INAUGURAL – DISSERTATION

zur

Erlangung der Doktorwürde

der

Gesamtfakultät für Mathematik, Ingenieur- und Naturwissenschaften

der

Ruprecht-Karls-Universität

Heidelberg

vorgelegt von

Waas, Lukas, M.Sc.

aus Chemnitz

Tag der mündlichen Prüfung: ....

Stratified Homotopy Theory and Generalized Simple Homotopy Theory: Foundations, Applications and Intersections

Lukas Waas

Betreuer: Prof. Dr. Markus Banagl

Abstract

This thesis is concerned with the homotopy theory of stratified spaces as well as generalized simple homotopy theory.
On the side of stratified homotopy theory, we establish a series of foundational results concerning several different versions of stratified homotopy theory that have been suggested in the literature. A central question in stratified homotopy theory is how the latter interacts with classical, geometrical examples of stratified spaces. With the aim of answering this question, we prove the existence of semi-model structures on the category of stratified spaces that present, respectively, the homotopy theories of stratified spaces suggested by Doteau and Henriques, Haine and Nand-Lal. Importantly, these structures are such that classical geometrical examples of stratified spaces are bifibrant, and they are furthermore strongly related to Quinn's approach to stratified homotopy theory.
To prove the existence of these structures, we perform a detailed investigation of combinatorial approaches to stratified homotopy theory, develop a theory of generalized regular neighborhoods in stratified spaces, and use the latter to obtain cellular models of generalized stratified homotopy links of stratified cell complexes.
Furthermore, we prove stratified analogues of the classical Kan-Quillen equivalence between simplicial sets and topological spaces. As a consequence of our investigations, we obtain a presentation of a stratified version of the homotopy hypothesis, as conjectured by Ayala, Francis and Rozenblyum: We prove that Lurie's construction of the infinity-category of Exit-paths defines a Quillen equivalence, between a semi-model category of stratified spaces and the left Bousfield localization of the Joyal model structure that presents the homotopy theory of such small infinity-categories in which every endomorphism is invertible.
We apply our theoretical results in the topological data analysis of stratified spaces, proving a sampling theorem that guarantees the recovery of persistent stratified homotopy-theoretic information from large classes of two-strata Whitney stratified spaces.

On the side of generalized simple homotopy theory, we develop an axiomatic framework that allows for the investigation of the latter in the context of (semi-)model categories that are equipped with appropriate notions of generating boundary inclusions and elementary expansions. We show that the resulting theory behaves much like the classical simple homotopy theory of spaces or chain complexes, and encompasses these frameworks.
We furthermore perform a detailed investigation of the simple homotopy theory of diagram categories, proving a decomposition theorem for their Whitehead groups and establishing results on the compatibility of simple equivalences with certain colimits.
We then apply our general framework to some of the (semi-)model categories for stratified homotopy theory which we investigated earlier in this thesis. In particular, we prove a decomposition theorem for the resulting stratified Whitehead groups associated to Douteau's theory in terms of classical Whitehead groups of strata and generalized homotopy links.

Zusammenfassung

Diese Dissertationsschrift befasst sich mit der Homotopietheorie stratifizierter Räume und generalisierter einfacher Homotopietheorie.
Auf Seiten der stratifizierten Homotopietheorie beweisen wir eine Reihe von grundlegenden Resultaten zu verschiedenen Ansätzen zu selbiger in der Literatur.
Eine zentrale Fragestellung in stratifizierter Homotopietheorie ist, wie diese mit klassischen geometrischen Beispielen stratifizierter Räume interagiert. Mit dem Ziel, diese Frage zu beantworten, beweisen wir die Existenz von Semi-Modellstrukturen auf Kategorien von stratifizierten topologischen Räumen, in welchen solche klassischen geometrischen Beispiele bifasernd sind. Diese präsentieren die Homotopietheorien stratifizierter Räume, die in den letzten Jahren von Douteau und Henriques, Haine und Nand-Lal vorgeschlagen wurden und stehen in enger Beziehung zu Quinns Ansatz zur stratifizierten Homotopietheorie.
Um die Existenz dieser Modellstrukturen zu beweisen, führen wir eine detaillierte Untersuchung kombinatorischer Modelle für stratifizierte Homotopietheorie durch, entwickeln eine Theorie verallgemeinerter regulärer Umgebungen im stratifizierten Kontext und beweisen die Äquivalenz von generalisierten Homotopielinks zu bestimmten einfachen zellulären Modellen.
Außerdem beweisen wir stratifizierte Analoga der klassischen Kan-Quillen-Äquivalenz zwischen simplizialen Mengen und topologischen Räumen. Insbesondere ergibt sich aus unseren Untersuchungen eine Präsentation der stratifizierten Homotopiehypothese durch Semi-Modellkategorien, wie sie von Ayala, Francis und Rozenblyum vermutet wurde: Wir beweisen eine Quillen-Äquivalenz – gegeben durch Lurie's Konstruktion der Unendlichkategorie der Exit-Pfade – zwischen einer Semi-Modellkategorie topologischer stratifizierter Räume und der Bousfield-Lokalisierung der Joyal-Modellstruktur, welche die Homotopietheorie solcher kleiner Unendlichkategorien repräsentiert, in denen alle Endomorphismen Isomorphismen sind.
Zudem präsentieren wir eine Anwendung unserer theoretischen Resultate in stratifizierter topologischer Datenanalyse. Konkret beweisen wir einen Samplingsatz, der es erlaubt, persistente stratifiziert-homotopietheoretische Informationen aus Samples von Whitney-stratifizierten Räumen mit zwei Strata zurückzugewinnen.

Auf Seiten der einfachen Homotopietheorie entwickeln wir ein allgemeines axiomatisches Framework, um selbige in einem semi-modellkategoriellen Kontext zu betreiben. Wir zeigen, dass sich die resultierende verallgemeinerte Theorie in vielerlei Hinsicht wie die klassische einfache Homotopietheorie von Räumen oder Kettenkomplexen verhält, und diese beinhaltet.
Weiterhin führen wir eine detaillierte Untersuchung der resultierenden einfachen Homotopietheorie von Diagrammkategorien durch, beweisen in diesem Kontext ein Zerlegungsresultat für die assoziierten Whitehead-Gruppen und zeigen Resultate zur Kompatibilität von verallgemeinerten einfachen Äquivalenzen mit bestimmten Kolimites.
Zum Abschluss wenden wir unser allgemeines Framework auf Semi-Modellstrukturen für stratifizierte Homotopietheorie an, die wir in der ersten Hälfte dieser Arbeit untersucht haben. Insbesondere beweisen wir ein Zerlegungsresultat für die resultierenden stratifizierten Whitehead-Gruppen in Douteaus Theorie, ausgedrückt durch klassische Whitehead-Gruppen von Strata und generalisierten Homotopielinks.

Acknowledgments

First off, my sincerest gratitude goes out to my collaborators and dear friends Tim Mäder and Sylvain Douteau: For all of the inspiring discussion, for the math we created, for the drinks and meals we shared, and for bearing with me when there was more than one opinion. You managed to make even the Covid years a memorable experience.
My deep gratitude goes out to my advisor Markus Banagl. I can still recall him writing the definition of de Rham cohomology on the board in 2015 awakening my interest in topology. It has not left me ever since, through the many years in which I had the wonderful opportunity to learn more about topology from him.
I also want to thank all of the people who I had the chance to discuss my and their work with and who gave helpful feedback along the way: Most prominently, David Chataur, for all of his support, Peter Haine, for the insightful discussions concerning his work, and Sylvain, for the numerous hours that we discussed stratified homotopy theory.
Then my deep thanks go out to all of the people who helped me in proofreading parts of this thesis: Tim Holzschuh, Jakob, and especially to Volker. Whatever mistakes are left are mine and mine alone.
On the financial side, my gratitude goes out to the *Landesgraduiertenförderung Baden-Württemberg* for financing my studies, as well as the *Young Researchers Convent* of the *Structures Cluster of Excellence Uni Heidelberg* for financing most of my travels and research visits.
Finally, I want to thank my friends and family. Thank you, Ricardo, Johannes, Tim, Sebastian, Timo, Kerstin, and especially Rike, for reminding me that there are other wonderful things in this world besides mathematics.

Contents

Foreword

This thesis presents my research on stratified homotopy theory and generalized simple homotopy theory, performed during my PhD at Heidelberg University from 2022 to the end of 2024. During my master's thesis in 2021, my advisor Markus Banagl suggested the following question to me: "*Is there a simple homotopy theory of stratified spaces?*" This question originally occurred in investigations of his into stratified methods in applied topology, but had evident relations to classification questions of stratified spaces. Innocent as it may seem at first glance, it has, of course, another two questions hidden in plain sight: "*What is the homotopy theory of stratified spaces, and what do we know about it?*" and "*What constitutes a simple homotopy theory?*". Pursuing these two questions, in particular the first one, has been the main bulk of my research for the duration of the last years. Before I get into a more mathematical account, let me say a few words on how the several different pieces in this thesis came to be, and how they relate to these original questions. First off, let me make clear that this is not a thesis about simple stratified homotopy theory. It is a thesis about stratified homotopy theory, and about a certain viewpoint on generalized simple homotopy theory (GSHT for short). These two theories have an overlap, and one can use general techniques from both theories to study that overlap, but I am personally not of the opinion that the results concerning this overlap, which I will present in the final chapter, are the ultimate goal of all of the other theory presented here. Rather, I took the questions about stratified simple homotopy theory as a guiding incentive to explore both theories, which I think are of independent interest.

When I began writing my thesis, the first question (concerned with the homotopy theory of stratified spaces) had just received a renewed wave of attention, with additional interest coming from higher categorical perspectives, and at least three different PhD theses being concerned with establishing the foundations of stratified homotopy theory (see the overview chapter below for details and references). All of these theses take related, but slightly different approaches to defining such a homotopy theory of stratified spaces and all of them made great advances in their own rights. They left open two crucial questions, however, answers to which seemed not only necessary for the approach to stratified simple homotopy that I had in mind, but also for a general foundational view on stratified homotopy theory:
First off, the question of an analogue of the Kan-Quillen equivalence between topological spaces and simplicial sets, which would freely allow one to pass between the stratified topological and the stratified simplicial world. Secondly, the question after the existence of certain model structures which enable one to relate the developed homotopy theory to classical, geometric examples of stratified spaces.

Part II of this thesis is concerned with tackling these two fundamental questions. The first of these two questions was answered in joint work with Sylvain Douteau (to appear in the *Mémoires de la Société Mathématique de France*), during the first half of my PhD years. In parallel to this, I was working on several collaborations concerned with applying algebro-homotopical methods in topological data analysis. Most importantly for the topics discussed here, my Ph.D. sibling Tim Mäder was pursuing the question of recovering stratified homotopic/homological information from (possibly stratified) point samples near a sufficiently regular stratified space. It turned out that the insights developed with Sylvain Douteau could be applied to these questions, which resulted in joint work (now published in the *Journal of Applied and Computational Topology*) that I also present here.
Following this, I tackled the second question on the existence of convenient model structures in independent work. My results on these explorations, which I cover here, have also been made preliminarily available in a series of preprints on the arXiv. These prove the existence of such (semi-model) structures, illuminate and establish the precise connections between the several different theories developed in recent years, and cover their relationships to ∞-categories and to more classical approaches to stratified homotopy theory. All of the investigations on stratified homotopy theory I have discussed so far are presented in the first half of this thesis.

Thus, having made progress on the foundations of stratified homotopy theory, in the remaining time, I returned to questions of generalized simple homotopy theories. At this point, it had become apparent to me, however, that there was a need to approach the question from a more big-picture perspective.
In my master's thesis, I had developed a simple homotopy theory for stratified simplicial sets, based on Douteau's model structure on stratified simplicial sets. It seemed likely that the arguments used there would work in significantly larger generality for all sorts of model categorical setups. Having several different versions (and models of these versions) for stratified homotopy theory available, there was clearly a need for an axiomatic approach to generalized simple homotopy theory. There were already two such general approaches on the market (again, see the overview chapters for proper references). However, both of them did not quite capture the perspective on generalized simple homotopy theory which I was looking to incarnate. Namely, the study of non-uniqueness of presentations of homotopy types (in some general homotopy theory) in terms of elementary building blocks, subject to certain elementary operations.
Taking this perspective, I develop such a generalized approach in Part III of this thesis, leading to a model categorical framework for generalized simple homotopy theory, investigations on interactions of simple homotopy theory with diagrams and homotopy colimits, and transfer theorems allowing one to switch between combinatorial and topological frameworks.
Equipped with such new tools in generalized simple homotopy theory, I then combined these insights with the results on stratified homotopy theory obtained in the first half of this thesis. In particular, this involved a computation of the Whitehead group constructed in my master's thesis, in terms of classical Whitehead groups of strata and links. This constitutes the final chapter of this thesis.

From what I have described about my work process so far, the reader may already suspect that I have a tendency towards establishing general theory, in order to tackle a specific problem. This tendency is certainly partially responsible for the length of this text. I am optimistic, however, that the degree of generality achieved here will prove useful in future investigations into stratified homotopy theory, simple homotopy theory, and the interaction of these disciplines with applied realms such as topological data analysis. Another reason for the length of this text is that the first half of the thesis is structured in terms of chapters which can be read independently and reflect the content of independent articles. This leads to a certain amount of redundancy in introductory and notational sections. At the same time, I expect that it will make chapters significantly more accessible to a reader only interested in particular parts of the theory.

Nevertheless, this does not solve the fact that a thesis of this length, especially when covering several different topics and structured into several partially independent pieces, written with slightly different goals and a high degree of generality in mind, may make for a somewhat lengthy reading experience. To address this, I wrote Part I of this thesis. It presents the material and main results of the remaining five hundred or so pages in a coherent context. The first chapter covers a general introduction to stratified homotopy theory, motivating and explaining the different approaches and presenting my results in this context. The second chapter surveys my results on generalized simple homotopy theory, with a specific focus on its applications to the stratified setting.

Guide to the reader

As already alluded to, this text covers several different topics which can (and probably should) to a certain degree be read independently. Let us now provide the reader with a general overview of the structure of this thesis and the dependency relations of its several parts. First off, this thesis does not feature an introduction in the classical sense of a few motivating words, followed by a short survey of the main results. Instead, it features two separate chapters

(Chapters 1 and 2 constituting Part I) which serve a similar purpose, but in more detail. Shorter and more specified introductions are featured in the separate chapters of Part II. Furthermore, this text contains two glossaries, one summarizing the most important recurring notation of the stratified setting in Chapter 1 (see Section 1.5) and one featuring notation used in Part III (see directly after Chapter 13).

Part I

Part I of this thesis is written in the style of an extended survey article. It summarizes the main results of Parts II and III and puts them into mathematical context. While formal definitions and theorems are given, the focus lies more on the conceptual side, providing intuition, examples and motivations for definitions and results. For proofs and technical details, we provide references to the latter parts. Chapter 1 also provides an introduction to stratified homotopy theory, starting essentially from scratch, and presenting several of the approaches in the literature. We expect that the quickest way to obtain a comprehensive overview of this thesis and an understanding of the main results is to first read Part I, and then move on to specific chapters in the latter parts for proofs and additional details.

The following two parts of the thesis can be read essentially independently, and both of them can be read entirely independently from Part I. Only in the last chapter of Part III, Chapter 13, are results from both parts used.

Part II

This part of the thesis is generally concerned with stratified homotopy theory and its applications. It is structured such that each chapter has the structure of an independent article. For example, notation is introduced separately in each of these chapters. Hence, while cross-referencing each other, each of them is accessible without having performed a detailed read of the other chapters. Two of these chapters, Chapters 3 and 4 are essentially (up to minor changes in notation) identical with the content of two articles which are, respectively, accepted and published ([DW22; MW24]). [DW22] is joint work with Sylvain Douteau and [MW24] is joint work with Tim Mäder. The remaining three chapters Chapters 5 to 7 present only my own work. While they can, in theory, be read separately, they strongly build on each other. Namely, Chapters 5 and 6 provides the technical groundwork for Chapter 7. The most insightful way of reading these chapters is, in all likelihood, to first read Section 1.3.1, and then read the chapters in order. Alternatively, one can jump into Chapter 7 right away, and return to the other chapters of II whenever necessary. We also recommend Section 1.5 as a list of some of the most important recurring notation.

Part III

Unlike the previous part, Part III, which is concerned with a model categorical approach to generalized simple homotopy theory, is intended to be read as one coherent piece, and notation is not introduced separately in each chapter. An index containing recurring notation can be found after Chapter 13. As already mentioned above, up to the final chapter, Chapter 13, none of the material relies on Part II though. Our model-categorical approach to simple homotopy theory required the development of a calculus of structured cell complexes in an abstract categorical setting. This language is covered in Chapter 8. This chapter is rather technical in nature, and, to a large part, re-frames existing theory on cell complexes in model categories in a setting that is suited to our purposes of simple homotopy theory. For a more motivated reading experience, we certainly recommend to first read Chapter 2. Furthermore, the subsequent chapters Chapters 9 and 10 do not rely on Section 8.3 of Chapter 8. We recommend to first skip the latter, and then return to it when reading Chapter 11. The next three chapters Chapters 9 to 12 all build upon each other, and are best read in linear order. Finally, Chapter 13 relies on results spread all over Parts II and III, and should probably only be read after reading Part III (or its summary in Chapter 2) and at least the summary of Part II in Chapter 1. A list of important recurring notation used in Part III can be found after Chapter 13.

Part I

A detailed overview of this thesis and its main results

Chapter 1

Stratified homotopy theory

Note to the reader: This chapter contains an introduction to the homotopy theory of stratified spaces and explains the main results of Part II in this context. Additional results, details, and proofs can be found in Part II. This chapter also features a glossary of some of the most important recurring notation in this chapter, which may also come in handy in latter parts of this text (see Section 1.5). To follow the material presented here in detail, basic familiarity with the language of model categories (see, for example, [Hir03]) and $(\infty, 1)$-categories (∞-categories, henceforth, see [Lur09]) is required. However, we will generally explain the conceptual role these formal theories play for our purposes in the text. When we say ∞-category, we will usually mean quasi-category, in the sense of Joyal and Lurie, but most of what we say below could just as well be phrased in any of the alternative, equivalent models for ∞-categories (see [Ber07b]).

From a tonal perspective, the following is phrased akin to what would constitute an introductory talk to the topic at a conference or an expository article. At times, we may adopt a less formal or rigorous tone than in the remainder of this work, referring to Part II for a completely rigorous treatment. We will also dedicate several pages to motivating the approach to stratified homotopy theory detailed here, explaining, for example, how certain homotopy-theoretic definitions naturally arise from geometric insights, or elaborating on the role that model categories play in our investigations.

In more detail, we will begin with an introduction into the basic notions of stratified homotopy theory in Section 1.1. We then present two slightly different approaches to stratified homotopy theory that have been pursued by several authors in recent years, the diagrammatic approach in Section 1.2 and the categorical approach in Section 1.3. We present the new results in this thesis and how they fit into the broader context in Sections 1.2.4, 1.2.5, 1.3.3 to 1.3.5 and 1.4.2 to 1.4.5.

Contents

1.1 Poset-stratified spaces and stratified homotopy equivalences

In the broadest sense, a stratification of a space X is a decomposition $X = \bigsqcup_{p \in P} X_p$ into set-theoretically disjoint pieces – called strata – fulfilling a series of requirements on the strata themselves and their topological interactions.

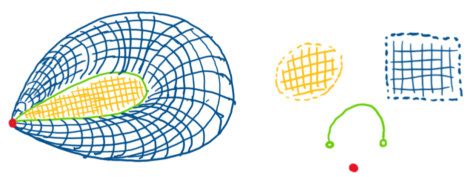


Figure 1.1: A stratified space (to the left) and separate sketches of its strata (to the right)

Historically, these types of decompositions primarily arose in the context of studying singular spaces, decomposing the latter into pieces that are manifolds (see [Whi65b; Mat12; Mat73] as well as the illustration in Fig. 1.1). For example, every algebraic variety can be decomposed into such manifold pieces in a way that equips the latter with a stratification particularly well suited to investigations of differential topology, a so-called Whitney stratification (see [Whi65b] and [Pfl01], for a good overview of the theory of such objects). From a highly conceptual perspective, stratified homotopy theory can be seen as the attempt to perform homotopy theory in a way that keeps track of homotopy-theoretic information of the strata as well as their interactions - the links. This direction of research, started most prominently by Quinn in [Qui88] and continued, for example, in [Hug99a; Mil13; Fri03; Woo09] has recently seen a renewed wave of attention and results, stemming to a large part from the model- and ∞-categorical perspective on abstract homotopy theory (see, for example, in no particular order [Lur17; AFR19; Nan19; BGH18; Dou19b; Dou21a; Dou21b; Jan24; KY21; Vol22; Hai23; HPT24; CST24].)

Having given a first, philosophical taste of what stratified homotopy theory is, let us now pass to a more rigorous context. Before we do so, let us begin with a caveat on the objects that we will refer to as stratified spaces here. When performing abstract homotopy theory based on some 1-category of objects that one is interested in studying, it is often preferable to begin with a 1-category that enjoys excellent categorical properties (such as having all limits and colimits, and being cartesian closed). This kind of requirement will generally come at the price of having a large class of objects, some of which may be rather pathological in nature. For example, the category of topological spaces certainly contains many examples that a geometrically minded person would prefer not to have in their category of allowable objects. From a homotopy theoretic perspective, however, this is usually not a major issue. Up to an

appropriate notion of weak equivalence one can usually even replace such highly pathological objects by something nicer. For example, it is a classical exercise in algebraic topology to prove that every topological space is weakly homotopy equivalent to a CW-complex. In this sense, the pathological objects are there to make the 1-category theory work, but they are ultimately irrelevant to the resulting homotopy theory (∞-category). The approach to stratified homotopy theory that we take here is analogous to this (we return to this question in Section 1.1.2).
The definition of stratified space that we will now give enjoys excellent categorical properties. However, a more geometrically minded person may possibly refuse to call them stratified spaces, as they lack many of the properties classically associated with geometric examples of such objects. Just as in the example of classical spaces, up to appropriate notions of weak equivalence, homotopy theoretic analogues of these classical properties will be restored.

1.1.1 The category of poset-stratified spaces

In geometric contexts, such as the theory of Whitney stratified spaces, the indexing set of the strata P usually comes with an inherent partial ordering that arises from the so-called *frontier condition.*

Definition 1.1.1.1. Given a topological space X, a set-theoretically disjoint decomposition $X = \bigsqcup_{p \in P} X_p$ is said to fulfill the *frontier condition* if the following holds:
For all $p, q \in P$, it holds that whenever the intersection of X_p with the closure of X_q, $\overline{X_q}$, is non-empty then $X_p \subset \overline{X_q}$.

Given the frontier condition and assuming that all strata are non-empty, one obtains a partial ordering of the indexing set P, by setting $p \leq q$, if X_p intersects the closure of X_q non-trivially. One may thus think of a stratification of a stratified space X over a poset P as a map $s\colon X \to P$, assigning to each point $x \in X$ the index of the stratum that it is contained in. In most geometrical examples, this map is continuous, if one equips the poset P with the so-called Alexandrov topology in which the downwards closed sets are the closed sets (see [WWY24] for the precise set-theoretic-topological details). In the case of a finite poset, continuity of the stratification map $s\colon X \to P$ just means that the unions $X_{\leq p} = \bigcup_{q \leq p} X_q$ are closed sets. These insights lead to the following abstract definition of a stratified space. In the following **Top** will always refer to the category of (compactly generated) topological spaces[1] and **Pos** will refer to the category of posets and order-preserving maps.

Definition 1.1.1.2. A *poset-stratified space* $\mathcal{X}$ is a triple $\mathcal{X} = (X, P_{\mathcal{X}}, s_{\mathcal{X}})$, consisting of a topological space $X \in \mathbf{Top}$, a partially ordered set $P_{\mathcal{X}} \in \mathbf{Pos}$ and a continuous map $s_{\mathcal{X}}\colon X \to P_{\mathcal{X}}$, called the *stratification map*, where $P_{\mathcal{X}}$ is equipped with the Alexandrov topology. A *stratified map* between poset-stratified spaces $\mathcal{X}$ and $\mathcal{Y}$ consists of a continuous map of the underlying spaces $f\colon X \to Y$ together with an order-preserving map $P_f\colon P_{\mathcal{X}} \to P_{\mathcal{Y}}$, making the diagram

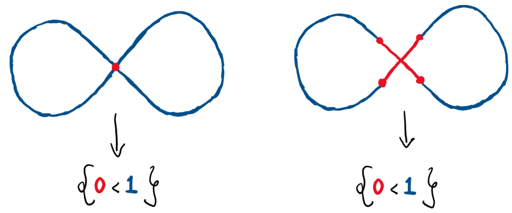
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s_{\mathcal{X}}"'] & Y \arrow[d, "s_{\mathcal{Y}}"] \\ P_{\mathcal{X}} \arrow[r, "P_f"'] & P_{\mathcal{Y}} \end{tikzcd} \tag{1.1}$$

commute.

[1]Compactly generated in the sense of having the final topology with respect to compact Hausdorff spaces. Whenever a space is not in this category, we apply the right adjoint of the inclusion of compactly generated spaces into topological spaces (see [Rez17]). Most of what we say here, also applies to the Δ-generated spaces of [Dug03], and any statement not involving internal mapping spaces even applies to general topological spaces. For some purposes, it is preferable to work with Δ-generated spaces, in order to obtain a combinatorial model category (see [Lur09]). For the homotopy theory that arises, these set theoretic questions are of no real importance.

Remark 1.1.1.3. One can translate back to the setting where a stratification is some type of decomposition of a space, by considering the decomposition $X = \bigsqcup_{p \in P_{\mathcal{X}}} s_{\mathcal{X}}^{-1}(p)$. We will thus also denote by $X_p := s_{\mathcal{X}}^{-1}(p)$ the fibers of the stratification map, and refer to the latter as strata.
In the case where the stratifications arise from the frontier condition and all strata are non-empty, every stratified map is uniquely determined by its underlying map of topological spaces $f\colon X \to Y$, a map of topological spaces $f\colon X \to Y$ defines a stratified map, if and only if the image of each stratum $f(X_p)$, $p \in P_{\mathcal{X}}$, fulfills $f(X_p) \subset Y_q$, for some $q \in P_{\mathcal{Y}}$.

Example 1.1.1.4. The following figures show two possible stratification maps of the figure eight-space over the poset with two elements $\{0 < 1\}$.



We used colors to indicate what stratum a point belongs to. Observe that the first stratification fulfills the frontier condition, while the second fails to have this property.

Example 1.1.1.5. Any filtration $X = \bigcup_{p \in \mathbb{N}} X_{\leq p}$ of a space X by a sequence $X_{\leq 0} \subset X_{\leq 1} \subset \dots$ of closed subspaces gives rise to a stratified space over $\mathbb{N}$, by setting $X_p := X_{\leq p} \setminus X_{\leq p-1}$, for $p \in \mathbb{N}$ (with $X_{\leq -1} = \varnothing$). Conversely, any stratified space $\mathcal{X}$ stratified over $P_{\mathcal{X}} = \mathbb{N}$ gives rise to a filtered space $(X, (X_{\leq p})_{p \in \mathbb{N}})$, by defining $X_{\leq p} := s_{\mathcal{X}}^{-1}(\{q \leq p\}) = \bigcup_{q \leq p} X_q$. These two constructions are inverse to each other. If one takes some care with local finiteness, then this bijection can be generalized to arbitrary posets. One needs to be careful, however, that while this construction defines a bijection on objects, it is only functorial in one direction on morphisms. Namely, every stratified map gives rise to a filtered (filtration-preserving) map, but the converse is evidently false. In this sense, our nomenclature of calling poset-stratified spaces *stratified* rather than *filtered* emphasizes the class of morphisms we consider.[2]

Notation 1.1.1.6. Poset-stratified spaces together with stratified maps form a category, which we denote by **Strat**. When we refer to a stratified space in the following context, we will also mean a poset-stratified space. We will generally use calligraphic notation to refer to stratified space $\mathcal{X}$ and use the same letter without the calligraphic font to refer to the underlying topological space. We will often also restrict ourselves to studying the subcategory of stratified spaces over some fixed poset P, where the maps on the poset level are given by the identity. The latter will be denoted $\mathbf{Strat}_P$, its objects are called *P-stratified spaces* and morphisms in this category are called *stratum-preserving maps.*

We have yet to give definitions of stratified spaces that behave more like the classical geometrical examples of stratified spaces. To this end, let us set up some additional general constructions on stratified spaces.

[2]The bijection on the object level has historically led many authors to refer to poset-stratified spaces as filtered spaces, and to reserve the term *stratified* for such filtered spaces which behave more like the classical geometrical examples people had in mind. The more recent shift of terminology to use the term poset-stratified is mainly because investigations have shifted from studying stratified spaces separately, to studying them in families, i.e., from a categorical point of view. By speaking of stratified spaces, one makes sure to emphasize that morphisms between them should not be filtered maps. Looking at the plethora of different definition of stratified spaces in the literature, it seems to us that one ultimately has to add a prefix such as *Whitney* or *homotopically* anyway, to emphasize what properties one expects the resulting stratified space to have. These type of linguistic difficulties will disappear, when we move over to the world of stratified homotopy types later on, as in this type of scenario stratified homotopy types will usually be represented by objects which have, homotopically speaking, properties much like the stratified spaces considered in more classical scenarios.

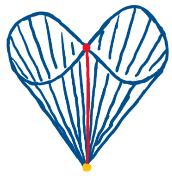


Figure 1.2: Stratified cone on the stratified figure eight, with cone-point stratum marked in orange.

Example 1.1.1.7. Given a stratified space $\mathcal{X}$ and a topological space T, we can consider the $P_{\mathcal{X}}$ stratified space $\mathcal{X} \times T$, obtained by equipping $X \times T$ with the stratification

$$X \times T \xrightarrow{\pi_X} X \xrightarrow{s_{\mathcal{X}}} P_{\mathcal{X}}.$$

This is, equivalently, the product in the category **Strat** of $\mathcal{X}$ with the stratified space obtained by treating T as trivially stratified over the poset with one element.

Example 1.1.1.8. Given a stratified space $\mathcal{X} \in$ **Top**, we can equip the topological cone $CX = X \times [0,1]/X \times \{0\}$ with a stratification over the poset $P_{\mathcal{X}}^{\triangleleft}$ obtained by adjoining a minimal element $*$ to $P_{\mathcal{X}}$. The stratification of CX is given by

$$[(x,t)] \mapsto \begin{cases} s_{\mathcal{X}}(x) & t > 0, \\ * & t = 0. \end{cases}$$

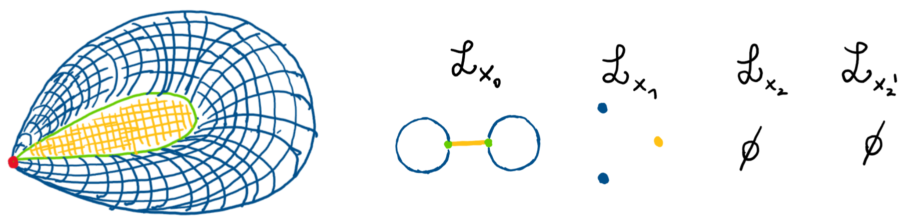
The resulting stratified space is denoted $C\mathcal{X}$ (see Fig. 1.2, for an illustration). One should note that in the setting of stratified spaces the cone CX is often not equipped with the quotient topology, but with the so-called teardrop topology (see [Qui88]). In the case where X is a compact Hausdorff space, however, the two topologies agree.

Notation 1.1.1.9. Given a poset P, and an element $p \in P$, we will use the notation $P_{>p}$ and $P_{\geq p}$ to refer to the poset of all elements that are, respectively, greater or greater equal than p.

Definition 1.1.1.10. [Lur17] A stratified space $\mathcal{X} \in \mathbf{Strat}_P$ is called *conically stratified*, if for every $p \in P$ and every point $x \in X_p$, there is a neighborhood U of x in X that is stratum-preserving homeomorphic (with respect to the stratification inherited from $\mathcal{X}$) to a stratified space of the form $C\mathcal{L} \times T$, where $\mathcal{L} \in \mathbf{Strat}_{P_{>p}}$, $T \in$ **Top**, and we treat $C\mathcal{L}$ as stratified over P by mapping the minimal element of $P_{>p}^{\triangleleft}$ to p. The stratified spaces $\mathcal{L}$ are often referred to as *(the local) links*[3] of $\mathcal{X}$ at $x \in \mathcal{X}$.

Most of the classical geometrical examples of stratified spaces – for example, Whitney stratified spaces and topological pseudomanifolds – are of this nature. We will not introduce these classes of stratified spaces here. For the reader unfamiliar with them, it will suffice to think of them as conically stratified spaces with manifold strata for now (see, for example [Pfl01; Ban07], for rigorous definitions). Any additional properties arising for such spaces will be explained as we move forward.

Example 1.1.1.11. Consider the poset-stratified space illustrated below, to the left. It is conically stratified over the poset $\{0 < 1 < 2, 2'\}$.



[3]In the topological setting, local links at a point x may generally not be unique.

To the right of it, we have illustrated local links at points that lie, respectively, in the 0, the 1, the 2, and the 2′-stratum.

We will see later on that the property of local conicality is not only geometrically meaningful, but also imparts the stratified space with certain desirable homotopical properties.

1.1.2 Stratified homotopy equivalences

If one follows the historical approach to homotopy theory, then the next step to defining a homotopy theory of stratified spaces is to expose a cylinder in the stratified category. There is an obvious cylinder in sight. Namely, given a stratified space $\mathcal{X}$, we can consider the product of the underlying topological space X with the unit interval $[0,1]$, and equip it with the stratification map given by

$$X \times [0,1] \xrightarrow{\pi_X} X \xrightarrow{s_{\mathcal{X}}} P_{\mathcal{X}}.$$

We will denote the resulting stratified space by $\mathcal{X} \times [0,1]$ (see Fig. 1.3, for an illustration). This stratification is such that the two end-point inclusions $i_0, i_1 : X \hookrightarrow X \times [0,1]$ as well as the projection $X \times [0,1] \to X$ define stratum-preserving maps. We hence end up with a cylinder

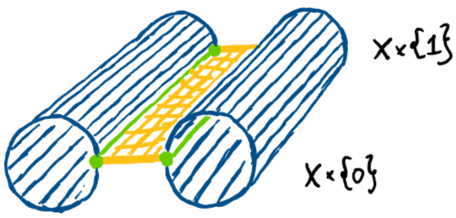


Figure 1.3: Stratified cylinder on the local link of x_0 in Example 1.1.1.11

functor on the category of stratified spaces, given by the functor $\mathcal{X} \mapsto \mathcal{X} \times [0,1]$ (acting in the obvious way on stratified maps) together with the natural commutative diagrams

$$\begin{tikzcd} \mathcal{X} \arrow[dr, hook, "i_0"] \arrow[drr, bend left] & & \\ & \mathcal{X} \times [0,1] \arrow[r, dashed, "\pi_{\mathcal{X}}"] & \mathcal{X}. \\ \mathcal{X} \arrow[ur, hook, "i_1"'] \arrow[urr, bend right] & & \end{tikzcd} \tag{1.2}$$

Any such cylinder functor naturally gives rise to a notion of homotopy and of homotopy equivalence, defined exactly analogously to the classical case of topological spaces (see Definition 3.2.3.4, for detailed definitions).

Notation 1.1.2.1. We will call homotopies with respect to the cylinder $- \times [0,1]$ *stratified homotopies* and homotopy equivalences with respect to this cylinder *stratified homotopy equivalences.* The stratified homotopy classes of stratified maps between two stratified spaces $\mathcal{X}$ and $\mathcal{Y}$ will be denoted by the notation $[\mathcal{X}, \mathcal{Y}]_s$. When we consider stratified spaces over the same poset P and only stratified homotopy classes of stratum-preserving maps, we will denote the resulting stratified homotopy classes of stratum-preserving maps by $[\mathcal{X}, \mathcal{Y}]_P$.

Let us make some first immediate observations on these notions of homotopy and equivalence.

Remark 1.1.2.2. First off, the cylinder $\mathcal{X} \times [0,1]$ is constructed precisely such that a stratified homotopy $H : \mathcal{X} \times [0,1] \to \mathcal{Y}$ is a continuous $[0,1]$-indexed family of stratified maps, such that the underlying map on posets is constant, as we move along $[0,1]$. In the case where all strata are non-empty, and we can think of stratified maps as being entirely defined on the level of spaces, the latter condition is equivalent to saying that

$$s_{\mathcal{Y}}(H_t(x)) = s_{\mathcal{Y}}(H_{t'}(x))$$

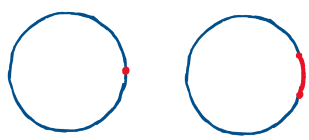
for $x \in X$ and $t, t' \in [0,1]$. Note that it also follows from the invariance of the poset map along the homotopy that any stratified homotopy equivalence defines an isomorphism of the underlying posets.

Remark 1.1.2.3. Fixing a class of equivalences W to be inverted in a 1-category $\mathbf{C}$ always defines a *homotopy theory*, given by the $(\infty,1)$-categorical localization $\mathbf{C}[W^{-1}]$ (see [Lur23, Tag 01M4]). We can ask ourselves what the resulting ∞-category obtained by localizing $\mathbf{Strat}$ at the class of stratified homotopy equivalences H_s, denoted $\mathbf{Strat}[H_s^{-1}]$, looks like. It follows from general theory developed in [DK87] that $\mathbf{Strat}[H_s^{-1}]$ admits a description in terms of a simplicial category. Namely, given two stratified spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\underline{\mathbf{Strat}}(\mathcal{X},\mathcal{Y})$ the simplicial mapping space between $\mathcal{X}$ and $\mathcal{Y}$, whose n-simplices are given by simplicial maps $\mathcal{X} \times |\Delta^n| \to \mathcal{Y}$, from the product of $\mathcal{X}$ with the topological n-simplex $|\Delta^n|$ into $\mathcal{Y}$. In particular, 0-simplices are given by stratified maps $\mathcal{X} \to \mathcal{Y}$ and 1-simplices are given by stratified homotopies of such maps. We may thus equivalently think of the set of stratified homotopy classes $[\mathcal{X},\mathcal{Y}]_s$ as the set of path-components $\pi_0\underline{\mathbf{Strat}}(\mathcal{X},\mathcal{Y})$.

Notation 1.1.2.4. The simplicial mapping spaces in Remark 1.1.2.3 give rise to a simplicial enrichment of $\mathbf{Strat}$, i.e., the structure of a simplicial category, which we denote by $\underline{\mathbf{Strat}}$ (see Recollection 3.2.2.4). Similarly, for $P \in \mathbf{Pos}$, we denote by $\underline{\mathbf{Strat}}_P$ the simplicial enrichment of $\mathbf{Strat}_P$ with the mapping spaces $\underline{\mathbf{Strat}}_P(\mathcal{X},\mathcal{Y})$ given by the sub-simplicial sets of $\underline{\mathbf{Strat}}(\mathcal{X},\mathcal{Y})$ whose n-simplices are such stratified maps $\mathcal{X} \times |\Delta^n| \to \mathcal{Y}$ that are stratum-preserving over P.

For classical examples of stratified spaces, stratified homotopy equivalence is a rather well-behaved notion. For example, it preserves the homotopy type of the strata of a stratified space, and more than that, it preserves the so-called homotopy links, roughly the homotopy-theoretic way the strata are connected. (These objects will be introduced in the next section, and will become of central importance). It also preserves important classical invariants such as intersection homology (of appropriately stratified topological pseudomanifolds, assuming that the stratified maps are compatible with the perversities, see [Fri20]). If one also considers poset-stratified spaces that do not arise from classical geometrical examples, stratified homotopy equivalence often turns out to be too rigid a notion of equivalence, however.

Example 1.1.2.5. Consider, for example, the two stratifications of S^1 sketched below. The one on the left is obtained from the one on the right by slightly thickening the lowest stratum.



We denote the stratified space on the left by $\mathcal{X}$ and the one on the right by $\mathcal{X}_\varepsilon$. It is a fairly straightforward exercise in elementary topology to show that these two stratified spaces are not stratified homotopy equivalent. For example, one can reason that this is the case by observing that any stratified homotopy equivalence would have to induce a homotopy equivalence of the underlying spaces, in particular a map of degree ± 1. However, no stratified map $\mathcal{X} \to \mathcal{X}_\varepsilon$ can be surjective, and, as every non-surjective map of spheres is contractible, also not of degree ± 1.

This rigidity of stratified homotopy equivalences makes the homotopy theory arising from them generally quite hard to work with. For example, just like classical homotopy equivalences between non-CW-complexes, stratified homotopy equivalences lack a concise algebraic recognition criterion. In fact, for many intents and purposes, one would like to regard the two stratified spaces in Example 1.1.2.5 to be treated as equal. Let us give two examples, one from applied topology (specifically topological data analysis, TDA) and one from the realm of algebraic topology (specifically sheaf theory):

- To be useful in applications in TDA, a crucial feature of the homotopy types of sufficiently regular spaces embedded in euclidean space is that they are invariant under small thickenings. This is of particular importance, as it allows for inference of homotopical information from point-samples or slightly noisy data (see Chapter 4 for details). If one is looking to use stratified methods in TDA, then one will often work with point samples, and can generally only expect to detect strata up to such small amounts of noise. Thus, one is generally interested in notions of equivalence that allow for such small thickenings.
- One of the core objects of study in the theory of stratified spaces are the categories of constructible sheaves (in the classical or ∞-categorical sense) associated to a specific stratification of a space (see [Ban07], for an introduction to the uses of such methods for stratified algebraic topology). The collapsing map $\mathcal{X}_\varepsilon \to \mathcal{X}$, collapsing the thickened stratum to a point, is such that it induces isomorphisms on these associated (∞-)categories of constructible sheaves (see Section 1.3.1). Hence, from this sheaf theoretic perspective, the two stratified spaces can be considered as equivalent.

There are two approaches to tackling this deficiency of stratified homotopy equivalences. One may call these the (geometrically minded) topologists' and the homotopy theorists' answer to the problem. Let us refer to these two fictive mathematicians by A and B, respectively. [4] A's answer goes something like this:

> "*One of the two stratified spaces you considered may not even deserve the name stratified space. At least it is not in the class of objects studied in geometrical investigations of stratified spaces. Reduce the class of objects you are considering, and you will end up with a well working homotopy theory of stratified spaces.*"

B's answer may go something along the following lines:

> "*The class of stratified homotopy equivalences is too small. You need to increase the class of morphisms you are inverting, while preserving the homotopical structure you are mainly interested in. Then you may end up with a well-working homotopy theory of stratified spaces.* "

Of course, if one asked either of the two about the other's opinion, they would likely answer that they are ultimately saying the same thing. It is a general phenomenon in (higher) category theory that often the localization of an (∞-)category $\mathcal{C}$ at a class of morphisms is equivalent to a fully faithful subcategory $\mathcal{C}$ (see classically, [DK80b, §7] for one rigorous incarnation of this phenomenon).[5] The most classical example in homotopy theory of this phenomenon is, of course, the case of localizing the homotopy theory of all topological spaces - arising from homotopy equivalences - at weak homotopy equivalences. It is ultimately a consequence of Whitehead's theorem that the homotopy theory one ends up with is the homotopy theory of CW-complexes. This localization at weak equivalences makes the homotopy theory of spaces significantly easier to handle (allowing, for example, the transition to the purely combinatorial model of simplicial sets) while at the same time losing almost no information: As long as one is interested in studying spaces with the homotopy type of a CW-complex, weak equivalence and homotopy equivalence are really the same thing.
A's approach to stratified homotopy theory of cutting down the class of stratified spaces is precisely the approach started by Quinn in [Qui88] and pursued further by, for example, in [Hug99b; Fri03; Mil13]. The approach we will pursue in this chapter is B's approach: We

[4] Clearly, these groups are far from being disjoint. One may certainly often encounter a person giving *both* answers, and in fact the two answers are equivalent, in a sense that we will make precise.

[5] Often, one even has a so-called *(co)reflective* localization, making the localization functor adjoint to the inclusion of the subcategory, see [nLa24j], for an overview.

study weaker notions of equivalence of stratified spaces. We will then see that the resulting localizations of stratified homotopy theory can equivalently be achieved by restricting to a subcategory of homotopically well-behaved stratified spaces. Interestingly, we will also see that this approach leads to results very similar to Quinn's approach.

1.2 The diagrammatic approach to stratified homotopy theory

Having decided on the approach of localizing at a larger class than just stratified homotopy equivalences, the obvious question arises to what that class should be. Any choice of class of weak equivalences W may lead to a different choice of stratified homotopy theory, given by the ∞-categorical localization $\mathbf{Strat}[W^{-1}]$. As we will see in Section 1.3, this question may have multiple legitimate answers. We can, however, begin by asking the question what properties a class of weak equivalences should fulfill.

(R1) Firstly, from a practical perspective, one would like the class of weak equivalences to be detectable, in the sense that there are reasonably easily verifiable, or at least conceptually illuminating criteria that can be used to verify that a map is a weak equivalence.

(R2) In a best case scenario, one would like to end up with a homotopy theory that one has a certain degree of control over, allowing for the explicit 1-categorical computation of higher categorical constructions. This kind of property is usually achieved by requiring that the class of weak equivalences extends to a *model structure*.

(R3) Furthermore, one would like the class of weak equivalences to be informed by the geometric theory of stratified spaces. They should preserve at least the most crucial homotopy theoretic features of classically relevant stratified spaces. Even more than that, in a best case scenario one would expect the mapping spaces $\mathbf{Strat}[W^{-1}](\mathcal{X}, \mathcal{Y})$ to agree (homotopically speaking) with the classical simplicial mapping spaces $\underline{\mathbf{Strat}}(\mathcal{X}, \mathcal{Y})$, when $\mathcal{X}$ and $\mathcal{Y}$ are nice, geometric examples of stratified spaces. In other words, one would want the classical stratified homotopy theory of geometric stratified spaces, using stratified homotopy equivalences, to be fully faithfully included in the resulting homotopy theory. In particular, this would imply a stratified Whitehead theorem for the class of weak equivalences and sufficiently geometric examples of stratified spaces.

1.2.1 Diagrammatic equivalences of stratified spaces

Let us now study a class of equivalences that ultimately meets these criteria, and that was suggested independently by Douteau and Henriques in [Dou21c; Hen], and extensively studied and developed by Douteau in [Dou21b; Dou21a; DW22].

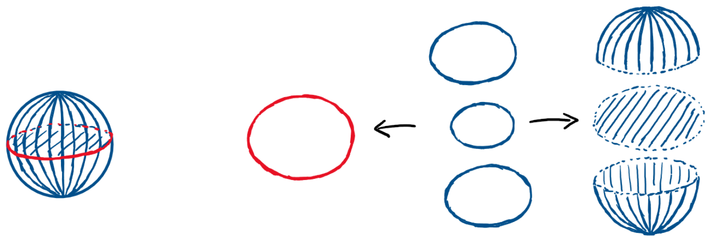
The goal here is to motivate how this class arises quite naturally from an inductive approach to stratified homotopy theory and geometrical observations about classical examples of stratified spaces. Let us begin with the case of stratified spaces with two strata, that is, the category of stratified spaces $\mathbf{Strat}_P$ where $P = \{p < q\}$ is a linear poset with two elements. It is an observation that already follows from work of Thom and Mather (see [Tho69; Mat12; Mat73]) that, for a Whitney stratified space $\mathcal{W}$ with two strata, the singular stratum $W_p \subset W$ admits a neighborhood that is (stratum-preserving) homeomorphic to a mapping cylinder on a topological fiber bundle $\xi\colon L \to W_p$, with fiber L_x a closed manifold. Here, the mapping cylinder M_ξ is stratified over $\{p < q\}$ by setting $W_p \subset M_\xi$ to be the p-stratum. In this fashion, one obtains from a Whitney stratified space a diagram of topological spaces

$$W_p \xleftarrow{\xi} L \hookrightarrow W_q,$$

with ξ a topological fiber bundle.

The manifold L is often called the *(global) link* of $\mathcal{W}$. Together with the two maps into the strata, it can be seen as the homotopical datum that determines how the two strata are glued together.

Example 1.2.1.1. To the left, below, is an illustration of a stratified space over the poset $\{1 < 2\}$, obtained by gluing a disk to a 2-sphere by identifying the boundary of the disk with the equator.



To the right is an illustration of the associated diagram consisting of the strata and the link.

It turns out that the diagrams constructed in this fashion (thought of as valued in the homotopy theory of spaces) entirely determine the stratified homotopy type of a Whitney stratified space with two strata (this will become apparent later in the text, see Chapter 4 for an application where we make central use of this insight). In particular, it is a fairly elementary exercise in using the homotopy lifting property of a fiber-bundle and the homotopy extension property of cofibrations that a map of Whitney stratified spaces $w: \mathcal{W} \to \mathcal{W}'$ that lifts to a commutative diagram

$$\begin{tikzcd} W_p \arrow[d, "w_p"'] & L \arrow[l, "\xi"'] \arrow[r, hook] \arrow[d, dotted] & W_q \arrow[d, "w_q"] \\ W'_p & L' \arrow[l, "\xi'"] \arrow[r, hook] & W'_q \end{tikzcd} \tag{1.3}$$

with all verticals (weak) homotopy equivalences is a stratified homotopy equivalence. We can see this claim as a proto-Whitehead theorem for stratified spaces. In fact, much more can be said, as we will see in a minute.
A similar scenario can be observed in the case of PL pseudomanifolds (roughly stratified polyhedra with manifold strata that are conically stratified in a PL sense), where L is given by the boundary of a regular neighborhood of W_p in W. Here, the topological fibration is replaced by the collapsing map of a so-called cone block-bundle, however (see [Sto72]). Finally, in the topological case of topological pseudomanifolds (see, for example, [Ban07]), the situation is somewhat more subtle, and there is no obvious way to extract a link fibration. To tackle this issue, Quinn (see [Qui88]) focused on the notion of homotopy links.

Definition 1.2.1.2. Let P be a general poset. Given a stratified space $\mathcal{X} \in \mathbf{Strat}_P$ and a pair of strata $p < q \in P$, we denote by $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X})$ the subspace of the path space $X^{[0,1]}$, given by such paths $\gamma: [0,1] \to X$, that fulfill $\gamma(0) \in X_p$ and $\gamma\big((0,1]\big) \subset X_q$ (see, Fig. 1.4, for an illustration of such a path). $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X})$ is called the $\{p,q\}$*-homotopy link of* $\mathcal{X}$. $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X})$ comes together with two evaluation maps

$$\mathrm{ev}_0: \mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to X_p;$$
$$\mathrm{ev}_1: \mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to X_q,$$

mapping a path to, respectively, its starting and its endpoint.

Homotopy links serve as a proxy for the links in the geometric scenarios detailed above. In fact, one can show that in these scenarios the homotopy link is homotopy equivalent to

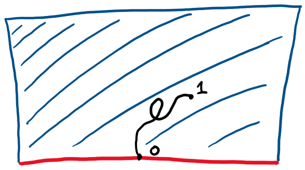


Figure 1.4: A path in a stratified space with two strata $\mathcal{X}$ that defines an element of $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X})$.

the geometric link (whenever the latter is defined; see [Qui88; Fri03]) and that for all of the scenarios mentioned above the evaluation map

$$\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to X_p$$

is a fibration (a Hurewicz fibration, to be precise; see [Qui88] for details). Homotopy links have the crucial advantage over geometrical links that they are functorial in arbitrary stratum-preserving maps (via postcomposition).
Inspired by the proto-Whitehead theorem above, the following class of weak equivalences for spaces stratified over $\{p < q\}$ is thus a natural candidate.

Definition 1.2.1.3. Let $P = \{p < q\}$ be a poset with two elements. A stratum-preserving map of P-stratified spaces $w\colon \mathcal{X} \to \mathcal{Y}$ is called a *diagrammatic equivalence*[6], if the induced maps

$$w_p\colon X_p \to Y_p;$$
$$w_q\colon X_q \to Y_q;$$
$$\mathcal{H}o\mathrm{Link}_{\{p<q\}}(w)\colon \mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{Y})$$

are weak homotopy equivalences.

This is already a promising candidate to meet Requirements (R1) to (R3). At least, weak equivalences are reasonably easy to detect (modulo the difficulty of computing homotopy links) and it fulfills a Whitehead theorem for a reasonably large class of classically relevant stratified spaces, namely Whitney stratified spaces.

Example 1.2.1.4. Consider again the two stratified spaces of Example 1.1.2.5. There is a collapsing map, $\mathcal{X}_\varepsilon \to \mathcal{X}$, that collapses the red stratum to a point and is given by a homeomorphism on the blue stratum. This map is a diagrammatic equivalence. Indeed, for both stratified spaces, the strata are contractible and the homotopy links have the homotopy type of the disjoint union of two points. The collapsing map $\mathcal{X}_\varepsilon \to \mathcal{X}$ induced a bijection on path components of the link, and hence weak equivalences on all strata and homotopy links.

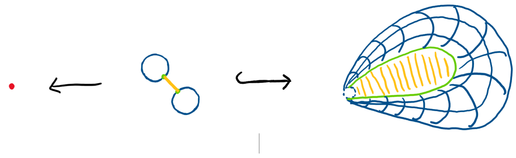
Suppose, for now, that we have accepted diagrammatic equivalences as a good definition of weak equivalence in the case of two strata and let us move on to the three strata case. Again, let us focus on the case of a Whitney stratified space $\mathcal{W}$, this time with three strata, i.e., stratified over a poset $P = \{p_0 < p_1 < p_2\}$. Given $p \in P$, we denote by $\mathcal{W}_{>p}$ the stratified space given by the union of all strata larger than p. Again, one can prove that a neighborhood around the lowest stratum is of the form of a cylinder on a fiber-bundle $\mathcal{L} \to W_{p_0}$ (see [Tho69; Mat12; Mat73]). This time, however, the total space of the bundle $\mathcal{L}$ and the fibers are themselves Whitney stratified spaces (with two strata) and stratification inherited from the inclusion $L \hookrightarrow \mathcal{W}_{>p_0}$. Thus, in the three strata scenario (or more generally in the n-strata scenario), one obtains a span

$$\mathcal{W}_{p_0} \leftarrow \mathcal{L} \to \mathcal{W}_{>p_0}$$

where the left arrow is a kind of *stratified fiber bundle.*

[6]The name *diagrammatic* comes from the fact that being a diagrammatic equivalence may be seen as inducing equivalences of the associated diagrams of the form $(X_p \leftarrow \mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to X_q)$ in the homotopy theory of diagrams of spaces of the form $\bullet \leftarrow \bullet \to \bullet$, obtained by localizing the 1-category of such diagrams in **Top** at pointwise weak homotopy equivalences.

Example 1.2.1.5. Consider the following illustration of the span arising from the stratified space in Example 1.1.1.11.



In this special case, the p_0-stratum is a point and hence the global link agrees with the local link at the unique point in the p_0-stratum.

Again, one can try to mirror this kind of structure homotopically, as was done in [Hug99b]. This requires treating the homotopy link itself as a stratified space.

Notation 1.2.1.6. Given a poset P and $p \in P$, we denote by $P_{>p}$ the poset consisting of all elements greater than p. Given $\mathcal{X} \in \mathbf{Strat}_P$, we denote by $\mathcal{X}_{>p} \in \mathbf{Strat}_{P_{>p}}$ the stratified space $s_{\mathcal{X}}^{-1}(P_{>p}) \to P_{>p}$ obtained by restricting the stratification map to $P_{>p} \subset P$. We will use analogous notation replacing ">" by other relations.

Notation 1.2.1.7. Let P be a general poset. Given a stratified space $\mathcal{X} \in \mathbf{Strat}_P$ and $p \in P$, we denote by $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$ the $P_{>p}$ stratified space, whose underlying space is the subspace of (with a slightly refined topology[7]) $X^{[0,1]}$, given by such paths $\gamma\colon [0,1] \to X$ that fulfill $\gamma(0) \in X_p$ and for which there exists a $q > p$ such that $\gamma\big((0,1]\big) \subset X_q$. The stratification of $\mathcal{H}o\mathrm{Link}_{>p}(\mathcal{X})$ is given by

$$\gamma \mapsto s_{\mathcal{X}}(\gamma(1)) \in P_{>p}.$$

The stratified space $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$ is called the p-th *stratified homotopy link of* $\mathcal{X}$. $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$ comes together with two evaluation maps

$$\mathrm{ev}_0\colon \mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to X_p;$$
$$\mathrm{ev}_1\colon \mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to \mathcal{X}_{>p},$$

the second of which is a stratum-preserving map over $P_{>p}$.

Just as in the case of two strata, one can show that the stratified homotopy link is stratified homotopy equivalent to the *stratified geometric link*, when one is working in a scenario where the latter is available (see [Fri03, Proposition A.1.] for a rigorous statement). One can now take the same approach to the three strata case, as we have taken in the two strata case, but proceeding inductively. Namely, define a stratum-preserving map $w\colon \mathcal{X} \to \mathcal{Y}$ of $\{p_0 < p_1 < p_2\}$-stratified spaces to be a *diagrammatic equivalence* if it is such that $w_{p_0}\colon X_{p_0} \to Y_{p_0}$ are weak homotopy equivalences and such that the induced stratum-preserving maps $\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{Y})$ and $\mathcal{X}_{>p_0} \to \mathcal{Y}_{>p_0}$ are themselves diagrammatic equivalences in the category $\mathbf{Strat}_{\{p_1<p_2\}}$. Let us explicitly decode what this means.

Example 1.2.1.8. Let $\mathcal{X}$ be a $\{p_0 < p_1 < p_2\}$-stratified space. If we first decompose $\mathcal{X}$ into a span in $\mathbf{Strat}$

$$X_{p_0} \leftarrow \mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X}) \to \mathcal{X}_{>p_0}$$

and then again decompose $\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})$ and $\mathcal{X}_{>p_0}$ in this manner, we end up with the following

[7] In the case of infinitely many strata, the compactly generated subspace topology needs to be slightly refined, in order to make the stratification map continuous (see Construction 7.2.2.6). We ignore this subtlety for now.

diagram of spaces (only the lower half of which commutes).

$$\begin{tikzcd}
 & & X_{p_0} & & \\
 & \mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})_{p_1} \arrow[ur] \arrow[dl] & \mathcal{H}o\mathrm{Link}_{\{p_1<p_2\}}(\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})) \arrow[l] \arrow[r] \arrow[d] & \mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})_{p_2} \arrow[ul] \arrow[dr] & \\
X_{p_1} & & \mathcal{H}o\mathrm{Link}_{\{p_1<p_2\}}(\mathcal{X}) \arrow[ll] \arrow[rr] & & X_{p_2}
\end{tikzcd} \tag{1.4}$$

Observe that $\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})_{p_1} = \mathcal{H}o\mathrm{Link}_{\{p_0<p_1\}}(\mathcal{X})$ and $\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})_{p_2} = \mathcal{H}o\mathrm{Link}_{\{p_0<p_2\}}(\mathcal{X})$. The most complicated space in this diagram seems to be the one centered at the barycenter of the simplex, namely the double homotopy link $\mathcal{H}o\mathrm{Link}_{\{p_1<p_2\}}(\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X}))$. Let us refer to this space as the $\{p_0 < p_1 < p_2\}$-homotopy link, for now. Having organized the situation as such, we now see that $w\colon \mathcal{X} \to \mathcal{Y}$ is a diagrammatic equivalence, if and only if it induces weak homotopy equivalences on all strata, all pairwise homotopy links, and on the $\{p_0 < p_1 < p_2\}$-homotopy link.

We could now keep pursuing this inductive approach, and define diagrammatic equivalences for n-strata stratified spaces. We hope that it is evident to the reader that decoding what these definitions would mean in practice would become increasingly complicated. There is, however, a systematic way of keeping track of the more and more involved iterated homotopy links occurring in Example 1.2.1.8, through the use of stratified simplices. This approach to defining weak equivalences of stratified spaces is (independently) due to Douteau and Henriques (see [Dou21c; Hen]). We will first have to introduce some language and notation.

Notation 1.2.1.9. We denote by Δ the full subcategory of the category of posets P, given by finite linear posets of the form $[n] := \{0, \cdots, n\}$, for $n \in \mathbb{N}$.

Notation 1.2.1.10. By a *flag* of a poset P, we mean a monotonous finite sequence $p_0 \leq \cdots \leq p_n$ of elements of the posets. Equivalently, we may think of a flag of P as a map of partially ordered sets $[n] \to P$. We will usually denote flags in the form $[p_0 \leq \cdots \leq p_n]$. By the category of flags of P, we mean the comma category $\Delta/_P$, under the inclusion $\Delta \hookrightarrow \mathbf{Pos}$ of Δ into the category of all posets $\mathbf{Pos}$. In other words, $\Delta/_P$ is the category whose objects are flags $[n] \to P$, and whose morphisms are commutative triangles

$$\begin{tikzcd}
{[n]} \arrow[rr] \arrow[dr] & & {[m]} \arrow[dl] \\
 & P. &
\end{tikzcd} \tag{1.5}$$

We will often just denote this category by Δ_P.
We say that a flag $\mathcal{I}\colon [n] \to P$ is *non-degenerate*, or *regular*, if it has no repetitions, i.e., if it is injective as a map of posets. We may just think of a non-degenerate flag as a finite linear subset of P. Hence, we will also denote regular flags in the form $\{p_0 < \cdots < p_n\}$.

Notation 1.2.1.11. We denote by $\mathrm{sd}(P)$ the full subcategory of Δ_P given by the non-degenerate flags. If we think of non-degenerate flags as subsets of P, then morphisms in $\mathrm{sd}(P)$ are just inclusions of subsets. Hence, the category $\mathrm{sd}(P)$ can equivalently be seen as the category given by the poset of finite linear subsets of P, ordered by inclusion. The latter is often referred to as the subdivision of P, hence the notation. We will often use the opposite category of $\mathrm{sd}(P)$, which should formally be denoted $\mathrm{sd}(P)^{\mathrm{op}}$. We will, at times, omit the brackets, settling on the convention that sd is to be performed before $(-)^{\mathrm{op}}$, always.

Remark 1.2.1.12. The notation $\mathrm{sd}(P)$ comes from the fact that, given a poset P, the simplicial set (or complex) given by the nerve of $\mathrm{sd}(P)$ is precisely the nerve of the barycentric

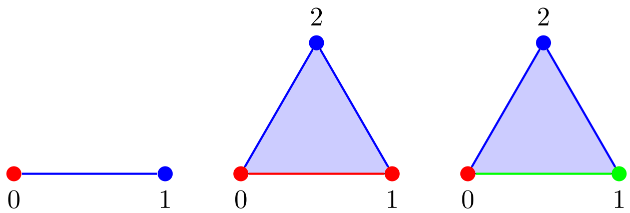


Figure 1.5: Illustrations of the stratified simplices over a poset with three elements $P = \{p_0 < p_1 < p_2\}$ associated to the flags $[p_0 < p_2]$, $[p_0 \leq p_0 < p_2]$, and $[p_0 < p_1 < p_2]$

subdivision of the nerve of P. Consider, for example, the subdivision of the poset with three elements $\{p_0 < p_1 < p_2\}$ illustrated below.

$$\begin{tikzcd}
 & & {[p_2]} \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & {[p_0 < p_2]} \arrow[r] & {[p_0 < p_1 < p_2]} & {[p_1 < p_2]} \arrow[l] & \\
{[p_0]} \arrow[ur] \arrow[urr] \arrow[rr] & & {[p_0 < p_1]} \arrow[u] & & {[p_1]} \arrow[ul] \arrow[ull] \arrow[ll]
\end{tikzcd} \tag{1.6}$$

Construction 1.2.1.13. Given a flag $\mathcal{J} = [p_0 \leq \dots \leq p_n]$ of a poset P, we denote by $|\Delta^{\mathcal{J}}|_s$ the P-stratified space obtained by equipping the topological n-simplex $|\Delta^n| \subset \mathbb{R}^{\{0,\dots,n\}}$ with the following stratification. Under the inclusion $|\Delta^n| \subset \mathbb{R}^{\{0,\dots,n\}}$, every element of $|\Delta^n|$ corresponds to a vector $t = (t_0, \dots, t_n)$ with positive real entries, such that $\sum t_i = 1$. Now define the stratification map of $|\Delta^{\mathcal{J}}|_s$ as

$$\begin{aligned} |\Delta^n| &\to P \\ t &\mapsto p_{\max\{i \in [n] \mid t_i > 0\}}. \end{aligned}$$

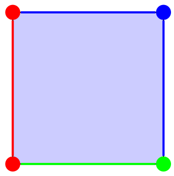
The stratified simplices of Construction 1.2.1.13 can be used to recover the iterated homotopy links of Example 1.2.1.8.

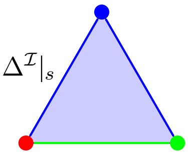
Notation 1.2.1.14. In the following, we will use the notation $C^0_P(\mathcal{X}, \mathcal{Y})$ to refer to the space of stratum-preserving maps between two P-stratified spaces $\mathcal{X}$ and $\mathcal{Y}$ (equipped with the compactly generated subspace topology inherited from the mapping space Y^X).

Example 1.2.1.15 (See Lemma 7.5.5.9.)**.** Let $P = \{p_0 < p_1 < p_2\}$. Given a P-stratified space $\mathcal{X}$ and a regular flag $\mathcal{I}$ of P, let us study the spaces of stratum-preserving maps from $|\Delta^{\mathcal{I}}|_s$ into $\mathcal{X}$, denoted $C^0_P(|\Delta^{\mathcal{I}}|_s, \mathcal{Y})$. In the case $\mathcal{I} = \{p\}$, these are the maps from a point into X, which map into the p-stratum. Hence, keeping track of the topology, we obtain X_p. In the case $\mathcal{I} = \{p < q\}$, we may identify $|\Delta^1|$ affinely with $[0,1]$ (with the obvious orientation). Under this identification, a map $|\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ is the same thing as a path starting in p and immediately exiting into q, i.e., an element of $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X})$. Hence, in this case we can identify the space of stratum-preserving maps $C^0_P(|\Delta^{\mathcal{I}}|_s, \mathcal{Y})$ with $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X})$. Finally, it remains to consider the case $\mathcal{I} = \{p_0 < p_1 < p_2\}$. Denote by $\mathcal{S}$ the stratified space obtained by equipping $[0,1] \times [0,1]$ with the stratification

$$(x, y) \mapsto \begin{cases} p_0 & , \text{ if } x = 0; \\ p_1 & , \text{ if } x > 0 \wedge y = 0; \\ p_2 & , \text{ if } y > 0 \wedge x > 0 \end{cases}$$

(see the illustration on the left directly below, with the p_0-stratum in red, the p_1-stratum in green, and the p_2-stratum in blue).





If we collapse the p_0-stratum in $\mathcal{S}$ to a point, then the resulting stratified space is stratum-preserving homeomorphic to $|\Delta^{\mathcal{I}}|_s$ (illustrated to the right, above). This collapsing map can be seen to be a stratified homotopy equivalence (see the proof of Lemma 7.5.5.9). It follows that up to homotopy equivalence we can identify $C^0_P(\mathcal{S}, \mathcal{X})$ with $C^0_P(|\Delta^{\mathcal{I}}|_s, \mathcal{X})$. Under the exponential law for mapping spaces, we can identify an element of $C^0_P(\mathcal{S}, \mathcal{X})$ with a path in the space of paths $X^{[0,1]}$. If we keep track of the stratification, we observe that there is a canonical homeomorphism

$$C^0_P(\mathcal{S}, \mathcal{X}) \cong \mathcal{H}o\mathrm{Link}_{\{p_1<p_2\}}(\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})).$$

Hence, it follows that up to natural homotopy equivalence, we may identify

$$C^0_P(|\Delta^{\mathcal{I}}|_s, \mathcal{X}) \simeq \mathcal{H}o\mathrm{Link}_{\{p_1<p_2\}}(\mathcal{H}o\mathrm{Link}^s_{p_0}(\mathcal{X})).$$

This leads to the following definition of Douteau and Henriques:

Definition 1.2.1.16 ([Dou21c; Hen])**.** Given a poset P and a regular flag $\mathcal{I} \in \mathrm{sd}(P)$, we denote by $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}$ the functor

$$\begin{aligned} \mathcal{H}o\mathrm{Link}_{\mathcal{I}} \colon \mathbf{Strat}_P &\to \mathbf{Top} \\ \mathcal{X} &\mapsto C^0_P(|\Delta^{\mathcal{I}}|_s, \mathcal{X}) \end{aligned}$$

acting on morphisms via postcomposition. Given a fixed stratified space $\mathcal{X}$, the topological space $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$ is called the *$\mathcal{I}$-th generalized homotopy link of $\mathcal{X}$*.

Given Example 1.2.1.15 and Definition 1.2.1.16, one may now give the following definition for a class of weak equivalences of stratified spaces based on generalized homotopy links. It provides a direct instead of an inductive definition, and was suggested in [Dou21c; Hen].

Definition 1.2.1.17. A stratum-preserving map $w \colon \mathcal{X} \to \mathcal{Y}$ in $\mathbf{Strat}_P$ is called a *diagrammatic equivalence*, if for each regular flag $\mathcal{I} \in \mathrm{sd}(P)$, the induced map

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y})$$

is a weak homotopy equivalence.

Notation 1.2.1.18. Henceforth, we will denote by $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ the ∞-category obtained by localizing $\mathbf{Strat}_P$ at the class of diagrammatic equivalences.

Remark 1.2.1.19. [Dou21c] also defined a variant of this theory where the poset is allowed to be flexible. We discuss this in more detail later on. It ultimately turns out that most investigations of this theory can be entirely reduced to the case of a fixed poset. For now, we will focus on the case of a fixed poset, however. Nevertheless, let us introduce the following language and notation.

Definition 1.2.1.20. A stratified map $w \colon \mathcal{X} \to \mathcal{Y}$ in $\mathbf{Strat}$ is called a *poset-preserving diagrammatic equivalence* if it induces an isomorphism on the stratification posets, and a diagrammatic equivalence in $\mathbf{Strat}_{P_{\mathcal{X}}}$ after treating $\mathcal{Y}$ as stratified over $P_{\mathcal{X}}$ via the isomorphism on the posets. The ∞-category obtained by localizing $\mathbf{Strat}$ at poset-preserving diagrammatic equivalences will be denoted $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$, with the $\mathfrak{p}$ standing for *poset-preserving*.

Notation 1.2.1.21. Henceforth, we will use two different notations to refer to the functor categories of 1– and ∞-categories. Given two (∞)-categories $\mathbf{C}$ and $\mathbf{D}$, we will denote the associated (∞-)category of functors from $\mathbf{C}$ to $\mathbf{D}$ by either $\mathbf{Fun}(\mathbf{C}, \mathbf{D})$ or, in exponential notation, by $\mathbf{D}^{\mathbf{C}}$.

One advantage of choosing generalized homotopy links over iterated applications of stratified homotopy links is that the generalized homotopy links of a stratified space assemble into a large commutative diagram.

Construction 1.2.1.22. The generalized homotopy link functors $\mathcal{H}o\mathrm{Link}_{\mathcal{I}} \colon \mathbf{Strat}_P \to \mathbf{Top}$ are also functorial in $\mathcal{I}$. Given an inclusion of regular flags $\mathcal{I} \subset \mathcal{I}'$, the associated face inclusion of stratified simplices $|\Delta^{\mathcal{I}}|_s \hookrightarrow |\Delta^{\mathcal{I}'}|_s$ induces a natural transformation $\mathcal{H}o\mathrm{Link}_{\mathcal{I}'} \Rightarrow \mathcal{H}o\mathrm{Link}_{\mathcal{I}}$, given by restricting a stratified singular simplex $|\Delta^{\mathcal{I}'}|_s \to \mathcal{X}$ along $|\Delta^{\mathcal{I}}|_s \hookrightarrow |\Delta^{\mathcal{I}'}|_s$. For example, in the case of $\mathcal{I}' = \{p < q\}$ and $\mathcal{I} = \{p\}$, identifying $\mathcal{H}o\mathrm{Link}_{\{p\}}(\mathcal{X}) \cong X_p$, the value of this natural transformation at $\mathcal{X}$, $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}_{\{p\}}(\mathcal{X})$, is simply the starting point evaluation map. Together, these natural transformations define a functor

$$\mathbf{Strat}_P \times \mathrm{sd}(P)^{\mathrm{op}} \to \mathbf{Top}$$

which, under the exponential law for categories, we may equivalently think of as a functor

$$\begin{aligned} \mathcal{H}o\mathrm{Link} \colon \mathbf{Strat}_P &\to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top}) \\ \mathcal{X} &\mapsto \big\{ \mathcal{I} \mapsto \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}). \end{aligned}$$

Example 1.2.1.23. It can be useful to visualize the homotopy link diagram $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ as a diagram of spaces whose shape is given by the first barycentric subdivision of the nerve of P. For example, in the case $P = \{p_0 < p_1 < p_2\}$, the generalized homotopy links arrange into a commutative diagram of the following shape

$$\begin{tikzcd} & & X_{p_2} & & \\ & \mathcal{H}o\mathrm{Link}_{\{p_0<p_2\}}(\mathcal{X}) \arrow[ur] \arrow[dl] & \mathcal{H}o\mathrm{Link}_{\{p_0<p_1<p_2\}}(\mathcal{X}) \arrow[l] \arrow[r] \arrow[u] \arrow[d] \arrow[dll] \arrow[drr] & \mathcal{H}o\mathrm{Link}_{\{p_1<p_2\}}(\mathcal{X}) \arrow[ul] \arrow[dr] & \\ X_{p_0} & & \mathcal{H}o\mathrm{Link}_{\{p_0<p_1\}}(\mathcal{X}) \arrow[ll] \arrow[rr] & & X_{p_1} \end{tikzcd} \tag{1.7}$$

In the following, we will denote the ∞-category obtained by localizing the category of topological spaces at weak homotopy equivalences by $\mathcal{S}\mathbf{paces}$. The category $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ has a natural notion of weak equivalence, namely such morphisms of diagrams that are given by weak homotopy equivalence for each $\mathcal{I} \in \mathrm{sd}(P)$. Localizing this class of weak equivalences, one equivalently obtains the ∞-category of functors $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$, or, in other words, the ∞-category of space valued presheaves on $\mathrm{sd}(P)$ (see Chapter 7, where such arguments are provided with more detailed references). By definition of diagrammatic equivalences, and the universal property of the localization, we obtain a canonical induced functor of ∞-categories

$$\begin{tikzcd} \mathbf{Strat}_P \arrow[r, "\mathcal{H}o\mathrm{Link}"] \arrow[d] & \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top}) \arrow[d] \\ \mathcal{S}\mathbf{trat}_P^{\mathfrak{d}} \arrow[r, dashed] & \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces}) \end{tikzcd} \tag{1.8}$$

making the diagram commute (up to natural isomorphism, depending on the precise model for localization one has in mind). We will also denote this functor by $\mathcal{H}o\mathrm{Link}$. $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ is an ∞-category that is reasonably easy to handle, being a presheaf category on a particularly simple category (a poset). This suggests the general approach of using the functor $\mathcal{H}o\mathrm{Link}$ to reduce questions about the homotopy theory $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ to questions about presheaves on $\mathrm{sd}(P)$. The following result, which follows from [Dou21c, Thm. 3] guarantees that this approach essentially loses no information.

Theorem 1.2.1.24 ([Dou21c])**.** *The functor of* ∞*-categories*

$$\mathcal{H}o\mathrm{Link}\colon \mathcal{S}\mathbf{trat}_P^{\mathfrak{d}} \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$$

is an equivalence of ∞*-categories.*

This result turns out to be an extremely powerful tool, not only for studying the homotopy theory $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$, but also for studying alternative approaches to stratified homotopy theory, which we will discuss in Section 1.3. When attempting to apply this result to classical examples of stratified spaces, one will observe the following two challenges.

(O1) First off, to apply the theorem to geometrical examples, one needs to have a strong understanding of how the ∞-category $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ interacts with the 1-category $\mathbf{Strat}_P$ - especially with such objects and morphisms arising from classical examples of stratified spaces - and the simplicial category $\underline{\mathbf{Strat}}_P$. For example, one important thing to know would be whether classical examples of stratified spaces fulfill a Whitehead theorem, identifying diagrammatic equivalences between them with stratified homotopy equivalences. This is essentially the question of whether the class of diagrammatic equivalences fulfills Requirement (R3).

(O2) Secondly, the effectiveness of this theorem clearly correlates strongly with one's understanding of the properties and the computability of the homotopy link functors and their interaction with 1-categorical and geometrical constructions.

We will address these challenges later on.

1.2.2 Remarks on the use of model structures

For now, let us again return to Requirements (R2) and (R3). In general, ∞-categories obtained through the localization of a category at some class of weak equivalences may be extremely complicated and difficult to understand. Probably the most well-established theory of getting better control over the resulting localizations, and connecting them with the 1-category one started with is the theory of model categories (first defined in [Qui67], see [Hov07; Hir03] for great introductions, as well as the appendix to [Lur09], for the connection to higher categories). In addition to a class of weak equivalences, one equips the (bicomplete) 1-category in question with two classes of morphisms, the so-called fibrations and cofibrations, which are assumed to fulfill a series of lifting and factorization properties. These additional pieces of data can then be used to perform ∞-categorical constructions, such as limits, colimits, or mapping spaces in the associated ∞-localization purely in terms of constructions in the respective 1-category (or some additional simplicial structure on it). Just to give an example, the classical Whitehead theorem for topological spaces - relating weak homotopy equivalences with homotopy equivalences - is a special instance of a general Whitehead theorem that holds in any model category (see [Hir03, p. 7.5.10]). We are not looking to give an introduction to the language of model categories here and expect a general degree of familiarity with basic terminology and methods of this language here (see [Hov07] for an introduction and [Hir03] for a rather exhaustive overview). Even for the reader not familiar with the basics of model categories, we think the following more philosophical point towards the role which they conceptually play in this work may be of interest.

The role that model categories take in the study of modern homotopy theory is often seen as akin to the role of an atlas in differential topology. They are not strictly speaking intrinsic to the theory, and the thing one is really interested in studying, but they can provide a useful framework to perform certain computations and make things more explicit. This is a good analogy, insofar as it illustrates two points:

- It emphasizes the computational usefulness of model categories, and the great deal of additional tractability over the behavior of the ∞-categorical localizations associated to them;
- It emphasizes that there may be many different sets of coordinates leading to the same intrinsic object of study;
- But it also emphasizes that only thinking of every construction for smooth manifolds in terms of coordinates can be rather cumbersome, and at times a conceptual hindrance to seeing the intrinsic nature of constructions.

The way the author usually interprets this analogy is that the role of smooth manifolds corresponds to ∞-categories, the 1-category corresponds to open subsets of Euclidean space, and the model category takes the role of the smooth charts. What seems to be slightly more difficult to capture in this analogy is the following: Often people using homotopy theory start with being interested in a specific 1-category of objects, and then want to know how these objects behave under the passage to homotopy theory. Attempting to stay in the analogy, they are very much interested in a specific point under some specific Euclidean chart. Keeping with the analogy, it seems to us that the power of model categories does not just lie in the fact that they allow for the computation of facts about some smooth manifold in charts, but that they allow for the choice of very specific charts particularly adapted to studying some very specific points in Euclidean space. For example, the fact that Serre fibrations are part of a model structure on topological spaces does not just serve to compute homotopy fibers, it also tells us that the ordinary fiber of some specific fiber bundle of manifolds we care about is *also* the homotopy fiber and thus can be studied through all of the tools concerned with homotopy fibers. In this kind of scenario, it is the fact that we can find a model structure for topological spaces that relates strongly to classical geometrical objects – manifolds are cofibrant, fiber bundles are fibrations, . . . – that enables us to apply the homotopy theory of topological spaces, to infer theorems about geometry and topology from homotopy theory.
An analogy that may be slightly adapted to expressing this specific advantage of model structures may be to compare the relationship of model categories with ∞-categories with the relationship of presentations of a group (in terms of generators and relations) with the group itself. Often the generators come with explicit geometric meaning (such as being reflections or rotations) and knowing how a specific element of a group can be expressed in terms of the generators and relations can be important information. Following this perspective, one often says that a model category *presents* the ∞-category obtained by localizing weak equivalences. This perspective will become particularly important in the second half of this thesis (Part III), when (in addition to the model structure) we also fix generating cofibrations and acyclic cofibrations, and perform investigations in generalized simple homotopy theory. Sticking to the analogy of groups and generators, we will concern ourselves with the question of whether an identity (weak equivalence) between two different expressions of a group element (homotopy type) in terms of generators (generating cofibrations) can be verified purely in terms of a finite sequence of modifications in terms of the relations (generating acyclic cofibrations).

1.2.3 Simplicial models for diagrammatic stratified homotopy theory

Having explained this, let us return to stratified spaces. To address Requirement (R3) and Obvervation (O1), i.e., to relate the homotopy theory $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ to geometrical examples of stratified spaces (and more classical approaches to stratified homotopy theory) it would be extremely useful to have a model structure on $\mathbf{Strat}_P$ that, at the very least, is such that classical geometrical examples, such as Whitney stratified spaces, are both fibrant and cofibrant. Already in [Dou21c] the existence of a model structure on $\mathbf{Strat}_P$ presenting $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ was proven. However, it was constructed in a way that makes all stratified spaces fibrant, and almost no classical geometrical example of a stratified space cofibrant. The construction of a model structure better adapted to geometrical examples turns out to be a surprisingly subtle (and technically difficult) question, which we discuss in Section 1.4. In many ways, the requirement for maps to be stratified often makes homotopy theoretic proofs in the topological category significantly more involved than similar arguments in a non-stratified setting. One case, however, where such arguments can often be simplified is the case of stratified spaces $\mathcal{X}$ that can be triangulated. By this we mean, that there exists a (non-stratified) triangulation that is compatible with the stratification in the sense that the subspaces $\mathcal{X}_{\leq p}$, for $p \in P_{\mathcal{X}}$, are themselves triangulated by a subcomplex. It is, for example, a classical fact in stratified topology that Whitney stratified spaces have this property ([Gor78]). Much about such triangulable stratified spaces can already be learned by considering model categories for stratified homotopy theory that are using *stratified simplicial sets* instead. The following definitions can be found in more detail in Chapter 3.

Notation 1.2.3.1. We will use the following language and notation.

1. We denote by **sSet** the category of simplicial sets, i.e., the category of set-valued presheaves on Δ (see, for example, [GJ12], for an introduction into the homotopy theory of such objects). We will use much of the notation standard for simplicial sets, found, for example, in [Lur09].

2. Given a poset P, we will often treat P as an element of **sSet**, by assigning to it its nerve $\mathrm{N}(P)$, i.e., the simplicial set whose n-simplices are given by the flags $[n] \to P$ (with the obvious face and degeneracy maps). Through the nerve functor, the category of posets **Pos** is embedded into **sSet** as a full (reflexive) subcategory. By abuse of notation, we will usually omit the nerve notation, and just treat posets as objects of **sSet**.

The category of stratified simplicial sets s**Strat** is now defined entirely analogously to the category of poset-stratified spaces in Definition 1.1.1.2, replacing spaces by simplicial sets, continuous maps by simplicial maps and the Alexandrov space associated to a poset by the nerve of a poset (see Chapter 5 for a detailed definition). In particular, a *stratified simplicial set* $\mathcal{X}$ is a triple $\mathcal{X} = (X, P_{\mathcal{X}}, s_{\mathcal{X}})$, consisting of a simplicial set $X \in \mathbf{sSet}$, a partially ordered set $P_{\mathcal{X}} \in \mathbf{Pos}$ and a simplicial map $s_{\mathcal{X}} : X \to P_{\mathcal{X}}$, called the *stratification map*. The subsimplicial set of X given by the fiber $s_{\mathcal{X}}^{-1}(p)$ at $p \in P$, is called the *p-stratum of $\mathcal{X}$*. We use the terminology *stratified simplicial map* and *stratum-preserving simplicial map* entirely analogously to the topological case.

Remark 1.2.3.2. The first important thing to observe about stratified simplicial sets is that the nerve functor $\mathrm{N} : \mathbf{Pos} \to \mathbf{sSet}$ admits a left adjoint, given by mapping a simplicial set X to the poset generated by equipping the set of vertices of X with the order generated by the relation $x \leq y$, whenever there is a 1-simplex $x \xrightarrow{\tau} y$ in X . It follows that a simplicial map $X \to P$ is the same thing as a map from the vertices of X, $s : X_0 \to P$, such that $s(x) \leq s(y)$ whenever there is a 1-simplex $x \xrightarrow{\tau} y$. In this sense, a stratification of a simplicial set X over a poset P is simply a choice of strata

$$x \mapsto s_{\mathcal{X}}(x) \in P$$

for the vertices of X, such that the paths in X, given by maps $\Delta^1 \to X$, do not descend in the stratification. From this perspective, a stratified simplicial map $\mathcal{X} \to \mathcal{Y}$ simply consists of a simplicial map $f : X \to Y$ together with a map of posets $P_f : P_{\mathcal{X}} \to P_{\mathcal{Y}}$, such that $s_{\mathcal{Y}} f(x) = P_f(s_{\mathcal{X}}(x))$ holds for every vertex $x \in X_0$.

Example 1.2.3.3. In more classical simplicial investigations of stratified objects, people often considered geometrical simplicial complexes K (in the sense of a subset of Euclidean space, together with subsets defining the simplices) together with a filtration $K_{\leq 0} \subset K_{\leq 1} \cdots \subset K_n$ by subcomplexes (see, for example, [Sto72]). If we take a first barycentric subdivision of such a filtered simplicial complex

$$\mathrm{sd}K_{\leq 0} \subset \cdots \subset \mathrm{sd}K_n$$

(treated as an abstract simplicial complex whose vertices are the simplices of K) then the vertices of $\mathrm{sd}K$ are naturally ordered (by thinking of them as simplices in K, and ordering them via inclusion). In fact, one can think of sd$K$ as the nerve of the poset of simplices of K ordered via face inclusion. Thus, sd$K$ can be treated as a simplicial set. sd$K$ comes with a canonical stratification, given by the map

$$\begin{aligned}(\mathrm{sd}K)_0 &\to [n]\\ \sigma &\mapsto \min\{k \in [n] \mid \sigma \subset K_{\leq k}\}.\end{aligned}$$

In this manner, all classical examples of stratified simplicial complexes or stratified PL objects can be treated in the context of stratified simplicial sets.

Notation 1.2.3.4. Given any kind of category $\mathbf{C}$ (1-category, enriched category or quasi-category), and two objects $X, Y \in \mathbf{C}$, we will refer to the object of morphisms from X to Y by the notation $\mathbf{C}(X, Y)$.

Analogously to the setting of stratified topological spaces, given a poset P, we denote by s$\mathbf{Strat}_P$ the category of stratified simplicial sets whose stratification poset is given by P, equipped with *stratum-preserving simplicial maps*. The category s$\mathbf{Strat}_P$ has one decisive technical advantage over $\mathbf{Strat}_P$. Namely, that since s$\mathbf{Strat}_P = \mathbf{sSet}_{/P}$ is the slice category of a presheaf category we may equivalently think of it as the category of presheaves over $\Delta_{/P} = \Delta_P$. The isomorphism of categories

$$\mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}({\Delta_P}^{\mathrm{op}}, \mathbf{Set})$$

is given by mapping a P-stratified simplicial set to the presheaf given by

$$\mathcal{J} \mapsto \mathrm{s}\mathbf{Strat}_P(\mathrm{N}([n]) \xrightarrow{\mathrm{N}(\mathcal{J})} \mathrm{N}(P), \mathcal{X}),$$

acting on morphisms of flags via precomposition. Under this identification, the presentable presheaves associated to a flag $\mathcal{J}$ are given by $\mathrm{N}([n]) \xrightarrow{\mathrm{N}(\mathcal{J})} \mathrm{N}(P)$. As $\mathrm{N}([n]) = \Delta^n$, we can think of these as stratified simplices, with the stratification specified by the flag $\mathcal{J}$. We thus use the notation $\Delta^{\mathcal{J}}$ to refer to stratified simplicial sets of this form. Explicitly, this means $\Delta^{\mathcal{J}}$, for $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, is given by stratifying Δ^n such that $k \in [n]$ lies in the p_k-stratum. The stratified subsimplicial set given by the boundary of the underlying simplex will be denoted $\partial\Delta^{\mathcal{J}}$.

