Journal Club on Topological Data Analysis

Edge Collapse and Persistence of Flag Complexes (by Boissonnat and Pritam)

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February 8, 2021

Outline

- Motivational introduction

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- Preliminaries

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- Simplicial collapses and flag complexes

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- Computational Experiments

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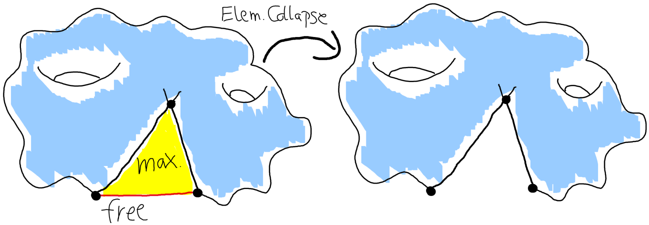


Example

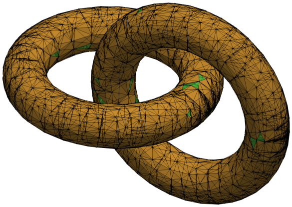


Figure: 23088 Simplices

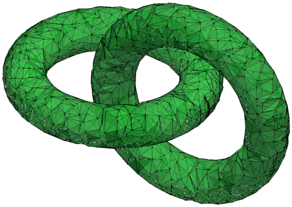
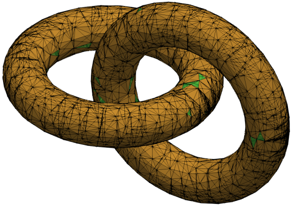


Figure: 9520 Simplices

Example



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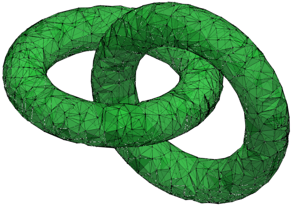

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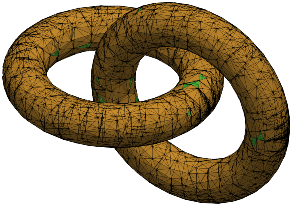
- Less than half of the simplices, same homotopy type, same homology groups.

Example



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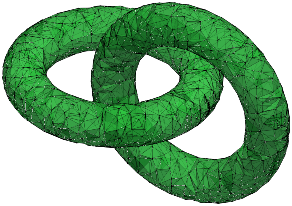

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- Less than half of the simplices, same homotopy type, same homology groups.
- Question: Is this a quick process?

Reference

J-D. Boissonnat, S. Pritam, *Edge Collapse and Persistence of Flag Complexes*, Proceedings of 36th International Symposium on Computational Geometry (2020), 19:1 – 19:15.

Preliminaries

- An **(abstract) simplicial complex** K is a collection of subsets of a non-empty finite set X, such that for every subset A in K, all the subsets of A are in K.

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- A map $f : K \to L$ between simplicial complexes is called a **simplicial map**, if it always maps a simplex in K to a simplex in L.
- A sequence of simplicial complexes

$$\{K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} K_n\}$$

  connected through simplicial inclusion maps f_i is called a **(simplicial) filtration**. We often write $\{K_i, f_i\}$ for such a filtration.

Preliminaries

- Computing simplicial homology over field coeffiecients for all K_i, yields a **persistence module**

$$\{H_p(K_1) \xrightarrow{f_{1,p}} H_p(K_2) \xrightarrow{f_{2,p}} \dots \xrightarrow{f_{n-1,p}} H_p(K_n)\}$$

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  in every degree p.
- Let X be a finte metric space and let $\epsilon > 0$. The *Vietoris-Rips complex of X at scale ϵ* is defined as the set

$$\mathrm{VR}_\epsilon(X) := \{\sigma \subset X | d(x,y) \leq 2\epsilon \text{ for all } x, y \in \sigma\}.$$

Simplicial Collapses

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- J. H. C. Whitehead, *Simplicial Spaces, Nuclei and m-groups*, Proceedings of the London mathematical society **2** (1939), no. 1, 243-327.
- M. M. Cohen, *A Course in Simple-Homotopy Theory*, Graduate Texts in Mathematics, Vol. 10, Springer-Verlag, New York, 1973.

