Magnitude Meets Persistence

- Magnitude
- Magnitude Homology
- Nerves, Blurring and Persistence

[Otter 18] arxiv: 1807.01540

[Cho 19] arxiv: 1910.02905

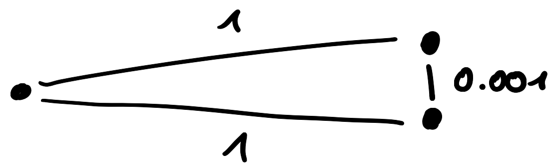
Magnitude

[Leinster 10] arxiv: 1012.5857

(X, d) finite metric space

Def. similarity matrix

$$(Z_X)_{ij} = e^{-d(x_i, x_j)}$$



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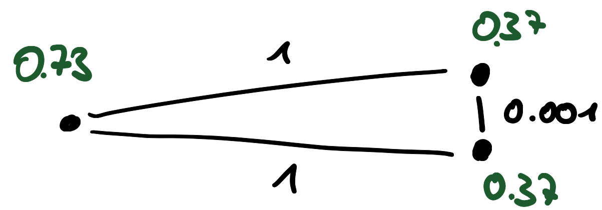

Def. a $\boxed{\text{weight of } Z_X}$ is a vector w st.

$$Z_X w = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

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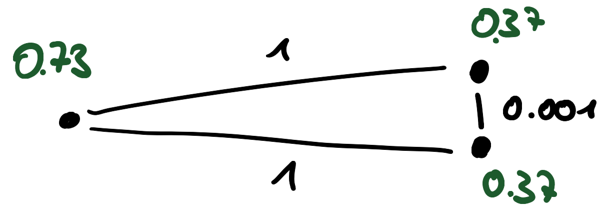

$|X| \sim 1.5$

Def. a $\boxed{\text{weight of } Z_X}$ is a vector w s.t.

$$Z_X w = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Def. The $\boxed{\text{magnitude of } X}$ is $|X| := \sum_i w_i$

Prop. If Z_X is invertible $|X| = \sum_{i,j} (Z^{-1})_{ij}$

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Magnitude is a measure for the effective number of points.
First appeared in biology (1995) to quantify biodiversity.

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For $t \in [0, \infty]$ write $tX := (X, t\,d(\cdot,\cdot))$

Def. magnitude function of X

$$t \longmapsto |tX|$$

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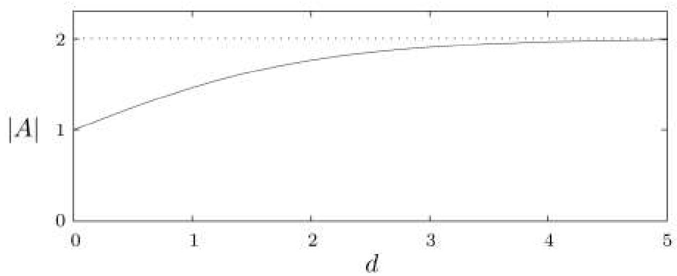
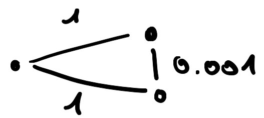
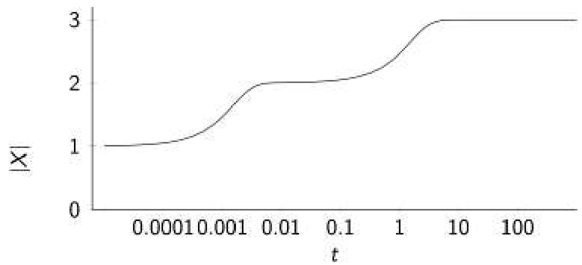
Ex. (i) $X = \bullet \overset{1}{\text{———}} \bullet$, $|tX| = 1 + \tanh\left(\frac{t}{2}\right)$