The theory of presheaf categories and how to construct model structures on the latter is well understood and studied (see, for example, [Cis06]). Hence, it is generally a plausible approach to understanding the simplicial stratified scenario first, and then transfer results from the simplicial side to the topological stratified world. This kind of transfer is made possible through the following adjunction, which is the stratified analogue of the classical realization and singular-simplicial set adjunction for simplicial sets and topological spaces.

Construction 1.2.3.5. By the general yoga of nerve and realization functors (see, for example, [Cis19]) it follows that any left-adjoint functor s$\mathbf{Strat}_P \to \mathbf{Strat}_P$ is uniquely defined (up to canonical natural isomorphism) by its restriction along the Yoneda-inclusion

$$\begin{aligned}\Delta_P &\hookrightarrow \mathrm{s}\mathbf{Strat}_P\\ \mathcal{J} &\mapsto \Delta^{\mathcal{J}}.\end{aligned}$$

Hence, to define a stratified realization functor from s$\mathbf{Strat}_P$ into $\mathbf{Strat}_P$, it suffices to define the realization functor for stratified simplices. We have already defined such a realization functor for stratified simplices in Construction 1.2.1.13 (extending to morphisms in the obvious way). The left Kan-extension of the functor

$$\begin{aligned}\Delta_P &\to \mathbf{Strat}_P\\ \mathcal{J} &\mapsto |\Delta^{\mathcal{J}}|_s\end{aligned}$$

to s$\mathbf{Strat}_P$ gives rise to an adjunction

$$|-|_s\colon \mathrm{s}\mathbf{Strat}_P \rightleftharpoons \mathbf{Strat}_P\colon \mathrm{Sing}_s$$

called the *stratified realization and stratified singular simplicial set adjunction.*

Remark 1.2.3.6. Explicitly, this means that the realization of a stratified simplicial set $\mathcal{X}$ is given by equipping the realization of its underlying simplicial set $|X|$ with the following stratification. $|\mathrm{N}(P)| = \bigcup_{\mathcal{I}\in\mathrm{sd}(P)} |\Delta^{\mathcal{I}}|_s$ is naturally stratified over P, by gluing the stratifications of the stratified realizations $|\Delta^{\mathcal{I}}|_s$, $\mathcal{I} \in \mathrm{sd}(P)$. One may then stratify $|X|$, via the composition

$$|X| \xrightarrow{|s_{\mathcal{X}}|} |\mathrm{N}(P)| \to P.$$

Conversely, the underlying simplicial set of $\mathrm{Sing}_s\mathcal{X}$, for a stratified space $\mathcal{X} \in \mathbf{Strat}_P$, can be identified with a subsimplicial set of the usual singular simplicial set $\mathrm{Sing}(X)$. Specifically, $\mathrm{Sing}_s(\mathcal{X})$ is given by such singular simplices $|\Delta^n| \to X$, for which the induced stratification

$$|\Delta^n| \to X \xrightarrow{s_{\mathcal{X}}} P$$

arises from some choice of flag $[n] \xrightarrow{\mathcal{J}} P$.

The adjunctions $|-|_s \dashv \mathrm{Sing}_s$, for varying $P \in \mathbf{Pos}$, assemble to a global adjunction

$$|-|_s\colon \mathrm{s}\mathbf{Strat} \rightleftharpoons \mathbf{Strat}\colon \mathrm{Sing}_s$$

which we denote the same, by abuse of notation.

Remark 1.2.3.7. The categories s$\mathbf{Strat}_P$ and s$\mathbf{Strat}$, much like the categories $\mathbf{Strat}_P$ and $\mathbf{Strat}$ inherit the structure of simplicial categories. Just as in the topological case, the simplicial structure is induced by a tensoring (see [nLa24k] for an overview over simplicial model categories), defined by setting $\mathcal{X} \otimes \Delta^n := (X \times \Delta^n, P_{\mathcal{X}}, X \times \Delta^n \xrightarrow{\pi_X} X \xrightarrow{s_{\mathcal{X}}} P)$. In particular, the n-simplices in the associated simplicial mapping spaces $\underline{\mathrm{s}\mathbf{Strat}}_P(\mathcal{X},\mathcal{Y})$ ($\underline{\mathrm{s}\mathbf{Strat}}(\mathcal{X},\mathcal{Y})$) are given by stratum-preserving (stratified) maps $\mathcal{X} \otimes \Delta^n \to \mathcal{Y}$.
In fact, in this fashion, one obtains not only simplicial categories, but simplicial categories tensored and cotensored over **sSet**, i.e., the basic ingredients needed for a simplicial model category. The adjunctions $|-|_s \dashv \mathrm{Sing}_s$ are compatible with these structures and lift to an adjunction of simplicial categories.

Notation 1.2.3.8. In the following, we will often deal with categories that extend to the structure of a simplicial category. In this case, we will denote the simplicial category in the form $\underline{\mathbf{C}}$ and the underlying 1-category by $\mathbf{C}$.

Recall that the unstratified analogue of this construction between simplicial sets and topological spaces is a Quillen equivalence, if one equips topological spaces with the Quillen model structure and simplicial sets with the Kan-Quillen model structure (see [Qui67], or Theorem 1.2.4.1 below). In fact, the model structure on topological spaces can be seen as transferred from the Kan-Quillen model structure, in the sense that a map of topological spaces $f\colon X \to Y$ is a weak homotopy equivalence (resp. Serre-fibration), if and only if $\mathrm{Sing}(f)\colon \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a (weak) homotopy equivalence (Kan-fibration). This classical fact suggested a similar approach in the stratified world: Define a model structure in the realm of stratified simplicial sets, and transfer it along the adjunction to the world of stratified topological spaces (see, for example, [Nan19], where a similar approach was pursued, and see Section 1.4 for more details). Such a model structure on the simplicial side was established by Douteau in [Dou21a] and studied independently by Henriques in [Hen]. This uses a general construction for model categories on presheaf categories constructed from the data of a cylinder and a class of generating anodyne extensions. In the case of stratified simplicial sets, one uses the cylinder functor given by the stratified cylinders $\mathcal{X} \otimes \Delta^1$, which is the obvious analogue to the cylinder in the stratified topological case.

Definition 1.2.3.9. The Douteau-Henriques model structure on $\mathrm{s}\mathbf{Strat}_P$ is the minimal model structure (in the sense of weak equivalences) such that the boundary inclusions $\mathcal{X} \hookrightarrow \mathcal{X} \otimes \Delta^1$ are weak equivalences, and such that the cofibrations are exactly given by the monomorphisms.

Notation 1.2.3.10. The model category obtained by equipping $\mathrm{s}\mathbf{Strat}_P$ with the Douteau-Henriques model structure will be denoted $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$.

Notation 1.2.3.11. Henceforth, we will use the notation convention that given a model category, denoted in the form $\mathbf{Category}^{\mathfrak{abc}}$, its associated ∞-category obtained by localizing at weak equivalences will be denoted in the form $\mathcal{C}\mathbf{ategory}^{\mathfrak{abc}}$, replacing the first letter with a calligraphic one.

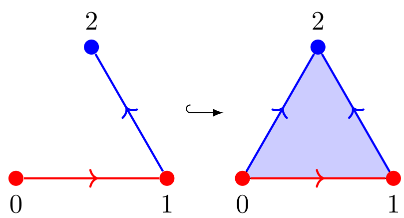
Definition 1.2.3.9 is, for now, a very abstract definition. The following result, using the general calculus developed in [Cis06], makes it significantly more explicit. Given a simplex $\Delta^n \in \mathbf{sSet}$, with $n \geq 1$, and $0 \leq k \leq n$, we denote by Λ^n_k its *k-th horn*, i.e., the sub-simplicial set obtained by removing from Δ^n the unique non-degenerate n-dimensional simplex and the $(n-1)$-dimensional face opposite to the k-th vertex. Recall that in the Kan-Quillen model structure for simplicial sets, the fibrations - i.e., the Kan-fibrations - can be characterized by having the right lifting property with respect to all horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$. It turns out that a similar statement can be made in the stratified case.

Definition 1.2.3.12 ([Dou21a])**.** Given a stratified simplex $\Delta^{\mathcal{J}}$, with $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, and $k \in [n]$, we denote by $\Lambda^{\mathcal{J}}_k$ the stratified sub-simplicial set of $\Delta^{\mathcal{J}}$ whose underlying simplicial set is the horn $\Lambda^n_k \subset \Delta^n$. A horn inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$ is called *admissible*, if one of the following two equivalent conditions holds (see Chapter 3 for more details).

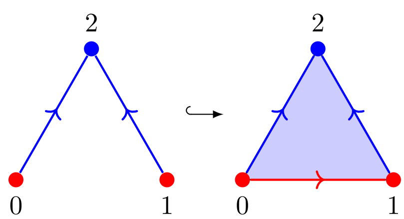
- The stratified realization $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$ is a stratified homotopy equivalence.
- The value p_k is repeated in $\mathcal{J}$.

Example 1.2.3.13. Consider the following three illustrations of stratified horn inclusions over $P = \{p_0 < p_1 < p_2\}$.

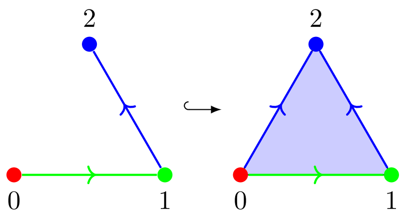
1. For $k = 1$ and $\mathcal{J} = [p_0 \leq p_0 < p_2]$ the associated inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$, illustrated directly below, is admissible.



2. For $k = 2$ and $\mathcal{J} = [p_0 \leq p_0 < p_2]$ the associated inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$, illustrated directly below, is not admissible.



3. For $k = 1$ and $\mathcal{J} = [p_0 < p_1 < p_2]$ the associated inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$, illustrated directly below, is not admissible.



Recall that one says that a set of morphisms S provides a generating set of (acyclic) cofibrations in a model category, if the (acyclic) cofibrations are generated from S under the operations of cobase change (i.e., pushout along some arbitrary arrow), transfinite composition and taking retracts (see, for example, [Hir03]). This is in turn equivalent to saying that a morphism is a trivial fibration (fibration) if and only if it has the right lifting property with respect to S. [Dou21a] showed the following result.

Theorem 1.2.3.14 ([Dou21a])**.** *The model category* $s\mathbf{Strat}_P^{\mathfrak{d}}$ *is simplicial (with respect to the canonical structure) and has the following properties:*

- *Cofibrations are generated by the set of stratified boundary inclusions*
$$\{\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}} \mid \mathcal{J} \in \Delta_P\}.$$
In particular, every object is cofibrant.
- *Acyclic cofibrations are generated by the set of admissible horn inclusions*
$$\{\Lambda_k^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}} \mid \mathcal{J} \in \Delta_P, k \text{ s.t. } \Lambda_k^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}} \text{ is admissible.}\}$$
In particular, the fibrant objects are precisely such stratified simplicial sets that have the horn filling property with respect to admissible horn inclusions.
- *A morphism between fibrant objects* $\mathcal{X} \to \mathcal{Y}$ *is a weak equivalence if and only if, for each regular flag* $\mathcal{I} \in \mathrm{sd}(P)$*, the associated map of simplicial mapping spaces*
$$\underline{s\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathcal{X}) \to \underline{s\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathcal{Y})$$
is a weak homotopy equivalence.

Remark 1.2.3.15. The characterization of weak equivalences in Theorem 1.2.3.14 can be seen as an analogue of the definition of diagrammatic equivalences in the topological setting. The restriction of this definition to fibrant objects is a priori not surprising. One should generally only expect simplicial mapping spaces in a simplicial model category to be homotopically well-behaved if the sources are cofibrant and the targets are fibrant, i.e., when they compute the mapping space in the associated ∞-category. We will see, however, that in this special instance, the fibrancy assumptions are actually superfluous, which will have far-reaching consequences for the connection between topological and simplicial models for stratified homotopy theory.

Construction 1.2.3.16. The model category $s\mathbf{Strat}_P^{\mathfrak{d}}$ also admits an extension to the case of flexible posets, denoted $s\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$, and called the *poset-preserving diagrammatic model structure* or *poset-preserving Douteau-Henriques model structure* (with the $\mathfrak{d}$ standing for diagrammatic, and the $\mathfrak{p}$ standing for *poset-preserving*). This uses a general construction of [CM20], that glues a family of model structures on a Grothendieck bifibration to a global model structure. In this case, the Grothendieck bifibration is the functor $s\mathbf{Strat} \to \mathbf{Pos}$, sending a stratified simplicial set to its underlying poset, whose fiber at $P \in \mathbf{Pos}$ is precisely the stratum-preserving category $s\mathbf{Strat}_P$. The details for this construction are not too surprising, and can be found in Chapter 5, for example. For the purpose of this introduction, it will suffice to know that weak equivalences are given by such stratified simplicial maps that induce isomorphisms on posets and weak equivalences in $s\mathbf{Strat}_P^{\mathfrak{d}}$, after identifying the posets of the source and target. Essentially anything one needs to know about these model structures can be reduced to fiberwise arguments.

These results already point towards the model categories $\mathrm{s}\mathbf{Strat}_P$ being well suited to connect the diagrammatic homotopy theory $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ to geometric examples of stratified spaces. This is for the following reasons:

1. All stratified simplicial sets being cofibrant ensures that every stratified space that admits a compatible triangulation (such as a Whitney stratified space or a PL pseudo manifold) is isomorphic to the stratified realization of a cofibrant object.

2. It follows by a result of Lurie's (see [Lur17, Theorem A.6.4.]) that the characterization for fibrant stratified simplicial sets is fulfilled by the stratified singular simplicial sets associated to conically stratified spaces. It was furthermore shown in [Nan19] that this also holds for the more general *homotopically stratified spaces* introduced by Quinn ([Qui88]).

3. The mapping spaces $\mathrm{s}\underline{\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathcal{X})$ provide simplicial analogue for homotopy links. Indeed, given a stratified space $\mathcal{Y}$ there is a canonical isomorphism

$$\mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})) \cong \mathrm{s}\underline{\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathrm{Sing}_s(\mathcal{X})).$$

   In particular, it follows by Theorem 1.2.3.14 that a map $w\colon \mathcal{X} \to \mathcal{Y}$ between two stratified spaces, whose image under Sing_s is fibrant in the Douteau-Henriques model structure, is a diagrammatic equivalence, if and only if $\mathrm{Sing}_s(w)$ is a weak equivalence in $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$.

These three facts can already be used to derive quite a bit about the diagrammatic homotopy theory $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$. For example, one consequence of these results is a Whitehead theorem for stratified spaces which are triangulable and whose stratified singular simplicial set is fibrant (for example, Whitney stratified spaces), first proven in [Dou21a].

1.2.4 Results: A Kan-Quillen equivalence without model structures

The obvious question arises on how strong of a link between the homotopy theories $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ and $\mathrm{s}\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ ($\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$) is actually established by the adjunctions $|-|_s \dashv \mathrm{Sing}_s$. Recall, for this purpose, the Kan-Quillen equivalence between topological spaces and simplicial sets, which is at the heart of large parts of modern homotopy theory.

Theorem 1.2.4.1 ([Qui67])**.** *The adjunction*

$$|-|\colon \mathbf{sSet} \rightleftharpoons \mathbf{Top}\colon \mathrm{Sing}$$

induces a Quillen equivalence between **sSet***, equipped with the Kan-Quillen model structure, and* **Top***, equipped with the Quillen model structure.*

Just saying that this adjunction is a Quillen equivalence actually slightly obscures how intimate the connection between these two theories established by the adjunction is. Recall that one says that a functor $F\colon \mathbf{C} \to \mathbf{D}$ between two categories with weak equivalences *creates weak equivalences*, if for every $w \in \mathbf{C}$ it holds that w is a weak equivalence if and only if $F(w)$ is a weak equivalence. Theorem 1.2.4.1 together with the fact that all simplicial sets are cofibrant and all topological spaces are fibrant actually implies that the following two equivalent conditions hold, which are (if one already has a Quillen adjunction) generally stronger than being a Quillen equivalence:

1. Sing_s and $|-|$ create weak equivalences;

2. Sing_s and $|-|$ preserve weak equivalences and unit and counit of the adjunction are weak equivalences.

In fact, the two conditions as stated above can be seen to be equivalent for general adjunctions between categories equipped with a class of weak equivalences fulfilling the two-out-of-three property and containing all isomorphisms. Such a pair, consisting of a category $\mathbf{C}$ together

with a class of morphisms $W \subset \mathbf{C}$ containing all isomorphisms and fulfilling two-out-of-three is called a *category with weak equivalences* (see [nLa24b], for an overview). In the language of categories with weak equivalences (or, more generally, so-called relative categories) we will refer to the second condition above by saying that the adjunction $|-| \dashv \mathrm{Sing}$ defines a *homotopy equivalence of categories with weak equivalences (of relative categories)* (see [BK12b], where this notion was defined). It follows immediately from the universal property of the localization that a homotopy equivalence of categories with weak equivalences descends to an equivalence of the associated ∞-categories, obtained by localizing weak equivalences.
One direct advantage of homotopy equivalences of categories with weak equivalences over Quillen equivalences, or more generally functors of categories with weak equivalences over left Quillen functors, is that there is no need to *derive*, i.e., fibrantly or cofibrantly replace, if one wants them to descend to functors on the associated ∞-categories. The obvious disadvantage is, of course, that these types of functors are generally not required to respect the extra structure of fibrations and cofibrations, which may contain relevant geometric or algebraic information.

Notation 1.2.4.2. In the following, we will often deal with categories with weak equivalences, or more generally relative categories, i.e., categories equipped with some wide subcategory of morphisms which are not necessarily stable under the two-out-of-three property and may not contain all isomorphisms. We will generally use the format ***C*ategory** to refer to relative categories, and denote the first letter in italics. If we are in the context of some model category or ∞-category obtained by localizing a class of weak equivalences, then replacing the first letter by an italic letter will indicate that we are referring to the respective relative category. When we start with a relative category and localize to pass to an ∞-category (modelled as quasi-categories) we replace the first letter by a calligraphic letter. For example, we denote by $\boldsymbol{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ the category with weak equivalences obtained by equipping $\mathbf{Strat}$ with the class of poset-preserving diagrammatic equivalences, and denote by $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ its associated ∞-category. We will refer to the homotopy 1-category associated to a model category, simplicial category or ∞-category by adding a prefix “ho”.

In [Dou21b], Douteau first proved a Quillen equivalence between $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ and $\mathbf{Strat}_P$, where the latter was equipped with a model structure that generally does not interact well with the geometry of classical examples of stratified spaces. The Quillen equivalence was, however, not constructed in terms of the adjunction $|-| \dashv \mathrm{Sing}_s$, but through a modified version of the latter, which, for example, does not produce triangulations of stratified spaces as they occur in more classical scenarios such as [Sto72]. This fact limits the usefulness of the adjunction when one is interested in studying geometrical examples of stratified spaces. For example, unlike the adjunction $|-| \dashv \mathrm{Sing}_s$, it is generally unclear how to extract a Whitehead theorem for PL pseudomanifolds from it.
In this sense, [Dou21b] produced the result that the resulting homotopy theories are equivalent, but not in the way one would have, a-priori, wanted or expected. Furthermore, the adjunction defined in [Dou21b] does not pass to a global equivalence between $\mathrm{s}\mathbf{Strat}$ and $\mathbf{Strat}$. Nevertheless, these results already served as a theoretical benchmark as well as a powerful tool used in further investigations.

In lieu of a model structure for the diagrammatic homotopy theory on $\mathbf{Strat}$ ($\mathbf{Strat}_P$) that is compatible with the adjunction $|-|_s \dashv \mathrm{Sing}_s$, the next best thing to ask for was whether the adjunctions defined homotopy equivalences of categories with weak equivalences. That this holds is the first core result presented in this thesis, which we developed in joint work with Douteau in [DW22]. This article is presented in Chapter 3 here. The following result can either be directly derived from Theorem 3.1.0.3 in Chapter 3 or from the statements and proofs of Lemmas 3.5.1.2 and 3.5.1.3. [8]

[8]The reader reading [DW22] and comparing it with the way this introduction is phrased will notice a clear difference in the language used to phrase things. Namely [DW22] is mostly expressed in the language of model categories and homotopy categories, only occasionally eluding to the ∞-categorical consequences, while this introduction starts from the perspective of ∞-categories and eludes to model categories mainly for the purpose

Main Result A_1. *The adjunctions*

$$|-|_s \colon s\boldsymbol{Strat}^{\mathfrak{d},\mathfrak{p}} \rightleftarrows \boldsymbol{Strat}^{\mathfrak{d},\mathfrak{p}} \colon \mathrm{Sing}_s,$$
$$|-|_s \colon s\boldsymbol{Strat}^{\mathfrak{d}}_P \rightleftarrows \boldsymbol{Strat}^{\mathfrak{d}}_P \colon \mathrm{Sing}_s$$

define homotopy equivalences of categories with weak equivalences. In particular, they descend to equivalences of ∞-categories $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}} \simeq \mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \simeq \mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$, after localizing at weak equivalences.

This result can already be used to obtain crucial insights about the interaction of the diagrammatic homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ with geometrical examples of stratified spaces. Let us denote by $\underline{\mathbf{Con}}$ the full simplicial subcategory of $\underline{\mathbf{Strat}}$ given by conically stratified spaces that are stratum-preserving homeomorphic to the realization of a stratified simplicial set. In particular, all PL pseudomanifolds or Whitney stratified spaces are within this category. In Chapter 3, we prove the following results (see Theorem 3.1.0.2 and the comment following it.)[9]

Corollary 1.2.4.3. *The inclusion of simplicial categories $\underline{\mathbf{Con}} \hookrightarrow \underline{\mathbf{Strat}}$ induces a fully faithful embedding of the ∞-category (given by the homotopy coherent nerve of) $\underline{\mathbf{Con}}$ into the ∞-category $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$. In particular, it follows that for $\mathcal{X},\mathcal{Y} \in \underline{\mathbf{Con}}$ the set of morphisms in the homotopy category $\mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$, between $\mathcal{X}$ and $\mathcal{Y}$, is in canonical bijection*

$$\mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}(\mathcal{X},\mathcal{Y}) \cong [\mathcal{X},\mathcal{Y}]_s$$

with the set of stratified homotopy classes.

In fact, it follows from the way this result is proven in Chapter 3 that this holds more generally for (appropriately triangulable) stratified spaces whose associated stratified singular simplicial set $\mathrm{Sing}_s(\mathcal{X})$ is fibrant in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$. In other words, at least if we assume the existence of triangulation, the requirements of Obvervation (O1) are fulfilled. When restricting to the conically stratified, triangulable case, the more classical approach to stratified homotopy theory investigated in [Qui88; Mil13] embeds fully faithfully into the homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$.

In Chapter 3, we also prove an analogue of Theorem 1.2.1.24 for the case of stratified simplicial sets: The simplicial homotopy link functor

$$\mathrm{HoLink} \colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$$

built from the simplicial mapping space functors $\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, -)$ admits a left adjoint, given by the coend formula

$$D \mapsto \int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes D_{\mathcal{I}}.$$

(See [RV13], for a short introduction to coend calculus). For the purpose of this introduction, knowing that $\mathrm{s}\mathbf{Strat}_P(\Delta^{\mathcal{I}}, -)$ admits a left adjoint will suffice. [Dou21a] already showed that this adjunction defines a Quillen equivalence between $\mathrm{s}\mathbf{Strat}_P$ and the projective model structure on $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ (this also holds for the injective model structure, see Recollection 5.2.2.1). More than this, in Chapter 3, we also show the following (See Theorem 3.1.0.3 in Chapter 3).

Main Result B. *The adjunction*

$$\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes - \colon \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet}) \rightleftarrows \mathrm{s}\mathbf{Strat}_P \colon \mathrm{HoLink}$$

of tying homotopy theory to 1-categorical or geometrical phenomena. We hope both takes on the topic can be useful to audiences with different preferences.

[9] In the context of this whole thesis, this is, in a sense, a preliminary result. In fact, we will discuss a significantly stronger version in Section 1.3. As to not overstate our number of results, we have chosen to enumerate this result and its derivatives in the form A_i, indicating that they build on the same mathematical insights.

defines a homotopy equivalence of the categories with weak equivalences $\mathbf{s}\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ *and* $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ *(where the latter is equipped with pointwise weak homotopy equivalences).*

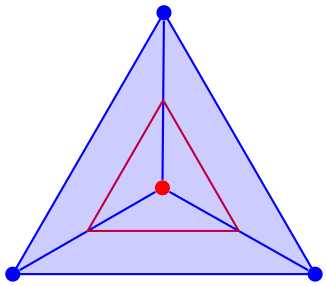
The crucial new fact here is that the characterization for weak equivalences in $\mathrm{s}\mathbf{Strat}_P$ in Theorem 1.2.3.14 also holds for stratified simplicial sets that are not fibrant. In other words, the functors $\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, -)$ always compute the *correct* ∞-categorical mapping spaces. In fact, this result is also centrally used in the proof of Main Result A_1.

Looking at the way $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ are defined, as well as the characterization of weak equivalence in $\mathrm{s}\mathbf{Strat}_P$ in Main Result B, it is not too surprising that these results ultimately rely on having as good as possible of an understanding of the homotopy links of stratified simplicial sets. They make use of several equivalent models for the homotopy links of a stratified simplicial set $\mathcal{X}$, each with their own technical or intuitive advantages. Let us survey a few of these. Let $\mathcal{I} \in \mathrm{sd}(P)$ be a regular flag. One may then consider:

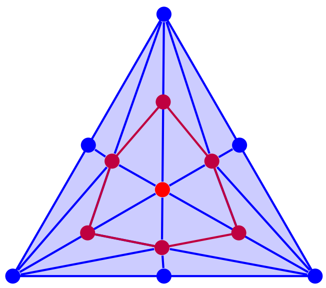
(D1) The *derived simplicial homotopy link* obtained by fibrantly replacing a stratified simplicial set $\mathcal{X}$, and then computing simplicial homotopy links.

(D2) The underived simplicial homotopy link $\mathrm{HoLink}_{\mathcal{I}}(\mathcal{X})$;

(D3) The geometric link (illustrated in purple directly below) given by taking the inverse image of the barycenter of $|\Delta^{\mathcal{I}}|_s \subset |\mathrm{N}(P)|_s$ under the realization of the stratification map $|X| \to |\mathrm{N}(P)|$.



(D4) The combinatorial link (illustrated in purple directly below), given by taking the inverse image in $\mathbf{sSet}$ of the vertex corresponding to the barycenter of $\Delta^{\mathcal{I}}$ under the first barycentric subdivision of the stratification map, $\mathrm{sd}X \to \mathrm{sd}\mathrm{N}(P)$.



In the case of two strata, $\mathcal{I} = \{p < q\}$, and a stratified simplicial complex, this is the classical construction for the boundary of a regular neighborhood of $X_p \subset X$.

(D5) The topological homotopy link $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|\mathcal{X}|_s)$.

In Chapter 3, we obtain the following result concerning these links (see Theorem 3.1.0.4), which may be phrased somewhat sloppily as follows:

Main Result C. *All of the definitions for homotopy links of stratified simplicial sets in (D1) to (D5) define the same homotopy type in* $\mathcal{S}$**paces** *(and this identification happens in a sufficiently natural way).*

This can be seen as a first step towards addressing Obvervation (O2). To just name one advantage of the combinatorial models for generalized homotopy links in (D4): It defines a left adjoint functor that preserves monomorphisms. This allows one to investigate its properties in a cell-by-cell approach, significantly simplifying many proofs. Main Result C can be seen as the first instantiation of a general paradigm of proof in stratified homotopy theory which we will apply all over this thesis: Ultimately, it is in the nature of the homotopy theories of stratified spaces we discuss in this thesis that many general statements can be reduced to an equivalent statement about homotopy links. One can then often construct models for these homotopy links that are specifically adapted to studying the statement in question.

1.2.5 Results: An application of stratified homotopy theory to stratified topological data analysis

A first interesting consequence of Corollary 1.2.4.3 and Theorem 1.2.1.24 is that one obtains a fully faithful embedding

$$\mathcal{C}\mathbf{on}_P \xhookrightarrow{\mathcal{H}o\mathrm{Link}} \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$$

of the homotopy theory defined by localizing triangulable conically stratified spaces over a fixed poset P at stratified homotopy equivalences into the homotopy theory of $\mathcal{S}\mathbf{paces}$ valued diagrams indexed over $\mathrm{sd}(P)^{\mathrm{op}}$. In [MW24], which is joint work with Tim Mäder and another article we present in this thesis, we leveraged this insight for the purpose of stratified topological data analysis. These results are presented in Chapter 4.

Chapter 4 is itself equipped with a detailed introduction and we refer the reader there for details. Let us just explain the core ideas of [MW24]. Considering the way the remainder of this chapter was written, we expect the possible reader to most likely come from an algebraic topology or homotopy theory background rather than from primarily the topological data analysis (TDA) side of things. We will thus phrase things in the language of homotopy theory and refer to Chapter 4 for a more TDA oriented account. We also refer to Chapter 4 for a list of relevant references.

Probably the most successful tool of topological data analysis is what is called *persistent homology*. Roughly speaking, the goal is to infer homological information from a data set, given, for example, in the form of a point cloud in some Euclidean space or a finite metric space. From an abstract, homotopy-theoretic point of view, these constructions can usually be interpreted as a two-step process. Beginning from finite point cloud data, one constructs a *persistent homotopy type*, which is just a functor from the category given by the poset of non-negative reals $\mathbb{R}_+$ into the ∞-category of spaces $\mathcal{S}\mathbf{paces}$, i.e., an element of the functor category $\mathcal{S}\mathbf{paces}^{\mathbb{R}_+}$. For example, one can associate to a point cloud $\mathbb{X}$ in $\mathbb{R}$ the functor given by $\varepsilon \mapsto \mathbb{X}_\varepsilon$, where $\mathbb{X}_\varepsilon$ denotes a closed thickening of $\mathbb{X}$ by ε, and the relation $\varepsilon \leq \varepsilon'$ is mapped to the obvious inclusion $\mathbb{X}_\varepsilon \subset \mathbb{X}_{\varepsilon'}$. In a second step, one then computes homology of the resulting spaces (usually with field coefficients) in some fixed dimension, thus obtaining a functor from $\mathbb{R}_+$ into vector spaces over some field k. Such an object is called a *persistence module*. It turns out that from the perspective of application, persistent homology has surprisingly favorable properties, being computable, interpretable, and admitting inference and stability results (see the introduction in Chapter 4 for detailed references). All in all, these results allow one to stably infer homological information from discrete point samples. Many of these properties are, from a theoretical point of view, not really properties only appearing at the homology level, but already properties of the persistent homotopy type constructions.

In practice, one often encounters data sets which are inherently singular, or come with some decomposition into several subtypes. Based on this observation, several authors have suggested the use of invariants of stratified spaces (such as intersection homology) for the purpose of topological data analysis. Naturally, these approaches first attempt to construct a *stratified persistent homotopy type*, in the sense that they refer to the homotopy theory obtained by localizing stratified homotopy equivalences. Usually, these stratified homotopy types are presented in terms of stratified simplicial complexes (i.e., stratified simplicial sets). There is a general issue with these approaches, which many of them share: namely the fact that stratified homotopy equivalences are generally too rigid to deal with the approximate nature and thickening construction inherent to any application. For example, in application, one will generally only be able to recover the singular stratum up to a controlled thickening, which, as we have seen in Example 1.1.2.5, will almost always change the stratified homotopy type (see Example 1.1.2.5). In Chapter 4, we thus suggest an alternative approach, based on the diagrammatic homotopy theory $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$. By passing to the weaker notion of diagrammatic equivalence, many of the rigidity problems associated to stratified homotopy equivalences can be circumvented. Furthermore, the equivalence of ∞-categories in Theorem 1.2.1.24, allows one to present persistent diagrammatic stratified homotopy types in terms of diagram $\mathbb{R}_+ \times \mathrm{sd}(P)^{\mathrm{op}} \to \mathcal{S}\mathbf{paces}$. It follows from the fully faithful embedding of conically stratified spaces into $\mathbf{Fun}(\mathrm{sd}(P)^{op}, \mathcal{S}\mathbf{paces})$ that as long as one is interested in recovering stratified homotopic information from geometrical examples such as Whitney stratified spaces, this approach essentially loses no stratified homotopic information.

In Chapter 4, we describe an explicit construction of persistent stratified homotopy types in terms of stratification diagrams in the two strata scenario and prove that, at least as long as one focuses on recovering information from compact Whitney stratified spaces, it has many of the favorable properties of the unstratified persistent homotopy types (see Main Result D in Chapter 4). We furthermore prove an inference theorem that gives theoretical conditions under which stratified homotopy theoretic information can be recovered from a non-stratified point sample (see Main Result E in Chapter 4).

1.2.6 The non-existence of *good* topological model structures

Let us end our treatment of the diagrammatic approach to stratified homotopy theory for now, by returning to the question of model categories. Corollary 3.5.2.4 very strongly points towards the existence of a model structure as we hoped for in Requirement (R3), more specifically, one transferred from the model structure on $\mathrm{s}\mathbf{Strat}_P$ in which a stratified space $\mathcal{X}$ is fibrant, if and only if $\mathrm{Sing}_s(\mathcal{X})$ is fibrant in $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ (see Section 1.3, for details on the transfer of model structures). Provided the existence of such a model structure, it would allow for a significant generalization of Corollary 1.2.4.3, also to the case of more general cell complexes and their retracts, instead of restricting to the triangulable case[10]. A related model structure was conjectured by Stephen Nand-Lal in [Nan19], but could ultimately only be proven after restricting to certain *fibrant* stratified spaces which are not closed under colimits, and hence do not form a model category. In fact, in joint work with Douteau, we have shown that no such model structure can exist (see Proposition 3.A.0.1 in Chapter 3).

Proposition 1.2.6.1. *There does not exist a model structure on* $\mathbf{Strat}$ *or* $\mathbf{Strat}_P$ *(for P not a discrete poset) for which every diagrammatic equivalence is a weak equivalence and the realizations of stratified boundary inclusions,* $|\partial\Delta^{\mathcal{J}}|_s \hookrightarrow |\Delta^{\mathcal{J}}|_s$ *are cofibrations.*

At first glance, this observation may be seen as a natural end point to the pursuit of model structures connecting the stratified homotopy theory with classical geometrical examples. We will return to this question in the next section, however, where we will explain that after a slight weakening of the axioms of a model category, namely passing to so-called *(left) semi-model*

[10] Generally, not even topological manifolds admit triangulations (see [Man16] for an overview).

categories (which has essentially no impacts on their usefulness) the existence of such structures can be proven.

1.3 The exit-path category approach

In the previous section, we have seen that if one takes an inductive approach to stratified homotopy theory and thinks of stratified homotopy types as inductively decomposable into strata and links, then one naturally ends up with the diagrammatic notion of equivalence described in Definition 1.2.1.17. In this sense, this is the minimal class of equivalences one ends up with, if one takes the perspective that strata and stratified homotopy links should detect weak equivalences. There is, however, a second perspective to stratified homotopy theory which arises from a foundational paradigm of much of modern higher category theory:

> *Assigning to a space its ∞-groupoid of paths induces an equivalence between the homotopy theory of spaces and the homotopy theory of ∞-groupoids.*

This paradigm is often referred to as Grothendieck's homotopy hypothesis (see [nLa24h], for an overview). One rigorous interpretation of this paradigm in terms of a theorem is the Kan-Quillen equivalence between topological spaces and simplicial sets (Theorem 1.2.4.1). Indeed, if we pass to the associated ∞-categories of bifibrant objects in the model categories in Theorem 1.2.4.1, then on the side of topological spaces we obtain the homotopy theory of retracts of topological cell complexes (a dense subcategory of which is given by CW-complexes) and on the simplicial side we obtain the homotopy theory of Kan complexes. The latter are precisely the quasi-categories (∞-categories) in which every morphism is an isomorphism – i.e., the ∞-groupoids – and the inclusion of Kan complexes into quasi-categories defines a fully faithful embedding of homotopy theories. Hence, if we denote by **CW** the category of CW-complexes, by **Kan** the (1-)category of Kan complexes and by H the respective classes of homotopy equivalences, then the functor of singular simplices induces an equivalence of ∞-categories

$$\Pi_\infty \colon \mathbf{CW}[H^{-1}] \simeq \mathcal{S}\mathbf{paces} \xrightarrow{\simeq} \mathbf{Kan}[H^{-1}] = \mathcal{G}\mathbf{rpd}_\infty,$$

with the thus obtained ∞-category of (small) ∞-groupoids, $\mathcal{G}\mathbf{rpd}_\infty$. From this perspective, $\mathrm{Sing}_s(T)$ can be seen as a model for the ∞-groupoid of paths $\Pi_\infty(T)$ of a space T.

A more colloquial version of the homotopy hypothesis can be phrased as follows:

> *From a homotopy theoretical perspective, a space is the same thing as an ∞-category in which every morphism is invertible.*

Syntactically, one may infer from this statement the following (naive) question:

> *What kind of ∞-category is a stratified space?*

It turns out that this question can also be made sense of from a rigorous mathematical perspective.

1.3.1 The topological stratified homotopy hypothesis

Let us first summarize several historical insights into this question in not necessarily historical order. To compare stratified spaces with (∞-)categories, one first needs an analogue of the fundamental (∞-)groupoid.

1 **and** 2**-categories of exit-paths:**

The idea that stratified spaces should have associated to them a *fundamental category* goes back to MacPherson (in unpublished work). It arises as follows: One of the core objects of study in classical algebraic topology are locally constant sheaves on a space X, or *local coefficient systems*, as the derived global sections of such objects give rise to cohomology with respect to local coefficients. Given a space X, and some category (1-category, for now) of coefficients $\mathbf{C}$ (let us say sets, abelian groups or modules over some fixed commutative ring R), we will denote by $\mathbf{Shv}^{\mathrm{loc}}_{\mathbf{C}}(X)$ the category of locally constant sheaves on X. For sufficiently regular topological spaces, let us say CW-complexes, there is an equivalence of categories, called the *monodromy correspondence*

$$\mathbf{Shv}^{\mathrm{loc}}_{\mathbf{C}}(X) \simeq \mathbf{Fun}(\Pi_1(X), \mathbf{C}),$$

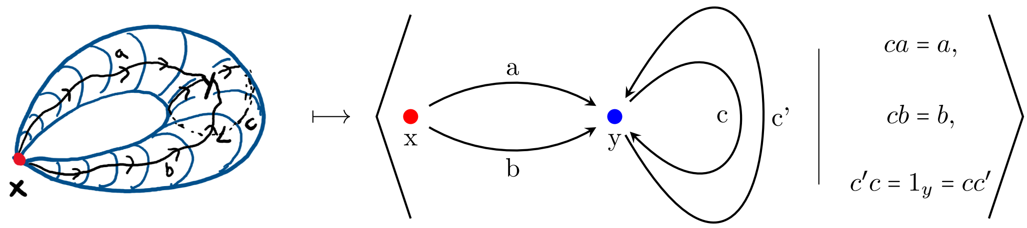
where $\Pi_1(X)$ denotes the fundamental groupoid of X. This equivalence is given by associating to a locally constant sheaf $\mathcal{F}$ on X a functor on $\Pi_1(X)$ mapping a point $x \in X$ to the stalk $\mathcal{F}_x$ at x and a path $x \to y$ to the monodromy action of this path on stalks. Most prominently, in the case of $\mathbf{C} = \mathbf{Set}$, where one can identify locally constant sheaves with covering spaces, this is just the classical monodromy correspondence for covering spaces (see, for example, [May99]). If we take $\mathbf{C} = \mathbf{Vec}_k$ the category of vector spaces over some field k, for example, and assume that X is path connected, then we may take this statement as saying that locally constant sheaves (with vector space coefficients) are the same things as presentations of the fundamental group of X, allowing for an extremely concise and well understood definition of locally constant sheaves.
If one is interested in studying cohomology theories of stratified spaces $\mathcal{X}$ – such as Goresky and MacPherson's *intersection (co)homology* (see [GM83] and [Ban07] for an overview) – then the role of locally constant sheaves is taken by *constructible sheaves*, i.e., such sheaves that become locally constant after restricting to the strata of $\mathcal{X}$[11]. Let us denote the category of such objects by $\mathbf{Shv}^{\mathrm{con}}_{\mathbf{C}}(\mathcal{X})$. MacPherson was interested in exposing a generalization of the monodromy correspondence for the case of stratified spaces, that is, to expose a notion of a *fundamental category* $\Pi_1(\mathcal{X})$ associated to a stratified space $\mathcal{X}$, such that one obtains an equivalence of categories

$$\mathbf{Shv}^{\mathrm{con}}_{\mathbf{C}}(\mathcal{X}) \simeq \mathbf{Fun}(\Pi_1(\mathcal{X}), \mathbf{C}). \tag{1.9}$$

In unpublished work, MacPherson constructed such a fundamental category $\Pi_1(\mathcal{X})$ for a class of conically stratified spaces with manifold strata (more precisely, topological pseudomanifolds that are allowed to have non-empty codimension one strata, see [Tre09]). Objects in this category are the points in $\mathcal{X}$, while morphisms are given by paths that only ascend in the stratification – so-called *exit-paths* – up to an appropriate stratified notion of homotopy.

Example 1.3.1.1. Consider the pinched torus (illustrated to the left, below). It is constructed by collapsing a meridian in a torus to a point, and taking it as the singular stratum in a stratification over $\{p_0 < p_2\}$. The associated category of exit-paths (fundamental category) is given, up to equivalence of categories, by the category defined by the generators and relations on the right-hand side.

[11] To be precise here, if one wants to think of intersection (co)homology to arise from an object in this category, one should really either think of sheaves in the ∞-categorical sense (see [Lur09; Lur17]), valued in some stable ∞-category, or of constructible objects in the classical derived sheaf categories, but we will return to this point in a minute.



For his class of *topological stratified spaces*, MacPherson proved an equivalence as in Eq. (1.9) for the case $\mathbf{C} = \mathbf{Set}$. The shift from loops (in fundamental groups) to exit-paths led to the name *exodromy correspondence*, for correspondences such as the one in Eq. (1.9) (coined in [BGH18]).

In [Tre09], Treuman constructed a 2-category $\Pi_2(\mathcal{X})$ which generalized the exodromy correspondence equivalence to constructible stacks instead of 1-categorical sheaves (taking $\mathbf{C}$ to be the $(2,1)$-category of categories). Furthermore, in [Woo09] (using a slightly different construction of the exit-path 1-category) Woolf generalized the case $\mathbf{C} = \mathbf{Set}$ to those of Quinn's homotopically stratified spaces (see [Qui88] and below for details), whose strata fulfill the usual local connectedness assumptions necessary for monodromy correspondences (this used a slightly different construction of a fundamental category, see [Woo09]).

∞-categories of Exit-paths:

Unlike the classical local coefficient systems of algebraic topology, the *constructible* coefficient systems defining intersection (co)homology cannot be expressed entirely in the 1-categorical language of constructible sheaves. Instead, one needs to either pass to derived categories of sheaves or (equivalently, see [Lur17]), allow for sheaves with coefficients in $(\infty,1)$-categories. More specifically, one wants versions of Eq. (1.9) for the case of coefficients in the stable derived ∞-category, $\mathcal{D}(R)$, of chain complexes of R-modules, where R is a field, or some sufficiently well-behaved ring. If one is looking to obtain an exodromy correspondence for such higher sheaves, then evidently a higher version of the fundamental groupoid $\Pi_1(\mathcal{X})$ is needed. Treuman's construction of a 2-category and generalization to stacks (i.e., sheaves valued in groupoids) already pointed towards the fact that such a thing should be possible. In [Lur17, A.6], Lurie proved that for a conically stratified space $\mathcal{X}$ the underlying simplicial set of $\mathrm{Sing}_s(\mathcal{X})$ is a quasi-category. The objects of this category are simply points in x. Its morphisms are given by such paths in $\mathcal{X}$ which either remain in a stratum or immediately exit into another stratum, i.e., by (particular) exit-paths. In this sense, the resulting ∞-category is very similar to the 1-categories constructed by MacPherson, Treuman, and Woolf. In fact, using the composition laws of a quasi-category, one can think of any ascending path in $\mathcal{X}$ crossing finitely many strata as a 1-simplex of $\mathrm{Sing}_s(\mathcal{X})$. Furthermore, Lurie proved that for a sufficiently regular space X (let us say a CW-complex, for the sake of simplicity) equipped with a stratification over a poset **Pos** that has no infinitely ascending chains, and such that the resulting stratified space $\mathcal{X}$ is conically stratified, there is an exodromy correspondence

$$\mathbf{Shv}^{\mathrm{con}}_{\mathcal{S}\mathbf{paces}}(\mathcal{X}) \simeq \mathbf{Fun}(\mathrm{Sing}_s(\mathcal{X}), \mathcal{S}\mathbf{paces}).$$

This correspondence was recently generalized by Haine, Porta and Teyssier in [HPT24] to significantly larger classes of stratified spaces (including those for which $\mathrm{Sing}_s(\mathcal{X})$ is not a quasi-category) and more general coefficients, allowing for stable ∞-categories such as $\mathcal{D}(R)$, which makes it a possible setting to study intersection (co)homology in.

The ∞-categorical exodromy correspondence pointed towards an alternative approach to stratified homotopy theory than the one taken in Section 1.2. Namely, observe that in the trivially stratified case (i.e., the case where $P_{\mathcal{X}}$ is a point) the quasi-category $\mathrm{Sing}_s(\mathcal{X})$ is the usual singular simplicial set, or, in other words, the fundamental ∞-groupoid associated to X, also denoted $\Pi_\infty(X)$. It follows from the classical Kan-Quillen equivalence that a map of topological spaces $w: X \to Y$ is a weak homotopy equivalence if and only if it induces an equivalence $\Pi_\infty(X) \to \Pi_\infty(Y)$ on ∞-groupoids. This suggested an alternative approach to defining weak equivalences of stratified topological spaces, namely to transfer the notion of weak equivalence along a fundamental ∞-category construction (see [Nan19], to which we will return later). Another series of insights that pointed in the direction that one may want to localize a different class of weak equivalences than diagrammatic equivalences was the following result of David Miller.

Quinn's homotopically stratified spaces and Miller's theorem:

In [Qui88], Quinn defined a class of stratified spaces which we will refer to as *homotopically stratified*. Roughly speaking, these were metrizable poset-stratified spaces for which the starting point evaluation maps $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to \mathcal{X}_p$ are Hurewicz fibrations and that additionally fulfill a cofibrancy condition relating to the inclusions $X_p \hookrightarrow X_p \cup X_q$, for $p < q$. For example, all Whitney stratified spaces and topological pseudomanifolds belong to this class (see, for example, [Rab22, Ex. 1.4.8]). Having in mind the way the iterative definition of diagrammatic equivalences in Section 1.2.1 also required the study of the homotopy theoretic interactions of more than two strata, it may at first glance be surprising that Quinn was able to obtain a well-working homotopy theory by purely focusing on two strata interactions. In fact, in [Mil13], Miller proved that a stratum-preserving map between homotopically stratified spaces is a stratified homotopy equivalence if and only if it induces (not weak but ordinary) homotopy equivalences on strata and pairwise links. [12] There is a conceptual reason for the fact that in Quinn's setting there was no need to consider generalized homotopy links.

Example 1.3.1.2. Let us explain why this is the case, for the special case of a triangulable conically stratified space (i.e., a Whitney stratified space or a PL pseudo manifold). A significantly more general statement holds, but requires the development of results not presented here yet. The following observation was made in [Hai23; Nan19].
Suppose that $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, is a stratified simplicial set, such that the underlying simplicial set is a quasi-category and such that all of the (simplicial) strata X_p, for $p \in P$, are Kan complexes. Then the associated (simplicial) homotopy link diagram has the property that the homotopy types of generalized links $\mathcal{I}$ are entirely determined by the strata and the pairwise homotopy links (and the maps between them). This is for the following reason:
Given a regular flag $\mathcal{I} = [p_0 < \cdots < p_n] \in \mathrm{sd}(P)$, denote by $\mathrm{Sp}(\mathcal{I}) \subset \Delta^{\mathcal{I}}$ the stratified simplicial set whose underlying simplicial set is the spine of Δ^n, i.e., the union of the edges $0 \to 1, 1 \to 2 \dots, n-1 \to n$. It follows by the universal property of the colimit that there is a

[12] In fact, this theorem was the origin point for Douteau's diagrammatic approach to stratified homotopy theory.

canonical isomorphism of simplicial sets

$$\begin{aligned}&\underline{\mathrm{s}\mathbf{Strat}}_P(\mathrm{Sp}(\mathcal{I}),\mathcal{X}) \cong \\ &\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\{p_0<p_1\}},\mathcal{X}) \times_{\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\{p_1\}},\mathcal{X})} \cdots \times_{\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\{p_n\}},\mathcal{X})} \underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\{p_{n-1}<p_n\}},\mathcal{X}).\end{aligned}$$

Under the assumption above, it is not hard to see that $\mathcal{X}$ is fibrant in the Douteau-Henriques model structure. In particular $\underline{\mathrm{s}\mathbf{Strat}}_P(\mathrm{Sp}(\mathcal{I}),\mathcal{X})$ is a Kan complex, and the iterated pullback we just described is even a homotopy limit. We have already explained in Section 1.2.4 that the simplicial sets $\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}},\mathcal{X})$ are simplicial analogues of the generalized homotopy links. Hence, we obtain an natural equivalence

$$\underline{\mathrm{s}\mathbf{Strat}}_P(\mathrm{Sp}(\mathcal{I}),\mathcal{X}) \simeq \mathrm{HoLink}_{\{p_0<p_1\}}(\mathcal{X}) \times^h_{X_{p_1}} \cdots \times^h_{X_{p_{n-1}}} \mathrm{HoLink}_{\{p_{n-1}<p_n\}}(\mathcal{X}),$$

where the $\times^h$ indicates a homotopy pullback. The inclusion $\mathrm{Sp}(\mathcal{I}) \hookrightarrow \Delta^{\mathcal{I}}$ has as its underlying simplicial map a Joyal equivalence (i.e., such a simplicial map that induces equivalences of quasi-categories, after fibrantly replacing by a quasi-category, see [Lur09]). As the strata of $\mathcal{X}$ are Kan complexes, it follows that the stratification map $s_{\mathcal{X}}: X \to P$ can be seen as a conservative functor of quasi-categories (i.e., a functor that creates isomorphisms). It follows from this, and the fact that P is a 1-category, that $s_{\mathcal{X}}: X \to P$ is a fibration in the Joyal model structure for quasi-categories. Combining this information, a standard argument (more specifically the fact that the Joyal model structure is cartesian, see [Lur09])) shows that the induced map of simplicial sets

$$\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}},\mathcal{X}) \to \underline{\mathrm{s}\mathbf{Strat}}_P(\mathrm{Sp}(\mathcal{I}),\mathcal{X})$$

is a Joyal equivalence and in particular a homotopy equivalence of Kan complexes. Consequently, it follows that the natural map

$$\mathrm{HoLink}_{\mathcal{I}}(\mathcal{X}) = \underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}},\mathcal{X}) \to \mathrm{HoLink}_{\{p_0<p_1\}}(\mathcal{X}) \times^h_{X_{p_1}} \cdots \times^h_{X_{p_{n-1}}} \mathrm{HoLink}_{\{p_{n-1}<p_n\}}(\mathcal{X}) \tag{1.10}$$

associated to the simplicial homotopy link diagram of $\mathcal{X}$ is a homotopy equivalence of Kan complexes. As a consequence, we obtain that a stratified map of two such simplicial sets $f: \mathcal{X} \to \mathcal{Y}$ is a weak equivalence in $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ if an only if it induces homotopy equivalences on strata and pairwise simplicial homotopy links.
Now, if $\mathcal{W}$ and $\mathcal{W}'$ are triangulable conically stratified spaces over P, then it follows by Douteau's Whitehead theorem (see [Dou21c, Thm. 4.23]) or by Corollary 1.2.4.3 that a map between them is a stratified homotopy equivalence, if and only if it is a diagrammatic equivalence, i.e., if it induces weak equivalences on all generalized homotopy links. Taking singular simplicial sets, we may identify the (topological) generalized homotopy links of $\mathcal{W}$ and $\mathcal{W}'$ with the simplicial generalized homotopy links of $\mathrm{Sing}_s(\mathcal{W})$ and $\mathrm{Sing}_s(\mathcal{W}')$. By Lurie's result on conically stratified spaces, it follows that $\mathrm{Sing}_s(\mathcal{W})$ and $\mathrm{Sing}_s(\mathcal{W}')$ fall into the class of stratified simplicial sets we have discussed in the previous part of this example. Consequently, it follows that any map $\mathrm{Sing}_s(\mathcal{W}) \to \mathrm{Sing}_s(\mathcal{W}')$ inducing weak equivalences on strata and pairwise homotopy links also induces weak equivalences on *all generalized homotopy links*. This proves (the non-trivial direction of) Miller's theorem for the special case of triangulable, conically stratified spaces (or more generally such triangulable stratified spaces for which $\mathrm{Sing}_s(\mathcal{X})$ is a quasi-category).

Diagrams $D \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ that fulfill the condition that the associated maps into the iterated (homotopy) pullbacks

$$D_{\mathcal{I}} \to D_{\{p_0<p_1\}} \times_{D_{p_1}} \cdots \times_{D_{p_{n-1}}} D_{\{p_{n-1}<p_n\}}$$

are isomorphisms in $\mathcal{S}\mathbf{paces}$ (i.e., arise from weak homotopy equivalences) were named *décollages* in [BGH18]. Readers familiar with the theory of (complete) Segal spaces will recognize this condition as very akin to one of the defining conditions of a complete Segal space (as introduced by Rezk in [Rez01]), another model for the theory of $(\infty, 1)$-categories. (See [Ber07b], for an overview of some of the most frequently used models). It provides another hint at the strong connection of stratified spaces and their invariants with ∞-categories. For stratified spaces as in Example 1.3.1.2, the equivalence in Eq. (1.10) gives us another way of thinking of generalized homotopy links associated to flags $\mathcal{I} = [p_0 < \cdots < p_n]$, with $n \geq 2$. Namely, an element of the generalized homotopy link $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$ can be seen as a composable sequence of exit-paths $\gamma_1, \dots, \gamma_n$ starting in the p_0-stratum of $\mathcal{X}$ and ending in the p_n stratum.

In fact, in [Mil13], Miller also constructed a kind of higher exit-path category *internal to Hurewicz homotopy theory* associated to a homotopically stratified space and proved an equivalence between a homotopy category of certain such categories and the category of stratified homotopy classes of homotopically stratified spaces. In a sense, by considering categories internal to topological spaces, Miller is performing some type of higher category theory, but using the Hurewicz model structure on spaces instead of the Serre model structure. This deviation from the standard approach to higher category theory makes it difficult to rigorously interpret Miller's result in the language we use here, but nevertheless Miller's theorem can be seen as a first instantiation of the paradigm that the homotopy theory of stratified spaces should be the same as the homotopy theory of certain higher categories.[13] We will henceforth refer to this paradigm as the *stratified homotopy hypothesis*, a name first popularized in [AFR19].

Conically smooth stratified spaces and the stratified homotopy hypothesis:

In [AFR19], Ayala, Francis and Rozenblyum studied the homotopy theory associated to a smooth notion of stratified space, so-called conically smooth stratified spaces. They asserted a fully faithful embedding of their theory of conically smooth stratified spaces into the homotopy theory of ∞-categories.[14] This is also the first explicit mention of the *topological stratified homotopy hypothesis*, which refers to the following conjecture. In the following, we denote by $\mathcal{C}\mathbf{at}_\infty$ the ∞-category of small quasi-categories, obtained by localizing the 1-category of small quasi-categories at equivalences of quasi-categories.

Conjecture 1.3.1.3 ([AFR19])**.** Topological exit-paths define a fully faithful functor

$$\mathbf{Exit}\colon \mathcal{S}\mathbf{trat} \hookrightarrow \mathcal{C}\mathbf{at}_\infty$$

from a homotopy theory of topological stratified spaces $\mathcal{S}\mathbf{trat}$ into ∞-categories, $\mathcal{C}\mathbf{at}_\infty$.

This statement is, of course, only a precise conjecture after one has made a choice of what the homotopy theory of topological stratified spaces should be. In fact, this situation is somewhat reminiscent of the situation of the classical homotopy hypothesis, which, when phrased by Grothendieck, pertained to a fairly different incarnation of ∞-groupoids than the now common model of ∞-groupoids in terms of Kan complexes (see [Gro21]). That the Kan-Quillen equivalence between spaces and simplicial sets (Theorem 1.2.4.1, which actually predates Grothendieck's conjecture) is now frequently taken as an answer to Grothendieck's hypothesis is mainly due to the fact that the work of Joyal and Lurie has demonstrated that quasi-categories present a powerful model for higher category theory, and that Kan complexes are exactly quasi-categories in which every morphism is an isomorphism. In the stratified situation, one instead needs to expose a homotopy theory of topological stratified spaces, and justify that this theory provides a good framework to perform stratified homotopy theory in.

1.3.2 Haine's proof of a topological stratified homotopy hypothesis

A first rigorous interpretation of the topological stratified homotopy hypothesis that fits into the commonly accepted frameworks for $(\infty, 1)$-categories was given by Haine in [Hai23]. The

[13] This result seems to be somewhat overlooked in the more recent literature. This may be due to the non-standard approach to higher categories taken in [Mil13].

[14] [AFR19] lacks a precise definition of the category of objects that is considered, which makes us unable to verify the correctness of the statement.

crucial intermediary results to prove such a result made in [Hai23] are the following two:

- The full subcategory $\mathcal{D}\mathbf{\acute{e}c}_P \subset \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ given by décollages is equivalent to the ∞-category of conservative functors from a quasi-category X into P. [Hai23] calls such stratified simplicial sets $\mathcal{X} = (X, X \to P)$ *abstract stratified homotopy types* over P, and denotes the ∞-category of such objects by $\mathcal{AS}\mathbf{trat}_P$ (see [Hai23] or Chapter 5, for a detailed definition).

- $\mathcal{D}\mathbf{\acute{e}c}_P$ is a reflective subcategory of $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$, i.e., the inclusion functor $\mathcal{D}\mathbf{\acute{e}c}_P \hookrightarrow \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ admits a left adjoint. In particular, there exists a class of morphisms in $W \subset \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ such that there is an induced equivalence
$$\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})[W^{-1}] \simeq \mathcal{D}\mathbf{\acute{e}c}_P.$$

[Hai23] then combines this result with the equivalence in Theorem 1.2.1.24
$$\mathcal{H}o\mathrm{Link}\colon \mathcal{S}\mathbf{trat}_P^{\mathfrak{d}} \xrightarrow{\simeq} \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces}).$$

Purely abstractly, one obtains a class of weak equivalences $W_{P,\mathfrak{c}} \subset \mathbf{Strat}_P$, together with an equivalence
$$\mathbf{Strat}_P[W_{P,\mathfrak{c}}^{-1}] \simeq \mathcal{AS}\mathbf{trat}_P$$
(see [Hai23, Thm. 0.2.1]). If one takes *a homotopy theory of topological stratified spaces* to mean *some localization of the category of poset-stratified spaces* $\mathbf{Strat}$ and takes the homotopy link functor $\mathcal{H}o\mathrm{Link}\colon \mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ to be a *Segal-space-style* interpretation of the Exit-path ∞-category, then this provides a rigorous (fixed poset) answer to Conjecture 1.3.1.3. [Hai23] also asserts a version for the case of varying posets from this. We will return to this question later.

For now, let us take a more detailed look at the class of weak equivalences $W_{P,\mathfrak{c}}$ that lead to the equivalence above. By definition, $W_{P,\mathfrak{c}}$ is the class of such stratum-preserving maps $w\colon \mathcal{X} \to \mathcal{Y}$, for which the transformation of diagrams $\mathcal{H}o\mathrm{Link}(w)\colon \mathcal{H}o\mathrm{Link}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}(\mathcal{Y}) \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ becomes an equivalence in the left Bousfield localization of the injective (or projective) model structure that presents décollages (see [Hir03], for an overview over Bousfield localization). In other words, $w\colon \mathcal{X} \to \mathcal{Y}$ is in $W_{P,\mathfrak{c}}$, if and only if, for every décollage $D \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$, the induced map of mapping spaces
$$\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})(\mathcal{H}o\mathrm{Link}(\mathcal{Y}), D) \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})(\mathcal{H}o\mathrm{Link}(\mathcal{X}), D)$$
is a weak homotopy equivalence.

This is still not a particularly concrete description. To become a little bit more concrete, let us make explicit at least some facts that are immediate from this description, and can be found in [Hai23].

Proposition 1.3.2.1. *Every stratum-preserving diagrammatic equivalence is in* $W_{P,\mathfrak{c}}$. *If* $\mathcal{X}$ *and* $\mathcal{Y}$ *in* $\mathrm{s}\mathbf{Strat}_P$ *are such that* Sing_s *maps them to quasi-categories*[15] *(for example, because* $\mathcal{X}$ *and* $\mathcal{Y}$ *are conically stratified) then the following are equivalent:*

1. $w\colon \mathcal{X} \to \mathcal{Y} \in \mathrm{s}\mathbf{Strat}_P$ *is in* $W_{P,\mathfrak{c}}$*;*

2. *w is a diagrammatic equivalence;*

3. *w induces weak equivalences on strata and pairwise homotopy links;*

[15] In [Hai23] such stratified spaces were called *exodromic*. [Hai23] also contains a version of the stratified homotopy hypothesis concerned with exodromic stratified spaces where the equivalence is given by Lurie's exit-path construction. The proof given there is, however, incorrect. See Section 7.D for details. It can be fixed, using our results in Chapter 7.

4. $\mathrm{Sing}_s(w)$ *is an equivalence of quasi-categories;*

5. $\mathrm{Sing}_s(w)$ *is a stratum-preserving simplicial homotopy equivalence, i.e., is invertible up to homotopies with respect to the stratified cylinder* $- \otimes \Delta^1$.

There is an obvious class of stratum-preserving maps that also fulfills Proposition 1.3.2.1. Namely, the class of such stratum-preserving maps w, for which the underlying simplicial map of $\mathrm{Sing}_s(w) \to \mathrm{Sing}_s(w)$ is a Joyal equivalence (also called categorical equivalences), i.e., becomes an equivalence of quasi-categories, after replacing $\mathrm{Sing}_s(\mathcal{X})$ and $\mathrm{Sing}_s(\mathcal{Y})$ fibrantly by quasi-categories, in the Joyal-model structure on **sSet**. Conjecturally, this suggests the following more direct characterization of the class $W_{P,\mathfrak{c}}$.

Conjecture 1.3.2.2. A stratum-preserving map is in $W_{P,\mathfrak{c}}$ if and only if the associated underlying simplicial map of stratified singular simplicial sets $\mathrm{Sing}_s(\mathcal{X}) \to \mathrm{Sing}_s(\mathcal{Y})$ is a Joyal equivalence.

If it indeed were the case that $W_{\mathfrak{p},\mathfrak{c}}$ is the class of maps that induce isomorphism on posets and Joyal equivalences after applying Sing_s, then this would have the beneficial consequence that one can actually describe the equivalence in terms of the adjunction $|-|_s \dashv \mathrm{Sing}_s$ (i.e., Lurie's version of the Exit-path construction) instead of passing through décollages.

1.3.3 Results: Extending Main Result A_1 and Main Result B to the categorical approach

Let us now explain why Conjecture 1.3.2.2 does indeed hold. This uses new results of Chapter 5 in which we investigate and define several model structures for stratified homotopy using only simplicial methods. To obtain an answer to Conjecture 1.3.2.2, which gives conjectures a description of the class $W_{P,\mathfrak{c}}$ in the language of model categories, the obvious approach is to first present abstract stratified homotopy types and décollages in terms of model structures.

[Hai23] defined a simplicial model structure on stratified simplicial sets that is obtained by equipping the slice category $\mathbf{sSet}_{/\mathrm{N}(P)}$ with the structure induced by the Joyal model structure (see, for example, [Hir03]), and then (left Bousfield) localizing at the stratified cylinder $\mathcal{X} \mapsto \mathcal{X} \otimes \Delta^1$ (see [Hai23], for details). It is called the *Joyal-Kan* model structure, and the resulting simplicial model category is denoted $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$. Bifibrant objects in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$ are quasi-categories together with a conservative functor into P and weak equivalences between bifibrant objects are precisely such stratified simplicial maps whose underlying map is a Joyal equivalence. In fact, $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$ presents the ∞-category of abstract stratified homotopy types over P (see [Hai23] and Proposition 5.3.1.8 for a proof of the statement). It will turn out that this 1-categorical description can be used to answer Conjecture 1.3.2.2.

Throughout this subsection, we will use Theorem 1.2.4.1 to instead think of the ∞-category $\mathcal{S}\mathbf{paces}$ as being given by localizing simplicial sets at weak homotopy equivalences. In this spirit, we will present décollages in terms of diagrams $\mathrm{sd}(P)^{\mathrm{op}} \to \mathbf{sSet}$. When we treat generalized homotopy links as simplicial sets instead of topological spaces, this will mean that we implicitly have applied the singular simplicial set functor.

Construction 1.3.3.1. Given a subcomplex $K \subset \mathrm{N}(P)$, the simplicial homotopy link $\mathrm{HoLink}_{\mathcal{I}}(K) \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ is given by the unique diagram with value Δ^0, at $\mathcal{I}$, with $\Delta^{\mathcal{I}} \subset K$, and $\varnothing$ otherwise. Given a simplicial set S, we use the shorthand $K \otimes_D S$, to denote the tensoring $\mathrm{HoLink}_{\mathcal{I}}(K) \otimes S$, with respect to the (pointwise) simplicial product in $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$. We are mainly interested in the following three cases of subcomplexes $K \subset \mathrm{N}(P)$. Let $\mathcal{I} = [p_0 < \cdots < p_n] \in \mathrm{sd}(P)$ be a regular flag. We may then consider the following associated subcomplexes of $\mathrm{N}(P)$:

1. A simplex $\Delta^{\mathcal{I}} \subset \mathrm{N}(P)$, for some regular flag $\mathcal{I} \in \mathrm{sd}(P)$;

2. Horns $\Lambda^{\mathcal{I}}_k \subset \Delta^{\mathcal{I}} \subset \mathrm{N}(P)$;
3. Spines $\mathrm{Sp}(\mathcal{I}) \subset \Delta^{\mathcal{I}}$.

If one (simplicially) left Bousfield localizes the injective model structure on $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$, denoted $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{inj}}$, at the class of spine inclusions

$$\{\mathrm{Sp}(\mathcal{I}) \otimes_D \Delta^0 \hookrightarrow \Delta^{\mathcal{I}} \otimes_D \Delta^0 \mid \mathcal{I} \in \mathrm{sd}(P)\}$$

then one obtains a model structure that presents the ∞-category of décollages (this follows by Proposition 5.2.2.19 and [Lur09, Prop. 4.2.4.4]). We denote this simplicial model category by $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{dé}}$. The left Quillen functor $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{inj}} \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{dé}}$ given by the identity presents the localization functor from space-valued presheaves on $\mathrm{sd}(P)$ to décollages. In particular, it follows that Haine's class of weak equivalences $W_{P,\mathfrak{c}}$ is given precisely by such stratum-preserving maps $w \colon \mathcal{X} \to \mathcal{Y}$ in $\mathbf{Strat}_P$, for which $\mathcal{H}o\mathrm{Link}(w) \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ (thought of as a diagram of Kan complexes) is a weak equivalence in $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{dé}}$. One then has the following result, relating several of the ∞-categories discussed in [BGH18] from the model categorical perspective. It summarizes Theorem 5.2.2.20, Proposition 5.2.1.3, and Corollary 5.2.2.2, and provides an extension of Main Result B.

Main Result F. *There is a diagram of simplicial Quillen adjunctions (with left and right part trivially commutative)*

$$\begin{tikzcd} \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \underline{\mathbf{sSet}})^{\mathrm{inj}} \arrow[r, shift left] \arrow[d, shift right, "1"'] & \underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P \arrow[l, shift left, "\mathrm{HoLink}"] \arrow[d, shift right, "1"'] \\ \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \underline{\mathbf{sSet}})^{\mathrm{dé}} \arrow[r, shift left] \arrow[u, shift right] & \underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P \arrow[l, shift left, "\mathrm{HoLink}"] \arrow[u, shift right] \end{tikzcd} \tag{1.11}$$

with the following properties.

- *The downward pointing verticals are given by the left Bousfield localizations at respectively the set of arrows* $\{\Lambda^{\mathcal{I}}_k \otimes_D \star \hookrightarrow \Delta^{\mathcal{I}} \otimes_D \star \mid \mathcal{I} = [p_0 < \dots < p_n] \in \mathrm{sd}(P), 0 < k < n\}$ *on the left and the class of stratified inner horn inclusions on the right.*
- *Both horizontals are simplicial Quillen equivalences that create weak equivalences in both directions.*

The lower horizontal Quillen equivalence presents the equivalence of ∞-categories between décollages and abstract stratified homotopy types. The crucial insight compared to the purely ∞-categorical statement is that the horizontals create weak equivalences between all, and not just appropriately fibrant objects [16]. As a corollary of this result, we obtain the following characterization of $W_{P,\mathfrak{c}}$ (see Lemma 7.3.3.3).

Corollary 1.3.3.2. *A stratum-preserving map* $w \colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ *is in* $W_{P,\mathfrak{c}}$ *if and only if the associated stratum-preserving simplicial map* $\mathrm{Sing}_s(w) \in \mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$ *is a weak equivalence in* $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$.

In [Hai23] it was shown that a stratum-preserving simplicial map $w \colon \mathcal{X} \to \mathcal{Y}$ between stratified simplicial sets whose strata are Kan complexes is a weak equivalence in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$ if and only if its underlying simplicial map is a Joyal equivalence. As this requirement is always fulfilled for stratified singular simplicial sets, we obtain an affirmative answer to Conjecture 1.3.2.2.

Corollary 1.3.3.3. *A stratum-preserving map* $w \colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ *is in* $W_{P,\mathfrak{c}}$ *if and only if the underlying simplicial map of* $\mathrm{Sing}_s(w)$ *is a Joyal-equivalence.*

[16] This is, at least in our opinion, the part of the theorem that is the most difficult to prove and that was not known at the time of writing of [Hai23]. The case of the lower horizontal can ultimately be derived from the case of the upper horizontal, which we have proven in Chapter 3.

This corollary justifies the following nomenclature:

Definition 1.3.3.4. We will call maps in $W_{P,\mathfrak{c}}$ *stratum-preserving categorical equivalences* (over P). We denote by $\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}}_P$ the category with weak equivalences defined by equipping $\mathbf{Strat}_P$ with stratum-preserving categorical equivalences.

We may use this corollary, together with Main Result A_1, to obtain an alternative proof of Haine's fixed poset answer to the stratified homotopy hypothesis (see Theorem 7.3.3.1 in Chapter 7).

Main Result $\mathbf{A}_2$. *The adjunction*

$$|-|_s\colon \mathbf{s}\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}}_P \rightleftarrows \boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}}_P\colon \mathrm{Sing}_s$$

defines a homotopy equivalence of categories with weak equivalences.

The ∞-category $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ associated to $\mathbf{s}\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}}_P$ is equivalent to the ∞-category of abstract stratified homotopy types over P, $\mathcal{AS}\mathbf{trat}_P$. It follows that this equivalence presents Haine's equivalence between $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ and $\mathcal{AS}\mathbf{trat}_P$ in terms of a homotopy equivalence of categories with weak equivalences. Note that, in this version, there is no need to pass through décollages and the equivalence is directly given through Lurie's stratified realization and functor of singular simplices adjunction.

1.3.4 Results: Global combinatorial models for stratified homotopy theory

The obvious question after a global version of Main Result A_2 that is also closer to the original wording of the stratified homotopy hypothesis in Conjecture 1.3.1.3 arises. In [Hai23], the author also defined a global version of the ∞-category of abstract stratified homotopy types. Namely, denote by $\mathcal{AS}\mathbf{trat}$ the full subcategory of the ∞-category of arrows $\mathbf{Fun}(\Delta^1, \mathcal{C}\mathbf{at}_\infty)$ given by conservative functors whose target is a partially ordered set. [Hai23] also states a version of a stratified homotopy hypothesis for this setting.[17]

To obtain an analogue of Main Result A_2 for this setting, we pursue a similar approach to the case of a fixed poset in the previous section. Namely, we present the categorical side in terms of model structures on stratified simplicial sets. Procuring such model structures is the content of the second half of Chapter 5. These model structures will serve an additional purpose later on. Namely, they will actually allow us to define something very close to model structures on the topological side of stratified homotopy theory.

The first approach to such global model structures is to glue the model structures over varying posets together to a global model structure, using a technique of [CM20] that was also employed in [Dou21c]. Our results in this line of investigation obtained in Section 5.3.1 can be summarized as follows:

Theorem 1.3.4.1. *The following classes determine the structure of a combinatorial, simplicial, cartesian closed model category, denoted* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ *on the simplicial category* $\underline{\mathrm{s}\mathbf{Strat}}$.

- *Cofibrations are precisely such maps for which the underlying simplicial map is a monomorphism.*
- *Weak equivalences are given by such stratified maps $w\colon \mathcal{X} \to \mathcal{Y}$, that induce isomorphisms on the underlying posets, and weak equivalences in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_{P_\mathcal{X}}$ after identifying $P_\mathcal{X} \cong P_\mathcal{Y}$ under this isomorphism.*

[17] While the statement in [Hai23] is correct, there are some difficulties with the way the statement is derived. See Remark 7.D.0.1, for details.

- *Fibrations are given by precisely such stratified maps that have the right lifting property with respect to admissible stratified horn inclusions.*

The analogous claim (omitting the statement on fibrations) where weak equivalences are defined via the model categories $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$ *holds, and the resulting simplicial model category is denoted* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$. *Here, fibrations between fibrant objects are given by precisely such stratified maps that have the right lifting property with respect to admissible horn inclusions.* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ *is obtained from* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ *by left-Bousfield localizing stratified inner horn inclusions and presents the* ∞*-category of abstract stratified homotopy types* $\mathcal{AS}\mathbf{trat}$.

Definition 1.3.4.2. Following the nomenclature in the diagrammatic case, we call a stratified map $f\colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}$ that induces isomorphisms on posets and is a stratum-preserving categorical equivalence after identifying the posets along the isomorphism, a *poset-preserving categorical equivalence*. The category with weak equivalences obtained by equipping $\mathbf{Strat}$ with such stratified maps is denoted by $\boldsymbol{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$.

As a corollary of Main Result A_2, we obtain the following global version of the latter (see, again Theorem 7.3.3.1 in Chapter 7).

Main Result A_3. *The adjunction*

$$|-|_s\colon \mathrm{s}\boldsymbol{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}} \rightleftarrows \boldsymbol{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}\colon \mathrm{Sing}_s$$

defines a homotopy equivalence of categories with weak equivalences. The ∞*-category* $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ *associated to* $\mathrm{s}\boldsymbol{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ *is equivalent to the* ∞*-category of abstract stratified homotopy types,* $\mathcal{AS}\mathbf{trat}$.

1.3.5 Results: Refined stratified simplicial sets and layered ∞-categories

Observe that Main Result A_3 does not quite provide one with a version of the stratified homotopy hypothesis as in Conjecture 1.3.1.3. Namely, the objects on the left-hand side of the equivalence are given not by a class of ∞-categories, but by ∞-categories together with a conservative functor.

As already alluded to in [BGH18], a version where no additional datum of a functor needs to be specified can be obtained by localizing further. To see this, let us take a short look at what kind of object an abstract stratified homotopy type – i.e., a quasi-category with a conservative functor into a poset – is.

Remark 1.3.5.1. Let X be a quasi-category. Suppose that there is a conservative functor $s\colon X \to P$, for some poset P. As every endomorphism in a poset is the identity, and hence an isomorphism, it follows that every endomorphism of X, $x \to x$ maps to an isomorphism $s(x) \to s(x)$. Since s is conservative, it follows that $x \to x$ is an isomorphism.

A quasi-category that has the property that every endomorphism is an isomorphism is called *layered*. We denote by $\mathcal{L}\mathbf{ay}_\infty \subset \mathcal{C}\mathbf{at}_\infty$ the full subcategory of the quasi-category of small quasi-categories given by layered quasi-categories (see [BGH18]).

Remark 1.3.5.2. Observe that layered quasi-categories are the only quasi-categories we can ever expect to arise from any Exit-path construction. Indeed, any endomorphism in $\mathcal{X}$ that only ascends in stratification is necessarily a loop contained in a single stratum. Such a path will always have an inverse, obtained by its opposite parametrization.

Notation 1.3.5.3. Recall from Remark 1.2.3.2 that the nerve functor $\mathbf{Pos} \hookrightarrow \mathbf{sSet}$ has a left adjoint. Given a simplicial set $X \in \mathbf{sSet}$, we denote the resulting poset - generated by the vertices of X, subject to relations arising from 1-simplices - by $P(X)$.

Remark 1.3.5.4. One then obtains an adjunction of ∞-categories

$$\mathcal{L}\mathbf{ay}_\infty \rightleftharpoons \mathcal{AS}\mathbf{trat}$$

with the left adjoint given by mapping a layered ∞-category X to $X \to P(X)$ and the right adjoint given by forgetting the stratification poset. The unit of this adjunction is an isomorphism (it is the identity, really), allowing us to think of $\mathcal{L}\mathbf{ay}_\infty$ as a coreflective subcategory of $\mathcal{AS}\mathbf{trat}$. In particular, $\mathcal{L}\mathbf{ay}_\infty$ is given by the localization of $\mathcal{AS}\mathbf{trat}$ at the counits of adjunction

$$(X, X \to P(X)) \to (X, X \to P_{\mathcal{X}}).$$

Let us present this adjunction in the language of model categories. We note that the following results are not new, from a purely ∞-categorical perspective. On this level, they were already explained in [BGH18]. However, on the model categorical level, they will provide a useful tool in the next section, to connect topological stratified homotopy theory with geometrical examples of stratified spaces. This step requires a right Bousfield localization, which generally involves fairly different, and less standard, methods than left Bousfield localizations. In the following, we summarize the results of Section 5.3.2. Let us begin with some examples that illustrate the change in perspective from posets being extrinsic to intrinsic:

Example 1.3.5.5. Consider the simplicial set $\partial\Delta^1 = \{0\} \sqcup \{1\}$. We can equip the latter with the following three possible stratifications:

- We can stratify $\partial\Delta^1$ over the poset with no relations $\{0\} \sqcup \{0'\}$, by mapping $0 \mapsto 0$ and $1 \mapsto 0'$.
- We can stratify $\partial\Delta^1$ over $\{0 < 1\}$, by mapping $0 \mapsto 0$ and $1 \mapsto 1$.
- We can stratify $\partial\Delta^1$ over the poset with one element $[0]$ (in the only way possible).

All of these stratifications are natural, depending on the perspective one starts from. The first one is the one obtained by thinking of $\partial\Delta^1$ as the coproduct of the terminal object with itself. The second is obtained by equipping $\partial\Delta^1$ with the inherited stratification from the stratified simplex $\Delta^{[1]}$ over $[1]$. The third is obtained by treating $\partial\Delta^1$ as the coproduct of the terminal object with itself in the category $\mathrm{s}\mathbf{Strat}_{\{0\}} \cong \mathbf{sSet}$. There are obvious comparison morphisms

$$(\partial\Delta^1, [0] \sqcup [0], \partial\Delta^1 \to [0] \sqcup [0]) \to (\partial\Delta^1, [1], \partial\Delta^1 \to [1]) \to (\partial\Delta^1, [0], \partial\Delta^1 \to [0]),$$

each of which is given by the identity on underlying simplicial sets, and none of which is a poset-preserving categorical equivalence. For many intents and purposes, however, for example when considering the categories of constructible sheaves on these objects, they are essentially identical. From this perspective, the objects of the ∞-category $\mathcal{AS}\mathbf{trat}$ of abstract stratified homotopy types still contain too much redundant data. This can become even more evident when one studies the case of empty stratified simplicial sets. Assigning to a poset P the stratified simplicial set $(\varnothing, P, \varnothing \to P)$ defines a fully faithful embedding of the 1-category of posets $\mathbf{Pos}$ into $\mathcal{AS}\mathbf{trat}$. In this sense, there are $\mathbf{Pos}$ many non-equivalent "*empty*" abstract stratified homotopy types. [18]

Notation 1.3.5.6. The notation $\Delta^{[n]} \in \mathrm{s}\mathbf{Strat}_{[n]}$ will always refer to the tautologically stratified simplex given by $1_{\Delta^n}\colon \Delta^n \to \Delta^n \cong \mathrm{N}([n])$. The previous example shows that there is a bit of ambiguity of what precise stratified simplicial set one refers to when writing $\partial\Delta^{[n]}$, for $n = 0, 1$. We will use the convention that the empty boundary $\partial\Delta^{[0]}$ is stratified over the empty poset, and that $\partial\Delta^{[1]}$ is stratified over the discrete poset $[0] \sqcup [0]$. These are the stratifications arising from applying the left adjoint to the forgetful functor $\mathrm{s}\mathbf{Strat} \to \mathbf{sSet}$ to $\partial\Delta^n$. For $n \geq 2$, there is no room for ambiguity and the stratification of $\partial\Delta^n$ induced by this left adjoint functor agrees with the stratification inherited from the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.

[18] To avoid such pathologies, Nand-Lal restricted to surjectively stratified spaces in [Nan19]. We will have no need for such a restriction on the level of 1-categories, however, as it will become automatic on the level of ∞-categories.

When concerned with explicit examples, this observation hardly ever seems to be an actual issue. From a conceptual perspective, however, one can obtain slightly cleaner results when using the following more intrinsic notion of stratification.

Construction 1.3.5.7 (see Section 5.3.2)**.** To a stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$, one can associate the poset $P_{\mathcal{X}^{\mathfrak{r}}}$, generated from the set of vertices of X subject to the generating relations

$$x \leq y$$

if there exists a 1-simplex $x \to y$ in X, or if $s_{\mathcal{X}}(x) = s_{\mathcal{X}}(y)$ and there exists a 1-simplex $y \to x$ in X. The elements of $P_{\mathcal{X}^{\mathfrak{r}}}$ can be identified with the path components of the strata of $\mathcal{X}$. This construction defines a functor $\mathrm{s}\mathbf{Strat} \to \mathbf{Pos}$ acting on morphisms in the obvious way. The stratification map of $\mathcal{X}$, $s_{\mathcal{X}}: X \to P_{\mathcal{X}}$, factors uniquely through the map $X \to P_{\mathcal{X}^{\mathfrak{r}}}$, mapping a vertex to its path component. We denote the resulting stratified simplicial set $(X, P_{\mathcal{X}^{\mathfrak{r}}}, X \to P_{\mathcal{X}^{\mathfrak{r}}})$ by $\mathcal{X}^{\mathfrak{r}}$. This construction defines a functor, $(-)^{\mathfrak{r}}: \mathrm{s}\mathbf{Strat} \to \mathrm{s}\mathbf{Strat}$, called the *refinement functor*, and $\mathcal{X}^{\mathfrak{r}}$ is called the *refinement of* $\mathcal{X}$. A stratified simplicial set $\mathcal{X}$, for which the natural map $P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{X}}$ is an isomorphism, is called *refined* and the refinement construction defines a right adjoint to the inclusion of refined stratified simplicial sets into $\mathrm{s}\mathbf{Strat}$, with counit given by the natural transformation $(X, P_{\mathcal{X}^{\mathfrak{r}}}, X \to P_{\mathcal{X}^{\mathfrak{r}}}) \to (X, P_{\mathcal{X}}, X \to P_{\mathcal{X}})$ defined by 1_X and $P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{X}}$.

Example 1.3.5.8. For any stratification of $\partial\Delta^1$, the resulting refinement is stratified isomorphic to $\Delta^{[0]} \sqcup \Delta^{[0]} \in \mathrm{s}\mathbf{Strat}$. For any stratified simplicial set of the form $(\varnothing, P, \varnothing \to P)$, the refinement is given by the initial object in $\mathrm{s}\mathbf{Strat}$, $(\varnothing, \varnothing, 1_{\varnothing})$.

One can use the refinement functor to further localize the model category $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ in order for the stratification poset of a fibrant object to become intrinsic (the analogous construction also works for $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$, but we omit it here, for the sake of conciseness). The most important facts about the resulting homotopy theory can then be summarized in the following theorem, which agglomerates the results of Section 5.3.2:

Main Result H. *Let S be the class of refinement morphisms $\{\mathcal{X}^{\mathfrak{r}} \to \mathcal{X} \mid \mathcal{X} \in \underline{\mathrm{s}\mathbf{Strat}}\}$. Then the right Bousfield localization of the simplicial model category $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ at S exists and is again combinatorial and cartesian. The defining classes of the resulting model category, denoted $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$ and called the categorical model structure, can be characterized as follows:*

1. *The cofibrations are generated by the set of stratified boundary inclusions $\{\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]} \mid n \in \mathbb{N}\}$, together with the inclusion of $\partial\Delta^{[1]} = \Delta^{[0]} \sqcup \Delta^{[0]}$ into the trivially stratified 1-simplex Δ^1, $\partial\Delta^{[1]} \hookrightarrow \Delta^1$. The cofibrant objects are precisely the refined stratified simplicial sets.*

2. *Weak equivalences are precisely those morphisms $f: \mathcal{X} \to \mathcal{Y}$ for which $f^{\mathfrak{r}}$ is a weak equivalence in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$.*

3. *As $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$ is a right Bousfield localization, it follows that fibrations are the same as in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$.*

In particular, it follows that bifibrant objects of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$ are precisely the abstract stratified homotopy types that are refined.

In Section 5.3.2, we show that the equivalence between refined abstract stratified homotopy types and layered ∞-categories can be presented in terms of the following Quillen equivalence, the ∞-categorical version of which was first stated in [BGH18].

Theorem 1.3.5.9 (Theorem 5.3.3.6)**.** *The forgetful functor*

$$\mathrm{s}\mathbf{Strat} \to \mathbf{sSet}$$

defines the right part of a Quillen equivalence between $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$ *and the left Bousfield localization of the Joyal model structure on* $\mathbf{sSet}$ *that presents the ∞-category of small layered infinity categories $\mathcal{L}\mathbf{ay}_{\infty}$.*

In this sense, it follows that the categorical model structure $s\mathbf{Strat}^{\mathfrak{c}}$ presents the ∞-category of small layered ∞-categories. The Quillen adjunction

$$\mathbf{sSet}^{\mathfrak{O}} \rightleftharpoons s\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}},$$

where $\mathbf{sSet}^{\mathfrak{O}}$ is the left Bousfield localization of the Joyal model structure whose bifibrant objects are precisely the layered quasi-categories, recovers the localization from abstract stratified homotopy types to layered ∞-categories discussed in Remark 1.3.5.4.