- The deformation retraction associated with the removal of σ and τ as before can be realized as follows:

$$
\begin{aligned}
H : |\tau| \times I &\to |\tau| \\
((x_1, \ldots, x_n), t) &\mapsto (1-t)(x_1, \ldots, x_n) \\
&\qquad + t(x_1 - \min_i x_i, \ldots, x_n - \min_i x_i)
\end{aligned}
$$

with simplices τ and σ parametrised as subsets of $\mathbb{R}^n$

$$
\begin{aligned}
|\tau| &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^{n} x_i \leq 1\}, \\
|\sigma| &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}.
\end{aligned}
$$

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- A simplicial complex K *collapses* (simplicially) to a subcomplex $L \subset K$, if L can be obtained by a finite sequence of elementary collapses. In that case we write $K \searrow L$.

We need some notation to formulate a class of collapses that are easy to identify algorithmically.

Star, Link, Simplicial Cone

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$$st_K(\sigma) := \{\tau \in K | \tau \cup \sigma \in K\}.$$

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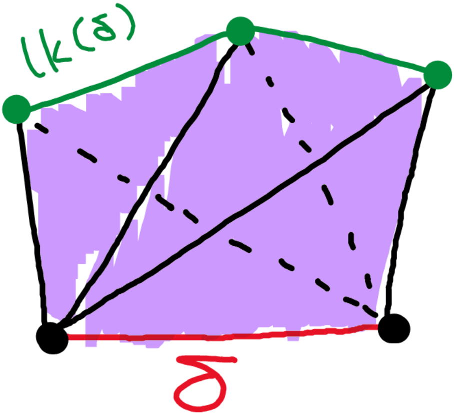
- Let L be a subcomplex of K and let v be a vertex in K but not in L. Then the set

$$vL := \{\tau | \tau \in L \text{ or } \tau = \sigma \cup v \text{ for } \sigma \in L\} \cup \{v\}$$

is called a **simplicial cone**.


lk(δ)
δ

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- A simplex σ in K is called a **dominated** if the link of σ in K is a simplicial cone, i.e. if there exists a vertex $v \notin \sigma$ and a subcomplex L of K, such that $lk_K(\sigma) = vL$.

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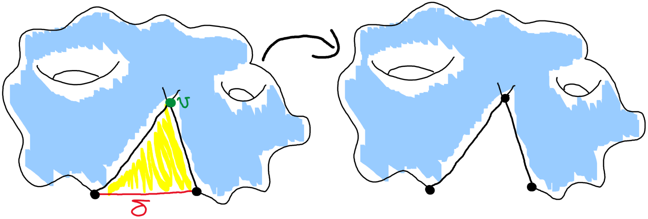
- A free simplex σ in K is dominated by $v = \tau \backslash \sigma$ where τ is the maximal coface:



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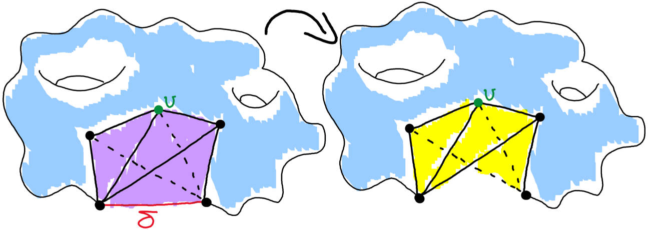
- Detecting dominated simplices in a complex can trigger a sequence of elementary collapses:



Lemma

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- For $lk_K(\sigma) = vL$ in K, these collapses can be associated with the removal of pairs $(\sigma \cup \alpha \cup \{v\}, \sigma \cup \alpha)$ in K that define elementary collapses.

How to identify the dominated simplices?

Lemma 1

A simplex $\sigma \in K$ is dominated by a vertex $v \in K$, $v \notin \sigma$, if and only if all the maximal simplices of K that contain σ also contain v.

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- Identifiying dominated simplices requires knowledge about maximal simplices in K.

Flag Complex and Neighborhood

- A complex K is a flag or a clique complex if, when a subset of its vertices form a clique (i.e. any pair of vertices is joined by an edge), they span a simplex

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- The open and closed neighborhoods of a k-simplex $\sigma = [v_1, \ldots, v_k]$ in K are defined as $N_K(\sigma) := \bigcap_{v_i \in \sigma} N_K(v_i)$ and $N_K[\sigma] := \bigcap_{v_i \in \sigma} N_K[v_i]$

Lemma 2

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" $\Leftarrow$ " :

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- Because K is a flag complex and τ is maximal, v must lie in τ. With Lemma 1 we conclude that $\sigma \prec v$.