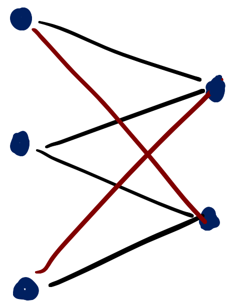
(ii) $X =$ 



Magnitude Function

Example of bad metric space

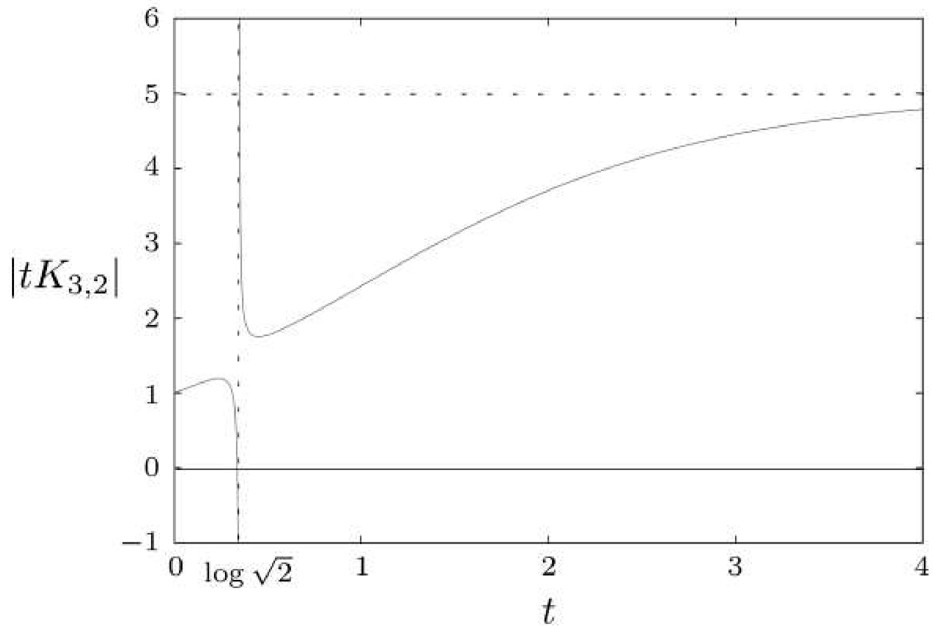
$K_{3,2} =$



edges = distance 1

+ shortest paths

$$|tK_{3,2}| = \frac{5 - 7e^{-t}}{(1+e^{-t})(1-2e^{-2t})}$$



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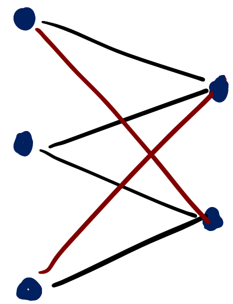

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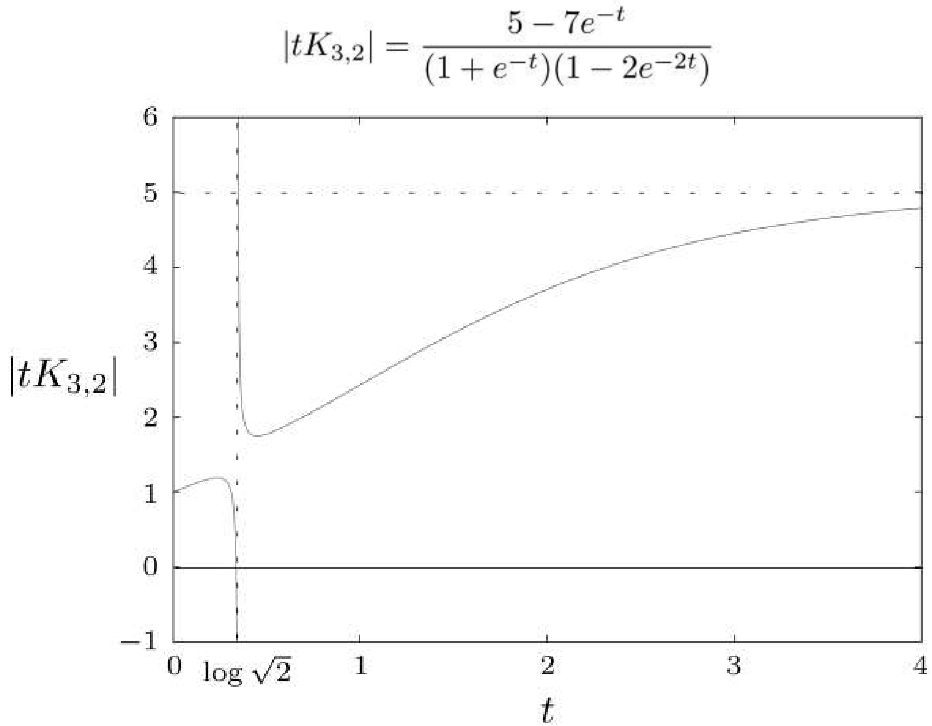

Prop. X finite metric space

(i) tX has magnitude for all but finitely many $t_i \in [0, \infty]$.