Remark 1.3.5.10. Working with $\underline{s\mathbf{Strat}}^{\mathfrak{c}}$ instead of $\mathbf{sSet}^{\mathfrak{O}}$ has some (minor) technical advantages. First off, unlike the Joyal model structure, the model category $\underline{s\mathbf{Strat}}^{\mathfrak{c}}$ is simplicial, and there is no need to pass to groupoid cores to obtain mapping spaces. Secondly, and in the same line of argument, the category $\underline{s\mathbf{Strat}}^{\mathfrak{c}}$ often allows for smaller models to present certain abstract stratified homotopy types. For example, to present the inclusion $\Delta^{[0]} \sqcup \Delta^{[0]} \hookrightarrow \Delta^1$ into a trivially stratified simplex in $\mathbf{sSet}$, one needs to glue in an additional inverse 1-simplex together with a homotopy into Δ^1, in order to ensure that both starting and end point lie in the same stratum. In $\underline{s\mathbf{Strat}}^{\mathfrak{c}}$, one can simply use the trivially stratified simplex $\Delta^1 \to [0]$. Finally, working with $\underline{s\mathbf{Strat}}^{\mathfrak{c}}$ has the added advantage that all homotopy theories of stratified spaces discussed here are presented in terms of the same underlying 1-category (simplicial category).

We can now define a third class of weak equivalences for stratified spaces, by considering the class of stratified maps that are mapped into weak equivalences in $\underline{s\mathbf{Strat}}^{\mathfrak{c}}$ under Sing_s. It turns out that there is a simpler description of this class of weak equivalences (which is part of the content of Theorem 7.3.3.1).

Proposition 1.3.5.11. *For a stratified map $f\colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}$ the following are equivalent:*

1. $\mathrm{Sing}_s(f) \in s\mathbf{Strat}^{\mathfrak{c}}$ *is a weak equivalence in* $s\mathbf{Strat}^{\mathfrak{c}}$.
2. *The underlying simplicial map of* $\mathrm{Sing}_s(f)$ *is a Joyal equivalence.*

Definition 1.3.5.12. A stratified map fulfilling any of the equivalent conditions in Proposition 1.3.5.11 will be called a *categorical equivalence.* The associated category with weak equivalences will be denoted by $\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}}$.[19]

In other words, categorical equivalences are such stratified maps that induce equivalences of ∞-categories on the associated ∞-categories of Exit-paths obtained by fibrantly replacing the associated stratified singular simplicial sets. It turns out that this definition of weak equivalences of stratified spaces is not new. In fact, weak equivalences of stratified spaces were defined like this by Nand-Lal in [Nan19]. We will return to the work of Nand-Lal in a minute. For now, let us observe the following final version of Main Result A_1 (see, again Theorems 7.3.3.1 and 7.4.4.4 in Chapter 7).

Main Result $\mathbf{A}_4$. *The adjunction*

$$|-|_s\colon s\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}} \rightleftharpoons \boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}}\colon \mathrm{Sing}_s$$

defines a homotopy equivalence of categories with weak equivalences. The composition of functors

$$\mathbf{Strat} \xrightarrow{\mathrm{Sing}_s} s\mathbf{Strat} \xrightarrow{\mathcal{X} \mapsto X} \mathbf{sSet}$$

(composed with Joyal-fibrant replacement) induces a fully faithful embedding

$$\mathcal{S}\mathbf{trat}^{\mathfrak{c}} \hookrightarrow \mathcal{C}\mathbf{at}_{\infty}.$$

The essential image of this embedding is given by $\mathcal{L}\mathbf{ay}_{\infty}$, the full subcategory of $\mathcal{C}\mathbf{at}_{\infty}$ given by layered ∞-categories.

[19]The nomenclature is compatible with the nomenclature for fixed posets, in the sense that a stratified map is a *stratum-preserving categorical equivalence* if and only if it is a *categorical equivalence* and a *stratum-preserving map.* It is also compatible with the poset-preserving nomenclature, in the sense that a stratified map is a *poset-preserving categorical equivalence* if and only if it is given by an isomorphism on the underlying posets, and is a *categorical equivalence.*

This theorem provides an affirmative answer to Conjecture 1.3.1.3.

1.4 Semi-model categories of stratified spaces

Main Result A_4 provides an answer to the topological stratified homotopy hypothesis that is already very close to the answer given by the classical Kan-Quillen equivalence (Theorem 1.2.4.1). To somebody interested in studying classical examples of stratified spaces, it may nevertheless take some additional convincing that the localization $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ deserves to be called *the homotopy theory of stratified spaces*. For example, in the case of the classical homotopy hypothesis, the Quillen model structure on topological spaces allows us to think of the homotopy theory obtained by localizing topological spaces at weak homotopy equivalences, $\mathbf{Top}[W^{-1}]$, as the homotopy theory obtained by localizing CW-complexes at actual homotopy equivalences, or rather as the simplicial category of CW-complexes. Let us now explain that in the world of stratified spaces a similarly convenient situation can be achieved. In other words, one can obtain an affirmative answer to Requirements (R2) and (R3).

1.4.1 Nand-Lals approach to stratified homotopy theory

In parallel work – independently from Douteau, Henriques and Haine – Nand-Lal (back then a student of Woolf) also aimed to establish a homotopy theory for stratified spaces around Lurie's stratified singular simplicial set construction (see [Nan19]). Instead of working with stratified simplicial sets, Nand-Lal considered the adjunction whose right adjoint is obtained by composing $\mathrm{Sing}_s\colon \mathbf{Strat} \to \mathrm{s}\mathbf{Strat}$ with the forgetful functor $\mathrm{s}\mathbf{Strat} \to \mathbf{sSet}$. The left adjoint is given by left Kan-extension of the functor mapping Δ^n to $|\Delta^{[n]} \xrightarrow{\cong} \mathrm{N}([n])|_s$.

Notation 1.4.1.1. By abuse of notation, we will also denote the resulting adjunction

$$\mathbf{sSet} \rightleftharpoons \mathbf{Strat}$$

by $|-|_s \dashv \mathrm{Sing}_s$. Which adjunction is meant will usually be clear from context. There is, however, a slight possibility of confusion when writing $|\Delta^n|_s$, for $n \in \mathbb{N}$, as it may not be clear whether this refers to the realization of a trivially stratified simplex or to the realization of the stratified simplex $\Delta^n \to [n]$. To distinguish these two cases, we will write $\Delta^{[n]}$ in the latter case. We use analogous notation for horn and boundary inclusions. One should take care, however, that the stratified boundary $|\partial\Delta^{[1]}|_s$ is necessarily the coproduct of $|\Delta^{[0]}|_s \sqcup |\Delta^{[0]}|_s$ and thus stratified over $[0] \sqcup [0]$, and not over $[0]$, and similarly $|\partial\Delta^{[0]}|_s$ is stratified over the empty poset.

[Nan19] defines the class of weak equivalences to be precisely what we have called categorical equivalences in the previous section.[20] One of the goals of [Nan19] was to establish that the homotopy theory defined in terms of this class of weak equivalences interacts well with classical examples of stratified spaces, most prominently Quinn's homotopically stratified spaces[21]. As we have already explained in Requirement (R3), one way of doing this is to expose a model structure in which the (co)fibrancy conditions interact well with the geometry (topology) of such classical examples of stratified spaces. To construct such a model structure, [Nan19] aimed to make use of the general machinery of transferred model structures (see [nLa23], for an overview). Recall that if $\mathbf{M}$ is a model category, $\mathbf{N}$ is a bicomplete category and $L\colon \mathbf{M} \rightleftharpoons \mathbf{N}\colon R$ is an adjunction, then one says that the model structure on $\mathbf{M}$ *right transfers* to $\mathbf{N}$, if the classes

$$\{w \in \mathbf{N} \mid R(w) \text{ is a weak equivalence.}\}$$

[20] To our best knowledge, Nand-Lal already pursued this definition before a proof of a stratified Homotopy hypothesis had been given by Haine.

[21] Strictly speaking, [Nan19] restricts to the case of stratified spaces with non-empty strata, but the difference is negligible, and we will ignore it in this chapter.

and

$$\{f \in \mathbf{N} \mid R(f) \text{ is a fibration.}\}$$

form the classes of, respectively, weak equivalences and fibrations of a model structure on $\mathbf{N}$. In the case of the adjunction $|-|_s\colon \mathbf{sSet} \rightleftharpoons \mathbf{Strat}\colon \mathrm{Sing}_s$, these classes necessarily determine the class of cofibrations to be the saturated class (i.e., the retracts of relative cell complexes) generated by the stratified boundary inclusions $|\partial\Delta^{[n]} \to \Delta^{[n]}|_s$.

Remark 1.4.1.2. Recall that a morphism $c\colon A \to X$ in a category $\mathbf{C}$ is a relative cell complex with respect to a class of morphisms I, if it can be written as a transfinite composition of cobase changes (pushouts along an arbitrary morphism) of morphisms in I. c is called an absolute cell complex, if A is initial in $\mathbf{C}$. In this case, one can just identify c with an object of $\mathbf{C}$.

Definition 1.4.1.3. A stratified space $\mathcal{X} \in \mathbf{Strat}_P$ is said to *admit the structure of a stratified cell complex* if it is an absolute cell complex with respect to the class of stratified boundary inclusions $\{|\partial\Delta^{\mathcal{J}} \to \Delta^{\mathcal{J}}|_s \mid \mathcal{I} \in \Delta_P\}$. In other words, the stratified map $\varnothing \to \mathcal{X}$ can be written as a transfinite composition of pushouts of stratified boundary inclusions.
A stratified space is called *triangularly cofibrant*, if it is a retract of a stratified space that admits the structure of a stratified cell complex.

The cofibrant objects in Nand-Lal's conjectured model structure are precisely the triangularly cofibrant stratified spaces that fulfill the additional condition that the stratification poset contains no redundant elements and relations in the following sense (see Section 1.4.3, for more details): All strata are nonempty and there is a relation $s_{\mathcal{X}}(x) \leq s_{\mathcal{X}}(y)$, if and only if there is a sequence of composable exit-paths from x to y. Such a stratified space is called *refined*. It is not hard to see that $\mathcal{X}$ is refined, if and only if $\mathrm{Sing}_s(\mathcal{X})$ is a refined stratified simplicial set (see Section 7.5.3 for details). In particular, being refined implies that all strata are path-connected. In a sense, this means that the stratification of triangularly stratified spaces is a lot closer to the stratifications classically considered, which were intrinsic to the topology of a decomposition of a space. This additional requirement on strata being path-connected is generally not a problem for classical examples, as it can generally be restored by slightly refining the stratification.

Example 1.4.1.4. For the realization of a stratified simplicial set $\mathcal{X}$ that has non-empty and path-connected strata, and where the stratification arises from the frontier condition, the stratified realization $|\mathcal{X}|_s$ is refined and triangularly cofibrant.

The fibrant objects in Nand-Lal's conjectured structure are precisely those objects that have the horn filling property with respect to all stratified inner horn inclusions $|\Lambda^{[n]}_k \hookrightarrow \Delta^{[n]}|_s$, i.e., for which the underlying simplicial set of $\mathrm{Sing}_s(\mathcal{X})$ is a quasi-category. This includes all conically stratified spaces, but by using a result of [Mil09], [Nan19] also proves that this applies to Quinn's homotopically stratified spaces, thus providing a strong connection with more classical approaches to stratified homotopy theory [22].

[Nan19, Sec. 8.4] makes very explicit that transferring a model structure along Sing_s turns out to be significantly harder than performing the analogous transfer in the unstratified case

[22] [Nan19] also asserted a partial converse. Namely that triangularly cofibrant stratified spaces fulfill Quinn's cofibrancy condition and that fibrant spaces fulfill Quinn's fibrancy condition, that the starting point maps $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to X_p$ are Hurewicz fibrations. The first of these statements should hold true (following from Chapter 6), albeit the proof in [Nan19] seems to produce a non-continuous map, where continuity is needed. The second statement does not seem plausible, since Quinn's theory manifestly relies on having Hurewicz fibrations, while fibrancy in the sense of [Nan19] only guarantees for Serre fibrations. Outside of the class of CW-complexes, these two classes may very well differ. One should generally expect any theory transferred from a combinatorial framework to produce Serre style fibrancy and cofibrancy conditions, as opposed to the more geometric Hurewicz style definitions. This generally means that fibrancy in a Serre-style framework is easier to achieve, while cofibrancy is a stronger condition. For many intents and purposes, in particular when working with examples from geometry, the difference seems to be negligible however.

(in fact, it follows from our result Proposition 1.2.6.1 that this is not possible at all). In [Nan19], it is explained that the core difficulty really lies with proving that the pushout of the stratified realization of a Joyal-acyclic cofibration is again a weak equivalence; as we demonstrate in Section 3.A and Proposition 7.4.0.1 this is, in fact, false. To alleviate this issue, [Nan19] conjectures that the classes he defined formed a so-called left semi-model category (*semi-model category* henceforth). Recall that a semi-model category (see [BW24]) fulfills almost the same axioms as a model category. The essential difference is that only morphisms with cofibrant source are assumed to factor into an acyclic cofibration followed by a fibration and that only acyclic cofibrations with cofibrant source lift against fibrations (see Definition 7.4.1.1, for the formal definition that we employ here). For the purpose of this overview section, it suffices to know that it has been demonstrated by several authors in recent years (see, for example [Spi04; Bar10; BW24; BDW23; WY18; Fre10]) that the slight lack of symmetry of axioms comes at almost no price when it comes to the powerfulness of the resulting framework of homotopical algebra. Namely, essentially any theorem about model categories either directly holds or has an analogue in semi-model categories (see, for example, [BW24, Rem 4, 5]). At the same time, semi-model categories have significantly better existence theorems, making them a practical alternative in most cases where model categories do not exist. In particular, semi-model categories are usually just as useful when the goal is to present a specific ∞-category and understand its interaction with some 1-category.

One way of proving that Nand-Lal's classes form a semi-model category is to show (among other things) that cobase changes of stratified realization of Joyal-acyclic cofibrations to a triangularly stratified space are weak equivalences. This still turns out to be a technically difficult to prove result (at least the only proof that we were able to achieve is rather involved from a technical point of view), a proof of which we give in Chapter 7. Instead [Nan19] proves the weaker statement that if one restricts to the category of fibrant objects, then the classes form a model structure ([Nan19, Thm. B]). The existence of a model structure on fibrant objects already turns out to be very useful. For example, it can be used to derive a strengthening of Douteau's Whitehead theorem. As the category of fibrant objects lacks colimits, one does not obtain a (semi-)model category, however. For the purposes of investigating the stratified homotopy hypothesis and of simple homotopy theory (which we investigate in the second half of this article) a semi-model category, together with the vast amount of literature pertaining to model categories and transferring to the semi-model setting, is preferable (or necessary even). [Nan19, Conj. 1] also conjectured that the units of adjunction $1 \to \mathrm{Sing}_s \circ |-|_s$ are categorical equivalences. By Main Result A_4, this is indeed the case.

Our strategy of proof for the existence of a semi-model structure is the following: We are looking to transfer the model structures developed in Chapter 5 from the combinatorial setting of stratified simplicial sets. The core obstruction to the transfer of semi-model categories is to show that the cobase change (between cofibrant objects) of a realization of an acyclic cofibration on the simplicial side remains a weak equivalence. It turns out (by a general transfer theorem for semi-model categories, found, for example, in [WY18]) that it suffices to study the case of stratified spaces that admit a stratified cell structure. To be able to deal with such spaces, we need to be able to control the generalized homotopy links of such objects. Doing so turns out to be significantly more technically involved than the investigations of homotopy links in Chapter 3. Nevertheless, similar results can be obtained. This is the content of Chapter 6, which we summarize in the following section, Section 1.4.2.

1.4.2 Results: On the homotopy links of stratified cell complexes

Let us now explain how having cellular models for the generalized homotopy links of stratified cell complexes can be used to obtain the desired stability result on acyclic cofibrations under pushouts. We will explain the case of a fixed poset P here. When we refer to a stratified space $\mathcal{X}$ as a *stratified cell complex*, we will really mean a stratified space $\mathcal{X}$ together with a choice of characteristic maps $\{\sigma_i : |\Delta^{\mathcal{I}_i}|_s \to \mathcal{X}\}$ that arise from a presentation of $\mathcal{X}$ in terms of a transfinite composition of pushouts of stratified boundary inclusions (see Section 6.2.3, for details, and Chapter 8 for a general theory of what it means to be a structured cell complex). By a subcomplex of a stratified cell complex, we will mean the obvious thing, i.e., a stratified subspace $\mathcal{A} \subset \mathcal{X}$, together with a subset of characteristic maps on $\mathcal{X}$ that define the structure of a stratified cell complex on $\mathcal{A}$. Now, suppose we are given a pushout square

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{B} \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & \mathcal{Y} \end{tikzcd} \tag{1.12}$$

where $\mathcal{A}, \mathcal{X}$ and $\mathcal{B}$ admit stratified cell structures, such that the upper horizontal defines the inclusion of a subcomplex. If the semi-model structure transferred from $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ to $\mathbf{Strat}_P^{\mathfrak{d}}$ existed, then it would follow that the above square defines a homotopy pushout diagram, i.e., a pushout diagram in the associated ∞-category $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$. It turns out that this single consequence of the existence of the transferred semi-model structure is essentially the only real technical obstruction to proving the existence of the latter. Indeed, assuming if $\mathcal{A} \to \mathcal{X}$ is an acyclic cofibration, then by stability of weak equivalences under homotopy pushouts, so would be its parallel arrow $\mathcal{B} \to \mathcal{Y}$.

As a consequence of Main Result B and Main Result A_1, together with the general fact that colimits in functor categories (of ∞-categories) can be detected point-wise, it follows that Diagram (1.12) is a homotopy pushout diagram if and only if it holds that for every regular flag $\mathcal{I} \in \mathrm{sd}(P)$ the associated commutative square of homotopy links

$$\begin{tikzcd} \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{A} \arrow[r] \arrow[d] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X} \arrow[d] \\ \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{B} \arrow[r] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{Y} \end{tikzcd} \tag{1.13}$$

is a homotopy pushout square. Suppose, for a second, that we are in the case of two-strata ($P = \{p < q\}$) and that Diagram (1.12) is given by inclusions of compact piecewise-linear stratified spaces, with the singular stratum given by a PL subspace. Then, up to weak homotopy equivalence, the pairwise homotopy link can equivalently be computed as the complement of X_p in a regular regular neighborhood $N_X(X_p)$ of X_p in X (this is a classical fact, see, for example, the appendix of [Fri03]). Taking compatible triangulations, one then obtains the following pushout diagram of complements of regular neighborhoods.

$$\begin{tikzcd} N_A(A_p) \setminus A_p \arrow[r, hook] \arrow[d] & N_X(X_p) \setminus X_p \arrow[d] \\ N_B(B_p) \setminus B_p \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & N_Y(Y_p) \setminus Y_p. \end{tikzcd} \tag{1.14}$$

As the diagram is a pushout and the horizontals are evidently Serre cofibrations, it follows that Diagram (1.13) is a homotopy pushout.

The PL case suggests that if one can obtain an appropriate theory of generalized regular neighborhoods for multi-strata interactions in stratified cell complexes, then the same argument can be replicated in the case of general stratified cell complexes.
The latter case turns out to be several magnitudes more complicated than the stratified PL case, however (at least that is our personal opinion of the proof we were able to obtain). This is essentially due to the fact that unlike in the PL case, the gluing maps involved in stratified cell complexes can be highly pathological, and there is a priori no reason to assume that (generalized) regular neighborhoods glue in a sufficiently coherent fashion. It is precisely the interaction of such highly pathological gluing maps, together with the generalization

to multi-strata homotopy links that makes it difficult to find a well-working definition of regular neighborhood that is general enough to apply to all stratified cell complexes (see Example 6.4.2.6, for some of the things that can possibly go wrong). Nevertheless, one can develop a theory of regular neighborhoods that is powerful enough to produce the following result (which is the main content of Chapter 6; see there for a more precise formulation of the result)).

Main Result I (Theorems 6.2.4.14 and 6.4.2.7 and Proposition 6.3.2.11)**.** *Let $\mathcal{X}$ be a stratified cell complex over a poset P and $\mathcal{I} = \{p_0 < \cdots < p_n\} \subset P$ a regular flag. There exists a (barycentric) subdivision of the cell structure on $\mathcal{X}$, and a (systematic construction of a) subcomplex $\mathcal{N}_{\mathcal{I}} \subset \mathcal{X}$ of the subdivision of $\mathcal{X}$, such that $X_{p_0} \subset \mathcal{N}_{\mathcal{I}}$ and such that there is a canonical weak homotopy equivalence*

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \simeq (\mathcal{N}_{\mathcal{I}})_{p_n}.$$

Furthermore, subdivisions can be chosen such that the construction of $\mathcal{N}_{\mathcal{I}}$ is compatible with stratum-preserving maps and pushouts along inclusions of subcomplexes.

In fact, we show that the subcomplexes $\mathcal{N}_{\mathcal{I}}$ can even be used to model the whole homotopy link diagram of [Dou21c].

Example 1.4.2.1. To have an example for the kind of subcomplexes arising in Main Result I, consider the following illustration (see Fig. 1.6) in the case of a cell structure for the pinched torus. Here, the pinched torus is stratified with three strata, one corresponding to the pinch point (red), and one to the equator with a single point removed (green), and one to the complement of the equator (blue).

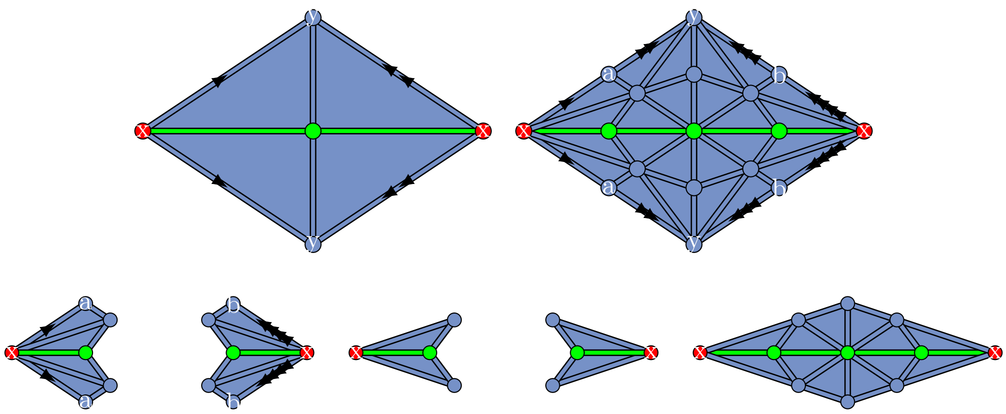


Figure 1.6: The upper left corner shows a stratified cell structure for the pinched torus, stratified over the poset $\{0 < 1 < 2\}$. Vertices with the same name, and edges with the same markings are being identified and the stratification is indicated by the coloring. To its right, a barycentric subdivision of this cell structure is shown. In the following row there are illustrations of the subcomplexes $\mathcal{N}_{\mathcal{I}}$ for $\mathcal{I} = [0 < 2], [0 < 1 < 2], [1 < 2]$.

Main Result I has the following corollary, which is central to the construction of semi-model categories of stratified spaces.

Theorem 1.4.2.2 (Corollary 6.4.2.8)**.** *Any pushout diagram of stratified spaces as in Diagram* (1.13) *descends to a pushout diagram in the ∞-category $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$.*

1.4.3 Results: The transferred semi-model structures for stratified spaces and their consequences

Then, having a good grasp on models for stratified homotopy theory on the combinatorial side as well as well-behaved models for the homotopy links of stratified cell complexes, we combine these insights to transfer the semi-model structures from the combinatorial side to the topological side. Doing so, and studying the properties and consequences of the resulting semi-model categories is the content of Chapter 7. Let us state our results here for the case of flexible posets. Analogous claims in the case of a fixed poset also hold.

Main Result J (Theorems 7.3.3.1, 7.4.2.7 and 7.4.2.10 and Corollaries 7.4.2.3 and 7.4.3.3)**.** *The simplicial model structures on* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$, $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$, *and* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$ *transfer to the simplicial category of poset-stratified spaces* $\underline{\mathbf{Strat}}$ *along the adjunction*

$$|-|_s : \underline{\mathrm{s}\mathbf{Strat}} \rightleftharpoons \underline{\mathbf{Strat}} : \mathrm{Sing}_s,$$

to simplicial, cofibrantly generated, cartesian closed semi-model categories.

In this way, we can equip the categories with weak equivalences $\boldsymbol{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}, \boldsymbol{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ and $\boldsymbol{S}\mathbf{trat}^{\mathfrak{c}}$ with the structure of a simplicial semi-model category. We denote the resulting simplicial semi-model categories of stratified topological spaces by $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$, $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ and $\underline{\mathbf{Strat}}^{\mathfrak{c}}$, and call their associated model structures, respectively, the *poset-preserving diagrammatic, poset-preserving categorical, and categorical semi-model structure* on $\underline{\mathbf{Strat}}$. We give detailed descriptions of all of these semi-model categories (as well as of a fourth one, given by the non-poset-preserving analogue of the poset-preserving diagrammatic structure) in Chapter 5. Let us give, as an example, an explicit description of the semi-model category $\underline{\mathbf{Strat}}^{\mathfrak{c}}$, whose existence was conjectured in [Nan19].

Theorem 1.4.3.1. *The simplicial category* $\underline{\mathbf{Strat}}$ *admits the structures of a cofibrantly generated simplicial semi-model category,* $\underline{\mathbf{Strat}}^{\mathfrak{c}}$ *- called the categorical semi-model structure - with the following classes:*

1. *Cofibrations are generated by the set of stratified boundary inclusions*
$$\{|\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]}|_s \mid n \in \mathbb{N}\}.$$
2. *Weak equivalences are the categorical equivalences of stratified spaces.*
3. *Fibrations are the stratum-preserving maps that have the right lifting property with respect to all acyclic cofibrations with cofibrant source.*

Furthermore, fibrant objects in $\underline{\mathbf{Strat}}^{\mathfrak{c}}$ *are precisely such stratified spaces* $\mathcal{X}$*, for which* $\mathrm{Sing}_s(\mathcal{X})$ *is a quasi-category and fibrations between fibrant objects are such stratified maps that have the right lifting property with respect to, equivalently, realizations of inner stratified horn inclusions or of admissible stratified horn inclusions.*

Now that we have several different (semi-)model categories for stratified homotopy theory available, it may be useful to observe how they are related. The situation can be described in the following result, which also (partially) subsumes Main Results A_1, A_3 and A_4.

Main Result A_5 (Theorem 7.4.2.11)**.** *There is a diagram of simplicial Quillen adjunctions (the right and the left adjoint parts of which are trivially commutative)*

$$\begin{tikzcd}
\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}} \arrow[r, shift left, "|-|_s"] \arrow[d, shift left] & \underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}} \arrow[l, shift left, "\mathrm{Sing}_s"] \arrow[d, shift left, "1"] \\
\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}} \arrow[r, shift left, "|-|_s"] \arrow[u, shift left, "1"] \arrow[d, shift left] & \underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}} \arrow[l, shift left, "\mathrm{Sing}_s"] \arrow[u, shift left] \arrow[d, shift left, "1"] \\
\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}} \arrow[r, shift left, "|-|_s"] \arrow[u, shift left, "1"] & \underline{\mathbf{Strat}}^{\mathfrak{c}} \arrow[l, shift left, "\mathrm{Sing}_s"] \arrow[u, shift left]
\end{tikzcd} \tag{1.15}$$

with the following properties.

1. *The upper downward verticals are given by left Bousfield localization of, respectively, the set of inner stratified horn inclusions to the left, and their realizations on the right.*

2. *The lower downward verticals are respectively given by right Bousfield localization at the refinement maps, and their topological analogues (see Section 7.5.3).*

3. *The horizontal adjunctions are Quillen equivalences that create weak equivalences in both directions.*

In particular, this elevates the answers to the topological stratified homotopy hypothesis obtained in Main Results A_3 and A_4 to (particularly well-behaved) simplicial Quillen equivalences.

The existence of these semi-model structures has several immediate consequences for the homotopy theories $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}, \mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$, which arise from the general theory of (semi-)model categories, and demonstrate that Requirements (R2) and (R3) are fulfilled. Let us just name two of the most prominent ones, which are strongly related. We state all results for the homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$, but analogous results for the alternative theories in Main Result J can be found in Chapter 7. First off, it follows that under appropriate fibrancy conditions, mapping spaces in the ∞-category $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ can be computed in terms of simplicial map spaces.

Corollary 1.4.3.2 (Corollary 7.4.4.2)**.** *Let $\mathcal{A} \in \mathbf{Strat}^{\mathfrak{c}}$ be a cofibrant stratified space and let $\mathcal{X} \in \mathbf{Strat}^{\mathfrak{c}}$ be a fibrant stratified space. Then there is a canonical weak equivalence of mapping spaces*

$$\mathcal{S}\mathbf{trat}^{\mathfrak{c}}(\mathcal{A}, \mathcal{X}) \simeq \underline{\mathbf{Strat}}(\mathcal{A}, \mathcal{X}).$$

In particular, it follows that there is a canonical bijection

$$\pi_0 \mathcal{S}\mathbf{trat}^{\mathfrak{c}}(\mathcal{A}, \mathcal{X}) \cong [\mathcal{A}, \mathcal{X}]_s$$

between morphisms in the homotopy category $\mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ *and stratified homotopy classes.*

Thus, at least as long as we restrict to bifibrant spaces, the associated homotopy category $\mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ is simply the naive homotopy category, obtained by identifying stratified homotopic maps. In fact, one obtains the following equivalent descriptions of the homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$. In the following, we denote by $\underline{\mathbf{Strat}}^{\mathfrak{c},o}$ the full simplicial subcategory of bifibrant objects in $\underline{\mathbf{Strat}}^{\mathfrak{c}}$.

Corollary 1.4.3.3 (Corollary 7.4.4.3)**.** *Denote by H_s the class of stratified homotopy equivalences between bifibrant stratified spaces in $\underline{\mathbf{Strat}}^{\mathfrak{c}}$. There are canonical equivalences of ∞-categories*

$$\underline{\mathbf{Strat}}^{\mathfrak{c},o} \simeq \mathbf{Strat}^{\mathfrak{c},o}[H_s^{-1}] \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{c}}$$

where equivalences between simplicial and quasi-categories are to be understood in terms of the Quillen equivalence between quasi-categories and simplicial categories of [Ber07a].

This allows one to think of the homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ as either the homotopy theory obtained by localizing a subcategory of **Strat** at stratified homotopy equivalences, or as a full simplicial subcategory of $\underline{\mathbf{Strat}}$. In this sense, the situation of the version of the stratified homotopy hypothesis in Main Result A_4 is very akin to the presentation of the classical homotopy hypothesis in terms of the Kan-Quillen equivalence between spaces and simplicial sets. Besides these immediate results, there is an enormous number of techniques and results concerned with the general theory of (semi-)model categories which can be accessed through Main Result J. Crucially for our purposes in the second half of this thesis, Main Result J enables one to investigate the homotopy theories of stratified spaces from the perspective of simple homotopy theory, which we discuss in in Part III. For more geometric applications, however, results such as Corollary 1.4.3.2 are generally only as useful as one's understanding of the class of bifibrant objects (or more generally fibrations and cofibrations) and its relation to geometrical examples.

1.4.4 Results: Bifibrant stratified spaces

Let us now address Requirement (R3), and explain how the (co)fibrancy conditions in the semimodel categories $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathbf{Strat}^{\mathfrak{c}}$ relate to geometric properties of stratified spaces, summarizing some of the results of Section 7.5 of Chapter 7.

As a first immediate consequence of the definition of the model structures in terms of transfers, one obtains that the cofibrant objects in $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ are precisely the triangularly cofibrant stratified spaces, i.e., the retracts of stratified cell complexes. As we have already noted, the cofibrant objects in $\mathbf{Strat}^{\mathfrak{c}}$ are precisely such triangularly cofibrant stratified spaces that have non-empty, path-connected strata, with all relations in the stratification poset arising from exit-paths (see Proposition 7.5.3.9). Given a triangularly cofibrant stratified space, this can always be achieved by subdividing strata into their path-components and removing redundant relations. As we observed before, the triangular cofibrancy condition is clearly fulfilled whenever the stratified space admits a triangulation that is compatible with the stratification (in an appropriate sense). In particular, Whitney stratified spaces and PL pseudo manifolds are triangularly cofibrant. Observe, however, that in the classical setting of the Quillen model structure, even non-triangulable manifolds are cofibrant (they are Euclidean neighborhood retracts, by [Han51], and hence retracts of triangulable objects). In the stratified setting, something similar can be said, as long as one assumes that the strata admit stratified mapping cylinder neighborhoods. Let us state the case of spaces stratified over $[n]$ here, for the sake of simplicity.

Definition 1.4.4.1. By a stratified mapping cylinder neighborhood of a stratum X_p, of $\mathcal{X} \in \mathbf{Strat}_{[n]}$, we mean a neighborhood of X_p that is stratum-preserving homeomorphic to a stratified mapping cylinder of a map $f\colon \mathcal{L} \to X_p$, with $\mathcal{L}$ stratified over $\{k > p\}$. More precisely, by the stratified mapping cylinder of f (over $P_{\mathcal{X}} = [n]$), we mean the stratified space obtained by equipping $M_f = L \times [0,1] \cup_{L\times\{0\}} X_p$ with the stratification

$$[l,t] \mapsto \begin{cases} s_{\mathcal{L}}(l) & ,t > 0 \\ p & ,\mathrm{t} = 0 \end{cases}$$
$$X_p \ni x \mapsto p.$$

We can then prove the following results (see Proposition 7.5.2.10 for the more general statement involving depth.)

Proposition 1.4.4.2. *Let $\mathcal{X} \in \mathbf{Strat}_{[n]}$ be a stratified space whose strata are cofibrant in the Quillen model structure on* **Top**. *Assume, furthermore, that every stratum of $\mathcal{X}$ admits a stratified mapping cylinder neighborhood. Then $\mathcal{X}$ is triangularly cofibrant.*

In particular, this proposition applies to topological pseudomanifolds that admit (stratified) cylinder neighborhoods. We note that this condition is stronger than the pairwise cofibrancy condition of Quinn in [Qui88][23]. From a conceptual perspective, this is not surprising. Quinn essentially performs in Hurewicz homotopy theory in [Qui88]. This means that cofibrancy conditions in Quinn's theory will generally be weaker, but fibrancy conditions will generally be stronger.

As we have explained, the differences in cofibrant objects between $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathbf{Strat}^{\mathfrak{c}}$ ultimately come down to rather non-invasive refinements of stratification. From the ∞-categorical perspective, $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ forms a (proper) subcategory of $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ (in the fixed poset case, this is just the fact that being a décollage is a nontrivial condition). One would thus generally expect there to be a major difference between the behavior of the two homotopy theories. Interestingly, in any geometric scenario, this difference essentially disappears. In

[23] Generally, there may be obstructions to the existence of such neighborhoods. See [Qui88, Thm. 1.7].

[Nan19], it was shown that Quinn's homotopically stratified spaces are fibrant in $\mathbf{Strat}^{\mathfrak{c}}$ and hence also in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$. Observe that Quinn's fibrancy condition, namely that the maps

$$\mathrm{HoLink}_{\{p<q\}}(\mathcal{X}) \to X_p$$

are Hurewicz fibrations, involves only pairwise conditions on strata. If one only considers stratified horns with two strata, then the defining conditions for fibrancy in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ agree. Up to the difference between Serre and Hurewicz fibrations, this shows that there is not much space left for stratified spaces that are fibrant in $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ but not in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ at all. It turns out that to produce such counterexamples, one needs to pass to a non-metrizable scenario. In fact, we prove the following result.

Main Result K (Proposition 7.5.1.4 and Theorem 7.5.1.6)**.** *Let $\mathcal{X} \in \mathbf{Strat}$ be a metrizable stratified space. Then the following conditions are equivalent:*

1. *$\mathcal{X}$ is fibrant in $\mathbf{Strat}^{\mathfrak{c}}$.*
2. *$\mathcal{X}$ is fibrant in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$.*
3. *$\mathcal{X}$ is fibrant in $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$.*
4. *For any pair $p < q \in P$, the starting point evaluation map $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(\mathcal{X}) \to \mathcal{X}_p$ is a Serre fibration.*

In particular, when restricted to metrizable stratified spaces, the homotopy theories defined by $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ and $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ (in terms of simplicial categories of bifibrant objects) agree.

One may read this result as stating that, as long as one works in a metrizable scenario, then all fibrancy conditions are the obvious Serre-homotopy theory analogue of the fibrancy condition for Quinn's homotopically stratified spaces.[24] Main Result K significantly simplifies the verification of fibrancy for most classical examples of stratified spaces. For example, one may derive from this result that fibrancy of metrizable stratified spaces is a local property that can be verified purely by investigating two-strata interactions. The fact that all metrizable conically stratified spaces (in particular, all topological pseudomanifolds) are fibrant follows from this quite readily.
Finally, let us give another characterization of bifibrant objects in $\mathbf{Strat}^{\mathfrak{c}}$. Namely, they are given by retracts of the following particularly nice class of stratified spaces:

Definition 1.4.4.3. A stratified space $\mathcal{X} \in \mathbf{Strat}$ is called *CFF-stratified*[25] if it fulfills the following conditions:

1. $\mathcal{X}$ admits the structure of a stratified cell complex.
2. $\mathcal{X}$ has (necessarily non-empty) path-connected strata, these fulfill the frontier condition and the stratification poset of $\mathcal{X}$ arises by equipping the set of strata with the relations arising from the frontier condition.
3. $\mathrm{Sing}_s\mathcal{X}$ is a quasi-category.

Using CFF-stratified spaces, one may equivalently describe the homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ as follows (see Corollary 7.5.4.9).

Corollary 1.4.4.4. *Denote by* **CFF** *the full subcategory of* **Strat** *given by CFF-stratified spaces and let H_s be the class of stratified homotopy equivalences in* **CFF**. *The inclusion of* $\mathbf{CFF} \hookrightarrow \mathbf{Strat}$ *induces an equivalence of ∞-categories*

$$\mathbf{CFF}[H_s^{-1}] \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{c}}.$$

[24] As already mentioned Nand-Lal proved in [Nan19] that Quinn's homotopically stratified spaces are fibrant in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$, and hence in all of the theories above. He made use of Millers results on homotopically stratified spaces, which manifestly used that the latter have *Hurewicz* fibrations of exit-paths. Our proof is entirely independent from this, and produces a stronger result as we only require Serre fibrations, and no cofibrancy assumptions on the interactions of strata.

[25] The "C" stands for cellular, the first "F" for fibrant and the second "F" for frontier.

1.4.5 Results: A final look at the stratified homotopy hypothesis

In the previous section, we have addressed Requirements (R1) to (R3), and hopefully illustrated that the semi-model categories of stratified spaces we defined present a useful tool to connect classical and recent approaches to stratified homotopy theory with geometrical examples of stratified spaces. Let us end this expository chapter with a final look at the stratified homotopy hypothesis and explain that we can now derive an answer that is very much akin to the classical (non-stratified) answer provided by Theorem 1.2.4.1. As a combination of Main Result A_5 and Theorem 1.3.5.9, one obtains the following formulation:

Theorem 1.4.5.1 (Theorem 7.4.4.4)**.** *The adjunction*

$$|-|_s \colon \mathbf{sSet} \rightleftharpoons \mathbf{Strat} \colon \mathrm{Sing}_s$$

defines a Quillen equivalence between $\mathbf{Strat}^{\mathfrak{c}}$ *and the left Bousfield localization of the Joyal model structure on* $\mathbf{sSet}$ *that presents the homotopy theory of small layered* ∞*-categories. This Quillen equivalence creates weak equivalences in both directions.*

Observe that, since not all objects of **Strat** are fibrant, the fact that both directions create weak equivalences is relevant additional information here. Having a Quillen equivalence that creates weak equivalences in both directions makes this a rather tractable correspondence. Corollary 1.4.4.4 or Corollary 1.4.3.3 then provide us with the setting that, much like in the classical scenario of topological spaces and CW-complexes, we can think of the homotopy theory defined by $\mathbf{Strat}^{\mathfrak{c}}$ as being given by localizing a class of stratified spaces (that contains most geometrically relevant examples) at stratified homotopy equivalences. Finally, as the equivalence is induced by Lurie's stratified singular simplicial set functor, we may think of it as presenting the ∞-category of exit-paths construction.

1.5 List of recurring notation used in this chapter

The following is a list of important recurring notation in this chapter (and partially in later parts of this text), together with page numbers referencing where it is introduced.

Important 1-categories, simplicial categories and operations on categories

Top	Category of (compactly generated or Δ-generated) topological spaces	P. 9
Pos	Category of partially ordered sets	P. 9
Strat	Category of poset-stratified spaces	P. 10
$\underline{\mathbf{Strat}}$	Simplicial category of poset-stratified spaces	P. 13
$\mathbf{Strat}_P$	Category of P-stratified spaces	P. 10
$\underline{\mathbf{Strat}}_P$	Simplicial category of P-stratified spaces	P. 13
$\mathbf{C}[W^{-1}]$	∞- or 1-categorical localization of $\mathbf{C}$ at W (depending on context)	P. 13
Δ	Category of finite linear posets of the form $[n] := \{0, \cdots, n\}$	P. 19
Δ_P	Category of flags of P	P. 19
$\mathrm{sd}(P)$	Category of regular flags of P	P. 19
$\mathbf{Fun}(\mathbf{C}, \mathbf{D})$, $\mathbf{D}^{\mathbf{C}}$	Notations for categories of functors from $\mathbf{C}$ to $\mathbf{D}$	P. 22
sSet	Category of simplicial sets	P. 25
s**Strat**	Category of stratified simplicial sets	P. 25
$\underline{\mathrm{s}\mathbf{Strat}}$	Simplicial category of stratified simplicial sets	P. 27
$\mathrm{s}\mathbf{Strat}_P$	Category of P-stratified simplicial sets	P. 26
$\underline{\mathrm{s}\mathbf{Strat}}_P$	Simplicial category of P-stratified simplicial sets	P. 27

Operations and functors on stratified spaces

$P_{\mathcal{X}}$	Stratification poset of $\mathcal{X}$	P. 9
$s_{\mathcal{X}}$	Stratification map of $\mathcal{X}$	P. 9
X_p	p-stratum of $\mathcal{X}$	P. 10
$\mathcal{X}_{>p}$	Stratified subspace given by strata greater than p	P. 18
$\underline{\mathbf{Strat}}(\mathcal{X}, \mathcal{Y})$	Simplicial mapping space of stratified maps from $\mathcal{X}$ to $\mathcal{Y}$	P. 13
$C^0_P(\mathcal{X}, \mathcal{Y})$	Space of stratum-preserving maps from $\mathcal{X}$ to $\mathcal{Y}$	P. 20
$[\mathcal{X}, \mathcal{Y}]]_s$	Set of stratified homotopy classes of stratified maps from $\mathcal{X}$ to $\mathcal{Y}$	P. 12
$[\mathcal{X}, \mathcal{Y}]]_P$	Set of stratified homotopy classes of stratum-preserving maps over P from $\mathcal{X}$ to $\mathcal{Y}$	P. 12
$\Delta^{\mathcal{J}}$	Stratified simplex associated to a flag $\mathcal{J}$	P. 26
$\partial\Delta^{\mathcal{J}}$	Boundary of the stratified simplex associated to a flag $\mathcal{J}$	P. 26
$\Lambda^{\mathcal{J}}_k$	k-th horn of the stratified simplex $\Delta^{\mathcal{J}}$	P. 28
$\lvert\Delta^{\mathcal{J}}\rvert_s$	Stratified topological simplex associated to a flag $\mathcal{J}$	P. 20
$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$	$\mathcal{I}$-th generalized homotopy link of $\mathcal{X}$	P. 21
$\mathcal{H}o\mathrm{Link}$	Homotopy link diagram functor	P. 21
$\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$	Stratified homotopy link with paths starting in the p-stratum	P. 18
$\lvert - \rvert_s$	Notation for (several) stratified realization functors	P. 27, 50
Sing_s	Notation for (several) functors of stratified singular simplices	P. 27, 50
HoLink	Simplicial homotopy link diagram functor	P. 32

In the following list of homotopical settings (such as quasi-categories, model categories . . .) we generally only list either the simplicial version of a semi-model category, or its non-simplicial counterparts. The other, respective, notation can be obtained by adding or removing the underlining. We also want to draw the reader's attention to the notational conventions of Notation 1.2.3.8, 1.2.3.11 and 1.2.4.2.

∞- and (semi-)model categories, and categories with weak equivalences

$\mathcal{S}\mathbf{paces}$	∞-category of spaces	P. 22
$\mathcal{C}\mathbf{at}_\infty$	∞-category of small ∞-categories (quasi-categories)	P. 41
$\mathcal{L}\mathbf{ay}_\infty$	∞-category of small layered ∞-categories (quasi-categories)	P. 46
$\mathcal{AS}\mathbf{trat}$	∞-category of abstract stratified homotopy types	P. 45
$\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{d}}_P$	Category with weak equivalences obtained by equipping $\mathbf{Strat}$ with stratum-preserving diagrammatic equivalences	P. 21, 31
$\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$	∞-category obtained by localizing $\mathbf{Strat}_P$ at stratum-preserving diagrammatic equivalences	P. 21
$\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$	Category with weak equivalences obtained by equipping $\mathbf{Strat}$ with poset-preserving diagrammatic equivalences	P. 31
$\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$	∞-category obtained by localizing $\mathbf{Strat}$ at poset-preserving diagrammatic equivalences	P. 21
$\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$	Model category of P-stratified simplicial sets equipped with the Douteau-Henriques model structure	P. 28, 29
$\mathbf{s}\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{d}}_P$	Category with weak equivalences obtained from $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$	P. 28, 31
$\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$	∞-category obtained by localizing $\mathrm{s}\mathbf{Strat}_P$ at weak equivalences in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$	P. 28
$\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$	Model category of stratified simplicial sets equipped with the (global) Douteau-Henriques model structure	P. 29, 45
$\mathrm{s}\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$	Category with weak equivalences obtained from $\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$	P. 29, 31, 45
$\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$	∞-category obtained by localizing $\mathrm{s}\mathbf{Strat}$ at weak equivalences in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$	P. 28, 29, 45
$W_{P,\mathfrak{c}}$	A class of weak equivalences of stratified spaces defined by Haine, later called categorical equivalences	P. 42
$\mathcal{D}\mathbf{\acute{e}c}_P$	∞-category of décollages associated to P	P. 42
$\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$	Model category of P-stratified simplicial sets equipped with the Joyal-Kan model structure	P. 43
$\mathbf{s}\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c}}_P$	Category with weak equivalences obtained from $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$	P. 31, 43
$\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$	∞-category obtained by localizing $\mathrm{s}\mathbf{Strat}_P$ at weak equivalences in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$	P. 28, 43
$\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{inj}}$	Injective model structure on simplicial presheaves on $\mathrm{sd}(P)$	P. 44
$\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{d\acute{e}}}$	Model structure for décollages on $\mathrm{sd}(P)$	P. 44
$\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$	Model category of stratified simplicial sets equipped with the global Joyal-Kan model structure	P. 45
$\mathrm{s}\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$	Category with weak equivalences obtained from $\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$	P. 31, 45
$\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$	∞-category obtained by localizing $\mathrm{s}\mathbf{Strat}$ at weak equivalences in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$	P. 28, 45
$\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$	Category with weak equivalences obtained by equipping $\mathbf{Strat}$ with poset-preserving categorical equivalences	P. 46
$\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$	∞-category obtained by localizing $\mathbf{Strat}$ at poset-preserving categorical equivalences	P. 28, 46

$s\mathbf{Strat}^{\mathfrak{c}}$	Model category of stratified simplicial sets equipped with the categorical model structure, defined by left Bousfield localizing $s\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ at the refinement maps	P. 48
$s\boldsymbol{S}\mathbf{trat}^{\mathfrak{c}}$	Category with weak equivalences obtained from $s\mathbf{Strat}^{\mathfrak{c}}$	P. 31, 48
$s\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$	∞-category obtained by localizing $s\mathbf{Strat}$ at weak equivalences in $s\mathbf{Strat}^{\mathfrak{c}}$	P. 28, 48
$\boldsymbol{S}\mathbf{trat}^{\mathfrak{c}}$	Category with weak equivalences obtained by equipping $\mathbf{Strat}$ with categorical equivalences	P. 49
$\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$	∞-category obtained by localizing $\mathbf{Strat}$ at categorical equivalences	P. 31, 49
$\underline{\mathbf{Strat}}_P^{\mathfrak{d}}$	Simplicial semi-model category obtained by equipping $\underline{\mathbf{Strat}}_P$ with the diagrammatic semi-model structure	P. 357
$\underline{\mathbf{Strat}}_P^{\mathfrak{c}}$	Simplicial semi-model category obtained by equipping $\underline{\mathbf{Strat}}_P$ with the categorical semi-model structure	P. 357
$\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$	Simplicial semi-model category obtained by equipping $\underline{\mathbf{Strat}}$ with the poset-preserving diagrammatic semi-model structure	P. 55, 357
$\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$	Simplicial semi-model category obtained by equipping $\underline{\mathbf{Strat}}$ with the poset-preserving categorical semi-model structure	P. 55, 357
$\underline{\mathbf{Strat}}^{\mathfrak{d}}$	Simplicial semi-model category obtained by equipping $\underline{\mathbf{Strat}}$ with the diagrammatic semi-model structure	P. 359
$\underline{\mathbf{Strat}}^{\mathfrak{c}}$	Simplicial semi-model category obtained by equipping $\underline{\mathbf{Strat}}$ with the categorical semi-model structure	P. 55, 359
$\underline{s\mathbf{Strat}}^{\mathfrak{d}}$	Simplicial semi-model category obtained by equipping $\underline{s\mathbf{Strat}}$ with the diagrammatic semi-model structure	P. 269

Chapter 2

Generalized simple homotopy theories

Note to the reader: This chapter presents the results of Part III of this thesis, which are concerned with generalized simple homotopy theory (GSHT for short). Our focus will be on presenting a model categorical approach to generalized simple homotopy theory. Hence, most of the results presented in this chapter can be understood without having read Chapter 1. Nevertheless, we will frequently return to the specific case of the stratified homotopy theories presented in Chapter 1 as a motivational example, and Section 2.5 will be entirely concerned with a simple homotopy theory associated to the diagrammatic stratified homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_{P}$. We will also make use of notation introduced in the previous chapter. We thus recommend reading Chapter 1 before reading this chapter and furthermore refer to the notation list in Section 1.5. Finally, we note that after Chapter 13 there is a list of notation used in Part III that can be of use when reading this chapter.

Contents

2.1 Introduction

A classical question in group theory is to decide whether two finite presentations, $\langle S|R\rangle$ and $\langle S'|R'\rangle$, present the same group. While this problem is generally undecidable (in the mathematical sense, see [MKS04], for an overview), at least in the affirmative case – when the two presentations do in fact present the same group – Tietze's theorem states that this identity can be verified by inductively modifying the presentations in terms of certain elementary operations, the so-called Tietze transformations (see, for example, [MKS04, Cor. 1.5], for details). A central question in the study of homotopy theory in the 30's and 40's, tackled most prominently in Whitehead's articles [Whi39; Whi50], was whether an analogous statement could be made in the world of homotopy theory. The analogy should be translated as follows:

1. The role of groups is taken by homotopy types and the role of group isomorphisms is taken by homotopy equivalences;
2. The role of finite group presentations is taken by finite (more or less) combinatorial objects, such as simplicial complexes or CW-complexes, which present homotopy types;
3. The role of elementary operations is taken by so-called *elementary expansions* (and their inverses, the *elementary collapses*) detailed below.

Let us give a precise version of the question for the case of CW-complexes.[1] Here, we will consider the decomposition into open cells of a CW-complex as part of its defining data. For $n \geq 0$, we denote by $D^{n+1} \subset \mathbb{R}^{n+1}$ the topological unit disk, by S^n its boundary, and by S^n_- the lower hemisphere of S^n, given by only such vectors whose $(n+1)$-th component is non-positive. The inclusion $S^n_- \hookrightarrow D^{n+1}$ has a natural relative cell structure, with the open cells given by the interior of D^{n+1} and the open upper hemisphere, i.e., the complement of S^n_- in S^n. Now, suppose we are given a finite CW-complex $\mathfrak{X}$, with underlying space X, together with a pushout diagram

$$\begin{tikzcd} S^n_- \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ D^{n+1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X' . \end{tikzcd} \tag{2.1}$$

Observe first that as the left hand vertical of this diagram is a cofibration and a homotopy equivalence, so is the right hand vertical. Furthermore, $X \hookrightarrow X'$ has a natural relative cell structure, given by the images of the two cells of $S^n_- \hookrightarrow D^{n+1}$ under the lower horizontal map. Now, finally, suppose that the upper horizontal is such that it maps the boundary of S^n_-

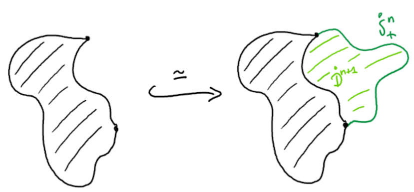


Figure 2.1: Illustration of an elementary expansion with the two new cells in different greens.

into the $(n-1)$-skeleton of $\mathfrak{X}$, and such that it maps S^n_- into the n-skeleton of $\mathfrak{X}$. Then X' inherits the structure of a CW-complex with its cell structure given by the cells of $\mathfrak{X}$, and the two additional open cells of the relative cell complex $X \hookrightarrow X'$. We denote the resulting CW-complex, consisting of X' together with this choice of cell structure, by $\mathfrak{X}'$. To summarize, we have obtained an inclusion of subcomplexes $\mathfrak{X} \hookrightarrow \mathfrak{X}'$ that is a homotopy equivalence. A (cell

[1]Note that presenting the question in terms of CW-complexes is slightly a-historical. The original question was phrased for simplicial complexes in [Whi39]. It was then rephrased for CW-complexes in [Whi50]. In fact, this was one of the original motivations for the definition of a CW-complex.

structure preserving) inclusion of finite CW-complex $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ is called an *elementary expansion*, if it can be constructed in this manner (see Fig. 2.1, for an illustration).
Analogously to the group theoretic picture, one can now ask the question whether one can verify that two finite CW-complexes $\mathfrak{X}$ and $\mathfrak{Y}$ present the same homotopy type purely in terms of elementary expansions, and their (homotopy) inverses. Slightly more refinedly, one may ask whether a specific homotopy equivalence $\varphi\colon \mathfrak{X} \simeq \mathfrak{Y}$ can be verified in terms of such operations, that is, whether its image in the homotopy category of spaces is given by a composition

$$\mathfrak{X} = \mathfrak{X}_0 \xrightarrow{e_0^{\pm 1}} \dots \xrightarrow{e_k^{\pm 1}} \mathfrak{X}_k = \mathfrak{Y}$$

of elementary expansions and their homotopy inverses. If this is the case, φ is called a *simple homotopy equivalence*. The study of questions concerned with this notion of simple homotopy equivalence is often referred to as *simple homotopy theory*.
In [Whi50], Whitehead defined a complete obstruction to a homotopy equivalence $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ being a simple homotopy equivalence, called the *Whitehead torsion* and denoted by $\langle \varphi \rangle$ here; it is an element of an abelian group, $\mathrm{Wh}(X)$, that can be constructed entirely in terms of the fundamental group $\pi_1(X)$ of X (for connected X). The abelian group $\mathrm{Wh}(X)$ is nowadays known as the *Whitehead group* of X. In particular, it follows that whenever $\pi_1(X)$ is such that $\mathrm{Wh}(X)$ vanishes, then every homotopy equivalence of finite CW-complexes with source $\mathfrak{X}$ is a simple homotopy equivalence. This holds, for example, if $\pi_1(X)$ vanishes or, more generally, is a free abelian group (see, for example, [BHS64]).
Let us now recall some equivalent descriptions of the Whitehead group, beginning with the algebraic perspective (see [Coh73; WJR13], for details). For the sake of simplicity, we will generally assume spaces to be connected, even if this is only necessary for some of the descriptions below.

(D1) Given a group G, we denote by $\mathrm{Gl}(\mathbb{Z}[G]) = \varinjlim \mathrm{Gl}_n(\mathbb{Z}[G])$ the infinite general linear group[2] of the group ring $\mathbb{Z}[G]$ (see, for example, [Coh73]). We then denote by $\mathrm{Wh}(G)$ the (abelian) quotient group of $\mathrm{Gl}(\mathbb{Z}[G])$ obtained by modding out the subgroup generated by:

- The (equivalence classes of) elementary matrices $1_{n\times n} + rE_{i,j}$, for $1 \le i, j \le n$, where $r \in \mathbb{Z}[G]$ and $E_{i,j}$ is the $n \times n$ matrix with 1 at (i,j) and 0 everywhere else;
- The (equivalence classes of) the diagonal matrices $1_{n\times n} - (1 \mp g)E_{n,n}$, for $g \in G$ and $n \ge 1$, which have $\pm g$ at the n-th entry of the diagonal, and 1 everywhere else.

If one only only mods out by the subgroup generated by elements as above, but requiring $g = 1$ in the second point, then one obtains what is called the first reduced K-group of $\mathbb{Z}[G]$, denoted $\tilde{K}_1(\mathbb{Z}[G])$. From the perspective of algebraic K-theory, this means that $\mathrm{Wh}(G)$ is the cokernel of the map $G \to \tilde{K}_1(\mathbb{Z}[G])$ which sends an element g of G to the diagonal matrix with g at its first entry, and 1 everywhere else. The Whitehead group of a connected space X can then be defined as $\mathrm{Wh}(X) := \mathrm{Wh}(\pi_1(X))$ [3]. In fact, the study of Whitehead groups of spaces was one of the original motivating questions for algebraic K-theory. This purely algebraic description of the Whitehead group in terms of the fundamental group makes is possible to compute Whitehead groups in certain cases, and much progress has been made in this direction over the years since Whitehead's original definition (see, for example, [Oli88]).

(D2) When X is a topological space with the weak homotopy type of a finite CW-complex, then $\mathrm{Wh}(X)$ is in bijection with the set of equivalence classes of presentations of X in terms of finite CW-complexes modulo simple homotopy equivalences. More specifically,

[2] The colimit is taken over the embeddings of general linear groups $\mathrm{Gl}_n(\mathbb{Z}[G]) \hookrightarrow \mathrm{Gl}_{n+1}(\mathbb{Z}[G])$ given by upper left blockmatrices.

[3] Choices of basepoint turn out to be irrelevant, even for a functorial formulation. One can also, instead, construct the Whitehead group in terms of the fundamental groupoid.

by a presentation of X we mean an isomorphism in the homotopy category of topological spaces ho**Top**, $w\colon X \xrightarrow{\simeq} \mathfrak{Y}$, where $\mathfrak{Y}$ is a finite CW-complex with fixed cell structure. Then the Whitehead group $\mathrm{Wh}(X)$ is in natural bijection with the set of equivalence classes of such presentations of X, subject to the relation that $(w\colon X \xrightarrow{\simeq} \mathfrak{Y}) \sim (w'\colon X \xrightarrow{\simeq} \mathfrak{Y}')$, if $w \circ w'^{-1}$ is given by a simple homotopy equivalence. In other words, the Whitehead group $\mathrm{Wh}(X)$ measures the non-uniqueness of presentations of the (weak) homotopy type of X in terms of finite CW-complexes, modulo elementary expansions.

(D3) If we fix the structure of a finite CW-complex $\mathfrak{X}$ on a space X, then this determines an isomorphism (induced by Whitehead torsions) between $\mathrm{Wh}(\pi_1(X))$ and a geometric description of the Whitehead group defined as follows: Consider the set of inclusions of finite CW-complexes $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ that are also homotopy equivalences.[4] Then, consider the quotient set of this set under the equivalence relation generated by $(\mathfrak{X} \hookrightarrow \mathfrak{Y}) \sim (\mathfrak{X} \hookrightarrow \mathfrak{Y} \overset{e}{\hookrightarrow} \mathfrak{Y}')$ where e is an elementary expansion. This set can be given a group structure, with neutral element given by the identity on $\mathfrak{X}$ and addition given by taking the diagonal in pushout squares of inclusions of cell complexes

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hook, "\simeq"] \arrow[d, hook, "\simeq"'] \arrow[dr, dotted, "\simeq"] & \mathfrak{Y} \arrow[d, hook, "\simeq"] \\
\mathfrak{Z} \arrow[r, hook, "\simeq"'] & \mathfrak{Y} \cup_{\mathfrak{X}} \mathfrak{Z}\,.
\end{tikzcd} \tag{2.2}$$

From this perspective, the Whitehead torsion of a homotopy equivalence $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is given by the equivalence class of the (appropriately cellularized) inclusion into the mapping cylinder of φ, $\mathfrak{X} \hookrightarrow M_\varphi$.

(D4) There are analogous descriptions of the Whitehead group as in the previous two points, where one replaces CW-complexes by simplicial complexes. In this case, the role of elementary expansions is taken by considering inclusions of simplicial complexes, $K \hookrightarrow L$, that are given by filling in a horn $\Lambda^n \subset K$ (see [Whi39] and [Waa21, p. 2.4.1], for a proof that this description agrees with the topological one).

(D5) In the case where $X = M$ is a closed smooth manifold of dimension greater or equal to 5, the s-cobordism theorem of Barden, Mazur and Stallings (see [LM24] for a great overview) provides the following interpretation of the Whitehead group of M: $\mathrm{Wh}(M)$ is in bijection with diffeomorphism classes of h-cobordisms $M \hookrightarrow W$ relative to M - i.e., cobordisms W from M to some closed manifold N, such that both inclusions $M, N \hookrightarrow W$ are homotopy equivalences. [5]

(D6) The Whitehead group of a space X also arises as the first homotopy group of a spectrum $\mathbf{Wh}(X)$ that is given by the homotopy cofiber of the assembly map in A-theory (see [WJR13] for details and definitions).

Together, these different perspectives provide intricate connections between algebraic K-theory, simple homotopy theory and the topology of manifolds. For example, in the topological case of Description (D5), where M is simply connected, one obtains that $\mathrm{Wh}(M) = \mathrm{Wh}(0) = \tilde{K}_1(\mathbb{Z}) = 0$, and hence that every h-cobordism $M \hookrightarrow W$ is homeomorphic to a cylinder $M \times [0,1]$. As one application, one can derive from this result an answer to the Poincaré conjecture in dimensions greater than or equal to 6 (see, for example, [LM24]).

In this thesis, we want to study generalized versions of simple homotopy theory. That is, we want to perform simple homotopy theory in other homotopy-theoretic frameworks than just the setting of spaces (or chain complexes), which are the classical settings for such

[4] Strictly speaking, this may be a proper class, but up to cell structure preserving isomorphisms there is only a set of such inclusions.

[5] Analogous claims hold in the PL and the topological world. See [LM24], for an overview.

investigations. The approach we take here is derived from Description (D2). Namely, we will focus on the following perspective:

> *Simple homotopy theory can be understood as the study of non-uniqueness of presentations of homotopy types, up to certain elementary operations.*

The original motivation to pursue generalized simple homotopy theory in this sense arose as follows:
In the author's master's thesis, we investigated the question of defining an analogue of the Whitehead group for stratified homotopy types (in this case presented by stratified simplicial sets). Having seen the several different approaches to stratified homotopy theory in Chapter 1, the reader may not be too surprised that the stratified setting presents one with several choices to be made, which may not necessarily lead to the same group. Clearly, there are choices to be made when it comes to the stratified homotopy theory that one considers, but there are also choices to be made when it comes to the perspective from which one wants to define a Whitehead group. In other words, one should have in mind the specific question of which of the several perspectives in Descriptions (D1) to (D6) one wants the Whitehead group to answer. For example, in [BQ79], Browder and Quinn defined a Whitehead torsion for stratified h-cobordisms for a class of PL stratified spaces with manifold strata (of dimension greater than or equal to 5), and showed that the obstruction to being stratum-preserving PL homeomorphic to a cylinder lies in the abelian group given by the sum of the Whitehead groups of the strata (see also [Wei94]).
On the other hand, the focus in [Waa21] was on answering the question of whether one can present stratified homotopy equivalences (or weaker notions of equivalences of stratified spaces) in terms of certain elementary combinatorial moves. In more recent years, this classical combinatorial perspective on simple homotopy theory has again gained in importance, mainly due to its applications in applied topology (see [For98; BP19; Bau21; CGN16]). In particular, Banagl, Mäder and Sadlo applied such elementary combinatorial operations in stratified topological data analysis in [BMS24]. To tackle the question of presenting stratified homotopy equivalences in terms of elementary combinatorial moves, the Whitehead group that one is concerned with should admit descriptions analogous to Descriptions (D2) to (D4). The Whitehead group of stratified simplicial sets we defined in [Waa21] had precisely this property. Let us recall the definition first.

Definition 2.1.0.1. Let P be a finite poset and let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ be a finite stratified simplicial set. The *diagrammatic stratified Whitehead group of* $\mathcal{X}$, $\mathrm{Wh}_P(\mathcal{X})$, is defined as follows.

- The underlying set is the set of acyclic cofibrations $\mathcal{X} \overset{\simeq}{\hookrightarrow} \mathcal{Y}$ of finite stratified simplicial sets in Douteau's model structure $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$, subject to the relation generated by

$$(\mathcal{X} \overset{\simeq}{\hookrightarrow} \mathcal{Y}) \sim (\mathcal{X} \overset{\simeq}{\hookrightarrow} \mathcal{Y} \overset{\simeq}{\hookrightarrow} \mathcal{Y} \cup_{\Lambda_k^{\mathcal{J}}} \Delta^{\mathcal{J}})$$

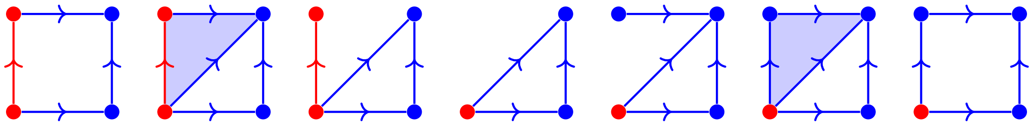
  where $\Lambda_k^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}$ is an admissible horn inclusion (i.e., a horn inclusion that realizes to a stratified homotopy equivalence).

- The sum of two equivalence classes $[(\mathcal{X} \hookrightarrow \mathcal{Y})]$ and $[(\mathcal{X} \hookrightarrow \mathcal{Y}')]$ is defined via diagonals in pushout squares

$$[(\mathcal{X} \overset{\simeq}{\hookrightarrow} \mathcal{Y})] + [(\mathcal{X} \overset{\simeq}{\hookrightarrow} \mathcal{Y}')] \coloneqq [(\mathcal{X} \overset{\simeq}{\hookrightarrow} \mathcal{Y} \cup_{\mathcal{X}} \mathcal{Y}')].$$

In this context, we define an isomorphism $w \colon \mathcal{X} \simeq \mathcal{Y}$ in $\mathrm{hos}\mathbf{Strat}_P^{\mathfrak{d}}$ between finite stratified simplicial sets to be a *simple diagrammatic equivalence*, if it can be written as a composition of morphisms that are presented by *elementary expansions* $\mathcal{A} \hookrightarrow \mathcal{A} \cup_{\Lambda_k^{\mathcal{J}}} \Delta^{\mathcal{J}}$, with $\Lambda_k^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}$ admissible, and their inverses in $\mathrm{hos}\mathbf{Strat}_P^{\mathfrak{d}}$. To any weak equivalence of finite stratified simplicial sets $w \colon \mathcal{X} \simeq \mathcal{Y}$ in $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ (or more generally to an isomorphism in $\mathrm{hos}\mathbf{Strat}_P^{\mathfrak{d}}$), one can associate an element $\langle w \rangle \in \mathrm{Wh}_P(\mathcal{X})$, called the *Whitehead torsion* of w, which vanishes if and only if w is a diagrammatic simple equivalence.

Example 2.1.0.2. Below, we have pictured a simple diagrammatic equivalence – between a stratification of a simplicial model of S^1 over $\{p < q\}$ (to the right) and an alternative stratification in which the p-stratum was thickened (to the left) – in terms of a sequence of stratified elementary expansions and their inverse operations.



Next, let us explain to what extent the stratified diagrammatic Whitehead group admits an analogue of Description (D2). In the following, given two stratified simplicial sets $\mathcal{X}, \mathcal{X}'$, we will identify $\mathrm{hos}\mathbf{Strat}^{\mathfrak{d}}_P(\mathcal{X}, \mathcal{X}') \cong \mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P(|\mathcal{X}|_s, |\mathcal{X}'|_s)$, under the equivalence of Main Result A_1. We will, furthermore, add an exponent "fin", to the notation for categories of stratified simplicial sets to indicate that we restrict to finite stratified simplicial sets.

Construction 2.1.0.3. Let $\mathcal{T} \in \mathbf{Strat}_P$ be a poset-stratified space. Let

$$\mathrm{Pres}(\mathcal{T}) \coloneqq \{(\mathcal{X}, \omega' \colon \mathcal{T} \simeq |\mathcal{X}|_s) \mid \mathcal{X} \in \mathrm{s}\mathbf{Strat}^{\mathrm{fin}}_P, \omega \in \mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P\}/\sim$$

be the quotient set of isomorphisms of $\mathcal{T}$ in $\mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ with the realization of a finite stratified simplicial set, subject to the relation that $(\mathcal{X}_1, \omega_1) \sim (\mathcal{X}_2, \omega_2)$ if

$$\omega_2 \circ {\omega_1}^{-1} \in \mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P(|\mathcal{X}_1|_s, |\mathcal{X}_2|_s) \cong \mathrm{hos}\mathbf{Strat}^{\mathfrak{d}}_P(\mathcal{X}_1, \mathcal{X}_2)$$

defines a simple diagrammatic equivalence between $\mathcal{X}_1$ and $\mathcal{X}_2$.

We can think of $\mathrm{Pres}(\mathcal{T})$ as the set of *presentations of the diagrammatic stratified homotopy type of $\mathcal{T}$ in terms of finite stratified simplicial sets*, subject to the relation that we identify two presentations if they only differ by a finite sequence of gluing in and removing admissible horn inclusions.
We then proved the following theorem.[6]

Theorem 2.1.0.4 ([Waa21])**.** *Let $\mathcal{T} \in \mathbf{Strat}_P$ be a poset-stratified space, let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ be a finite stratified simplicial set, and let $\omega \colon \mathcal{T} \simeq |\mathcal{X}|_s$ be an isomorphism in $\mathrm{ho}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$. Then the map*

$$\begin{aligned} \mathrm{Pres}(\mathcal{T}) &\to \mathrm{Wh}_P(\mathcal{X}) \\ [(\mathcal{X}', \omega')] &\mapsto \langle \omega' \circ \omega^{-1} \rangle \end{aligned}$$

is a bijection.

This bijection can be seen as a stratified analogue of the identity between the descriptions of the classical Whitehead group in Descriptions (D2) and (D3). In particular, it follows that the associated torsion elements in $\mathrm{Wh}_P(\mathcal{X})$ provide complete obstructions to the question of whether a stratum-preserving diagrammatic equivalence of stratified spaces can be presented in terms of elementary expansions arising from admissible boundary inclusions.

The obvious question arises, whether one can also find analogues of the alternative descriptions of the Whitehead group in Descriptions (D2) to (D4) in the stratified scenario. More specifically, one may ask:

(Q1) Is there a topological version of $\mathrm{Wh}_P(\mathcal{X})$ using stratified cell complexes instead of stratified simplicial sets?

[6] This result of [Waa21] predated the proofs of the equivalence between the stratified simplicial and stratified topological homotopy theories, which meant we had to resort to a rather technical proof of fully-faithfulness of homotopy categories in the finite setting in terms of a stratified simplicial approximation theorem. With the techniques now available, a proof becomes significantly easier.

(Q2) Is there an algebraic description of $\mathrm{Wh}_P(\mathcal{X})$? Or, in a similar vein of investigation, can $\mathrm{Wh}_P(\mathcal{X})$ be computed in terms of classical Whitehead groups associated to invariants of $\mathcal{X}$?

(Q3) How does $\mathrm{Wh}_P(\mathcal{X})$ relate to stratified h-cobordisms and to the Whitehead torsion of Browder and Quinn?

(Q4) Can one construct similar Whitehead groups for the alternative approaches to stratified homotopy theory we presented in Chapter 1?

The approach which we will take to address these questions will be quite general. In fact, we only touch on stratified spaces at the very end of the investigation. Already Questions (Q1) and (Q4) made it apparent to us that instead of constructing simple homotopy theories by hand, for each theory one is interested in studying, an axiomatic approach is in order.

The notions of *presentation* and *elementary expansions* can essentially be made sense of in any context of a (semi-)model category, equipped with a set of generating cofibrations (thought of as boundary inclusions of cells) and generating acyclic cofibrations (thought of as horn inclusions into a simplex, or hemisphere inclusions into a disk). It will turn out, that given a solid grasp on the simple homotopy theories arising from such operations, as well as their diagram categories and homotopy colimits, answers to Questions (Q1), (Q2) and (Q4) are rather straightforward to obtain. Describing such a model-categorical approach to generalized simple homotopy theory, and investigating its properties is the content of Part III. We think that the main value of our investigations into model categorical approaches to simple homotopy theory does not necessarily lie in their applications to the stratified scenario, but in their general applicability to all sorts of homotopy theoretic frameworks. Nevertheless, we will use the insights gained in this investigation to obtain a description of the diagrammatic Whitehead groups $\mathrm{Wh}_P(\mathcal{X})$ purely in terms of classical Whitehead groups of strata and generalized homotopy links.

We will now move on to explaining the main results of Part III. As we have already mentioned, a large part of these results is quite general, and can be understood without having any knowledge of stratified homotopy theory. Nevertheless, we will regularly return to the case of stratified homotopy theory, in order to give some immediate applications of the abstract results. Let us finish with a remark on higher simple homotopy theory.

Remark 2.1.0.5. Whitehead groups have higher analogues, arising from the higher homotopy groups of Whitehead spectra (see Description (D6)).[7] The study of these higher Whitehead groups started in [Hat75] by Hatcher, and was continued, for example, by Waldhausen (see [WJR13])[8]. It is, at times, referred to as *higher simple homotopy theory.* We will explicitly not be performing *higher generalized simple homotopy theory* here, at least not intentionally. Rather, we restrict ourselves to the case $n = 1$, i.e., the study of simple equivalences in some abstract homotopy theoretic setting and Whitehead groups in degree 1. This does not mean, of course, that we will not use methods of higher category theory; in fact, they will be used on a regular basis. It rather means that, while there are several natural candidates for Whitehead spaces arising in our theory, we will only study their path components (or maybe fundamental groups, depending on degree shifts). There are several reasons why we have chosen to not dive into the *higher* setting here. On the one hand, we figure that one should first have a series of examples of how simple homotopy theory (in the 1-truncated sense) works in scenarios different from the classical, i.e., the topological or the algebraic one, before one attempts to

[7] In fact, they also have lower analogues. Namely, one can think of the 0-th reduced K-group $\tilde{K}_0(\pi_1 X)$ of a finitely dominated space X as the group of Wall's finiteness obstructions, which are the obstructions to a finitely dominated space having the homotopy type of a finite CW-complex (see [FR01], for an overview).

[8] There is a degree shift in what people have referred to as the Whitehead spectrum, which one should be aware of. For example, what Lurie calls the Whitehead spectrum in his lecture notes on simple homotopy theory is a de-looping of what Hatcher calls the Whitehead spectrum

define what a *generalized higher simple homotopy theory* is. Furthermore, at the very least, we expect one should have an elaborate understanding of higher algebraic K-theory before one attempts to generalize. At the point of writing this, the author cannot claim to have such a deep understanding.
It may very well turn out that the model categorical approach to simple homotopy theory which we present here will ultimately also cover the higher picture, but, at the same time, it may also very well turn out that the notion of equivalences between simple homotopy theories which we consider in the following text fails to preserve some relevant features beyond dimension 1.

2.2 Cell complexes, presentations and expansions

As we have already explained in the previous section, the approach to generalized simple homotopy theory we take is that we are looking to study the uniqueness of presentations of homotopy types in some generalized homotopy theory, up to certain elementary operations. Roughly speaking, when we speak of a presentation of a homotopy type, then we mean *presentation in terms of a weak equivalence with a finite cell complex.* We develop such a general theory of cell complexes in Chapter 8.

2.2.1 Results: Cell complexes and cellularized categories

Let us first give a rigorous discussion of what we mean by a cell complex in a more general categorical framework. To do so, we will quickly give some of the central definitions of Chapter 8. To this end, recall the notion of a transfinite composition (see, [Hir03], for details). Given an ordinal number λ, we will denote by $\lambda + 1$ its successor ordinal, given by the union of λ with $\{\lambda\}$. Recall that a *transfinite composition*, in a category $\mathbf{C}$ with small colimits, is a diagram $X^\bullet \colon \lambda + 1 \to \mathbf{C}$,[9] such that for each limit ordinal $\beta \in \lambda + 1$, the natural morphism $\varinjlim_{\alpha<\beta} X^\alpha \to X^\beta$ is an isomorphism.

Definition 2.2.1.1. Let $\mathbf{C}$ be a category that has small colimits. Let $\mathbb{B}$ be a set of morphisms in $\mathbf{C}$ (which we refer to as the set of *generating boundary inclusions*). By a *structured relative* $\mathbb{B}$*-cell complex*, $\mathfrak{c}$, we mean the following data:

- A morphism $c \colon A \to X \in \mathbf{C}$;
- A subset $\mathfrak{C}_\mathfrak{c} \subset \mathbb{B} \times \bigsqcup_{(\partial D \to D) \in \mathbb{B}} \mathbf{C}(D, X)$, called the set of *characteristic maps*, consisting of pairs of the form $(\partial D \to D, D \to X)$, such that the following holds[10]:
  To keep notation concise, we will refer to an elements of $e \in \mathfrak{C}_\mathfrak{c}$ in the fixed notation $(\partial D_e \to D_e, \sigma_e \colon D_e \to X)$.
  - There exists an ordinal λ, a decomposition
  $$\mathfrak{C}_\mathfrak{c} = \bigsqcup_{\alpha \in \lambda} I_\alpha$$
  and a transfinite composition
  $$A = X^0 \to X^1 \to \cdots \to X^\lambda = X,$$
  composing to $c \colon A \to X$.
  - This transfinite composition is such that for each $\alpha \in \lambda$, there exists a family of pairs of morphisms $(D_e \to X^{\alpha+1}, \partial D_e \to X^\alpha)_{e \in I_\alpha}$, where $D_e \to X^{\alpha+1}$ is a factorization of $\sigma_e \colon D_e \to X$ through $X^{\alpha+1} \to X$ and $\partial D_e \to X^\alpha$ is a factorization of $\partial D_e \to D_e \to$

[9] We think of $\lambda + 1$ as a category with morphisms given by relations.
[10] The ∂D notation is purely formal. We will always use it to refer to the source of morphisms in $\mathbb{B}$.

$X^{\alpha+1}$ through $X^\alpha \to X^{\alpha+1}$. These factorizations are furthermore required to be such that the resulting square

$$\begin{tikzcd} \coprod_{e\in I_\alpha} \partial D_e \arrow[r] \arrow[d, dashed] & \coprod_{e\in I_\alpha} D_e \arrow[d, dashed] \\ X^\alpha \arrow[r] & X^{\alpha+1} \end{tikzcd} \tag{2.3}$$

is a pushout square.

An *absolute structured* $\mathbb{B}$*-cell complex* is a relative structured cell complex where $A = \varnothing$ is an initial object in $\mathbf{C}$. In this case, we just think of $\varnothing \to X$ as the object $X \in \mathbf{C}$.

Observe that structured relative cell complexes are simply relative cell complexes in the sense in which they also show up in the definition of cofibrantly generated model categories (for example, in [Hir03]) together with the additional datum of a choice of characteristic maps.

Notation 2.2.1.2. We will use the general notational convention of referring to structured relative cell complexes by small fraktur letters, i.e. $\mathfrak{c}$; and referring to absolute structured cell complexes by capital fraktur letters, i.e. $\mathfrak{X}$. The underlying morphism (object) in $\mathbf{C}$ will be denoted with the same letter, but using regular font. The associated set of characteristic maps will be denoted in the form $\mathfrak{C}_\mathfrak{c}$ or $\mathfrak{C}_\mathfrak{X}$. We will also refer to the set of characteristic maps of a (relative) cell complex as *the set of cells* of $\mathfrak{X}$. When we refer to a structured absolute or relative complex as finite, we mean that its set of cells is finite.

A similar definition of cell complexes in a general category was given in [Hir03][11]. For our investigation of simple homotopy theory, it will be crucial that the set of characteristic maps is part of the data of a relative structured cell complex (hence the "*structured*"). When a set of boundary inclusion $\mathbb{B}$ is specified, then we will usually omit the $\mathbb{B}$ from "$\mathbb{B}$*-cell complex*" and just speak of structured cell complexes. In order to obtain a well-behaved theory of cell complexes, some additional assumptions need to be made. These are encoded in the following definition:

Definition 2.2.1.3. A *celullarized category* consists of the data of

1. a category $\mathbf{C}$ that has small colimits,
2. a set $\mathbb{B}$ of morphisms in $\mathbf{C}$,

such that the following holds:

P(i) $\mathbb{B}$ contains no isomorphisms;

P(ii) All relative $\mathbb{B}$-cell complexes $a\colon A \to B$ have the property that every pushout square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \tag{2.4}$$

in $\mathbf{C}$ is also a pullback square.

The elements of $\mathbb{B}$ will be called *generating boundary inclusions* and denoted in the form $\partial D \to D$. Property P(i), which states that no generating boundary inclusion is an isomorphism, is a rather minor assumption. Clearly, it can always be ensured by removing some isomorphisms from $\mathbb{B}$. It is essentially only there because we prefer to work in a theory where isomorphisms

[11] Hirschhorn considered the transfinite composition and the pushout squares to be part of the data. We will take the approach that only the set of characteristic maps is part of the data.

of absolute structured cell complexes (which we define in a second) can be detected in terms of bijections on the sets of cells. The second assumption, a slightly weaker version of which was assumed in the definition of a cellular model category in [Hir03], has major consequences. In particular, it ensures that every relative cell complex is a monomorphism, but significantly more can be derived from it (see Section 8.1, for details). Let us first give some seminal examples of cellularized categories and their structured cell complexes in order to convince the reader of the naturality of these assumptions. Proofs that these examples form cellularized categories are provided in Section 8.1.

Example 2.2.1.4. The category of sets, **Set**, equipped with the single boundary inclusion $\varnothing \to *$, is easily verified to be a cellularized category. The relative cell complexes are precisely the inclusions of sets. Every inclusion of sets $A \hookrightarrow X$ admits exactly one cell structure, namely the one where the cells are given by the elements of $X \setminus A$.

Example 2.2.1.5. The category of topological spaces (or one of its appropriately generated derivatives) **Top**, equipped with the set of boundary inclusions given by inclusions of the boundaries of disks $\mathbb{B} = \{\partial D^n \hookrightarrow D^n \mid n \geq 0\}$ is a cellularized category. The structured relative cell complexes are essentially relative CW-complexes, together with choices of characteristic maps, where the assumption that gluing maps $\partial D^n \to X$ need to map into the $(n-1)$-skeleton are dropped.

Example 2.2.1.6. If we are looking to recover CW-complexes, we need to introduce additional data which ensures that dimensions are appropriately respected in the gluing process. Consider the category of filtered topological spaces **Filt**, defined as follows. An object of **Filt** is a space $T \in \mathbf{Top}$, together with a family of subsets $T^0 \subset \cdots \subset T^n \subset \cdots \subset T^\infty = T$. We will usually just write T to refer to the whole filtered space. Morphisms from $(T_0, (T_0^n)_{n\in\mathbb{N}})$ to $(T_1, (T_1^n)_{n\in\mathbb{N}})$ are given by continuous maps $f \colon T_0 \to T_1$, such that $f(T_0^n) \subset T_1^n$, for all $n \in \mathbb{N}$. This category has all small colimits. They are given by taking the colimit of the underlying spaces, and then equipping it with the smallest filtration that makes the structure maps into the colimit filtration preserving. For $n \geq 0$, denote by E^n the filtered space obtained by equipping D^n with the filtration.

$$k \mapsto \begin{cases} \varnothing & , k < n-1 \\ \partial D^n & , k = n-1 \\ D^n & , k \geq n\,. \end{cases}$$

Furthermore, denote by ∂E^n the filtered subspace of E^n given by

$$k \mapsto \begin{cases} \varnothing & , k < n-1 \\ \partial D^n & , k \geq n-1. \end{cases}$$

If we set $\mathbb{B} = \{\partial E^n \hookrightarrow E^n \mid n \geq 0\}$, then a structured $\mathbb{B}$-complex $\mathfrak{X} \colon \varnothing \to X$ specifies exactly the same data as a classical CW-complex, equipped with choices of characteristic maps. Furthermore, the morphisms in **Filt** between two such complexes $\mathfrak{X}$ and $\mathfrak{Y}$ are precisely the cellular maps.

Remark 2.2.1.7. We will not discuss the case of classical CW-complexes much from here on out. This is for the following reason: When we move on to simple homotopy theory, this will be done through the perspective of cofibrantly generated semi-model categories. We suspect that one can define semi-model structures on filtered spaces that present the homotopy theory of spaces in which generating cofibrations are as in Example 2.2.1.6. While this is certainly an interesting avenue to pursue, for all intents and purposes we have in mind, using either simplicial sets or topological spaces is entirely sufficient. Furthermore, in the way simple homotopy theory of CW-complexes is classically performed, for example in [Coh73; Whi50], the definition of cell complex that is used does not recall the characteristic map, but instead only the decomposition into open cells. In this sense, the exact geometric definition of Whitehead groups in [Coh73] does not really fit into the framework of cellularized categories anyway. We will use a more general framework for simple homotopy theory later on, to also incorporate examples that do not arise from cellularized categories, in order to compare to the classical definitions.

Example 2.2.1.8. The category of simplicial sets **sSet** equipped with the set of boundary inclusions $\mathbb{B} = \{\partial\Delta^n \to \Delta^n \mid n \geq 0\}$ is a cellularized category. The relative cell complexes are precisely the inclusions of simplicial sets $A \hookrightarrow X$. Every such inclusion admits exactly one cell structure, namely the one given by the set of non-degenerate simplices in X that are not in A, $X_{n.d.} \setminus A_{n.d.}$.

Example 2.2.1.9. The category of non-negatively graded chain complexes of (left)modules over some (not-necessarily commutative) ring R, $\mathbf{Ch}_{\geq 0}(R)$, admits the structure of a cellularized category. In the following, we use shift notation in the form $A_\bullet[-n]$, to indicate the chain complex given in degree k by A_{k-n}. For $n \geq 0$, we denote by $D^n_\bullet$ the chain complex

$$\cdots \to 0 \to R \xrightarrow{1} R \to 0 \cdots \to 0$$

which is non-zero exactly in degree n and $n-1$ (n, if $n = 0$). Observe that maps $D^n_\bullet \to X_\bullet$, for $X_\bullet \in \mathbf{Ch}_{\geq 0}(R)$ are in natural one-to-one correspondence with the elements of X_n. Denote by $\partial D^n_\bullet = R[1-n]$, the subcomplex given by R at $n-1$ (supposing $n \geq 1$) and 0 everywhere else (given by $\partial D^n_\bullet = 0$ if $n = 0$). Then set

$$\mathbb{B} = \{\partial D^n_\bullet \hookrightarrow D^n_\bullet \mid n \geq 0\}.$$

An (absolute) structured cell complex $\mathfrak{X}\colon 0 \to X_\bullet$ specifies the same data as a choice of basis for a free chain complex $X_\bullet$ in each degree. A structured relative cell complex $\mathfrak{c}\colon A_\bullet \hookrightarrow X_\bullet$ specifies the data of elements $\{b_i\} \subset X_n$, for each $n \geq 0$, such that $\{[b_i]\}$ is a basis of X_n/A_n. In particular, it follows that an absolute structured cell complex $\mathfrak{X}$ is essentially the same thing as a free chain complex $X_\bullet$ together with a choice of basis in each degree.

