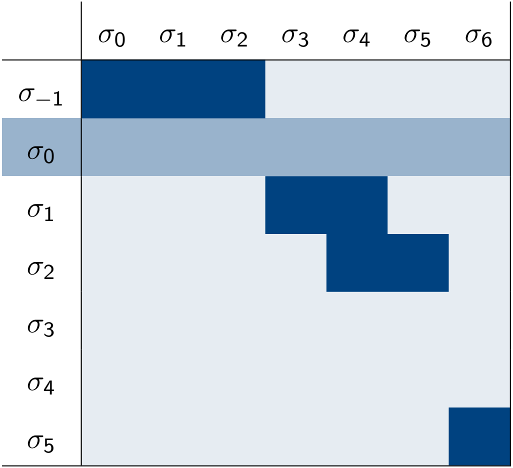
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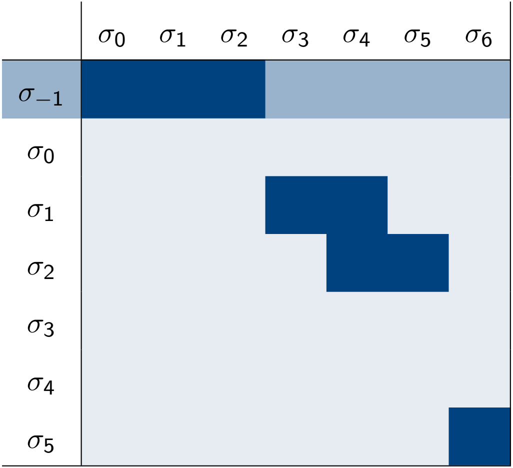

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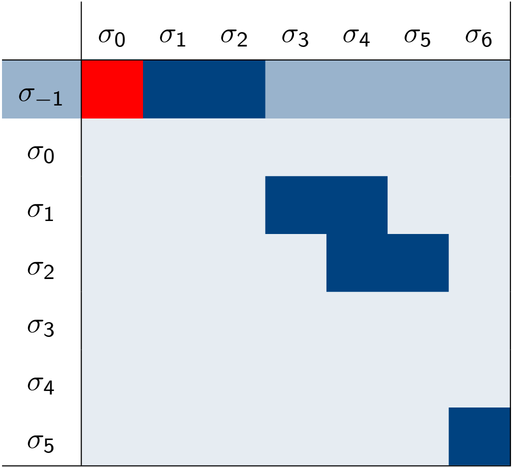
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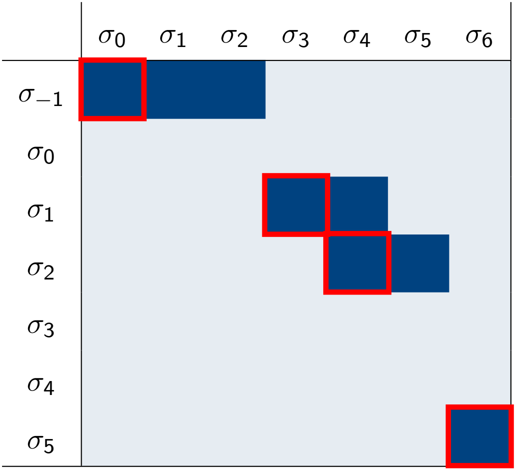
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Number of operations

Let X be a filtered Vietoris-Rips complex, whose quasi-minimal set of generators is ordered first by degree. Let v be the number of generators in degree 0 and N the maximal non-trivial degree of X.