" $\Rightarrow$ " :

- $\sigma \prec v \xLongrightarrow{Lemma 1}$ all maximal simplices that contain σ also contain v. This implies $N_K[\sigma] \subseteq N_K[v]$.

□

Theorem

Let $f : K \to L$ be a simplicial map between two complexes K and L and let $K' \subset K$ and $L' \subset L$ be subcomplexes of K and L such that $K \searrow K'$ and $L \searrow L'$. Then there exists a map $f' : K' \to L'$, induced by f, such that the persistence of $f_* : H_p(K) \to H_p(L)$ and $f'_* : H_p(K') \to H_p(L')$ are the same for any integer $p \geq 0$.

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Proof

$$
\begin{array}{ccc}
|K| & \xrightarrow{|f|} & |L| \\
|i_K| \uparrow \downarrow |r_K| & & |i_L| \uparrow \downarrow |r_L| \\
|K'| & \xrightarrow{|f'|} & |L'|
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
H_p(|K|) & \xrightarrow{|f|_*} & H_p(|L|) \\
|i_K|_* \uparrow \downarrow |r_K|_* & & |i_L|_* \uparrow \downarrow |r_L|_* \\
H_p(|K'|) & \xrightarrow{|f'|_*} & H_p(|L'|)
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  - Dominated edges can easily be recognized on the 1-skeleton

Edge Collapse Algorithm

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- Let $\{K_1 \to K_2 \to \cdots \to K_n\}$ be a filtration of flag complexes and let $\{G_1 \to G_2 \to \cdots \to G_n\}$ be the associated filtration of 1-skeleta.

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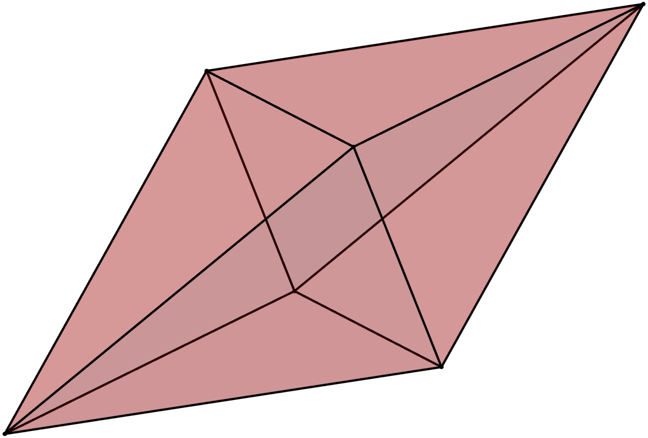
Example Edge Collapse



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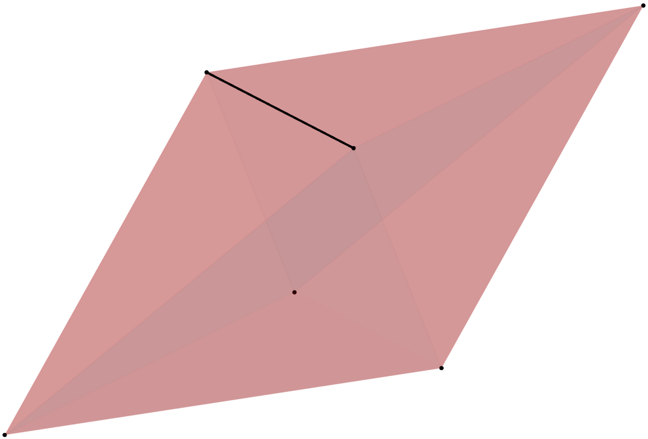

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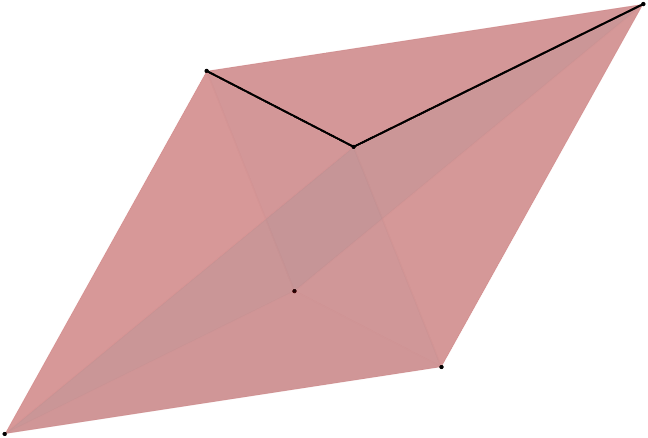