Remark 2.2.1.10. These examples already illustrate well the different roles that cell structures can take in simple homotopy theory. When working in the realm of simplicial sets, the cell structure is always intrinsic and there is no need to explicitly specify characteristic maps. When working with CW-complexes, the cell structure (a priori) needs to be specified. However, it is a deep result due to Chapman (see [Cha74]) that every homeomorphism of finite CW-complexes is a simple homotopy equivalence, which makes it generally acceptable to omit cell structures. When working in the world of algebra, then remembering the cell structure (i.e., the choice of basis) is of utmost importance. Consider an isomorphism of free modules over a group ring $\phi\colon \mathbb{Z}[G]^n \to \mathbb{Z}[G]^n$. We can either think of ϕ as an isomorphism of chain complexes concentrated in degree 0, or as a matrix in $\mathrm{Gl}_n(\mathbb{Z}[G])$, specifying an element in the Whitehead group $\mathrm{Wh}(G)$. Clearly, this assignment relies on the choice of basis on $\mathbb{Z}[G]^n$, and if we are allowed to choose an arbitrary alternative basis on the target, then we may very well take the associated matrix to be the identity matrix, making the associated element in the Whitehead group 0.

Finally, for our investigations of stratified simple homotopy theory, the following two examples are important.

Example 2.2.1.11. Given a fixed poset P, we obtain a cellularized category by equipping $\mathrm{s}\mathbf{Strat}_P$ with the set of stratified boundary inclusions $\{\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}} \mid \mathcal{J} \in \Delta_P\}$. Just as in the non-stratified simplicial setting, each inclusion of stratified simplicial sets $\mathcal{A} \hookrightarrow \mathcal{X}$ has a unique-cell structure, given by the non-degenerate simplices of $\mathcal{X}$ not contained in $\mathcal{A}$. On the topological side, $\mathbf{Strat}_P$, we may consider the set of stratified boundary inclusions $\{|\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P\}$. The resulting structured cell complexes are precisely the structured stratified cell complexes we used in Chapter 6.

Notation 2.2.1.12. We will generally denote a cellularized category just by its underlying category, and keep the set of generating boundary inclusions $\mathbb{B}$ implicit. When multiple cellularized categories are involved, we will use the notation $\mathbb{B}_{\mathbf{C}}$ to refer to the generating boundary inclusions of $\mathbf{C}$. We will also assume that no two morphisms in $\mathbb{B}$ have the same target (clearly, this can always be achieved by an immaterial set-theoretic modification). This has the advantage that the characteristic maps $(\partial D \to D, D \to X) \in \mathfrak{C}_{\mathfrak{c}}$, for a structured relative cell complex $\mathfrak{c}\colon A \to X$, are uniquely specified by the map $D \to X$, and we will regularly use this to treat them as morphisms in $\mathbf{C}$ with target X, not as tuples. In particular, it allows us to think of $\mathfrak{C}_{\mathfrak{c}}$ as a subset of $\bigsqcup_{\partial D \to D \in \mathbb{B}} \mathbf{C}(D, X)$.

Definition 2.2.1.13. Let $\mathbf{C}$ be a cellularized category. By a *structure preserving morphism* of two such structured relative cell complexes $A_0 \overset{\mathfrak{c}_1}{\hookrightarrow} X_0$ and $A_1 \overset{\mathfrak{c}_1}{\hookrightarrow} X_1$, we mean a pair of morphisms $(f_A\colon A_0 \to A_1, f_X\colon X_0 \to X_1)$ in $\mathbf{C}$ fitting into a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "f_A"] \arrow[d, hook, "c_0"] & A_1 \arrow[d, hook, "c_1"] \\ X_0 \arrow[r, "f_X"] & X_1 , \end{tikzcd} \tag{2.5}$$

in $\mathbf{C}$ such that, for every characteristic map $\sigma \in \mathfrak{C}_{\mathfrak{c}_0}$ the induced map $f \circ \sigma$ is in $\mathfrak{C}_{\mathfrak{c}_1}$. We denote by $\mathbf{RCell(C)}$ the category of structured relative $\mathbb{B}$-complexes and structure preserving morphisms, with the obvious notions of identity and composition. Furthermore, given $A \in \mathbf{C}$, we denote by $\mathbf{RCell(C)}_A$ the subcategory of $\mathbf{RCell(C)}$ given by all relative structured cell complexes with source A and with morphisms given by such morphisms of structured relative cell complexes that are given by the identity on A.

Clearly, we are not only interested in studying (proper) relative cell complexes, but also absolute cell complexes, obtained by letting the source be the initial object $\varnothing \in \mathbf{C}$.

Notation 2.2.1.14. We denote by $\mathbf{Cell(C)}$ the full subcategory of $\mathbf{RCell(C)}$, given by such structured relative cell complexes $A \overset{\mathfrak{c}}{\hookrightarrow} X$, for which $A \cong \varnothing$ is initial in $\mathbf{C}$. As the maps on initial objects are entirely redundant, we will just refer to objects of $\mathbf{Cell(C)}$ in the form $\mathfrak{X}$.

Remark 2.2.1.15. Observe that the category $\mathbf{Cell(C)}$ is rather discrete in nature. To be more precise, just like in most examples of combinatorial categories, there is only a finite set of morphisms between two finite structured cell complexes $\mathfrak{X}$ and $\mathfrak{Y}$ in $\mathbf{Cell(C)}$. Indeed, it is not hard to see from the construction of absolute cell complexes in terms of colimits of generating boundary inclusions that every structure preserving map $\mathfrak{X} \to \mathfrak{Y}$ is entirely determined by the induced map $\mathfrak{C}_{\mathfrak{X}} \to \mathfrak{C}_{\mathfrak{Y}}$.

Next, let us consider two fundamental operations on relative cell complexes, well known from the calculus of cofibrantly generated model categories.

Construction 2.2.1.16. Given a structured relative cell complex $A_0 \overset{\mathfrak{c}_0}{\hookrightarrow} X_0$, and an arrow $g\colon X_0 \to X_1$, we denote

$$g\mathfrak{C}_{\mathfrak{c}_0} := \{g \circ \sigma \mid \sigma \in \mathfrak{C}_{\mathfrak{c}_0}\} \subset \bigsqcup_{\partial D \to D \in \mathbb{B}} \mathbf{C}(D, X).$$

Generally, there is no reason why $g\mathfrak{C}_{\mathfrak{c}_0}$ should define the structure of a cell complex on X_1, or on some relative cell complex $A_1 \hookrightarrow X_1$. However, in the following situations, this is indeed the case.

1. Suppose that the diagram

$$\begin{tikzcd} A_0 \arrow[r, "f_A"] \arrow[d, "c_0"] & A_1 \arrow[d, "c_1"] \\ X_0 \arrow[r, "f'"] & X_1 \end{tikzcd} \tag{2.6}$$

   is a pushout square. Then it follows from the compatibility relations of transfinite compositions and pushouts that c_1 is a relative cell complex and $f'\mathfrak{C}_{\mathfrak{c}_0}$ defines a cell structure on c_1, such that (f, f') defines a morphism of structured relative cell complexes. These types of morphisms in $\mathbf{RCell(C)}$ will be called *cobase change morphisms*. We will also call the associated squares with verticals given by structured relative cell complexes *cobase change squares*.

2. Consider a transfinite composition

$$c\colon A = X^0 \to X^1 \to \cdots \to X^\lambda = X$$

of relative cell complexes, together with, for each $\alpha \in \lambda$, a cell structure $\mathfrak{C}^\alpha$ on $X^\alpha \to X^{\alpha+1}$. Denote by f^α the canonical morphism $X^\alpha \to X$. Then it follows from the composability of transfinite compositions, that the set $\bigcup_{\alpha\in\lambda} f^{\alpha+1}\mathfrak{C}^{\alpha+1}$ defines a cell structure on c. We call the resulting cell complex the *vertical transfinite composition* of the structured relative cell complexes $\mathfrak{c}^\alpha\colon X^\alpha \hookrightarrow X^{\alpha+1}$. Vertical compositions with $\lambda = 2$ will be denoted in the form $\mathfrak{c}_1 \circ \mathfrak{c}_0$.

Remark 2.2.1.17. Cobase changes will be of particular importance when investigating the functoriality of Whitehead groups. Consider the forgetful functor

$$\begin{aligned} \mathbf{RCell(C)} &\to \mathbf{C} \\ (\mathfrak{c}\colon A \to X) &\mapsto A. \end{aligned}$$

Any arrow $f\colon A \to A'$ in $\mathbf{C}$ together with a structured relative cell complex $\mathfrak{c}\colon A \to X$ lifts to a cobase change morphism

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "\mathfrak{c}"] & A' \arrow[d, "\mathfrak{c}'"] \\ X \arrow[r, "f'"] & X' \end{tikzcd} \tag{2.7}$$

unique up to canonical isomorphism over f_A. We denote the structured relative cell complex on the right by $f_{\mathrm{i}}\mathfrak{c} := \mathfrak{c}'$. Cobase changes have a universal property, which extends this construction to a functor

$$f_{\mathrm{i}}\colon \mathbf{RCell(C)}_A \to \mathbf{RCell(C)}_{A'}\,.$$

In fact, $\mathbf{RCell(C)} \to \mathbf{C}$ is a cocartesian fibration (see [nLa24g]), which means we can think of cobase change as a pseudo-functor into the $(2,1)$-category of categories $\mathbf{Cat}$,

$$\begin{aligned} \mathbf{RCell(C)} &\to \mathbf{Cat} \\ A &\mapsto \mathbf{RCell(C)}_A \\ f &\mapsto f_{\mathrm{i}}. \end{aligned}$$

Much of Section 8.1 is dedicated to simply verifying that the theory of structured cell complexes associated to a relative category behaves as one would expect a theory of cell complexes to behave. As is often the case with such purely category-theoretic investigations, the difficulty is mainly one of finding the right definition, and then most of the expected results *fall into place* with rather straightforward proofs. Let us just name a few such statements, without going into rigorous details, to give a flavor of the elementary results obtained in Section 8.1. We will appeal to concrete results later on, when we need them.

- Section 8.1.3: There is a notion of subcomplex of a structured relative cell complex $\mathfrak{c}\colon A \to X$, given by morphisms $\tilde{\mathfrak{c}} \to \mathfrak{c}$ in $\mathbf{RCell(C)}_A$ that induce injections on the sets of cells. Equivalently, these can be characterized as the monomorphisms in $\mathbf{RCell(C)}_A$. A subcomplex is uniquely specified by its set of cells (up to canonical isomorphism). If $\tilde{\mathfrak{c}} \hookrightarrow \mathfrak{c}$ is the inclusion of a subcomplex, then the associated underlying map of objects $\tilde{X} \hookrightarrow X$ inherits the structure of a relative cell complex from $\mathfrak{c}$.

- Section 8.1.5: Subcomplexes admit intersections and unions.

- Section 8.1.6: There is a theory of compactness, which ultimately ensures that morphisms $Y \to \mathfrak{X}$ (from an appropriately compact object Y) into a structured cell complex $\mathfrak{X}$ factor through some finite subcomplex of $\mathfrak{X}$.
- Section 8.2: There is a theory of *cellularized functors*, given by equipping colimit preserving functors between cellularized categories with additional data, such that they lift to functors between the associated categories of relative cell complexes, in a manner that is compatible with the basic operations on relative cell complexes. Cellularization of functors are uniquely determined by making choices of cell structure for the images of each generating boundary inclusion.

2.2.2 Presentations and elementary expansions

Now that we have explained what we mean by a *structured cell complex*, let us explain what we mean by a presentation. Suppose that $\mathbf{C}$ is a cellularized category and suppose that $\mathbf{C}$ is additionally equipped with a class of morphisms $W \subset \mathbf{C}$, which we think of as weak equivalences. We write ho$\mathbf{C}$ for the 1-categorical localization of $\mathbf{C}$ at W.

By a *(finite) presentation of the homotopy type of an object* $Y \in \mathbf{C}$, we mean a finite structured cell complex $\mathfrak{X}$, together with an isomorphism $\omega: Y \simeq X$ in ho$\mathbf{C}$. For the sake of simplicity of notation, we will just write presentations in the form $\omega: Y \simeq \mathfrak{X}$. Now, suppose we are given two presentations $\omega_1: Y \simeq \mathfrak{X}_1$ and $\omega_2: Y \simeq \mathfrak{X}_2$. Composing ω_1^{-1} with ω_2, we obtain an isomorphism $\mathfrak{X}_1 \simeq \mathfrak{X}_2$ in ho$\mathbf{C}$. We can then ask whether this isomorphism can be verified purely in terms of certain elementary operations on the structured cell complexes $\mathfrak{X}_1$ and $\mathfrak{X}_2$.[12] The elementary operations are defined as follows:

Definition 2.2.2.1. A *cellularized category with expansions* consists of:

1. A cellularized category $\mathbf{C}$;
2. A set $\mathbb{E}_{\mathbf{C}} \subset \mathbf{RCell}(\mathbf{C})$ of finite structured relative cell complexes.

Elements of $\mathbb{E}_{\mathbf{C}}$ are called *generating elementary expansions*.

Clearly, any set of finite structured relative cell complexes in a cellularized category $\mathbf{C}$ will define a category with expansions. Here are some more natural examples to keep in mind:

Example 2.2.2.2. The cellularized category of simplicial sets **sSet** (with the standard set of boundary inclusions) can be equipped with the set of horn inclusions

$$\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1, 0 \leq k \leq n\}.$$

We will see that these expansions lead to classical simple homotopy theory.

Example 2.2.2.3. The cellularized category of topological spaces **Top** (with the boundary inclusions given by the boundary inclusions of disks) can be equipped with a class of generating elementary expansions as follows. First off, note that we may replace the boundary inclusions of disks by boundary inclusions $|\partial\Delta^n \hookrightarrow \Delta^n|$, without really changing the theory (see Remark 8.2.3.3). We may then take

$$\mathbb{E}_{\mathbf{C}} = \{|\Lambda^n_k \hookrightarrow \Delta^n| \mid n \geq 1, 0 \leq k \leq n\},$$

[12] The name *presentation* is also justified insofar as we could use the framework to discuss presentations of groups. This can be achieved by localizing the category of filtered topological spaces T, which fulfill $T^0 = \star$, at such maps which induce isomorphisms on fundamental groups. The resulting homotopy category is the category of groups. One may then consider cell complexes with respect to the boundary inclusions $\star \hookrightarrow (\star \subset S^1 \subset S^1 \subset \dots)$ and $(\star \subset S^1 \subset S^1 \subset \dots) \hookrightarrow (\star \subset S^1 \subset D^2 \subset D^2 \subset \dots)$. The resulting structured cell complexes, which are simply 2-dimensional CW-complexes with a unique 0-cell, provide presentations of the associated fundamental groups in terms of generators and relations. If we are looking to compare presentations in this setting, we also need to allow for cells of the form $(\star \subset \star \subset S^2 \subset S^2 \subset \dots) \hookrightarrow (\star \subset \star \subset S^2 \subset D^3 \subset \dots)$, in order to define expansions that modify relations.

with the cell structure on $|\Lambda^n_k \hookrightarrow \Delta^n|$ given by the two cells $|\Delta^n| \xrightarrow{1} |\Delta^n|$ and the inclusion of the k-th face $|\Delta^{n-1}| \hookrightarrow |\Delta^n|$. We could certainly also take a cubical approach to defining expansions. We prefer the simplicial route here, for its evidently preferable interaction with simplicial sets.

Example 2.2.2.4. The category of positively graded chain complexes $\mathbf{Ch}_{\geq 0}(R)$ over some not-necessarily commutative ring R, cellularized as in Example 8.1.1.15, can be equipped with the following sets of generating expansions: We can consider the set of expansions

$$\mathbb{E}' = \{0 \to D^n_\bullet \mid n \geq 1\}$$

where $D^n_\bullet$ is equipped with the cell structure arising from the obvious basis. This leads to a rather small class of simple equivalences (see Example 2.2.2.9).
A more interesting simple homotopy theory arises if we consider the following class of expansions:
Denote by $I_\bullet$ the chain complex given by

$$\ldots 0 \to R \xrightarrow{(-1,1)} R \oplus R \to 0$$

with $R \oplus R$ in degree 0. Denote by e_0 the inclusions of $R[0]$ into $I_\bullet$ via inclusion in the left component and analogously by e_1 the inclusion in the right component. Given a unit $g \in R$, we denote by $\mathfrak{e}_{0,g}$ the structured relative cell complex obtained by equipping e_0 with the cell structure given by the elements $1 \in \mathbf{R}$ in degree 1 and $(0,g) \in \mathbf{R} \oplus \mathbf{R}$ in degree 0. $\mathfrak{e}_{1,g}$ is defined analogously, using $(g,0)$ as the second basis element. Then, given a subgroup G of the group of units of R, we can consider the set of expansions

$$\mathbb{E}_G := \{\mathfrak{e}_{i,g}[-n] \colon \mathbf{R}[-n] \hookrightarrow I_\bullet[-n] \mid i = 0,1, n \in \mathbb{N}, g \in G\}\,.$$

Example 2.2.2.5. We can equip the cellularized category of P-stratified simplicial sets $\mathrm{s}\mathbf{Strat}_P$ with a class of elementary expansions by considering the set of admissible horn inclusions

$$\{\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}} \mid \mathcal{J} = [p_0 \leq \cdots \leq p_n] \in \Delta_P, 0 \leq k \leq n, |\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s \text{ is a stratified homotopy equivalence}\}\,.$$

However, we could also use the set of admissible horn inclusions and inner stratified horn inclusions. Similarly to Example 2.2.2.3, we obtain the structures of a cellularized category with expansions on $\mathbf{Strat}_P$, by equipping it with the stratified realizations of these horn inclusions.

Notation 2.2.2.6. Just as in the case of cellularized categories, we will omit the expansions from the notation for a cellularized category with expansions, and just write $\mathbf{C}$, to refer to the latter. The associated set of generating expansions is always denoted by $\mathbb{E}_{\mathbf{C}}$.

Definition 2.2.2.7. Let $\mathbf{C}$ be a cellularized category with expansions.

1. A relative cell complex $\mathfrak{e}' \in \mathbf{RCell}(\mathbf{C})$ that fits into a cobase change square

$$\begin{tikzcd} \Lambda \arrow[r, "f"] \arrow[d, hook, "\mathfrak{e}"'] & A \arrow[d, hook, "\mathfrak{e}'"] \\ D \arrow[r] & X \end{tikzcd} \tag{2.8}$$

   where $\mathfrak{e} \in \mathbb{E}_{\mathbf{C}}$, or $\mathfrak{e}$ is an empty relative cell complex is called an *elementary expansion*.

2. A transfinite vertical composition of elementary expansions is called an *expansion*.

3. The inclusion of a subcomplex

$$i \colon (A \overset{\tilde{\mathfrak{c}}}{\hookrightarrow} \tilde{X}) \hookrightarrow (A \overset{\mathfrak{c}}{\hookrightarrow} X)$$

   in $\mathbf{RCell}(\mathbf{C})_A$, for $A \in \mathbf{C}$, is called an *expansion* (an *elementary expansion*), if the associated structured relative cell complex $\mathfrak{i} \colon \tilde{X} \hookrightarrow X$ is an expansion (an elementary expansion).

4. A finite structured relative cell complex $\mathfrak{s}\colon A \hookrightarrow X$ is called a *structured simple equivalence*, if there exists a finite expansion $\mathfrak{e}\colon X \hookrightarrow Y$, such that the vertical composition $\mathfrak{e} \circ \mathfrak{s}\colon A \hookrightarrow X$ is an expansion. An inclusion of finite structured cell complexes $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ is called a *simple equivalence*, if the induced structured relative cell complex $\mathfrak{i}\colon X \hookrightarrow Y$ is a structured simple equivalence.

Remark 2.2.2.8. Also allowing for empty relative cell complexes in the definition of an elementary expansion ensures that all isomorphisms of structured cell complexes define elementary expansions.

Example 2.2.2.9. Let us make explicit what an elementary expansion is in the case of free chain complexes of Example 2.2.2.4. If we take the first definition of generating elementary expansions, then an elementary expansion is (either an isomorphism) or an inclusion

$$A_\bullet \hookrightarrow A_\bullet \oplus D^n_\bullet,$$

for $n \geq 1$, equipped with the obvious choice of basis. General expansions are simply inclusions

$$A_\bullet \hookrightarrow A_\bullet \oplus \bigoplus_{i \in I} D^{n_i}_\bullet.$$

It is not too surprising that this notion of expansion ultimately leads to a notion of simple equivalence which is rather rigid. In particular, these operations can essentially only extend the differential matrices of a based chain complex by taking the sum with an identity matrix. The situation becomes a bit more interesting if we consider the more general set of expansions $\mathbb{E}_G$, for G a subgroup of the group of units of R. Then, a (non-trivial) elementary expansion of a chain complex $A_\bullet$ amounts to adding two free generators b_n and b_{n+1} in two degrees n and $n+1$, and extending the boundary operator d of $A_\bullet$ via

$$d(b_n) = 0$$

and

$$d(b_{n+1}) = \pm g b_n + a,$$

for some $a \in A_n$ and $g \in G$.

2.3 Whitehead model categories

Let us now present a set of axioms that guarantee that a cellularized category with expansions induces a well-behaved simple homotopy theory. Roughly speaking, our approach will mirror the geometric approach to defining Whitehead groups presented in [Coh73], albeit in the framework of a general cofibrantly generated semi-model category, equipped with notions of generating boundary inclusions and expansions. Before we do so, let us remark on two different approaches to generalized simple homotopy theory in the literature.

(A minor change of axioms) Recovering an approach of Eckmann and Siebenmann

Already in [Eck06; Sie70], Eckmann and Siebenmann (independently) gave a very general axiomatic categorical framework, in which one can make sense of the notions of expansions, simple equivalence and Whitehead group. The framework uses as its input data a small category $\mathbf{C}$ admitting pushouts, as well as a class of morphisms $E \subset \mathbf{C}$ that contains all isomorphisms, is stable under cobase change and composition, and fulfills an additional property simplifying the structure of the (1-categorical) localization $\mathbf{C}[E^{-1}]$. [Eck06] then defines the Whitehead group associated to an element $X \in \mathbf{C}$ as the set of arrows

$$\{f\colon X \to Y \in \mathbf{C} \mid f \text{ descends to an isomorphism in } \mathbf{C}[E^{-1}]\}$$

subject to the relation generated by composition with morphisms in E, with addition given by taking pushouts. [Eck06; Sie70] demonstrate that many of the elementary properties of Whitehead groups can already be derived under this minimal set of assumptions.

The intended use of this framework, suggested in this form in [Eck06], is that one should take $\mathbf{C}$ to be a category in which the morphisms are given by inclusions of some notion of cell complexes and should take E to be the class of expansions, associated to the respective framework. These types of categories will essentially never have pushouts. To see this, observe the following category-theoretical lemma, the proof of which is a straightforward exercise in elementary category theory.

Lemma 2.3.0.1. *Suppose that* $\mathbf{C}$ *is a category such that*

- $\mathbf{C}$ *has finite coproducts;*
- *every arrow in* $\mathbf{C}$ *is a monomorphism.*

Then, for every $X, Y \in \mathbf{C}$*, the set* $\mathbf{C}(X,Y)$ *is empty or a singleton. In other words, up to equivalence of categories,* $\mathbf{C}$ *is a poset.*

Any of the categories of inclusions of cell complexes or simplicial complexes that one could have in mind have their morphisms given by monomorphisms, and have an initial object. Hence, it follows that if they had pushouts, they would also have coproducts. They certainly also generally do have morphism sets of cardinality greater or equal to 2, as there will generally be multiple ways to include one cell complex into another.

We think what [Eck06; Sie70] had in mind was not to take the pushout in the category of inclusion of subcomplexes, but instead take the pushout in a larger category containing more morphisms. In any case, the arguments presented in [Eck06] really do not make use of the full universal property of the pushout. All that is really needed is a sufficiently coherent family of squares that behaves *almost like pushout squares*. In Chapter 9, we present an alternative (more general) set of axioms that applies to both the classical scenarios which Eckmann and Siebenmann had in mind as well as to the more general frameworks we presented in the previous section. Modulo the change in axioms and the usage of some more modern category theoretical methods, the results and proofs are really the same as presented in [Eck06]. We will not go into detail here and refer the interested reader to Chapter 9. One should note, however, that many of the results presented in the following sections at least partially rely on the axiomatic results of Eckmann and Siebenmann.

Kamps and Porter's approach to abstract simple homotopy theory

In [KP86], Kamps and Porter presented an alternative approach to abstract simple homotopy theory in the framework of a *category with cofibrations equipped with a generating cylinder* (see [KP86]). In this framework, one starts with a fixed cylinder functor $- \times I \colon \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$ (together with natural boundary inclusions $X \sqcup X \hookrightarrow X \times I$ and projections $X \times I \to X$, which together factor the fold map $X \sqcup X \to X$) and considers the homotopy theory given by homotopy equivalences with respect to this cylinder. Simple equivalences are generated under the two-out-of-three law and pushouts along cofibrations from the cylinder projections $X \times I \to X$ and isomorphisms. This approach has the advantage that the notion of simple equivalence is intrinsic, after one has decided on a choice of cylinder. The authors demonstrate that their axioms lead to a notion of simple homotopy theory that behaves much like the classical case of CW-complexes. For our purposes, this framework is too restrictive for the following reasons:

- In the scenarios which we consider, weak equivalences do not arise as the homotopy equivalences with respect to a cylinder. While one could obtain such a framework by passing to bifibrant objects, this would generally destroy the finiteness assumptions which are essential to our investigations of simple homotopy theory.

- Allowing for all isomorphisms to be simple equivalences does only make sense for our purposes when we work in categories of cell structure preserving morphisms. Otherwise, as is already apparent from the algebraic case, isomorphisms that ignore the cell structure may generally not be simple equivalences in the sense which we want to study. More than that, we really want to work in a framework where morphisms preserve cell structures. The categories of structured cell complexes which we consider are generally too rigid to admit cylinder projections, however.

- Finally, we are explicitly looking to work in a framework were we have control over the generating expansions that we feed into the theory, both in the sense that we want to know that every simple equivalence can be expressed as a zig-zag of elementary expansions, and in the sense that we want to add in additional expansions, to enlarge the class of simple equivalences. That such a change in homotopy theory can generally be done without changing the underlying homotopy theory can be seen in Example 2.3.1.2, for example.

Nevertheless, we will partially incorporate the perspective of Kamps and Porter into our theory, namely, we will require the *existence* of a well-behaved cylinder. Requiring such a cylinder has numerous convenient consequences. For example, it can be used to verify the (modified) axioms of Eckmann and Siebenmann. Furthermore, many of the proofs we provide in our framework are inspired by the ones in [KP86], which are in turn, in all likelihood, inspired by the ones in [Whi50; Coh73].

2.3.1 Results: Elementary properties of Whitehead model categories

In this subsection, we explain our model categorical approach to generalized simple homotopy theory and give the first foundational results that justify the usage of the theory. All definitions and results in this subsection can be found in Chapter 10. Let us now give the axioms of what we call a *Whitehead model category* and give the basic results on the resulting framework. We will first spell them out, and then give definitions of the terminology used.

Definition 2.3.1.1. A cellularized category with expansions $\mathbf{C}$ is called a *Whitehead model category* if $\mathbf{C}$ admits the structure of a cofibrantly generated (left) semi-model category, such that the following holds:

(A1) $\mathbb{B}_{\mathbf{C}}$ provides a set of generating cofibrations and the source and target of every morphism in $\mathbb{B}_{\mathbf{C}}$ are filtration compact.

(A2) For every generating elementary expansion $\mathfrak{e}\colon A \hookrightarrow X \in \mathbb{E}_{\mathbf{C}}$, it holds that the underlying morphism $e\colon A \hookrightarrow X$ in $\mathbf{C}$ is an acyclic cofibration whose source, A, is cofibrant and filtration compact.

(A3) An object $X \in \mathbf{C}$ is fibrant if and only if $X \to \star$ has the right lifting property with respect to the underlying morphisms in $\mathbb{E}_{\mathbf{C}}$.

(A4) $\mathbf{C}$ admits a simple cylinder.

Let us now explain the role of these axioms:

- Requiring that $\mathbf{C}$ admits the structure of a semi-model category ensures that we are in a convenient framework to perform homotopy theory from a 1-categorical perspective. While the definition is phrased in terms of existence, it is not hard to see (Remark 10.2.2.6) that the remaining axioms are such that weak equivalences are uniquely determined by the classes of generating boundary inclusions and elementary expansions. When we refer to $\mathbf{C}$ as a semi-model category in the following, it will be with respect to this uniquely determined structure.

- Recall that a set of morphisms $\mathbb{B}$ in a semi-model category $\mathbf{C}$ generates cofibrations, if every cofibration is retract of a relative cell complex in $\mathbb{B}$. More than that, Axiom (A1) ensures that every object in $\mathbf{C}$ is weakly equivalent to a $\mathbb{B}$-cell complex, allowing us to perform homotopy theory from the perspective of cell complexes.
- The filtration compactness assumptions in Axiom (A2), for $D \in \mathbf{C}$, imply that every morphism $D \to \mathfrak{X}$ into a finite cell complex factors through a finite subcomplex $\tilde{\mathfrak{X}} \subset \mathfrak{X}$ (see Section 8.1.6 for rigorous definitions). For CW-complexes, it is a well known classical fact that compact spaces D have this property. Together with Axiom (A3), this essentially guarantees that one can perform homotopy theory of finite cell complexes without passing through the transfinite setting. For example, it follows that every morphism in the homotopy category ho$\mathbf{C}$ between finite structured cell complexes $\mathfrak{X}$ and $\mathfrak{Y}$ can be written as a composition of a morphism between finite cell complexes $\mathfrak{X} \to \mathfrak{Z}$ with the inverse of an expansion $\mathfrak{Y} \hookrightarrow \mathfrak{Z}$.
- Axiom (A3) deserves special attention, in the sense that we do not require expansion to generate acyclic cofibration, but only to be able to detect fibrant objects. In particular, this assumption is often fulfilled in the setting of Cisinksi model structures on presheaves (see [Cis06]).
- The crucial final ingredient, required by Axiom (A4), is the notion of admitting a so-called *simple cylinder functor*. Before we explain what this is, note that Axiom (A4) is again phrased as an existence statement and the cylinder will not be part of the data of a Whitehead model category.

Let us now explain what we mean by a simple cylinder functor. To this end, recall the following classical fact about CW-complexes (see, for example, [Coh73]). Given a CW-complex $\mathfrak{X}$, the cylinder $\mathfrak{X} \times [0,1]$ inherits a cell structure from $\mathfrak{X}$, with open cells of the form $e \times (0,1), e \times \{0\}$ and $e \times \{1\}$, for e an open cell in $\mathfrak{X}$. This cell structure is essentially uniquely determined by the cell structure on

$$D^n \times \{0,1\} \cup_{\partial D^n \times \{0,1\}} \partial D^n \times [0,1] \hookrightarrow D^n \times [0,1],$$

for $n \in \mathbb{N}$ with one cell, and requiring certain compatibility conditions with cobase changes and vertical transfinite compositions (see Section 8.2 for details). Given an inclusion of finite CW-complexes $\mathfrak{A} \hookrightarrow \mathfrak{X}$, one can consider the inclusions of subcomplexes

$$\mathfrak{A} \times [0,1] \cup \mathfrak{X} \times \{i\} \hookrightarrow \mathfrak{X} \times [0,1],$$
$$\mathfrak{A} \times [0,1] \cup \mathfrak{X} \times \{0,1\} \hookrightarrow \mathfrak{X} \times [0,1],$$

for $i \in \{0,1\}$. Then the following holds:

(O1) The first of these inclusions is always a simple equivalence;

(O2) The second inclusion is a simple equivalence if $\mathfrak{A} \hookrightarrow \mathfrak{X}$ is an expansion.

In fact, to prove this, one can easily reduce to the cases where $\mathfrak{A} \hookrightarrow \mathfrak{X}$ is the boundary inclusion of a cell, or an elementary expansion.

A *simple cylinder functor* on a category with expansions $\mathbf{C}$ consists of a colimit preserving functor (suggestively denoted) $- \otimes [0,1] \colon \mathbf{C} \to \mathbf{C}$, together with a natural factorization of the fold maps

$$\begin{tikzcd} X \sqcup X \arrow[rr] \arrow[dr] & & X \\ & X \otimes [0,1] \arrow[ur] & \end{tikzcd} \tag{2.9}$$

and choices of cell structure on

$$(D \sqcup D) \cup_{\partial D \sqcup \partial D} \partial D \times [0,1] \otimes D \to D \otimes [0,1]$$

for each arrow $\partial D \to D \in \mathbb{B}$, such that analogues of Observations (O1) and (O2) hold (see Section 10.2.1, for a rigorous definition).

Example 2.3.1.2. All of the cellularized categories with expansions we presented in Examples 2.2.2.2 to 2.2.2.5, with the exception of the first class of generating expansions in Example 2.2.2.4, are a Whitehead model category. In the cases of Examples 2.2.2.2, 2.2.2.3 and 2.2.2.5, one can use the simple cylinder given by $- \otimes \Delta^1$ from the tensor action of **sSet** on the respective simplicial semi-model category. In the case of the second class of expansions in Example 2.2.2.4, a simple cylinder is obtained by tensoring with the chain complex

$$I_\bullet = (\cdots \to 0 \to R \xrightarrow{(-1,1)} R \oplus R \to 0)$$

with cell structures induced by the standard bases for tensor products.

Let us now begin by stating some of our main results on Whitehead model categories, which demonstrate that they behave much like the classical setting of simple homotopy theory. For the remainder of this subsection, we fix some Whitehead model category $\mathbf{C}$. We will first need some notation:

Notation 2.3.1.3. Let $\mathbf{C\ddot{e}ll}(\mathbf{C}) \subset \mathbf{Cell}(\mathbf{C})$ be the wide subcategory given by inclusions of subcomplexes. Let $\mathbf{C\ddot{e}ll}_c(\mathbf{C}) \subset \mathbf{C\ddot{e}ll}(\mathbf{C})$ be the full subcategory given by finite structured cell complexes. We denote by $\mathfrak{ho}\mathbf{C}$ the (1-categorical) localization of $\mathbf{C\ddot{e}ll}(\mathbf{C})$ at expansions. We denote by $\mathfrak{ho}_c\mathbf{C}$ the localization of $\mathbf{C\ddot{e}ll}_c(\mathbf{C})$ at finite expansions.

Firstly, as long as one works on the level of homotopy categories, one may just as well work entirely in the setting of inclusions of structured cell complexes:

Main Result L (Theorem 10.2.2.1 and Lemma 9.1.3.2)**.** *Let* $\mathbf{C}$ *be a Whitehead model category. Then the following holds:*

1. *The canonical functor* $\mathfrak{ho}\mathbf{C} \to \mathrm{ho}\mathbf{C}$ *is an equivalence of categories.*[13]
2. *The canonical functor* $\mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}\mathbf{C}$ *is fully faithful.*

Furthermore, every morphism in $\mathfrak{ho}_c\mathbf{C}$ *can be expressed in terms of a composition of an inclusion of finite cell complexes with the inverse of an expansion.*

In particular, Main Result L allows us to equivalently think of morphisms $X \to Y$ in $\mathrm{ho}\mathbf{C}$, between the underlying objects of two cell complexes $\mathfrak{X}$ and $\mathfrak{Y}$ in terms of a morphism $\mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{ho}\mathbf{C}$ (or even in $\mathfrak{ho}_c\mathbf{C}$, if $\mathfrak{X}$ and $\mathfrak{Y}$ are finite). We will often pass back and forth through this equivalence, without making explicit mention of it.
We may now define what it means for a morphism in $\mathfrak{ho}_c\mathbf{C}$ to be a simple equivalence.

Definition 2.3.1.4. A morphism $s : \mathfrak{X} \to \mathfrak{Y} \in \mathfrak{ho}_c\mathbf{C}$ is called a *simple equivalence*, if it can be written in the form $s = e_2^{-1} \circ e_1$, with e_1 and e_2 finite expansions.

Notation 2.3.1.5. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be finite structured cell complexes. If we refer to a morphism $f : X \to Y$ as a simple equivalence, we mean that the associated morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{ho}_c\mathbf{C}$ is a simple equivalence. We will at times also write things along the lines of "*let* $f : \mathfrak{X} \to \mathfrak{Y}$ *be a morphism in* $\mathbf{C}$" to refer to a morphism of the underlying objects in $\mathbf{C}$.

It will turn out that this definition is compatible with the analogous language for inclusions of subcomplexes. Let us list some of the expected elementary properties of simple equivalences.

Proposition 2.3.1.6. *Simple equivalences have the following properties:*

[13] We expect that this result has a higher analogue, which holds at least in the case where we can extend the simple cylinder to a whole *simple simplicial structure on* $\mathbf{C}$. For our purposes, the 1-categorical statement will suffice.

- *They satisfy the two-out-of-three property and contain identities, and hence are stable under inversion.*

- *Given a cobase change square of finite structured cell complexes*[14]

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d, hook, "a"'] & \mathfrak{X}' \arrow[d, hook, "a'"] \\ \mathfrak{Y} \arrow[r, "f'"'] & \mathfrak{Y}' \end{tikzcd} \tag{2.10}$$

  with a an inclusion of a subcomplex and f a morphism in $\mathbf{C}$, it holds that if a is a simple equivalence, then so is a' and if f is a simple equivalence, then so is f'.

Just as in the classical scenario, there is a Whitehead group whose elements are precisely the obstructions to being a simple equivalence.

Construction 2.3.1.7. Given a finite structured cell complex $\mathfrak{X} \in \mathfrak{ho}_c\mathbf{C}$, we denote by $\widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X})$ the abelian monoid defined as follows:

1. Objects are equivalence classes of inclusions of subcomplexes $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ subject to the equivalence relation generated by composition with (elementary) expansions.

2. The neutral element is given by the identity $\mathfrak{X} \to \mathfrak{X}$.

3. Given two such equivalence classes $\langle \mathfrak{X} \hookrightarrow \mathfrak{Y}_1 \rangle$ and $\langle \mathfrak{X} \hookrightarrow \mathfrak{Y}_2 \rangle$, the sum is given by the diagonal in the pushout square

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Y}_2 \arrow[d, hook] \\ \mathfrak{Y}_1 \arrow[r, hook] & \mathfrak{Y}_1 \cup_{\mathfrak{X}} \mathfrak{Y}_2 \end{tikzcd} \tag{2.11}$$

   in $\mathbf{Cell}(\mathbf{C})$.

We denote by $\mathrm{Wh}_{\mathbf{C}}(\mathfrak{X})$ the abelian sub-group of $\widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X})$ given by the invertible elements. $\widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X})$ will be referred to as *the Whitehead monoid* of $\mathfrak{X}$ and $\mathrm{Wh}_{\mathbf{C}}(\mathfrak{X})$ will be referred to as the *Whitehead group* of $\mathfrak{X}$.

Remark 2.3.1.8. One should note that there are no size issues in the definition of the Whitehead group and Whitehead monoid. Indeed, as finite cell complexes are glued from a set of generating cofibrations, along a set of morphisms, up to isomorphism, there is only a set of finite structured cell complexes. Furthermore, between any two structured cell complexes, there is only a set of morphisms. Consequently, choosing appropriate representatives of isomorphism classes, the Whitehead monoid can be exposed as a quotient of a set of morphisms.

Whitehead group and Whitehead monoid define contravariant functors on the associated homotopy categories of finite structured cell complexes (see Chapter 9 and Proposition 10.2.3.12, for the following insight:)

Proposition 2.3.1.9. *Whitehead group and Whitehead monoid extend to covariant functors on $\mathfrak{ho}_c\mathbf{C}$ valued in abelian monoids (groups). Given two finite structured cell complexes $\mathfrak{X}$ and $\mathfrak{Y}$, and a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{C}$, the induced morphism*

$$f_*\colon \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X}) \to \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{Y})$$

[14]This means that the relative cell structure on a' is the cobase change of the one on a along f.

is given by mapping an equivalence class $\langle a\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}\rangle$ to the equivalence class of the parallel arrow a' in a cobase change square

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d, hook, "a"'] & \mathfrak{X}' \arrow[d, hook, "a'"] \\ \mathfrak{Y} \arrow[r, "f'"'] & \mathfrak{Y}' . \end{tikzcd} \tag{2.12}$$

By Main Result L, this entirely determines the functoriality of Whitehead group and monoid.

As we already suggested at the beginning of this chapter, the goal of our approach to abstract simple homotopy theory was to answer the question of uniqueness of presentations up to simple equivalence, or, equivalently, to obtain criteria when an equivalence can be expressed in terms of a sequence of finite elementary moves. It turns out that Whitehead groups precisely provide answers to this question.

Construction 2.3.1.10. By Main Result L, we may express any morphism $\alpha\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{ho}_c\mathbf{C}$ in terms of an inclusion of a subcomplex $a\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}'$ into a finite complex $\mathfrak{Y}'$, followed by the inverse of an expansion $\mathfrak{Y} \hookrightarrow \mathfrak{Y}'$. We denote by

$$\langle \alpha \rangle \in \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X})$$

the equivalence class of a in $\widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X})$. This construction is well-defined (see Section 10.2.3). We call this element the *Whitehead torsion of α*.

Supposing that one knows that the torsion is well-defined (which is the more difficult part to show) then one immediately obtains:

Corollary 2.3.1.11. *A morphism $\alpha\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{ho}_c\mathbf{C}$ is a simple equivalence if and only if $\langle \alpha \rangle = 0$.*

Remark 2.3.1.12. The cell structure on a structured cell complex $\mathfrak{X}$ is essentially entirely irrelevant to the construction of $\widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X})$. In fact, if one uses relative structured cell complexes instead of inclusions of subcomplexes, then one can replicate Construction 2.3.1.7 completely analogously and define a Whitehead group and monoid for an arbitrary object of $\mathbf{C}$. The cell structure becomes relevant only when we associate Whitehead torsions to a morphism (see Remark 10.1.2.10). Hence, when we are mainly interested in the isomorphism type of the Whitehead group, we will at times simply write $\mathrm{Wh}_{\mathbf{C}}(X)$, without specifying the structure of a cell complex $\mathfrak{X}$ on X.

Let us now relate the Whitehead group to presentations. To this end, we first explicitly spell out the quotient sets of presentations:

Construction 2.3.1.13. Let $\mathbf{C}$ be a Whitehead model category and let $X \in \mathrm{ho}\mathbf{C}$. We denote by $\mathrm{Pres}_{\mathbf{C}}(X)$ the quotient set of presentations

$$\mathrm{Pres}_{\mathbf{C}}(X) \coloneqq \{(\mathfrak{Y}, \omega\colon X \to Y) \mid \mathfrak{Y} \in \mathfrak{ho}_c\mathbf{C}, \omega \in \mathrm{ho}\mathbf{C}(X,Y) \text{ is an isomorphism }\}/\sim_s ,$$

where the equivalence relation $\sim_s$ is given by $(\mathfrak{Y}_1, \omega_1) \sim_s (\mathfrak{Y}_2, \omega_2)$, if and only if $\mathfrak{Y}_1 \xrightarrow{\omega_2\omega_1^{-1}} \mathfrak{Y}_2$ is a simple equivalence. This construction is contravariantly functorial in isomorphisms in $\mathrm{ho}\mathbf{C}$, the wide subcategory of which is denoted $(\mathrm{ho}\mathbf{C})^{\simeq}$, with functoriality given by precomposition, inducing a functor

$$\mathrm{Pres}_{\mathbf{C}}\colon (\mathrm{ho}\mathbf{C})^{\simeq,\mathrm{op}} \to \mathbf{Set} .$$

We may extend $\mathrm{Pres}_{\mathbf{C}}(X)$ to a larger set, functorial in arbitrary morphisms, as follows: Define

$$\widetilde{\mathrm{Pres}}_{\mathbf{C}}(X) \coloneqq \{(\mathfrak{Y}, \omega\colon X \to Y) \mid \mathfrak{Y} \in \mathfrak{ho}_c\mathbf{C}, \alpha \in \mathrm{ho}\mathbf{C}(X,Y)\}/\sim_{\tilde{s}} ,$$

where, $\sim_{\tilde{s}}$ is the extension of $\sim_s$, given by $(\mathfrak{Y}_1, \omega_1) \sim_s (\mathfrak{Y}_2, \omega_2)$, if and only if there exists a simple equivalence $\gamma: \mathfrak{Y}_1 \to \mathfrak{Y}_2$, such that $\gamma \circ \omega_1 = \omega_2$. This construction is contravariantly functorial in morphisms in ho$\mathbf{C}$, with functoriality given by precomposition, inducing a functor

$$\widetilde{\mathrm{Pres}_{\mathbf{C}}}: (\mathrm{ho}\mathbf{C})^{\mathrm{op}} \to \mathbf{Set}\,.$$

The following theorem provides the relationship between presentations and the Whitehead group.

Main Result M (Theorem 10.2.3.9)**.** *Let* $\mathbf{C}$ *be a Whitehead model category and let* $X \in \mathrm{ho}\mathbf{C}$. *Let* $\omega_0: X \xrightarrow{\simeq} \mathfrak{X}_0$ *be a finite presentation of* X. ω_0 *induces a bijection*

$$\begin{aligned} \widetilde{\mathrm{Pres}_{\mathbf{C}}}(X) &\xrightarrow{1:1} \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X}_0) \\ (\mathfrak{Y}, \alpha) &\mapsto \langle \mathfrak{X}_0 \xrightarrow{\alpha \circ \omega_0^{-1}} \mathfrak{Y} \rangle\,, \end{aligned}$$

which restricts to a bijection

$$\begin{aligned} \mathrm{Pres}_{\mathbf{C}}(X) &\xrightarrow{1:1} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}_0) \\ (\mathfrak{Y}, \omega) &\mapsto \langle \mathfrak{X}_0 \xrightarrow{\omega \circ \omega_0^{-1}} \mathfrak{Y} \rangle\,. \end{aligned}$$

As an immediate corollary of Main Result M, one obtains the following.

Corollary 2.3.1.14. *Given a finite cell complex* $\mathfrak{X}$*, the Whitehead group* $\mathrm{Wh}_{\mathbf{C}}(\mathfrak{X})$ *consists of precisely the equivalence classes of such inclusions of finite subcomplexes* $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ *that define weak equivalences in* $\mathbf{C}$.

Most of what we have presented so far ultimately follows from the calculus of Eckmann and Siebenmann (or, more precisely, its adapted version in Chapter 9) together with Main Result L. Before we move on to examples as well as more involved results on Whitehead model categories, let us recall one of the crucial formulas in classical simple homotopy theory in this framework: Observe that Main Result M tells us that Whitehead groups (their underlying sets, to be more precise) are functorial in isomorphisms in $\mathfrak{ho}_c\mathbf{C}$ in two different ways. First off, there is the covariant functoriality described in Proposition 2.3.1.9. Secondly, there is the contravariant functoriality arising from the functoriality of $\mathrm{Pres}_{\mathbf{C}}(-)$. We will denote the latter functoriality in the form ω^*. Then one has the following composition formula.

Proposition 2.3.1.15 (Lemma 9.1.3.11)**.** *Let* $\omega: \mathfrak{X} \to \mathfrak{Y}$ *be an isomorphism in* $\mathfrak{ho}_c\mathbf{C}$. *Then the following identity holds:*

$$\omega_* \omega^* = 1_{\mathrm{Wh}_{\mathbf{C}}(\mathfrak{Y})} + \omega_* \langle \omega \rangle.$$

In particular, it follows that

$$(\omega_*)^{-1} = \omega^* - \langle \omega \rangle.$$

In the special case where ω *is simple, we obtain*

$$(\omega_*)^{-1} = \omega^*.$$

If we insert elements of the Whitehead group a, b into this formula, we obtain

$$\begin{aligned} \omega^*(a+b) &= (\omega_*)^{-1}(a) + (\omega_*)^{-1}(b) + \langle \omega \rangle \\ &= \omega^*(a) - \langle \omega \rangle + (\omega^*)(b) - \langle \omega \rangle + \langle \omega \rangle \\ &= \omega^*(a) + \omega^*(b) - \langle \omega \rangle. \end{aligned}$$

This provides another interesting interpretation of the Whitehead torsion: It is the obstruction to ω^* being a group homomorphism.

2.3.2 Results: Some examples of Whitehead groups

Let us now discuss some explicit examples of the Whitehead group functors associated to a Whitehead model category. In the following, when we refer to the categories **sSet** and **Top** as a Whitehead model category, it will be with respect to the choices of generating boundary inclusions and extensions of Examples 2.2.2.2 and 2.2.2.3. As a minimum requirement for the setting we have introduced in the previous subsection to provide a good abstract framework to study Whitehead model categories, we would expect that the Whitehead groups associated to **sSet** and **Top** agree with the classical Whitehead groups. We will explain why this is the case in Section 2.3.4. Instead, let us start with some examples on the algebraic side.

Example 2.3.2.1. Given a not necessarily commutative unital ring R and a subgroup G of its group of units, let us denote by $\mathrm{Wh}_G(-)$ the Whitehead group functor associated to the Whitehead model category on $\mathbf{Ch}_{\geq 0}(R)$ described in Example 2.2.2.4, using the set of expansions $\mathbb{E}_G$. This simple homotopy theory has the peculiar property that the Whitehead group functor $\mathfrak{ho}_c\mathbf{Ch}_{\geq 0}(R) \to \mathbf{Ab}$ is constant, in the sense that the terminal morphism $A_\bullet \to 0$ induces an isomorphism

$$\mathrm{Wh}_G(A_\bullet) \cong \mathrm{Wh}_G(0).$$

To see this, observe that under Main Result M, we can identify the elements of $\mathrm{Wh}_G(0)$ with simple equivalence classes of acyclic based chain complexes. Fix some choice of basis on $A_\bullet$ inducing a cell structure on $\mathfrak{A}$ on $A_\bullet$. The map

$$0_*\colon \mathrm{Wh}_G(\mathfrak{A}) \to \mathrm{Wh}_G(0)$$

is given by mapping the class of a (basis element preserving) inclusion of based chain complexes $\mathfrak{A} \hookrightarrow \mathfrak{B}$ to the class of the associated quotient complex $B_\bullet/A_\bullet$, equipped with the basis given by the classes of such basis elements in the basis of $B_\bullet$ that are not in $A_\bullet$. We denote this based free chain complex by $\mathfrak{B}/\mathfrak{A}$. 0_* admits a section $s\colon \mathrm{Wh}_G(0) \to \mathrm{Wh}_G(\mathfrak{A})$, induced by $0 \to A_\bullet$. Explicitly, this section is given by mapping the class of an acyclic based chain complex $\mathfrak{X}$ to the class of the inclusion of subcomplexes $\mathfrak{A} \xhookrightarrow{(1,0)} \mathfrak{A} \oplus \mathfrak{X}$. A priori, the image of s is given by the equivalence classes of such inclusions of subcomplexes $\mathfrak{A} \to \mathfrak{B}$, for which there is a basis preserving splitting

$$\mathfrak{B} \cong \mathfrak{A} \oplus \mathfrak{B}/\mathfrak{A},$$

under $\mathfrak{A}$. Observe that on the level of graded abelian groups, such a splitting trivially exists. Simply map a basis element $[b] \in B_n/A_n$ to b. This splitting defines a splitting of chain complexes, if and only if, under this isomorphism, the differential of $B_\bullet$ is of the form

$$\begin{pmatrix} d_A & 0 \\ 0 & d_{B/A}. \end{pmatrix}$$

It turns out that, up to expansions, every inclusion of based chain complexes $\mathfrak{A} \hookrightarrow \mathfrak{B}$ can be brought into this form (compare [Whi50, Lem. 2]). This is a consequence of the *cell trading argument*, which can be found in [Coh73, p. 7.3], for example. The completely analogous argument in chain complexes allows one to modify $\mathfrak{B}$ such that all basis elements in the complement of A are in dimension $n \gg \max\{m \in \mathbb{N} \mid A_m \neq 0\}$. Then, the differential of B clearly has the form described above. It follows that s is surjective, making s and 0_* isomorphisms.

Example 2.3.2.2. Suppose, again, we are in the situation of the previous example. To any based acyclic chain complex $\mathfrak{B}$, one can associate a torsion element $\tau(\mathfrak{B})$ in the first reduced K-group of R

$$\tilde{K}_1(R) \cong \mathrm{Ab}(\mathrm{Gl}(R))/\{1,-1\}$$

given taking the quotient of the abelianization of $\mathrm{Gl}(R)$, denoted $K_1(R)$, by the subgroup $\{1,-1\}$. Roughly, one trades cells until $\mathfrak{B}$ is concentrated in two degrees, and then treats the (appropriately signed) differential as an element of $\mathrm{Gl}(R)$ (see, for example, [Coh73]). If $G = 1$, then this assignment induces an isomorphism

$$\mathrm{Wh}_1(0) \cong \tilde{K}_1(R).$$

If G is some larger group, then one only obtains a well-defined morphism after taking the quotient of $K_1(R)$ by $\pm G$ instead of $\{-1, 1\}$. One then obtains an isomorphism

$$\mathrm{Wh}_G(0) \cong K_1(R)/\pm G.$$

In particular, in the special case where R is the group ring $\mathbb{Z}[G]$, one obtains

$$\mathrm{Wh}_G(0) \cong K_1(\mathbb{Z}[G])/\pm G \cong \mathrm{Wh}(G).$$

We will not provide a proof of these statements here, as the computation is only supposed to illustrate how our framework reproduces more classical approaches to simple homotopy theory. A proof can be obtained by using arguments very much similar to the ones provided in [Whi50; Coh73].

Another important point illustrated by Example 2.3.2.2 is that one should probably treat the phenomenon that simple homotopy type is a homeomorphism invariant for CW-complexes as a lucky coincidence, and not as something that one can generally expect for other simple homotopy theories. Indeed, any isomorphism $R^n \to R^n$ that defines a non-trivial element in $\tilde{K}_1(R)$ will provide a counterexample. We provide another example of isomorphisms in the underlying category that do not induce simple homotopy equivalences for the case of stratified spaces in Section 13.3.2, for the setting of stratified homotopy theory. Let us now return to the case of stratified homotopy theory.

Example 2.3.2.3. In Example 2.2.2.5, we gave sets of generating boundary inclusions and expansions for the category of stratified simplicial sets over some fixed poset P, $\mathrm{s}\mathbf{Strat}_P$ (namely the set of stratified boundary inclusions and the set of admissible horn inclusions). These equip the category $\mathrm{s}\mathbf{Strat}_P$ with the structure of a Whitehead model category (see Theorem 13.1.1.2). The associated model category is precisely the model category $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ of Theorem 1.2.3.14. Just like the associated model category, we will denote this Whitehead model category by $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$. The associated notions of simple equivalence, Whitehead group, and Whitehead torsion agree with the ones for the diagrammatic simple homotopy theory of stratified simplicial sets (defined in [Waa21]) which we discussed in the introduction of this chapter. In particular, given a finite stratified simplicial set $\mathcal{X}$, we may interpret the stratified diagrammatic Whitehead group $\mathrm{Wh}_P(\mathcal{X})$ as the Whitehead group $\mathrm{Wh}_{\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}}(\mathcal{X})$. For obvious reasons of conciseness, we will stick to the notation $\mathrm{Wh}_P(\mathcal{X})$.

The results we presented in Chapter 1, specifically the existence of semi-model categories on the topological side which we prove in Chapter 7, also allow us to define a topological version of the stratified diagrammatic Whitehead group. As we did not explicitly describe the semi-model structures on $\mathbf{Strat}_P$ for a fixed poset in Chapter 1, we will state the result here, for the convenience of the reader:

Theorem 2.3.2.4 (Theorem 7.4.2.6 and Lemma 7.5.1.1)**.** *Let P be a poset. The simplicial category $\underline{\mathbf{Strat}}_P$ admits the structure of two simplicial, cofibrantly generated left semi-model categories, denoted $\underline{\mathbf{Strat}}_P^{\mathfrak{d}}$ and $\underline{\mathbf{Strat}}_P^{\mathfrak{c}}$ and called, respectively, the diagrammatic model structure and the categorical model structure, such that the following holds:*

1. *Weak equivalences in $\underline{\mathbf{Strat}}_P^{\mathfrak{d}}$ are precisely the stratum-preserving diagrammatic equivalences.*
2. *Weak equivalences in $\underline{\mathbf{Strat}}_P^{\mathfrak{c}}$ are precisely the stratum-preserving categorical equivalences.*
3. *In both model structures, a set of generating cofibrations is given by the set of stratified boundary inclusions*

$$\{|\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P\}.$$

4. *A set of generating acyclic cofibrations for $\underline{\mathbf{Strat}}^{\mathfrak{d}}_P$ is given by the set of realizations of admissible boundary inclusions*

$$\{|\Lambda^{\mathcal{J}}_k \hookrightarrow \Lambda^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P, k \text{ is s.t. } \Lambda^{\mathcal{J}}_k \hookrightarrow \Lambda^{\mathcal{J}} isadmissible\}.$$

5. *A stratified space is fibrant in $\underline{\mathbf{Strat}}^{\mathfrak{c}}_P$, if and only if it has the horn filling property with respect to all realizations of stratified inner horn inclusions over P.*

Example 2.3.2.5. It follows from Theorem 2.3.2.4 that we can also obtain a Whitehead model structure on $\mathbf{Strat}_P$, by using the classes of generating boundary inclusions and elementary expansions of Example 2.2.2.5 (see Theorem 13.1.2.5). That is, the generating boundary inclusions are given by the cofibrant generators in $\underline{\mathbf{Strat}}^{\mathfrak{d}}_P$, and the generating elementary expansions are given by realizations of admissible horn inclusions (equipped with the cell structure inherited from the simplicial structure). We will also denote this Whitehead model category by $\mathbf{Strat}^{\mathfrak{d}}_P$.

Note that to define this theory and for the results in Chapter 10 to apply, it is crucial to know that $\mathbf{Strat}^{\mathfrak{d}}_P$ forms a semi-model category. Hence, the results on stratified homotopy theory presented in Chapter 1 are fundamental to having a topological picture for simple stratified homotopy theory available.

Remark 2.3.2.6. We can proceed similarly with the semi-model structure on $\mathbf{Strat}^{\mathfrak{c}}_P$ that presents the categorical stratified homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ defined by Haine, obtaining an answer to Question (Q4). In this case, generating elementary expansions are given by realizations of admissible stratified horn inclusions and inner horn inclusions. We will only focus on the diagrammatic Whitehead model category $\mathbf{Strat}^{\mathfrak{d}}_P$ in this thesis, however. The main reason we have focused on the diagrammatic instead of the categorical model structure here is that it also provides an optimal starting point for any further investigation into the categorical setting. Indeed, as $\mathbf{Strat}^{\mathfrak{c}}_P$ is obtained by increasing the class of generating expansions in $\mathbf{Strat}^{\mathfrak{d}}_P$, any simple equivalence in the diagrammatic scenario will also produce a simple equivalence in the categorical scenario.

We do now return to some of the motivating questions described in the introduction of this chapter. Namely, we can ask what the precise relationship between the Whitehead model categories s$\mathbf{Strat}^{\mathfrak{d}}_P$ and $\mathbf{Strat}^{\mathfrak{d}}_P$ is, and whether we can compute the stratified diagrammatic Whitehead groups $\mathrm{Wh}_P(\mathcal{X})$ algebraically (Questions (Q1) and (Q2)). There is a rather conceptual approach to answering these questions. Namely, using Main Result B and Main Result A_1 we obtain Quillen equivalences

$$\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet}) \simeq \mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P \simeq \mathbf{Strat}^{\mathfrak{d}}_P,$$

where the left-hand side is equipped with the injective model structure. $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ has a natural notion of generating cofibrations and acyclic cofibrations (which we discuss in detail later on) leading to an associated Whitehead model category. Suppose, for a second, that we had a good notion of *equivalence of Whitehead model categories*, preserving all of the relevant structures such as simple equivalences and Whitehead groups. Then one may hope to lift these Quillen equivalences to such equivalences of Whitehead model categories. In particular, this would allow for the free transition between the topological, the simplicial and the diagrammatic theory. As the latter arises from a theory of presheaves, valued in simplicial sets, we can furthermore expect that much of the theory can be understood by having a good understanding of the simple homotopy theory of simplicial sets. This is precisely the approach we make rigorous in the following subsections.

2.3.3 Results: Equivalence and transfer of Whitehead model categories

Let us begin by exposing a notion of functors of Whitehead model categories. Implicitly, we have already referred to such objects when we discussed simple cylinders. First, we will give a

definition of functors of cellularized categories. It turns out that, just like the theory of cell complexes works best in a relative framework, the same holds true for cellularized functors. For the sake of simplicity, we only give a definition of the absolute case here. The basic properties of the relative case are covered in Section 8.2.
In the following, we denote by $[1]$ the category with two objects 0 and 1 and one non-identity arrow $0 \to 1$. In particular, given a category $\mathbf{C}$, we denote by $\mathbf{C}^{[1]}$ the category of arrows in $\mathbf{C}$.

Definition 2.3.3.1. Let $\mathbf{C}$ and $\mathbf{D}$ be cellularized categories. A *cellularization* of a colimit preserving functor $F\colon \mathbf{C} \to \mathbf{D}$ is a lift

$$\begin{tikzcd} \mathbf{RCell}(\mathbf{C}) \arrow[r, dashed, "\mathfrak{F}"] \arrow[d] & \mathbf{RCell}(\mathbf{D}) \arrow[d] \\ \mathbf{C}^{[1]} \arrow[r, "F^{[1]}"] & \mathbf{D}^{[1]} \end{tikzcd} \tag{2.13}$$

such that for any pair of structured relative cell complexes $\mathfrak{c}\colon A \to X$ and $\mathfrak{d}\colon X \to Y$ in $\mathbf{RCell}(\mathbf{C})$, the vertical composition law

$$\mathfrak{F}(\mathfrak{d} \circ \mathfrak{c}) = \mathfrak{F}(\mathfrak{d}) \circ \mathfrak{F}(\mathfrak{c})$$

holds. A colimit preserving functor $F\colon \mathbf{C} \to \mathbf{D}$ together with a choice of cellularization $\mathfrak{F}$ will be called a *cellularized functor.*

Notation 2.3.3.2. We abuse notation insofar as we generally denote cellularized functors in the form $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$, and refer to the underlying functor by $F\colon \mathbf{C} \to \mathbf{D}$. We will, at times, also refer to $\mathfrak{F}$ as the *cellularization.*

Remark 2.3.3.3. The most basic facts about cellularized functors one can prove are that they preserve both cobase changes as well as transfinite vertical compositions (see Section 8.2, for details). As every relative cell complex can be built under these operations from generating boundary inclusions, it follows that cellularizations of a colimit preserving functor F are in bijection with families $(\mathfrak{F}(b))_{b \in \mathbb{B}_{\mathbf{C}}}$ consisting of a relative cell structure on $F(b) \in \mathbf{D}$, for every generating boundary inclusion $b \in \mathbb{B}_{\mathbf{C}}$ (see Proposition 8.3.3.6, for a precise statement).

Example 2.3.3.4. By the definition of the cellularizations of $\mathbf{Strat}_P$, for $P \in \mathbf{Pos}$, it follows that the realization functors $|-|_s\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_P$ admit a canonical cellularization.

Example 2.3.3.5. The simplicial chain complex functor $C_\bullet(-)\colon \mathbf{sSet} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$, mapping a simplicial set to the free chain complex on its non-degenerate simplices, admits a canonical cellularization arising from the obvious bases.

Definition 2.3.3.6. A cellularized functor $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ between Whitehead model categories is called a *Whitehead functor (W-functor, for short)*, if one of the following equivalent conditions holds:

1. For every generating boundary inclusion $b \in \mathbb{B}$, it holds that the associated structured relative cell complex $\mathfrak{F}(\mathfrak{b})$ is finite and for every generating elementary expansion $\mathfrak{e} \in \mathbb{E}_{\mathbf{C}}$, it holds that the associated structured relative cell complex $\mathfrak{F}(\mathfrak{e})$ is a structured simple equivalence.

2. $\mathfrak{F}\colon \mathbf{RCell}(\mathbf{C}) \to \mathbf{RCell}(\mathbf{D})$ preserves finite relative cell complexes and structured simple equivalences.

Remark 2.3.3.7. Assume, for a second, that all Whitehead model categories are locally presentable (see [nLa25d]). Supposing that one takes **Top** to be the category of Δ-generated spaces, then all of the examples in Examples 2.2.2.2 to 2.2.2.5 have this property. It follows by the adjoint functor theorem (see [nLa24a]) that any cellularized functor admits a right adjoint. In Section 10.3, we show that under this condition every W-functor $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ induces a left Quillen functor of the associated semi-model categories.

Notation 2.3.3.8. Given a Whitehead model category $\mathbf{C}$, we denote by $\mathcal{C}\mathbf{ell}(\mathbf{C})$ the quasi-category whose objects are the absolute structured cell complexes in $\mathbf{C}$ and whose morphism spaces are the morphism spaces of underlying objects in the associated quasi-categorical localization $\mathcal{C} = \mathbf{C}[W^{-1}]$, where W is the class of weak equivalences in $\mathbf{C}$. The forgetful functor $\mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}$ is an equivalence of categories. We only use $\mathcal{C}\mathbf{ell}(\mathbf{C})$ to additionally keep track of cell structures for the purpose of simple homotopy theory. In particular, there is a canonical isomorphism of homotopy categories $\mathrm{ho}\mathcal{C}\mathbf{ell}(\mathbf{C}) = \mathfrak{ho}\mathbf{C}$. We denote by $\mathcal{S}\mathbf{im}(\mathbf{C}) \subset \mathcal{C}\mathbf{ell}(\mathbf{C})$ the subgroupoid given by finite cell complexes and simple equivalences (i.e., such morphisms that map to simple equivalences in $\mathfrak{ho}_c\mathbf{C}$).

Using the fact that cell complexes in a Whitehead model category are always cofibrant, it follows that a W-functor $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ induces functors

$$\begin{aligned} \mathcal{C}\mathbf{ell}(\mathbf{C}) &\to \mathcal{C}\mathbf{ell}(\mathbf{D}) \\ \mathcal{S}\mathbf{im}(\mathbf{C}) &\to \mathcal{S}\mathbf{im}(\mathbf{D}) \end{aligned}$$

which then descend to functors

$$\begin{aligned} \mathfrak{ho}_c\mathbf{C} &\to \mathfrak{ho}_c\mathbf{D}; \\ \mathfrak{ho}\mathbf{C} &\to \mathfrak{ho}\mathbf{D}; \\ \mathrm{ho}\mathcal{S}\mathbf{im}(\mathbf{C}) &\to \mathrm{ho}\mathcal{S}\mathbf{im}(\mathbf{D}) \end{aligned}$$

without any need to derive. We also denote these functors by $\mathfrak{F}$ by abuse of notation. Furthermore, as every W-functor preserves cobase changes and expansions, $\mathfrak{F}$ also induces a natural transformation of Whitehead monoids

$$\begin{aligned} \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X}) &\to \widetilde{\mathrm{Wh}}_{\mathbf{D}}(\mathfrak{F}(\mathfrak{X})) \\ [\mathfrak{X} \hookrightarrow \mathfrak{Y}] &\mapsto [\mathfrak{F}(\mathfrak{X}) \hookrightarrow \mathfrak{F}(\mathfrak{Y})] \end{aligned}$$

which restricts to a natural transformation of Whitehead groups. We will denote the resulting natural transformations by $\widetilde{\mathrm{Wh}}_{\mathfrak{F}}\colon \widetilde{\mathrm{Wh}}_{\mathbf{C}} \Rightarrow \widetilde{\mathrm{Wh}}_{\mathbf{D}} \circ \mathfrak{F}$ and $\mathrm{Wh}_{\mathfrak{F}}\colon \mathrm{Wh}_{\mathbf{C}} \Rightarrow \mathrm{Wh}_{\mathbf{D}} \circ \mathfrak{F}$.

For the remainder of this section, we will generally assume local presentability of Whitehead model categories. This is mainly in order to be in a scenario in which we can refer to the familiar techniques of Quillen functors. In this case, the following conditions on a W-functor $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ turn out to be equivalent (see Section 10.3):

1. F defines the left part of a Quillen equivalence;
2. The induced (left derived) functor $\mathrm{ho}\mathbf{C} \to \mathrm{ho}\mathbf{D}$ is an equivalence of categories;
3. The induced functor $\mathfrak{ho}\mathbf{C} \to \mathfrak{ho}\mathbf{D}$ is an equivalence of categories;
4. The induced functor of ∞-categories $\mathfrak{F}\colon \mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{D})$ is an equivalence of ∞-categories.

Weak equivalences of Whitehead model categories are then defined as follows.

Definition 2.3.3.9 (See Section 10.3)**.** Let $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ be a W-functor of (locally presentable) Whitehead model categories. Then $\mathfrak{F}$ is called a *weak equivalence of Whitehead model categories* if F defines the left part of a Quillen equivalence, and one of the following equivalent conditions holds:

1. The induced functor $\mathrm{ho}\mathcal{S}\mathbf{im}(\mathbf{C}) \to \mathrm{ho}\mathcal{S}\mathbf{im}(\mathbf{D})$ is an equivalence of categories, or more explicitly:
   Every finite cell complex in $\mathbf{D}$ has the simple homotopy type of a finite cell complex of the form $\mathfrak{F}(\mathfrak{X})$, for some finite cell complex $\mathfrak{X}$ in $\mathbf{C}$. A morphism $\omega\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{ho}_c\mathbf{C}$ is a simple equivalence, if and only if $\mathfrak{F}(\omega) \in \mathfrak{ho}_c\mathbf{C}$ is a simple equivalence.

2. The natural transformation of Whitehead groups $\mathrm{Wh}_{\mathfrak{F}}$ is an isomorphism.

We think that it is evident to the reader from this definition, that for most intents and purposes of simple homotopy theory (for example, for the computation of Whitehead groups), weakly equivalent Whitehead model categories can be treated as equal. We will also consider a slightly stronger notion of equivalence, namely that one can construct an inverse in terms of cellularized functors.

Example 2.3.3.10. Suppose that $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ is a W-functor and suppose that there is another W-functor $\mathfrak{G}\colon \mathbf{D} \to \mathbf{C}$ together with natural transformations of the associated functor of ∞-categories

$$\eta\colon 1_{\mathcal{C}\mathbf{ell}(\mathbf{C})} \simeq \mathfrak{G} \circ \mathfrak{F}$$

and

$$\varepsilon\colon 1_{\mathcal{C}\mathbf{ell}(\mathbf{D})} \simeq \mathfrak{F} \circ \mathfrak{G},$$

which define simple equivalences, at every finite cell complex in, respectively, $\mathbf{C}$ and $\mathbf{D}$. Then $\mathfrak{F}$ is a weak equivalence of Whitehead model categories. We will call such a W-functor an *(∞-categorical) homotopy equivalence of Whitehead model categories.*[15]

We may now use the language we have just developed to lift the equivalences between the several different models for the homotopy theory $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ described in Chapter 1 to the level of simple homotopy theory.

First off, in Section 12.1 of Chapter 12, we prove the following comparison theorem for transferred Whitehead model categories. This will require the notion of a *simplicial Whitehead model category*, which, roughly speaking, refers to a Whitehead model category $\mathbf{C}$ that is equipped with the structure of a simplicial model category, such that the action of $\mathbf{sSet}$ on $\mathbf{C}$ can be cellularized in a manner that is compatible with simple equivalences (see Section 12.1). All of the examples defined in Examples 2.2.2.2 to 2.2.2.5 that define Whitehead model categories fall into this class. The seminal example to have in mind for the following theorem is the case of the Whitehead model categories $\mathbf{sSet}$ and $\mathbf{Top}$.

Main Result N (Theorem 12.1.0.4)**.** *Let $\underline{\mathbf{C}}$ be a simplicial Whitehead model category. Suppose that every generating boundary inclusion $b \in \mathbb{B}_{\mathbf{C}}$ has cofibrant source. Furthermore, let $\underline{\mathbf{D}}$ be a simplicial semi-model category, and*

$$L\colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$$

be a simplicial left Quillen functor. Now, suppose that the following holds:

- *$L(\mathbb{B}_{\mathbf{C}})$ defines the structure of a cellularized category on $\mathbf{D}$;*
- *Equipping this cellularized category with the class of expansions $L(\mathbb{E}_{\mathbf{C}})$ (with the cell structures induced by L) defines the structure of a Whitehead model category on $\mathbf{D}$ (compatible with the semi-model structure);*
- *The functor of ∞-categories $\mathcal{C} \to \mathcal{D}$ induced by L is fully faithful.*

Then L canonically inherits the structure of a W-functor $\mathfrak{L}\colon \mathbf{C} \to \mathbf{D}$ that is a weak equivalence of Whitehead model categories, with respect to the induced structure on $\mathbf{D}$.

As an immediate corollary of this result together with the fixed poset version of Main Result A_5 (see Theorem 7.4.2.11) we obtain the following example:

Main Result O. *Let $P \in \mathbf{Pos}$. The stratified realization functor*

$$|-|_s\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_P,$$

equipped with the obvious cellularization, induces a weak equivalence of Whitehead model categories between $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ and $\mathbf{Strat}_P^{\mathfrak{d}}$.

[15]We add the suffix ∞-categorical, as opposed to the case where one can present the natural transformations above by zig-zag sequence of 1-categorical natural transformations of functors.

This answers Question (Q1), and allows us to freely pass between the topological and the simplicial diagrammatic stratified simple homotopy theories. In the special case where $P = \star$, this produces a weak equivalence of Whitehead model categories between **sSet** and **Top**.

2.3.4 Results: Comparison with classical simple homotopy theory

For the approach to abstract simple homotopy theory that we take here to be justified at all, a minimum requirement that one would expect to be fulfilled is that the Whitehead groups associated to **sSet** and **Top** agree with the classical Whitehead groups of a finite CW-complex. There is an obvious construction of the classical Whitehead group to compare our Whitehead group with, namely the definition of the Whitehead group in terms of inclusions of CW-complexes defined in [Coh73] (see Description (D3)). It turns out that the comparison with the classical scenario is slightly more involved than one would expect at first glance. This is mainly due to the fact that the definition of CW-complex in [Whi50] and [Coh73] specifies only open cell decompositions, and not choices of characteristic maps.

Recollection 2.3.4.1. In [Coh73], Cohen used a definition of finite CW-complex, in which CW-complexes consist of a space together with a decomposition into open cells. In particular, Cohen's definition of CW-complexes differs from the one we used in Example 2.2.1.6, insofar as the characteristic maps of cells are *not part of the data.* Only the decomposition into open cells is considered to be a part of the defining data of a CW-complex. Cohen then defines his *geometric Whitehead group* $\mathrm{Wh}_{\mathrm{Co}}(\mathfrak{X})$, of a finite, not-necessarily connected CW-complex $\mathfrak{X}$ as in Description (D3).

The fact that Cohen does not make choices of characteristic maps part of the data makes the theory, a priori, significantly more flexible and less combinatorial in nature than the theory arising from the Whitehead model category $\mathbf{Strat}_P$. Observe, for example, that given two CW-complexes in this sense $\mathfrak{X}$ and $\mathfrak{Y}$, the set of inclusions $\{\mathfrak{X} \hookrightarrow \mathfrak{Y}\}$ is generally infinite. Given any cell preserving inclusion $\mathfrak{X} \hookrightarrow \mathfrak{Y}$, any perturbation of the interior of a cell in $\mathfrak{X}$ by a homeomorphism that descends to the identity on the boundary defines a new inclusion $\mathfrak{X} \hookrightarrow \mathfrak{Y}$. With CW-complexes in the sense of Example 2.2.1.6, the set of morphisms $\mathbf{Cell}(\mathbf{C})(\mathfrak{X}, \mathfrak{Y})$ can be seen as a subset of $\mathbf{Set}(\mathfrak{C}_{\mathfrak{X}}, \mathfrak{C}_{\mathfrak{Y}})$ and thus is finite if $\mathfrak{X}$ and $\mathfrak{Y}$ are finite. For our purposes, where we are looking for the simple homotopy theory to be as combinatorial as possible, the latter behavior is much preferable. At the same time, this makes the comparison of Cohen's Whitehead group with the Whitehead group functor $\mathrm{Wh}_{\mathbf{Top}}(-)$ slightly difficult: Cell complexes in **Top** are, at the same time, more rigid than Cohen's CW-complexes since they require fixed characteristic maps, and more flexible, as they allow for arbitrary gluings of cells. There is, however, a direct comparison map from the setting of simplicial sets.
Namely, given a simplicial set X, one can equip $|X|$ with the cell structure whose open cells are given by the open non-degenerate simplices in $|X|$. Denote by $\mathrm{ho}\mathbf{sSet}^{\mathrm{fin}}$ the homotopy category of finite simplicial sets. In Chapter 12, specifically Section 12.2, we prove the following theorem (phrased somewhat differently there)[16].

Main Result P (Theorem 12.2.0.4)**.** *Topological realization of simplicial sets (equipped with the obvious cell structures) induces a natural isomorphism*

$$\phi \colon \mathrm{Wh}_{\mathbf{sSet}}(X) \cong \mathrm{Wh}_{\mathbf{Co}}(|X|),$$

for X a finite simplicial set. ϕ is compatible with Whitehead torsion in the sense that

$$\phi(\langle \alpha \colon X \to Y \rangle) = \langle |\alpha| \rangle$$

holds, for $\alpha \colon X \to Y \in \mathrm{ho}\mathbf{sSet}^{\mathrm{fin}}$ with source and target a finite simplicial set. In particular, it follows that $\alpha \colon X \to Y \in \mathrm{ho}\mathbf{sSet}^{\mathrm{fin}}$ is a simple equivalence in the Whitehead model category **sSet***, if and only if $|\alpha|$ is a simple homotopy equivalence in the classical sense.*

[16] A version of this result was also obtained in [Waa21]. However, the latter used the computation of the Whitehead group in terms of the fundamental group. Our new proof has no need for this, and is more conceptual in nature.

It is a classical fact that every finite CW-complex has the simple homotopy type of a finite simplicial set (simplicial complex even, see [Coh73]). Together with the classical Kan-Quillen equivalence and Main Result P, one can take this as stating that there is an equivalence between the simple homotopy theory of simplicial sets and the simple homotopy theory of CW-complexes. In fact, the way we handle this equivalence in Section 12.2 is through a notion of Whitehead framework slightly different than the setting of Whitehead model categories (roughly Eckmann's and Siebenmann's axiomatic framework) in which one can write out a comparison functor between the two theories, which then turns out to be an equivalence, in a sense specified in Section 9.2.

2.4 Simple homotopy theory of diagrams

Suppose that $\mathbf{C}$ is a Whitehead model category and that $\mathbf{R}$ is a reasonably well-behaved small indexing category (we will make this explicit in a second). We may then ask the following question: What is a good notion of presentation of a homotopy coherent diagram $F \in \mathbf{Fun}(\mathbf{R}, \mathcal{C})$ and what is a good notion of simple equivalence? There are two rather natural approaches:

- One could define a presentation of a diagram F to consist of choices of presentation $F^r \simeq \mathfrak{X}^r$, at each $r \in \mathbf{R}$. Then, given two such choices of presentations $(F^r \simeq \mathfrak{X}^r)_{r\in\mathbf{R}}$ and $(F^r \simeq \mathfrak{X}'^r)_{r\in\mathbf{R}}$, one could consider them as related, if the induced equivalences $\mathfrak{X}^r \simeq \mathfrak{X}'^r$ are simple. The resulting set of presentations of the homotopy type of F, $\mathrm{Pres}(F)$, would then simply be the product $\Pi_{r\in\mathbf{R}}\mathrm{Pres}_{\mathbf{C}}(F^r)$.
- Alternatively, one could present the ∞-category $\mathbf{Fun}(\mathbf{R}, \mathcal{C})$ in terms of the structure of a Whitehead model category on $\mathbf{Fun}(\mathbf{R}, \mathbf{C})$. In particular, one then obtains associated notions of structured cell complexes and expansions in the setting of diagrams.

While the former approach may be conceptually cleaner, we will pursue the latter approach here. This is mainly due to the fact that we are specifically interested in the interaction of the 1-categories of (structured) cell complexes with the resulting simple homotopy theories, which makes a model categorical approach seem preferable. It will turn out, however, that at least when it comes to the resulting notions of simple equivalence and presentation set, the two approaches are essentially equivalent (for appropriate finite indexing categories). Let us provide another motivation for studying the simple homotopy theory of diagrams aside from stratified homotopy theory. Namely, the theory of simple homotopy colimits. The following paragraph is lifted from the introduction of Chapter 11.

It is a classical question in homotopy theory what precise shape colimit diagrams need to have in order for them to preserve (weak) homotopy equivalences. For example, a classical statement is what is sometimes called the cube lemma (see [KP86], for example): Suppose we are given a commutative diagram of spaces

$$\begin{tikzcd}
\bullet \arrow[rr, "f"] \arrow[dr, "w_0"] \arrow[dd, hook, "a"'] & & \bullet \arrow[dr, "w_2"] \arrow[dd, hook, "a'" near end] & \\
& \bullet \arrow[rr] \arrow[dd, hook] & & \bullet \arrow[dd] \\
\bullet \arrow[rr, "f'" near end] \arrow[dr, "w_1"'] & & \bullet \arrow[dr, "w"] & \\
& \bullet \arrow[rr] & & \bullet
\end{tikzcd} \tag{2.14}$$

with the front and back square pushout, all hooked arrows closed Hurewicz cofibrations, and w_0, w_1, w_2 homotopy equivalences. Then w is also a homotopy equivalence. Similarly, if the hooked arrows are Serre-cofibrations and w_0, w_1, w_2 are weak homotopy equivalences, then so is w. From a modern perspective, these claims may be interpreted as the back and front square not just being pushout squares, but *homotopy pushout squares* or, even more modernly put, pushout squares in the associated ∞-categories (of general spaces and, respectively, of spaces with the homotopy type of a CW-complex; see, for example, [Lur09], specifically Theorem 4.2.4.1). Now, suppose that all objects in the above cube are equipped with cell structures in **Top**, and with respect to these cell structures, w_0, w_1, w_2 are simple equivalences. One may then again repeat the question and ask whether w is a simple equivalence. In other words, what shape do the front and the back face need to have in order for simple equivalences to be preserved under pushout. More generally, we may ask the question whether we can compute the Whitehead torsion of w in terms of the torsions of w_0, w_1, w_2. In [Coh73, Prop.22], Cohen gives such a criterion for the classical simple homotopy theory of CW-complexes. Namely, if both the front and back face are given by pushout squares such that all arrows are given by inclusions of subcomplexes, then

$$\langle w \rangle = f'_* \langle w_1 \rangle + a'_* \langle w_2 \rangle - (f' \circ a)_* \langle w_0 \rangle \, .$$

In particular, if w_0, w_1 and w_2 are simple equivalences then the expression above is 0, and w is also a simple equivalence. In [KP86, Theorem (4.34)], the authors gave such a formula (under somewhat stronger conditions) for more general simple homotopy theories. Their frameworks do not cover the combinatorial examples which we are interested in, however.

To study these kinds of questions, we will now study the simple homotopy theory of diagrams. In particular, we will provide similar results for finite colimit diagrams of significantly more general shape, reprove the sum-formula for general Whitehead model categories under weaker assumptions than [Coh73], and compute the general Whitehead groups associated to diagram categories.

2.4.1 Reedy categories and Reedy model structures

To study diagram categories from the perspective of Whitehead model categories, we first need to establish a notion of cell complex in a diagram category. It turns out that, modulo keeping track of cell structures, this is already a well understood question, studied for example by Riehl and Verity in [RV13] and most prominently by Hirschhorn in [Hir03] in the context of model structures on Reedy categories. Let us recall some of the basic definitions in this context. First off, one should note that given some indexing category $\mathbf{I}$, there may generally be different types of diagrams of cell complexes that one could consider. For example, in the case of spans, i.e., $\mathbf{I} = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$ one can consider diagrams where one, or where both legs are inclusions of sub-complexes. These choices may differ depending on the concrete application that one has in mind. We encode what types of diagrams one allows for in an additional structure on the indexing category of the diagrams. Let us recall some basic facts about Reedy categories, which can be found, for example in [RV13; Hir03].

Definition 2.4.1.1. A *Reedy category* consists of the data of

1. a small category $\mathbf{R}$;
2. a map $\deg\colon \mathrm{Ob}(\mathbf{R}) \to \mathbb{N}$ on the objects of $\mathbf{R}$, called the *degree function*;
3. two wide subcategories $\mathbf{R}^+, \mathbf{R}^- \subset \mathbf{R}$. Morphisms in $\mathbf{R}^+$ are sometimes called *face maps*, and morphisms in $\mathbf{R}^-$ are sometimes called *degeneracy maps*;

such that the following conditions hold:

1. For every non-identity morphism $f\colon r \to r'$ in $\mathbf{R}^+$, it holds that $\deg(r) < \deg(r')$.
2. For every non-identity morphism $f\colon r \to r'$ in $\mathbf{R}^-$, it holds that $\deg(r') < \deg(r)$.
3. Every morphism $f \in \mathbf{R}$ admits a unique factorization $f = f^+ \circ f^-$ with $f^+ \in \mathbf{R}^+$ and $f^- \in \mathbf{R}^-$.

Remark 2.4.1.2. For most intents and purposes, the concrete choice of degree function is immaterial, and it suffices that such a function exists. We will thus usually not specify it.

Example 2.4.1.3. Let us give some guiding examples to keep in mind when thinking about Reedy categories:

1. The terminal category $\star$ is a Reedy category, with the degree function taking the unique object to 0, and $\star^+ = \star^- = \star$.

2. The category Δ, equipped with the obvious degree function $[n] \mapsto n$, and with Δ^+ the sub-category of order preserving injections and Δ^- the sub-category of order preserving surjections is a Reedy category. Analogously, given a poset P, one obtains a Reedy structure on the categories of flags Δ_P.

3. The poset $\mathbb{N}$, with $\deg(n) = n$, and $\mathbb{N}^+ = \mathbb{N}$ (and consequently $\mathbb{N}^-$ a discrete category with only identity arrows) is a Reedy category.

4. The category
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \tag{2.15}$$
with any (disjoint) choice of assignments $+$ and $-$ to the non-identity arrows defines a Reedy category.

5. The poset categories $\mathrm{sd}(P)$ form Reedy categories, with the degree function given by mapping a flag $\mathcal{I} = [p_0 < \dots < p_n] \in \mathrm{sd}(P)$ to n, and with every morphism a face map.

6. The opposite of a Reedy category $\mathbf{R}$ is a Reedy category, with the roles of $\mathbf{R}^+$ and $\mathbf{R}^-$ exchanged.