(ii) the magnitude function of X is analytic at all $t \notin \{t_i\}$

(iii) For $t \gg 0$ the magnitude function of X is increasing

(iv) $|tX| \to \# X$ as $t \to \infty$

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Fact If $X \hookrightarrow \mathbb{R}^n$ isometrically, the magnitude function does not have singularities

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How do these singularities arise? $|tX| = \dfrac{a_1 q^n + \cdots + a_m q^{n+m}}{b_1 q^r + \cdots + b_s q^{r+s}}$, $q = e^{-t}$

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Schematically

$$z_X^{-1} = \frac{1}{z_X} = \frac{1}{1-(z_X-1)} \overset{\text{Taylor}}{=} \sum_n (-1)^n (z_X - 1)^n$$

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where

$$(Z_X - \mathbb{1})(x,x) = 0$$

$$(Z_X - \mathbb{1})^n (x_0, x_n) = \sum_{x_i \neq x_j} Z_X(x_0, x_1) \cdots Z_X(x_{n-1}, x_n)$$

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$$(Z_X - 1)(x,x) = 0$$

$$Z_{tX}(x_i, x_j) = e^{-t\,d(x_i,x_j)} \equiv q^{d(x_i,x_j)}$$

$$(Z_X - 1)^n (x_0, x_n) = \sum_{x_i \neq x_j} Z_X(x_0,x_1) \cdots Z_X(x_{n-1},x_n) = \sum_\ell \sum_{\substack{x_i \neq x_j \\ \ell = \sum d(x_i,x_{i+1})}} q^\ell = \sum_\ell a_{n,\ell}\, q^\ell$$

Magnitude Function

In summary

$$\boxed{|X| = \sum_{i,j} Z_X^{-1}(x_i, x_j) = \sum_{n=1}^{\infty} \sum_{\ell \in [0,\infty)} (-1)^n \, a_{n,\ell} \, q^\ell}$$

Magnitude ~~Function~~ Homology

[Leinster-Shulman 17] arxiv: 1711.00802

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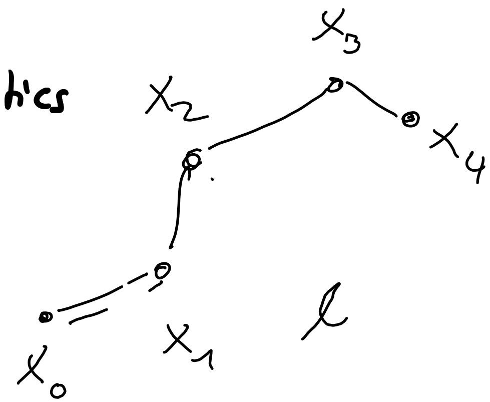
$$\boxed{|X| = \sum_{i,j} \zeta_X^{-1}(x_i, x_j) = \sum_{n=1}^{\infty} \sum_{\ell \in [0,\infty)} (-1)^n a_{n,\ell}\, q^{\ell}} = \chi(A_{\bullet,\bullet})$$

$\mathrm{rk}(A_{n,\ell})$?

Does there exist a homology theory whose Euler characteristics gives the magnitude of X when evaluated at $q = e^{-1}$?