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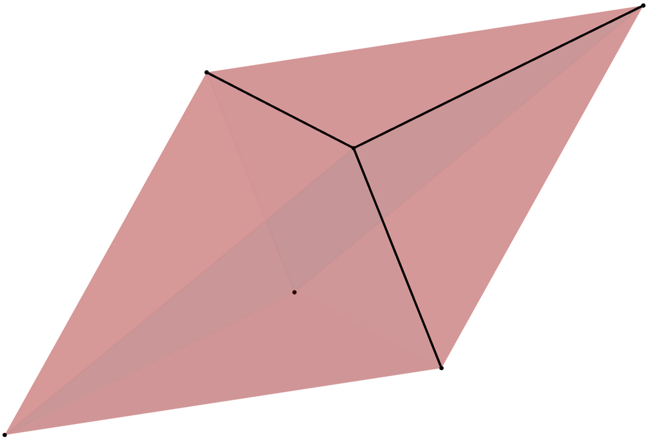

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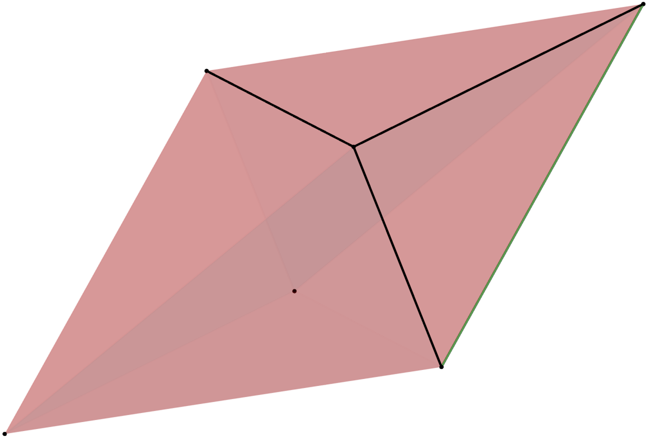

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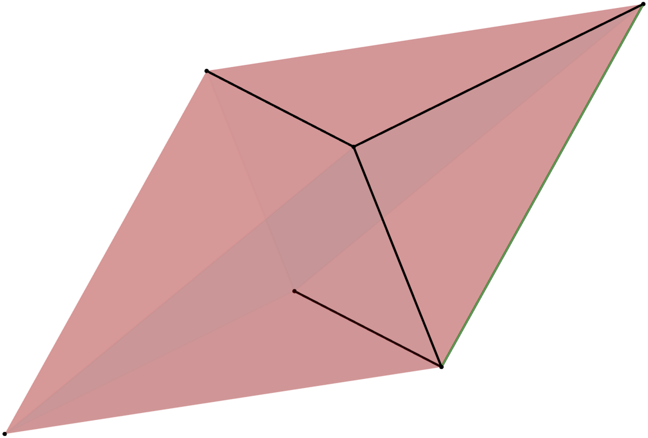

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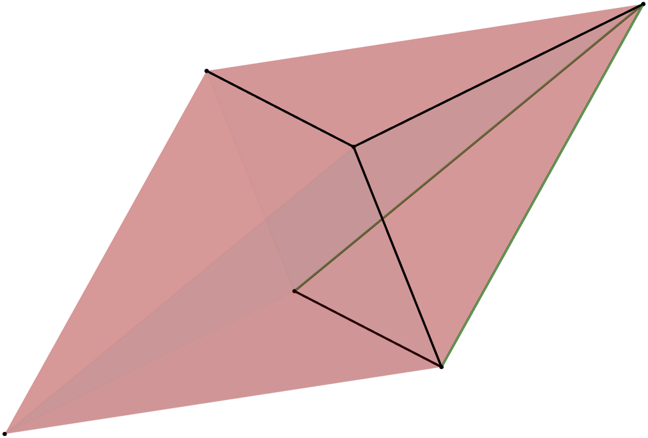