In the following, we will always use the notation $\mathbf{C}^{\mathbf{I}}$ to refer to the 1-category of functors from a category $\mathbf{I}$ into $\mathbf{C}$. The evaluation of a functor $F \in \mathbf{C}^{\mathbf{I}}$ at $i \in \mathbf{I}$ will be denoted by F^i, in the covariant and by F_i in the contravariant case. Given a cofibrantly generated (semi-)model category $\mathbf{C}$ and a Reedy category $\mathbf{R}$, the functor category $\mathbf{C}^{\mathbf{R}}$ inherits the structure of a cofibrantly generated model category. Generating cofibrations and fibrations are given as follows.

Construction 2.4.1.4. Given $U \in \mathbf{Set}^{\mathbf{R}}$ and $D \in \mathbf{C}$, we can consider the functor

$$\begin{aligned} \mathbf{R} &\to \mathbf{C} \\ r &\mapsto \bigsqcup_{x \in U^r} D \end{aligned}$$

acting on morphisms in $\mathbf{R}$ via the universal property of the coproduct. We will denote the resulting diagram in the form $U \circledcirc D$ here. This construction defines a bivariate functor

$$- \circledcirc - \colon \mathbf{Set}^{\mathbf{R}} \times \mathbf{C} \to \mathbf{C}^{\mathbf{R}}$$

which is cocontinuous in both arguments.

Notation 2.4.1.5. In the context of a Reedy category $\mathbf{R}$, we will denote the associated hom sets $\mathbf{R}(r, s)$ by $\mathbf{R}^s_r$. Following this notation, we denote the value of the covariant Yoneda embedding $\mathbf{R}^{\mathrm{op}} \hookrightarrow \mathbf{Set}^{\mathbf{R}}$ at $r \in \mathbf{R}$ by $\mathbf{R}^r$, and by $\mathbf{R}_r$ the analogous covariant object. We denote by $\partial \mathbf{R}^r$ the subfunctor of $\mathbf{R}^r$ given by only such morphisms $f \colon s \to r$ for which $f^+ \neq 1$. Dually, in the covariant case, we denote by $\partial \mathbf{R}_r$ the subfunctor of $\mathbf{R}_r$, given by such morphisms f with source r for which $f^- \neq 1$. The inclusions $\partial \mathbf{R}_r \to \mathbf{R}_r$ will be denoted by ι_r (and we use analogous notation for the contravariant case).

We will mainly state results for the setting of covariant functors $\mathbf{R} \to \mathbf{C}$ here. The contravariant cases can be derived from this by dualizing and passing to opposite Reedy categories.
The bifunctor $- \circledcirc -$ embeds well into a whole calculus of weighted colimits which is discussed, for example, in [RV13]. There, the authors prove that, given sets of generating cofibrations $\mathbb{B}_{\mathbf{C}}$ and acyclic cofibrations $\mathbb{E}_{\mathbf{C}}$ of a model category $\mathbf{C}$, one obtains generators for the so-called Reedy model structure on $\mathbf{C}^{\mathbf{R}}$ in terms of the following sets. Before we spell them out, we will need some additional notation.

Notation 2.4.1.6. Given any bivariate functor $- \otimes -: \mathbf{D}_1 \times \mathbf{D}_2 \to \mathbf{D}$, and two morphisms $i_1: X_1 \to Y_1 \in \mathbf{D}_1$ and $i_2: X_2 \to Y_2 \in \mathbf{D}_2$, we denote by $i_1 \hat{\otimes} i_2$, the morphism

$$i \hat{\otimes} j: X_1 \otimes Y_2 \cup_{X_1 \otimes X_2} Y_1 \otimes X_2 \to Y_1 \otimes Y_2$$

induced by the universal property of the product. We also call $i_1 \hat{\otimes} i_2$ the *Leibniz tensor of* i_1 *and* i_2.

The following theorem is shown in the model category case in [RV13]. The semi-model category case can be shown entirely analogously (see [Bar10]).

Theorem 2.4.1.7 ([RV13])**.** *Let* $\mathbf{R}$ *be a Reedy category and* $\mathbf{C}$ *be a cofibrantly generated model category with cofibrant generators* $\mathbb{B}_{\mathbf{C}}$ *and acyclic cofibrant generators* $\mathbb{E}_{\mathbf{C}}$. *Then* $\mathbf{C}^{\mathbf{R}}$ *inherits a model structure, in which the weak equivalences are the pointwise weak equivalences, and which is cofibrantly generated by the two sets*

$$\{(\partial \mathbf{R}_r \to \mathbf{R}_r) \hat{\circledcirc} b \mid b \in \mathbb{B}_{\mathbf{C}}\}$$

and

$$\{(\partial \mathbf{R}_r \to \mathbf{R}_r) \hat{\circledcirc} e \mid e \in \mathbb{E}_{\mathbf{C}}\}.$$

The resulting model structure is called the *Reedy model structure on* $\mathbf{C}^{\mathbf{R}}$. This result provides the basis for our approach to the simple homotopy theory of diagram categories. Before we move on to homotopy theory, let us say a few words on the resulting notion of cell complex.

2.4.2 Results: Cell complexes on Reedy categories

For the remainder of this subsection, fix a cellularized category $\mathbf{C}$. In the following, given a Reedy category $\mathbf{R}$, when we refer to $\mathbf{C}^{\mathbf{R}}$ as a cellularized category, it will be with respect to the generating boundary inclusions in Theorem 2.4.1.7. In Section 8.3, we perform a detailed investigation of the cellularized categories $\mathbf{C}^{\mathbf{R}}$. Much of the results there build and expand upon results in [RV13]. In particular, we interpret the calculus of weighted colimits in Reedy categories defined in [RV13] in a cellularized framework, and investigate the question of when colimits, or more generally left Kan extension functors and their right adjoints, admit cellularizations. Our results are quite technical, and (modulo the emphasis on cell structures) probably known. We will thus not present them here (see Section 8.3 for details), and instead refer to them when we use them later on. Let us just present one crucial insight (from a more elementary point of view), which we think is integral to understanding what a structured cell complex in the cellularized categories $\mathbf{C}^{\mathbf{R}}$ is. Details can be found in Section 8.3 and Chapter 11.

Construction 2.4.2.1 (See Construction 11.1.1.11)**.** A characteristic map in a cell complex $\mathfrak{X}$ in $\mathbf{C}^{\mathbf{R}}$ is just a map $\mathbf{R}_r \circledcirc D \to X$, where $r \in \mathbf{R}$ and D is the target of a generating boundary inclusion in $\mathbb{B}_{\mathbf{C}}$. We will denote the set of characteristic maps of $\mathfrak{X}$, corresponding to some fixed $r \in \mathbf{R}$, by $\mathfrak{C}_{\mathfrak{X},r}$. The functor $\mathbf{R}_r \circledcirc -$ is left adjoint to the evaluation at r functor. Hence, we may equivalently treat such a characteristic map as a morphism $D \to X^r$ and $\mathfrak{C}_{\mathfrak{X},r}$ as a set of morphisms with target X_r. Then the set of morphisms

$$\bigcup_{f: r' \to r \in \mathbf{R}^+} X(f) \mathfrak{C}_{\mathfrak{X},r'}$$

defines a cell structure on X^r. Hence, we may think of $\mathfrak{X}$ as a diagram $\mathbf{R} \to \mathbf{C}$, together with a choice of cell structure on $\mathfrak{X}^r$, for each $r \in \mathbf{R}$. These cell structures fulfill two additional properties. To state them, we need to introduce some additional notation. For $r \in \mathbf{R}$, denote by $L^r(X)$ the latching object of X, given by the colimit $\varinjlim_{f: r' \to r \in \mathbf{R}^+, f \neq 1_r} X^{r'}$, over the subcategory of the overcategory $\mathbf{R}^+_{/r}$ given by all morphisms but the identity morphism. There is a canonical morphism $L^r(X) \to X^r$, and it is a classical fact that X is cofibrant in a Reedy model structure if and only if this morphism is a cofibration. Then, for every $r \in \mathbf{R}$, the following two statements hold:

1. The union of the images of the cell structures on $\mathfrak{X}^{r'}$ under $X^{r'} \to L^r(X)$, for $f: r' \to r \in \mathbf{R}^+$, $f \neq 1$, form a cell structure on $L^r(X)$. We denote the associated structured cell complex by $L^r(\mathfrak{X})$.

2. The induced map $L^r(\mathfrak{X}) \to \mathfrak{X}^r$ is a subcomplex inclusion.

It turns out that mapping a cell structure on $X: \mathbf{R} \to \mathbf{C}$ to a family of cell structures $(\mathfrak{X}^r)_{r \in \mathbf{R}}$, as described above, induces a bijection between cell structures on X in $\mathbf{C}^{\mathbf{R}}$ and families of cell structures $(\mathfrak{X}^r)_{r \in \mathbf{R}}$ on $(X^r)_{r \in \mathbf{R}}$ fulfilling these two properties. Hence, we may equivalently think of a structured cell complex in $\mathbf{C}^{\mathbf{R}}$ as a diagram of structured cell complexes in $\mathbf{C}$ (with morphisms in $\mathbf{C}$) fulfilling some additional conditions.

Let us look at some elementary examples to illustrate this.

Example 2.4.2.2. If we equip the span category $\bullet \leftarrow \bullet \rightarrow \bullet$ with the Reedy structure $\bullet \xleftarrow{-} \bullet \xrightarrow{+} \bullet$ (where the signs indicate that a morphism is in $\mathbf{R}^+$ or $\mathbf{R}^-$) then the resulting diagrams in Construction 2.4.2.1 are diagrams of structured cell complexes of the form $\mathfrak{Y} \leftarrow \mathfrak{X} \hookrightarrow \mathfrak{Z}$, with the right arrow the inclusion of a subcomplex. If we take the Reedy structure $\bullet \xleftarrow{+} \bullet \xrightarrow{+} \bullet$, then we obtain diagrams of the form $\mathfrak{Y} \hookleftarrow \mathfrak{X} \hookrightarrow \mathfrak{Z}$, with both arrows inclusions of subcomplexes.

Example 2.4.2.3. If we equip the poset category $\mathbb{N}$ with the Reedy structure of Example 2.4.1.3, then the resulting diagrams of structured cell complexes are of the form

$$\mathfrak{X}^0 \hookrightarrow \mathfrak{X}^1 \hookrightarrow \dots$$

with all arrows inclusions of subcomplexes.

Example 2.4.2.4. If we equip $\mathrm{sd}(P)^{\mathrm{op}}$ with the opposite structure inherited from $\mathrm{sd}(P)$ with all morphisms face maps, then the associated $\mathrm{sd}(P)^{\mathrm{op}}$ indexed diagrams of structured cell complexes fulfill no additional conditions, i.e., a cell structure on a diagram $D: \mathrm{sd}(P)^{\mathrm{op}} \to \mathbf{C}$ is simply a choice of cell structure for $D_{\mathcal{I}}$, for each $\mathcal{I} \in \mathrm{sd}(P)^{\mathrm{op}}$.

2.4.3 Results: Whitehead model structures on functor categories

Next, let us add expansions to the cellularized categories $\mathbf{C}^{\mathbf{R}}$, for a Reedy category $\mathbf{R}$. For the remainder of this subsection, suppose that $\mathbf{C}$ is equipped with the structure of a Whitehead model category. The cell structures on elementary expansions $\mathfrak{e} \in \mathbb{E}_{\mathbf{C}}$ induce canonical cell structures on the relative cell complexes $(\iota_r: \partial \mathbf{R}_r \to \mathbf{R}_r) \hat{\circledcirc} e$, given by the morphisms $1_{\mathbf{R}_r} \circledcirc \sigma$, for $\sigma \in \mathfrak{C}_{\mathfrak{e}}$. We denote the resulting structured relative cell complexes by $\iota_r \hat{\circledcirc} \mathfrak{e}$. We can now equip $\mathbf{C}$ with the set of generating elementary expansions given by

$$\{\iota_r \hat{\circledcirc} \mathfrak{e} \mid e \in \mathbb{E}_{\mathbf{C}}, r \in \mathbf{R}\}.$$

In this fashion, the cellularized category $\mathbf{C}^{\mathbf{R}}$ is equipped with the structure of a cellularized category with expansions. For this construction to define a Whitehead model category, one needs the elementary expansion $\mathbb{E}_{\mathbf{C}}$ to actually define a set of generating acyclic cofibrations. A Whitehead model category with this property will be called *properly generated*. Furthermore, to ensure that the compactness criteria in Axioms (A1) and (A2) are fulfilled, one needs the

Reedy category $\mathbf{R}$ to be such that for every $r \in \mathbf{R}$ there are only finitely many $r' \in \mathbf{R}$, such that there exists a morphism $r \to r' \in \mathbf{R}^-$. Such a Reedy category will be said to have *locally finitely many degeneracies*. The dual notion will be referred to as *having locally finitely many faces*. In Chapter 11 we show the following result:

Proposition 2.4.3.1 (Proposition 11.1.1.8)**.** *Let $\mathbf{C}$ be a properly generated Whitehead model category, and let $\mathbf{R}$ be a Reedy category which has locally finitely many degeneracies. Equip the cellularized category $\mathbf{C}^{\mathbf{R}}$ with the set of expansions*

$$\{\iota_r \hat{\circledcirc} \mathfrak{e} \mid e \in \mathbb{E}_{\mathbf{C}}, r \in \mathbf{R}\}\,.$$

Then, equipped with this class of expansions, $\mathbf{C}^{\mathbf{R}}$ is a properly generated Whitehead model category.

Hence, we can now perform simple homotopy theory on diagram categories indexed over a Reedy category and valued in a properly generated Whitehead model category.

Example 2.4.3.2. Given a poset P, the resulting Whitehead model category on $\mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$ has as its underlying model category the injective model structure on simplicial presheaves. It has the following generating boundary inclusions. Given $\mathcal{I} \in \mathrm{sd}(P)$, the Yoneda diagram $\mathrm{sd}(P)^{\mathcal{I}}$ is the diagram given by $*$, at $\mathcal{I}' \subset \mathcal{I}$, and by $\varnothing$ otherwise. The inclusion $\iota^{\mathcal{I}}\colon \partial(\mathrm{sd}(P))^{\mathcal{I}} \to \mathrm{sd}(P)^{\mathcal{I}}$ is the inclusion of the subdiagram given by $*$, at $\mathcal{I}'$ a proper subflag of $\mathcal{I}$, and by $\varnothing$ otherwise. It follows that the resulting generating boundary inclusions $\iota^{\mathcal{I}} \hat{\circledcirc}(\partial\Delta^n \to \Delta^n)$ are simply the inclusions of the diagram

$$\mathcal{I}' \mapsto \begin{cases} \Delta^n & \text{if } \mathcal{I}' \subsetneq \mathcal{I}, \\ \partial\Delta^n & \text{if } \mathcal{I}' = \mathcal{I}, \\ \varnothing & \text{else} \end{cases}$$

into the diagram

$$\mathcal{I}' \mapsto \begin{cases} \Delta^n & \text{if } \mathcal{I}' \subset \mathcal{I}, \\ \varnothing & \text{else.} \end{cases}$$

The analogous description for generating expansions involving horn inclusions holds. One special feature of the cell structures associated to a simple homotopy theory of simplicial presheaves is that cell structures are always intrinsic to the isomorphism type of a diagram in these theories (if they exist). It follows by Example 2.4.2.4 that every diagram $D \in \mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$ admits a unique cell structure. Furthermore, the model structure associated to the Whitehead model category $\mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$ is the injective model structure on simplicial presheaves. These results hold, more generally, whenever $\mathrm{sd}(P)$ is replaced with a so-called *elegant Reedy category* (see Section 8.3.5 for a definition).

Recall that weak equivalences in the Reedy model structures on diagrams are always given by pointwise weak equivalences. For the Whitehead model structures on diagram categories to be useful to the investigation of diagrams in a simple homotopy theory, one would of course want this characterization on the pointwise level to also extend to simple equivalences. Indeed, this turns out to be the case, at least under some appropriate finiteness assumptions:
Given a Whitehead model category $\mathbf{C}$ and a Reedy category $\mathbf{R}$, the evaluation functors $(-)^r\colon \mathbf{C}^{\mathbf{R}} \to \mathbf{C}$ admit a canonical cellularization (see, Example 8.3.6.9; we have already seen the absolute case in Construction 2.4.2.1). With respect to this cellularization, the evaluation functors become W-functors (Remark 11.1.2.3).
Being a W-functor does, in particular, imply that evaluation preserves simple equivalences. Hence, evaluation defines morphisms on Whitehead groups $\mathrm{Wh}_{\mathbf{C}^{\mathbf{R}}}(\mathfrak{X}) \to \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r)$. More than this, we prove the following result, computing the Whitehead group of a cell complex in the diagram category in terms of the pointwise Whitehead groups.

Main Result Q (Theorem 11.1.2.6)**.** *Let* $\mathbf{C}$ *be a properly generated Whitehead model category and let* $\mathbf{R}$ *be a finite Reedy category. The cellularized evaluation functors* $(-)^r : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}$*, for* $r \in \mathbf{R}$*, induce a natural isomorphism*

$$\mathrm{Wh}_{\mathbf{C}^{\mathbf{R}}}(\mathfrak{X}) \xrightarrow{(\mathrm{Wh}_{(-)^r})_{r\in\mathbf{R}}} \prod_{r\in\mathbf{R}} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r)$$

for finite structured cell complexes $\mathfrak{X}$ *in* $\mathbf{C}^{\mathbf{R}}$*.*

Example 2.4.3.3. As a consequence of Main Results P and Q we obtain that for a finite poset P the Whitehead groups associated to the Whitehead model category $\mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$ are given by

$$\mathrm{Wh}_{\mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}}(D) \cong \prod_{\mathcal{I}\in\mathrm{sd}(P)} \mathrm{Wh}(D_{\mathcal{I}})$$

where $\mathrm{Wh}(D_{\mathcal{I}})$ is the classical Whitehead group of (the realization of) $D_{\mathcal{I}}$.

Note that Main Result Q does, in particular, imply that the simple equivalences in $\mathbf{C}^{\mathbf{R}}$ are exactly the pointwise simple equivalences, if we think of cell complexes in $\mathbf{C}^{\mathbf{R}}$ as diagrams of cell complexes as in Construction 2.4.2.1. More than this, we obtain the Whitehead group in terms of the products of pointwise presentation sets as we have alluded to in the introduction of this section. The characterization of simple equivalences also holds beyond the finite, in the locally finite case. Here we need the notion of a Reedy category having locally finitely many faces, which is defined dually to the case of degeneracies:

Corollary 2.4.3.4 (Corollary 11.1.2.8)**.** *Let* $\mathbf{C}$ *be a properly generated Whitehead model category and let* $\mathbf{R}$ *be a Reedy category which locally has finitely many faces and degeneracies. Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be finite cell complexes in* $\mathbf{C}^{\mathbf{R}}$*. Finally, let* $\omega : \mathfrak{X} \to \mathfrak{Y}$ *be a morphism in* $\mathfrak{ho}_c(\mathbf{C}^{\mathbf{R}})$*. Then* ω *is a simple equivalence if and only if, for every* $r \in \mathbf{R}$*, the associated morphism*

$$\omega^r : \mathfrak{X}^r \to \mathfrak{Y}^r$$

in $\mathfrak{ho}_c\mathbf{C}$ *is a simple equivalence.*

2.4.4 Results: Simple homotopy colimits

One important corollary of the characterization of simple equivalences in Corollary 2.4.3.4 is that it allows one to produce compatibility results of simple homotopy equivalences with certain colimits and left Kan extensions. Essentially, the situation is such that a functor of Reedy categories $F : \mathbf{R} \to \mathbf{S}$ (i.e., a functor preserving both degeneracies and faces) has the property that the associated left Kan extension functor

$$F_! : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{S}}$$

is a W-functor (and hence preserves simple equivalences) under exactly the same conditions under which it is generally recognized to be a left Quillen functor, provided that some additional finiteness assumptions hold(see [Bar07] and Corollary 11.1.2.2 for our result). Let us explicitly state the case of colimits, i.e., the case where $\mathbf{S}$ is the terminal category. Then $\mathbf{R}$ is called *left fibrant*, if, for every $r \in \mathbf{R}$, the category $\partial(\mathbf{R}^-)_{r/}$, consisting of such arrows $r \to r'$ in $\mathbf{R}^-$ that are not the identity, is empty or connected.

Example 2.4.4.1 (Example 8.3.7.11)**.** Let $\mathbf{R}$ be a left fibrant Reedy category and $\mathbf{C}$ be a cellularized category. Then the colimit functor

$$\varinjlim : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}$$

is naturally a cellularized functor. For a structured cell complex $\mathfrak{X}$ in $\mathbf{C}^{\mathbf{R}}$, the cell structure on $\varinjlim X$ is then explicitly given by

$$\{D \xrightarrow{\sigma} X^r \to \varinjlim X \mid r \in \mathbf{R}, \sigma \in \mathfrak{C}_{\mathfrak{X},r} : \nexists f : r \to r' \text{s.t.} f^- \neq 1\}.$$

Take, for example, the left fibrant Reedy category $\bullet \xleftarrow{-} \bullet \xrightarrow{+} \bullet$. In this case, the colimit functor is given by mapping a diagram $Y \xleftarrow{f} A \xhookrightarrow{i} X$ to the lower right corner in the pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d] \\ X \arrow[r] & X \cup_A Y\,. \end{tikzcd} \tag{2.16}$$

Given an associated diagram $\mathfrak{Y} \xleftarrow{f} \mathfrak{A} \xhookrightarrow{i} \mathfrak{X}$, the cell structure on $Y \cup_X Z$ is precisely the cell structure inherited from the cobase change

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "f"] \arrow[d, hook, "i"'] & \mathfrak{Y} \arrow[d, hook, "i'"] \\ \mathfrak{X} \arrow[r, "f'"] & \mathfrak{X} \cup_{\mathfrak{A}} \mathfrak{Y}. \end{tikzcd} \tag{2.17}$$

In other words, the cells of $\mathfrak{X} \cup_{\mathfrak{A}} \mathfrak{Y}$ are given by

$$i'\mathfrak{C}_{\mathfrak{Y}} \sqcup f'(\mathfrak{C}_{\mathfrak{X}} \smallsetminus i\mathfrak{C}_{\mathfrak{A}}).$$

It turns out that, under mild finiteness conditions, the cellularized colimit functors also define W-functors.

Corollary 2.4.4.2 (Corollary 11.2.1.1)**.** *Let* $\mathbf{C}$ *be a properly generated Whitehead model category. Let* $\mathbf{R}$ *be a left fibrant Reedy category that has locally finitely many degeneracies. Then the cellularized colimit functor*

$$\varinjlim : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}$$

of Example 8.3.7.11 is a W-functor. In particular, if $\mathbf{R}$ *has locally many faces and degeneracies, it follows that a morphism of diagrams*

$$\alpha : \mathfrak{X} \to \mathfrak{Y}$$

in $\mathfrak{ho}_c(\mathbf{C}^{\mathbf{R}})$*, such that*

$$\alpha^r : \mathfrak{X}^r \to \mathfrak{Y}^r$$

is a simple equivalence, for each $r \in \mathbf{R}$*, induces a simple equivalence*

$$\varinjlim \mathfrak{X}^r \to \varinjlim \mathfrak{Y}^r$$

with respect to the induced cell structures.

Example 2.4.4.3. In the special case where $\mathbf{R} = \{\bullet \xleftarrow{-} \bullet \xrightarrow{+} \bullet\}$ one obtains a gluing theorem for simple equivalences. Namely, suppose one is given two diagrams $\{\mathfrak{Y}_0 \leftarrow \mathfrak{A}_0 \hookrightarrow \mathfrak{X}_0\}$ and $\{\mathfrak{Y}_1 \leftarrow \mathfrak{A}_1 \hookrightarrow \mathfrak{X}_1\}$ of structured finite cell complexes in some Whitehead model category $\mathbf{C}$. Suppose, in addition to this, one is given a morphism

$$\omega : \{\mathfrak{Y}_0 \leftarrow \mathfrak{A}_0 \hookrightarrow \mathfrak{X}_0\} \to \{\mathfrak{Y}_1 \leftarrow \mathfrak{A}_1 \hookrightarrow \mathfrak{X}_1\}$$

in $\mathrm{ho}(\mathbf{C}^{\mathbf{R}})$, that is given by simple equivalences, at each $r \in \mathbf{R}$. Then the induced morphism (given by the derived colimit functor)

$$\varinjlim \omega : \mathfrak{Y}_0 \cup_{\mathfrak{A}_0} \mathfrak{X}_0 \to \mathfrak{Y}_1 \cup_{\mathfrak{A}_1} \mathfrak{X}_1$$

is a simple homotopy equivalence.

In fact, one can also derive a generalization of the well-known sum formula for simple equivalences (see [Coh73, p. 23.1]). In the following, we denote by $\mathbf{Q}$ the Reedy category

$$\begin{tikzcd} (0,0) \arrow[r, "-"] \arrow[d, "+"'] & (1,0) \arrow[d, "+"] \\ (0,1) \arrow[r, "-"'] & (1,1). \end{tikzcd} \tag{2.18}$$

Main Result R (Theorem 11.2.1.7). *Let $\mathbf{C}$ be a properly generated Whitehead model category. Let $\mathfrak{X}, \mathfrak{Y}$ be two finite structured cell complexes in $\mathbf{C}^{\mathbf{Q}}$, such that the associated diagrams*

$$\begin{tikzcd} \mathfrak{X}^{(0,0)} \arrow[r, "f"] \arrow[d, hook, "a"'] \arrow[dr, "d"] & \mathfrak{X}^{(1,0)} \arrow[d, hook, "a'"] \\ \mathfrak{X}^{(0,1)} \arrow[r, "f'"'] & \mathfrak{X}^{(1,1)} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{Y}^{(0,0)} \arrow[r] \arrow[d, hook] & \mathfrak{Y}^{(1,0)} \arrow[d, hook] \\ \mathfrak{Y}^{(0,1)} \arrow[r] & \mathfrak{Y}^{(1,1)} \end{tikzcd} \tag{2.19}$$

are cobase change. Suppose that we are given a morphism

$$\omega\colon \mathfrak{X} \to \mathfrak{Y}$$

in $\mathfrak{ho}_c(\mathbf{C}^{\mathbf{Q}})$. Suppose, furthermore, that $\omega^{(0,0)}, \omega^{(1,0)}, \omega^{(0,1)}$ are isomorphisms in $\mathfrak{ho}_c\mathbf{C}$ (i.e., come from zig-zags of weak equivalences in $\mathrm{ho}\mathbf{C}$). Then $\omega^{(1,1)}\colon \mathfrak{X}^{(1,1)} \to \mathfrak{Y}^{(1,1)}$ is also an isomorphism in $\mathfrak{ho}_c\mathbf{C}$, and the identity of Whitehead torsions

$$\langle w^{(1,1)} \rangle = f'_*\langle w^{(0,1)} \rangle + a'_*\langle w^{(1,0)} \rangle - d_*\langle w^{(0,0)} \rangle$$

holds.

This result provides a generalization of the classical sum theorem in simple homotopy theory (see [Coh73, p. 23.1]) in the following three ways:

1. The theorem applies to a general Whitehead model category $\mathbf{C}$, not just the setting of classical simple homotopy theory.

2. More general diagrams are allowed. ([Coh73] requires the diagram to consist entirely of inclusions.)

3. The formula applies for morphism in $\mathrm{ho}(\mathbf{C}^{\mathbf{R}})$, i.e., ultimately to morphism in the *homotopy coherent* setting.

Finally, we may use the results on simple homotopy colimits to derive a very general recognition criterion for two cellularized functors on presheaf categories valued in simple homotopy theories to agree up to simple equivalence. We state a slightly weakened version here, for the case of elegant Reedy categories (see Section 8.3.5). For the reader not familiar with this notion, it will suffice to know that Δ, Δ_P and $\mathrm{sd}(P)$ are elegant.

Main Result S (Theorem 11.2.3.13). *Let $\mathbf{R}$ be an elegant Reedy category which locally has finitely many degeneracies and faces. Let $\mathbf{C}$ be a properly generated Whitehead model category, such that the underlying semi-model category is simplicial and locally presentable. Suppose we are given two cellularized functors $\mathfrak{F}, \mathfrak{G}\colon \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \to \mathbf{C}$ that preserve finite cell complexes. Suppose furthermore that the following conditions hold:*

- *For each pair $r, r' \in \mathbf{R}$ and $I, J \in \{F, G\}$, the derived mapping spaces $\mathcal{C}(I(\mathbf{R}^r), J(\mathbf{R}^{r'}))$ are empty or contractible and the derived mapping spaces $\mathcal{C}(F(\mathbf{R}^r), G(\mathbf{R}^r))$ and $\mathcal{C}(G(\mathbf{R}^r), F(\mathbf{R}^r))$ are non-empty.*

Then there exists an essentially unique natural transformation of the associated functors of ∞-categories

$$\begin{tikzcd} \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \arrow[r, bend left, "\mathfrak{F}"] \arrow[r, bend right, "\mathfrak{G}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{C}\mathbf{ell}(\mathbf{C}). \end{tikzcd} \tag{2.20}$$

This natural transformation is an isomorphism of functors of ∞-categories. Suppose, now, that additionally the following holds:

- *For each $r \in \mathbf{R}$, the unique morphism in $\mathfrak{ho}_c\mathbf{C}$, $\mathfrak{F}(\mathbf{R}^r) \to \mathfrak{G}(\mathbf{R}^r)$, is a simple equivalence (for example, this is the case if $\mathrm{Wh}_{\mathbf{C}}(\mathfrak{F}(\mathbf{R}^r)) = 0$ holds).*

Then, for each finite structured cell complex X in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, the induced morphism $\eta_X : \mathfrak{F}(X) \to \mathfrak{G}(X)$ in $\mathcal{C}\mathbf{ell}(\mathbf{C})$ is a simple equivalence (i.e., descends to a simple equivalence in $\mathfrak{ho}_c\mathbf{C}$).

A natural isomorphism of the associated functors valued in ∞-categories as in Main Result S will be called a *simple equivalence of cellularized functors*.

The first part of this theorem, concerning the existence of an essentially unique natural transformation, has nothing to do with simple homotopy theory, and is likely known to experts in some shape or form. On its own, it already provides a useful criterion to compare functors, which can be seen as a version of the theorem of acyclic models. For example, the theorem applies if the two functors $\mathfrak{F}$ and $\mathfrak{G}$ take representable presheaves into objects which are homotopically speaking terminal, or close to it (subterminal to be precise) and if there are *some* necessarily essentially unique equivalences between the respective values of $\mathfrak{F}$ and $\mathfrak{G}$ on representable presheaves. Then one gets a comparison transformation of the associated functors of ∞-categories for free, and that transformation is necessarily an isomorphism. This technique turns out to be very useful when one is attempting to verify that two evidently related constructions are homotopically speaking the same, and it is hard to expose an explicit natural transformation or verify that the latter defines a weak equivalence.
The crucial new information that we contribute in this result is that the resulting isomorphism also produces a *simple equivalence*, as long as one is evaluating on finite structured cell complexes. We use this result in the computation of diagrammatic stratified Whitehead groups, which we describe in the next section. Let us give a toy example, to illustrate how the result can be used.

Example 2.4.4.4. It is a well-known classical fact that the last vertex map $\mathrm{l.v.} : \mathrm{sd}X \to X$ from a barycentric subdivision of a finite simplicial set to X is a weak equivalence, and a simple equivalence if X is finite (a simple morphism, i.e., a morphism with contractible fibers, even; see [WJR13]). Suppose, for a second, that we did not know of this classical fact, and in fact, we had no idea how to construct a natural transformation $\mathrm{sd}X \to X$. Then we can apply Main Result S. We only need to verify that Δ^n and $\mathrm{sd}\Delta^n$ are weakly contractible, for each $n \in \mathbb{N}$. Both simplicial sets are given by the nerves of a poset with a terminal element. Hence, they are even contractible. It thus follows by Main Result S that there exists a natural isomorphism of functors of ∞-categories $\mathrm{sd} \Rightarrow 1$, which evaluates to simple equivalences at finite simplicial sets. If somebody now told us about the classical last vertex map, Main Result S would immediately imply that the last vertex map *is that essentially unique natural transformation* (up to higher coherence). In particular, it follows that the last vertex map is a simple equivalence, for X finite.

2.5 Simple diagrammatic stratified homotopy theory

Let us now return to the diagrammatic stratified Whitehead group of [Waa21]. The results we present in this section can be found in detail in Chapter 13.

2.5.1 Results: Computation of the diagrammatic stratified Whitehead group

By Main Result O, we already know that the realization functor

$$|-|_s\colon \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \to \mathbf{Strat}_P^{\mathfrak{d}}$$

induces a weak equivalence of Whitehead model categories. This allows us to freely switch between the topological cellular setting and the setting of stratified simplicial sets in which cell structures are intrinsic. At the end of Section 2.3.2, we already alluded to the idea of using the Quillen equivalence $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet}) \simeq \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ to compute diagrammatic stratified Whitehead groups.[17] The left adjoint functor of this equivalence $\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -\colon \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet}) \to \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ is a cellularized functor. However, for our purposes, we are primarily interested in replacing the right adjoint, i.e., the simplicial homotopy link, by a cellularized functor.

Remark 2.5.1.1. $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$, equipped with the cellularization of Example 2.4.3.2, can itself be seen as a category of **Set** valued presheaves with respect to an induced Reedy structure on $\mathrm{sd}(P) \times \Delta$. In Section 8.3.5, we explain that in these types of cellularized categories, every object has an intrinsic cell structure and the inclusions of subcomplexes are precisely the monomorphisms ($\mathrm{s}\mathbf{Strat}_P$ is also a special case of this phenomenon). It follows from this that being a cellularized functor with target such a cellularized category is a property, and not an additional structure. Such a functor is cellularized if and only if it sends relative cell complexes in the source into monomorphisms and preserves colimits.

Definition 2.5.1.2 (See also Corollary 13.2.1.14)**.** A cellularized functor

$$L\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$$

is called a *cellular link functor* if the following holds:

1. L preserves finite cell complexes;
2. L is weakly equivalent to $\mathrm{HoLink}\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$. (That is, the two functors become isomorphic, after ∞-categorically localizing weak equivalences in the target category).

As a consequence of Main Result S we obtain the following equivalent characterization of cellular link functors

Proposition 2.5.1.3 (Corollary 13.2.1.14)**.** *A cellularized functor*

$$L\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$$

that preserves finite cell complexes is a cellular link functor if and only if the following holds: For every $\mathcal{I}' \in \mathrm{sd}(P)$ and $\mathcal{J} \in \Delta_P$ a flag degenerating from a regular flag $\mathcal{I} \in \mathrm{sd}(P)$, it holds that

$$L(\Delta^{\mathcal{J}})_{\mathcal{I}'} \simeq \begin{cases} * & \text{, if } \mathcal{I}' \subset \mathcal{I}, \\ \varnothing & \text{, else.} \end{cases}$$

This criterion makes cellular link functors rather easy to detect. Let us give some examples of cellularized functors, all of which may easily be seen to be cellular link functors through Proposition 2.5.1.3.

Example 2.5.1.4. We have already seen one example of a cellular link functor in Model (D4) at least of its evaluations at regular flags $\mathcal{I} \in \mathrm{sd}(P)$. Namely, given a stratified simplicial set

[17] For reasons of formatting and to be coherent with the notation in Chapter 1, we write $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ instead of $\mathbf{sSet}^{(\mathrm{sd}P)^{\mathrm{op}}}$ in this section.

$\mathcal{X}$, take the first barycentric subdivision $\mathrm{sd}(s_{\mathcal{X}})\colon \mathrm{sd}X \to \mathrm{sdN}(P)$ of the stratification map, and then map

$$\mathcal{I} \mapsto \mathrm{Link}^{\mathrm{sd}}_{\mathcal{I}}(\mathcal{X}) = \big(\mathrm{sd}(s_{\mathcal{X}})\big)^{-1}(\mathcal{I})$$

where we treat $\mathcal{I}$ as a vertex in $\mathrm{sdN}(P) \cong \mathrm{N}(\mathrm{sd}(P))$. The structure maps of $\mathrm{Link}^{\mathrm{sd}}$ (between different flags $\mathcal{I}$) are most easily seen from the following equivalent description. Namely, observe that since the functors $\mathrm{Link}^{\mathrm{sd}}_{\mathcal{I}}$ described above preserve colimits, we may just as well consider them as the left Kan extension of their restriction to stratified simplices. Here $\mathrm{Link}^{\mathrm{sd}}_{\mathcal{I}}(\Delta^{\mathcal{J}})$ is equivalently given by the nerve of the following poset. (We treat $\mathcal{J}$ as a map $[n] \to P$, in the following).

$$\{S \subset [n] \mid \mathcal{J}(S) = \mathcal{I}\}.$$

The structure map $\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}}) \to \mathrm{Link}_{\mathcal{I}'}(\Delta^{\mathcal{J}})$, associated to $\mathcal{I}' \subset \mathcal{I}$, is given by

$$S \mapsto S \cap \mathcal{J}^{-1}(\mathcal{I}').$$

This construction defines a functor $\Delta_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ which then left Kan extends to a functor $\mathrm{Link}^{\mathrm{sd}}\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ that agglomerates the link functors of Model (D4). Observe that $\mathrm{Link}^{\mathrm{sd}}_p(\mathcal{X}) = \mathrm{sd}(X_p)$ and that in the two strata case $\mathrm{Link}^{\mathrm{sd}}_{\{p<q\}}(\mathcal{X})$ recovers the classical construction of the boundary of a regular neighborhood of $X_p \subset X$ from PL topology.

There is another, combinatorially more minimal, cellular link functor. It makes use of the classical PL topology fact that the link of X in a join $X \star Y$, with X and Y compact polyhedra is PL homeomorphic to $X \times Y$. We have already used this link functor in Chapter 5.

Example 2.5.1.5. Given $p \in \mathbf{Pos}$ and a flag $\mathcal{J} \in \Delta_P$, we will denote by $\mathcal{J}_p$ the subflag of $\mathcal{J}$ given by the inverse image of p under $\mathcal{J}\colon [n] \to P$. The cellular link functor

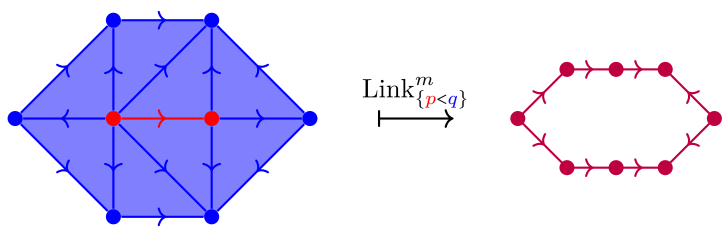
$$\mathrm{Link}^m\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$$

is defined as the left Kan extension of the functor

$$\Delta^{\mathcal{J}} \mapsto \{\mathcal{I} \mapsto \Pi_{p \in \mathcal{I}} \Delta^{\mathcal{J}_p}\}$$

acting on morphisms in the obvious way. One advantage of this version of a link functor is that it does not perform any subdivisions on the level of strata, i.e., we have $\mathrm{Link}^m_p \mathcal{X} = X_p$.

Example 2.5.1.6. Below we have illustrated a stratified simplicial set $\mathcal{X}$ over $\{p < q\}$ and the link $\mathrm{Link}^m_{\{p<q\}}(\mathcal{X})$.



The importance of cellular link functors is due to the following result, which we derive from Main Result S.

Main Result T (Theorem 13.2.1.15 and Proposition 13.2.1.12)**.** *Let $P \in \mathbf{Pos}$. Cellular link functors $L\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ have the following properties:*

1. *Any natural transformation of cellular link functors defines a simple equivalence of cellularized functors.*
2. *Any two cellular link functors are simply equivalent, through an essentially unique natural transformation of functors of ∞-categories.*

3. *Any cellular link functor is a W-functor.*

4. *Any cellular link functor L defines an ∞-categorical homotopy equivalence of Whitehead model categories $L\colon \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \simeq \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$.*

In particular, it follows that, given any finite stratified simplicial set $\mathcal{X}$, there is a natural isomorphism of Whitehead groups

$$\mathrm{Wh}_P(\mathcal{X}) \cong \mathrm{Wh}_{\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})}(\mathrm{Link}(\mathcal{X})),$$

where Link is any choice of cellular link functor. We may then combine this result with Main Result Q to obtain the following result.

Main Result U. *Let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ be a finite stratified simplicial set and let* Link *be any cellular link functor. Then* Link *induces an isomorphism of Whitehead groups*

$$\mathrm{Wh}_P(\mathcal{X}) \cong \bigoplus_{\mathcal{I} \in \mathrm{sd}(P)} \mathrm{Wh}(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})).$$

This answers Question (Q2) affirmatively. In particular, Main Result U provides an answer to the original question of [Waa21], of what types of stratified homotopy equivalences $\omega\colon |\mathcal{X}|_s \to |\mathcal{Y}|_s$ between realizations of stratified simplicial sets (or more generally diagrammatic equivalences) can be presented in terms of a zig-zag of pushouts of admissible horn inclusions. By Main Result A_1, ω lifts to a morphism $\tilde{\omega}\colon \mathcal{X} \to \mathcal{Y}$ in $\mathrm{ho}\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$. The obstructions to ω being simple in this sense are given precisely by the classical Whitehead torsion of the contributions of ω to strata and generalized homotopy links

$$\langle \mathrm{Link}_{\mathcal{I}}(\tilde{\omega}) \rangle \in \mathrm{Wh}(\mathrm{Link}_{\mathcal{I}}\mathcal{X}) \cong \mathrm{Wh}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|\mathcal{X}|_s))$$

for $\mathcal{I} \in \mathrm{sd}(P)$.[18] It also follows that not every stratum-preserving homeomorphism between realizations of simplicial complexes is a simple equivalence in $\mathbf{Strat}_P^{\mathfrak{d}}$ (see Section 13.3.2).

At first glance, the occurrence of obstructions in the Whitehead groups of generalized homotopy links provides a different answer than the one obtained for the more geometric approaches to stratified simple homotopy theory of Browder and Quinn (see [BQ79]), where the obstructions lie in the Whitehead groups of the simplicial strata. Let us explain why this is the case, providing an answer to Question (Q3) in the next and final subsection.

2.5.2 Outlook: The relationship with the geometric Whitehead torsion of a stratified h-cobordism

In [BQ79], Browder and Quinn defined a notion of h-cobordism and Whitehead torsion for several different categories of stratified spaces, in particular, the setting of certain PL stratified spaces. In this subsection, we will discuss the relationship of this geometric take on simple stratified homotopy theory with the combinatorial take we discussed so far. We will make use of some results of ours which are not contained in this text and will appear in a later article. We hope that, even without having access to the proofs yet, the elaborations here can be insightful. As the following mainly serves as an outlook, we will be slightly loose with the required definitions in PL topology and refer to [Sto72] for a good source of stratified PL investigations.

To make the comparison with the simplicial world somewhat cleaner, we will purposefully confuse piecewise linear objects with simplicial objects that triangulate them in the following. More specifically, we work with triangulations of compact PL stratified spaces that are given

[18] We have, strictly speaking, not defined the Whitehead group of a non-connected space that is not a CW-complex. It is given by the coproduct of the Whitehead groups of the fundamental groups of the path components.

by finite stratified simplicial complexes $\mathcal{M} \in \mathrm{s}\mathbf{Strat}_{[n]}$ – i.e., finite stratified simplicial sets such that every non-degenerate simplex is uniquely determined by its set of vertices. When we refer to a finite stratified simplicial complex $\mathcal{M} \in \mathrm{s}\mathbf{Strat}_{[n]}$ as a *compact PL manifold stratified space (with or without boundary)*, we inductively mean the following:
For $n = -1$, i.e. $[n] = \varnothing$, we mean an empty stratified simplicial complex. For $n+1$, we mean a stratified simplicial complex, $\mathcal{M}$, such that, for each $k \in [n+1]$ and each $x \in (|\mathcal{M}|_s)_k$, there exists a closed PL neighborhood of x that is stratum-preserving PL homeomorphic to the product of a PL disk of dimension k with the realization of a simplicial cone $\Delta^0 \star \mathcal{L}_x$ on a PL manifold stratified space $\mathcal{L}_x \in \mathrm{s}\mathbf{Strat}_{[n-k-1]}$ without boundary. Here, $\Delta^0 \star \mathcal{L}_x$ is the stratified simplicial complex over $[n]$ obtained by taking the simplicial join of Δ^0 with $\mathcal{L}_x$, mapping the cone-point Δ^0 to k, and shifting the stratum of the remaining vertices in $\mathcal{L}_x$ by $k+1$ (see Fig. 2.2, for an illustration). By definition, it follows that the strata of $|\mathcal{M}|_s$ are PL manifolds, possibly with boundary. We say that $\mathcal{M}$ is *without boundary,* if the strata of $|\mathcal{M}|_s$ have empty boundary.

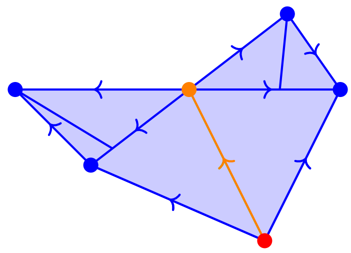


Figure 2.2: Illustration of a stratified simplicial cone, which defines a compact stratified PL manifold stratified space (with boundary) over the poset $\{0 < 1 < 2\}$.

By construction, it holds that for every compact PL manifold stratified space, $\mathcal{M}$, the realization is a conically stratified space that furthermore admits a stratified cell structure. In particular, $|\mathcal{M}|_s$ is bifibrant in all of the model structures on $\mathbf{Strat}_P$ (see Theorem 2.3.2.4), and hence well-suited to being investigated in terms of the homotopy theory we discussed in Chapter 1. The points in the boundary of $|\mathcal{M}|_s$ – i.e., such points which lie in the boundaries of the manifold strata – then inherit a triangulation in terms of a subcomplex $\partial\mathcal{M}$ of $\mathcal{M}$, which defines a PL manifold stratified space without boundary. One can prove that under these assumptions $|\partial\mathcal{M}|_s \hookrightarrow |\mathcal{M}|_s$ always admits an appropriately stratified PL collar neighborhood (see, for example, [Sto72]).

In this language, a stratified h-cobordism in Browder's and Quinn's sense is an inclusion of compact PL manifold stratified spaces $\mathcal{M} \hookrightarrow \mathcal{W}$ that induces a stratum-preserving simplicial isomorphism of $\mathcal{M}$ onto a component of the boundary of $\mathcal{W}$, such that both the inclusion of $\mathcal{M}$ as well as the inclusion of the remaining boundary component induce weak homotopy equivalences on all (simplicial) strata. In [BQ79], Browder and Quinn provide an s-cobordism theorem for such stratified h-cobordisms (see also [Wei94]). Namely, if $\mathcal{W}$ has empty strata in dimension less than or equal to 5 (or more generally, if $\mathcal{W}_{\leq 5}$ is already stratum preserving PL-homeomorphic to a cylinder), then $\mathcal{M} \hookrightarrow \mathcal{W}$ is stratum preserving PL-homeomorphic to $i_0\colon \mathcal{M} \hookrightarrow \mathcal{M} \times \Delta^1$ relative to $\mathcal{M}$, if and only if the Whitehead torsions of the simplicial inclusions $\mathcal{M}_p \to \mathcal{W}_p$ vanish, for all $p \in [n]$.

In light of Main Result U, this result may surprise in two ways. First off, one may be surprised that the definition of h-cobordism only requires equivalences on strata, and not on strata and links. Secondly, one may be surprised that the obstruction to being a cylinder seems to have no contribution associated to links.

The local and the global part of a stratified homotopy equivalence

Let us first explain why a stratified h-cobordism, as defined above, realizes to a stratified homotopy equivalence. In Chapter 7, we prove the following result. Recall from Notation 1.2.1.7 the stratified homotopy links $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$, which come together with evaluation maps $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to \mathcal{X}_p$. Furthermore, throughout this section, we will denote the fibers of maps $f: E \to B$ at $x \in B$ in the form E_x.

Proposition 2.5.2.1 (Corollary 7.5.5.14)**.** *Let $w: \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ be a stratum-preserving map of stratified spaces that are fibrant in $\mathbf{Strat}^{\mathfrak{d}}_P$. Then the following are equivalent.*

1. *w is a diagrammatic equivalence;*
2. *For each $p \in P$, the induced map $w_p: \mathcal{X}_p \to \mathcal{Y}_p$ is a weak homotopy equivalence, and for each $x \in \mathcal{X}_p$ (or just for a representative system of path components) the induced map on the stratified homotopy link fibers*

$$\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})_x \to \mathcal{H}o\mathrm{Link}^s_p(\mathcal{Y})_{w(x)}$$

is a diagrammatic equivalence.

The analogous equivalence holds for categorically fibrant spaces (i.e., stratified spaces fibrant in $\mathbf{Strat}^{\mathfrak{c}}_P$) and categorical equivalences. Even more, in this case it follows from the décollage condition that these two statements are furthermore equivalent to the following condition:

3. For each $p \in P$, the induced map $w_p: \mathcal{X}_p \to \mathcal{Y}_p$ is a weak homotopy equivalence, and for each $x \in \mathcal{X}_p$ (or just for a representative system of path components) the induced map on stratified homotopy link fibers

$$\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})_x \to \mathcal{H}o\mathrm{Link}^s_p(\mathcal{Y})_{w(x)}$$

induces weak equivalences on strata.

This condition is generally even easier to verify, as it only involves strata and unstratified, pairwise homotopy links. The important point to make here is that the fibers of the homotopy link fibrations $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to \mathcal{X}_p$ are entirely local invariants. On the π_0 level, this follows from the fact that exit paths starting in x can be retracted into small neighborhoods of x (see Lemma 7.5.5.15, for the general statement). If we apply the Whitehead theorem in $\mathbf{Strat}^{\mathfrak{d}}_P$, it follows that for bifibrant stratified spaces being a stratified homotopy equivalence can be verified in two steps: There is a global condition to verify, on the level of strata, and there is a local condition to verify, on the level of local links. Locally, using the existence of collar neighborhoods, every stratified h-cobordism is of the form

$$\mathcal{U} \times \{0\} \hookrightarrow \mathcal{U} \times [0,1],$$

and thus a stratified homotopy equivalence. Hence, one only needs to verify the global, i.e., the strata-wise weak equivalence condition.

The local and the global part of diagrammatic stratified Whitehead torsion

It turns out that the local plus global decomposition for stratified homotopy equivalences also holds for Whitehead torsion, in a sense. To explain this, we will need transfer morphisms for Whitehead groups. In [And74], Anderson defined a geometric transfer for Whitehead groups and PL fiber bundles. Let $\xi: E \to B$ be a PL fiber bundle of finite polyhedra. Then $\xi^!: \mathrm{Wh}(E) \to \mathrm{Wh}(B)$ is defined as follows[19]. Represent an element $\alpha \in \mathrm{Wh}(B)$ by the

[19]As homeomorphisms are simple homotopy equivalences, the choice of triangulation on B is irrelevant for Whitehead torsions.

Whitehead torsion of an inclusion of polyhedra $B \hookrightarrow B'$. Then, choose a PL homotopy inverse $B' \to B$, and define $\xi^!(\alpha) = \langle w'^{-1} \rangle$, where w'^{-1} is a homotopy inverse to the PL basechange.

$$\begin{tikzcd} E' \arrow[r, "w'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\xi"] \\ B' \arrow[r, "w"'] & B \end{tikzcd} \tag{2.21}$$

of w along ξ. It was later observed that significantly less was necessary to construct a transfer (see [Lüc87]). For example, in [Lüc87] Lück showed that it suffices to have a fibration of CW-complexes whose fiber has the homotopy type of a finite CW-complex and whose monodromy action on the fiber can be constructed purely in terms of simple homotopy equivalences (fixing some presentation of the fiber). The generalization in [Lüc87] was constructed entirely on the algebraic side of simple homotopy theory.
In upcoming work, we will explain another approach to such a generalization, namely a construction of a Whitehead group transfer that is analogous to Anderson's construction, but for a class of maps of finite simplicial sets. Specifically, we consider such maps of finite simplicial sets for which the homotopy fibers of the realization agree with the classical fibers (i.e., the realization is a quasi-fibration, see [DT58]), and which are furthermore such that the fundamental groupoid of the base space acts on these fibers in terms of simple homotopy equivalences. Let us call such maps *SQS-fibrations* here. SQS-fibrations provide a significantly larger class of maps than (triangulations of) PL fiber bundles or PL fibrations, even, and are generally easier to construct. They are classified by the faithful sub ∞-groupoid $\mathcal{S}\mathbf{im(sSet)}$ of $\mathcal{S}\mathbf{paces}$ given by simple equivalences of finite simplicial sets.

Using geometric methods, Anderson obtained the following formula for the Whitehead torsion of a fiber homotopy equivalence of PL fiber bundles (for the sake of simplicity, we present the connected case here). Given a commutative square

$$\begin{tikzcd} E \arrow[r, "\tilde{w}"] \arrow[d, "\xi"] & \hat{E} \arrow[d] \\ B \arrow[r, "w"'] & \hat{B} \end{tikzcd} \tag{2.22}$$

with verticals PL fiber bundles and with w and $\tilde{w}$ PL homotopy equivalences of connected polyhedra, Anderson showed an identity of Whitehead torsions

$$\langle \tilde{w} \rangle = \xi^! \langle w \rangle + \chi(B)(i_x)_* \langle \tilde{w}_x \rangle$$

where $i_x\colon E_x \to E$ is the inclusion of the fiber at some $x \in B$, χ denotes the Euler characteristic and $\tilde{w}_x\colon E_x \to \hat{E}_{w(x)}$ is the restriction of $\tilde{w}$ to fibers over x and $w(x)$. Through methods of simple simplicial homotopy theory, one can obtain a generalization of this formula to SQS-fibrations.

For the purpose of investigating compact PL manifold stratified spaces $\mathcal{M} \in \mathrm{s}\mathbf{Strat}_{[n]}$, this generalization is quite useful: Namely, making use of Stone's theory of cone-block bundles (see [Sto72]), one can prove that, for each $\mathcal{I} \in \mathrm{sd}(P)$, the natural map $f_{\mathcal{I}}\colon \mathrm{Link}^m_{\mathcal{I}} \mathcal{M} \to \mathcal{M}_{\min \mathcal{I}}$ – from the cellular link of Example 2.5.1.4 into the minimal stratum occurring in $\mathcal{I}$ – is an SQS-fibration (it does generally not realize to a PL fibration, and certainly not to a PL fiber bundle). It follows that one can compute the link contributions, for $\mathcal{I} \in \mathrm{sd}[n]$, to Whitehead torsions of a stratified homotopy equivalence of PL manifold stratified spaces $\omega\colon \mathcal{M} \to \mathcal{N} \in \mathrm{s}\mathbf{Strat}_{[n]}$ (given by a stratified simplicial map)[20] in terms of the following formula:
Using analogous notation as in Anderson's formula, it holds that

$$\langle \mathrm{Link}^m_{\mathcal{I}}(\omega) \rangle = f^!_{\mathcal{I}} \langle \omega_{\min \mathcal{I}} \rangle + \sum_x \chi(\mathcal{M}^x_{\min \mathcal{I}})(i_x)_* \langle (\mathrm{Link}^m_{\mathcal{I}}(\omega))_x \rangle.$$

[20]Being simplicial is no real restriction. There are stratified simplicial approximation theorems (see [Waa21; Sch71]), which allow any stratified homotopy class of stratified maps to be presented by simplicial maps

Here $x \in \mathcal{M}_{\min \mathcal{I}}$ ranges over a representative system of the path components $\bigsqcup \mathcal{M}^x_{\min \mathcal{I}} = \mathcal{M}_{\min \mathcal{I}}$ of $\mathcal{M}_{\min \mathcal{I}}$. Using Main Result U and potentially a version of stratified simplicial approximation as in [Sch71], one may read this result as stating the following.
The Whitehead torsion of a stratified homotopy equivalence between compact PL manifold stratified spaces (presented by a stratified simplicial map ω) consists of a global part

$$\sum_{\mathcal{I} \in \mathrm{sd}(P)} f^!_{\mathcal{I}} \langle \omega_{\min \mathcal{I}} \rangle$$

determined under transfers by the torsions on strata, and a sum of local parts

$$\sum_{\mathcal{I} \in \mathrm{sd}(P)} \sum_x \chi(\mathcal{M}^x_{\min \mathcal{I}})(i_x)_* \langle (\mathrm{Link}^m_{\mathcal{I}}(\omega))_x \rangle$$

given by torsions of the induced maps of fibers of generalized link SQS-fibrations. In case of a stratified h-cobordism $\omega \colon \mathcal{M} \hookrightarrow \mathcal{W}$, it is not too hard to see that the local parts vanish.[21] It follows, using Main Result U, that then the torsion $\langle \omega \rangle \in \mathrm{Wh}_P(\mathcal{M})$ is entirely determined by its strata-wise components.

[21] Weinberger called such maps *simple homotopy transverse* in [Wei94].

Part II

Stratified homotopy theory: Foundations and applications

Chapter 3

From homotopy links to stratified homotopy theories

Note to the reader: The following chapter is almost verbatim identical with the article [DW22], which was written in joint work with Sylvain Douteau. It is accepted to be published in the *Memoires de la Société Mathématique de France.* Compared to [DW22], we have made several minor corrections, which will also appear in the final published version. We have also adapted notation for the main categories in order to be consistent with Part I and the following chapters. Nevertheless, notation here differs slightly from the one in the single authors parts of this thesis. For example, the hom sets in a category $\mathbf{C}$ are denoted in the form $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ instead of $\mathbf{C}(X.Y)$ as is the case everywhere else, and slightly different notation for stratified realization functors and under-categories is used. The main notation difference is that in this chapter we often refer to stratified spaces by their underlying object, while in the other chapters, we take care to use calligraphic letters for stratified objects. As we will not run into the case of having multiple stratifications on the same space in this chapter, this will not be an issue. We note that there are no straight-up notational conflicts. Furthermore, the chapter is entirely self contained when it comes to notation, meaning that any notation used will be introduced, or provided with a reference. Furthermore, as this chapter only deals with one specific type of stratified homotopy theory, we do not add prefixes, such as *diagrammatic*, to the classes of weak equivalences (as we do in the chapters discussing multiple theories) and do not add the additional superscripts that distinguish between the different model structures.

In previous work, Sylvain Douteau defined homotopy theories for stratified spaces from a simplicial and a topological perspective. In both frameworks stratified weak equivalences are detected by suitable generalizations of homotopy links. These two frameworks are connected through a stratified version of the classical adjunction between the realization and the functor of singular simplices. Using a modified version of this adjunction, Douteau showed that over a fixed poset of strata the two homotopy theories were equivalent. Building on this result we now show that the unmodified adjunction induces an equivalence between the global homotopy theories of stratified spaces and of stratified simplicial sets. We do so through an in depth study of the homotopy links. As a consequence, we prove that the classical homotopy theory of conically stratified spaces embeds fully-faithfully into the homotopy theory of all stratified spaces.

3.1 Introduction

Stratifications and stratified spaces were first introduced by Whitney [Whi65b], Thom [Tho69] and Mather to describe manifolds with singularities. In this context, stratifications correspond to decompositions into strata that are themselves manifolds and satisfy certain compatibility conditions. Stratified spaces of this kind are usually called pseudo-manifolds, since they provide an extension of the class of manifolds over which many invariants can be extended while retaining their key properties. Most notably, intersection cohomology, introduced by Goresky and MacPherson in [GM80] is an extension of singular cohomology to pseudo-manifolds that still satisfies Poincaré Duality. Intersection cohomology is only an invariant up to stratum-preserving homotopy equivalence, not arbitrary homotopy equivalences, which motivates the development of a homotopy theory for stratified spaces. This question was already asked by Goresky and MacPherson (see for example [Bor+08, Problems 4 and 11]).

Motivated by the study of the homotopical properties of stratified objects, Quinn [Qui88] introduced a more general notion of stratified spaces: Homotopically stratified sets. While for pseudo-manifolds compatibility conditions between the strata are characterized by geometric links, homotopically stratified sets make use of homotopy links (or holinks), which are spaces of paths going from one stratum to another. As Miller later showed in [Mil13, Theorem 6.3], stratified homotopy equivalences between homotopically stratified sets can be fully characterized as those maps inducing homotopy equivalences on all strata and homotopy links. This inspired the idea that homotopy links and strata should be the basic blocks for defining a stratified homotopy type.

Let us now recall a few classical results of homotopy theory: In [Qui67] Quillen introduced the notion of a model category and immediately gave two seminal examples. The model category of topological spaces, **Top**, and the model category of simplicial sets, **sSet**. He also showed that the adjunction relating those two categories $|-|\colon \mathbf{sSet} \leftrightarrow \mathbf{Top}\colon \mathrm{Sing}$ was in fact a Quillen equivalence, meaning that the two homotopy theories are equivalent. This is a key result in homotopy theory since it allows one to prove results about the homotopy theory of spaces while working in a purely combinatorial setting.

In this paper, we consider stratified spaces in the broadest sense - those are just spaces with a continuous map towards a poset, $X \to P$, as defined by Woolf in [Woo09] and popularized by Lurie in [Lur17]. This point of view gives rise to two kind of categories. First, the categories of stratified objects and stratum-preserving map over a fixed poset. Here, objects can be taken to mean either topological spaces or simplicial sets, producing the categories $\mathbf{Strat}_P$ (resp $\mathrm{s}\mathbf{Strat}_P$), of topological spaces (resp simplicial sets) stratified over the poset P. Secondly, there are the categories of stratified objects over all posets, where maps are given by commutative squares. Those are denoted **Strat** and s**Strat** for the topological and simplicial versions. Note that the categories $\mathbf{Strat}_P$, correspond to the fibers of the functor $\mathbf{Strat} \to \mathrm{Poset}$ over the discrete categories $\{P\}$, and similarly for $\mathrm{s}\mathbf{Strat}_P$ and s**Strat**.

The emerging field of stratified homotopy theory has seen a lot of recent activity (See for example [AFR19; Nan19; Hai23], and [Lur17, Appendix A]) with applications to algebraic geometry [BGH18], G-isovariant homotopy theory [KY21], and to the study of classical invariants of stratified spaces [CT20]. In this context, the first author showed that there exist two independent model structures on the category $\mathbf{Strat}_P$, of spaces [Dou21c] and $\mathrm{s}\mathbf{Strat}_P$, of simplicial sets [Dou21a], stratified over a fixed poset P. See also [Dou19a]. Both are defined from appropriate notions of homotopy links (and generalization of those to tuples of strata), i.e. weak equivalences between stratified objects are maps inducing weak equivalences between all strata and homotopy links. In addition, it is shown in [Dou21c] that the model structures on $\mathbf{Strat}_P$ and $\mathrm{s}\mathbf{Strat}_P$, for varying P, assemble to form model structures on **Strat** and s**Strat** respectively. Furthermore, the classical adjunction $|-|\colon \mathbf{sSet} \leftrightarrow \mathbf{Top}\colon \mathrm{Sing}$ admits stratified versions, $|-|_P\colon \mathrm{s}\mathbf{Strat}_P \leftrightarrow \mathbf{Strat}_P\colon \mathrm{Sing}_P$, for all posets P. Those can be glued together, producing a global adjunction $|-|_s\colon s\mathbf{Strat} \leftrightarrow \mathbf{Strat}\colon \mathrm{Sing}_s$.

The adjunction $|-|_s\colon s\mathbf{Strat} \leftrightarrow \mathbf{Strat}\colon \mathrm{Sing}_s$ is not a Quillen equivalence, however. In fact, if P is not discrete, the adjunction $|-|_P\colon \mathrm{s}\mathbf{Strat}_P \leftrightarrow \mathbf{Strat}_P\colon \mathrm{Sing}_P$ is not even a Quillen adjunction. Nevertheless, by composing the above adjunction with a suitably defined stratified subdivision, the first author showed in [Dou21b] that there exists a modified Quillen equivalence

$$|\mathrm{sd}_P(-)|_P\colon \mathrm{s}\mathbf{Strat}_P \leftrightarrow \mathbf{Strat}_P\colon \mathrm{Ex}_P\mathrm{Sing}_P, \tag{3.1}$$

meaning that the homotopy theory of spaces, and simplicial sets, stratified over the same poset, coincide. On the other hand the adjunction (3.1) is no longer compatible with the gluing process producing **Strat** and s**Strat**, which means that it does not allow for a direct comparison between the homotopy theory associated to the two global model categories.

Relatedly, one would want to interpret Miller's theorem [Mil13, Theorem 6.3] - which characterizes stratum-preseving homotopy equivalences between suitably nice stratified spaces - as a statement about cofibrant-fibrant objects in a model category. But in fact, the objects appearing in Miller's theorem are almost never cofibrant as objects of **Strat**.

Both of those observations might motivate one to consider some other model structure for the category of stratified spaces, to remedy these problems. It turns out however that such a structure does not exist, without making major changes to the topological setup, as we show in Section 3.A. What we show instead is that even though the adjunctions $(|-|_P, \mathrm{Sing}_P)$ and $(|-|_s, \mathrm{Sing}_s)$ are not Quillen equivalences, they still induce - in a very strong sense - isomorphisms between the homotopy theories of stratified simplicial sets and stratified spaces (see Theorem 3.5.1.1 and Remark 3.5.1.4).

Theorem 3.1.0.1. *The adjoint pairs* $|-|_P \dashv \mathrm{Sing}_P$ *and* $|-|_s \dashv \mathrm{Sing}_s$ *descend to equivalences of homotopy categories*

$$|-|_P\colon \mathrm{ho}s\mathbf{Strat}_P \leftrightarrow \mathrm{ho}\mathbf{Strat}_P\colon \mathrm{Sing}_P,$$
$$|-|_s\colon \mathrm{ho}s\mathbf{Strat} \leftrightarrow \mathrm{ho}\mathbf{Strat}\colon \mathrm{Sing}_s.$$

Note that an even stronger statement holds. Instead of the homotopy categories, one can consider the simplicial localizations in the sense of Dwyer and Kan [DK80a]. Then, one sees that in addition to inducing equivalences between the homotopy categories, the pairs $|-|_P \dashv \mathrm{Sing}_P$ and $|-|_s \dashv \mathrm{Sing}_s$ induce Dwyer-Kan equivalences between the simplicial localizations. In particular, this means that the ∞-categories associated to stratified spaces and stratified simplicial sets are equivalent.

We also consider the case of triangulable conically stratified spaces (see Remark 3.2.4.4). As mentioned earlier, stratum-preserving homotopy equivalences between those spaces can be explicitly characterized (see [Mil13, Theorem 6, 3]). In particular, for those objects weak equivalences and stratum-preserving homotopy equivalences coincide. In fact, even though those objects are not fibrant-cofibrant, they behave as if they were. What this means is that in order to study their homotopy theory, one does not need to work with their hom-set in the homotopy category ho**Strat**, but one can work instead with the much more explicit set of stratified maps up to stratum-preserving homotopies, because the two coincide. This can be phrased more rigorously as follows (Corollary 3.5.2.4 and Remark 3.5.2.5):

Theorem 3.1.0.2. *Let* $\mathrm{Con} \subset \mathbf{Strat}$ *be the full subcategory of triangulable conically stratified spaces, and* $\simeq$ *the relation of stratified homotopy. Then the induced functor*

$$\mathrm{Con}/\simeq \hookrightarrow \mathrm{ho}\mathbf{Strat}$$

is a fully faithful embedding.
Let $\mathrm{Con}_P \subset \mathbf{Strat}_P$ *be the full subcategory of triangulable conically stratified spaces over* P*, and* $\simeq_P$ *be the relation of stratum-preserving homotopy. Then the induced functor*

$$\mathrm{Con}_P/\simeq_P \hookrightarrow \mathrm{ho}\mathbf{Strat}_P$$

is a fully faithful embedding.

Again, this statement can also be strengthened to a fully faithful inclusion of infinity categories. In the proofs contained in this paper, we mainly focus on stratified objects over a fixed poset, P. In this context, our approach is twofold. On one hand, we give a precise comparison of three model categories, and in particular of their classes of weak equivalences. Those model categories are those of spaces and simplicial sets stratified over P, as well as the category of diagrams of simplicial sets, indexed by strictly increasing chains in P (which we call regular flags). An object in this last category corresponds to the data of the strata and (generalized) homotopy links of a stratified object. This is summed up in the following theorem (Corollaries 3.4.5.2 to 3.4.5.4 and 3.6.0.2 and Theorem 3.3.2.1).

Theorem 3.1.0.3. *All functors in the following diagram of adjunctions (the right and left parts of which are commutative up to natural isomorphism) preserve and characterize weak equivalences between arbitrary objects.*

$$\begin{tikzcd}
 & \mathrm{s}\mathbf{Strat}_P \arrow[dl, shift left, "D_P"] \arrow[dr, shift left, "|-|_P"] & \\
\mathbf{Diag}_P \arrow[ur, shift left, "C_P"] \arrow[rr, shift left] & & \mathbf{Strat}_P \arrow[ul, shift left, "\mathrm{Sing}_P"] \arrow[ll, shift left]
\end{tikzcd} \tag{3.2}$$

Furthermore, all of these functors descend to equivalences of the underlying homotopy categories.

Again, Theorem 3.1.0.3 strengthens to a statement about the corresponding infinity categories. Theorem 3.1.0.3 suggests a somewhat richer picture than what one has in the non-stratified case. Indeed, in the case where $P = \{*\}$, the categories $\mathbf{Diag}_P$ and $\mathrm{s}\mathbf{Strat}_P$ are both canonically equivalent to **sSet**, turning this triangle into the familiar adjoint pair $\mathbf{sSet} \leftrightarrow \mathbf{Top}$. In general however, the categories $\mathbf{Diag}_P$ and $\mathrm{s}\mathbf{Strat}_P$ are very different, and the usefulness of thinking in terms of diagrams is illustrated in Sections 3.B and 3.6.2, through the lens of vertical objects.

Furthermore, the fact that the functors in Diagram (3.2) preserve and reflect all weak equivalences is a stronger property than one might expect. Indeed, four of those six functors are part of Quillen equivalences. One expects such a functor to only reflect weak equivalences either between cofibrant or between fibrant objects. In fact, weak equivalences in $\mathrm{s}\mathbf{Strat}_P$ can even be defined as those maps that are sent to weak equivalences in $\mathbf{Diag}_P$, **after** a suitable fibrant replacement. Theorem 3.1.0.3 then implies that this fibrant replacement is not needed. This gives some insight as to why the model structure on $\mathrm{s}\mathbf{Strat}_P$ described by Henriques in [Hen] and the one studied here coincide (see Remark 3.6.0.3).

On the other hand, we also have a more topological approach. Recall that pseudo-manifolds (and more generally conically stratified objects) can be described locally via the data of their strata and their (local) links. Indeed, in such a space, any stratum has a neighborhood that is locally homeomorphic to the product of the stratum and a cone on the (local) link. This is the phenomenon that we want to generalize to be able to reconstruct a stratified homotopy type, from the data of strata and homotopy links.

First note that the geometric definitions for the (global) link in terms of a boundary of a regular neighborhood readily extend to arbitrary stratified simplicial objects. This can be done either through the simplicial structure (by making use of the subdivision), or by working topologically. On the other hand, the homotopy links - which are nothing more than spaces of exit-paths - can be defined, as certain mapping spaces, for arbitrary stratified objects. Though, *a priori*, the result might depend on the category in which we compute those mapping spaces.

Note also that while the historical definition of homotopy links, given by Quinn in [Qui88], is only concerned with pairs of strata $[p < q]$, we study a generalized version of those homotopy links, which is defined for any increasing chain of strata $\mathcal{I} = [p_0 < \cdots < p_n]$. In this case, instead of exit-paths, the elements of the $\mathcal{I}$-th homotopy link of a stratified object can be thought of as "exit-simplices".

We show that, for a stratified simplicial set, all definitions of links and homotopy links coincide up to weak equivalence (see Theorems 3.4.5.1 and 3.6.0.1 and Remark 3.6.0.3). This is summed up in the following theorem, where the geometric interpretations given for the different definitions hold when investigating pairs of strata, i.e. when $\mathcal{I} = [p < q]$. The two notions of links are illustrated in Fig. 3.1 and elements of the three different homotopy links are represented in Fig. 3.2.

Theorem 3.1.0.4. *Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set, $\mathcal{I}$ a flag and $b \in |\Delta^{\mathcal{I}}|$ the barycenter. Then the following spaces are weakly equivalent:*

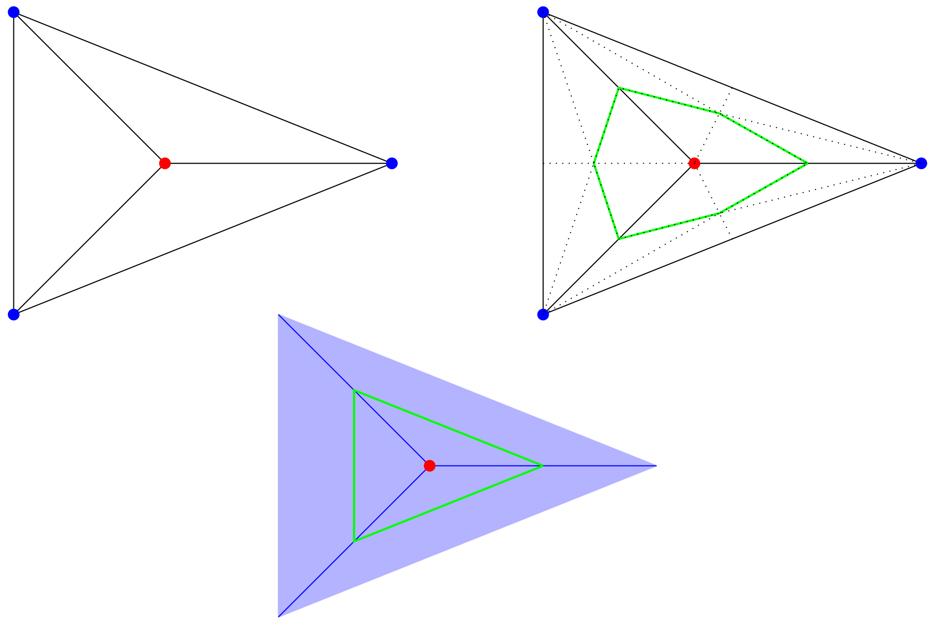


Figure 3.1: A simplicial set K stratified over $P = \{0 < 1\}$, its simplicial link, $\mathrm{Link}_{[0<1]}(K)$, and the topological link, $|K|_b$, with b in the interior of the interval $N(\{0 < 1\})$.

- $|\mathrm{HoLink}_{\mathcal{I}}(K)|$, *a simplicial version of the space of exit-paths,*
- $|\mathrm{HoLink}_{\mathcal{I}}(K^{\mathrm{Fib}})|$, *same as the above, but computed on a fibrant replacement,*
- $|\mathrm{Link}_{\mathcal{I}}(K)|$, *the usual simplicial link, defined in terms of the subdivision,*
- $|K|_b$, *a geometric notion of link, defined from the realization,*
- $Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$, *a space of exit-paths with extra conditions,*
- $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$, *the space of exit-paths defined by Quinn.*

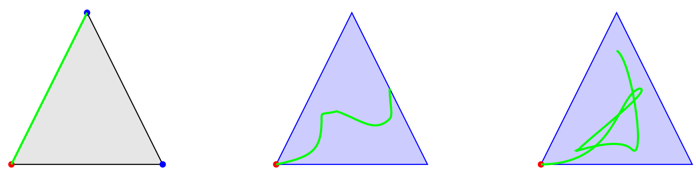


Figure 3.2: In green, from left to right, a 0-simplex in $\mathrm{HoLink}_{\mathcal{I}}(\Delta^{\mathcal{J}})$, an exit path in $Ho\mathrm{Link}_{\mathcal{I}}(|\Delta^{\mathcal{J}}|_{N(P)})$ and an exit-path in $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|\Delta^{\mathcal{J}}|_P)$, with $P = \{0 < 1\}$, $\mathcal{I} = [0 < 1]$, and $\mathcal{J} = [0 \leq 1 \leq 1]$

While Theorems 3.1.0.3 and 3.1.0.4 might look very different, they are closely related. Indeed, in the model categories we study, weak equivalences are defined in terms of strata and homotopy links, in the sense that a map is a weak equivalence if and only if it induces weak equivalences between all strata and homotopy links. This means that a comparison between different notions of homotopy links immediatly induces a comparison between the corresponding class of weak equivalences. The converse is also true, albeit in a less straightforward way. In fact, we prove Theorems 3.1.0.3 and 3.1.0.4 in parallel throughout the paper, going back and forth between the topological and the homotopical point of view.

The article is organized as follows.

Section 3.2 contains a recollection of all the necessary notions from [Dou21a; Dou21c; Dou21b], as well as definitions for all notions of link and homotopy link appearing in this article.

In Section 3.3, we prove that Sing_P characterizes all weak equivalences. We do so by first proving in Section 3.3.1 that the natural map $K \to \mathrm{Ex}_P(K)$ is a strong anodyne extension. In particular, this completes the proof that Ex_P^∞ is a fibrant replacement functor for stratified simplicial sets, see Corollary 3.3.1.2. The proof relies on the notion of strong anodyne extensions, introduced by Moss in [Mos19] and already applied to stratified simplicial sets in [Dou21a].

In Section 3.4 we prove that the functor $|-|_P\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_P$ characterizes weak equivalences between arbitrary objects. We do so by showing that, for a stratified simplicial set K, $|\mathrm{Link}_{\mathcal{I}}(K)|_P$, $|K|_b$, $\mathit{Ho}\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$ and $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$ are all weakly-equivalent. The key technical part, which consists mostly of point-set topology, is the proof that $\mathit{Ho}\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$ and $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$ are weakly equivalent.

In Section 3.5, we use the results of Sections 3.3 and 3.4 to show that the adjunctions $|-|_s \dashv \mathrm{Sing}_s$ and $|-|_P \dashv \mathrm{Sing}_P$ descend to equivalences between the homotopy categories of stratified simplicial sets and stratified spaces. We then use this result to show that the homotopy theory of triangulable conically stratified objects embeds fully faithfully in the homotopy category of stratified spaces (Corollary 3.5.2.4). Finally, in Section 3.5.3, we deduce a stratified simplicial approximation theorem from the comparison between the homotopy categories.

In Section 3.6, we prove that for a stratified simplicial set K, the holinks $|\mathrm{HoLink}_{\mathcal{I}}(K)|$, $|\mathrm{HoLink}_{\mathcal{I}}(K^{\mathrm{Fib}})|$ and $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$ are all weakly-equivalent. We do so by investigating a well-behaved classes of stratified spaces, vertically stratified CW-complexes and simplicial sets, in Section 3.6.2 and proving a series of approximation theorems (Propositions 3.6.2.9 and 3.6.2.16). Along the way, we show that the class of vertically stratified CW-complex, with their vertical maps and homotopies, model the homotopy category of stratified spaces (see Theorem 3.6.2.18). Finally, Section 3.6.3 gives the proof of Proposition 3.6.1.4 which is the key technical argument in comparing the homotopy links.

In Section 3.A, we give an example of a particular stratum-preserving map which obstructs the transport of the model structure from $\mathrm{s}\mathbf{Strat}_P$ to $\mathbf{Strat}_P$, and constrains the kind of model structures that one can hope to obtain on $\mathbf{Strat}_P$. We also discuss how this example translates for other work on the subject such as [Nan19] and [Hai23].

Section 3.B expands on Section 3.6.2, making the relationship between vertical objects and diagrams more precise, and giving an alternate higher level proof of Theorem 3.6.2.18.

3.2 Preliminaries

We begin by recalling the framework of stratified homotopy theory used in [Dou21a; Dou21c; Dou21b] as well as several of the central results of the theory. These preliminaries are intended to be rather exhaustive, making the remainder of the paper generally accessible to the reader with general knowledge of model categories and abstract homotopy theory. While introducing the necessary language and notation, these preliminaries also serve to put the results discussed in Section 3.1 into rigorous context. The reader already familiar with the framework used in [Dou21a; Dou21c; Dou21b] can safely skip most of those preliminaries, reading only Section 3.2.5 for the definitions of generalized links and homotopy links and Sections 3.2.10 to 3.2.12 for a precise recollection of the needed results.

First, we introduce the categories of stratified objects we are studying (Sections 3.2.1 to 3.2.3). We equip these objects with notions of generalized homotopy links and introduce the basic relationships between the latter (Sections 3.2.4 to 3.2.6). These generalized homotopy links entail notions of weak equivalence on categories of stratified objects in question, which are integrated into model structures in Sections 3.2.7 to 3.2.9. Finally, the homotopical properties of the functors connecting the resulting model categories are discussed in Sections 3.2.10 to 3.2.12.

3.2.1 Partially ordered sets

We begin by recalling a few general construction for partially ordered sets, which will serve as the indexing sets for stratifications.

Definition 3.2.1.1. Recall that a partially ordered set (or poset) is the data of a set P, equipped with an irreflexive, transitive and asymmetric relation, that is generically denoted "$<$". We will also consider the weak relation, "$\leq$", defined in the usual way via $p \leq q$ if $p < q$ or $p = q$. In this paper, a poset map is a map $\alpha\colon P \to Q$ such that if $p \leq p'$ in P, then $\alpha(p) \leq \alpha(p')$ in Q. The category of posets and poset maps is denoted **Pos**.

Definition 3.2.1.2. Given a poset, P, define the order topology on P as follows. A subset $A \subset P$ is closed if and only if it is a downset, i.e. it fulfills the condition:

$$p \in A, q \leq p \Rightarrow q \in A.$$

Remark 3.2.1.3. A basis of closed sets for this topology is given by the sets A_p, $p \in P$, defined as follows:

$$A_p = \{q \in P \mid q \leq p\}.$$

Remark 3.2.1.4. Note that given a map of sets $\alpha\colon P \to Q$ between posets, the map α is a map of posets if and only if it is a continuous map between the posets equipped with their order topologies. This means that the order topology defines a fully-faithful functor $\mathbf{Pos} \hookrightarrow \mathbf{Top}$.

Definition 3.2.1.5. Let P be a poset.

- A *flag* of P, $\mathcal{J}$, is a finite sequence in P, $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$.
- A *regular flag* of P, $\mathcal{I}$, is a finite sequence in P with no repeated entries $\mathcal{I} = [q_0 < \cdots < q_k]$. We will usually reserve the letter $\mathcal{I}$ for regular flags.
- Given a flag $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, we denote its underlying regular flag (obtained by deleting repeated entries) by $\{p_0 \leq \cdots \leq p_n\}$.

A map of flags $f\colon [p_0 \leq \cdots \leq p_n] \to [q_0 \leq \cdots \leq q_m]$ is a non decreasing map $f\colon \{0, \dots, n\} \to \{0, \dots, m\}$ such that $q_{f(i)} = p_i$, for all $0 \leq i \leq n$. Let Δ_P be the category of flags and maps of flags and $\mathrm{sd}(P) \subset \Delta_P$ be the full subcategory of regular flags.

Definition 3.2.1.6. The *nerve of a poset* P is the simplicial set $N(P)$ whose simplices are the flags of P. The k-th face operation is given by omitting the k-th entry in a flag while the k-th degeneracy is given by repeating the k-th entry. For $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, we will write $\Delta^{\mathcal{J}}$ for the associated n-simplex of $N(P)$, $\Delta^n \to N(P)$, seen as a simplicial set.

Remark 3.2.1.7. Note that, as was the case for spaces, a map of sets $\alpha\colon P \to Q$ between posets is a map of posets if and only if it extends to a simplicial map $N(\alpha)\colon N(P) \to N(Q)$. In other words, there is a fully-faithful embedding $N\colon \mathbf{Pos} \hookrightarrow \mathbf{sSet}$.

Remark 3.2.1.8. It is also classical to see a poset as a (small) category where there are no non-identity endomorphisms, and where there is at most one map between two objects. Starting from a set-theoretic poset P one gets such a category by taking P to be the set of objects, and having an arrow from $p \to q$ whenever $p \leq q$. Note that from this point of view, the nerve of the poset is nothing more than the nerve of the corresponding category. Similarly, the category of regular flags $\mathrm{sd}(P)$ is the subdivision of P, seen as a category.

Definition 3.2.1.9. Let P be a poset. Define $\varphi_P\colon |N(P)| \to P$ as follows. For $\mathcal{I} = [p_0 < \cdots < p_n]$ a regular flag, using the identification

$$|\Delta^{\mathcal{I}}| \cong \{(t_0, \dots, t_n) \mid 0 \leq t_i \leq 1,\ 0 \leq i \leq n,\ \sum_i t_i = 1\} \subset \mathbb{R}^{n+1},$$

we set

$$\varphi_P(t_0, \dots, t_n) = p_m,\ m = \max\{i \mid t_i \neq 0\}.$$

3.2.2 Stratified spaces

Before we define stratified spaces, a technical remark about the nature of the topological spaces considered is in order.

Remark 3.2.2.1. In this paper, **Top** stands for the category of Δ-generated spaces (see [Dug03] and [FR08]) and all continuous maps between them. This is to ensure that the category is locally presentable, which is needed to get the model structure described in Theorem 3.2.8.1 (see also Remark 3.2.8.5). It shouldn't be of much concern however, since all the usual examples of stratified spaces, as well as posets themselves, are Δ-generated spaces. The implications of this choice of category for mapping spaces are discussed in Remark 3.2.2.5.

In this paper, we consider the most general notion of stratified spaces among those commonly used.

Definition 3.2.2.2. A *stratified space* is the data of:

- a topological space X,
- a poset P,
- a continuous map $\varphi_X\colon X \to P$, called the *stratification.*

By abuse of notation, we will often refer to the above data just by the space X. The *strata* of X are the subspaces $\varphi_X^{-1}(p) = X_p \subset X$ for $p \in P$. A *stratified map* between two stratified spaces $X \to P$ and $Y \to Q$ is a commutative square of continuous maps

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_X"'] & Y \arrow[d, "\varphi_Y"] \\ P \arrow[r, "\bar{f}"] & Q \end{tikzcd}$$

We will often refer to such a square just by the map f. The category of stratified spaces and stratified maps is denoted **Strat**.

Given a poset P, a *stratum-preserving* map between two spaces stratified over P, X and Y, is a continuous map $f\colon X \to Y$ such that the following triangle commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi_X"'] & & Y \arrow[dl, "\varphi_Y"] \\ & P\,. & \end{tikzcd}$$

The category of spaces stratified over P and stratum-preserving maps is denoted $\mathbf{Strat}_P$.

Remark 3.2.2.3. Consider the functor $\mathbf{Strat} \to \mathbf{Pos}$ sending $X \to P$ to P and a stratified map f to $\bar{f}$. This functor is a bifibration (see [CM20, Section 2.1]), and the fiber over a poset P is the subcategory $\mathbf{Strat}_P$. This situation is pictured in the following diagram:

$$\begin{tikzcd} \mathbf{Strat}_P \arrow[r, hook] \arrow[d] & \mathbf{Strat} \arrow[d] \\ \{P, 1_P\} \arrow[r, hook] & \mathbf{Pos}\,. \end{tikzcd}$$

Recollection 3.2.2.4. The category $\mathbf{Strat}_P$ admits the structure of a simplicial category. The tensoring of $\mathbf{Strat}_P$ for $X \in \mathbf{Strat}_P$ and $S \in \mathbf{sSet}$ is given by $X \times |S|$ with the stratification induced by first projecting to the first component. By abuse of notation, this stratified space will be denoted $X \times S$. Consequently, for $X, Y \in \mathbf{Strat}_P$ the simplicial hom is defined via

$$\mathrm{Map}(X,Y)_n = \mathbf{Strat}_P(X \times \Delta^n, Y).$$

Furthermore, **Strat**$_P$ is also a tensored and cotensored category over **Top**, again with the tensoring induced by taking the product and stratifying via projection to the stratified component. The mapping space

$$\mathcal{C}^0_P(X,Y) \subset \mathcal{C}^0(X,Y)$$

is obtained by considering the topology on $\mathbf{Strat}_P(X,Y)$ induced by the inclusion into $\mathcal{C}^0(X,Y)$. By construction, we obtain natural isomorphisms

$$\mathrm{Sing}\big(\mathcal{C}^0_P(X,Y)\big) \cong \mathrm{Map}(X,Y)$$

relating the simplicial and the topological enrichment of **Strat**$_P$.