↳ $\boxed{\text{Magnitude Homology}}$

Magnitude Homology

Def. Magnitude Complex

$$MC_{n,\ell}(X) = \mathbb{Z}\left[\left\{ \langle x_0, \dots, x_n \rangle \;\middle|\; \ell = \sum_{i=0}^{n-1} d(x_i, x_{i+1}),\ x_i \neq x_{i+1} \right\}\right]$$

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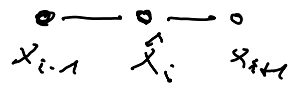
$$(-1)^i$$

Def Differential $d_n = \sum_{i=1}^{n-1} d_n^i \;:\; MC_{n,\ell}(X) \to MC_{n-1,\ell}(X)$



$$d_n^i(\langle x_0, \dots, x_n \rangle) = \begin{cases} \langle x_0, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_n \rangle & \text{if } d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

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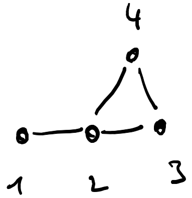
Thm [LS 17] $\quad |tX| = \mathrm{Poin}(MH_{\bullet,\bullet}(X))(t=-1, q) = \sum \mathrm{rk}\, MH_{n,\ell}(X)\, t^n q^\ell \Big|_{t=-1}$

Magnitude Homology - Example

$X =$ 

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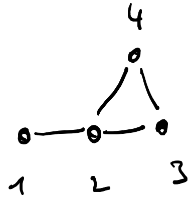

$d_0 \equiv 0$, $d_1 \equiv 0$

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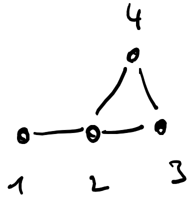
$X =$ 

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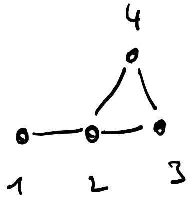
$X =$ 

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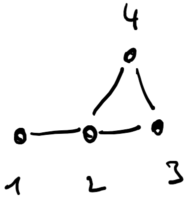
$X =$ 

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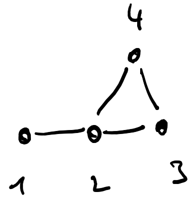
$X =$ 

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Magnitude Homology - Geometric Interpretation

More generally:

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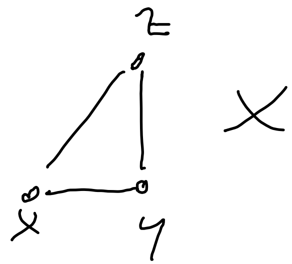

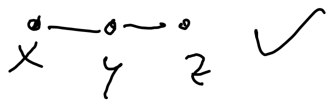

Say y is **between** x and z, if $d(x,y) + d(y,z) = d(x,z)$.
If there is no y between x and z then they are **adjacent**.

$$MH_{1,\ell} = \begin{cases} \mathbb{Z}[\langle x_i, x_j \rangle] & \ell = d(x_i, x_j) \text{ and } x_i, x_j \text{ adjacent} \\ 0 & \end{cases}$$

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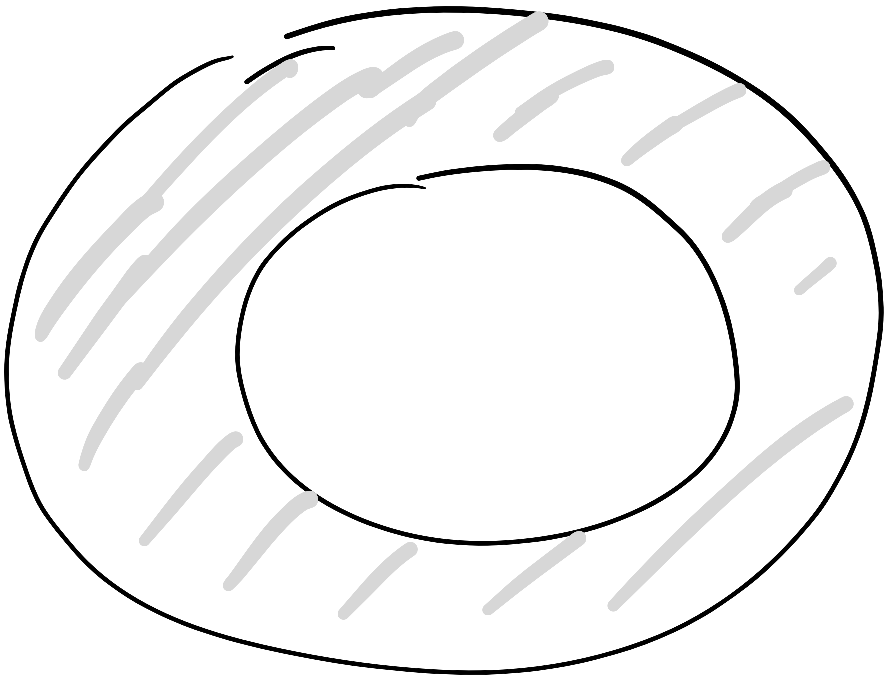
Corollary If $X \subset \mathbb{R}^n$ is a closed convex subset (not necessarily finite) then $MH_{1,\ell}(X) = 0$.