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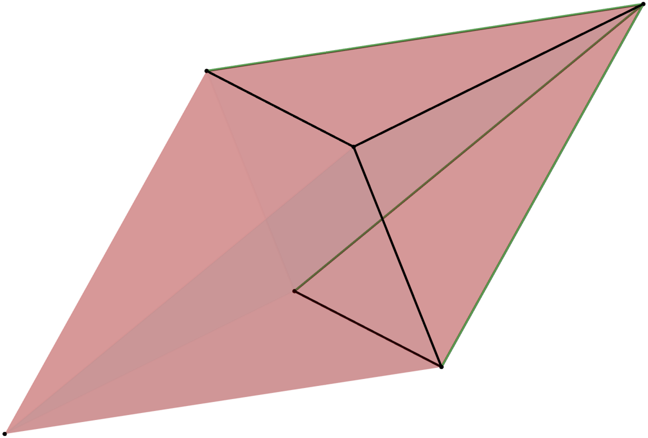

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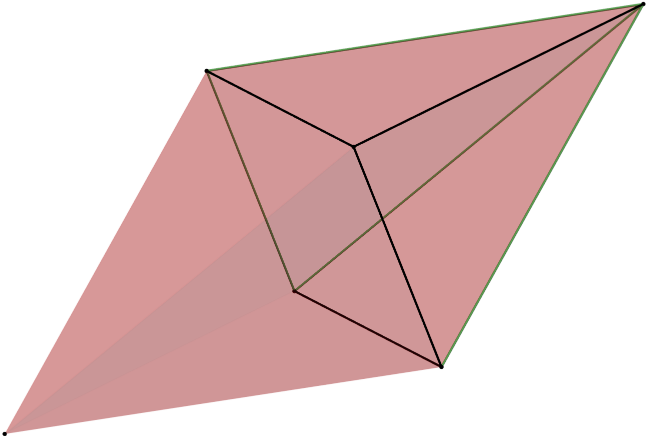

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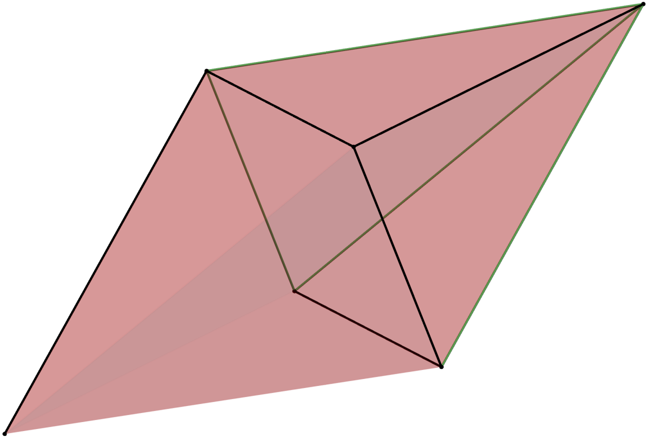
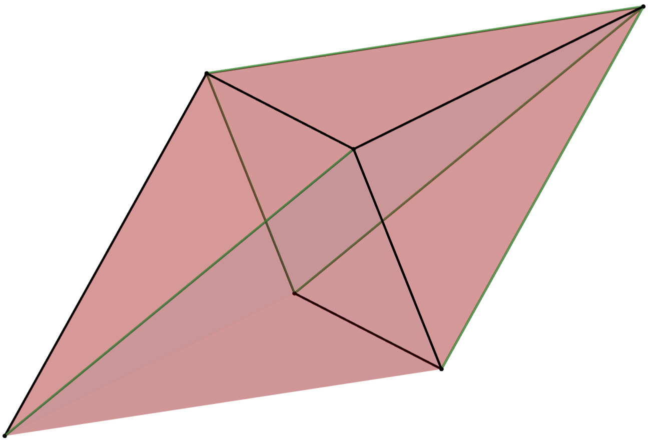
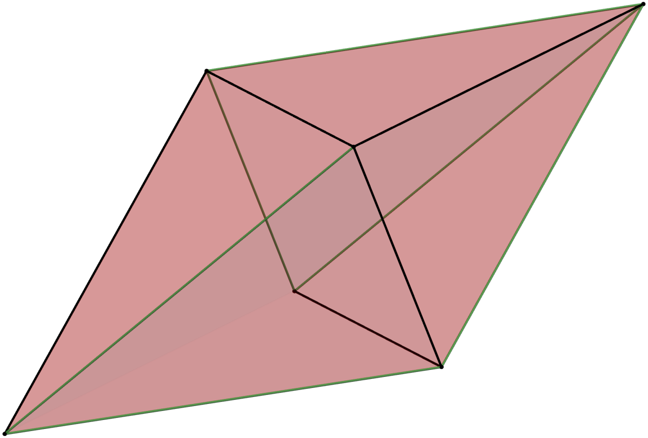