Note that the category **Strat** also admits simplicial and topological enrichment, defined in a similar way, as well as internal hom-sets (see [Nan19, section 6.2]).

Remark 3.2.2.5. There is some ambiguity on the topology that should be assigned to the mapping space $\mathcal{C}^0_P(X,Y)$. In this work, the category **Top** stands for the category of Δ-generated space, (see [FR08; Dug03]), and so $\mathcal{C}^0_P(X,Y)$ should be equipped with the Δ-ification of the compact open topology. On the other hand, note that $\mathrm{Sing}(\mathcal{C}^0_P(X,Y))$ gives the same simplicial set whether one uses the compact open topology or its Δ-ification, and in particular, the map $\mathcal{C}^{0,\Delta}_P(X,Y) \to \mathcal{C}^{0,\mathrm{c.o.}}_P(X,Y)$ given by the change in topology is a weak equivalence. Since we are only interested in the (weak) homotopy type of those objects, we will always consider $\mathcal{C}^0_P(X,Y)$ as equipped with the compact open topology.

Furthermore, if X is some compact stratified space and K is a locally finite stratified simplicial set, then $\mathcal{C}^0_P(X,|K|_P)$ will be metrizable. In particular, $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P) = \mathcal{C}^0_P(|\Delta^{\mathcal{I}}|_P,|K|_P)$, will be metrizable if K is a locally finite stratified simplicial set (see Definition 3.2.5.1). This observation will be useful in the proof of Theorem 3.4.4.1.

The simplicial structure immediately induces a notion of homotopy.

Definition 3.2.2.6. Let $f,g\colon X \to Y$ be two maps in **Strat**. A *stratified homotopy* between f and g is the data of a stratified map $H\colon X \times \Delta^1 \to Y$ such that the restriction to $X \times \{0\}$ and $X \times \{1\}$ are equal to f and g respectively. We denote this relation by $f \simeq_s g$. A *stratified homotopy equivalence* is a stratified map $f\colon X \to Y$ such that there exists a stratified map $g\colon Y \to X$, such that $g \circ f \simeq_s 1_X$ and $f \circ g \simeq_s 1_Y$. For $X, Y \in \mathbf{Strat}_P$, we will write $[X,Y]_P$ for the set of stratified homotopy classes of stratum-preserving maps between X and Y.

Remark 3.2.2.7. One can rephrase the above definition as follows: given two stratified spaces $X \to P$ and $Y \to Q$ and two maps between them, f and g, a stratified homotopy between f and g corresponds to a commutative square in **Top**:

$$\begin{tikzcd} X \times [0,1] \arrow[r, "H"] \arrow[d, "\varphi_X \circ \mathrm{pr}_X"'] & Y \arrow[d, "\varphi_Y"] \\ P \arrow[r, "\bar{H}"] & Q\,. \end{tikzcd}$$

There are several things to note here.

- If two maps are stratified homotopic, the underlying maps of spaces are homotopic, since H is also a homotopy in the usual sense.
- Since the homotopy is constant at the level of posets, one must have $\bar{H} = \bar{f} = \bar{g}$. In particular, any stratified homotopy between stratum-preserving maps is given by a stratum-preserving homotopy, H. For this reason we will not make a distinction between stratum-preserving and stratified homotopies, and just use "stratified homotopy" as a generic term.
- Since the homotopy is constant at the poset level, any stratified homotopy equivalence must be over an isomorphism of posets.

We will also make use of a stronger notion of stratified spaces, for which, in addition to a decomposition into strata, the stratification also encodes information about the neighborhoods of strata.

Definition 3.2.2.8. A *strongly stratified space* is the data of

- a space X,
- a poset P,
- a continuous map $\varphi_X : X \to |N(P)|$ called the *(strong) stratification*.

Stratum-preserving maps between strongly stratified spaces are defined analogously to Definition 3.2.2.2 as commutative triangles

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi_X"'] & & Y \arrow[dl, "\varphi_Y"] \\ & {|N(P)|} & \end{tikzcd} .$$

This leads to the category $\mathbf{Top}_{N(P)}$, of spaces strongly stratified over P and stratum-preserving maps. By abuse of language, we will call the objects of $\mathbf{Top}_{N(P)}$ spaces stratified over $N(P)$.

Recollection 3.2.2.9. The map $\varphi_P : |N(P)| \to P$, of Definition 3.2.1.9 induces a functor:

$$\varphi_P \circ - : \mathbf{Top}_{N(P)} \to \mathbf{Strat}_P$$
$$(X, \varphi_X : X \to |N(P)|) \mapsto (X, \varphi_P \circ \varphi_X : X \to P).$$

This functor admits a right-adjoint, $- \times_P |N(P)| : \mathbf{Strat}_P \to \mathbf{Top}_{N(P)}$, defined on objects as the following pull-back:

$$\begin{tikzcd} Y \times_P |N(P)| \arrow[r] \arrow[d] & Y \arrow[d, "\varphi_Y"] \\ {|N(P)|} \arrow[r, "\varphi_P"] & P \end{tikzcd},$$

where the strong stratification on $Y \times_P |N(P)|$ is given by the projection on the second factor. Note that $\mathbf{Top}_{N(P)}$ also admits a simplicial and topological enrichment, as in Recollection 3.2.2.4, which analogously to the stratified setting, induces a notion of *strongly stratified homotopies*. These enrichment structures are compatible with the adjunction $\varphi_P \circ - \dashv - \times_P |N(P)|$. In particular, this adjoint pair preserves the respective notions of stratified homotopies.

3.2.3 Stratified simplicial sets

We will also consider a simplicial version of stratified spaces, the stratified simplicial sets.

Definition 3.2.3.1. A *stratified simplicial set* is the data of:

- a simplicial set K,
- a poset P,
- a simplicial map $\varphi_K : K \to N(P)$, called the *stratification*.

By abuse of notation, we will often refer to the above data just by the simplicial set K. A *stratified map* between two stratified simplicial sets $K \to N(P)$, $L \to N(Q)$ is a commutative square of simplicial maps.

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "\varphi_K"'] & L \arrow[d, "\varphi_L"] \\ N(P) \arrow[r, "N(\bar{f})"] & N(Q) \end{tikzcd}$$

The category of stratified simplicial sets and stratified maps is denoted $s\mathbf{Strat}$.

Given a poset P, a *stratum-preserving map* between two simplicial sets stratified over P, K and L, is a simplicial map $f: K \to L$, such that the following triangle commutes:

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "\varphi_K"'] & & L \arrow[dl, "\varphi_L"] \\ & N(P). & \end{tikzcd}$$

The category of simplicial sets stratified over P is denoted $\mathrm{s}\mathbf{Strat}_P$.

Remark 3.2.3.2. The category $\mathrm{s}\mathbf{Strat}_P$ can alternatively be defined as a presheaf category. Indeed, there is an equivalence of categories $\mathrm{s}\mathbf{Strat}_P \cong \mathbf{Fun}(\Delta_P^{\mathrm{op}}, \mathbf{Set})$ (see Definition 3.2.1.5). Note that under this identification, a flag $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ is sent to the stratified simplicial set $\varphi_{\mathcal{J}}: \Delta^n \to N(P)$, satisfying $\varphi_{\mathcal{J}}(k) = p_k$, where k stands for the k-th vertex of Δ^n. We will denote this stratified simplicial set $\Delta^{\mathcal{J}}$, and call all such objects stratified simplices. See [Dou21a, Proposition 1.3]. Note that this is consistent with the convention of Definition 3.2.1.6.

Recollection 3.2.3.3. The category $\mathrm{s}\mathbf{Strat}_P$ admits the structure of a simplicial category. The tensoring between $K \in \mathrm{s}\mathbf{Strat}_P$ and $S \in \mathbf{sSet}$ is given by the product $K \times S$ with the stratification given by the composition $K \times S \to K \to N(P)$. For $K, L \in \mathrm{s}\mathbf{Strat}_P$ the simplicial hom is defined via

$$\mathrm{Map}(K, L)_n = \mathrm{s}\mathbf{Strat}_P(K \times \Delta^n, L).$$

Just as for stratified spaces, the simplicial structure induces a notion of homotopy equivalences.

Definition 3.2.3.4. Let $f, g: K \to L$ be two maps in $s\mathbf{Strat}$. An *elementary stratified homotopy* between f and g is the data of a stratified map $H: K \times \Delta^1 \to L$, whose restrictions to $K \times \{0\}$ and $K \times \{1\}$ give f and g respectively. The maps f and g are said to be *stratified homotopic* if there exists a finite sequence of maps f_i, $0 \leq i \leq n$ such that $f_0 = f$, $f_n = g$, and for all $0 \leq i \leq n-1$, f_i and f_{i+1} are related by an elementary stratified homotopy. A *stratified homotopy equivalence* is a stratified map $f: K \to L$ such that there exists a stratified map $g: L \to K$ such that $g \circ f$ and $f \circ g$ are respectively stratified homotopic to 1_K and 1_L. For $K, L \in \mathrm{s}\mathbf{Strat}_P$, we will write $[K, L]_P$ for the set of stratum-preserving maps between K and L up to stratified homotopy.

Recollection 3.2.3.5. The categories of stratified simplicial sets and stratified spaces are connected through a $|-| \dashv \mathrm{Sing}$ style adjunction, just as in the unstratified context. It can be described explicitly as follows. Given some fixed poset P, we define a realization style functor

$$\begin{aligned} |-|_{N(P)}: \mathrm{s}\mathbf{Strat}_P &\to \mathbf{Top}_{N(P)} \\ (\varphi_K: K \to N(P)) &\mapsto (|\varphi_K|: |K| \to |N(P)|). \end{aligned}$$

Composing with the functor $\varphi_P \circ -: \mathbf{Top}_{\mathrm{N}(P)} \to \mathbf{Strat}_P$ (see Recollection 3.2.2.9) gives a functor $|-|_P: \mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_P$. Both functors admit right adjoints, $\mathrm{Sing}_P: \mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P$ and $\mathrm{Sing}_{N(P)}: \mathbf{Top}_{\mathrm{N}(P)} \to \mathrm{s}\mathbf{Strat}_P$, defined as follows. Given a stratified space $\varphi_X: X \to P$, its *stratified singular simplicial set*, $\mathrm{Sing}_P(X)$, is given by the pullback

$$\begin{tikzcd} \mathrm{Sing}_P(X) \arrow[r] \arrow[d] & \mathrm{Sing}(X) \arrow[d, "\mathrm{Sing}(\varphi_X)"] \\ N(P) \arrow[r, hook] & \mathrm{Sing}(P), \end{tikzcd}$$

where the bottom map is the adjoint map to φ_P from Definition 3.2.1.9 under $|-| \dashv \mathrm{Sing}$. Equivalently, as a presheaf on Δ_P, $\mathrm{Sing}_P(X)$ can be constructed via

$$\mathrm{Sing}_P(X)_{\mathcal{J}} = \mathbf{Strat}_P(|\Delta^{\mathcal{J}}|_P, X).$$

Note that the collection of adjunctions $|-|_P \dashv \mathrm{Sing}_P$ extend to an adjunction

$$|-|_s\colon s\mathbf{Strat} \leftrightarrow \mathbf{Strat}\colon \mathrm{Sing}_s.$$

The definition of $\mathrm{Sing}_{N(P)}$ is entirely analogous.

Note also that the adjunctions $|-|_P \dashv \mathrm{Sing}_P$ and $|-|_s \dashv \mathrm{Sing}_s$ are compatible with the simplicial structures (see [Dou21a, Proposition 4.9]).

Proposition 3.2.3.6. *The adjunctions $|-|_P \dashv \mathrm{Sing}_P$ and $|-|_s \dashv \mathrm{Sing}_s$ are simplicial, and preserve stratified homotopies (see Definitions 3.2.2.6 and 3.2.3.4).*

3.2.4 Classical links and homotopy links

Recall that pseudo-manifolds are particular kinds of stratified spaces whose strata are manifolds satisfying gluing conditions. Those conditions can be expressed through the use of some smooth structures, as in [Whi65b] or [Tho69]. Or they can be expressed in a purely topological fashion by asking that all points should have neighborhoods that are stratum-preserving homeomorphic to products of appropriately stratified cones with trivially stratified spaces. We recall the more recent and more general definition of *conically stratified spaces*, which will be sufficient for our purpose. Note that pseudo-manifolds are examples of conically stratified spaces.

Definition 3.2.4.1 ([Lur17, Definition A.5.5])**.** Let P be a poset, and define $c(P) = \{*\} \coprod P$, with $* < p$ for all $p \in P$. Let $\varphi_L\colon L \to P$ be a stratified space. Its (stratified) *cone* is the stratified space

$$c(L) = L \times [0,1)/L \times \{0\} \to c(P)$$
$$(l,t) \mapsto \begin{cases} \varphi_L(l) & \text{, if } t > 0 \\ * & \text{, if } t = 0. \end{cases}$$

equipped with the teardrop topology, (see Remark 3.2.4.2). A stratified space $X \to P$ is *conically stratified* if, for every $x \in X$, in some stratum X_p, there exists

- a stratified space $L \to P_{>p} = \{q \in P \mid q > p\}$, the *(local) link at x*,
- an open neighborhood $x \in U \subset X$,
- a space Z,
- and a stratified homeomorphism $Z \times c(L) \cong U$, over the poset identification $c(P_{>p}) \cong P_{\geq p} \subset P$.

Remark 3.2.4.2. In the above definition, one considers the teardrop topology on the cone $c(L)$ (see [Lur17, Definition A.5.3]). Note that for a compact (stratified) space, L, it coincides with the quotient topology. Since pseudo-manifolds are always assumed to have compact links, this subtlety does not come into play when studying those objects. However, when studying more general examples of conically stratified spaces, this distinction is crucial. In fact, we will use throughout a result of Lurie [Lur17, Theorem A.6.4] (see Theorem 3.2.10.2) whose proof relies on the properties of the teardrop topology.

An even more general notion, more suited for the study of stratified homotopies, was introduced by Quinn in [Qui88], that of homotopically stratified sets. For those objects, instead of considering local links, one considers pairwise homotopy links, which provide a homotopy-theoretic global approach to the former.

Definition 3.2.4.3 ([Qui88, Definition 2.1])**.** Let $X \to P$ be a stratified space and $p < q \in P$. The *homotopy link* of the p-stratum in the q-stratum is the topological space

$$\mathcal{H}o\mathrm{Link}_{p<q}(X) = \{\gamma\colon [0,1] \to X \mid \gamma(0) \in X_p,\ \gamma(t) \in X_q,\ \forall t > 0\},$$

whose topology is induced by the inclusion $\mathcal{H}o\mathrm{Link}_{p<q}(X) \subset C^0([0,1], X)$.

Quinn then defined (see [Qui88, Definition 3.1]) *homotopically stratified sets* as those (metric) stratified spaces satisfying:

- evaluation at 0, $\mathrm{ev}_0\colon \mathcal{H}o\mathrm{Link}_{p<q}(X) \to X_p$ is a Hurewicz-fibration in **Top**, for all $p < q \in P$,
- inclusions $X_p \hookrightarrow X_p \cup X_q$ are tame, for all $p < q \in P$.

Remark 3.2.4.4. In this paper, we will not be overly concerned with the distinctions between those classes of spaces. As we will see in Theorem 3.2.10.2, pseudo-manifolds, conically stratified spaces, and homotopically stratified sets share a common homotopical property. It is this property - and not their geometrical definitions directly - that is leveraged in the proof of Theorem 3.1.0.2. We only state results about conically stratified spaces - since it is both a large class, and reasonably easy to define - but Theorem 3.1.0.2 also holds for pseudo-manifolds or homotopically stratified sets.

Homotopy links are known to characterize the stratified homotopy equivalences between homotopically stratified sets, as was shown in [Mil13].

Theorem 3.2.4.5 ([Mil13, Theorem 6.3])**.** *A stratified map $f\colon X \to Y \in \mathbf{Strat}_P$, between two homotopically stratified sets is a stratified homotopy equivalence if and only if the induced maps of spaces*

- $X_p \to Y_p$,
- $\mathcal{H}o\mathrm{Link}_{p<q}(X) \to \mathcal{H}o\mathrm{Link}_{p<q}(Y)$,

are homotopy equivalences, for all $p \in P$ and $q \in P$ with $p < q$.

It is also possible to get a similar statement, while only asking the maps between strata and homotopy links to be *weak*-equivalences. In this case, one has to ask that X and Y are realizations of stratified simplicial sets, (see [Dou21a, Theorem 5, 4]). In any case, these results suggest that homotopy links are a good starting point to build a general homotopy theory of stratified spaces. On the other hand, consider the following example.

Example 3.2.4.6. Consider the stratified simplex $\Delta^{\mathcal{J}}$ for the (regular) flag $\mathcal{J} = [p_0 < p_1 < p_2]$ in P as well as its boundary $\partial\Delta^{\mathcal{J}} \subset \Delta^{\mathcal{J}}$ with the induced stratification (see Fig. 3.3). The inclusion

$$i\colon |\partial\Delta^{\mathcal{J}}|_P \hookrightarrow |\Delta^{\mathcal{J}}|_P$$

induces homotopy equivalences on all strata and pairwise homotopy links, since for both spaces all of these are contractible. However, i is clearly not a stratified homotopy equivalence since if we forget the stratification, then the underlying map of topological spaces is not even a weak equivalence. Note that this is not in contradiction to Theorem 3.2.4.5 as $|\partial\Delta^{\mathcal{J}}|_P$ is not a homotopically stratified set. Indeed, the natural map $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|\partial\Delta^{\mathcal{J}}|_P) \to (|\partial\Delta^{\mathcal{J}}|_P)_{p_1}$, for $\mathcal{I} = \{p_1 < p_2\}$, is not a fibration as it has a nonempty source but is not surjective. If we want the stratified homotopy type to be finer than the homotopy type of the underlying space, $\partial\Delta^{\mathcal{J}}$ and $\Delta^{\mathcal{J}}$ need to have a distinct stratified homotopy type, which can not be defined through strata and homotopy links alone.
On the other hand, note that the homotopy links of Definition 3.2.4.3 can be equivalently defined as $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) = \mathcal{C}^0_P(|\Delta^{\mathcal{I}}|_P, X)$, for $\mathcal{I} = \{p < q\}$ a regular flag of P. Note that this definition readily extends to *arbitrary* regular flags $\mathcal{I} = \{p_0 < \cdots < p_n\}$. In particular, for the example at hand, one checks that the inclusion $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|\partial\Delta^{\mathcal{J}}|_P) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|\Delta^{\mathcal{J}}|_P)$ is not a weak equivalence for $\mathcal{I} = \mathcal{J}$. Indeed, the domain is empty, as there exists no stratified maps $|\Delta^{\mathcal{J}}|_P \to |\partial\Delta^{\mathcal{J}}|_P$, while the codomain contains the identity map. This illustrates the necessity to consider *generalized homotopy links*. Indeed, at least when restricting to stratified spaces that admit (stratified) triangulations, stratified maps inducing weak equivalences between all strata and generalized homotopy links are weak equivalence of the underlying spaces (see Example 3.2.8.3 and Remark 3.2.8.4).

Figure 3.3: The simplex $\Delta^{\mathcal{J}}$ and its boundary $\partial\Delta^{\mathcal{J}}$, with $\mathcal{J} = [p_0 < p_1 < p_2]$. The first has a non-empty $\mathcal{J}$-holink, while $\mathrm{HoLink}_{\mathcal{J}}(\partial\Delta^{\mathcal{J}}) = \emptyset$. For all other flags $\mathcal{I} \neq \mathcal{J}$, $\Delta^{\mathcal{J}}$ and $\partial\Delta^{\mathcal{J}}$ have equivalent $\mathcal{I}$-holinks.

3.2.5 Generalized links and homotopy links

We now turn to the more general notion of homotopy links, that was suggested in Example 3.2.4.6. Note that we will simply call those *homotopy links*, instead of generalized homotopy links, and refer to the homotopy links of Quinn as *classical homotopy links*. Since the latter is just a particular case of the former, as pointed out in Example 3.2.4.6, this should not cause too much confusion.

Definition 3.2.5.1. Let $K \in \mathrm{s}\mathbf{Strat}_P$, $X \in \mathbf{Top}_{N(P)}$, $Y \in \mathbf{Strat}_P$ and $\mathcal{I} = [p_0 < \cdots < p_n]$ a regular flag. We define $\mathcal{I}$-th *homotopy links*, in each of the three categories, as follows:

- $\mathrm{HoLink}_{\mathcal{I}}(K) = \mathrm{Map}(\Delta^{\mathcal{I}}, K) \in \mathbf{sSet}$,
- $\mathit{Ho}\mathrm{Link}_{\mathcal{I}}(X) = \mathbf{C}^0_{N(P)}(|\Delta^{\mathcal{I}}|_{N(P)}, X) \in \mathbf{Top}$,
- $\mathcal{H}\!o\mathrm{Link}_{\mathcal{I}}(Y) = \mathbf{C}^0_{P}(|\Delta^{\mathcal{I}}|_{P}, Y) \in \mathbf{Top}$.

When there is a need to distinguish between them, the first two will be respectively called the *simplicial homotopy link* and the *strong homotopy link*, while the last one will always just be called the homotopy link.

Remark 3.2.5.2. There is a simple mnemonic underlying the choice of font for $\mathrm{HoLink}_{\mathcal{I}}$, $\mathit{Ho}\mathrm{Link}_{\mathcal{I}}$ and $\mathcal{H}\!o\mathrm{Link}_{\mathcal{I}}$. Vertices of $\mathrm{HoLink}_{\mathcal{I}}(K)$ are given by simplicial maps from a stratified simplex into K. Meanwhile, elements of $\mathit{Ho}\mathrm{Link}_{\mathcal{I}}(X)$ are given by strongly stratum-preserving maps from (the realization) of a stratified simplex into a space X and elements of $\mathcal{H}\!o\mathrm{Link}_{\mathcal{I}}(Y)$ by stratum-preserving ones. Hence, the degree of rigidity required from these maps decreases in the order of listing, see Fig. 3.2. This is reflected by the font used for the "Ho" part of the respective notation.

Remark 3.2.5.3. Let $X \in \mathbf{Strat}_P$ be a P-stratified space and $\mathcal{I}$ a regular flag. Then, by Recollection 3.2.2.4 and Recollection 3.2.2.9 we have natural isomorphisms:

$$\mathrm{HoLink}_{\mathcal{I}}\big(\mathrm{Sing}_P(X)\big) \cong \mathrm{Sing}\big(\mathcal{H}\!o\mathrm{Link}_{\mathcal{I}}(X)\big) \cong \mathrm{Sing}\big(\mathit{Ho}\mathrm{Link}_{\mathcal{I}}(X \times_P |N(P)|)\big).$$

Before considering a converse statement with $K \in \mathrm{s}\mathbf{Strat}_P$, we first introduce a generalization of the classical notion of links of sub-complex of a simplicial complex.

Definition 3.2.5.4. Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set, and $\mathcal{I}$ a regular flag. *The $\mathcal{I}$-th simplicial link of K*, $\mathrm{Link}_{\mathcal{I}}(K)$, is defined via the following pullback diagram:

$$\begin{tikzcd} \mathrm{Link}_{\mathcal{I}}(K) \arrow[r, hook] \arrow[d] & \mathrm{sd}(K) \arrow[d, "\mathrm{sd}(\varphi_K)"] \\ \Delta^0 \arrow[r, "i_{\mathcal{I}}"] & \mathrm{sd}(N(P)), \end{tikzcd}$$

where sd is the usual barycentric subdivision, and $i_{\mathcal{I}}$ is the map sending the point to the unique vertex in $\mathrm{sd}(N(P))$ corresponding to the regular flag $\mathcal{I}$. This construction induces a functor

$$\mathrm{Link}_{\mathcal{I}} : \mathrm{s}\mathbf{Strat}_P \to \mathbf{sSet}.$$

Remark 3.2.5.5. In the case when $K \in \mathrm{s}\mathbf{Strat}_P$ comes from a simplicial complex and $P = \mathcal{I} = \{p_0 < p_1\}$, the simplicial set $\mathrm{Link}_{\mathcal{I}}(K)$ is the simplicial link of the p_0 stratum in K. In other words, it is the complex obtained by taking the boundary of the simplicial complex given by simplices which contain a vertex in the p_0 stratum. As already noted in [Qui88], from this it follows that $|\mathrm{Link}_{\mathcal{I}}(K)|$ and $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$ are weakly equivalent. One of the key result in this paper (Theorem 3.1.0.4), is that this generalizes to arbitrary regular flags and stratified simplicial sets, and to all the notions of homotopy links considered here.

Definition 3.2.5.6. Let $\varphi_K : K \to N(P)$ be a stratified simplicial set, $\mathcal{I}$ a regular flag and $b \in |\Delta^{\mathcal{I}}|$ be the barycenter. Let $|K|_b \subset |K|$ be the subspace given by $|\varphi_K|^{-1}(b)$. In other words, $|K|_b$ is the space defined via the following pullback:

$$\begin{tikzcd} {|K|_b} \arrow[r, hook] \arrow[d] & {|K|} \arrow[d, "|\varphi_K|"] \\ \{b\} \arrow[r, hook] & {|N(P)|}. \end{tikzcd}$$

Remark 3.2.5.7. One can also think of $|K|_b$ as a *geometric link* (see, [Mil68, Theorem 2.10]). Consider the case where $P = \{0 < 1\} = \mathcal{I}$, with the identification $|N(P)| \simeq [0,1]$, and $K \in \mathrm{s}\mathbf{Strat}_P$ is a triangulation of some pseudo-manifold with an isolated singularity. Then, $|\varphi_X|^{-1}_{N(P)}([0,1[)$ is a neighborhood of the singular point, and must be isomorphic to some cone, $c(L)$, for some compact manifold L. The subspace $|K|_b = |\varphi_X|^{-1}_{N(P)}(\frac{1}{2})$ is then homeomorphic to the subspace $L \times \{\frac{1}{2}\}$, or in other words, to the link. See Fig. 3.1 for a side by side comparison between $\mathrm{Link}_{\mathcal{I}}(K)$ and $|K|_b$.

Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set, $\mathcal{I}$ a regular flag and $b \in |\Delta^{\mathcal{I}}|$ the barycenter. We will be interested in the following maps:

$$\begin{tikzcd} {|\mathrm{HoLink}_{\mathcal{I}}(K)|} \arrow[r, "1"] & {Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})} \arrow[r, "2"] \arrow[d, "3"] & {\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)} \\ & {|K|_b} & \\ & {|\mathrm{Link}_{\mathcal{I}}(K)|} \arrow[u, "*"] & \end{tikzcd} \tag{3.3}$$

The map labeled $*$ is a non-natural homeomorphism and will be defined at the begining of Section 3.4, Proposition 3.4.1.1. On the other hand the other maps are both easily defined, and seen to be natural. Using the structure described so far, the first map is equivalently given as a map $\mathrm{Map}(\Delta^{\mathcal{I}}, X) = \mathrm{HoLink}_{\mathcal{I}}(X) \to \mathrm{Sing}(Ho\mathrm{Link}_{\mathcal{I}}(|X|_{N(P)})) = \mathrm{Map}(|\Delta^{\mathcal{I}}|_{N(P)}, |X|_{N(P)})$. It is the map sending a map $\Delta^{\mathcal{I}} \times \Delta^n \to X$ to its realization. The second map just corresponds to the inclusion $\mathcal{C}^0_{N(P)}(|\Delta^{\mathcal{I}}|_{N(P)}, |X|_{N(P)}) \hookrightarrow \mathcal{C}^0_{N(P)}(|\Delta^{\mathcal{I}}|_P, |X|_P)$. Finally, the third map is given by evaluating maps of the form $|\Delta^{\mathcal{I}}|_{N(P)} \to |X|_{N(P)}$ at $b \in |\Delta^{\mathcal{I}}|$. Section 3.4 is devoted to proving that the second and the third map in Diagram (3.3) are weak equivalences, while Section 3.6 handles the first map (or rather, equivalently, it deals with the composition of the first and the second).

3.2.6 Homotopy links and diagrams

Recollection 3.2.6.1. Let K be a stratified simplicial set, and $\mathcal{I} \subset \mathcal{I}'$ two regular flags. The inclusion $\Delta^{\mathcal{I}} \to \Delta^{\mathcal{I}'}$ induces a map $\mathrm{HoLink}_{\mathcal{I}'}(K) \to \mathrm{HoLink}_{\mathcal{I}}(K)$. In fact, the data of the holinks can be synthesized as the following diagram

$$\begin{aligned} \mathrm{sd}(P)^{\mathrm{op}} &\to \mathbf{sSet} \\ \mathcal{I} &\mapsto \mathrm{HoLink}_{\mathcal{I}}(K), \end{aligned}$$

where $\mathrm{sd}(P)$ is the category of regular P-flags (see Definition 3.2.1.5). We write $\mathbf{Diag}_P = \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ for the category of such diagrams. Note that $\mathbf{Diag}_P$ can equivalently be described as the category of presheaves over $\mathrm{sd}(P) \times \Delta$, i.e. $\mathbf{Diag}_P \cong \mathbf{Fun}((\mathrm{sd}(P) \times \Delta)^{\mathrm{op}}, \mathbf{Set})$, though we will mostly consider objects of $\mathbf{Diag}_P$ as diagrams of simplicial sets.

We have a functor

$$\begin{aligned} D_P \colon \mathrm{s}\mathbf{Strat}_P &\to \mathbf{Diag}_P \\ K &\mapsto \big(\mathcal{I} \mapsto \mathrm{HoLink}_{\mathcal{I}}(K)\big) \end{aligned}$$

The functor admits a left adjoint, $C_P \colon \mathbf{Diag}_P \to \mathrm{s}\mathbf{Strat}_P$. Since $\mathbf{Diag}_P$ is a presheaf category over $\mathrm{sd}(P) \times \Delta$, it is enough to define C_P on pairs $(\mathcal{I}, n) \in \mathrm{sd}(P) \times \Delta$ and then take the left Kan extension along the Yoneda embedding $\mathrm{sd}(P) \times \Delta \hookrightarrow \mathbf{Diag}_P$. On $\mathrm{sd}(P) \times \Delta$, one defines $C_P(\mathcal{I}, n) = \Delta^{\mathcal{I}} \times \Delta^n$. A more explicit definition can be found in [Dou21c, Definition 2.5]. Similarily, there are functors $D_P^{\mathbf{Top}} \colon \mathbf{Strat}_P \to \mathbf{Diag}_P$, as well as $D_P^{\mathbf{Top}_{N(P)}} \colon \mathbf{Top}_{N(P)} \to \mathbf{Diag}_P$ defined as follows

$$\begin{aligned} D_P^{\mathbf{Top}} \colon \mathbf{Strat}_P &\to \mathbf{Diag}_P \\ Y &\mapsto \big(\mathcal{I} \mapsto \mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y))\big) \\ D_P^{\mathbf{Top}_{N(P)}} \colon \mathbf{Top}_{N(P)} &\to \mathbf{Diag}_P \\ X &\mapsto \big(\mathcal{I} \mapsto \mathrm{Sing}(Ho\mathrm{Link}_{\mathcal{I}}(X))\big) \end{aligned}$$

$D_P^{\mathbf{Top}}$ and $D_P^{\mathbf{Top}_{N(P)}}$ both admit left adjoints, $C_P^{\mathbf{Top}} \colon \mathbf{Diag}_P \to \mathbf{Strat}_P$, and $C_P^{\mathbf{Top}_{N(P)}} \colon \mathbf{Diag}_P \to \mathbf{Top}_{N(P)}$. Note that the six functors defined above satisfy several relations, which can be summed up by the following diagram of adjunctions:

$$\begin{tikzcd}
& \mathrm{s}\mathbf{Strat}_P \arrow[ddl, shift left, "D_P"] \arrow[dr, shift left, "|-|_{N(P)}"] \arrow[ddrr, bend left, shift left, "|-|_P"] & & \\
& & \mathbf{Top}_{\mathrm{N}(P)} \arrow[ul, shift left, "\mathrm{Sing}_{N(P)}"] \arrow[dll, shift left] \arrow[dr, shift left, "\varphi_P \circ -"] & \\
\mathbf{Diag}_P \arrow[uur, shift left, "C_P"] \arrow[urr, shift left] \arrow[rrr, shift left, "C_P^{\mathbf{Top}}"] & & & \mathbf{Strat}_P \arrow[lll, shift left, "D_P^{\mathbf{Top}}"] \arrow[ul, shift left, "- \times_P |N(P)|"] \arrow[uull, bend right, shift left, "\mathrm{Sing}_P"']
\end{tikzcd} \tag{3.4}$$

In the above diagram, all pairs of parallel maps are adjoint pairs. Furthermore, if one restricts to left adjoints, then one gets a commutative diagram (up to natural isomorphisms), and similarly for the right adjoints. Given the compatibility between all those functors, we will mostly omit the superscript $\mathbf{Top}$ and $\mathbf{Top}_{N(P)}$.

The values of the functors D_P on stratified objects compile all the information about their (generalized) homotopy links. This suggests that diagrams form a reasonable basis to build stratified homotopy types from.

3.2.7 The model category of diagrams

Recall that any category of functors into a model category admits a projective model structure. Note that throughout the paper, we consider the category of simplicial sets with the Kan-Quillen model structure.

Theorem 3.2.7.1 ([Hir03, Theorem 11.6.1]). *There exists a projective model structure on* $\mathbf{Diag}_P = \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$*, such that:*

- *fibrations are maps $f\colon F \to G$, such that $f(\mathcal{I})\colon F(\mathcal{I}) \to G(\mathcal{I})$ is a Kan fibration for all $\mathcal{I} \in \mathrm{sd}(P)$,*
- *weak equivalences are maps $f\colon F \to G$, such that $f(\mathcal{I})\colon F(\mathcal{I}) \to G(\mathcal{I})$ is a weak equivalence of simplicial sets, for all $\mathcal{I} \in \mathrm{sd}(P)$.*

This model structure is cofibrantly generated, and we need the following definition to make explicit the sets of generating (trivial) cofibrations.

Definition 3.2.7.2. Let S be a simplicial set and $\mathcal{I}$ a regular flag. Let $S^{\mathcal{I}} \in \mathbf{Diag}_P$ be the diagram defined as follows:

$$S^{\mathcal{I}}(\mathcal{I}') = \begin{cases} S & \text{if } \mathcal{I}' \subset \mathcal{I}, \\ \varnothing & \text{else.} \end{cases}$$

Proposition 3.2.7.3. *The set of generating cofibrations for the model structure of Theorem 3.2.7.1 is given by:*

$$\{(\partial\Delta^n)^{\mathcal{I}} \to (\Delta^n)^{\mathcal{I}} \mid n \geq 0,\ \mathcal{I} \in \mathrm{sd}(P)\}.$$

The set of generating trivial cofibrations is given by:

$$\{(\Lambda^n_k)^{\mathcal{I}} \to (\Delta^n)^{\mathcal{I}} \mid n \geq 1,\ 0 \leq k \leq n,\ \mathcal{I} \in \mathrm{sd}(P)\}.$$

Where Λ^n_k is the k-th horn on Δ^n, i.e. $\Lambda^n_k = \cup_{i\neq k} d_i(\Delta^n) \subset \Delta^n$.

It is difficult to give an explicit description of the cofibrant objects in a projective model structure in general, but in the case of $\mathbf{Diag}_P$ there exists a simple characterization. Since cofibrant diagrams are related to the vertical objects we will use in Section 3.6.2 (see Remark 3.6.2.8), we give such a description here.

Proposition 3.2.7.4. *Assume that P is of finite length, and let $F \in \mathbf{Diag}_P$ be a diagram. Then F is cofibrant if and only if the following two conditions are satisfied:*

1. *For all regular flags $\mathcal{I} \subset \mathcal{I}'$, the map $F(\mathcal{I}') \to F(\mathcal{I})$ is a monomorphism,*
2. *If $\mathcal{I}_1, \mathcal{I}_2$ are two regular flags such that $\mathcal{I}_0 = \mathcal{I}_1 \cap \mathcal{I}_2 \neq \varnothing$, then either*
   - *$\mathcal{I}_3 = \mathcal{I}_1 \cup \mathcal{I}_2$ is a flag, and $F(\mathcal{I}_1) \cap F(\mathcal{I}_2) = F(\mathcal{I}_3) \subset F(\mathcal{I}_0)$,*
   - *or $\mathcal{I}_1 \cup \mathcal{I}_2$ is not a flag, and $F(\mathcal{I}_1) \cap F(\mathcal{I}_2) = \varnothing$.*

Proof. We give a brief sketch of the proof. For the direct implication, note that a projectively cofibrant diagram must satisfy Condition (1) whenever the indexing category is a poset. For Condition (2), one easily checks that it is satisfied by the domains and codomains of generating cofibrations. Then, it is a matter of checking that this property is preserved by taking disjoint unions, pushouts and retracts.

For the reverse implication, consider the following characterization of projectively cofibrant diagrams of simplicial sets [Dug01, Corollary 9.4]. A diagram $F\colon \mathrm{sd}(P)^{\mathrm{op}} \to \mathbf{sSet}$ is projectively cofibrant if and only if the following two conditions are satisfied:

(a) For all $n \geq 0$, the functor of n-simplicies $F_n\colon \mathrm{sd}(P)^{\mathrm{op}} \to \mathbf{Set}$ splits into two sub-functors $F_n = F_n^{\text{degen}} \coprod F_n^{\text{non-degen}}$ where the former reaches exactly the degenerate n-simplices of $F(\mathcal{I})$, for all $\mathcal{I}$.

(b) For all $n \geq 0$, F_n can be further decomposed into a coproduct of representables.

We will first prove that (1) $\Rightarrow$ (a). Let $\sigma \in F(\mathcal{I})_n$ for some $\mathcal{I} \in \mathrm{sd}(P)$. Define the set $F_\sigma \subset \coprod_{\mathcal{I}} F(\mathcal{I})_n$ as the smallest subset such that

- $\sigma \in F_\sigma$,

- $\forall\ \mathcal{I},\mathcal{I}' \in \mathrm{sd}(P)$, if $\tau \in F(\mathcal{I}) \cap F_\sigma$ and $\mathcal{I}' \subset \mathcal{I}$, then the image of τ in $F(\mathcal{I}')$ is in F_σ
- $\forall\ \mathcal{I},\mathcal{I}' \in \mathrm{sd}(P)$, if $\tau \in F(\mathcal{I})$ and $\mathcal{I}' \subset \mathcal{I}$, and the image of τ in $F(\mathcal{I}')$ is in F_σ, then τ is in F_σ.

First note that by construction, if $\sigma \in F(\mathcal{I})$ and $\tau \in F(\mathcal{I}')$, then either $F_\sigma \cap F_\tau = \varnothing$ or $F_\sigma = F_\tau$. Next, by (1), if σ is non-degenerate, then so are all the simplices in F_σ, since cofibrations send non-degenerate simplices to non-degenerate simplices, and similarly, if σ is degenerate all simplices in F_σ are degenerate. Now, note that F_σ defines a subfunctor of F_n by setting $F_\sigma(\mathcal{I}) = F_\sigma \cap F(\mathcal{I})$. To prove that $(1)+(2) \Rightarrow (b)$, it is enough to show that the F_σ defined above are representable. Let $Q_\sigma = \{(\tau,\mathcal{I}) \mid \tau \in F_\sigma(\mathcal{I})\}$, and equip it with the order relation $(\tau_1,\mathcal{I}_1) < (\tau_2,\mathcal{I}_2)$ if $\mathcal{I}_1 \subset \mathcal{I}_2$ and the image of τ_2 in $F(\mathcal{I}_1)$ is τ_1. Then, by construction of F_σ, the poset Q_σ is connected. In particular, if $(\tau,\mathcal{I})$ and $(\tau',\mathcal{I})$ are two elements of Q_σ, then they must be related by a zigzag. Note that by (2), any zigzag of the form $(\tau_1,\mathcal{I}_1) > (\tau_0,\mathcal{I}_0) < (\tau_2,\mathcal{I}_2)$ can be replaced by a zigzag $(\tau_1,\mathcal{I}_1) < (\tau_3,\mathcal{I}_3) > (\tau_2,\mathcal{I}_2)$. Then, by repeated application of (2), one can assume that there exists $(\tau'',\mathcal{I}')$, such that, $(\tau,\mathcal{I}) < (\tau'',\mathcal{I}') > (\tau',\mathcal{I})$, which implies that $\tau = \tau'$. This has two consequences. First, Q_σ is a subposet of $\mathrm{sd}(P)^{\mathrm{op}}$, and second, $F_\sigma(\mathcal{I})$ is either empty or is a point. To show that F_σ is reprensentable, it remains to be showed that Q_σ admits a maximal element. Once again, by repeated application of (2), any two elements $(\tau_1,\mathcal{I}_1),(\tau_2,\mathcal{I}_2)$ must admit an upper bound $(\tau_3,\mathcal{I}_3) \geq (\tau_1,\mathcal{I}_1),(\tau_2,\mathcal{I}_2)$, but now, $\mathcal{I}_3$ is a regular flag of length greater or equal than the length of $\mathcal{I}_1$ and $\mathcal{I}_2$, with possible equality only if $\mathcal{I}_1 = \mathcal{I}_2$. Since P is assumed to be of finite length, the length of regular flags is bounded, and thus Q_σ admits a maximal element, $\mathcal{I}_\sigma$. The functor $F_\sigma\colon \mathrm{sd}(P)^{\mathrm{op}} \to \mathbf{Set}$ is then represented by $\mathcal{I}_\sigma$. □

3.2.8 Model categories for stratified spaces

We can now describe the model category of stratified spaces as introduced in [Dou21c].

Theorem 3.2.8.1 ([Dou21c, Theorem 2.15])**.** *There exists a model structure on* $\mathbf{Strat}_P$*, where a map* $f\colon X \to Y$ *is*

- *a fibration if the induced maps* $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y)$ *are fibrations for all* $\mathcal{I} \in \mathrm{sd}(P)$*,*
- *a weak equivalence if the induced maps* $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y)$ *are weak equivalences for all* $\mathcal{I} \in \mathrm{sd}(P)$*.*

This model structure is cofibrantly generated and the set of generating cofibrations is given by

$$\{|\Delta^{\mathcal{I}}|_P \times S^{n-1} \hookrightarrow |\Delta^{\mathcal{I}}|_P \times B^n \mid n \geq 0,\ \mathcal{I} \in \mathrm{sd}(P)\},$$

while the set of generating trivial cofibrations is given by

$$\{|\Delta^{\mathcal{I}}|_P \times B^n \times \{0\} \hookrightarrow |\Delta^{\mathcal{I}}|_P \times B^n \times [0,1] \mid n \geq 0,\ \mathcal{I} \in \mathrm{sd}(P)\},$$

where B^n *and* S^{n-1} *are respectively the* n*-dimensional ball and the* $(n-1)$*-dimensional sphere bounding it.*

Remark 3.2.8.2. The model category $\mathbf{Strat}_P$ has the notable property that all of its objects are fibrant, which is a property also enjoyed by the classical model structure on **Top**. To see this, note that for all $\mathcal{I} \in \mathrm{sd}(P)$, $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(P) = *$, in particular, for all $X \in \mathbf{Strat}_P$, the map $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(P) = *$ is a fibration in **Top**, which implies that the map $X \to P$ is a fibration in $\mathbf{Strat}_P$.

Example 3.2.8.3. For arbitrary stratified spaces in $\mathbf{Strat}_P$ it is not the case that the forgetful functor $\mathbf{Strat}_P \to \mathbf{Top}$ preserves weak equivalences. Indeed, let E be the Hawaiian Earring (see for ex. [EK00]) and $f : E' = \bigvee_{n\in\mathbb{N}} S^1 \to E$ the bijection obtained by decomposing H

into its circles, and gluing these along the basepoint. f is far from being a weak equivalence of topological spaces. Indeed, [EK00, Thm. 3.1] shows that it does not even induce an isomorphism on singular homology in degree 1. Nevertheless, if we consider E and E' as stratified over $P = \{0 < 1\}$, by sending the singular point to 0, then f induces a weak equivalence of stratified spaces. Indeed, it even induces a homeomorphism on strata and homotopy links. This should not be excessively surprising. The forgetful functor $\mathbf{Strat}_P \to \mathbf{Top}$ is left Quillen, which means it is only expected to preserve weak equivalences between **cofibrant** objects. However, not every object in $\mathbf{Strat}_P$, in particular not the Hawaiian Earring, is cofibrant.

Remark 3.2.8.4. Despite Example 3.2.8.3, it holds true that the forgetful functor $\mathbf{Strat}_P \to \mathbf{Top}$ preserves weak equivalences for a much larger class than that of cofibrant objects. In fact, it preserves weak equivalences between realizations of stratified simplicial sets. This is a consequence of Theorem 3.5.1.1, together with the fact that $\mathrm{s}\mathbf{Strat}_P \to \mathbf{sSet}$ preserves weak equivalences, by a direct application of Ken Brown's Lemma ([Hir03, Cor. 7.7.2].

Remark 3.2.8.5. The model structure defined above was originally obtained by transporting the model structure from $\mathbf{Diag}_P$ along the adjunction $C_P\colon \mathbf{Diag}_P \leftrightarrow \mathbf{Strat}_P\colon D_P$ (see Recollection 3.2.6.1). This uses the transport theorem [Hes+17, Corollary 3.3.4], and this is where the necessity to restrict to Δ-generated spaces arose, since the theorem requires $\mathbf{Strat}_P$ to be locally presentable (see Remark 3.2.2.1). Using the alternative transfer theorem [Ste13, A.1] instead, one can prove the existence of an identically defined model structure on the category of P-stratified spaces defined using *all* topological spaces. Denote the larger model category constructed in this fashion by $\mathcal{T}_P$. Then $\mathcal{T}_P$ has identically defined acyclic cofibrations and fibrations as $\mathbf{Strat}_P$. In particular, the adjunction

$$\mathcal{T}_P \leftrightarrow \mathbf{Strat}_P$$

given by inclusion and Δ-generation is a Quillen adjunction. It follows immediately from the universal property of Δ-generation and the definition of homotopy links that both unit and counit of this adjunction are weak equivalences. In particular, the two model categories are Quillen-equivalent. Hence, from the perspective of homotopy theory, the restriction to Δ-generated spaces can be neglected. At the same time, $\mathbf{Strat}_P$ enjoys the additional benefit of being a combinatorial model category enriched over $\mathbf{Top}$, (and over $\mathbf{sSet}$, through the functor Sing) making it generally preferable to work with. Even more, one easily checks that with its simplicial structure described in Recollection 3.2.2.4, $\mathbf{Strat}_P$ is a simplicial model category.

There is a similar model structure for $\mathbf{Top}_{\mathrm{N}(P)}$, also obtained by transporting along the adjunction with $\mathbf{Diag}_P$.

Theorem 3.2.8.6 ([Dou21c, Theorem 2.8])**.** *There exists a model structure on* $\mathbf{Top}_{\mathrm{N}(P)}$*, where a map* $f\colon X \to Y$ *is*

- *a fibration if the induced maps* $Ho\mathrm{Link}_{\mathcal{I}}(X) \to Ho\mathrm{Link}_{\mathcal{I}}(Y)$ *are fibrations for all* $\mathcal{I} \in \mathrm{sd}(P)$,
- *a weak equivalence if the induced maps* $Ho\mathrm{Link}_{\mathcal{I}}(X) \to Ho\mathrm{Link}_{\mathcal{I}}(Y)$ *are weak equivalences for all* $\mathcal{I} \in \mathrm{sd}(P)$.

This model structure is cofibrantly generated and the set of generating cofibrations is given by

$$\{|\Delta^{\mathcal{I}}|_{N(P)} \times S^{n-1} \hookrightarrow |\Delta^{\mathcal{I}}|_{N(P)} \times B^n \mid n \geq 0,\ \mathcal{I} \in \mathrm{sd}(P)\},$$

while the set of generating trivial cofibrations is given by

$$\{|\Delta^{\mathcal{I}}|_{N(P)} \times B^n \times \{0\} \hookrightarrow |\Delta^{\mathcal{I}}|_{N(P)} \times B^n \times [0,1] \mid n \geq 0,\ \mathcal{I} \in \mathrm{sd}(P)\},$$

where B^n *and* S^{n-1} *are respectively the* n*-dimensional ball and the* $(n-1)$*-dimensional sphere bounding it.*

The model categories from Theorems 3.2.8.1 and 3.2.8.6 are Quillen-equivalent, through the adjunction $\varphi_P \circ - \dashv - \times_P |N(P)|$ described in Recollection 3.2.2.9, see [Dou21c, Theorem 2.15] together with the correction in [Dou21b, Theorem 3.15] for a proof.

Theorem 3.2.8.7. *The adjunction $\varphi_P \circ -: \mathbf{Top}_{\mathrm{N}(P)} \leftrightarrow \mathbf{Strat}_P: - \times_P |N(P)|$ is a Quillen equivalence.*

The model categories $\mathbf{Top}_{\mathrm{N}(P)}$ and $\mathbf{Strat}_P$ are also Quillen-equivalent to $\mathbf{Diag}_P$.

Theorem 3.2.8.8 ([Dou21c, Theorem 2.12])**.** *The adjunctions $C_P: \mathbf{Diag}_P \leftrightarrow \mathbf{Top}_{\mathrm{N}(P)}: D_P$ and $C_P: \mathbf{Diag}_P \leftrightarrow \mathbf{Strat}_P: D_P$ are Quillen equivalences.*

Leveraging the structure described in Remark 3.2.2.3, one can construct a model structure on **Strat** through the theory of Quillen-bifibrations, see [CM20]. Note that any isomorphism of posets $\alpha: P \to Q$ induces an equivalence of categories $\mathbf{Diag}_Q \to \mathbf{Diag}_P$. In particular, given a map in **Strat**

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_X"'] & Y \arrow[d, "\varphi_Y"] \\ P \arrow[r, "\bar{f}"] & Q, \end{tikzcd} \tag{3.5}$$

where $\bar{f}$ is an isomorphism, one can consider the diagram associated to Y as a diagram in $\mathbf{Diag}_P$, and consider f as inducing a map in $\mathbf{Diag}_P$. From this, it follows that a weak equivalence induces a map of homotopy links. The original version of the following result in [Dou21c] and [DW22] is lacking two cofibrant generators. We have added them below. That these are the correct generators follows from Lemma 5.3.1.9.

Theorem 3.2.8.9 ([Dou21c, Theorem 3.6])**.** *There exists a model structure on* **Strat** *where a map $f: (X \to P) \to (Y \to Q)$ is a weak equivalence if and only if $\bar{f}: P \to Q$ is an isomorphism, and the maps induced by f*

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \to \mathcal{H}o\mathrm{Link}_{\bar{f}(\mathcal{I})}(Y),$$

are weak equivalences for all $\mathcal{I} \in \mathrm{sd}(P)$.
This model category is cofibrantly generated with the set of generating cofibrations given by

$$\{|\Delta^{\mathcal{I}}|_{\mathbb{N}} \times S^{n-1} \hookrightarrow |\Delta^{\mathcal{I}}|_{\mathbb{N}} \times B^n \mid n \geq 0,\ \mathcal{I} \in \Delta_{\mathbb{N}}\},$$

together with the following two maps of stratified spaces with empty underlying space:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ \varnothing \arrow[r] & {[0]}, \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ {[0] \sqcup [0]} \arrow[r, hook] & {[1]}. \end{tikzcd} \tag{3.6}$$

The set of generating trivial cofibrations is given by

$$\{|\Delta^{\mathcal{I}}|_{\mathbb{N}} \times B^n \times \{0\} \hookrightarrow |\Delta^{\mathcal{I}}|_{\mathbb{N}} \times B^n \times [0,1] \mid n \geq 0,\ \mathcal{I} \in \Delta_{\mathbb{N}}\}.$$

3.2.9 The model category of stratified simplicial sets

Before defining a model structure for stratified simplicial sets, we need to define a particular set of stratified horn inclusions.

Definition 3.2.9.1. A *stratified horn inclusion* is an inclusion $\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}} \in \mathrm{s}\mathbf{Strat}_P$, where $\mathcal{J}$ is a flag and $\Lambda^{\mathcal{J}}_k = \cup_{i \neq k} d_i(\Delta^{\mathcal{J}})$, i.e. it is the union of all proper faces of $\Delta^{\mathcal{J}}$ other than its k-th face. An *admissible horn inclusion* is a stratified horn inclusion $\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}}$ such that any of the following equivalent conditions is satisfied:

- The map $\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}}$ is a stratified homotopy equivalence (see Definition 3.2.3.4).
- There exists a flag $\mathcal{J}'$ such that $\Delta^{\mathcal{J}}$ is either the k-th or $(k+1)$-th degeneracy of $\Delta^{\mathcal{J}'}$.
- $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ and $p_k = p_{k+1}$ or $p_k = p_{k-1}$.

For a proof that the above conditions are equivalent, see [Dou21a, Proposition 1.13].

Theorem 3.2.9.2 ([Dou21a, Theorem 2.14])**.** *There exists a cofibrantly generated model structure on* s**Strat**$_P$ *where the generating set of cofibrations is given by*

$$\{\partial\Delta^{\mathcal{J}} \to \Delta^{\mathcal{J}} \mid \Delta^{\mathcal{J}} \in \Delta_P\},$$

and the generating set of trivial cofibrations is given by

$$\{\iota^{\mathcal{J}}_k \colon \Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}} \mid \iota^{\mathcal{J}}_k \textit{ is an admissible horn inclusion}\}.$$

Weak equivalences in this category are the maps $f\colon K \to L$ such that for all $\mathcal{I} \in \mathrm{sd}(P)$, the induced maps

$$\mathrm{HoLink}_{\mathcal{I}}(K^{\mathrm{Fib}}) \to \mathrm{HoLink}_{\mathcal{I}}(L^{\mathrm{Fib}}) \tag{3.7}$$

are weak equivalences, where $(-)^{\mathrm{Fib}}$ means passing to a fibrant replacement.

Remark 3.2.9.3. As is usual for a model category of presheaves, all objects of s**Strat**$_P$ are cofibrant. In fact, the class of cofibrations is exactly the class of monomorphisms in s**Strat**$_P$. Indeed, the set of generating cofibrations is the set of boundary inclusions of (stratified) simplices, which can easily be seen to generate all monomorphisms.

Remark 3.2.9.4. Note that stratified homotopy equivalences in the sense of Definition 3.2.3.4 are weak equivalences in the model category described above. Furthermore, this model structure is minimal among model structures satisfying this property and having monomorphisms as cofibrations, i.e. it has the smallest possible class of weak equivalences. This comes from the fact that the model structure of Theorem 3.2.9.2 is built using Cisinski's theory [Cis06]. By [Cis06, Théorème 1.3.22], such a model category is specified by the data of a cylinder (Definition 3.2.3.4) and a class of anodyne extension. In the case of s**Strat**$_P$, since admissible horn inclusions are stratified homotopy equivalences, the class of anodyne extensions is the one generated by the empty set, see [Cis06, 1.3.12, Proposition 1.3.13]. This implies in particular that if a model structure on s**Strat**$_P$ is such that

- the cofibrations are monomorphisms,
- stratified homotopy equivalences are weak equivalences,

then it is a right Bousfield localization of the structure of Theorem 3.2.9.2. One particular example of interest is the model structure s**Strat**$_P^{\text{Joyal-Kan}}$ defined by Haine in [Hai23]. Haine starts from the Joyal model structure, and then localizes the structure along the inclusions of stratified simplicial sets into their cylinders. In particular, s**Strat**$_P^{\text{Joyal-Kan}}$ is a right Bousfield localization of s**Strat**$_P$.

Remark 3.2.9.5. Note that, contrary to the case of stratified spaces (Theorem 3.2.8.1), the weak equivalences for stratified simplicial sets are not *defined* as the maps inducing weak equivalences between homotopy links, but only as those maps satisfying this property *after a suitable fibrant replacement.* In particular, the model structure on s**Strat**$_P$ is not transported from the category of diagrams **Diag**$_P$. On the other hand, consider the model structure described by Henriques in [Hen], in which weak equivalences are defined as those maps inducing weak equivalences between all simplicial homotopy links. The two model structures are known to be the same for very general reasons (for example, consider the previous remark), but it is not immediately clear why the classes of weak equivalences coincide. We investigate this in Section 3.6, see also Remark 3.6.0.3.

Similarly to the case of stratified spaces, there is a model structure on all stratified simplicial sets. The original version of the following result in [DW22] is lacking two cofibrant generators. We have added them below. That these are the correct generators follows from Lemma 5.3.1.9. See Corollary 5.3.1.10.

Theorem 3.2.9.6. *There exists a model structure on* $s\mathbf{Strat}$ *where a map* $f\colon (K \to N(P)) \to (L \to N(Q))$ *is a weak equivalence if and only if* $\bar{f}\colon P \to Q$ *is an isomorphism and the maps induced by* f

$$\mathrm{HoLink}_{\mathcal{I}}(K^{\mathrm{Fib}}) \to \mathrm{HoLink}_{\bar{f}(\mathcal{I})}(L^{\mathrm{Fib}})$$

are weak equivalences for all $\mathcal{I} \in \mathrm{sd}(P)$*, where* $(-)^{\mathrm{Fib}}$ *is a fibrant replacement in* $s\mathbf{Strat}_P$*.*

The model category is cofibrantly generated, and the set of generating cofibrations is given by

$$\{\partial\Delta^{\mathcal{I}} \hookrightarrow \Delta^{\mathcal{I}} \mid \mathcal{I} \in \mathrm{sd}(\mathbb{N})\}$$

together with the following two maps of stratified simplicial sets with empty underlying simplicial set:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ \varnothing \arrow[r] & {[0]}, \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ {[0] \sqcup [0]} \arrow[r, hook] & {[1]}. \end{tikzcd} \tag{3.8}$$

The set of generating trivial cofibrations is given by

$$\{\iota_k^{\mathcal{I}}\colon \Lambda_k^{\mathcal{I}} \hookrightarrow \Delta^{\mathcal{I}} \mid \iota_k^{\mathcal{I}} \textit{ is admissible in } s\mathbf{Strat}_{\mathbb{N}}\}.$$

3.2.10 Failure to be a Quillen equivalence

Now that we have two homotopy theories, one for stratified simplicial sets, and one for stratified spaces, we want to show that they coincide through the adjunction $|-|_s\colon s\mathbf{Strat} \leftrightarrow \mathbf{Strat}\colon \mathrm{Sing}_s$. In the language of model categories, this would mean showing that the adjunction is a Quillen equivalence. However, this adjunction already fails to be a Quillen adjunction, since the right adjoint Sing_s does not preserve fibrancy, as we show in the following recollection.

Recollection 3.2.10.1. Let $X \to P$ be a stratified space. For X to be fibrant, one has to check that for all regular flags $\mathcal{I} \in \mathrm{sd}(P)$, $D_P(X)(\mathcal{I})$ is a Kan-complex. But note that by definition, those are equal to $\mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X))$, and so, are Kan-complexes, which means that any stratified space is fibrant as already observed in Remark 3.2.8.2. On the other hand, for $\mathrm{Sing}_P(X)$ to be fibrant means that in any diagram of the form

$$\begin{tikzcd} \Lambda_k^{\mathcal{J}} \arrow[r] \arrow[d] & \mathrm{Sing}_P(X) \arrow[d] \\ \Delta^{\mathcal{J}} \arrow[r] \arrow[ur, dashed] & N(P), \end{tikzcd}$$

where $\Lambda_k^{\mathcal{J}}$ is admissible, there exists a lift. Using the adjunction $|-|_P \dashv \mathrm{Sing}_P$, and the fact that $\mathrm{Sing}_P(P) = N(P)$, this is equivalent to asking for a lift in diagrams of the form

$$\begin{tikzcd} {|\Lambda_k^{\mathcal{J}}|_P} \arrow[r] \arrow[d] & X \arrow[d] \\ {|\Delta^{\mathcal{J}}|_P} \arrow[r] \arrow[ur, dashed] & P. \end{tikzcd}$$

Now consider the poset $P = \{0 < 1\}$, and the stratified space $X = |\Lambda_1^{\mathcal{J}}|_P$ with $\mathcal{J} = [0 \le 0 \le 1]$. Then, $\Lambda_1^{\mathcal{J}}$ is admissible, and so for $\mathrm{Sing}_P(X)$ to be fibrant, there must be a lift in the following diagram

$$\begin{tikzcd} {|\Lambda^{\mathcal{J}}_1|_P} \arrow[r, "1"] \arrow[d] & {|\Lambda^{\mathcal{J}}_1|_P} \arrow[d] \\ {|\Delta^{\mathcal{J}}|_P} \arrow[r] \arrow[ur, dashed] & P, \end{tikzcd}$$

but there exist no stratified section $|\Delta^{\mathcal{J}}|_P \to |\Lambda^{\mathcal{J}}_1|_P$, as illustrated in Fig. 3.4, which means that $\mathrm{Sing}_P(|\Lambda^{\mathcal{J}}_1|_P)$ is not fibrant. In particular, the functor Sing_P does not preserve fibrations.

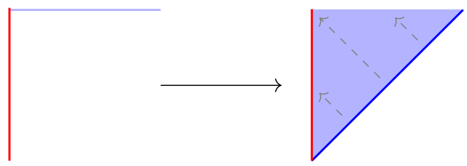


Figure 3.4: The inclusion $|\Lambda^{\mathcal{J}}_1|_P \hookrightarrow |\Delta^{\mathcal{J}}|_P$, with $\mathcal{J} = [0 \le 0 \le 1]$, admits no stratified retraction, since any retraction would send some part of the interior of $|\Delta^2|$ to the singular stratum.

On the other hand, stratified spaces that are in one of the three classes mentioned in Section 3.2.4, pseudo-manifolds, conically stratified spaces and homotopically stratified sets, have better behaved Sing_P.

Theorem 3.2.10.2 ([Lur17, Theorem A.6.4][Nan19, Proposition 8.1.2.6])**.** *Let $X \to P$ be a stratified space. If it is either a conically stratified space or if P is finite and X is a homotopically stratified set, then the simplicial set underlying $\mathrm{Sing}_P(X)$ is a quasi-category.*

In those cases, this implies that $\mathrm{Sing}_P(X)$ is a fibrant object in $\mathrm{s}\mathbf{Strat}_P$, see [Dou21a, Proposition 4.12]. Note that while Recollection 3.2.10.1 implies that the adjunction $|-|_P \dashv \mathrm{Sing}_P$ is not a Quillen adjunction, we can still use it to deduce results about objects for which Sing_P *is* fibrant, such as those mentioned above. In fact, it is possible to recover an independent proof of Miller's Theorem (Theorem 3.2.4.5), this way (see [Dou21a, Section 5]). Relatedly, Theorem 3.1.0.2 (Corollary 3.5.2.4) states that those objects do behave like fibrant objects in the homotopy category $\mathrm{ho}\mathbf{Strat}$.

3.2.11 The stratified subdivision and its adjoint

As we have seen in the previous section, the model structure on $\mathbf{Strat}_P$ and $\mathrm{s}\mathbf{Strat}_P$ cannot be compared directly through the adjunction $|-|_P \dashv \mathrm{Sing}_P$. To achieve a comparison, one needs to work with a suitably defined stratified subdivision. Note that this stratified subdivision was already used to characterize the model structure of Theorem 3.2.9.2 in [Dou21a], and that, through its adjoint, it allows for the definition of a fibrant replacement functor in $\mathrm{s}\mathbf{Strat}_P$, see Section 3.3.1 and Corollary 3.3.1.2.

Remark 3.2.11.1. In this subsection and through the remainder of this paper, we will often abuse notation by identifying an n-dimensional stratified simplex $\Delta^{\mathcal{J}}$ with its underlying simplex Δ^n. This allows for notation such as $\Delta^{\mathcal{J}}_0$ to refer to the set of vertices of Δ^n.

Recollection 3.2.11.2. Denote by $\mathrm{sd}\colon \mathbf{sSet} \to \mathbf{sSet}$ the classical barycentric subdivision [Kan57, Section 2]. Given a stratified simplex $\Delta^{\mathcal{J}}$, $\mathcal{J} = [p_0 \le \cdots \le p_n]$ its stratified subdivision is the subcomplex $\mathrm{sd}_P(\Delta^{\mathcal{J}}) \subset \mathrm{sd}(\Delta^n) \times \Delta^{\mathcal{J}}$, given by such simplices $[(\sigma_0, q_0), \dots, (\sigma_k, q_k)]$ fulfilling $\{q_0, \dots, q_k\} \subset \{p_i \mid i \in \sigma_0\}$. It is stratified via projection to the second component. This construction left Kan extends to a functor

$$\mathrm{sd}_P\colon \mathrm{s}\mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P.$$

For a stratified simplicial set $K \in \mathrm{s}\mathbf{Strat}_P$, we can also describe the stratified simplicial set $\mathrm{sd}_P(K)$ more explicitly. Note that we have an inclusion $\mathrm{sd}_P(N(P)) \subset \mathrm{sd}(N(P)) \times N(P)$, composing it with the projection on the first factor gives a map $\mathrm{sd}_P(N(P)) \to \mathrm{sd}(N(P))$. The stratified simplicial set $\mathrm{sd}_P(K)$ can then be seen as the following pullback:

$$\begin{tikzcd}
\mathrm{sd}_P(K) \arrow[r] \arrow[d] & \mathrm{sd}(K) \arrow[d, "\mathrm{sd}(\varphi_K)"] \\
\mathrm{sd}_P(N(P)) \arrow[r] & \mathrm{sd}(N(P))\,,
\end{tikzcd}$$

and the stratification is given by the composition $\mathrm{sd}_P(K) \to \mathrm{sd}_P(N(P)) \to N(P)$.

The functor sd_P naturally comes with a natural transformation, $\mathrm{l.v.}_P\colon \mathrm{sd}_P \to 1_{\mathrm{s}\mathbf{Strat}_P}$, the *stratified last vertex map*. We define it on stratified simplices by giving its value on vertices:

$$\begin{aligned}
\mathrm{l.v.}_P\colon \mathrm{sd}_P(\Delta^{\mathcal{J}})_0 &\to \Delta^{\mathcal{J}}_0 \\
([q_0 \leq \dots \leq q_k], q) &\mapsto \max\{i \mid q_i = q\}.
\end{aligned}$$

Then, since the above is a map between simplicial complexes, its value on vertices uniquely defines a simplicial map $\mathrm{l.v.}_P\colon \mathrm{sd}_P(\Delta^{\mathcal{J}}) \to \Delta^{\mathcal{J}}$.

The functor sd_P admits a right adjoint, defined as

$$\begin{aligned}
\mathrm{Ex}_P : \mathrm{s}\mathbf{Strat}_P &\to \mathrm{s}\mathbf{Strat}_P \\
K &\mapsto \mathrm{s}\mathbf{Strat}_P(\mathrm{sd}_P(\Delta^{-}), K)
\end{aligned}$$

where again, by Remark 3.2.3.2, we consider $\mathrm{s}\mathbf{Strat}_P$ as a category of presheaves over Δ_P. The adjoints of the maps $\mathrm{l.v.}_P\colon \mathrm{sd}_P(K) \to K$ give natural maps $\iota_K\colon K \hookrightarrow \mathrm{Ex}_P(K)$, which assemble into a natural transformation $\iota\colon 1_{\mathrm{s}\mathbf{Strat}_P} \to \mathrm{Ex}_P$. Finally, we denote by sd_P^n, Ex_P^n, $\mathrm{l.v.}_P^n, \iota^n$ the obvious functors obtained by iteration and by Ex_P^∞ the colimit of the diagram

$$1_{\mathrm{s}\mathbf{Strat}_P} \hookrightarrow \mathrm{Ex}_P \hookrightarrow Ex_P^2 \hookrightarrow \dots.$$

Given $K \in \mathrm{s}\mathbf{Strat}_P$ the stratified simplicial set $\mathrm{Ex}_P^\infty(K)$ is fibrant [Dou21a, Lemma 2.10]. It is the content of Corollary 3.3.1.2 that $X \hookrightarrow \mathrm{Ex}_P^\infty(X)$ in fact defines a fibrant replacement of K.

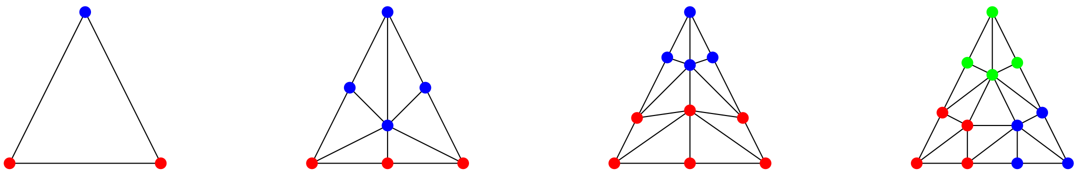


Figure 3.5: From left to right, the simplex $\Delta^{[0\leq 0\leq 1]}$, its "naive" subdivision, its stratified subdivision $\mathrm{sd}_P(\Delta^{[0\leq 0\leq 1]})$, and the stratified subdivision of $\Delta^{[0\leq 1\leq 2]}$

We will also use some properties of the subdivision functor.

Proposition 3.2.11.3. *For all stratified simplicial sets $K \in \mathrm{s}\mathbf{Strat}_P$, the last vertex map $\mathrm{l.v.}_P\colon \mathrm{sd}_P(K) \to K$ is a weak equivalence. In particular, if $f\colon K \to L$ is a map in $\mathrm{s}\mathbf{Strat}_P$, then f is a weak equivalence if and only if $\mathrm{sd}_P(f)$ is a weak equivalence.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{sd}_P(K) \arrow[r, "\mathrm{sd}_P(f)"] \arrow[d, "\mathrm{l.v.}_P"'] & \mathrm{sd}_P(L) \arrow[d, "\mathrm{l.v.}_P"] \\
K \arrow[r, "f"] & L\,.
\end{tikzcd}$$

The first part of the proposition asserts that the vertical morphisms are weak equivalences. Thus, by the two out of three law, the first assertion of the proposition implies the second assertion. For the first assertion, consider [Dou21a, Lemma A.3], which applies to sd_P and $\mathrm{l.v.}_P$ as shown in the proof of [Dou21a, Theorem 2.14]. One can also prove this directly by checking that it holds on representables, then using the cube lemma in an inductive argument. ☐

3.2.12 Recovering a Quillen equivalence

Composing the adjoint pairs $\mathrm{sd}_P \dashv \mathrm{Ex}_P$ and $|-|_P \dashv \mathrm{Sing}_P$, one gets a Quillen equivalence.

Theorem 3.2.12.1 ([Dou21b, Theorem 1.15])**.** *The adjoint pair*

$$|\mathrm{sd}_P(-)|_P\colon \mathrm{s}\mathbf{Strat}_P \leftrightarrow \mathbf{Strat}_P\colon \mathrm{Ex}_P\mathrm{Sing}_P \tag{3.9}$$

is a Quillen equivalence.

In particular, this means that the homotopy theories of spaces and simplicial sets stratified over P coincide.

Remark 3.2.12.2. The Quillen equivalence obtained above is only partially satisfactory for several reasons.

- First, the adjunction (3.9) is not very well suited for the study of conically stratified objects, since even reasonable PL-pseudomanifolds are not cofibrant objects of $\mathbf{Strat}_P$. In particular, they are not in the image of $|\mathrm{sd}_P(-)|_P$. Thus, it makes it difficult to relate the homotopy theory of those classical objects with that of stratified spaces. On the other hand, by working with the $|-|_P \dashv \mathrm{Sing}_P$ adjunction, we are able to show that the classical homotopy theory of conically stratified spaces embeds fully faithfully in that of stratified spaces (Corollary 3.5.2.4). The proof relies on the fact that triangulated conically stratified spaces are in the image of $|-|_P$ and have fibrant Sing_P. This illustrates the usefulness of working with the more natural adjunction $|-|_P \dashv \mathrm{Sing}_P$.
- The functors sd_P and Ex_P are not at all compatible with the Quillen adjunctions $\mathrm{s}\mathbf{Strat}_P \leftrightarrow \mathbf{sSet}_Q$, induced by maps of posets $\alpha\colon P \to Q$. In particular, one cannot recover a global adjunction $s\mathbf{Strat} \leftrightarrow \mathbf{Strat}$ by gluing the adjunctions (3.9) for all P. This means that in order to compare the homotopy theory of all stratified spaces and stratified simplicial sets, then one needs to work directly with the adjunction $|-|_s \dashv \mathrm{Sing}_s$.
- Finally, the stratified setting is very similar to the classical setting. In the latter, the $|-| \dashv \mathrm{Sing}$ adjunction is already a Quillen equivalence, with no need for subdivision. Given the fact that the homotopy theory associated to $s\mathbf{Strat}$ and $\mathbf{Strat}$ can actually be related in a meaningful way (Theorem 3.5.1.1), through the $|-|_s \dashv \mathrm{Sing}_s$ adjunction, the appearance of sd_P and Ex_P in (3.9) can appear as artificial.

With that said, Theorem 3.2.12.1 can be seen as the first step to obtain a comparison of the topological and simplicial setting through the $|-|_P \dashv \mathrm{Sing}_P$ adjunction. Since the latter adjunction is not a Quillen adjunction, we will need to work with categories with weak equivalences instead of the full model structures, see Section 3.5. In particular, we will need to prove that the functors $|-|_P$ and Sing_P characterize all weak equivalences (Corollary 3.4.5.2 and Theorem 3.3.2.1). By that, we mean that those functor preserve and reflect weak equivalences. By Theorem 3.2.12.1, this is already known for the functors $|\mathrm{sd}_P(-)|_P$ and $\mathrm{Ex}_P\mathrm{Sing}_P$, since they are part of a Quillen equivalence and all objects of $\mathrm{s}\mathbf{Strat}_P$ are cofibrant and all objects of $\mathbf{Strat}_P$ are fibrant. Thus, it suffices to show that for any $K \in \mathrm{s}\mathbf{Strat}_P$, $|\mathrm{sd}_P(K)|_P \to |K|_P$ is a weak equivalence, (see Section 3.4, and the proof of Corollary 3.4.5.2), and that for any $X \in \mathbf{Strat}_P$, the map $\mathrm{Sing}_P(X) \to \mathrm{Ex}_P(\mathrm{Sing}_P(X))$ is a weak equivalence (Proposition 3.3.1.1).

3.3 Ex_P and Sing_P characterize weak equivalences

In the category of simplicial sets, an explicit and particularly convenient fibrant replacement functor is given by the Kan fibrant replacement $S \hookrightarrow \mathrm{Ex}^\infty(S)$ (see [Kan57; GJ12]). The inclusion map is a trivial cofibration $S \hookrightarrow \mathrm{Ex}^\infty(S)$ which can be obtained (through transfinite composition) by glueing on simplices along horns (see [Mos19]). In this section, we show that the same can be said for Ex^∞_P, in particular that it gives a fibrant replacement functor in the category $\mathrm{s}\mathbf{Strat}_P$. From this, we then also obtain that the functor $\mathrm{Sing}_P : \mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P$ characterizes weak equivalences.

3.3.1 Fibrant replacement a la Kan, in $\mathrm{s}\mathbf{Strat}_P$

The main content of this subsection is proving the following result.

Proposition 3.3.1.1. *Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set, then the map $K \to \mathrm{Ex}_P(K)$ is a trivial cofibration.*

Together with the results from [Dou21a], we will then obtain as an immediate consequence:

Corollary 3.3.1.2. *The functor $\mathrm{Ex}_P^\infty : \mathrm{s}\mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P$ is a fibrant replacement functor.*

We prove Proposition 3.3.1.1 by adapting a proof of the equivalent statement in the non-stratified case from [Mos19]. This relies on the notion of a stratified strong anodyne extension, already used in [Dou21a]. Some intermediary results of [Mos19] immediately extend to the stratified case and their proofs will be omitted. We refer the reader to the proof of [Mos19, Theorem 22] for more details. Nevertheless, a significant amount of technical preparation is required to replicate the necessary arguments in the stratified framework.

Definition 3.3.1.3. A *(stratified) strong anodyne extension, (S)SAE* for short, is a morphism $A \hookrightarrow B$ in $\mathrm{s}\mathbf{Strat}_P$ that is given by a transfinite composition of cobase changes of admissible horn inclusions.

For the sake of brevity, we are just going to omit the *stratified* and just refer to SAEs from here on out.

Remark 3.3.1.4. By definition, all SAEs are trivial cofibrations. In particular, an SAE $A \to B$ is a monomorphism, and can be safely identified with the inclusion of a (stratified) sub-simplicial set.

Strong anodyne extensions enjoy an entirely combinatorial characterization, given as follows. Given a stratified simplicial set K, let $K_{\text{n.d.}}$ be its set of non-degenerate simplices.

Definition 3.3.1.5. Let $m: A \hookrightarrow B$ be a cofibration in $\mathrm{s}\mathbf{Strat}_P$.

1. A *pairing* on m is a partition of $B_{n.d.} \setminus A_{n.d.}$ into two sets B_I and B_{II} together with a bijection $T : B_{II} \to B_I$. The elements of B_I and B_{II} are referred to as *type I* and *type II* simplices respectively.

Now let $T: B_{II} \to B_I$ be a pairing on m.

2. T is called *proper* if for each $\sigma \in B_{II}$, σ is a codimension one face of $T(\sigma)$ in a unique way.

3. T is called *admissible*, if in addition, for any type II simplex, σ, such that $T(\sigma): \Delta^{\mathcal{J}} \to B$, and $\sigma = d_k(T(\sigma))$, $\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}}$ is an admissible horn inclusion.

4. Given a pairing T on m, the *ancestral relation* is the transitive binary relation on B_{II} generated by $\sigma <_T \tau$ if $\sigma \neq \tau$ and σ is a face of $T(\tau)$

5. T is called *regular* if the ancestral relation $<_T$ is well founded.

Proposition 3.3.1.6. *A cofibration $A \hookrightarrow B$ in $\mathrm{s}\mathbf{Strat}_P$ is a SAE if and only if it admits an admissible proper regular pairing.*

Proof. The proof of [Mos19, Proposition 12] directly generalizes. The extra admissibility hypothesis guarantees that only admissible horns are being filled. □

Now, since SAEs are trivial cofibrations, we can prove Proposition 3.3.1.1 by exhibiting a suitable pairing. We will achieve this by decomposing the map $K \to \mathrm{Ex}_P(K)$ into two maps, and exhibiting a presentation for each of the factors.

Definition 3.3.1.7. Let $K \in s\mathbf{Strat}_P$ be a stratified simplicial set. Its naive subdivision is $\mathrm{sd}_P^{\mathrm{naiv}}(K) = (\mathrm{sd}(K), \varphi_K \circ \mathrm{l.v.})$. This defines a functor

$$\mathrm{sd}_P^{\mathrm{naiv}} : s\mathbf{Strat}_P \to s\mathbf{Strat}_P$$

which admits a right adjoint, $\mathrm{Ex}_P^{\mathrm{naiv}} : s\mathbf{Strat}_P \to s\mathbf{Strat}_P$. The latter can also be described as a simplicial subset $\mathrm{Ex}_P^{\mathrm{naiv}}(K) \subset \mathrm{Ex}(K)$

$$\mathrm{Ex}_P^{\mathrm{naiv}}(K) = \{\sigma : \mathrm{sd}(\Delta^{\mathcal{J}}) \to K \mid \sigma \text{ is stratum-preserving}\}$$

Now, let $\mathcal{J}$ be some flag $[p_0 \le \cdots \le p_n]$, define the following map on vertices:

$$\begin{aligned} t_{\mathcal{J}} : \mathrm{sd}_P(\Delta^{\mathcal{J}}) &\to \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}}) \\ (q_0 \le \cdots \le q_m, r) &\mapsto (q_0 \le \cdots \le q_l) \end{aligned}$$

where $q_l = r$ and $q_{l+1} > r$. This extends to a map of stratified simplicial sets, $t_{\mathcal{J}} : \mathrm{sd}_P(\Delta^{\mathcal{J}}) \to \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}})$, and further into a natural transformation $t : \mathrm{sd}_P \to \mathrm{sd}_P^{\mathrm{naiv}}$.

Note that the natural transformation $t : \mathrm{sd}_P \to \mathrm{sd}_P^{\mathrm{naiv}}$ fits into a commutative diagram

$$\begin{tikzcd} \mathrm{sd}_P \arrow[rr, "\mathrm{l.v.}_P"] \arrow[dr, "t"'] & & 1_{s\mathbf{Strat}_P} \\ & \mathrm{sd}_P^{\mathrm{naiv}} \arrow[ur, "\mathrm{l.v.}"'] & \end{tikzcd} \tag{3.10}$$

which, by adjunction, gives the following diagram

$$\begin{tikzcd} 1_{s\mathbf{Strat}_P} \arrow[rr] \arrow[dr] & & \mathrm{Ex}_P \\ & \mathrm{Ex}_P^{\mathrm{naiv}} \arrow[ur] & \end{tikzcd} , \tag{3.11}$$

where the horizontal arrow is the map of interest. We will show that the maps to, and from $\mathrm{Ex}_P^{\mathrm{naiv}}$ are SAEs. In particular, this will imply that their composition is an SAE, which in turn implies Proposition 3.3.1.1. We start by showing that they are cofibrations.

Lemma 3.3.1.8. *All natural transformations in Diagram* (3.11) *are cofibrations.*

Proof. It suffices to show, that all of the natural transformations in (3.10) are epimorphisms of simplicial sets when evaluated at $\Delta^{\mathcal{J}}$. For l.v. this is well known. Thus, by closedness of epimorphisms under composition, it remains to show that $t_{\mathcal{J}}$ is an epimorphism, for each flag $\mathcal{J} = [p_0 \le \cdots \le p_n]$. Label $e_0, \ldots, e_n$ the vertices of $\Delta^{\mathcal{J}}$, and let $\sigma = (\sigma_0, \ldots, \sigma_k)$ be a simplex in $\mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}})$. For $0 \le i \le k$, let $q_i := \max\{p_j \mid e_j \in \sigma_i\}$. Note that q_i specifies the stratum to which the vertex σ_i of $\mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}})$ belongs. For $q \in P$, denote $j_q = \min\{i \mid q_i = q\}$. Next, define $\widetilde{\sigma} \subset \Delta^{\mathcal{J}}$ as $\widetilde{\sigma} = \{e_i \mid e_i \in \sigma_{j_{p_i}}\}$. In other words, $\widetilde{\sigma}$ is the simplex given by such vertices e, which lie in the smallest σ_i which, considered as a simplex of $\mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}})$, has the same stratum as e. Note, that $\widetilde{\sigma}$ is built such that for each q_i it contain some vertex of stratum q_i. In particular,

$$\sigma' = ((\sigma_0 \cup \widetilde{\sigma}, q_0), \ldots, (\sigma_k \cup \widetilde{\sigma}, q_k))$$

defines a simplex of $\mathrm{sd}_P(\Delta^{\mathcal{J}})$. An elementary computation shows that $t(\sigma') = \sigma$. $\square$

Proposition 3.3.1.9. *Let $K \in s\mathbf{Strat}_P$ be a stratified simplicial set, the map $K \hookrightarrow \mathrm{Ex}_P^{naiv}(K)$ is an SAE.*

Proof. By Lemma 3.3.1.8 we already know that it is a cofibration, and so we need to exhibit a proper regular admissible pairing. Consider the (non-stratified) composition

$$K \to \mathrm{Ex}_P^{\mathrm{naiv}}(K) \to \mathrm{Ex}(K).$$

We know of a proper regular pairing for the composition from [Mos19, Theorem 22]. By Lemma 3.3.1.10, it is enough to show that this pairing correctly restricts to $\mathrm{Ex}_P^{\mathrm{naiv}}(K)$, and that the restricted pairing is admissible. Note that in [Mos19], the pairing T is defined by pre-composing with certain maps $r_n^k\colon \mathrm{sd}(\Delta^{n+1}) \to \mathrm{sd}(\Delta^n)$, for $0 \leq k \leq n$ defined on vertices as follows (see [Mos19, Definition 25], the pairing is defined immediately afterwards). First, for vertices of $\mathrm{sd}(\Delta^{n+1})$ of the form $\{i\}$, one has :

$$r_n^k(\{i\}) = \begin{cases} \{i\} & \text{if } 0 \leq i \leq k \\ \{0,\dots,k\} & \text{if } i = k+1 \\ \{i-1\} & \text{if } i > k+1 \end{cases}$$

And, then for an arbitrary vertex $\sigma \in \mathrm{sd}(\Delta^{n+1})$, one has

$$r_n^k(\sigma) = \bigcup_{i\in\sigma} r_n^k(\{i\})$$

Now let $\mathcal{J} = [p_0 \leq \dots \leq p_n]$ be some flag, and $\mathcal{J}^k$ the flag obtained by repeating p_k. Then, note that r_n^k gives a stratum-preserving map

$$r_n^k\colon \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}^k}) \to \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}})$$

Now, if $f\colon \mathrm{sd}(\Delta^n) \to K$ is some simplex of type II which happens to be in $\mathrm{Ex}_P^{\mathrm{naiv}}(K)$, then there is some flag $\mathcal{J}$ such that $f\colon \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}}) \to K$ is a stratum-preserving map. Furthermore, one has $T(f) = f \circ r_n^k$ for some $0 \leq k \leq n$, but then, $f \circ r_n^k\colon \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}^k}) \to K$ is also a stratum-preserving map. Which implies that T restricts to a proper and regular pairing to the inclusion $K \to \mathrm{Ex}_P^{\mathrm{naiv}}(K)$. Finally, the pairing is admissible since the horn inclusion $\Lambda_k^{\mathcal{J}^k} \to \Delta^{\mathcal{J}^k}$ is admissible, by definition of $\mathcal{J}^k$. □

Lemma 3.3.1.10. *Suppose one is given two cofibrations of simplicial sets $m_1 : B_0 \hookrightarrow B_1$, $m_2 : B_1 \hookrightarrow B_2$. Then, a proper regular pairing T on $m_2 \circ m_1$ restricts to proper regular pairings on m_1 and on m_2 if and only if we have $T(B_{II} \cap B_{1,n.d.}) \subset B_{1,n.d.}$. This also holds for stratified simplicial sets and admissible pairings.*

Proof. Note first that properness and admissibility are clearly sustained under restriction. As any subset of a well founded set is well founded, the same holds for regularity. Hence, the only thing to verify is that T restricts to a bijection. The condition on T guarantees that the restriction of T to $B_{II} \cap B_{1,n.d.} \to B_I \cap B_{1,n.d.}$ is well defined. Injectivity is automatic, and surjectivity follows from the fact that for $\sigma \in B_I$, $T^{-1}\sigma$ is always a face of σ. In particular, if $\sigma \in B_{1,I}$, then $T^{-1}(\sigma) \subset \sigma \subset B_1$. Finally, the restriction of T on $B_{1,\mathrm{n.d.}}$ gives a proper regular pairing. This also implies that the restriction of T on $B_2 \setminus B_1$ is a proper regular pairing. □

We are left with showing, that $\mathrm{Ex}_P^{\mathrm{naiv}}(K) \hookrightarrow \mathrm{Ex}_P(K)$ is an SAE for $K \in \mathrm{s}\mathbf{Strat}_P$. We will do so in two steps. We will first construct some sub-object of $\mathrm{Ex}_P^{\mathrm{naiv}}(K)$, $\widehat{\mathcal{J}}$, such that $\widehat{\mathcal{J}} \hookrightarrow \mathrm{Ex}_P(K)$ is an SAE, following Moss's proof in [Mos19], and then use Lemma 3.3.1.10 to show that it restricts to the desired SAE. To describe $\widehat{\mathcal{J}}$ and the pairing, we need to introduce some maps on subdivisions.