Magnitude Homology - Geometric Interpretation

Example



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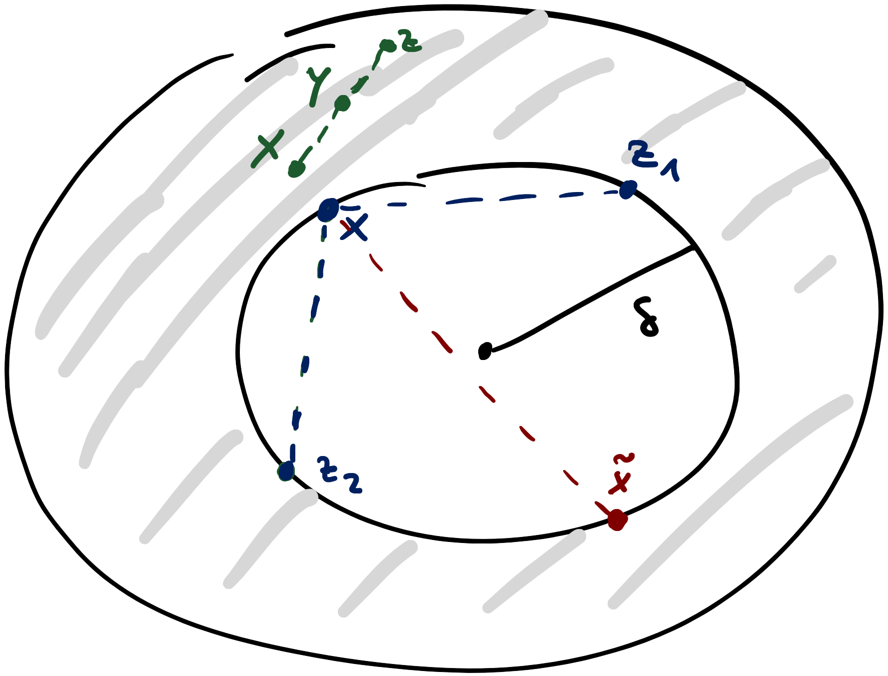

$$MH_{1,\ell} = \begin{cases} \mathbb{Z}[S^1 \cdot 2] & 0 < \ell < \delta \\ \mathbb{Z}[S^1] & \ell = \delta \\ 0 & \text{else} \end{cases}$$

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Thm Suppose

- X is geodetic $\simeq$ any 2 pts are connected by a unique shortest path

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  either X is Menger convex $\approx$ any 2 pts are non-adjacent

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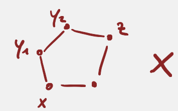
and has no 4-cuts

$$\begin{aligned} d(x,y_1)+d(y_1,y_2) &= d(x,y_2) \\ d(y_1,y_2)+d(y_2,z) &= d(y_1,z) \end{aligned} \Rightarrow d(x,z) = d(x,y_1)+d(y_1,y_2)+d(y_2,z)$$



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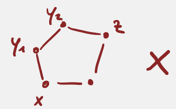

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Magnitude Homology – Geometric Interpretation

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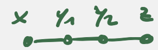
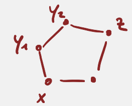
  

  ✓

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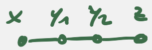

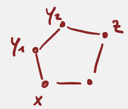

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$\leadsto MH_{2,\bullet}$ obstructs any (or both) of these properties

Magnitude Homology – Geometric Interpretation

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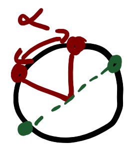

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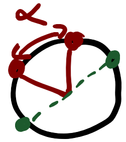

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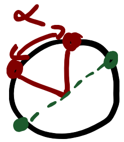
Examples • $X = S^1$ w/ arc-distance. 