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$$\sum_{n=0}^{N} \underbrace{\binom{v}{n+1}}_{\dim(C_n X)} = \sum_{n=0}^{N} \underbrace{\binom{v-1}{n}}_{\dim(B_{n-1} X)} + \sum_{n=0}^{N} \underbrace{\binom{v-1}{n+1}}_{\dim(Z_n X)}$$

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Using the compress optimisation, this decreases to:

$$\underbrace{\binom{v-1}{0}}_{\dim(B_{-1} X)} + \sum_{n=0}^{N} \underbrace{\binom{v-1}{n+1}}_{\dim(Z_n X)} = \sum_{n=0}^{N+1} \binom{v-1}{n}$$

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The number of columns to be processed is:

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Using the clear optimisation, this decreases to:

$$\sum_{n=0}^{N} \underbrace{\binom{v-1}{n+1}}_{\dim(B^{n+1} X)} + \underbrace{\binom{v-1}{0}}_{\dim(Z^0 X)} = \sum_{n=0}^{N+1} \binom{v-1}{n}$$

Algorithm for filtered chain complexes

Decomposition into interval spheres

Input: Totally ordered quasi-minimal set of generators $\{x_1, \ldots, x_m\}$
Output: List of interval spheres

```
List = ∅
D = BOUNDARY MATRIX(x_1, ..., x_m)
while exists non-zero row i in D do
    x_i, x_j = PAIR(D, i)
    Append I^{deg(x_i)} [ent(x_i), ent(x_j)) to List
    SPLIT(x_i, x_j, D)
for all indices i of remaining rows in D do
    Append I^{deg(x_i)} [ent(x_i), ∞) to List
Return List
```

Algorithm for filtered chain complexes

SPLIT

Input: Boundary matrix D, pair of splittable generators x_i, x_j

Output: Reduced boundary matrix D

for k *with* $D_j^k \neq 0$ **do**
    Add to the k-th row the i-th row multiplied by $-(D_j^i)^{-1} D_j^k$
Delete row j from D
Delete row i and column j from D
Delete column i from D

Algorithm for filtered chain complexes

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Input: Boundary matrix D, pair of splittable generators x_i, x_j

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Delete row j from D $\Longrightarrow$ COMPRESS

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Algorithm for filtered chain complexes

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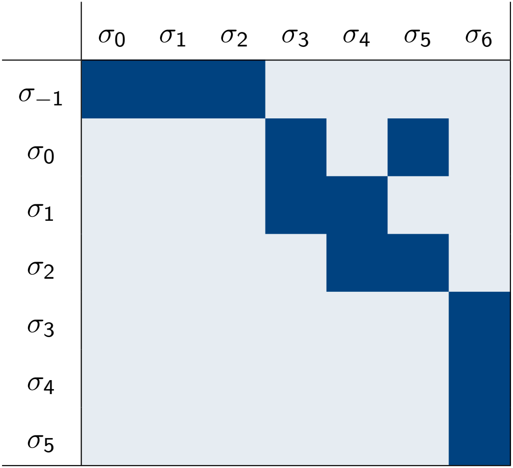
Delete row j from D $\Longrightarrow$ COMPRESS

Delete row i and column j from D

Delete column i from D $\Longrightarrow$ CLEAR

Example



σ_0 σ_1 σ_2 σ_3 σ_4 σ_5 σ_6
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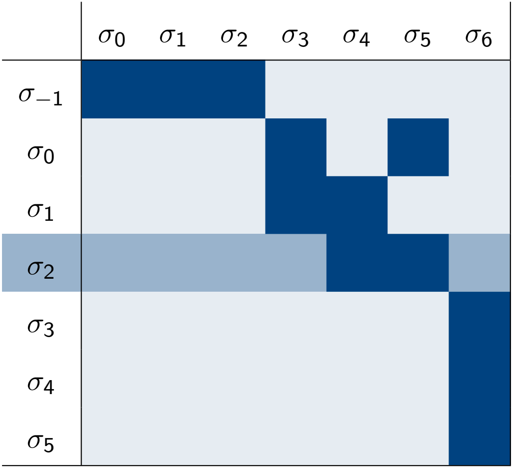
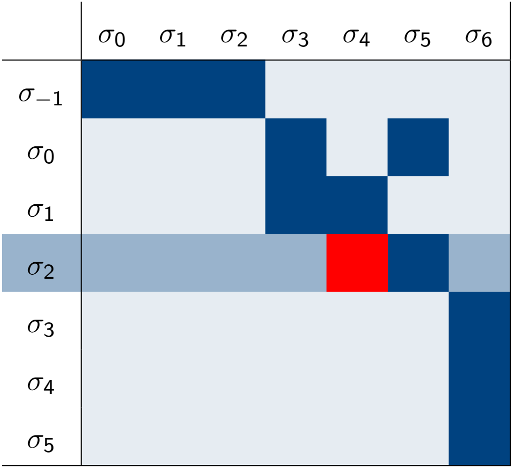
Example