Definition 3.3.1.11. Let $n \geq 0$, and $0 \leq k \leq n$. Define maps $\widetilde{\jmath}_n^k\colon \mathrm{sd}(\Delta^n) \to \mathrm{sd}(\Delta^n)$ and $\widetilde{r}_n^k\colon \mathrm{sd}(\Delta^{n+1}) \to \mathrm{sd}(\Delta^n)$ as follows. On vertices of the form $\{i\}$, they are defined as

$$\widetilde{\jmath}_n^k(\{i\}) = \begin{cases} \{i,\dots,n\} & \text{if } i < k \\ \{i\} & \text{if } i \geq k \end{cases} \quad \text{and } \widetilde{r}_n^k(\{i\}) = \begin{cases} \{i\} & \text{if } i < k \\ \{k,\dots,n\} & \text{if } i = k \\ \{i-1\} & \text{if } i > k \end{cases}$$

On vertices σ, they are defined as

$$\widetilde{j}_n^k(\sigma) = \bigcup_{i\in\sigma} \widetilde{j}_n^k(\{i\}) \text{ and } \widetilde{r}_n^k(\sigma) = \bigcup_{i\in\sigma} \widetilde{r}_n^k(\{i\})$$

And then, they are extended linearly to maps of simplicial sets. Then, given a flag $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, the product maps

$$\widetilde{j}_n^k \times 1 : \mathrm{sd}(\Delta^n) \times N(P) \to \mathrm{sd}(\Delta^n) \times N(P)$$
$$\widetilde{r}_n^k \times 1 : \mathrm{sd}(\Delta^{n+1}) \times N(P) \to \mathrm{sd}(\Delta^n) \times N(P)$$

Restrict to stratum-preserving maps

$$j_n^k : \mathrm{sd}_P(\Delta^{\mathcal{J}}) \to \mathrm{sd}_P(\Delta^{\mathcal{J}});$$
$$r_n^k : \mathrm{sd}_P(\Delta^{\mathcal{J}^k}) \to \mathrm{sd}_P(\Delta^{\mathcal{J}}),$$

where $\mathcal{J}^k$ is the flag obtained from $\mathcal{J}$ by repeating p_k.

For K some fixed stratified simplicial set, $n \geq 0$, and $0 \leq k \leq n$, let J_n^k be the subset of $\mathrm{Ex}_P(K)_n$ defined as follows

$$J_n^k = \{\sigma : \mathrm{sd}_P(\Delta^{\mathcal{J}}) \to K \mid \sigma \circ j_n^k = \sigma\}$$

The maps j_n^k and r_n^k satisfy a lot of relations, somewhat similar to the simplicial relations.

Proposition 3.3.1.12. *The morphisms j^k, r^k fulfill the following equations (for any fixed flag $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ as the target of all the compositions)*

$$r^k \circ \mathrm{sd}_P(d^k) = 1_{\Delta^{\mathcal{J}}} \qquad 0 \leq k \leq n \qquad (1')$$
$$j^k \circ r^k \circ \mathrm{sd}_P(d^i) \circ j^{k+1} = j^k \circ r^k \circ \mathrm{sd}_P(d^i) \qquad 0 \leq k < i \leq n \qquad (2')$$
$$r^k \circ \mathrm{sd}_P(d^i) = \mathrm{sd}_P(d^i) \circ r^{k-1} \qquad 0 \leq i < k \leq n \qquad (3')$$

$$r^k \circ j^{h+1} = j^h \circ r^k \qquad 0 \leq h \leq k \leq n \qquad (4')$$
$$j^k \circ \mathrm{sd}_P(d^i) \circ j^{k-1} = j^k \circ \mathrm{sd}_P(d^i) \qquad 1 \leq k \leq n,\ 0 \leq i \leq n \qquad (5')$$
$$j^h \circ r^k = j^h \circ \mathrm{sd}_P(s^k) \qquad 1 \leq k < h \leq n \qquad (6')$$

$$j^k \circ r^k \circ r^{k+1} = j^k \circ r^k \circ \mathrm{sd}_P(s^k) \qquad 0 \leq k \leq n \qquad (7')$$
$$\mathrm{sd}_P(s^h) \circ j^k \circ r^k = j^k \circ r^k \circ \mathrm{sd}_P(s^{h+1}) \qquad 0 \leq k \leq h \leq n \qquad (8')$$
$$\mathrm{sd}_P(s^h) \circ j^{k+1} \circ r^{k+1} = j^k \circ r^k \circ \mathrm{sd}_P(s^h) \qquad 0 \leq h \leq k \leq n \qquad (9')$$

Proof. The maps j_n^k and r_n^k are obtained from the corresponding maps in [Mos19] by conjugating them with the automorphisms $D_n : \mathrm{sd}(\Delta^n) \to \mathrm{sd}(\Delta^n)$ sending $\{i\}$ to $\{n-i\}$. Equation $(1')$ through $(9')$ are then obtained by conjugating Moss' equations (1) through (9) [Mos19, Lemma 26]. □

Lemma 3.3.1.13. *The subset $\widehat{J} \subset \mathrm{Ex}_P(K)$, defined as*

$$\widehat{J}_n = J_n^n$$

is a simplicial subset. Furthermore, $\widehat{J} \subset \mathrm{Ex}_P^{naiv}(K)$.

Proof. For the first part, it suffices to show that if $\sigma : \mathrm{sd}_P(\Delta^{\mathcal{J}}) \to K$ is in J_n^n, then for all $0 \leq k \leq n$, we have both

$$\sigma \circ \mathrm{sd}(d^k) \in J_{n-1}^{n-1} \text{ and } \sigma \circ \mathrm{sd}(s^k) \in J_{n+1}^{n+1}.$$

The former comes from equality (5'), while the later follows from the following equality:

$$j_n^n \circ \mathrm{sd}_P(s^k) \circ j_{n+1}^{n+1} = j_n^n \circ \mathrm{sd}_P(s^k) \tag{10'}$$

Given the definitions of the j^k, it is enough to check that Eq. (10') holds on vertices of $\mathrm{sd}(\Delta^{n+1})$ of the form $\{i\}$, for the $\widetilde{j}$ and $\mathrm{sd}(s^k)$. But evaluating the transformed equation on $\{i\}$ gives on both sides either $\{i,\dots,n\}$, if $k \geq i$, or $\{i-1,\dots,n\}$ if $k < i$, which concludes the first part of the proof.

For the second part, Let $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, and consider the factorization problem

$$\begin{tikzcd} \mathrm{sd}_P(\Delta^{\mathcal{J}}) \arrow[rr, "j^n"] \arrow[dr, "t_{\mathcal{J}}"'] & & \mathrm{sd}_P(\Delta^{\mathcal{J}}) \\ & \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}}) \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

One checks that the map f, defined on vertices as $f(\sigma) = (\widetilde{j}_n^n(\sigma), \max\{p_i \mid i \in \sigma\})$, makes the diagram commute. But then, any simplex $\sigma \in \widehat{J}$ must satisfy $\sigma = \sigma \circ j^n = \sigma \circ f \circ t_{\mathcal{J}}$. In particular, such a simplex is in $\mathrm{Ex}_P^{\mathrm{naiv}}(K)$, which concludes the proof. □

Lemma 3.3.1.14. *The inclusion $\widehat{J} \to \mathrm{Ex}_P(K)$ is an SAE.*

Proof. Moss' proof of [Mos19, Theorem 22] directly translates into a proof that the inclusion of Lemma 3.3.1.14 is an SAE. Note that due to the conjugation with D_n, the inclusion relations between the J_n^k are reversed from those in [Mos19]. Additionally, the key difference is that in our setting, $\widehat{J}$ does not coincide with K. Nevertheless, [Mos19, Lemmas 27, 28 and 29] generalize in our setting since they are direct consequences of the equalities of [Mos19, Lemma 26], which also hold in our context in the form of Proposition 3.3.1.12.

To see that the pairing from [Mos19] is admissible, let $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ be some flag, and $\sigma \colon \mathrm{sd}_P(\Delta^{\mathcal{J}}) \to K$ a type II simplex. Then, there is some $k \geq 0$ such that $T(\sigma) = \sigma \circ r^k$, and $\sigma = d_k T(\sigma)$. In particular, $T(\sigma)$ is of the form $\mathrm{sd}_P(\Delta^{\mathcal{J}^k}) \to K$, with $\mathcal{J}^k$ obtained from $\mathcal{J}$ by repeating p_k. This means that $\Lambda_k^{\mathcal{J}^k} \to \Delta^{\mathcal{J}^k}$ is an admissible horn inclusion, and so the pairing is admissible. □

Proposition 3.3.1.15. *Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set, the map $\mathrm{Ex}_P^{naiv}(K) \to \mathrm{Ex}_P(K)$ is an SAE.*

Proof. By Lemma 3.3.1.10, it is enough to show that the pairing given in the proof of Lemma 3.3.1.14 correctly restricts to $\mathrm{Ex}_P^{\mathrm{naiv}}(K)$. Let $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ be a flag, and $\sigma \colon \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}}) \to K$ a type II simplex in $\mathrm{Ex}_P^{\mathrm{naiv}}(K)$. Its image under the pairing T is the simplex of $\mathrm{Ex}_P(K)$, $\sigma \circ t_{\mathcal{J}} \circ r^k$, for some $0 \leq k \leq n$, and we need to find some $\tau \in \mathrm{Ex}_P^{\mathrm{naiv}}(K)$ such that $\sigma \circ t_{\mathcal{J}} \circ r^k = \tau \circ t_{\mathcal{J}^k}$, where $\mathcal{J}^k$ is obtained from $\mathcal{J}$ by repeating p_k. Consider the following diagram:

$$\begin{tikzcd} \mathrm{sd}_P(\Delta^{\mathcal{J}^k}) \arrow[r, "r^k"] \arrow[d, "t_{\mathcal{J}^k}"'] & \mathrm{sd}_P(\Delta^{\mathcal{J}}) \arrow[d, "t_{\mathcal{J}}"] \\ \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}^k}) \arrow[r, dashed, "g"] & \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}}) \end{tikzcd}$$

It suffices to find a stratum-preserving map $g \colon \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}^k}) \to \mathrm{sd}_P^{\mathrm{naiv}}(\Delta^{\mathcal{J}})$ making the diagram commute, since then $\tau = \sigma \circ g$ would satisfy $\sigma \circ t_{\mathcal{J}} \circ r^k = \tau \circ t_{\mathcal{J}^k}$. We define g on vertices as follows.

$$g(\mu) = \{i \in \widetilde{r_n^k}(\mu) \mid p_i \leq \max\{p_j \mid j \in \mu\}\}$$

for μ a vertex of $\mathrm{sd}(\Delta^{n+1})$. An elementary computation now gives commutativity. □

Proposition 3.3.1.9 and Proposition 3.3.1.15 together complete the proof of Proposition 3.3.1.1. Indeed, we have proven that the map $K \to \mathrm{Ex}_P^{\mathrm{naiv}}(K) \to \mathrm{Ex}_P(X)$ is the composition of two SAE, hence it is a SAE.

3.3.2 Sing_P characterizes weak equivalences

Proposition 3.3.1.1 has the following immediate consequence.

Theorem 3.3.2.1. *Let $f: X \to Y$ be a map in $\mathbf{Strat}_P$. It is a weak equivalence if and only if $\mathrm{Sing}_P(f): \mathrm{Sing}_P(X) \to \mathrm{Sing}_P(Y)$ is a weak equivalence in $s\mathbf{Strat}_P$. The analgous result holds for $\mathrm{Sing}_{N(P)}$ and maps in $\mathbf{Top}_{N(P)}$*

Proof. By [Dou21b] we know that $\mathrm{Ex}_P\mathrm{Sing}_P$ is the right functor of a Quillen equivalence. In particular, if $f: X \to Y$ is a map in $\mathbf{Strat}_P$, since all objects of $\mathbf{Strat}_P$ are fibrant, it is a weak equivalence if and only if $\mathrm{Ex}_P\mathrm{Sing}_P(f): \mathrm{Ex}_P\mathrm{Sing}_P(X) \to \mathrm{Ex}_P\mathrm{Sing}_P(Y)$ is a weak equivalence in $s\mathbf{Strat}_P$. Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Sing}_P(X) \arrow[r] \arrow[d] & \mathrm{Sing}_P(Y) \arrow[d] \\
\mathrm{Ex}_P\mathrm{Sing}_P(X) \arrow[r] & \mathrm{Ex}_P\mathrm{Sing}_P(Y)
\end{tikzcd}$$

By Proposition 3.3.1.1, the vertical maps are trivial cofibrations and in particular weak equivalences. By two out of three, this implies that $\mathrm{Sing}_P(f)$ is a weak equivalence if and only if $\mathrm{Ex}_P\mathrm{Sing}_P(f)$ is a weak equivalence if and only if f is a weak equivalence. The proof for $\mathrm{Sing}_{N(P)}$ is identical. ☐

3.4 Realizations characterize weak equivalences

In this section, we prove that a stratum-preserving simplicial map $f : K \to L$ in $s\mathbf{Strat}_P$ is a weak equivalence if and only if $|f|_P$ is a weak equivalence in $\mathbf{Strat}_P$. In light of the results in [Dou21b] which establish that $|\mathrm{sd}_P(-)|_P$ is the left functor in a Quillen equivalence, and those in [Dou21a] which imply that $\mathrm{sd}_P(K) \to K$ is a weak equivalence in $s\mathbf{Strat}_P$, the main part is to show that $|-|_P$ does preserve weak equivalences. We show this result by obtaining all but the first of the weak equivalences in Diagram (3.3).

3.4.1 Comparing the simplicial links and the geometrical links

We start with a comparison between the simplicial link, $|\mathrm{Link}_{\mathcal{I}}(-)|$, and the geometrical link, $|-|_b$, see Definitions 3.2.5.4 and 3.2.5.6.

Proposition 3.4.1.1. *Let $K \in s\mathbf{Strat}_P$ be a stratified simplicial set and $A \subset K$ a stratified simplicial subset. Let $\mathcal{I}$ be a regular flag in P, and $b \in |\Delta^{\mathcal{I}}|$ the barycenter. Then, there is a commutative diagram in* $\mathbf{Top}$

$$\begin{tikzcd}
|\mathrm{Link}_{\mathcal{I}}(A)| \arrow[r, hook] \arrow[d, "\sim"] & |\mathrm{Link}_{\mathcal{I}}(K)| \arrow[d, "\sim"] \\
|A|_b \arrow[r, hook] & |K|_b
\end{tikzcd}$$

where the vertical maps are homeomorphisms.

The proof of the proposition relies on the following lemma, which is a direct consequence of [WJR13, Theorem 2.3.2].

Lemma 3.4.1.2. *Let $h_P: |\mathrm{sd}(N(P))| \to |N(P)|$ be the usual homeomorphism between a simplicial complex and its barycentric subdivision. Let $(K, \varphi_K) \in s\mathbf{Strat}_P$ be a stratified simplicial set. There exists a homeomorphism $h_K: |\mathrm{sd}(K)| \to |K|$, which induces an isomorphism in $\mathbf{Top}_{N(P)}$:*

$$h_K: (|\mathrm{sd}(K)|, h_P \circ |\mathrm{sd}(\varphi_K)|) \to |(K, \varphi_K)|_{N(P)}$$

Furthermore, the homeomorphism h_K restricts to homeomorphisms on all non-degenerate simplices. In particular, if $(A, \varphi_A) \subset (K, \varphi_K)$ is a stratified simplicial subset, then the restriction of h_K induces an isomorphism in $\mathbf{Top}_{N(P)}$

$$(h_K)_{|A}: (|\mathrm{sd}(A)|, h_P \circ |\mathrm{sd}(\varphi_A)|) \to |(A, \varphi_A)|_{N(P)}$$

Proof. Consider the simplicial map $\varphi_K: K \to N(P)$. Its codomain is a non-singular simplicial set, and so by [WJR13, Theorem 2.3.2], there exists a homeomorphism h_K such that the following diagram commutes:

$$\begin{tikzcd} |\mathrm{sd}(K)| \arrow[r, "h_K"] \arrow[d, "|\mathrm{sd}(\varphi_K)|"'] & |K| \arrow[d, "|\varphi_K|"] \\ |\mathrm{sd}(N(P))| \arrow[r, "h_P"] & |N(P)| \end{tikzcd}$$

The commutativity of the diagram implies that h_K is an isomorphism in $\mathbf{Top}_{N(P)}$, which is the first part of the lemma. The second part of the lemma is the content of [WJR13, Proposition 2.3.23]. ☐

Proof of Proposition 3.4.1.1. Let $A \subset K$ be an inclusion of stratified simplicial sets. Since the realization functor $|-|: \mathbf{sSet} \to \mathbf{Top}$ preserves pullbacks, $|\mathrm{Link}_{\mathcal{I}}(K)|$ is given by the following pullback:

$$\begin{tikzcd} |\mathrm{Link}_{\mathcal{I}}(K)| \arrow[r, hook] \arrow[d] & |\mathrm{sd}(K)| \arrow[d, "|\mathrm{sd}(\varphi_K)|"] \\ \{*\} \arrow[r, hook, "|i_{\mathcal{I}}|"] & |\mathrm{sd}(N(P))| \end{tikzcd}$$

On the other hand, $|K|_b$ is defined as the following pullback:

$$\begin{tikzcd} |K|_b \arrow[r, hook] \arrow[d] & |K| \arrow[d, "|\varphi_K|"] \\ \{b\} \arrow[r, hook] & |N(P)| \end{tikzcd}$$

By Lemma 3.4.1.2, h_K and h_P give an isomorphism between the arrows on the right-hand side of both squares. On the other hand, $h_P(|i_{\mathcal{I}}|(*)) = b$, since b is the barycenter of $|\Delta^{\mathcal{I}}|$, which means that there is an isomorphism between the pullback squares, producing the isomorphism $|\mathrm{Link}_{\mathcal{I}}(K)| \to |K|_b$. Since h_K correctly restricts to simplicial subsets, by Lemma 3.4.1.2, we obtain the left side of the square in Proposition 3.4.1.1 as well as its commutativity. ☐

3.4.2 From links to homotopy links

We begin by showing that, up to homotopy, the geometric link given by $|K|_b$ agrees with $\mathit{Ho}\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$.

Lemma 3.4.2.1. *Let K be a stratified simplicial set, $\mathcal{I}$ a regular flag, and $y = (y_0, \dots, y_n)$ a point in the interior of $|\Delta^{\mathcal{I}}|$. The inclusion $\{y\} \hookrightarrow |\Delta^{\mathcal{I}}|_{N(P)}$ induces a weak equivalence*

$$r_y: \mathit{Ho}\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \to |K|_y. \tag{3.12}$$

The proof will take the remainder of the subsection. However, we begin with an immediate corollary.

Corollary 3.4.2.2. *Let K and L be two stratified simplicial sets, and $f: |K|_{N(P)} \to |L|_{N(P)}$ any map in $\mathbf{Top}_{N(P)}$. Then f is a weak equivalence if and only if f induces weak equivalences $|K|_b \to |L|_b$ for all barycenters of simplices in $|N(P)|$.*

Proof. By definition of the model structure on $\mathbf{Top}_{N(P)}$, $f : X \to Y$ is a weak equivalence if and only if f induces weak equivalences $Ho\mathrm{Link}_{\mathcal{I}}(X) \to Ho\mathrm{Link}_{\mathcal{I}}(Y)$, for all regular flags, $\mathcal{I}$. Thus, it is enough to show that the evaluation at b map

$$Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \to |K|_b$$

is a weak equivalence. This is the content of Lemma 3.4.2.1. □

We will actually prove that the map (3.12) is a homotopy equivalence. To define an inverse, we need a way to define a map $|\Delta^{\mathcal{I}}|_{N(P)} \to |K|_{N(P)}$ from the data of a point in $|K|_b$. This is the purpose of the next lemma.

Lemma 3.4.2.3. *Let K be a stratified simplicial set, and $\mathcal{I}$ a regular flag. Let $|K|_{\mathrm{Int}(|\Delta^{\mathcal{I}}|)}$ be the pre-image of* $\mathrm{Int}(|\Delta^{\mathcal{I}}|)$ *under $|\varphi_K|_{N(P)}$ and consider $|K|_{\mathrm{Int}(|\Delta^{\mathcal{I}}|)} \times |\Delta^{\mathcal{I}}|_{N(P)}$ as a strongly stratified space via the projection on the second factor. Then there exists a map in* $\mathbf{Top}_{N(P)}$

$$\rho\colon |K|_{\mathrm{Int}(|\Delta^{\mathcal{I}}|)} \times |\Delta^{\mathcal{I}}|_{N(P)} \to |K|_{N(P)},$$

such that the restriction of ρ to the fiber-product $|K|_{\mathrm{Int}(|\Delta^{\mathcal{I}}|)} \times_{|\varphi_K|} |\Delta^{\mathcal{I}}|_{N(P)}$ coincides with the projection on the first factor, and extends in this way to $|K| \times_{|\varphi_K|} |\Delta^{\mathcal{I}}|_{N(P)}$.

Proof. Any point in $|K|_{\mathrm{Int}(|\Delta^{\mathcal{I}}|)}$ can be described uniquely as a pair $(\sigma, (t_0, \dots, t_m))$, where $\sigma\colon \Delta^m \to K$ is a non-degenerate simplex such that $\varphi_K(\Delta^m)$ is some degeneracy of $\Delta^{\mathcal{I}}$, and $(t_0, \dots, t_m)$ is a point in the standard m-simplex (that is, a $(m+1)$-tuple satisfying $0 \le t_i \le 1$, for all i and $\sum t_i = 1$). We can relabel the $(m+1)$-tuple $(t_0, \dots, t_m)$ as $(t_0^0, \dots, t_{k_0}^0, t_0^1, \dots, t_{k_n}^n)$ so that the coordinates of $|\varphi_K|(\sigma, (t_0^0, \dots, t_{k_0}^0, t_0^1, \dots, t_{k_n}^n))$ are $(\sum t_i^0, \sum t_i^1, \dots, \sum t_i^n)$. The condition that $|\varphi_K|(\sigma, (t_0^0, \dots, t_{k_0}^0, t_0^1, \dots, t_{k_n}^n)) \in \mathrm{Int}|\Delta^{\mathcal{I}}|$ thus guarantees that $\sum t_i^k > 0$, for all $0 \le k \le n$. Now, given a point in $|\Delta^{\mathcal{I}}|_{N(P)}$, $(q_0, \dots, q_n)$, let ρ be the continuous map:

$$\begin{aligned} \rho\colon |K|_{\mathrm{Int}(|\Delta^{\mathcal{I}}|)} \times |\Delta^{\mathcal{I}}|_{N(P)} &\to |K|_{N(P)} \\ ((\sigma, (t_0, \dots, t_m)), (q_0, \dots, q_n)) &= (\sigma, (\frac{q_0}{\sum_i t_i^0} t_0^0, \dots, \frac{q_0}{\sum_i t_i^0} t_{k_0}^0, \frac{q_1}{\sum_i t_i^1} t_0^1, \dots, \frac{q_n}{\sum_i t_i^n} t_{k_n}^n)) \end{aligned}$$

For a point in the fibered product, one must have $\sum_i t_i^l = q_l$ for all $0 \le l \le n$, and ρ extends on it as:

$$\begin{aligned} |K| \times_{|\varphi_K|} |\Delta^{\mathcal{I}}| &\to |K| \\ ((\sigma, (t_0^0, \dots, t_{k_0}^0, t_0^1, \dots, t_{k_n}^n)), (q_0, \dots, q_n)) &\mapsto (\sigma, (t_0^0, \dots, t_{k_0}^0, t_0^1, \dots, t_{k_n}^n)) \end{aligned}$$

□

proof of Lemma 3.4.2.1. Let $y = (y_0, \dots, y_n)$ be a point in $\mathrm{Int}(|\Delta^{\mathcal{I}}|)$. We show that the evaluation at y

$$r_y\colon Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) = \mathcal{C}^0_{N(P)}(|\Delta^{\mathcal{I}}|_{N(P)}, |K|_{N(P)}) \to |K|_y$$

is a homotopy equivalence. We define a section $|K|_y \to Ho\mathrm{Link}_{\mathcal{I}}(K)$ as follows

$$\begin{aligned} c_y\colon |K|_y &\to Ho\mathrm{Link}_{\mathcal{I}}(K) \\ x &\mapsto \left\{ \begin{array}{ccc} |\Delta^{\mathcal{I}}|_{N(P)} & \to & |K|_{N(P)} \\ (q_0, \dots, q_n) & \mapsto & \rho(x, (q_0, \dots, q_n)) \end{array} \right. \end{aligned}$$

By the previous lemma, one has $r_y \circ c_y = 1_{|K|_y}$. To construct the homotopy in the other direction, we will use the following (non-stratified) identification:

$$\begin{aligned} |\partial\Delta^{\mathcal{I}}| \times [0,1] / |\partial\Delta^{\mathcal{I}}| \times \{0\} &\simeq |\Delta^{\mathcal{I}}| \\ ((a_0, \dots, a_n), s) &\mapsto (sa_0 + (1-s)y_0, \dots, sa_n + (1-s)y_n). \end{aligned}$$

We now define the homotopy from $c_y \circ r_y$ to the identity.

$$H\colon \mathit{Ho}\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \times [0,1] \to \mathit{Ho}\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$$
$$(f,t) \mapsto \begin{cases} |\Delta^{\mathcal{I}}|_{N(P)} & \to \quad |K|_{N(P)} \\ (\bar{a},s) & \mapsto \quad \rho(f(\bar{a},st),(\bar{a},s)) \end{cases}$$

Note that H is well defined, since when $t \neq 1$, $f(\bar{a},st)$ is in $|K|_{\mathrm{Int}(|\Delta^{\mathcal{I}}|)}$, for all $(\bar{a},s) \in |\Delta^{\mathcal{I}}|$, and for $t = 1$, either $s \neq 1$, and $f(\bar{a},s) \in |K|_{\mathrm{Int}(|\Delta^{\mathcal{I}})|}$, or $s = 1$ and $(f(\bar{a},1),(\bar{a},1)) \in |K| \times_{|\varphi_K|} |\Delta^{\mathcal{I}}|$, and ρ is defined as the projection on the first factor. ☐

3.4.3 $|-|_{N(P)}$ preserves weak equivalences

Next, we use Corollary 3.4.2.2 to derive the following result.

Proposition 3.4.3.1. *The functor*

$$|-|_{N(P)} : \mathrm{s}\mathbf{Strat}_P \to \mathbf{Top}_{\mathrm{N}(P)}$$

preserves weak equivalences.

Proof. Note first, that by Ken Brown's Lemma, it suffices to show that $|-|_{N(P)}$ sends trivial cofibrations into weak equivalences. Now assume that $f\colon A \hookrightarrow K$ is a trivial cofibration. By Corollary 3.4.2.2 it is enough to show that for any $b \in |N(P)|$, the barycenter of some simplex $\Delta^{\mathcal{I}}$, the inclusion $|A|_b \to |K|_b$ is a weak equivalence. By Proposition 3.4.1.1 this is equivalent to showing that the inclusion $|\mathrm{Link}_{\mathcal{I}}(A)| \hookrightarrow |\mathrm{Link}_{\mathcal{I}}(K)|$ is a weak equivalence. But since the functor $|-|$ preserves weak equivalences, it is enough to show that $\mathrm{Link}_{\mathcal{I}}(A) \hookrightarrow \mathrm{Link}_{\mathcal{I}}(K)$ is a weak equivalence, which follows from Lemma 3.4.3.2. ☐

Lemma 3.4.3.2. *The functor* $\mathrm{Link}_{\mathcal{I}} : \mathrm{s}\mathbf{Strat}_P \to \mathbf{sSet}$ *preserves trivial cofibrations.*

Proof. It is sufficient to prove that $\mathrm{Link}_{\mathcal{I}}$ sends strong anodyne extensions to strong anodyne extensions (see Section 3.3) because in both model structures, trivial cofibrations are given by retracts of strong anodyne extensions. The functor $\mathrm{Link}_{\mathcal{I}}$ is constructed by first applying a left adjoint (subdivision) and then pulling back to a vertex of $\mathrm{sd}(N(P))$. In particular, it is given by the composition of two functors preserving colimits and therefore preserves all colimits itself. Thus, it suffices to show that $\mathrm{Link}_{\mathcal{I}}$ sends admissible horn inclusions into (strong) anodyne extensions in **sSet**. Further, if we show the statement for admissible horn inclusions up to a certain dimension n, then it automatically follows that $\mathrm{Link}_{\mathcal{I}}$ sends all stratified strong anodyne extensions $A \to B$ with B of dimension lesser or equal to n into strong anodyne extensions.

We proceed via induction over n. The case $n = 0$ is trivial. So let $\mathcal{J}$ be a flag in P of length $n+1$ and $k \leq n+1$ such that $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$ is an admissible horn inclusion. We show that $\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k) \hookrightarrow \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$ is a strong anodyne extension. Let $\tau\colon \Delta^{\mathcal{J}} \to \Delta^{\mathcal{J}}$ be the maximal non-degenerate simplex of $\Delta^{\mathcal{J}}$ and $\tau_k\colon \Delta^{\mathcal{J}'} \hookrightarrow \Delta^{\mathcal{J}}$ be its k-th face. If $\mathcal{J}$ does not degenerate from $\mathcal{I}$, then by construction neither does $\mathcal{J}'$, since $\Lambda^{\mathcal{J}}_k$ is admissible. In particular, $\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$ does not contain the vertices corresponding to τ and τ_k. These are precisely the vertices of $\mathrm{sd}(\Delta^{\mathcal{J}})$ that are missing in $\mathrm{sd}(\Lambda^{\mathcal{J}}_k)$. Hence, $\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}})$ is a bijection and there is nothing to show.

Now, if $\mathcal{J}$ degenerates from $\mathcal{I}$, then so does $\mathcal{J}'$ and hence both τ_k and τ define vertices in $\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$. Let D be the full subcomplex of $\mathrm{sd}(\Delta^{\mathcal{J}})$ spanned by all vertices, but the one corresponding to τ_k, and let $D_{\mathcal{I}}$ be its intersection with $\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$. We obtain a pairing on the inclusion $A := D_{\mathcal{I}} \hookrightarrow \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}}) =: B$ by taking B_I to be the set of non-degenerate simplices in $B_{n.d.} \setminus A_{n.d.}$, of the form $[\sigma_0, ..., \sigma_{m-1}, \tau_k, \tau]$, and B_{II} the set of simplices of the form $[\sigma_0, \dots, \sigma_{m-1}, \tau_k]$, for $m \geq 0$. Then, the map sending $[\sigma_0, ..., \sigma_{m-1}, \tau_k]$ to $[\sigma_0, ..., \sigma_{m-1}, \tau_k, \tau]$ is a proper regular pairing $B_{II} \to B_I$. Thus, $D_{\mathcal{I}} \hookrightarrow \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$ is a SAE. Now, $D_{\mathcal{I}}$ is a cone on $\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k)$ with conepoint τ. Hence by a simplicial set version of [Whi39, Corollary

p. 249] it is enough to find a strong anodyne extension $\Delta^0 \to \mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k)$ to conclude that $\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k) \hookrightarrow D_{\mathcal{I}}$ is also a strong anodyne extension. Note that since $\Lambda^{\mathcal{J}}_k$ is an admissible horn and $\mathcal{J}$ degenerates from $\mathcal{I}$, any stratum-preserving inclusion $\Delta^{\mathcal{I}} \hookrightarrow \Lambda^{\mathcal{J}}_k$ is a strong anodyne extension. Passing to the links, we get a map $\Delta^0 = \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{I}}) \hookrightarrow \mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k)$, which is a strong anodyne extension by the inductive hypothesis. Finally, we have found two strong anodyne extensions $\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k) \to D_{\mathcal{I}}$ and $D_{\mathcal{I}} \to \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$, which compose to give the desired strong anodyne extension. □

3.4.4 Comparing homotopy links in $\mathbf{Top}_{N(P)}$ and $\mathbf{Strat}_P$

In this section, we prove the following theorem.

Theorem 3.4.4.1. *Let K be a stratified simplicial set, and $\mathcal{I}$ be some regular flag. The natural map*

$$HoLink_{\mathcal{I}}(|K|_{N(P)}) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$$

is a weak equivalence.

Given that the proof of the above theorem is fairly long and somewhat involved, we give a brief summary here. First, we show Lemmas 3.4.4.2 and 3.4.4.3, allowing us to make two simplifying assumptions: that $\mathcal{I} = P$, and that K is locally finite. Then, we show (in Lemma 3.4.4.5) that up to homotopy, the map $HoLink_{\mathcal{I}}(|K|_{N(P)}) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$ factors through $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}})$, where $|K|_P^{\mathrm{red}} \subset |K|_P$ is a subspace obtained by taking away some of the least singular simplices of K (see Definition 3.4.4.4 and Fig. 3.6). We then prove that the map $HoLink_{\mathcal{I}}(|K|_{N(P)}) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}})$ is a homotopy equivalence (Lemma 3.4.4.5). This simplifies the problem because now we need to compare two homotopy links in $\mathbf{Strat}_P$, $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}})$ and $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$. To compare these two, we will decompose the inclusion $|K|_P^{\mathrm{red}} \subset |K|_P$ into $|K|_P^{\mathrm{red},k} \subset \cdots \subset |K|_P^{\mathrm{red},0} = |K|_P$, see Definition 3.4.4.8, and prove that each of those induces a homotopy equivalence between homotopy links. Intuitively, homotopy inverses are obtained by sending a map $f\colon |\Delta^{\mathcal{I}}|_P \to |K|_P^{\mathrm{red},l}$ to some homotopic map $g\colon |\Delta^{\mathcal{I}}|_P \to |K|_P^{\mathrm{red},l+1} \subset |K|_P^{\mathrm{red},l}$, where the homotopy between f and g comes from precomposing f by some homotopy $H\colon |\Delta^{\mathcal{I}}|_P \times [0,1] \to |\Delta^{\mathcal{I}}|_P$. This homotopy should stratifiedly homotope the identity on $|\Delta^{\mathcal{I}}|_P$ into a smaller space, which f maps into $|K|_P^{\mathrm{red},l+1}$. As one might expect, there exists no single homotopy H that works for arbitrary $f \in \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$. Instead, we first define some homotopy $S^l\colon |\Delta^{\mathcal{I}}|_P \times [0,1] \to |\Delta^{\mathcal{I}}|_P$ (Definition 3.4.4.10), as well as a parameter $\alpha\colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l}) \times [0,1] \to [0,1]$ (Lemma 3.4.4.13). For any given f, the homotopy we are looking for is then obtained by combining the homotopy S^l and the parameter α. Note that a partition of unity in $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$ appears in the construction of α. Such a partition of unity exists thanks to the assumption that K is locally finite, see Remark 3.2.2.5.

Before moving on to the proof, we will need the following simplifying assumptions:

As.(1) K is a locally finite simplicial set. This ensures that

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P) = \mathbf{C}^0_P(|\Delta^{\mathcal{I}}|_P, |K|_P)$$

is metrizable (when equipped with the compact open topology, see Remark 3.2.2.5). Lemma 3.4.4.2 implies that we can assume that K is locally finite without loss of generality.

As.(2) $P = \mathcal{I}$. Lemma 3.4.4.3 implies that we can assume this is true without loss of generality.

Lemma 3.4.4.2. *Let K be a stratified simplicial set, and let A be the category of finite stratified simplicial subsets, $K^\alpha \subset K$, and inclusions. Then, for all $n \geq 0$ and all pointings, the* *following natural maps are isomorphisms:*

$$\varinjlim_{\alpha\in A} \pi_n(Ho\mathrm{Link}_{\mathcal{I}}(|K^\alpha|_{N(P)}),*) \to \pi_n(Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}),*),$$
$$\varinjlim_{\alpha\in A} \pi_n(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K^\alpha|_{P}),*) \to \pi_n(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_{P}),*).$$

Proof. Through adjunction, one sees that an element in the homotopy group $\pi_n(Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}),x)$ is nothing more than a map $S^n\times|\Delta^{\mathcal{I}}|_{N(P)}\to|K|_{N(P)}$, in $\mathbf{Top}_{N(P)}$, sending $*\times|\Delta^{\mathcal{I}}|_{N(P)}$ to x. In particular, such a map only reaches a finite simplicial subset of K, which implies the surjectivity of the map under study. The same observation for homotopies implies the injectivity. □

Lemma 3.4.4.3. *Let K be a stratified simplicial set. Let $\widetilde{K}$ be the simplicial subset defined as the following pullback:*

$$\begin{tikzcd} \widetilde{K} \arrow[r] \arrow[d] & K \arrow[d, "\varphi_K"] \\ \Delta^{\mathcal{I}} \arrow[r] & N(P)\,. \end{tikzcd}$$

The inclusion $\widetilde{K}\to K$ induces weak equivalences

$$Ho\mathrm{Link}_{\mathcal{I}}(|\widetilde{K}|_{N(P)}) \to Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}),$$
$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|\widetilde{K}|_{P}) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_{P}).$$

Proof. The first map is even an isomorphism. Indeed, Let $f\colon|\Delta^{\mathcal{I}}|_{N(P)}\to|K|_{N(P)}$ be a map in $\mathbf{Top}_{N(P)}$. Since geometric realization preserves pullbacks, such a map must factor through $|\widetilde{K}|_{N(P)}$.

For the second map, consider the following subspace of $|K|_P$:

$$Z=\{x\in|K|_P \mid \varphi_P\circ|\varphi_K|(x)\in\mathcal{I}\}$$

By construction, one has $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Z)\cong\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$, so it is enough to show that the inclusion $\iota\colon|\widetilde{K}|_P\to Z$ is a stratified homotopy equivalence. Now, recall that the points of $|K|_P$ can be described by pairs $(\sigma\colon\Delta^{\mathcal{J}}\to K,\xi=(\xi_0,\dots,\xi_n))$, where $\mathcal{J}=[p_0\le\dots\le p_n]$ is some flag. With those notations, one can give an alternative description of Z as

$$Z=\{(\sigma,\xi)\in|K|_P \mid p_{m_\xi}\in\mathcal{I}\},$$

where $m_\xi=\max\{i\mid\xi_i\neq0\}$. Given a point (σ,ξ) in Z, let $\xi_{\mathcal{I}}=\sum_{p_i\in\mathcal{I}}\xi_i$, and let ϵ_i be 0 if $p_i\notin\mathcal{I}$ and 1 if $p_i\in\mathcal{I}$. With this notation, we can define a stratified retract, $r\colon Z\to\widetilde{K}$, at the level of simplices :

$$r_\sigma\colon Z\cap\sigma(\Delta^{\mathcal{J}})\to|\widetilde{K}|\cap\sigma(\Delta^{\mathcal{J}})$$
$$(\xi_0,\dots,\xi_n)\mapsto\frac{1}{\xi_{\mathcal{I}}}(\epsilon_0\xi_0,\dots,\epsilon_n\xi_n)$$

One checks easily that those maps are compatible with faces and degeneracies, and produce a global retract $r\colon Z\to\widetilde{K}$, satisfying $r\circ i=1_{\widetilde{K}}$. Furthermore, if $(\xi_0,\dots,\xi_n)\in Z\cap\sigma(\Delta^{\mathcal{J}})$, then $p_{m_\xi}\in\mathcal{I}$ and $\epsilon_{m_\xi}=1$. This implies that

$$p_{m_\xi}=\varphi_P\circ|\varphi_K|(\xi_0,\dots,\xi_n)=\varphi_P\circ|\varphi_K|(r(\xi_0,\dots,\xi_n))=p_{m_\xi},$$

and so r is a stratified map. Finally, the straight-line homotopies between $\iota_\sigma\circ r_\sigma$ and $1_{Z\cap\sigma(\Delta^{\mathcal{J}})}$ assemble to produce a stratified homotopy between $\iota\circ r$ and 1_Z □

In the remainder of the subsection, we use the following notations: $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ is a flag, and $\mathcal{I} = [q_0 < \cdots < q_k]$ is the regular flag such that $\mathcal{I} = \{p_0 \leq \cdots \leq p_n\}$. In other words, $\Delta^{\mathcal{J}}$ is degenerated from $\Delta^{\mathcal{I}}$. Furthermore, if $p \in P$, let $\mathcal{J}_p = \{i \mid 0 \leq i \leq n,\ p_i = p\}$. Then, for a point $(\xi_0, \dots, \xi_n) \in |\Delta^{\mathcal{J}}|$, let $\xi_p = \sum_{i \in \mathcal{J}_p} \xi_i$. Note that by construction, the stratification $|\Delta^{\mathcal{J}}| \to |N(P)|$ is given by the map $(\xi_0, \dots, \xi_n) \mapsto (\xi_{q_0}, \dots, \xi_{q_k})$.

Definition 3.4.4.4. Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set. We define the subspace $|K|_P^{\mathrm{red}} \subset |K|_P$ as follows:

$$|K|_P^{\mathrm{red}} = \{(\sigma, \xi) \mid \xi_p = 0 \Rightarrow \xi_{p'} = 0\ \ \forall p' \geq p\}$$

Equivalently, $|K|_P^{\mathrm{red}}$ is defined as the following pullback

$$\begin{tikzcd} |K|_P^{\mathrm{red}} \arrow[r, hook] \arrow[d] & |K|_P \arrow[d, "|\varphi_K|"] \\ |\Delta^{\mathcal{I}}|_P^{\mathrm{red}} \arrow[r, hook] & |N(P)|_P . \end{tikzcd}$$

Figure 3.6: The stratified space $|\Delta^{\mathcal{I}}|_P$ and the subspace $|\Delta^{\mathcal{I}}|_P^{\mathrm{red}}$, with $\mathcal{I} = [p_0 < p_1 < p_2]$. The dashed blue lines and the green circle indicate the missing faces.

Lemma 3.4.4.5. *Let K be a stratified simplicial set. There exists a homotopy equivalence, $\lambda^*\colon Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{red})$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd} Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \arrow[rr] \arrow[dr, dashed, "\lambda^*"] & & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P) \\ & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{red}) \arrow[ur, hook] & \end{tikzcd} \tag{3.13}$$

Proof. Let $\mathcal{I} = [q_0 < \cdots < q_k]$ be a regular flag, and consider the linear map $\lambda\colon |\Delta^{\mathcal{I}}| \to |\Delta^{\mathcal{I}}|$ sending the vertex q_i to the barycenter of $\Delta^{[q_0 < \cdots < q_i]} \subset \Delta^{\mathcal{I}}$. We will show that the map

$$\begin{aligned} \lambda^*\colon Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}}) \\ f &\mapsto \Big\{x \mapsto (f \circ \lambda)(x)\Big\} \end{aligned}$$

has the desired properties. First, note that the map $\lambda\colon |\Delta^{\mathcal{I}}|_P \to |\Delta^{\mathcal{I}}|_P$ is a stratum-preserving map, which means that for any $f \in Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$, $f \circ \lambda \in \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$. Furthermore, by construction, $\lambda(|\Delta^{\mathcal{I}}|) \subset |\Delta^{\mathcal{I}}|^{\mathrm{red}}$. In particular, λ^* takes values in $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}})$, and it is well-defined.

Now, let $H\colon |\Delta^{\mathcal{I}}|_P \times [0,1] \to |\Delta^{\mathcal{I}}|_P$ be the straight-line homotopy between λ and $1_{|\Delta^{\mathcal{I}}|}$. Then the map

$$\begin{aligned} H^*\colon Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \times [0,1] &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P) \\ (f, s) &\mapsto \Big\{\xi \mapsto f(H(\xi, s))\Big\} \end{aligned}$$

gives the homotopy between the two paths in Diagram (3.13).

It remains to be shown that λ^* is a homotopy equivalence. Consider the map $\rho\colon |K|_P^{\mathrm{red}} \times_P |N(P)| \to |K|_{N(P)}$ from Lemma 3.4.4.6. Using the natural isomorphism $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}}) \simeq Ho\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}} \times_P |N(P)|)$, ρ induces a map

$$\rho_*\colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}}) \cong Ho\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}} \times_P |N(P)|) \to Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$$

We will show that ρ_* is a homotopy inverse to λ^*. We first consider the composition $\rho_* \circ \lambda^*$. Consider the homotopy H^* defined above. For $f \in Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$, $H^*(f,s)$ lands in $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}}) \cong Ho\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}} \times_P |N(P)|)$ if $s < 1$ and lands in $Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})$ for $s = 1$. Let $\widehat{|K|}$ be the strongly stratified space defined as the following union:

$$\widehat{|K|} = |K|_P^{\mathrm{red}} \times_P |N(P)| \cup |K|_{N(P)} \times_{|N(P)|} |N(P)| \subset |K| \times_P |N(P)| \subset |K| \times |N(P)|$$

where the stratification is given by projecting on the second factor. The homotopy H^* factors through a map

$$H^*\colon Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \times [0,1] \to Ho\mathrm{Link}_{\mathcal{I}}(\widehat{|K|}).$$

By Lemma 3.4.4.6, ρ extends to $\widehat{|K|}$ as the identity on $|K|_{N(P)}$. In particular, the composition $\rho_* \circ H^*$ gives a homotopy between $\rho_* \circ \lambda^*$ and 1.

For the other composition, $\lambda^* \circ \rho_*$, consider the homotopy in $\mathbf{Strat}_P$, from Lemma 3.4.4.6

$$R\colon \left(|K|_P^{\mathrm{red}} \times_P |N(P)|\right) \times [0,1] \to |K|_P.$$

Given a map in $\mathbf{Top}_{N(P)}$, $f\colon |\Delta^{\mathcal{I}}|_{N(P)} \to |K|_P^{\mathrm{red}} \times_P |N(P)|$, and $s \in [0,1]$, consider the composition

$$|\Delta^{\mathcal{I}}|_P \xrightarrow{H_s} |\Delta^{\mathcal{I}}|_P \xrightarrow{f} |K|_P^{\mathrm{red}} \times_P |N(P)| \xrightarrow{R_s} |K|_P$$

If $s > 0$, R_s takes value in $|K|_P^{\mathrm{red}}$. On the other hand, if $s = 0$, $R_0 = \rho$, $H_0 = \lambda$, and $f \circ \lambda$ takes value in $|K|_P^{\mathrm{red}} \times_P |\Delta^{\mathcal{I}}|_P^{\mathrm{red}}$. By Remark 3.4.4.7, this implies that $R_0 \circ f \circ H_0$ takes value in $|K|_P^{\mathrm{red}}$. Now consider the following homotopy:

$$\begin{aligned} G\colon Ho\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}} \times_P |N(P)|) \times [0,1] &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}}) \\ (f,s) &\mapsto R_s \circ f \circ H_s \end{aligned}$$

At $s = 1$, $H_1 = 1$, and R_1 is the projection to $|K|_P^{\mathrm{red}}$. In particular, G_1 is the natural homeomorphism $Ho\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}} \times_P |N(P)|) \simeq \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}})$. At $s = 0$, G_0 is $\lambda^* \circ \rho_*$, precomposed with the aforementioned natural homeomorphism. In particular, λ^* is a homotopy equivalence. $\square$

Lemma 3.4.4.6. *There exists a map in* $\mathbf{Top}_{N(P)}$

$$\rho\colon |K|_P^{red} \times_P |N(P)| \to |K|_{N(P)},$$

which extends continuously to $|K|_{N(P)} \times_{|N(P)|} |N(P)|$ *as* $1_{|K|}$*, where the stratification on the domain is given by the projection on the second factor. Furthermore, there exists a homotopy in* $\mathbf{Strat}_P$

$$R\colon \left(|K|_P^{red} \times_P |N(P)|\right) \times [0,1] \to |K|_P$$

between ρ *and the projection to* $|K|_P^{red}$*, such that for all* $s > 0$*,* R_s *takes value in* $|K|_P^{red}$*.*

Remark 3.4.4.7. Note that, since the map $\rho\colon |K|_P^{\mathrm{red}} \times_P |N(P)| \to |K|_{N(P)}$ from Lemma 3.4.4.6 is a strongly stratified map, its restriction to $|K|_P^{\mathrm{red}} \times_P |\Delta^{\mathcal{I}}|_P^{\mathrm{red}}$ must take value in $|K|_P^{\mathrm{red}} = |\varphi_K|^{-1}(|\Delta^{\mathcal{I}}|_P^{\mathrm{red}})$.

Proof of Lemma 3.4.4.6. Recall that if $(\xi_0,\dots,\xi_n)\in|\Delta^{\mathcal{J}}|$, with $\mathcal{J}=[p_0\leq\dots\leq p_n]$, and $p\in P$, then $\xi_p=\sum_{p_i=p}\xi_i$. Now, write $\mathcal{I}=[q_0<\dots<q_k]$. By As.(2), $P=\mathcal{I}$, which means that for all $0\leq i\leq n$ there exists $0\leq j\leq k$ such that $p_i=q_j$. Furthermore, we will label the points in $|N(P)|=|\Delta^{\mathcal{I}}|$ as $(t_{q_0},\dots,t_{q_k})$. We will construct ρ on simplices, let $\sigma\colon\Delta^{\mathcal{J}}\to K$ be a stratified simplex. Define $\rho_{\mathcal{J}}$ as :

$$\begin{aligned}\rho_{\mathcal{J}}\colon|\Delta^{\mathcal{J}}|_P^{\mathrm{red}}\times_P|N(P)|&\to|\Delta^{\mathcal{J}}|_{N(P)}\\ \big((\xi_0,\dots,\xi_n),(t_{q_0},\dots,t_{q_k})\big)&\mapsto(\alpha_0\xi_0,\dots,\alpha_n\xi_n)\end{aligned}$$

where $\alpha_i=\frac{t_{p_i}}{\xi_{p_i}}$, if $\xi_{p_i}\neq0$, and $\alpha_i=0$ if $\xi_{p_i}=0$. Now, if $\xi_q=0$ for some $q\in P$, since $(\xi_0,\dots,\xi_n)$ is in $|\sigma(\Delta^{\mathcal{J}})|_P^{\mathrm{red}}$, this implies that $\xi_{q'}=0$, for $q'\geq q$. In turn, $\varphi_P(t_{q_0},\dots,t_{q_k})=\varphi_P\circ|\varphi_K|(\xi_0,\dots,\xi_n)<q$. In particular, in this case, one must have $t_{q'}=0$ for $q'\geq q$, and $\sum\alpha_i\xi_i=\sum t_{q_j}=1$, which implies that $\rho_{\mathcal{J}}$ is well defined. If one sets $(\xi'_0,\dots,\xi'_n)=(\alpha_0\xi_0,\dots,\alpha_n\xi_n)$, one has $\xi'_q=t_q$ for all $q\in P$. In particular, $\rho_{\mathcal{J}}$ is a map in $\mathbf{Top}_{N(P)}$. Furthermore, if we restrict $\rho_{\mathcal{J}}$ to pairs $((\xi_0,\dots,\xi_n),(t_{q_0},\dots,t_{q_k}))$ such that $|\varphi_K|(\xi_0,\dots,\xi_n)=(t_{q_0},\dots,t_{q_k})$, then, for all $0\leq i\leq n$, either $\xi_{p_i}\neq0$ and $\alpha_i=1$, or $\xi_{p_i}=0$, and in both cases, $\alpha_i\xi_i=\xi_i$. Using this $\rho_{\mathcal{J}}$ can be extended to $|\Delta^{\mathcal{J}}|_{N(P)}\times_{|N(P)|}|N(P)|$ by the identity (or, more precisely, the projection on the first factor). Assembling all the $\rho_{\mathcal{J}}$ together gives the desired map.

Now, to produce the homotopy R, consider the following straight-line homotopy defined on simplices, where the α_i are defined as above:

$$\begin{aligned}R_{\mathcal{J}}\colon\Big(|\Delta^{\mathcal{J}}|_P^{\mathrm{red}}\times_P|N(P)|\Big)\times[0,1]&\to|\Delta^{\mathcal{J}}|_P\\ ((\xi_0,\dots,\xi_n),(t_{q_0},\dots,t_{q_k}),s)&\mapsto((s+(1-s)\alpha_0)\xi_0,\dots,(s+(1-s)\alpha_n)\xi_n)\end{aligned}$$

To see that $R_{\mathcal{J}}$ is a stratum-preserving map, notice that if $s=0$, $R_{\mathcal{J}}$ is just $\rho_{\mathcal{J}}$ which is a map in $\mathbf{Top}_{N(P)}$. If $s>0$, $(s+(1-s)\alpha_i)\neq0$ for all $0\leq i\leq n$ and so $(s+(1-s)\alpha_i)\xi_i=0\Leftrightarrow\xi_i=0$ for all $0\leq i\leq n$. This implies two things. First, $\varphi_P\circ|\varphi_K|(\xi_0,\dots,\xi_n)=\varphi_P\circ|\varphi_K|((s+(1-s)\alpha_0)\xi_0,\dots,(s+(1-s)\alpha_n)\xi_n)$, and so $R_{\mathcal{J}}$ is a stratified homotopy, and second, $R_{\mathcal{J}}$ takes value in $|\Delta^{\mathcal{J}}|_P^{\mathrm{red}}$ for all $s>0$. Finally, notice that for $s=1$, $R_{\mathcal{J}}$ is the projection on $|\Delta^{\mathcal{J}}|_P$, and so $R_{\mathcal{J}}$ is a homotopy between $\rho_{\mathcal{J}}$ and this projection. Assembling the $R_{\mathcal{J}}$ gives the desired homotopy. □

In order to prove Theorem 3.4.4.1, it remains to be shown that the map induced by the inclusion $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red}})\to\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$ is a weak equivalence. We will do so by further decomposition.

Definition 3.4.4.8. Write $\mathcal{I}=[q_0<\dots<q_k]$, and let $0\leq l\leq k$. Define the following subspace of $|K|_P$:

$$|K|_P^{\mathrm{red},l}=\{(\sigma,\xi)\mid\xi_p=0\Rightarrow\xi_{p'}=0,\ \forall p'\geq p,\ \forall p<q_l\}.$$

We then have a sequence of inclusions

$$|K|_P^{\mathrm{red}}=|K|_P^{\mathrm{red},k}\subset\dots\subset|K|_P^{\mathrm{red},0}=|K|_P.$$

Theorem 3.4.4.1 then follows from the following lemma.

Lemma 3.4.4.9. *Let $0\leq l\leq k-1$, the inclusion $|K|_P^{red,l+1}\to|K|_P^{red,l}$ induces a homotopy equivalence*

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{red,l+1})\to\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{red,l})$$

In order to prove Lemma 3.4.4.9, we will need a few more technical results.

Definition 3.4.4.10. Let $0\leq l\leq k-1$, define the map S^l as

$$\begin{aligned}S^l\colon|\Delta^{\mathcal{I}}|\times[0,1]&\to|\Delta^{\mathcal{I}}|\\ ((t_{q_0},\dots,t_{q_k}),s)&\mapsto(t_{q_0},\dots,t_{q_{l-1}},t_{q_l}+(1-s)\sum_{j=l+1}^{k}t_{q_j},st_{q_{l+1}},\dots,st_{q_k}).\end{aligned}$$

Lemma 3.4.4.11. *The map S^l is stratum-preserving outside of $s = 0$, and it restricts to the projection on the first factor on the subspace*

$$|\Delta^{[q_0<\dots<q_l]}| = \{(t_{q_0}, \dots, t_{q_k}) \mid t_q = 0 \ \forall q > q_l\}.$$

Proof. Let $t = (t_{q_0}, \dots, t_{q_k})$, and $s \in [0,1]$, and write $t' = S^l(t,s)$. If $\varphi_P(t) = q \leq q_l$, then $t_{q'} = t'_{q'} = 0$ for all $q' > q$, and $S^l(t,s) = t$ for all s, this addresses the second part of the lemma. If $\varphi_P(t) = q > q_l$, then for $s > 0$, one has $t'_q = st_q > 0$, and $t'_{q'} = 0$ for all $q' > q$. In particular, $\varphi_P(t') = q$, and $S^l(-,s)$ is a stratum-preserving map. □

Lemma 3.4.4.12. *For all $0 \leq l \leq k-1$, $|K|_P^{red,l+1}$ is a neighborhood of the q_l-stratum of $|K|_P^{red,l}$*

Proof. Note that the subspace $|K|_P^{\mathrm{red},l}$ can be defined as the following pullback square:

$$\begin{tikzcd}
|K|_P^{\mathrm{red},l} \arrow[r, hook] \arrow[d] & |K|_P \arrow[d, "|\varphi_K|_P"] \\
|\Delta^{\mathcal{I}}|_P^{\mathrm{red},l} \arrow[r] & |\Delta^{\mathcal{I}}|_P
\end{tikzcd}$$

Now since $|\varphi_K|_P$ is a continuous, stratum-preserving map, it is enough to show that $|\Delta^{\mathcal{I}}|_P^{\mathrm{red},l+1}$ is a neighborhood of the q_l-stratum of $|\Delta^{\mathcal{I}}|_P^{\mathrm{red},l}$. Now, a point in the q_l-stratum of $|\Delta^{\mathcal{I}}|_P^{\mathrm{red},l}$ is of the form $(t_{q_0}, \dots, t_{q_k})$, with $t_{q_l} \neq 0$ and $t_q = 0$ for all $q > q_l$. Furthermore, the defining condition of $|\Delta^{\mathcal{I}}|_P^{\mathrm{red},l}$ implies that $t_q \neq 0$ for all $q < q_l$. This means that there exists a neighborhood of t, $U \subset |\Delta^{\mathcal{I}}|$, such that for any $t' \in U$, $q \leq q_l \Rightarrow t'_q \neq 0$. But this means that all points in U satisfy the defining condition of $|\Delta^{\mathcal{I}}|_P^{\mathrm{red},l+1}$. In particular, $U \subset |\Delta^{\mathcal{I}}|_P^{\mathrm{red},l+1}$, and the latter is a neighborhood of the q_l-stratum of $|\Delta^{\mathcal{I}}|_P^{\mathrm{red},l}$. □

Lemma 3.4.4.13. *There exists a continuous map $\alpha: \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{red,l}) \times [0,1] \to [0,1]$, such that*

1. $\alpha(f,s) = 0 \Rightarrow s = 0$,
2. $f(S^l(t, \alpha(f, \sum_{j=l+1}^k t_{q_j}))) \in |K|_P^{red,l+1}$, *for all* $t = (t_{q_0}, \dots, t_{q_k}) \in |\Delta^{\mathcal{I}}|$ *and for all* $f \in \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{red,l})$.

Proof. Let $\epsilon > 0$, and define $|\Delta^{\mathcal{I}}|^{\epsilon,l}$ as the following subset of $|\Delta^{\mathcal{I}}|$:

$$|\Delta^{\mathcal{I}}|^{\epsilon,l} = \{t \in |\Delta^{\mathcal{I}}| \mid \sum_{i=l}^k t_{q_i} \geq \epsilon\}$$

Now, pick some stratified map $f: |\Delta^{\mathcal{I}}|_P \to |K|_P^{\mathrm{red},l}$, and consider the intersection of the following nested family of compact subsets:

$$\bigcap_{\alpha>0} f(S^l(|\Delta^{\mathcal{I}}|^{\epsilon,l} \times [0,\alpha])) = f(S^l(|\Delta^{\mathcal{I}}|^{\epsilon,l} \times \{0\}))$$

Note that if $t \in |\Delta^{\mathcal{I}}|^{\epsilon,l}$, $S^l(t,0)$ is in the q_l-stratum of $|\Delta^{\mathcal{I}}|_P$. In particular, $f(S^l(|\Delta^{\mathcal{I}}|^{\epsilon,l} \times \{0\}))$ is a compact subset of the q_l-stratum of $|K|_P^{\mathrm{red},l}$. Since, by Lemma 3.4.4.12, $|K|_P^{\mathrm{red},l+1}$ is a neighborhood of the q_l-stratum of $|K|_P^{\mathrm{red},l}$, there must exist $\alpha^\epsilon > 0$ such that $f(S^l(|\Delta^{\mathcal{I}}|^{\epsilon,l} \times [0,\alpha^\epsilon])) \subset |K|_P^{\mathrm{red},l+1}$. By the definition of the compact-open topology, this also holds for any g in a neighborhood U of f in $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$. Hence, we can cover $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$ by a family of opens U_i such that there exists $\alpha_i^\epsilon > 0$ satisfying $g \in U_i \Rightarrow g(S^l(|\Delta^{\mathcal{I}}|^{\epsilon,l} \times [0,\alpha_i^\epsilon])) \subset |K|_P^{\mathrm{red},l+1}$. By Lemma 3.4.4.2, we may assume that K is locally finite. But this implies that $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$ is metrizable, and hence, paracompact (see also Remark 3.2.2.5). In particular, one can find a partition of unity (ϕ_i) subordinated to the open cover (U_i). We can now define the following continuous map:

$$\begin{aligned}\alpha^\epsilon \colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l}) &\to [0,1] \\ f &\mapsto \sum_i \phi_i(f)\alpha_i^\epsilon\end{aligned}$$

Note that by construction, for all $f \in \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$, we have $\alpha^\epsilon(f) \leq \alpha_i^\epsilon$, for some i such that $f \in U_i$. In particular, we have

$$f(S^l(|\Delta^{\mathcal{I}}|^{\epsilon,l} \times [0, \alpha^\epsilon(f)])) \subset |K|_P^{\mathrm{red},l+1} \tag{3.14}$$

Now, to construct α, we will use the family of functions α^ϵ, for $\epsilon = \frac{1}{2^n}$, $n \geq 1$. Consider the covering of $(0,1]$ given by the family $I_n = (\frac{1}{2^{n+1}}, \frac{1}{2^{n-2}})$, $n \geq 1$, and the family of closed intervals $J_n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$, $n \geq 1$. Pick a family of bump functions on $[0,1]$, ψ_n, $n \geq 1$ satisfying :

- $\psi_n(s) \in [0,1]$, for all $s \in (0,1]$,
- $\psi_n(s) = 1$ if $s \in J_n$,
- $\psi_n(s) = 0$ if $s \notin I_n$,

and define the continuous map:

$$\begin{aligned}\alpha \colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l}) \times [0,1] &\to [0,1] \\ (f,s) &\mapsto s\prod_{n\geq 1}\left(1 - \psi_n(s)(1 - \alpha^{\frac{1}{2^n}}(f))\right)\end{aligned}$$

Note that for $s \in [0,1]$, there is only a finite number of $n \geq 1$ such that $\psi_n(s) \neq 0$, which means that the above product only has a finite amount of non-trivial terms. We need to check that it satisfies both conditions of Lemma 3.4.4.13. Let $f \in \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$. The first part is clear, since $\alpha(f,s) = 0$ implies that either $s = 0$, or that the product is 0, which is not possible, since it only has a finite number of non-trivial terms which are all non-zero. For the second part, first note that if $s \in J_n$

$$\alpha(f,s) \leq \alpha^{\frac{1}{2^n}}(f) \tag{3.15}$$

Now, if $t \in |\Delta^{\mathcal{I}}|$, then

- either $\sum_{j=l+1}^k t_{q_j} = 0$, but then $t \in |\Delta^{q_0 < \dots < q_l}|$, and by Lemma 3.4.4.11, in this case $S^l(t,u) = t$ for any $u \in [0,1]$. In particular, the expression in Lemma 3.4.4.13 reduces to $f(t)$. Now, $f(t)$ is of the form $(\sigma, (\xi_0, \dots, \xi_n))$, but since f is stratum-preserving, it must satisfy $\xi_p = 0$ for all $p > q_l$. Additionally, since f takes value in $|K|_P^{\mathrm{red},l}$, we have $\xi_p = 0 \Rightarrow \xi_{p'} = 0$ for $p' \geq p$ and $p < q_l$. Now, since we already now that $\xi_{p'} = 0$ for $p' > q_l$, we trivially have the implication $\xi_{q_l} = 0 \Rightarrow \xi_{p'} = 0$ for $p' \geq q_l$. In particular $f(t) \in |K|_P^{\mathrm{red},l+1}$.

- or $\sum_{j=l+1}^k t_{q_j} > 0$. Then let n be such that $\sum_{j=l+1}^k t_{q_j} \in J_n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$. By definition, $t \in |\Delta^{\mathcal{I}}|^{\epsilon,l}$, for $\epsilon = \frac{1}{2^n}$. Combining equations (3.15) and (3.14), we get :

$$f(S^l(t, \alpha(f, \sum_{j=l+1}^k t_{q_j}))) \in f(S^l(|\Delta^{\mathcal{I}}|^{\epsilon,l} \times [0, \alpha^\epsilon(f)])) \subset |K|_P^{\mathrm{red},l}$$

which concludes the proof.

□

Proof of Lemma 3.4.4.9. Using the map $\alpha\colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})\times[0,1]\to[0,1]$ from Lemma 3.4.4.13, we define the following homotopy:

$$
\begin{aligned}
H\colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l}) \times [0,1] &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l}) \\
(f,s) &\mapsto \left\{ \begin{array}{cl} |\Delta^{\mathcal{I}}|_P & \to |K|_P^{\mathrm{red},l} \\ t=(t_{q_0},\dots,t_{q_k}) & \mapsto f(S^l(t,(1-s)+s\alpha(f,\sum_{j=l+1}^k t_{q_j}))) \end{array}\right.
\end{aligned}
$$

We first check that $H(f,s)$ is a stratum-preserving map. By Lemma 3.4.4.11, $f(S^l(t,u))$ has the correct stratification, except maybe when $u=0$. In the latter case, one must have $s=1$ and $\alpha(f,\sum_{j=l+1}^k t_{q_j})=0$, but by Lemma 3.4.4.13, this is only possible when $\sum_{j=l+1}^k t_{q_j}=0$. This corresponds to $t\in|\Delta^{[q_0<\dots<q_l]}|$, and by Lemma 3.4.4.11, in this case one has $S^l(t,u)=t$ for all values of u. We conclude that $H(f,s)$ is indeed a stratum-preserving map. Furthermore, its image lies in $|K|_P^{\mathrm{red},l}$, by construction, which means that H is well-defined. But now, H_0 is the identity map, since $S^l(t,1)=t$ for all $t\in|\Delta^{\mathcal{I}}|$. By the second part of Lemma 3.4.4.13, for all $f\in\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$, the image of $H(f,1)\colon\Delta^{\mathcal{I}}\to|K|_P^{\mathrm{red},l}$ lies in $|K|_P^{\mathrm{red},l+1}$. This implies that H_1 lands in $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l+1})$. On the other hand, if $f\in\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l+1})$, then $H(f,s)$ lies in $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l+1})$ for all $s\in[0,1]$. This shows that the inclusion $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l+1})\hookrightarrow\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P^{\mathrm{red},l})$ defines a homotopy equivalence (with inverse induced by H_1), which concludes the proof. ☐

3.4.5 Realizations characterize weak equivalences

We summarize the results of the previous subsections in the following theorem.

Theorem 3.4.5.1. *Let K be a stratified simplicial set, $\mathcal{I}$ a regular flag and b a point in the interior of $|\Delta^{\mathcal{I}}|$. Then all maps in the following diagram are weak equivalences:*

$$
\begin{tikzcd}
Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)}) \arrow[r] \arrow[dr] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P) \\
|\mathrm{Link}_{\mathcal{I}}(K)| \arrow[r] & |K|_b
\end{tikzcd}
$$

Corollary 3.4.5.2. *Let $f\colon K\to L$ be in $\mathrm{s}\mathbf{Strat}_P$. The following assertions are equivalent:*

- *f is a weak equivalence in $\mathrm{s}\mathbf{Strat}_P$,*
- *$|f|_{N(P)}\colon|K|_{N(P)}\to|L|_{N(P)}$ is a weak equivalence in $\mathbf{Top}_{N(P)}$,*
- *$|f|_P\colon|K|_P\to|L|_P$ is a weak equivalence in $\mathbf{Strat}_P$.*

Furthermore, if $g\colon|K|_{N(P)}\to|L|_{N(P)}$ is a map in $\mathbf{Top}_{N(P)}$, then it is a weak equivalence if and only if its image by the functor $\varphi_P\circ-\colon\mathbf{Top}_{N(P)}\to\mathbf{Strat}_P$ is a weak equivalence in $\mathbf{Strat}_P$.

Proof. Let us first prove the second part. Let $g\colon|K|_{N(P)}\to|L|_{N(P)}$ be a map in $\mathbf{Top}_{N(P)}$. It is a weak equivalence if and only if, for all regular flags $\mathcal{I}$, the maps induced by g,

$$Ho\mathrm{Link}_{\mathcal{I}}(|K|_{N(P)})\to Ho\mathrm{Link}_{\mathcal{I}}(|L|_{N(P)})$$

are weak equivalences. But, by Theorem 3.4.4.1, this is equivalent to asking that the following maps are weak equivalences:

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)\to\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|L|_P)$$

This, in turn, is equivalent to $(\varphi_P\circ-)(g)$ being a weak equivalence in $\mathbf{Strat}_P$.

For the first part of the lemma, we know from [Dou21b, Theorem 1.15] that the functor $|\mathrm{sd}_P(-)|_P\colon\mathrm{s}\mathbf{Strat}_P\to\mathbf{Top}_P$ is the left part of a Quillen equivalence. In particular, it characterizes weak equivalences between cofibrant objects. Since all objects of $\mathrm{s}\mathbf{Strat}_P$ are cofibrant, this implies that f is a weak equivalence if and only if $|\mathrm{sd}_P(f)|_P$ is a weak equivalence. Now consider the following commutative diagram:

$$\begin{tikzcd}
{|\mathrm{sd}_P(K)|_{N(P)}} \arrow[r, "{|\mathrm{sd}_P(f)|_{N(P)}}"] \arrow[d, "{|\mathrm{l.v.}_P|_{N(P)}}"'] & {|\mathrm{sd}_P(L)|_{N(P)}} \arrow[d, "{|\mathrm{l.v.}_P|_{N(P)}}"] \\
{|K|_{N(P)}} \arrow[r, "{|f|_{N(P)}}"] & {|L|_{N(P)}}
\end{tikzcd}$$

By [Dou21a, Lemma A.3], we know that $\mathrm{l.v.}_P\colon \mathrm{sd}_P(K) \to K$ is a weak equivalence for all $K \in \mathrm{s}\mathbf{Strat}_P$, and by Proposition 3.4.3.1, we also know that $|-|_{N(P)}$ preserves weak equivalences. This implies that the vertical arrows in the previous diagram are weak equivalences. By two out of three, this means that f is a weak equivalence if and only if $|f|_{N(P)}$ is a weak equivalence, which concludes the proof. ☐

Corollary 3.4.5.3. *The functor $|C_P(-)|_P\colon \mathbf{Diag}_P \to \mathbf{Strat}_P$ characterizes all weak equivalences.*

Proof. A map in $\mathbf{Diag}_P$, $f\colon F \to G$ is a weak equivalence if and only if, for all regular flags $\mathcal{I}$, $F(\mathcal{I}) \to G(\mathcal{I})$ is a weak equivalence. On the other hand, the map $|C_P(f)|_P\colon |C_P(F)|_P \to |C_P(G)|_P$ is a weak equivalence in $\mathbf{Strat}_P$ if and only if the map $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|C_P(F)|_P) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|C_P(G)|_P)$ is a weak equivalence for all $\mathcal{I}$. In particular, it is enough to show that for any $F \in \mathbf{Diag}_P$, the natural map $F(\mathcal{I}) \to \mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|C_P(F)|_P))$ is a weak equivalence for all regular flags $\mathcal{I}$. Consider the following commutative diagram, where b is the barycenter of $|\Delta^{\mathcal{I}}|$:

$$\begin{tikzcd}
& & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|C_P(F)|_P) \\
{|F(\mathcal{I})|} \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & Ho\mathrm{Link}_{\mathcal{I}}(|C_P(F)|_{N(P)}) \arrow[ur] \arrow[dr] & \\
& & {|C_P(F)|_b\,.}
\end{tikzcd}$$

We need to prove that the top map is a weak equivalence, but by Theorem 3.4.5.1, it is enough to show that the bottom map is a weak equivalence. Now, note that $|C_P(F)|_b \cong |F(\mathcal{I})| \times \{b\}$, as can be computed directly from the definition of C_P. In particular, the bottom map is an isomorphism, which concludes the proof. ☐

Corollary 3.4.5.4. *The functor $C_P\colon \mathbf{Diag}_P \to \mathrm{s}\mathbf{Strat}_P$ characterizes all weak equivalences.*

Proof. Let $f\colon F \to G$ be a map in $\mathbf{Diag}_P$. By Corollary 3.4.5.3, f is a weak equivalence if and only if $|C_P(f)|_P$ is a weak equivalence. But by Corollary 3.4.5.2, the latter is true if and only if $C_P(f)$ is a weak equivalence. ☐

3.5 Equivalence of homotopy categories and applications

The goal of this section is to show that the adjoint functors

$$|-|_s\colon s\mathbf{Strat} \leftrightarrow \mathbf{Strat}\colon \mathrm{Sing}_s,$$

as well as the fiberwise pairs $|-|_P\colon \mathrm{s}\mathbf{Strat}_P \leftrightarrow \mathbf{Strat}_P\colon \mathrm{Sing}_P$ descend to equivalences between the homotopy categories of stratified simplicial sets and stratified spaces. We deduce from this that the naive homotopy theory of conically stratified spaces embeds fully faithfully in the homotopy theory of stratified spaces, as well as a simplicial approximation theorem.

3.5.1 Equivalence between homotopy theories

In this subsection, we prove that the adjunctions $|-|_P \dashv \mathrm{Sing}_P$ and $|-|_s \dashv \mathrm{Sing}_s$ induce equivalences of homotopy categories. As neither is part of a Quillen adjunction, this is not to be understood in terms of derived functors in the sense of model categories. Instead, consider the homotopy categories as explicitly constructed by formally inverting the weak equivalences. All functors involved are shown to preserve weak equivalences and hence induce functors of the homotopy categories. The following theorem then states that these induced functors give equivalences of categories.

Theorem 3.5.1.1. *The adjoint pairs $|-|_P \dashv \mathrm{Sing}_P$, and $|-|_s \dashv \mathrm{Sing}_s$ induce well-defined equivalences between homotopy categories,*

$$|-|_P\colon \mathrm{hos}\mathbf{Strat}_P \leftrightarrow \mathrm{ho}\mathbf{Strat}_P\colon \mathrm{Sing}_P$$
$$|-|_s\colon \mathrm{ho}s\mathbf{Strat} \leftrightarrow \mathrm{ho}\mathbf{Strat}\colon \mathrm{Sing}_s.$$

We first prove that the functors of Theorem 3.5.1.1 pass to the homotopy categories.

Lemma 3.5.1.2. *The adjoint pairs $|-|_P \dashv \mathrm{Sing}_P$ and $|-|_s \dashv \mathrm{Sing}_s$ induce well-defined functors at the level of homotopy categories.*

Proof. By Corollary 3.4.5.2, the functor $|-|_P\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_P$ preserves weak equivalences for all posets P. By construction of the model structures on $\mathbf{Strat}$ and $s\mathbf{Strat}$, this implies that $|-|_s$ also preserves all weak equivalences. Since the homotopy categories are nothing more than the localization of the categories at the classes of weak equivalences, this implies that $|-|_P$ and $|-|_s$ both induce functors between homotopy categories. By Theorem 3.3.2.1, $\mathrm{Sing}_P\colon \mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P$ also preserves all weak equivalences, and so the same is true for $\mathrm{Sing}_s\colon \mathbf{Strat} \to s\mathbf{Strat}$, which means that they also induce well-defined functors at the level of homotopy categories. □

To conclude the proof of Theorem 3.5.1.1, we need the following lemma.

Lemma 3.5.1.3. *Let X be a space stratified over P, then the co-unit of the adjunction $|-|_P \dashv \mathrm{Sing}_P$,*

$$|\mathrm{Sing}_P(X)|_P \to X$$

is a weak equivalence in $\mathbf{Strat}_P$. Let K be a simplicial set stratified over P, then the unit of the adjunction $|-|_P \dashv \mathrm{Sing}_P$,

$$K \to \mathrm{Sing}_P(|K|_P)$$

is a weak equivalence in $\mathrm{s}\mathbf{Strat}_P$.

This also holds for the unit and co-unit of the adjunction $|-|_s \dashv \mathrm{Sing}_s$.

Proof. Consider first the adjunction $\mathrm{sd}_P \dashv \mathrm{Ex}_P$. For any simplicial set stratified over P, K, we have the commutative diagram

$$\begin{tikzcd} \mathrm{sd}_P(K) \arrow[rr, "\mathrm{l.v.}_P"] \arrow[dr, "\mathrm{sd}_P(\iota_K)"'] & & K, \\ & \mathrm{sd}_P\mathrm{Ex}_P(K) \arrow[ur, "\epsilon_K"'] & \end{tikzcd}$$

where the map ϵ_K is the co-unit. Now, by Proposition 3.2.11.3, $\mathrm{l.v.}_P$ is a weak equivalence, and sd_P preserves weak equivalences. By Proposition 3.3.1.1, ι_K is a weak equivalence, which means that $\mathrm{sd}_P(\iota_K)$ is a weak equivalence. By two out of three, this implies that $\epsilon_K\colon \mathrm{sd}_P\mathrm{Ex}_P(K) \to K$ is a weak equivalence.

Now let X be a space stratified over P, and consider the commutative diagram:

$$\begin{tikzcd} |\mathrm{sd}_P\mathrm{Ex}_P\mathrm{Sing}_P(X)|_P \arrow[rr] \arrow[dr, "|\epsilon_{\mathrm{Sing}_P(X)}|_P"'] & & X \\ & |\mathrm{Sing}_P(X)|_P \arrow[ur] & \end{tikzcd}$$

By Corollary 3.4.5.2, the functor $|-|_P$ preserves weak equivalences, which means that the map $|\epsilon_{\mathrm{Sing}_P(X)}|_P$ is a weak equivalence. Furthermore, the co-unit of the adjunction $|\mathrm{sd}_P(-)|_P \dashv \mathrm{Ex}_P\mathrm{Sing}_P$, is also a weak equivalence, since the adjunction is a Quillen equivalence [Dou21b, Theorem 1.15], and all objects in $\mathbf{Strat}_P$ (resp. s$\mathbf{Strat}_P$) are fibrant (resp. cofibrant). This means that by two out of three the co-unit $|\mathrm{Sing}_P(X)|_P \to X$ is a weak equivalence.

Now, let K be a simplicial set stratified over P, and consider the following composition:

$$|K|_P \to |\mathrm{Sing}_P(|K|_P)|_P \to |K|_P,$$

where the first map is the realization of the unit of the adjunction $|-|_P \dashv \mathrm{Sing}_P$, and the second map is the co-unit evaluated at $|K|_P$. The composition gives the identity, and we have proved that the second map is a weak equivalence in $\mathbf{Strat}_P$, which means that the map $|K|_P \to |\mathrm{Sing}_P(|K|_P)|_P$ is a weak equivalence in $\mathbf{Strat}_P$, by two out of three. But since, by Corollary 3.4.5.2, the realization functor characterizes weak equivalences, this means that the unit $K \to \mathrm{Sing}_P(|K|_P)$ is a weak equivalence in s$\mathbf{Strat}_P$.

For the case of $|-|_s \dashv \mathrm{Sing}_s$, note that if X is a space stratified over P, then $|\mathrm{Sing}_s(X)|_s = |\mathrm{Sing}_P(X)|_P$, by definition, which immediately gives the proof. □

Proof of Theorem 3.5.1.1. Consider the natural transformations $|\mathrm{Sing}_P(-)|_P \to 1_{\mathbf{Strat}_P}$ and $1_{\mathrm{s}\mathbf{Strat}_P} \to \mathrm{Sing}_P(|-|_P)$. By Lemma 3.5.1.3 they take value in weak equivalences. Since the functors $|-|_P$ and Sing_P pass to the homotopy categories, so do the natural transformations $|\mathrm{Sing}_P(-)|_P \to 1_{\mathrm{ho}\mathbf{Strat}_P}$ and $1_{\mathrm{hos}\mathbf{Strat}_P} \to \mathrm{Sing}_P(|-|_P)$. Those now take value in isomorphisms, meaning that $(|-|_P, \mathrm{Sing}_P)$ gives an equivalence between the homotopy categories. The same argument gives that $(|-|_s, \mathrm{Sing}_s)$ induces an equivalence between the homotopy categories. □

Remark 3.5.1.4. Theorem 3.5.1.1 is only stated in terms of homotopy categories because that is all that is needed for the applications of Sections 3.5.2 and 3.5.3, but a much stronger version holds. Consider the simplicial localization defined by Dwyer and Kan in [DK80a], in terms of hammocks. Lemmas 3.5.1.2 and 3.5.1.3 give precisely the hypothesis needed to apply [DK80a, Corollary 3.6]. In particular, the functors $(|-|_s, \mathrm{Sing}_s)$ and $(|-|_P, \mathrm{Sing}_P)$ induce Dwyer-Kan equivalences between the simplicial localizations:

$$L^H s\mathbf{Strat} \leftrightarrow L^H\mathbf{Strat} \ \text{ and } \ L^H \mathrm{s}\mathbf{Strat}_P \leftrightarrow L^H\mathbf{Strat}_P.$$

With the stronger result, one can say that the pairs of adjoint functors, $(|-|_s, \mathrm{Sing}_s)$ and $(|-|_P, \mathrm{Sing}_P)$ induce equivalences between the homotopy **theories** of stratified spaces and stratified simplicial sets. In particular, they induce equivalences between the underlying ∞-categories, which can be explicitly described as the Dwyer-Kan localizations.

3.5.2 Embedding the classical stratified homotopy category

It follows from Proposition 3.A.0.1 that there exists no model structure on $\mathbf{Strat}_P$ which is transported from s$\mathbf{Strat}_P$, along $\mathrm{Sing}_P\colon \mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P$. Nevertheless, it turns out that stratified spaces with fibrant Sing_P and realizations of stratified simplicial sets behave respectively much like fibrant and cofibrant objects of a model category.

Recollection 3.5.2.1. Recall that particularly nice stratified spaces, such as pseudo manifolds or more generally homotopically and conically stratified spaces, have the right lifting property with respect to realizations of admissible horn inclusions (see Theorem 3.2.10.2, and [Lur17, Theorem A.6.4][Nan19, Proposition 8.1.2.6]). In other words, such spaces map to fibrant objects under Sing_P.

Recall that for $X, Y \in \mathbf{Strat}_P$, $[X,Y]_P$ stands for the set of stratified homotopy classes of stratified maps between X and Y (Definition 3.2.2.6). Similarly, if $K, L \in \mathrm{s}\mathbf{Strat}_P$, $[K,L]_P$ stands for the set of stratified homotopy classes of stratified simplicial maps between K and L.

Lemma 3.5.2.2. *Let $X \in \mathbf{Strat}_P$ be a stratified space such that $\mathrm{Sing}_P(X) \in \mathrm{s}\mathbf{Strat}_P$ is fibrant. Then, for any weak equivalence of stratified simplicial sets $f\colon K \to L$ in $\mathrm{s}\mathbf{Strat}_P$, the induced map*

$$[|L|_P, X]_P \to [|K|_P, X]_P$$

is a bijection.

Proof. We have a commutative diagram

$$\begin{tikzcd} {[|L|_P, X]_P} \arrow[r] \arrow[d, "\cong"] & {[|K|_P, X]_P} \arrow[d, "\cong"] \\ {[L, \mathrm{Sing}_P(X)]_P} \arrow[r, "f^*"] & {[K, \mathrm{Sing}_P(X)]_P}, \end{tikzcd}$$

where the vertical maps are bijections, thanks to the fact that the adjunction $(|-|_P, \mathrm{Sing}_P)$ is simplicial [Dou21a, Proposition 4.9]. Since $f\colon K \to L$ is a weak equivalence and $\mathrm{Sing}_P(X)$ is fibrant, the lower horizontal is a bijection. Thus, by commutativity of the diagram, so is the upper horizontal, as was to be shown. □

As an immediate corollary of this lemma, we obtain:

Theorem 3.5.2.3. *Let $K \in \mathrm{s}\mathbf{Strat}_P$ and $X \in \mathbf{Strat}_P$ such that $\mathrm{Sing}_P(X)$ is a fibrant object of $\mathrm{s}\mathbf{Strat}_P$. Then, the natural map*

$$[|K|_P, X]_P \to \mathrm{ho}\mathbf{Strat}_P(|K|_P, X)$$

is a bijection.

Proof. Consider the realization of the last vertex map $|\mathrm{sd}_P(K)|_P \xrightarrow{|\mathrm{l.v.}_P|_P} |K|_P$. Since $\mathrm{l.v.}_P$ is a weak equivalence (Proposition 3.2.11.3) by Corollary 3.4.5.2 so is its realization. Furthermore, it follows from [Dou21b, Theorem 1.15] that $|\mathrm{sd}_P(K)|_P$ is a cofibrant object in $\mathbf{Strat}_P$. Thus, it follows that $|\mathrm{l.v.}_P|_P\colon |\mathrm{sd}_P(K)|_P \to |K|_P$ defines a cofibrant replacement of $|K|_P$. We obtain a commutative diagram

$$\begin{tikzcd} {[|K|_P, X]_P} \arrow[r] \arrow[d, "\cong"'] & \mathrm{ho}\mathbf{Strat}_P(|K|_P, X) \arrow[d, "\cong"] \\ {[|\mathrm{sd}_P(K)|_P, X]_P} \arrow[r, "\cong"] & \mathrm{ho}\mathbf{Strat}_P(|\mathrm{sd}_P(K)|_P, X]_P) \end{tikzcd}$$

Since $|\mathrm{l.v.}_P|_P$ is a weak equivalence, the right vertical map is a bijection. By Lemma 3.5.2.2, the same holds for the left vertical map. Since $|\mathrm{sd}_P(K)|_P$ is cofibrant, and all objects of $\mathbf{Strat}_P$ are fibrant, the bottom horizontal map is also a bijection. Hence, by commutativity of the diagram, the upper horizontal is a bijection, as was to be shown. □

By Recollection 3.5.2.1 we obtain the following immediate corollary of Theorem 3.5.2.3.

Corollary 3.5.2.4. *Let $\mathrm{Con}_P \subset \mathbf{Strat}_P$ the full subcategory of conically P-stratified which are triangulable (stratum-preserving homeomorphic to the realization of a stratified simplicial set). Denote by $\mathrm{Con}_P/\simeq_P$ the category obtained by identifying stratum-preserving homotopic maps. Then, the inclusion*

$$\mathrm{Con}_P \hookrightarrow \mathbf{Strat}_P$$

induces a fully faithful embedding

$$\mathrm{Con}_P/\simeq_P \hookrightarrow \mathrm{ho}\mathbf{Strat}_P.$$

Remark 3.5.2.5. Theorem 3.5.2.3 and Corollary 3.5.2.4 also hold over varying posets, i.e., in $\mathbf{Strat}$. Indeed, it follows from the equivalence between the homotopy categories of $\mathbf{Strat}$ and $\mathrm{s}\mathbf{Strat}$ which is Theorem 3.5.1.1, and the fibrancy property of conically stratified spaces, which also holds in $s\mathbf{Strat}$ since fibrant objects of $s\mathbf{Strat}$ are characterized fiberwise.