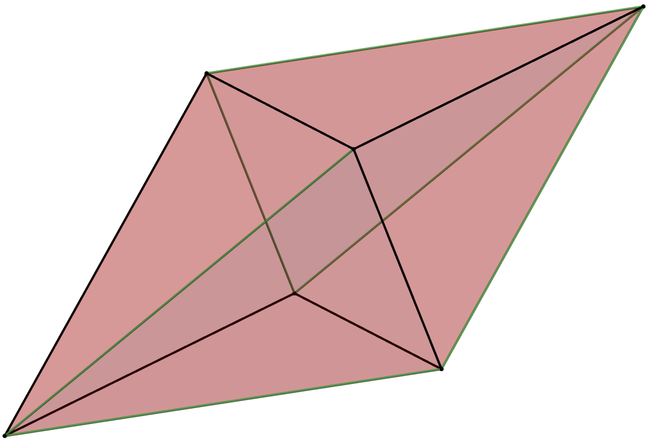
Example Edge Collapse



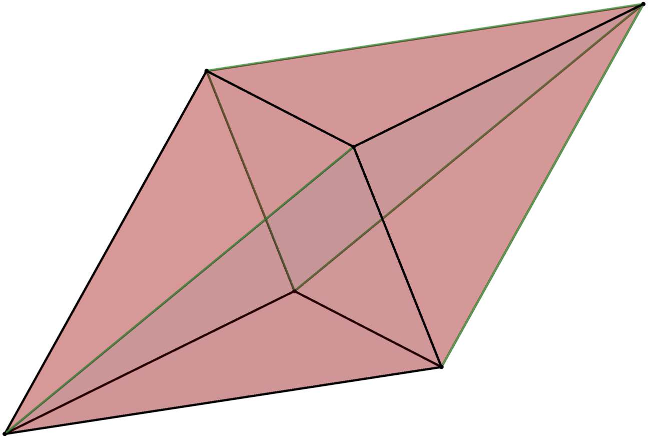
Example Edge Collapse



Example Edge Collapse



Example Edge Collapse



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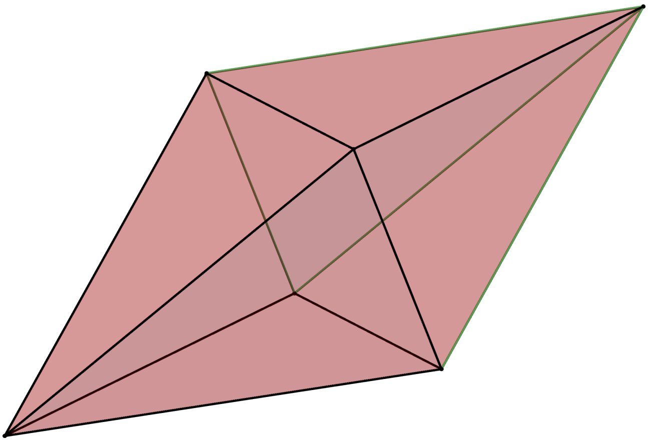

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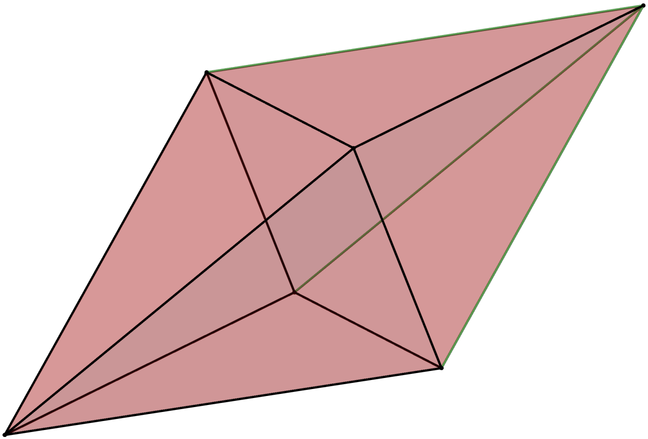