Example



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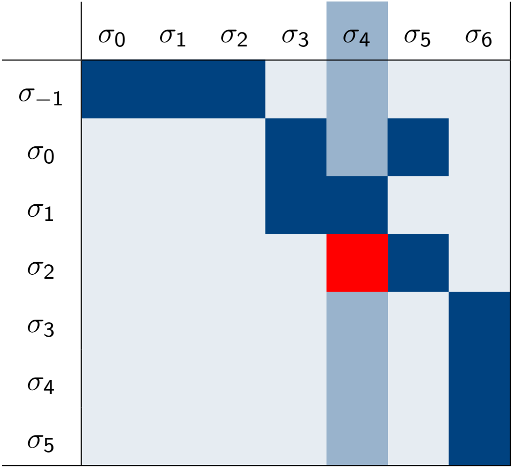
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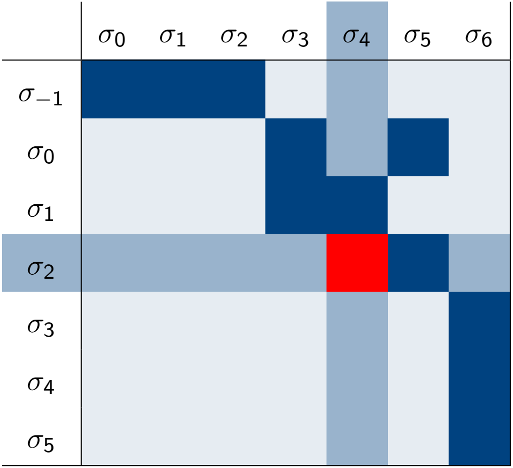
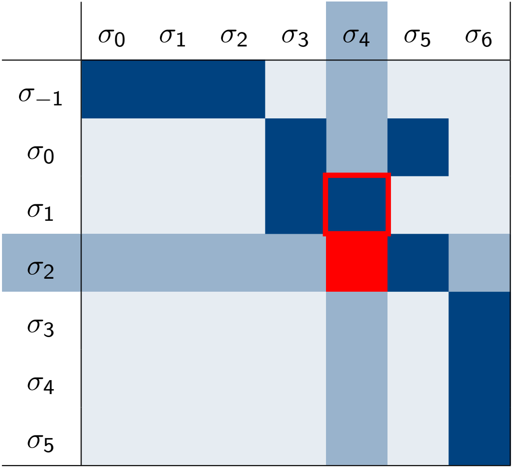