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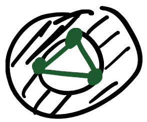
• $X =$  is geodetic and does not have 4-cuts.

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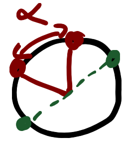
Examples • $X = S^1$ w/ arc-distance. 

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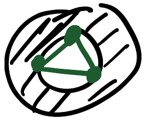
• $X =$  is geodetic and does not have 4-cuts.

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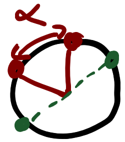
Examples • $X = S^1$ w/ arc-distance. 

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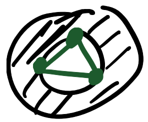
• $X =$  is geodetic and does not have 4-cuts.

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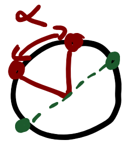
Examples • $X = S^1$ w/ arc-distance. 

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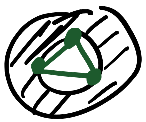
• $X =$  is geodetic and does not have 4-cuts.

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measures failure of "simultaneous convexity" of triangle's edges

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Nerves, Blurring and Persistence

[Otter 18] arxiv: 1807.01540

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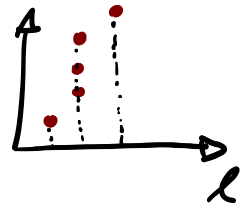
2. Magnitude Homology only contains "ephemeral" interval modules

   ⇝ MH is "fully localized" at ℓ,

   we are looking for "blurred" magnitude homology that has "observable" bars to make contact w/ persistence.

These two steps are easiest described in categorical language

Nerves, Blurring and Persistence - Lawvere metric spaces

Recall the following generalizations of metric spaces

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- Quasi-metric drops symmetry: $d(x,y) \neq d(y,x)$
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$$X(x,y) \times X(y,z) \to X(x,z)$$
$$\Updownarrow$$
$$d(x,y) + d(y,z) \geq d(x,z)$$

Thm [Lawvere 74]

A $([0,\infty], +, 0)$-enriched category X is an extended pseudo-quasi-metric space.

Nerves, Blurring and Persistence - ℓ^p-Nerves

Note: We could also use $a +_p b := (a^p + b^p)^{1/p}$ as composition in $(V, \otimes, 0)$

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Now we note [Cho 19]

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Nerves, Blurring and Persistence - Blurring and Localizing

Def [otter 18] Blurred Magnitude Homology is the persistent homology of the ℓ^1-Nerve of X: $\widetilde{MH}_{\bullet,\bullet} := H_\bullet(\widetilde{MC}_{\bullet,\bullet}, d)$

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Def

$$\mathrm{Loc}_J(\widetilde{MC}_{\bullet,\bullet})(r) = \widetilde{MC}_{\bullet,r} \Big/ \bigcup_{s \in J_r} \mathrm{im}(\widetilde{MC}_{\bullet,s} \to \widetilde{MC}_{\bullet,r})$$

Choosing the maximal non-trivial sieve J_{max}, i.e.

$$J_r = \{ s \in [0,\infty) \mid s < r \}$$

leads to

Thm [Cho 19]

Maximally localized blurred magnitude homology is magnitude homology

$$\mathrm{Loc}_{J_{max}}(\widetilde{MC}_{\bullet,\bullet}) = MC_{\bullet,\bullet}$$

Nerves, Blurring and Persistence – Summary

$$
\begin{tikzcd}[column sep=small]
 & \text{Global } \ell^1\text{-homology} \arrow[dl, Rightarrow, "\eta"', "\ell^1 \subset \ell^\infty"] \arrow[dr, Rightarrow, "\rho", "\mathrm{Loc}_{J_{max}}"'] & \\
\begin{array}{c} \text{Global } \ell^\infty\text{-homology} \\ \parallel \\ \text{Persistent Homology} \\ \text{of VR-Nerve} \end{array} \arrow[dr, Rightarrow, "\rho"', "\mathrm{Loc}_{J_{max}}"] & & \begin{array}{c} \text{Local } \ell^1\text{-homology} \\ \cong \\ \text{Magnitude Homology} \end{array} \arrow[dl, Rightarrow, "\eta", "\ell^1 \subset \ell^\infty"'] \\
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