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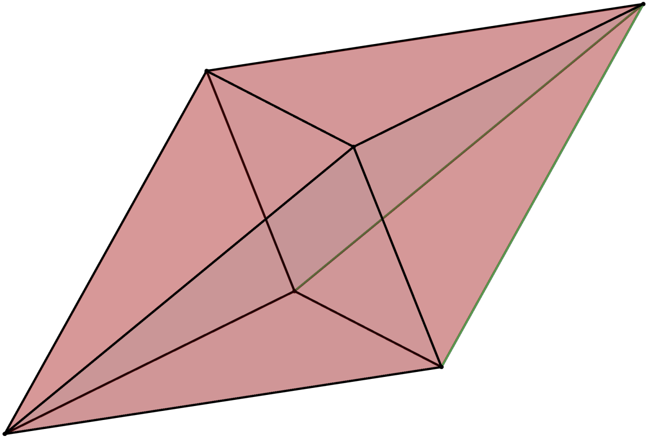

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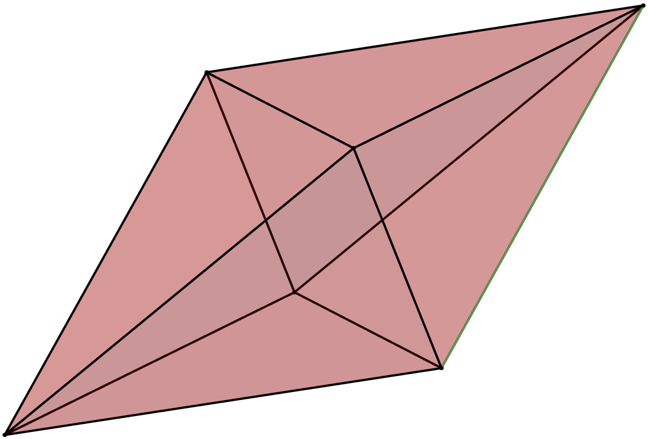

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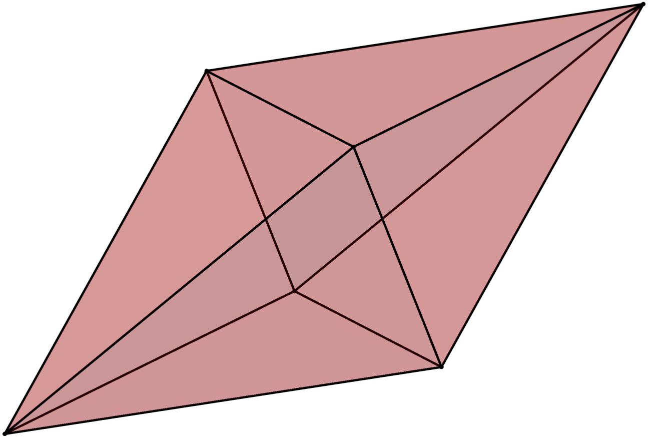

Computational Experiments

Data	Pnt	Thrsld	EdgeCollapser +PD				
			Edge(I)/Edge(C)	Size/Dim	*dim*	Pre-Time	Tot-Time
netw-sc	379	5.5	8.4K/417	1K/6	∞	0.62	0.73
senate	103	0.415	2.7K/234	663/4	∞	0.21	0.24
eleg	297	0.3	9.8K/562	1.8K/6	∞	1.6	1.7
HIV	1088	1050	182K/6.9K	86.9M/?	6	491	2789
torus	2000	1.5	428K/14K	44K/3	∞	288	289

Data	Pnt	Threshold	Ripser		Ripser		Ripser	
			dim	Time	*dim*	Time	*dim*	Time
netw-sc	379	5.5	4	25.3	5	231.2	6	∞
senate	103	0.415	3	0.52	4	5.9	5	52.3
"	"	"	6	406.8	7	∞		
eleg	297	0.3	3	8.9	4	217	5	∞
HIV	1088	1050	2	31.35	3	∞		
torus	2000	1.5	2	193	3	∞		

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- Questions and other remarks?