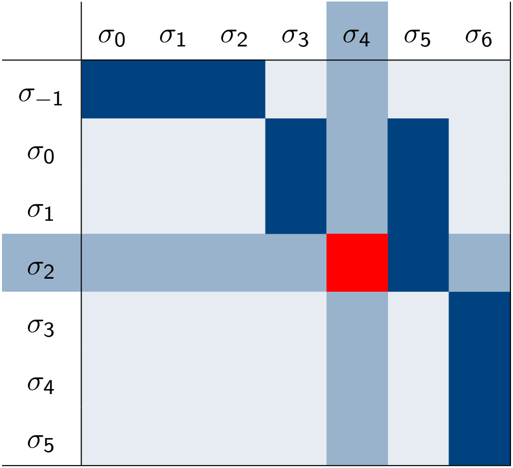
I^0 $[2, 4)$

Example



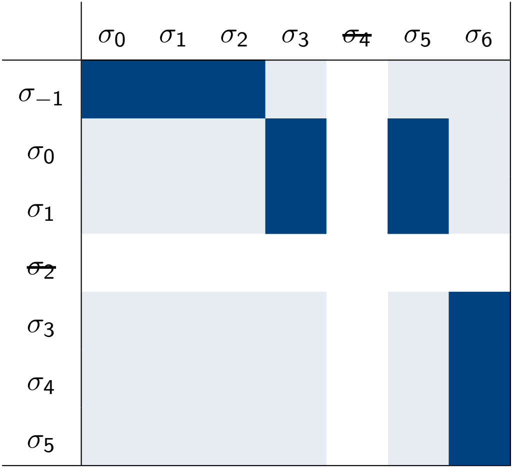
$I^0\ [2,4)$

Example



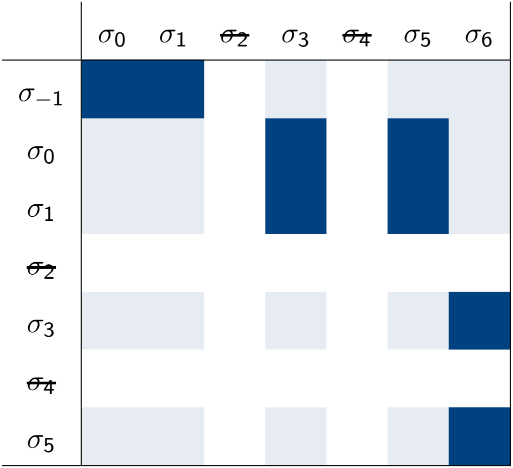
$I^0\ [2, 4)$

Example



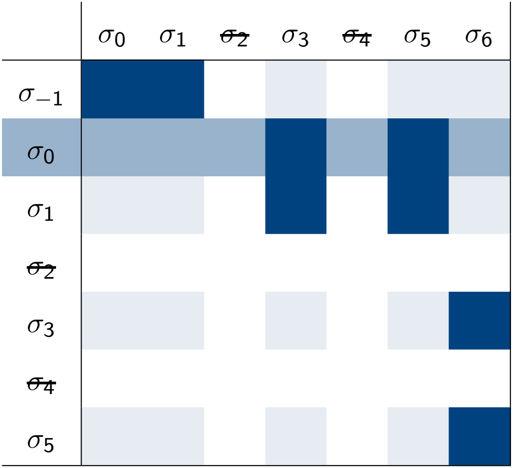
$I^0\ [2,4)$

Example



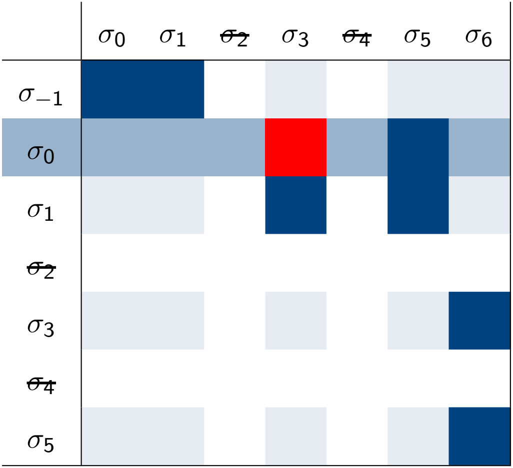
$I^0\ [2,4)$

Example



$I^0\ [2, 4)$

Example



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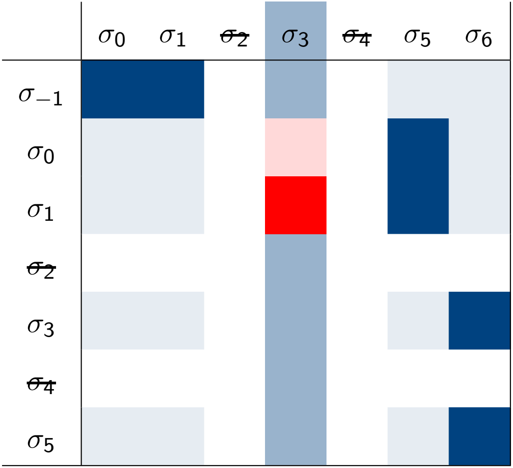

$I^0\ [2,4)$

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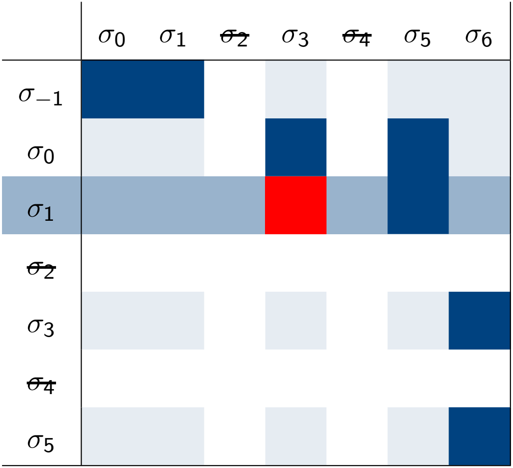

$I^0\ [2,4)$

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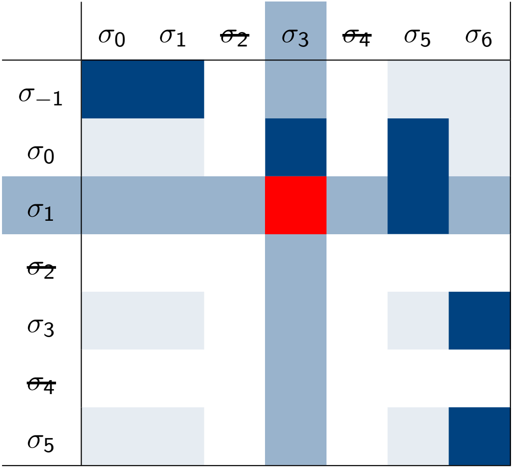
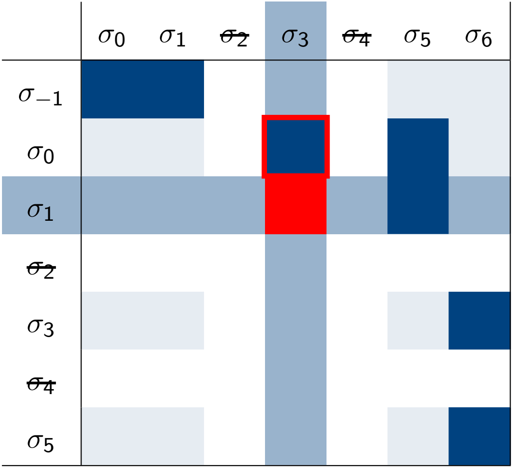


$I^0\left[2,4\right), I^0\left[1,3\right)$

Example



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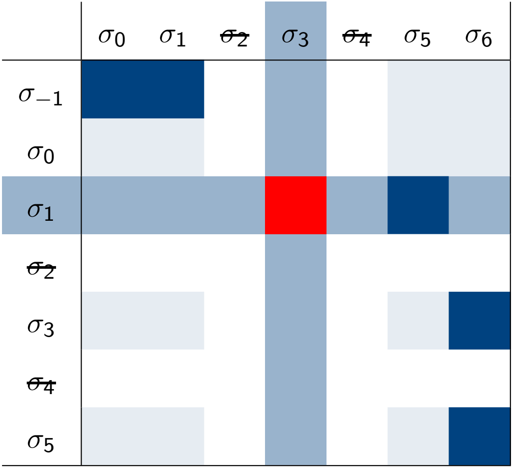
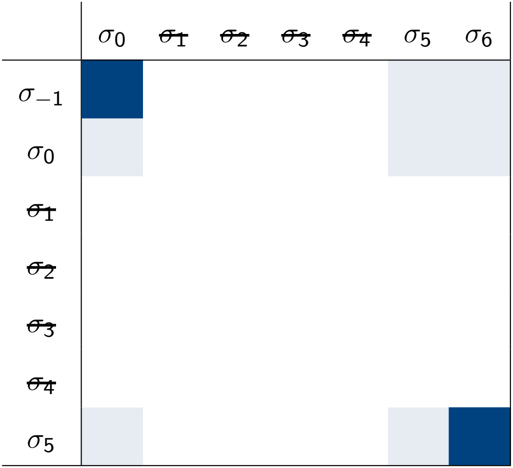
Example



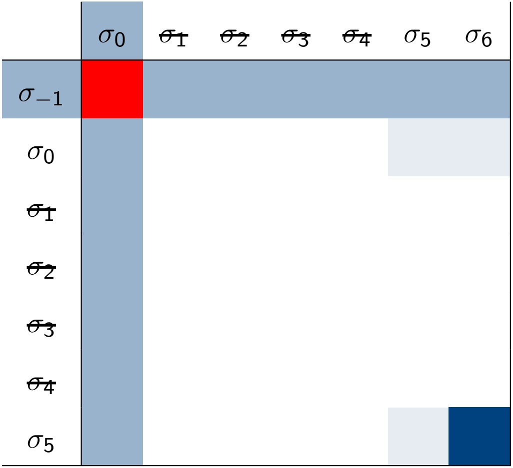
$I^0\,[2,4)\,,\,I^0\,[1,3)$

Example



$I^0\,[2,4)\,,\,I^0\,[1,3)$

Example



$I^0\left[2,4\right), I^0\left[1,3\right), I^0\left[0,\infty\right)$

Example

	$\cancel{\sigma_0}$	$\cancel{\sigma_1}$	$\cancel{\sigma_2}$	$\cancel{\sigma_3}$	$\cancel{\sigma_4}$	σ_5	σ_6
$\cancel{\sigma_{-1}}$							
$\cancel{\sigma_0}$							
$\cancel{\sigma_1}$							
$\cancel{\sigma_2}$							
$\cancel{\sigma_3}$							
$\cancel{\sigma_4}$							
σ_5							

$I^0\,[2,4)\,,\; I^0\,[1,3)\,,\; I^0\,[0,\infty)$

Example

	$\cancel{\sigma_0}$	$\cancel{\sigma_1}$	$\cancel{\sigma_2}$	$\cancel{\sigma_3}$	$\cancel{\sigma_4}$	σ_5	σ_6
$\cancel{\sigma_{-1}}$							
$\cancel{\sigma_0}$							
$\cancel{\sigma_1}$							
$\cancel{\sigma_2}$							
$\cancel{\sigma_3}$							
$\cancel{\sigma_4}$							
σ_5							

$I^0\ [2,4)\ ,\ I^0\ [1,3)\ ,\ I^0\ [0,\infty)\ ,\ I^1\ [5,6)$

Example

	$\cancel{\sigma_0}$	$\cancel{\sigma_1}$	$\cancel{\sigma_2}$	$\cancel{\sigma_3}$	$\cancel{\sigma_4}$	$\cancel{\sigma_5}$	$\cancel{\sigma_6}$
$\cancel{\sigma_{-1}}$							
$\cancel{\sigma_0}$							
$\cancel{\sigma_1}$							
$\cancel{\sigma_2}$							
$\cancel{\sigma_3}$							
$\cancel{\sigma_4}$							
$\cancel{\sigma_5}$							

$I^0\left[2,4\right), I^0\left[1,3\right), I^0\left[0,\infty\right), I^1\left[5,6\right)$

Conclusions

- Algorithm to decompose filtered chain complexes into interval spheres
- The algorithm does not require a fixed order on generators, intrinsically adopts both the clear and compress, and produces the barcode decomposition as a byproduct
- "Novel" algorithm to retrieve the barcode decomposition using row operations, dualising the standard persistence algorithm

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Invariants for tame parametrised chain complexes,
W. Chachólski, B. Giunti, and C. Landi, 2020

Algorithmic decomposition of filtered chain complexes,
W. Chachólski, B. Giunti, A. Jin, and C. Landi, 2020

Notes on pivot pairings, B. Giunti, 2021

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Future directions:

Test the performances of the different algorithms

Optimise the PAIR algorithm

- Using local properties of filtered simplicial complexes
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Thank you for your attention!