Remark 3.5.2.6. Note, that Lemma 3.5.2.2, Theorem 3.5.2.3 and Corollary 3.5.2.4 admit a strengthening in terms of simplicial categories. More precisely, if for $X, Y \in \mathbf{Strat}_P$ and $K, L \in s\mathbf{Strat}_P$ one replaces the set of homotopy classes of maps $[X, Y]_P$ and $[K, L]_P$, by the simplicial mapping spaces, and the sets $\mathrm{ho}\mathbf{Strat}_P(X, Y)$ and $\mathrm{hos}\mathbf{Strat}_P(K, L)$ by derived mapping spaces, the statements remain true with analogous proofs. In practice, this means that when working with spaces satisfying the hypothesis of Theorem 3.5.2.3 the classical mapping spaces has the correct homotopy type and there is no need to derive. This applies to triangulable conically stratified spaces as detailed in Remark 3.5.2.7.

Remark 3.5.2.7. The simplicial perspective described in Remark 3.5.2.6 can be used to strengthen Corollary 3.5.2.4 to a statement about infinity categories. Indeed, Con_P, as a subcategory of $\mathbf{Strat}_P$, inherits the structure of a simplicial category, while $\mathbf{Strat}_P$ itself is a simplicial model category. This means that the inclusion of the full simplicial sub-category $\mathbf{Con}_P \hookrightarrow \mathbf{Strat}_P$ descends to a functor

$$\mathbf{Con}_P \hookrightarrow L^H_{\mathrm{simp}}(\mathbf{Strat}_P)$$

Where $L^H_{\mathrm{simp}}(\mathbf{Strat}_P)$ is the diagonal hammock localization of a simplicial model category, described in [DK80b, Prop. 4.8]. Corollary 3.5.2.4 then generalizes to the statement that the above map is a fully faithful embedding of simplicial categories. Combining this with the fact that $L^H_{\mathrm{simp}}(\mathbf{Strat}_P) \simeq L^H(\mathbf{Strat}_P)$ ([DK80b, Prop. 4.8]), this means that the "naive" infinity category of conically P-stratified and triangulable stratified spaces embeds fully faithfully in that of P-stratified spaces. The analogous statement for **Con** and **Strat** also holds by the same argument.

3.5.3 A simplicial approximation theorem

Exposing Ex^∞_P as a fibrant replacement functor in $s\mathbf{Strat}_P$ allows one to study the homotopy category of $s\mathbf{Strat}_P$ through actual maps in $s\mathbf{Strat}_P$ from some subdivision of the domain. Using Theorem 3.5.1.1, one can then transport those results to $\mathbf{Strat}_P$ and its homotopy category to obtain stratified versions of the classical simplicial approximation theorems.

Proposition 3.5.3.1. *Let $K \in s\mathbf{Strat}_P$ be finite and $L \in s\mathbf{Strat}_P$. Then, for any morphism $\phi\colon K \to L$ in the homotopy category* $\mathrm{hos}\mathbf{Strat}_P$ *there exists an $n \in \mathbb{N}$ and a morphism $f\colon \mathrm{sd}^n_P(K) \to L$ such that the diagram*

$$\begin{tikzcd} & \mathrm{sd}^n_P K \arrow[dl, "\mathrm{l.v.}^n_P"'] \arrow[dr, "f"] & \\ K \arrow[rr, "\phi"] & & L \end{tikzcd}$$

commutes in $\mathrm{hos}\mathbf{Strat}_P$.

Remark 3.5.3.2. Proposition 3.5.3.1 also holds as a relative version. To be more precise, this involves the following replacements: Replace K by a pair $A \hookrightarrow K$ and L by some stratum-preserving simplicial map $g : A \to L$. The role of $s\mathbf{Strat}_P$ is then taken by the under category $s\mathbf{Strat}^A_P$ with the induced model structure from [Hir03, Theorem 7.6.5]. However, one needs to be mindful of the fact that, for the relative version, sd^n_P maps into $s\mathbf{Strat}_P^{\mathrm{sd}^n_P(A)}$. In particular, the final commutativity condition holds in the homotopy category of the latter. Aside from this, the proof is formally identical.

As an immediate corollary of this result (its relative version), and Theorem 3.5.1.1 we obtain the following corollary, which we prove first.

Proposition 3.5.3.3. *Suppose we are given a commutative diagram in $\mathbf{Strat}_P$:*

$$\begin{tikzcd} {|A|_P} \arrow[r, "{|g|_P}"] \arrow[d, hook, "{|i|_P}"'] & {|L|_P} \\ {|K|_P ,} \arrow[ur, "\phi"'] & \end{tikzcd}$$

where A, K and L are stratified simplicial sets, with A and K finite, $i: A \hookrightarrow K$ is an inclusion, $g: A \to L$ some arbitrary map in s**Strat**$_P$ *and $\phi: |K|_P \to |L|_P$ some map in* **Strat**$_P$*. Then, there exists an $n \in \mathbb{N}$ and a map $\hat{f}: \mathrm{sd}^n_P(K) \to L \in$* s**Strat**$_P$ *such that:*

- *the following diagram commutes in* s**Strat**$_P$

$$\begin{tikzcd} \mathrm{sd}^n_P(A) \arrow[d, "\mathrm{l.v.}^n_P"'] \arrow[dr, "\hat{f}_{|\mathrm{sd}^n_P(A)}"] & \\ A \arrow[r, "g"'] & L \end{tikzcd}$$

- *the following diagram commutes in* ho**Strat**$_P^{|\mathrm{sd}^n_P(A)|_P}$

$$\begin{tikzcd} |\mathrm{sd}^n_P(K)|_P \arrow[d, "|\mathrm{l.v.}^n_P|_P"'] \arrow[dr, "|\hat{f}|_P"] & \\ |K|_P \arrow[r, "\phi"'] & |L|_P \end{tikzcd} \tag{3.16}$$

In particular, for $n \geq 1$, Diagram (3.16) can even be assumed to commute up to stratified homotopy relative to $|\mathrm{sd}^n_P(A)|_P$, since then $|\mathrm{sd}^n_P(i)|_P$ is a cofibrant object in **Top**$_P^{|\mathrm{sd}^n_P(A)|_P}$.

Proof. Note that, by subdividing the source once, we may without loss of generality assume that i is such that its realization is cofibrant in **Strat**$_P$. As every object is fibrant, this means that morphisms in the homotopy category ho**Strat**$_P^{|A|}$ from $|i|_P$ to $|g|_P$ agree with homotopy classes of maps in **Strat**$_P^{|A|_P}$ (i.e. homotopy classes rel $|A|_P$). Using this and (a relative version of) Theorem 3.5.1.1 we obtain

$$[|i|_P, |g|_P]_P^{|A|_P} = \mathrm{ho}\mathbf{Strat}_P^{|A|_P}(|i|_P, |g|_P) \cong \mathrm{hos}\mathbf{Strat}_P{}^A(i, g).$$

where the left hand side denotes relative stratum-preserving homotopy classes and the right hand side bijection is given by realization. Now, apply Proposition 3.5.3.1, to obtain the result. ☐

We now move on to the proof of Proposition 3.5.3.1. We are going to prove the non-relative version. The relative proof is structurally almost identical. First, we need two equations involving subdivision and Ex_P which are easily verified. Recall that we denote $\iota^n : L \hookrightarrow Ex^n_P(L)$ the natural inclusion induced by pulling back a simplex along $\mathrm{l.v.}^n_P$.

Lemma 3.5.3.4. *Denote by η and ε the unit and counit of $\mathrm{sd}^n_P \dashv \mathrm{Ex}^n_P$ respectively. Then, the equations*

$$\varepsilon \circ \mathrm{sd}^n_P(\iota^n) = \mathrm{l.v.}^n_P$$
$$\mathrm{Ex}^n_P(\mathrm{l.v.}^n_P) \circ \eta = \iota^n$$

hold.

As an immediate corollary, we obtain:

Corollary 3.5.3.5. *Let $K, L \in$* s**Strat**$_P$*. Then, the following diagram of bijections commutes:*

$$\begin{tikzcd} & \mathrm{hos}\mathbf{Strat}_P(K, L) \arrow[dl] \arrow[dr] & \\ \mathrm{hos}\mathbf{Strat}_P(K, \mathrm{Ex}^n_P(L)) \arrow[rr] & & \mathrm{hos}\mathbf{Strat}_P(\mathrm{sd}^n_P(K), L) \end{tikzcd},$$

where the right diagonal is given by pre-composing with $\mathrm{l.v.}^n$*, the left diagonal by post-composing with* ι^n *and the bottom horizontal is the adjunction map.*

Proof. Let $\phi \in \mathrm{hos}\mathbf{Strat}_P(K,L)$. Let $\hat{\phi}$ be the adjoint morphism to $\iota^n \circ \phi$. It is given by $\varepsilon \circ \mathrm{sd}^n(\iota^n \circ \phi)$. By Lemma 3.5.3.4 we have

$$\begin{aligned} \varepsilon \circ \mathrm{sd}^n(\iota^n \circ \phi) &= \mathrm{l.v.}_P^n \circ \mathrm{sd}_P^n(\phi) \\ &= \phi \circ \mathrm{l.v.}_P^n. \end{aligned}$$

In particular, this shows the required commutativity. □

We now have everything necessary available to derive Proposition 3.5.3.1.

Proof of Proposition 3.5.3.1. By Corollary 3.3.1.2, $\iota^\infty : L \hookrightarrow \mathrm{Ex}_P^\infty L$ defines a fibrant replacement of L. Now, consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{s}\mathbf{Strat}_P(K,L) \arrow[r] \arrow[d] & \mathrm{hos}\mathbf{Strat}_P(K,L) \arrow[d] \\ \mathrm{s}\mathbf{Strat}_P(K,\mathrm{Ex}_P^\infty(L)) \arrow[r] & \mathrm{hos}\mathbf{Strat}_P(K,\mathrm{Ex}_P^\infty(L)) \end{tikzcd}$$

with the verticals given by postcomposition with ι^∞. Since Ex_P^∞ is fibrant, the lower horizontal is surjective. As K is finite, $\mathrm{s}\mathbf{Strat}_P(K,\mathrm{Ex}_P^\infty(L)) = \varinjlim \mathrm{s}\mathbf{Strat}_P(K,\mathrm{Ex}_P^n(L))$. In particular, for any $\phi \in \mathrm{hos}\mathbf{Strat}_P(K,L)$, we find some $f' \colon K \to \mathrm{Ex}_P^n(L)$ mapping to the same element as ϕ in $\mathrm{hos}\mathbf{Strat}_P(K,\mathrm{Ex}_P^\infty L)$. As ι^∞ is a weak equivalence and thus postcomposing with it gives a bijection in the homotopy category, we get that in particular $f' = \iota^n \circ \phi$ in $\mathrm{hos}\mathbf{Set}_P(K,\mathrm{Ex}_P^n(L))$. Next, consider the following diagram, which is commutative by Corollary 3.5.3.5:

$$\begin{tikzcd} \mathrm{s}\mathbf{Strat}_P(K,L) \arrow[rr] \arrow[d] & & \mathrm{hos}\mathbf{Strat}_P(K,L) \arrow[dl, "\cong"] \arrow[dd, "\cong"] \\ \mathrm{s}\mathbf{Strat}_P(\mathrm{sd}_P^n(K),L) \arrow[r] \arrow[d, "\cong"] & \mathrm{hos}\mathbf{Set}_P(\mathrm{sd}_P^n(K),L) \arrow[dr, "\cong"] & \\ \mathrm{s}\mathbf{Strat}_P(K,\mathrm{Ex}_P^n(L)) \arrow[rr] & & \mathrm{hos}\mathbf{Strat}_P(K,\mathrm{Ex}_P^n(L)) \end{tikzcd}$$

Bijections are marked by $\cong$. We have shown that $f' \in \mathrm{s}\mathbf{Strat}_P(K,\mathrm{Ex}_P^n(L))$ and $\phi \in \mathrm{hos}\mathbf{Strat}_P(K,L)$ have the same image in $\mathrm{hos}\mathbf{Strat}_P(K,\mathrm{Ex}_P^n(L))$. But the commutativity of the diagram gives that the map $f \colon \mathrm{sd}_P^n(K) \to L \in \mathrm{s}\mathbf{Strat}_P$, adjoint to f', and $\phi \in \mathrm{hos}\mathbf{Strat}_P(K,L)$ must have the same image in $\mathrm{hos}\mathbf{Strat}_P(\mathrm{sd}_P^n(K),L)$, which concludes the proof. □

3.6 Simplicial homotopy links and vertically stratified complexes

Consider again the following diagram of categories with weak equivalences:

$$\begin{tikzcd} & \mathbf{Diag}_P \arrow[dl, shift left] \arrow[dr, shift left] & \\ \mathrm{s}\mathbf{Strat}_P \arrow[ur, shift left, "D_P"] \arrow[rr, shift left, "|-|_P"] & & \mathbf{Strat}_P \arrow[ll, shift left, "\mathrm{Sing}_P"] \arrow[ul, shift left, "D_P^{\mathbf{Top}}"'] \end{tikzcd} .$$

In Section 3.4 we have seen that weak equivalences in $\mathrm{s}\mathbf{Strat}_P$ are detected by (combinatorial) links and that (up to realization), those naturally have the same weak homotopy type as the homotopy links in $\mathbf{Strat}_P$. As a particular consequence, we obtained the fact that the realization functor $|-|_P$ characterizes weak equivalences. The functor $D_P^{\mathbf{Top}}$ characterizes weak equivalences by definition and it is the content of Section 3.3 that Sing_P characterizes weak equivalences. We have also shown (Corollaries 3.4.5.3 and 3.4.5.4) that C_P and thus $C_P^{\mathbf{Top}} = |C_P(-)|_P$ characterize weak equivalences. To complete this picture, it remains to investigate the functor $D_P\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$. Rephrasing this question in terms of links, we need to compare the combinatorial homotopy link HoLink to the other notions of (homotopy) link, which we have already shown to agree (up to the Quillen equivalence $|-| \dashv \mathrm{Sing}$). There is an obvious natural comparison map

$$\mathrm{HoLink}_{\mathcal{I}}(K) = \mathrm{Map}(\Delta^{\mathcal{I}}, K) \to \mathrm{Map}(|\Delta^{\mathcal{I}}|_P, |K|_P) = \mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)$$

given by realization. The goal of this section is to prove the following theorem.

Theorem 3.6.0.1. *Let $K \in \mathrm{s}\mathbf{Strat}_P$. Then the natural inclusion*

$$\mathrm{HoLink}_{\mathcal{I}}(K) \hookrightarrow \mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P))$$

is a weak equivalence of simplicial sets. Hence, equivalently, so is the adjoint map

$$|\mathrm{HoLink}_{\mathcal{I}}(K)| \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P).$$

As an immediate corollary of this result and Corollary 3.4.5.2, one obtains:

Corollary 3.6.0.2. *The functor $D_P : \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$ characterizes weak equivalences.*

Proof. By Corollary 3.4.5.2, a map $f\colon K \to L$ in $\mathrm{s}\mathbf{Strat}_P$ is a weak equivalence if and only if $|f|_P\colon |K|_P \to |L|_P$ is a weak equivalence in $\mathbf{Strat}_P$. This implies that f is a weak equivalence if and only if for all regular flags $\mathcal{I}$, f induces weak equivalences $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|L|_P)$. By Theorem 3.6.0.1 this is equivalent to asking that f induces weak equivalences $D_P(K)(\mathcal{I}) = \mathrm{HoLink}_{\mathcal{I}}(K) \to \mathrm{HoLink}_{\mathcal{I}}(L) = D_P(L)(\mathcal{I})$, which concludes the proof. ☐

Remark 3.6.0.3. Corollary 3.6.0.2 applies in particular to fibrant replacement maps $K \to K^{\mathrm{Fib}}$ (for example, the map $K \hookrightarrow \mathrm{Ex}_P^{\infty}(K)$). This means that the map $\mathrm{HoLink}_{\mathcal{I}}(K) \to \mathrm{HoLink}_{\mathcal{I}}(K^{\mathrm{Fib}})$ is a weak equivalence, or in other words, that the naïve simplicial homotopy link and the fibrantly defined homotopy link coincide. This result provides insight as to why the model structure on $\mathrm{s}\mathbf{Strat}_P$ described in [Hen] and the one studied in this paper coincide, even though they have very different descriptions. Philosophically, the former has a class of weak equivalences defined from the naïve homotopy links $\mathrm{HoLink}_{\mathcal{I}}(K)$ while the latter has a class of weak equivalences defined from $\mathrm{HoLink}_{\mathcal{I}}(K^{\mathrm{Fib}})$, and the weak equivalence between those mapping spaces implies that the two model structures indeed have the same class of weak equivalences. In fact, they coincide.

3.6.1 Sketch of proof of Theorem 3.6.0.1

We are going to prove Theorem 3.6.0.1 through a comparison of homotopy groups using simplicial approximation style results. Note that using simplicial approximation to produce homotopies between simplicial maps leads to an uncommon type of cylinder object. To handle these cylinder objects, we define sd-homotopies. Remark 3.6.1.5, at the end of this subsection, provides some insight into sd-homotopies.

Definition 3.6.1.1. Let S, S' be simplicial sets, $k \geq 0$, and $f, f'\colon \mathrm{sd}^k(S) \to S'$ be two simplicial maps. Then f and f' are *sd-homotopic*, if there exists a simplicial map

$$H\colon \mathrm{sd}^k(S \times \Delta^1) \to S'$$

such that H restricts on either end of the cylinder to f and f'. Such a map is called an *sd-homotopy*. If f and f' are pointed maps, a *pointed sd-homotopy* is instead a pointed map

$$H\colon \mathrm{sd}^k(S \wedge \Delta^1_+) \to S',$$

where $\wedge$ stands for the smash product, and Δ^1_+ stands for the 1-simplex with a freely adjoined base-point.
Now, let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set, $\mathcal{I} \in \mathrm{sd}(P)$ a regular flag and $f, f' \colon \mathrm{sd}^k(S) \times \Delta^{\mathcal{I}} \to K$ two stratified simplicial maps. The maps f and f' are called *(stratified) sd-homotopic* if there exists a stratum-preserving simplicial map

$$H \colon \mathrm{sd}^k(S \times \Delta^1) \times \Delta^{\mathcal{I}} \to K,$$

called a *(stratified) sd-homotopy in* $\mathrm{s}\mathbf{Strat}_P$, such that H restricts on either end of the cylinder to f and f'. If in addition $* \in S$ and $\Delta^{\mathcal{I}} \to K$ are pointed, and f and f' are pointed maps, then a *stratified, pointed sd-homotopy* is instead a pointed stratified map

$$H \colon \mathrm{sd}^k(S \wedge \Delta^1_+) \times \Delta^{\mathcal{I}} \to K.$$

(Stratified) pointed sd-homotopies generate equivalence relations on the sets of pointed maps $\mathbf{sSet}^*(\mathrm{sd}^k(S), S')$ and $\mathrm{s}\mathbf{Strat}_P^{\Delta^{\mathcal{I}}}(\mathrm{sd}^k(S) \times \Delta^{\mathcal{I}}, K)$. We write $\sim_*$ for both equivalence relations defined this way.

Remark 3.6.1.2. Let $f, f' \colon \mathrm{sd}^k(S) \to S'$ be two simplicial map, related by a simplicial homotopy $H \colon \mathrm{sd}^k(S) \times \Delta^1 \to S'$. Note that there is a simplicial map

$$Q \colon \mathrm{sd}^k(S \times \Delta^1) \to \mathrm{sd}^k(S) \times \mathrm{sd}^k(\Delta^1) \xrightarrow{1 \times \mathrm{l.v.}^k} \mathrm{sd}^k(S) \times \Delta^1.$$

Precomposing H with Q gives a sd-homtopy

$$H \circ Q \colon \mathrm{sd}^k(S \times \Delta^1) \to S'$$

between f and f'. In particular, if two maps are homotopic, they are also sd-homotopic. The analogous statement holds for stratified homotopies as well as in the pointed case.

Lemma 3.6.1.3. *Let $\Delta^{\mathcal{I}} \to K \in \mathrm{s}\mathbf{Strat}_P^{\Delta^{\mathcal{I}}}$ be an $\mathcal{I}$ pointed stratified simplicial set. Consider* $\mathrm{HoLink}_{\mathcal{I}}(K)$ *with the induced pointing. Then, for each $n \geq 0$ there are natural bijections*

$$\begin{aligned} \varinjlim \mathrm{s}\mathbf{Strat}_P^{\Delta^{\mathcal{I}}}(\mathrm{sd}^k(S^n) \times \Delta^{\mathcal{I}}, K)/\sim_* &\cong \varinjlim \mathbf{sSet}^*(\mathrm{sd}^k(S^n), \mathrm{HoLink}_{\mathcal{I}}(K))/\sim_* \\ &\cong \pi_n(\mathrm{HoLink}_{\mathcal{I}}(K)) \end{aligned}$$

where S^n is some triangulation of the n-sphere. The first bijection is induced by the adjunction $- \times \Delta^{\mathcal{I}} \dashv \mathrm{Map}(\Delta^{\mathcal{I}}, -)$, and the second by composing with an inverse to $\mathrm{l.v.}^k$ *(in the homotopy category).*

Proof. The first map is a bijection since, by construction, the simplicial adjunction $- \times \Delta^{\mathcal{I}} \dashv \mathrm{Map}(\Delta^{\mathcal{I}}, -)$ preserves sd-homotopies. The second follows from the adjunction $\mathrm{sd} \dashv \mathrm{Ex}$, the fact that Ex^∞ defines a fibrant replacement in $\mathbf{sSet}$, and that S^n is a finite simplicial set (see Remark 3.6.1.5 for more details). Note that the commutativity conditions needed to ensure that the map from the colimit is well defined were already checked in the proof of Proposition 3.5.3.1, specifically, in Corollary 3.5.3.5. ☐

We can now conclude the proof of Theorem 3.6.0.1.

proof of Theorem 3.6.0.1. Let K be a stratified simplicial set, together with some pointing $\phi \colon \Delta^{\mathcal{I}} \to K$ corresponding to a basepoint in $\mathrm{HoLink}_{\mathcal{I}}(K)$. Consider the commutative diagram

$$\begin{tikzcd}
\varinjlim \mathrm{s}\mathbf{Strat}_P^{\Delta^{\mathcal{I}}}(\mathrm{sd}^k(S^n) \times \Delta^{\mathcal{I}}, K)/\sim_* \arrow[r] \arrow[d, "\cong"] & \left[|S^n \times \Delta^{\mathcal{I}}|_P, |K|_P\right]_P^{|\Delta^{\mathcal{I}}|_P} \arrow[d, "\cong"] \\
\varinjlim \mathbf{sSet}^*(\mathrm{sd}^k(S^n), \mathrm{HoLink}_{\mathcal{I}}(K))/\sim_* \arrow[d, "\cong"] & \left[|S^n|, \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P)\right]^* \arrow[d, "\cong"] \\
\pi_n(\mathrm{HoLink}_{\mathcal{I}}(K)) \arrow[r] & \pi_n(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P))
\end{tikzcd},$$

where the bottom horizontal map is induced by $\mathrm{HoLink}_{\mathcal{I}}(K) \to \mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(|K|_P))$ and the top horizontal is given by first realizing and then precomposing with a homotopy inverse to $|\mathrm{l.v.}^k| \times 1_{\Delta^{\mathcal{I}}}$. Note that the top horizontal map is well-defined since (stratified) sd-homotopies realize to (stratified) homotopies, by the fact that $|\mathrm{sd}^k(S^n \times \Delta^1) \times \Delta^{\mathcal{I}}|$ is a cylinder for $|\mathrm{sd}^k(S^n) \times \Delta^{\mathcal{I}}|$.

All the vertical maps are already known to be bijections. To finish the proof of Theorem 3.6.0.1, it suffices to show that the top horizontal map is a bijection. This is a consequence of Proposition 3.6.1.4. Indeed, the direct statement of Proposition 3.6.1.4 gives surjectivity, while the homotopy statement shows that if two pointed maps $f, f'\colon \mathrm{sd}^k(S^n)\times\Delta^{\mathcal{I}} \to K$ realize to homotopic maps, then they are related by a pointed sd-homotopy $\mathrm{sd}^{k'}(sd^k(S^n)\wedge\Delta^1_+)\times\Delta^{\mathcal{I}} \to K$. By Remark 3.6.1.2, this also means that $f \circ (\mathrm{l.v.}^{k'} \times 1_{\Delta^{\mathcal{I}}}) \sim_* f' \circ (\mathrm{l.v.}^{k'} \times 1_{\Delta^{\mathcal{I}}})$, which proves injectivity. □

Proposition 3.6.1.4. *Let $S \in \mathbf{sSet}^\star$ be a pointed finite simplicial set, $\mathcal{I}$ a regular flag and $K \in \mathrm{s}\mathbf{Strat}_P^{\Delta^{\mathcal{I}}}$ a pointed stratified simplicial set. Then, for any pointed stratum-preserving map $\phi\colon |S \times \Delta^{\mathcal{I}}|_P \to |K|_P$ and $k \gg 0$, there exists a pointed stratified simplicial map $f\colon \mathrm{sd}^k(S)\times\Delta^{\mathcal{I}} \to K$ such that*

$$\phi \circ |\mathrm{l.v.}^k \times 1_{\Delta^{\mathcal{I}}}|_P \simeq_P |f|_P \ \mathit{rel}\ * \times|\Delta^{\mathcal{I}}|_P.$$

Conversely, if any two pointed stratified simplicial maps $f_0, f_1 : S \times \Delta^{\mathcal{I}} \to K$ fulfill

$$|f_0|_P \simeq_P |f_1|_P \ \mathit{rel}\ * \times|\Delta^{\mathcal{I}}|_P,$$

then, for $k \gg 0$, there exists a pointed sd-homotopy $H\colon \mathrm{sd}^k(S \wedge \Delta^1_+) \times \Delta^{\mathcal{I}} \to K$ between $f_0 \circ (\mathrm{l.v.}^k \times 1_{\Delta^{\mathcal{I}}})$ and $f_1 \circ (\mathrm{l.v.}^k \times 1_{\Delta^{\mathcal{I}}})$.

Note, that in comparison to Proposition 3.5.3.3, the left hand side only subdivides in the nonstratified part of $S^n \times \Delta^{\mathcal{I}}$. That such a more efficient subdivision suffices, is a consequence of the particularly simple shape of this stratified simplicial set. It is a special example of a vertically stratified object. We will study those in details in Section 3.6.2. This will serve a dual purpose. First off, we use these objects to obtain a proof of Proposition 3.6.1.4 in Section 3.6.3. Secondly, they give a simple and convenient model for the homotopy category of stratified spaces (see Theorem 3.6.2.18).

Remark 3.6.1.5. Simplicial approximation theorems, such as Proposition 3.5.3.3, allow one to produce a simplical maps $f\colon \mathrm{sd}^k(S) \to S'$ from the data of a continuous map $\phi\colon |S| \to |S'|$. The two maps will then be related by a (topological) homotopy

$$H\colon |\mathrm{sd}^k(S)| \times [0,1] \to |S'|$$

relating $|f|$ and $\phi \circ |\mathrm{l.v.}^k|$. In its relative version, one can in addition assume that if ϕ was already simplicial on some subobject $A \subset S$, i.e. $\phi_{||A|} = |g|$, with $g\colon A \to S'$, then f can be chosen such that $f_{\mathrm{sd}^k(A)} = g \circ \mathrm{l.v.}^k$, and the homotopy H can then be taken relative to $|\mathrm{sd}^k(A)|$. One common use of the relative statement is when one has a pair of simplicial maps $f, f'\colon S \to S'$ whose realizations happen to be homotopic, through some map $H'\colon |S \times \Delta^1| \cong |S| \times [0,1] \to |S'|$. Through relative simplicial approximations, one gets a map

$$H\colon \mathrm{sd}^k(S \times \Delta^1) \to S',$$

which restricts to $f \circ \mathrm{l.v.}^k$ and $f' \circ \mathrm{l.v.}^k$ on either side of the cylinder $\mathrm{sd}^k(S \times \Delta^1)$. Now, note that H is not an elementary simplicial homotopy (see Definition 3.2.3.4), but instead an sd-homtopy. In particular, unless one is given an appropriate simplicial map $\mathrm{sd}^{k'}(S)\times\mathrm{sd}^{k''}(\Delta^1) \to \mathrm{sd}^k(S\times\Delta^1)$, one cannot deduce that, after enough subdivisions, the maps $f\circ\mathrm{l.v.}^k$ and $f'\circ\mathrm{l.v.}^k$ become simplicially homotopic in the usual sense. It does not seem unreasonable to assume that such a map can generally be exposed. In fact in the unordered simplicial complex case, such a result follows from the material presented in [Spa89, Chapter 3, Section 4]. We have no need for such a result in the following proof, however, and will work with the alternative cylinder instead. Of course, f and f' are still equal in the homotopy category, and if S' is fibrant, this is enough to show that they must be related through a simplicial homotopy.

One way to derive these type of simplicial approximation results, which we already employed in the proof of Proposition 3.5.3.1, is by leveraging the Ex^∞ functor. Let us again illustrate this for homotopies. Let $f, f' \colon S \to S'$ be two maps who become equal in the homotopy category (this is equivalent to asking that they realize to homotopic maps). $\iota \colon S' \to \mathrm{Ex}^\infty(S')$ be the usual inclusion. Then, since Ex^∞ is a fibrant replacement functor, $f \circ \iota$ and $f' \circ \iota$ must be related through a simplicial homotopy $\hat{H} \colon S \times \Delta^1 \to \mathrm{Ex}^\infty(S')$. Now, if S is finite, the map $\hat{H}$ must factor through $\mathrm{Ex}^n(S')$ for some $n \geq 0$. Using the adjunction $\mathrm{sd}^n \dashv \mathrm{Ex}^n$, we get an sd-homtopy

$$H \colon \mathrm{sd}^n(S \times \Delta^1) \to S'$$

which restricts on either side of the cylinder $\mathrm{sd}^n(S \times \Delta^1)$ to $f \circ \mathrm{l.v.}^n$ and $f' \circ \mathrm{l.v.}^n$. Again, note that, a priori, this is not enough to conclude that after enough subdivisions, $f \circ \mathrm{l.v.}^n$ and $f' \circ \mathrm{l.v.}^n$ become simplicially homotopic in the usual sense.

3.6.2 Vertically stratified objects

Throughout this subsection, if $\sigma \colon \Delta^n \to S$ is a possibly degenerate simplex, $\widehat{\sigma}$ stands for the unique non-degenerate simplex of S, of which σ is a degeneracy.

Definition 3.6.2.1. A *P-labelled simplicial set* is the data of

- a simplicial set S;
- a labelling map, $\lambda_S \colon S_{\text{n.d.}} \to N(P)_{\text{n.d.}}$,

such that, for any $\sigma \subset \tau$ in S, $\lambda_S(\tau) \subset \lambda_S(\sigma)$. A *label-preserving map* $f \colon (S, \lambda_S) \to (S', \lambda_{S'})$ is a simplicial map $f \colon S \to S'$, such that for all $\sigma \in S_{\text{n.d.}}$, $\lambda_S(\sigma) \subset \lambda_{S'}(\widehat{f(\sigma)})$. We denote by P-**sSet** the category of P-labelled simplicial sets with label-preserving maps.

Example 3.6.2.2. Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set. Consider the (non-stratified) simplicial set $\mathrm{sd}(K)$. Define a labelling map, $\lambda \colon \mathrm{sd}(K)_{\text{n.d.}} \to N(P)_{\text{n.d.}}$ as follows. If (μ, σ) is a vertex in $\mathrm{sd}(K)$, with $\mu \subset \Delta^n$ and $\sigma \colon \Delta^n \to K$, let $\lambda(\mu, \sigma) = \widehat{\varphi_K \circ \sigma \circ \mu}$. For higher dimensional simplices, set $\lambda((\mu_0, \ldots, \mu_k), \sigma) = \lambda(\mu_0, \sigma)$. Then $(\mathrm{sd}(K), \lambda)$ is a P-labelled simplicial set, and for any stratum-preserving simplicial map $f \colon K \to L$, $\mathrm{sd}(f) \colon (\mathrm{sd}(K), \lambda) \to (\mathrm{sd}(L), \lambda)$ is a label-preserving map.

Definition 3.6.2.3. Let (S, λ_S) be a P-labelled simplicial set. Its *verticalization* is the inclusion of stratified simplicial sets $V(S, \lambda_S) \hookrightarrow S \times N(P)$, where $V(S, \lambda_S)$ is defined as the following subset:

$$V(S, \lambda_S) = \bigcup_{\sigma \colon \Delta^n \to S \text{ , n.d.}} \mathrm{Im}(\sigma) \times \lambda_S(\sigma) \subset S \times N(P)\,,$$

where the union is taken over all non-degenerate simplices of S. If $f \colon (S, \lambda_S) \to (S', \lambda_{S'})$ is a label-preserving map, then the map $f \times 1_{N(P)}$ restricts to a stratified map $V(f) \colon V(S, \lambda_S) \to V(S', \lambda_{S'})$ and V defines a functor $P\text{-}\mathbf{sSet} \to \mathrm{s}\mathbf{Strat}_P$ in this fashion.

Next, let us pay some more attention to the stratified simplicials sets which lie in the essential image of the verticalization functor.

Definition 3.6.2.4. A *pre-verticalization* on a stratified simplicial set K is the data of

- a simplicial set $\bar{K}$;
- a monomorphism in $\mathrm{s}\mathbf{Strat}_P$, $K \hookrightarrow \bar{K} \times N(P)$.

A *vertical map* between stratified simplicial sets equipped with pre-verticalizations $K \subset \bar{K} \times N(P)$ and $L \subset \bar{L} \times N(P)$ is a stratum-preserving map $f\colon K \to L$ such that there exists a simplicial map $\bar{f}\colon \bar{K} \to \bar{L}$ making the following diagram commute:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, hook] & L \arrow[d, hook] \\ \bar{K} \times N(P) \arrow[r, "\bar{f} \times 1_{N(P)}"] & \bar{L} \times N(P)\,. \end{tikzcd}$$

Definition 3.6.2.5. A *vertically stratified simplicial set* is a stratified simplicial set, K, equipped with a pre-verticalization, $K \hookrightarrow \bar{K} \times N(P)$, which is isomorphic to the verticalization of a P-labelled simplicial set through some vertical map:

$$\begin{tikzcd} K \arrow[r, "\simeq"] \arrow[d, hook] & V(S, \lambda_S) \arrow[d, hook] \\ \bar{K} \times N(P) \arrow[r, "\simeq"] & S \times N(P)\,. \end{tikzcd}$$

Remark 3.6.2.6. Note that for a stratified simplicial set, having a pre-verticalization is not enough to be a vertically stratified simplicial set. All stratified simplicial sets admit a tautological pre-verticalization, $K \to K \times N(P)$. However, we are only interested in those pre-verticalizations that come from taking the verticalizations of P-labelled simplicial sets. Furthermore, when considering vertical maps between those, one need not keep track of the associated simplicial map $\bar{f}\colon \bar{K} \to \bar{L}$ since in those cases, if such a map exists, it is unique. To see this, observe that for a vertically stratified simplicial set $K \to \bar{K} \times N(P)$, the composition $K \to \bar{K} \times N(P) \to \bar{K}$ must be surjective, leaving at most one choice for $\bar{f}$ (see also the fully faithfulness part of Proposition 3.B.0.3).

Example 3.6.2.7. Let $K \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set, and $(\mathrm{sd}(K), \lambda)$ the P-labelled simplicial set from Example 3.6.2.2. Then $V(\mathrm{sd}(K), \lambda) = \mathrm{sd}_P(K) \subset \mathrm{sd}(K) \times N(P)$. Furthermore, if $f\colon K \to L$ is a stratified map, then $\mathrm{sd}_P(f) = V(\mathrm{sd}(f))$ is a vertical map. See Fig. 3.7.

Remark 3.6.2.8. Vertically stratified simplicial sets can be characterized in another way. They are precisely the images of cofibrant objects in $\mathbf{Diag}_P$ under the functor $C_P\colon \mathbf{Diag}_P \to \mathrm{s}\mathbf{Strat}_P$. Indeed, one can go from a P-labelled simplicial set, (S, λ_S), to a (cofibrant) diagram, F, by setting $F(\mathcal{I}) = \{\sigma \in S \mid \Delta^{\mathcal{I}} \subset \lambda_S(\widehat{\sigma})\}$, for regular flags $\mathcal{I}$,. Conversely, given a cofibrant diagram F, set $S = \cup_{\mathcal{I}} F(\mathcal{I})$, and $\lambda_S(\sigma) = \max\{\Delta^{\mathcal{I}} \mid \sigma \in F(\mathcal{I})\}$. To see why this labelling map is well-defined, recall from Proposition 3.2.7.4 that cofibrant objects in $\mathbf{Diag}_P$ are precisely the diagrams, F, such that:

- For all $\mathcal{I}' \subset \mathcal{I}$, the map $F(\mathcal{I}) \to F(\mathcal{I}')$ is a monomorphism.
- If $\mathcal{I}_0 \subset \mathcal{I}_1, \mathcal{I}_1'$, then $F(\mathcal{I}_0) \supset F(\mathcal{I}_1) \cap F(\mathcal{I}_1') \neq \varnothing$ if and only if there exist $\mathcal{I}_2$ such that $\mathcal{I}_1, \mathcal{I}_1' \subset \mathcal{I}_2$, and in this case, $F(\mathcal{I}_1) \cap F(\mathcal{I}_1') = F(\mathcal{I}_2)$, for the smallest such $\mathcal{I}_2$.

One then checks that the verticalization process corresponds to applying C_P to the corresponding diagram.

This relation between vertically stratified simplicial sets and diagrams fits into a larger picture, which is explored in more details in Section 3.B, more specifically Proposition 3.B.0.2.

Proposition 3.6.2.9. *Let $f\colon S \times \Delta^{\mathcal{I}} \to L$ in $\mathrm{s}\mathbf{Strat}_P$ be a stratum-preserving simplicial map between vertically stratified simplicial sets. There exists a stratum-preserving homotopy, $H\colon (S \times \Delta^{\mathcal{I}}) \times \Delta^1 \to L$ such that $H_0 = f$ and H_1 is a vertical map. Furthermore, if f is already vertical on $A \times \Delta^{\mathcal{I}} \subset S \times \Delta^{\mathcal{I}}$, then the homotopy H, can be taken relative to $A \times \Delta^{\mathcal{I}}$.*

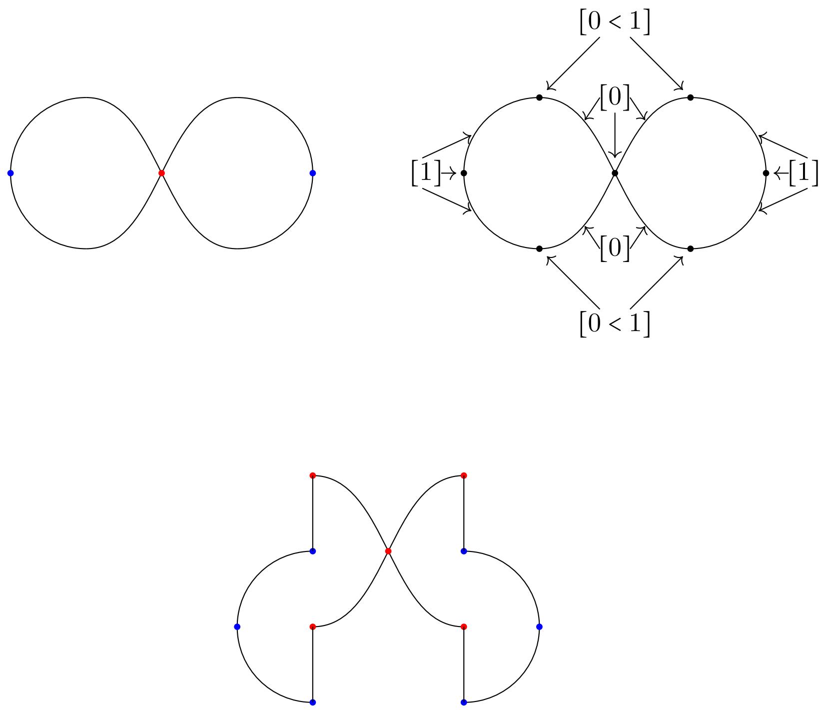


Figure 3.7: The figure 8 as a stratified simplicial set over $P = \{0 < 1\}$, its subdivision as a P-labelled simplicial set, and the associated vertically stratified simplicial set, $\mathrm{sd}_P(K)$.

Proof. We give an explicit definition of the homotopy for the case $S = \Delta^n$ which is natural in n and hence extends to the general case. Note that f factors through $S' \times \Delta^{\mathcal{I}} \subset L$, where $L = V(\bar{L}, \lambda_{\bar{L}})$, and $S' \subset \bar{L}$ is spanned by the simplices σ such that $\Delta^{\mathcal{I}} \subset \lambda_{\bar{L}}(\sigma)$. Hence, we may without loss of generality assume L to be of the form $S' \times \Delta^{\mathcal{I}}$ for some $S' \in \mathbf{sSet}$. In particular, as f is stratum-preserving, it is uniquely determined by the map $\hat{f} : \Delta^n \times \Delta^{\mathcal{I}} \to S'$ obtained by composing with the projection to S', and we may equivalently show the existence of a simplicial map (natural in Δ^n and S') $\hat{H}\colon \Delta^n \times \Delta^{\mathcal{I}} \times \Delta^1 \to S'$ such that $\hat{H}_0 = \hat{f}$ and $\hat{H}_1$ factors through $\Delta^n \times \Delta^{\mathcal{I}} \to \Delta^n$. Any k-simplex in $\Delta^n \times \Delta^{\mathcal{I}} \times \Delta^1$ is given by the data of a flag

$$(x_0, p_0) \le \dots \le (x_k, p_k)$$

with $x_i \in [n]$ and $p_i \in \mathcal{I}$, together with some $l \in [k+1]$ indicating whether the vertices (x_i, p_i) project to 0 or $1 \in \Delta^1$. More explicitely, l stands for the k-simplex of Δ^1, $[0, \dots, 0, 1, \dots 1]$, where the first 1 appears at the l-th position. Note that a simplex $([(x_0, p_0), \dots (x_k, p_k)], l)$ lies in the sub-object $\Delta^n \times \Delta^{\mathcal{I}} \times \{0\}$ (resp. $\Delta^n \times \Delta^{\mathcal{I}} \times \{1\}$) if and only if $l = k+1$ (resp. $l = 0$). Using this, consider the (not stratum-preserving) simplicial map

$$\begin{aligned} R^n\colon \Delta^n \times \Delta^{\mathcal{I}} \times \Delta^1 &\to \Delta^n \times \Delta^{\mathcal{I}} \\ ([(x_0, p_0), \dots (x_k, p_k)], l) &\mapsto [(x_0, p_0), \dots, (x_l, p_l), (x_{l+1}, p_m), \dots, (x_k, p_m)], \end{aligned}$$

where p_m is the maximum of $\mathcal{I}$. It is immediate from the definition, that R^n is natural in n and furthermore, that $R^n_0 = 1$ and R^n_1 factors through $\Delta^n \times \Delta^{\mathcal{I}} \to \Delta^n$. Now, finally define $\hat{H}$ as the composition

$$\hat{H} : \Delta^n \times \Delta^{\mathcal{I}} \times \Delta^1 \xrightarrow{R^n} \Delta^n \times \Delta^{\mathcal{I}} \xrightarrow{\hat{f}} S',$$

which fufills the requirements by the respective properties of R^n, and is natural in n. Finally, for the relative statement, note that if $\hat{f}$ already factors through $\Delta^n \times \Delta^{\mathcal{I}} \to \Delta^n$, i.e. f is vertical on $\Delta^n \times \Delta^{\mathcal{I}}$, then we have a commutative diagram

$$\begin{tikzcd}
\Delta^n \times \Delta^{\mathcal{I}} \times \Delta^1 \arrow[r, "R^n"] \arrow[d, "\pi_{\Delta^n \times \Delta^{\mathcal{I}}}"] & \Delta^n \times \Delta^{\mathcal{I}} \arrow[d] \arrow[dr, "\hat{f}"] & \\
\Delta^n \times \Delta^{\mathcal{I}} \arrow[r] & \Delta^n \arrow[r, dashed] & S'
\end{tikzcd}$$

In other words, $\hat{H}$ factors through $\pi_{\Delta^n \times \Delta^{\mathcal{I}}} : \Delta^n \times \Delta^{\mathcal{I}} \times \Delta^1 \to \Delta^n \times \Delta^{\mathcal{I}}$ and thus is a constant homotopy, which shows the relative statement. $\square$

The theory of P-labelled and of vertically stratified simplicial sets has a topological counterpart, which we briefly introduce here.

Definition 3.6.2.10. A *P-labelled CW-complex* is the data of

- a CW-complex, T;
- a labelling map, $\lambda_T\colon \{\text{cells of } T\} \to N(P)_{\text{n.d.}}$.

Such that, for any pair of cells e_α, e_β such that $e_\alpha \cap \overline{e_\beta} \neq \emptyset$, $\lambda_T(e_\beta) \subset \lambda_T(e_\alpha)$. A *label-preserving map* $f\colon (T, \lambda_T) \to (T', \lambda_{T'})$ is a continuous map $f\colon T \to T'$, such that for any cells $e_\alpha \in T, e_\beta \in T'$ such that $f(e_\alpha) \cap e_\beta \neq \emptyset$, one has $\lambda_T(e_\alpha) \subset \lambda_{T'}(e_\beta)$. We denote the category of P-labeled CW-complexes by P-**CW**.

Example 3.6.2.11. Given a P-labelled simplicial set (T, λ_T), its realization admits the structure of a P-labelled CW-complex with a cell for each non-degenerate simplex, which is given the same label as the simplex.

As for P-labelled simplicial sets, we are interested in the verticalization of a P-labelled CW-complex.

Definition 3.6.2.12. Let (T, λ_T) be a P-labelled CW-complex. Its *verticalization* is the inclusion $V(T, \lambda_T) \hookrightarrow T \times |N(P)|_P$, where $V(T, \lambda_T)$ is the following subset:

$$V(T, \lambda_T) = \bigcup_{e_\alpha \in \{\text{cells of } T\}} e_\alpha \times |\lambda_T(e_\alpha)|_P \subset T \times |N(P)|_P.$$

Extending to morphisms in the obvious way, this construction defines a functor $V\colon P\text{-}\mathbf{CW} \to \mathbf{Strat}_P$, which factors through $V\colon P\text{-}\mathbf{CW} \to \mathbf{Top}_{N(P)}$.

Again, it can be useful to study the objects in the essential image of the verticalization functor. These lead to a particularly convenient class of stratified spaces which admit a cell decomposition similar to classical CW-complexes (see Remark 3.6.2.14). We give a brief outlook into the resulting theory here, without going too much into detail. The definitions require a definition of a *pre-verticalization* and *vertical map*, which we omit here, since they are entirely analogous to the simplicial definition in Definition 3.6.2.4, replacing $N(P)$ by $|N(P)|_P$.

Definition 3.6.2.13. A *vertically stratified CW-space* is a stratified space equipped with a pre-verticalization, which is vertically isomorphic to the verticalization of a P-labeled CW-complex. When such an isomorphism is fixed we speak of a *vertically stratified CW-complex*. We will consider those objects in the categories $\mathbf{Strat}_P$ and $\mathbf{Top}_{N(P)}$.

Remark 3.6.2.14. Vertically stratified CW-spaces can also be characterized explicitly through the existence of a cell decomposition. Though, in this case, one has to be careful to also impose verticality conditions on the gluing of cells. Since the cells are of shape $B^n \times |\Delta^{\mathcal{I}}|_{N(P)}$, for $n \geq 0$, and regular flags $\mathcal{I}$, and are glued along their boundaries $S^{n-1} \times |\Delta^{\mathcal{I}}|_{N(P)}$, this implies that vertically stratified CW-spaces are actual cell complexes in $\mathbf{Top}_{N(P)}$ (and in $\mathbf{Strat}_P$). In particular, they are cofibrant objects.

Example 3.6.2.15. Following Examples 3.6.2.7 and 3.6.2.11, given a stratified simplicial set $K \in s\mathbf{Strat}_P$, we have a P-labelled CW-complex $(|\mathrm{sd}(K)|, |\lambda|)$. Its verticalization gives a vertically stratified CW-structure on $|\mathrm{sd}_P(K)|_P$:

$$|\mathrm{sd}_P(K)|_P \hookrightarrow |\mathrm{sd}(K)| \times |N(P)|_P.$$

See Proposition 3.B.0.3 for a complete picture of the relationship between labelled objects, vertical objects and diagrams indexed over $\mathrm{sd}(P)$.

Proposition 3.6.2.16. *Let $f: X \to Y$ be a map in $\mathbf{Top}_{N(P)}$ between vertically stratified CW-spaces, $X \cong V(\bar{X}, \lambda_{\bar{X}})$ and $Y \cong V(\bar{Y}, \lambda_{\bar{Y}})$. Then f is strongly stratified homotopic to a vertical map. Furthermore, if $\bar{A} \subset \bar{X}$ is a subcomplex equipped with the induced labelling, then the homotopy can be taken relative to $A = V(\bar{A}, \lambda_{\bar{A}}) \subset X$. The analogous statement also holds for maps and homotopies in $\mathbf{Strat}_P$.*

Proof. Here, we give a proof using the cellular structure of vertically stratified CW-spaces. A proof going through abstract homotopy theory is given in Section 3.B. We first prove the statement for maps in $\mathbf{Top}_{N(P)}$ by the usual induction on skeletons argument. Let us write $X^n = V(\bar{A} \cup \bar{X}^n, \lambda_{\bar{X}})$, for $n \geq -1$. We will construct a sequence of maps, $g_i: X \to Y$, $i \geq -1$, with $g_{-1} = f$, and g_n vertical on X^n, and g_n and g_{n-1} homotopic relative to X^{n-1}, for $n \geq 0$, through some homotopy in $\mathbf{Top}_{N(P)}$, H_n. By subdividing the interval, one can then concatenate the homotopies H_n, to a homotopy $H: X \times [0,1) \to Y$. By construction, the map $g: x \mapsto \lim_{t \to 1} H(x,t)$ is well defined, vertical and extends H to $X \times [0,1]$. Hence, g is as required in the statement of the proposition.

Now, let $n \geq 0$, and assume g_{n-1} has been constructed. Let e_α be some n-cell of X of shape $B^n \times \Delta^{\mathcal{I}}$ and define the homotopy

$$\begin{aligned} H'_{n,\alpha}: B^n \times |\Delta^{\mathcal{I}}|_{N(P)} \times [0,1] &\to Y \subset \bar{Y} \times |N(P)|_{N(P)} \\ ((x,\xi), s) &\mapsto (\mathrm{pr}_{\bar{Y}}(g_{n-1}(x, s(1,0,\dots,0) + (1-s)\xi)), \xi) \end{aligned}$$

Assembling the $H'_{n,\alpha}$ gives a homotopy on the n-skeleton $H'_n: X^n \times [0,1] \to Y$, which is constant on the $n-1$ skeleton, by the induction hypothesis. But now, by Remark 3.6.2.14, the inclusion $X^n \to X$ is a cofibration in $\mathbf{Top}_{N(P)}$, and thus has the homotopy extension property. This means that H'_n extends to the desired homotopy $H_n: X \times [0,1] \to Y$, between g_{n-1} and some map g_n with the required properties, which concludes the proof for $\mathbf{Top}_{N(P)}$.

For maps in $\mathbf{Strat}_P$, first note that, as noticed in Remark 3.6.2.14, X and Y are cofibrant objects, in $\mathbf{Strat}_P$, and in fact come from cofibrant objects in $\mathbf{Top}_{N(P)}$. Since all objects in $\mathbf{Strat}_P$ and $\mathbf{Top}_{N(P)}$ are fibrant, and since $\varphi_P \circ -: \mathbf{Top}_{N(P)} \to \mathbf{Strat}_P$ is a Quillen equivalence between the two model categories, this implies that we have canonical bijections

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Strat}_P}(X,Y)/{\simeq_P} &\cong \mathrm{Hom}_{\mathrm{ho}\mathbf{Strat}_P}(X,Y) \\ &\cong \mathrm{Hom}_{\mathrm{ho}\mathbf{Top}_{N(P)}}(X,Y) \\ &\cong \mathrm{Hom}_{\mathbf{Top}_{N(P)}}(X,Y)/{\simeq_{N(P)}}. \end{aligned}$$

Where $\simeq_P$ and $\simeq_{N(P)}$ stand for the homotopy relations in $\mathbf{Strat}_P$ and $\mathbf{Top}_{N(P)}$ respectively. In particular, this implies that there must exist $f': X \to Y$ a map in $\mathbf{Top}_{N(P)}$ homotopic to f as a map in $\mathbf{Strat}_P$. For the relative statement, consider the relative categories $\mathbf{Strat}_P^A$, and $\mathbf{Top}_{N(P)}^A$. The Quillen equivalence between $\mathbf{Strat}_P$ and $\mathbf{Top}_{N(P)}$ passes to the relative categories, and the above argument then gives the existence of a map in $\mathbf{Top}^A_{N(P)}$, $f': X \to Y$, homotopic to f as a map of $\mathbf{Strat}_P^A$. In other words, it is homotopic to f relative to A. We then conclude the proof by applying the first part of the proposition to f'. □

Definition 3.6.2.17. Two vertical maps between vertically stratified CW-spaces, $f, g: X \to Y$, are said to be *vertically homotopic* if there exists a vertical map $H: X \times [0,1] \to Y$ which is a homotopy between f and g. Write $f \simeq_V g$ when f and g are vertically homotopic, and let

$\mathrm{V}\mathbf{CW}_P/{\simeq_V}$ be the category whose objects are vertically stratified CW-spaces and whose set of morphisms between X and Y is the set of vertical maps quotiented by the equivalence relation $\simeq_V$.

Now, as a corollary of Proposition 3.6.2.16, we obtain that the homotopy category $\mathrm{ho}\mathbf{Strat}_P$ may equivalently be described as the homotopy category of vertically stratified CW-spaces $\mathrm{V}\mathbf{CW}_P/{\simeq_V}$, and vertical maps and homotopies. The comparison is given by the forgetful functor, sending a vertically stratified CW-space $X \hookrightarrow \bar{X} \times |N(P)|_P$ to the stratified space X.

Theorem 3.6.2.18. *the forgetful functor induces an equivalence of categories*

$$\mathrm{V}\mathbf{CW}_P/{\simeq_V} \cong \mathrm{ho}\mathbf{Strat}_P.$$

Proof. By classical results, we know that $\mathrm{ho}\mathbf{Strat}_P \cong \mathbf{Strat}_P^{\mathrm{Cof}}/{\simeq_P}$. Furthermore, for all $X \in \mathbf{Strat}_P$, X is weakly equivalent to $|\mathrm{sd}_P(\mathrm{Sing}_P(X))|_P$, which admits the structure of a vertically stratified CW-complex by Example 3.6.2.15. Since stratified spaces admitting the structure of a vertically stratified CW-complexes are cofibrant objects, when X is also a cofibrant object it is in fact homotopy equivalent to the vertically stratified CW-complex $|\mathrm{sd}_P(\mathrm{Sing}_P(X))|_P$. Thus, one can restrict from the subcategory of cofibrant objects, to the subcategory of vertically stratified CW-spaces (while still retaining all stratum-preserving maps and homotopies). But then, by Proposition 3.6.2.16, it is enough to consider only vertical maps, and, from the relative case, we see that it is also enough to consider only vertical homotopies between vertical maps, which concludes the proof. $\square$

Remark 3.6.2.19. Theorem 3.6.2.18 implies that vertically stratified CW-complexes and vertical maps (and equivalently labeled CW-complex with label-preserving maps, see Section 3.B, specifically Corollary 3.B.0.6) are a model for the homotopy category of stratified spaces. This is particularly convenient for a few reasons.

- As illustrated in the proof of Proposition 3.6.2.16, vertically stratified CW-complexes allow for inductive arguments on their skeletons, giving access to reasonably elementary and topological proofs.
- Vertically stratified CW complexes allow for a nice interpretation of the homotopy links, and the stratified homotopy groups. Indeed, let $X \cong V(\bar{X}, \lambda_{\bar{X}}) \subset \bar{X} \times |N(P)|_P$ be a vertically stratified CW-complex, let $\mathcal{I}$ be a regular flag, and let $\bar{X}_{\mathcal{I}} \subset \bar{X}$ be the subcomplex containing all cells $e_\alpha \subset \bar{X}$, such that $\Delta^{\mathcal{I}} \subset \lambda_{\bar{X}}(e_\alpha)$. Then, there is a homotopy equivalence
$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \simeq \bar{X}_{\mathcal{I}}$$
where the map from $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X)$ to $\bar{X}_{\mathcal{I}}$ is given by evaluating at the 0-th vertex, then projecting to $\bar{X}$, while its homotopy inverse is the map sending a point $x \in \bar{X}_{\mathcal{I}}$ to the composition $\{x\} \times |\Delta^{\mathcal{I}}|_P \hookrightarrow \bar{X}_{\mathcal{I}} \times |\Delta^{\mathcal{I}}|_P \hookrightarrow X$. In particular, the p-stratum of X is homotopy equivalent to the subobject $\bar{X}_p \subset \bar{X}$, and for all regular flags $\mathcal{I} = \{p_0 < \cdots < p_n\}$, the above equivalence can be rewritten as
$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \simeq \bar{X}_{p_0} \cap \cdots \cap \bar{X}_{p_n}.$$
In particular, for a vertically stratified CW-complex, its strata - and homotopy links - can be interpreted as subobjects - and intersections of those subobjects - of the associated P-labelled CW-complex. Even more, a pointing x of $\bar{X}$ in a cell with label $\Delta^{\mathcal{I}}$ gives a stratified pointing $\phi\colon |\Delta^{\mathcal{I}}|_P \to X$ (and all pointing of X come from restrictions of such pointings, up to homotopy). Thus, the stratified homotopy groups of [Dou21c] associated to the pointing ϕ are nothing more than the data of the homotopy groups of the $\bar{X}_{\mathcal{I}'}$, for $\mathcal{I}' \subset \mathcal{I}$, with respect to the pointing x.
- Given a vertically stratified CW-complex $X \cong V(\bar{X}, \lambda_{\bar{X}})$ its underlying (non-stratified) homotopy type is that of the CW-complex $\bar{X}$. Indeed, to see this, consider the diagram

of spaces associated to $V(X,\lambda_X)$, $F:\mathcal{I}\mapsto \bar{X}_{\mathcal{I}}$ defined as in the previous bulletpoint (in the appendix this construction is denoted $U^{\mathbf{Top}}$, see Proposition 3.B.0.2 for more details). Then, the underlying space of $X \cong V(\bar{X},\lambda_{\bar{X}})$ is a simplicial model for the homotopy colimit of F (compare for ex [Hir03, Sec. 18.1]). At the same time $\bar{X}$ is the regular colimit of F. Furthermore, it follows immediately from the construction of F, that it is a cofibrant diagram in the projective model structure, on the category of functors from $\mathrm{sd}(P)^{\mathrm{op}}$ to **Top** (Proposition 3.B.0.2). In particular (see for example [Dug08, Prop 9.11]), we have weak equivalences of topological spaces:

$$X = \mathrm{hocolim}\, F \xrightarrow{\sim} \varinjlim F = \bar{X}.$$

The analogous result holds for vertically stratified simplicial sets.

3.6.3 One last approximation theorem

We now move on to the proof of Proposition 3.6.1.4. We first need a technical lemma on how to concatenate sd-homotopies.

Lemma 3.6.3.1. *Let S,S' be simplicials sets and Let $H:\mathrm{sd}^k(S\times\Delta^1)\to S'$ as well as $H':\mathrm{sd}^k(S\times\Delta^1)\to S'$ be sd-homotopies between simplicial maps f and g, and g and h respectively. Then, there exists a concatenated sd-homotopy $H'':\mathrm{sd}^{k+2}(S\times\Delta^1)$ from $f\circ \mathrm{l.v.}^2$ to $h\circ\mathrm{l.v.}^2$. The analogous result for pointed and stratified sd-homotopies also holds.*

Proof. We prove the non-pointed non-stratified case. The other cases work completely analogously. First consider a gluing

$$H\cup H':\mathrm{sd}^k(S\times\Delta^1)\cup_{\mathrm{sd}^k(S)}\mathrm{sd}^k(S\times\Delta^1)\to S',$$

gluing H to H' along g. The left hand side of this map is equivalently given by

$$\mathrm{sd}^k(S\times(\Delta^1\cup_{1,0}\Delta^1)) = \mathrm{sd}^k(S\times(\Delta^1\cup_{1,0}\Delta^1)).$$

Note, that we may naturally identify

$$\mathrm{sd}^2(\Delta^1) = \Delta^1\cup_{1,1}\Delta^1\cup_{0,0}\Delta^1\cup_{1,1}\cup\Delta^1.$$

Using this identification, one obtains a map

$$c:\mathrm{sd}^2(\Delta^1)\to(\Delta^1\cup_{1,0}\Delta^1)$$

given by collapsing the second and the fourth interval of $\mathrm{sd}^2(\Delta^1)$ to a point. In particular, this map maps left endpoint to left endpoints and right endpoints to right endpoints. From c, we in turn now obtain the composition

$$c':\mathrm{sd}^2(S\times\Delta^1)\to\mathrm{sd}^2(S)\times\mathrm{sd}^2(\Delta^1)\xrightarrow{\mathrm{l.v.}^2\times c} S\times(\Delta^1\cup_{1,0}\Delta^1)$$

mapping the left boundary of the subdivided cylinder $\mathrm{sd}^2(S\times\Delta^1)$ to the left boundary of the glued cylinder $S\times(\Delta^1\cup_{1,0}\Delta^1)$ and analogously for the right boundary. Precomposing $H'\cup H'$, with $\mathrm{sd}^k(c')$ produces an sd-homotopy H'' as required. ☐

Next, we show the following.

Lemma 3.6.3.2. *Let $S\in\mathbf{sSet}$ be a finite simplicial set with simplicial subset $S_0\subset S$. Further, let $K\cong V(\bar{K},\lambda_{\bar{K}})\in \mathrm{s}\mathbf{Strat}_P$ be a vertically stratified simplicial set, and $\mathcal{I}$ a regular flag of P. Assume that we are given a map in $\mathbf{Strat}_P$, $\phi:|S\times\Delta^{\mathcal{I}}|_P\to|K|_P$ and a vertical map in $\mathrm{s}\mathbf{Strat}_P$ $g:S_0\times\Delta^{\mathcal{I}}\to K$, such that*

$$\phi|_{|S_0\times\Delta^{\mathcal{I}}|_P} = |g|_P.$$

Then for $k \gg 0$ there exists a vertical, simplicial map $f\colon \mathrm{sd}^k(S) \times \Delta^{\mathcal{I}} \to K$, such that

$$f|_{\mathrm{sd}^k(S_0)\times\Delta^{\mathcal{I}}} = g \circ (\mathrm{l.v.}^k \times 1_{\Delta^{\mathcal{I}}})$$

and such that $|f|_P$ is stratum-preserving homotopic to $\phi \circ |\mathrm{l.v.}^k \times 1_{\Delta^{\mathcal{I}}}|_P$ relative to $|\mathrm{sd}^k(S_0) \times \Delta^{\mathcal{I}}|_P$.

Proof. By Proposition 3.6.2.16, there exists a vertical map $h\colon |S \times \Delta^{\mathcal{I}}|_P \to |K|_P$, that is stratified homotopic to ϕ relative to $|S_0 \times \Delta^{\mathcal{I}}|_P$. The image of $\bar{h}$ must be contained in the cells with labels containing $\Delta^{\mathcal{I}}$. We write $K_{\mathcal{I}}$ for the corresponding subsimplicial set of $\bar{K}$. The situation can be summed up by the following commutative diagram:

$$\begin{tikzcd} |S_0| \arrow[r, "|\bar{g}|"] \arrow[d, hook] & |K_{\mathcal{I}}| \\ |S| \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

Now, by the non-stratified, relative version of Proposition 3.5.3.1 (see also Remark 3.5.3.2), there exists some $k \geq 0$ and some simplicial map $h'\colon \mathrm{sd}^k(S) \to K_{\mathcal{I}}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{sd}^k(S_0) \arrow[r, "\bar{g} \circ \mathrm{l.v.}^k"] \arrow[d, hook] & K_{\mathcal{I}} \\ \mathrm{sd}^k(S) \arrow[ur, "h'"'] & \end{tikzcd}$$

and such that $|h'|$ is homotopic to $\bar{h} \circ |\mathrm{l.v.}^k|$ relative to $\mathrm{sd}^k(S_0)$. But now the composition

$$\mathrm{sd}^k(S) \times \Delta^{\mathcal{I}} \xrightarrow{h' \times 1_{\Delta^{\mathcal{I}}}} K_{\mathcal{I}} \times \Delta^{\mathcal{I}} \hookrightarrow K$$

is the desired map. ☐

We may now combine the previous two lemmas, to obtain the following proposition.

Proposition 3.6.3.3. *Let S be a finite pointed simplicial set and $\mathcal{I}$ some regular flag in P. Then, for any $S_0 \subset S$ a simplicial subset containing the base point, there exists a pointed vertical simplicial map $\ell\colon \mathrm{sd}^k(S) \times \Delta^{\mathcal{I}} \to \mathrm{sd}_P(S \times \Delta^{\mathcal{I}})$ which restricts to a pointed vertical simplicial map $\ell_0\colon \mathrm{sd}^k(S_0) \times \Delta^{\mathcal{I}} \to \mathrm{sd}_P(S_0 \times \Delta^{\mathcal{I}})$ such that the diagram*

$$\begin{tikzcd} \mathrm{sd}^k(S_0) \times \Delta^{\mathcal{I}} \arrow[r, "\ell_0"] \arrow[dr, "\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}"'] & \mathrm{sd}_P(S_0 \times \Delta^{\mathcal{I}}) \arrow[d, "\mathrm{l.v.}_P"] \\ & S_0 \times \Delta^{\mathcal{I}} \end{tikzcd} \tag{3.17}$$

commutes up to pointed sd-homotopy.

Before we begin with the proof, let us quickly remark on the somewhat specific wording of Proposition 3.6.3.3.

Remark 3.6.3.4. Proposition 3.6.3.3 is at the heart of the proof of Proposition 3.6.1.4. We effectively use it in two ways. First, for the existence part of Proposition 3.6.1.4, one uses the case $S_0 = S$ guaranteeing the existence of a global l with certain commutativity properties. Then, for the uniqueness part (up to sd-homotopy) one considers the case where S is a cylinder with boundary S_0. In that case, it is crucial that the restriction of the global map ℓ to $\mathrm{sd}^k(S_0) \times \Delta^{\mathcal{I}}$ has its image in $\mathrm{sd}_P(S_0 \times \Delta^{\mathcal{I}})$, however, the commutativity property is then only needed for that restriction.

Proof of Proposition 3.6.3.3. To simplify notation, we omit all exponents from last vertex maps. Note, that $|\mathrm{l.v.}_P|_P\colon |\mathrm{sd}_P(S \times \Delta^{\mathcal{I}})|_P \to |S \times \Delta^{\mathcal{I}}|_P$ is a weak equivalence between cofibrant, fibrant objects with respect to the model structure on $\mathbf{Strat}_P^{|\Delta^{\mathcal{I}}|_P}$. In particular, it has an inverse, up to stratum-preserving pointed homotopy equivalence. Denote these by γ_S and γ_{S_0} respectively. By naturality of $\mathrm{l.v.}_P$ the diagram

$$\begin{tikzcd}
{|S_0 \times \Delta^{\mathcal{I}}|_P} \arrow[r, "\gamma_{S_0}"] \arrow[d, hook] & {|\mathrm{sd}_P(S_0 \times \Delta^{\mathcal{I}})|_P} \arrow[d, hook] \\
{|S \times \Delta^{\mathcal{I}}|_P} \arrow[r, "\gamma_S"] & {|\mathrm{sd}_P(S \times \Delta^{\mathcal{I}})|_P}
\end{tikzcd}$$

is commutative in $\mathrm{ho}\mathbf{Strat}_P^{\Delta^{\mathcal{I}}}$. Now, since all objects involved are cofibrant, the diagram is commutative up to pointed stratum-preserving homotopy. Now, first apply Lemma 3.6.3.2 to γ_{S_0} with the role of the simplicial subset taken by the basepoint. For some $k' \geq 0$, we obtain a pointed vertical simplicial map $\ell^1_{S_0}\colon \mathrm{sd}^{k'}(S_0) \times \Delta^{\mathcal{I}} \to \mathrm{sd}_P(S_0 \times \Delta^{\mathcal{I}})$ such that the following solid diagram commutes up to pointed, stratum-preserving homotopy. To reduce the overload of notation, we omit the indices P for stratified realizations.

$$\begin{tikzcd}
{|\mathrm{sd}^{k'+k''}(S_0) \times \Delta^{\mathcal{I}}|} \arrow[r, dotted, "|\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}|"] \arrow[d, dotted, hook] & {|\mathrm{sd}^{k'}(S_0) \times \Delta^{\mathcal{I}}|} \arrow[r, "|\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}|"] \arrow[d, hook] \arrow[rr, bend left, "|\ell^1_{S_0}|"] & {|S_0 \times \Delta^{\mathcal{I}}|} \arrow[r, "\gamma_{S_0}"] \arrow[d, hook] & {|\mathrm{sd}_P(S_0 \times \Delta^{\mathcal{I}})|} \arrow[d, hook] \\
{|\mathrm{sd}^{k'+k''}(S) \times \Delta^{\mathcal{I}}|} \arrow[r, dotted, "|\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}|"] \arrow[rrr, dotted, bend right, "|\ell^2_S|"'] & {|\mathrm{sd}^{k'}(S) \times \Delta^{\mathcal{I}}|} \arrow[r, "|\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}|"] \arrow[rr, dashed, bend right, "\tilde{\gamma}_S"'] & {|S \times \Delta^{\mathcal{I}}|} \arrow[r, "\gamma_S"] & {|\mathrm{sd}_P(S \times \Delta^{\mathcal{I}})|.}
\end{tikzcd}$$

Now, in this diagram the vertical left solid arrow is a cofibration in $\mathbf{Strat}_P^{|\Delta^{\mathcal{I}}|_P}$ and all objects are cofibrant fibrant. Since the outer solid arrow diagram is commutative up to stratum-preserving pointed homotopy, the homotopy extension property of cofibrations against fibrant objects gives the existence of a dashed map $\tilde{\gamma}_S$, making the square commute on the nose, and making the lower right triangle commute up to stratum-preserving homotopy. Now, apply Lemma 3.6.3.2 to $\tilde{\gamma}_S$ and $\ell^1_{S_0}$ to obtain a pointed vertical simplicial map ℓ^2_S in the dotted part of the diagram, for k'' sufficently large. Again, the most outer part of the complete diagram commutes on the nose (in $s\mathbf{Strat}_P$) and the lower left triangle commutes up to stratum-preserving pointed homotopy.

Now ℓ^2_S is not yet the map we are looking for, since we still need to check that (3.17) commutes up to pointed sd-homotopy. Write $\ell^2_{S_0} = \ell^1_{S_0} \circ (\mathrm{l.v.}^{k''} \times 1_{\Delta^{\mathcal{I}}})$, for the restriction of ℓ^2_S to $\mathrm{sd}^{k'+k''}(S_0) \times \Delta^{\mathcal{I}}$. We want to construct a simplicial homotopy between $\mathrm{l.v.}_P \circ \ell^2_{S_0}$ and $\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}$, but to apply Lemma 3.6.3.2 we need two vertical simplicial maps. What we will do instead is, first replace $\mathrm{l.v.}_P \circ \ell^2_{S_0}$ by a vertical map, and then, exhibit a sd-homotopy from the latter to $\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}$. Finally we will concatenate the two sd-homotopies using Lemma 3.6.3.1. By Proposition 3.6.2.9 there exists a simplicial pointed stratum-preserving homotopy

$$H_1\colon \mathrm{sd}^{k'+k''}(S_0) \wedge \Delta^1_+ \times \Delta^{\mathcal{I}} \to S_0 \times \Delta^{\mathcal{I}}$$

from $\mathrm{l.v.}_P \circ \ell^2_{S_0}$ to a basepoint preserving vertical simplicial map $f\colon \mathrm{sd}^{k'+k''}(S_0) \times \Delta^{\mathcal{I}} \to S_0 \times \Delta^{\mathcal{I}}$. Note that, by construction, $\mathrm{l.v.}_P \circ \ell^2_{S_0}$ realizes to a map that is stratum-preserving pointed homotopic to the realization of $\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}$, and so it must also be true of f. More specifically, there must exist a pointed stratified homotopy

$$H' \colon |\text{sd}^{k'+k''}(S_0) \wedge \Delta^1_+ \times \Delta^{\mathcal{I}}|_P \to |S_0 \times \Delta^{\mathcal{I}}|_P$$

Between $|f|_P$ and $|\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}|_{N(P)}$. Since f and $\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}$ are both pointed vertical maps, they define a vertical map

$$f \vee (\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}) \colon \text{sd}^{k'+k''}(S_0) \times \Delta^{\mathcal{I}} \vee \text{sd}^{k'+k''}(S_0) \times \Delta^{\mathcal{I}} \to S_0 \times \Delta^{\mathcal{I}}$$

We can now apply the relative version of Lemma 3.6.3.2 to H' and $f \vee (\text{l.v.} \times 1_{\Delta^{\mathcal{I}}})$ to get a simplicial map

$$H_2 \colon sd^{k'''}(\text{sd}^{k'+k''}(S_0) \wedge \Delta^1_+) \times \Delta^{\mathcal{I}} \to S_0 \times \Delta^{\mathcal{I}}$$

Now consider boundary preserving maps

$$\alpha_1 \colon \text{sd}^{k'+k''}(S_0 \wedge \Delta^1_+) \to \text{sd}^{k'+k''}(S_0) \wedge \Delta^1_+$$
$$\alpha_2 = \text{sd}^{k'''}(\alpha_1) \colon \text{sd}^{k'+k''+k'''}(S_0 \wedge \Delta^1_+) \to \text{sd}^{k'''}(\text{sd}^{k'+k''}(S_0) \wedge \Delta^1_+).$$

The compositions $H_1' = H_1 \circ (\alpha_1 \times 1_{\Delta^{\mathcal{I}}}) \circ (\text{l.v.} \times 1_{\Delta^{\mathcal{I}}})$ and $H_2' = H_2 \circ (\alpha \times 1_{\Delta^{\mathcal{I}}})$ give pointed sd-homotopies respectively between $\text{l.v.}_P \circ \ell^2_{S_0} \circ (\text{l.v.} \times 1_{\Delta^{\mathcal{I}}})$ and $f \circ (\text{l.v.} \times 1_{\Delta^{\mathcal{I}}})$ and between $f \circ (\text{l.v.} \times 1_{\Delta^{\mathcal{I}}})$ and $\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}$. Using Lemma 3.6.3.1 we may concatenate these sd-homotopies to a sd-homotopy

$$H \colon \text{sd}^{k'+k''+k'''+2}(S_0 \wedge \Delta^1_+) \times \Delta^{\mathcal{I}} \to S_0 \times \Delta^{\mathcal{I}},$$

between $\text{l.v.}_P \circ \ell^2_{S_0} \circ (\text{l.v.} \times \Delta^{\mathcal{I}})$ and $\text{l.v.} \times \Delta^{\mathcal{I}}$.

Finally, we take $k = k' + k'' + k''' + 2$, and ℓ to be the composition

$$\text{sd}^k(S) \times \Delta^{\mathcal{I}} \xrightarrow{\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}} \text{sd}^{k'+k''}(S) \times \Delta^{\mathcal{I}} \xrightarrow{\ell^2_S} \text{sd}_P(S \times \Delta^{\mathcal{I}}).$$

As, we have just proven, the restriction of this map to $\text{sd}^k(S_0) \times \Delta^{\mathcal{I}}$ satisfies the homotopy commutativity of (3.17). □

We now have all the necessary technical results to prove Proposition 3.6.1.4.

proof of Proposition 3.6.1.4. We first prove the direct statement. Note that a pointing $\Delta^{\mathcal{I}} \to K$ canonically lifts along l.v._P to a pointing $\Delta^{\mathcal{I}} \to \text{sd}_P(K)$. The lift is given by the composition

$$\Delta^{\mathcal{I}} \to \text{sd}_P(\Delta^{\mathcal{I}}) \to \text{sd}_P(K)$$

where the first map is specified by sending the the maximal non-degenate simplex μ of $\Delta^{\mathcal{I}}$ to $[(\mu, p_0), \ldots, (\mu, p_n)]$, for $\mathcal{I} = [p_0 < \cdots < p_n]$. We then have a diagram in the pointed category

$$\begin{tikzcd} |S \times \Delta^{\mathcal{I}}|_P \arrow[r, dashed, "\hat{\phi}"] \arrow[dr, "\phi"'] & |\text{sd}_P(K)|_P \arrow[d, "|\text{l.v.}_P|_P"] \\ & |K|_P. \end{tikzcd}$$

Note, that $|\text{l.v.}_P|_P$ is a weak equivalence between fibrant objects in $\mathbf{Strat}_P^{|\Delta^{\mathcal{I}}|_P}$, and thus, ϕ admits a lift, $\hat{\phi}$, making the above diagram commutative up to stratum-preserving pointed homotopy. Now, apply the relative version of Lemma 3.6.3.2 to $\hat{\phi}$, to produce a pointed map $\hat{f} \colon \text{sd}^k(S) \times \Delta^{\mathcal{I}} \to \text{sd}_P(K)$. The map $f = \text{l.v.}_P \circ \hat{f} \colon \text{sd}^k(S) \times \Delta^{\mathcal{I}} \to K$ is then the desired map. Indeed, we have the following pointed stratum-preserving homotopies:

$$|f|_P = |\text{l.v.}_P \circ \hat{f}|_P \simeq_P |\text{l.v.}_P|_P \circ \hat{\phi} \circ |\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}|_P \simeq_P \phi \circ |\text{l.v.} \times 1_{\Delta^{\mathcal{I}}}|_P.$$

Now, for the homotopy statement, take $S_0' := S \vee S \hookrightarrow S \wedge \Delta^1_+ =: S'$, and set

$$g' = f_0 \cup_{\Delta^{\mathcal{I}}} f_1 : {S_0}' \times \Delta^{\mathcal{I}} \to K$$

and let $H' \colon |S' \times \Delta^{\mathcal{I}}|_P \to |K|_P$ be the stratified pointed homotopy between $|f_0|_P$ and $|f_1|_P$. Next, consider the commutative diagram

$$\begin{tikzcd} |\mathrm{sd}_P(S_0' \times \Delta^{\mathcal{I}})|_P \arrow[r, "|\mathrm{sd}_P(g')|_P"] \arrow[d, hook] & |\mathrm{sd}_P K|_P \arrow[d, "|\mathrm{l.v.}_P|_P"] \\ |\mathrm{sd}_P(S' \times \Delta^{\mathcal{I}})|_P \arrow[r, "H' \circ |\mathrm{l.v.}_P|_P"] \arrow[ur, dashed, "\phi'"] & |K|_P . \end{tikzcd}$$

Since the left hand vertical is a cofibration in $\mathbf{Strat}_P$ and the right hand vertical is a weak equivalence between fibrant objects, there must exist some lift up to homotopy, ϕ', making the upper left triangle commute on the nose and the lower right triangle commute up to stratified homotopy relative to $|\mathrm{sd}_P(S_0' \times \Delta^{\mathcal{I}})|_P$. Now, precompose this diagram with ℓ from Proposition 3.6.3.3, for $k \gg 0$ to obtain the following commutative diagram:

$$\begin{tikzcd} |\mathrm{sd}^k(S_0') \times \Delta^{\mathcal{I}}|_P \arrow[r, "|\ell_0|_P"] \arrow[d, hook] & |\mathrm{sd}_P(S_0' \times \Delta^{\mathcal{I}})|_P \arrow[r, "|\mathrm{sd}_P(g')|_P"] \arrow[d, hook] & |\mathrm{sd}_P K|_P \\ |\mathrm{sd}^k(S') \times \Delta^{\mathcal{I}}|_P \arrow[r, "|\ell|_P"] & |\mathrm{sd}_P(S' \times \Delta^{\mathcal{I}})|_P \arrow[ur, "\phi'"] & \end{tikzcd}$$

Set $\phi = \phi' \circ |\ell|_P$ and $g = \mathrm{sd}_P(g') \circ \ell_0$. Now, apply Lemma 3.6.3.2 to ϕ and g. For $k' \gg 0$, we obtain a simplicial map

$$H^1 \colon \mathrm{sd}^{k+k'}(S') \times \Delta^{\mathcal{I}} \to \mathrm{sd}_P(K)$$

whose restriction to $\mathrm{sd}^{k+k'}(S_0') \times \Delta^{\mathcal{I}}$ is given by $\mathrm{sd}_P(g') \circ \ell_0 \circ (\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}})$. By naturality of $\mathrm{l.v.}_P$ one has $\mathrm{l.v.}_P \circ \mathrm{sd}_P(g') = g' \circ \mathrm{l.v.}_P$. In particular, $\mathrm{l.v.}_P \circ H^1 \colon \mathrm{sd}^{k+k'}(S') \times \Delta^{\mathcal{I}} \to K$ restricts on the boundary to $g' \circ \mathrm{l.v.}_P \circ \ell_0 \circ (\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}})$. By Proposition 3.6.3.3, this map is sd-homotopic to $g' \circ (\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}})$ through a map of the form

$$H^2 \colon \mathrm{sd}^{k+k'}(S_0' \wedge \Delta^1_+) \times \Delta^{\mathcal{I}} \to K$$

Now, the two sd-homotopies $\mathrm{l.v.}_P \circ H^1$ and H^2 can be concatenated. More explicitely, consider the inclusion $i_1 \colon S_0' \to S'$ and $i_2 \colon S_0' = S_0' \wedge \{0\}_+ \to S_0' \wedge \Delta^1_+$. These maps induce inclusions

$$i_1' \colon \mathrm{sd}^{k+k'}(S_0') \times \Delta^{\mathcal{I}} \hookrightarrow \mathrm{sd}^{k+k'}(S') \times \Delta^{\mathcal{I}}$$
$$i_2' \colon \mathrm{sd}^{k+k'}(S_0') \times \Delta^{\mathcal{I}} \hookrightarrow \mathrm{sd}^{k+k'}(S_0' \wedge \Delta^1_+) \times \Delta^{\mathcal{I}}$$

The restrictions $\mathrm{l.v.}_P \circ H^1 \circ i_1'$ and $H^2 \circ i_2'$ are both equal to $g' \circ \mathrm{l.v.}_P \circ \ell_0 \circ (\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}})$, by construction. In particular, $\mathrm{l.v.}_P \circ H^1$ and H^2 can be glued along $\mathrm{sd}^{k+k'}(S_0') \times \Delta^{\mathcal{I}}$ to obtain a map

$$H^3 \colon \mathrm{sd}^{k+k'}(S' \cup_{S_0'} (S_0' \wedge \Delta^1_+)) \times \Delta^{\mathcal{I}} \to K,$$

which restricts to $g' \circ (\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}}) = (f_0 \cup_{\Delta^{\mathcal{I}}} f_1) \circ (\mathrm{l.v.} \times 1_{\Delta^{\mathcal{I}}})$ on the boundary. Now to turn this map into an sd-homotopy, consider the following composition:

$$\alpha \colon \mathrm{sd}^{k+k'+2}(S \wedge \Delta^1_+) \to \mathrm{sd}^{k+k'}(S \wedge \mathrm{sd}^2(\Delta^1_+)) \to \mathrm{sd}^{k+k'}(S' \cup_{S_0'} (S_0' \wedge \Delta^1_+)),$$

where the left hand map comes from the natural map $\mathrm{sd}^2(S \wedge \Delta^1_+) \to S \wedge \mathrm{sd}^2(\Delta^1_+)$ and the right hand map is the one suggested by Fig. 3.8. To construct the latter explicitely, it is enough to note that there is an isomorphism

$$S' \cup_{S_0'} (S_0' \wedge \Delta^1_+) \cong S \wedge (\Delta^1 \cup_{1,0} \Delta^1 \cup_{1,1} \Delta^1)_+,$$

and a boundary preserving map $\mathrm{sd}^2(\Delta^1) \to \Delta^1 \cup_{1,0} \Delta^1 \cup_{1,1} \Delta^1$. Now, the desired pointed sd-homotopy is given by the composition

$$H = H^3 \circ (\alpha \times 1_{\Delta^{\mathcal{I}}}) \colon \mathrm{sd}^{k+k'+2}(S \wedge \Delta^1_+) \times \Delta^{\mathcal{I}} \to K.$$

□

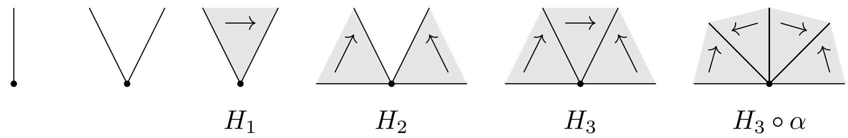


Figure 3.8: From left to right, schematic pictures of the simplicial sets, S, S'_0, S', $S'_0 \wedge \Delta^1_+$, the union $S' \cup_{S'_0} (S'_0 \wedge \Delta^1_+)$ and the smash product $S \wedge \mathrm{sd}^2(\Delta^1_+)$. Pointing are indicated by large dots, and arrows indicate direction of homotopies.

3.A Incompatible criteria for model structures of stratified spaces

The goal of this section is to prove the following proposition, and its corollary.

Proposition 3.A.0.1. *For P a non-discrete poset, there exists no model structure on $\mathbf{Strat}_P$ satisfying all of the following properties:*

1. *Realizations of monomorphisms are cofibrations.*
2. *Stratified homotopy equivalences are weak equivalences.*
3. *For a weak equivalence, the induced map between classical homotopy links is also a weak-equivalence.*

Remark 3.A.0.2. In fact, we will prove a slightly stronger version of Proposition 3.A.0.1, where condition 3 is replaced by the following:

(3') For some fixed flag, $\mathcal{I} = \{p < q\}$, the homotopy link functor $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}$ sends weak equivalences to weak equivalences of topological spaces.

Corollary 3.A.0.3. *If P is a non-discrete poset, then the model structure on $\mathbf{Strat}_P$ transported from $\mathrm{s}\mathbf{Strat}_P$ along the adjunction $|-|_P \colon \mathrm{s}\mathbf{Strat}_P \leftrightarrow \mathbf{Strat}_P \colon \mathrm{Sing}_P$ does not exist.*

Let us first deduce the corollary from the proposition.

Proof. Assume that the transported model structure on $\mathbf{Strat}_P$ exists. Then $|-|_P \dashv \mathrm{Sing}_P$ is a Quillen adjunction and so the functor $|-|_P$ must preserve cofibrations, which implies that the model structure satisfies (1). Furthermore, Sing_P preserves stratified homotopy equivalences, so the model structure must satisfy (2). Finally, if $f \colon X \to Y$ is a weak-equivalence, then, by definition, $\mathrm{Sing}_P(f)$ must be a weak-equivalence in $\mathrm{s}\mathbf{Strat}_P$. By Corollary 3.6.0.2, this implies that f induces weak-equivalences $\mathrm{HoLink}_{\mathcal{I}}(\mathrm{Sing}_P(X)) \to \mathrm{HoLink}_{\mathcal{I}}(\mathrm{Sing}_P(Y))$ for all regular flags $\mathcal{I}$. By Remark 3.2.5.3, this implies that f induces weak-equivalences $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y)$ for all regular flags $\mathcal{I}$. In particular, the model category must also satisfy (3). But then, it satisfies simultaneously (1), (2) and (3), which is a contradiction. □

Proposition 3.A.0.1 also has implications for other model structures.

Remark 3.A.0.4. Consider Haine's model stucture on the category of stratified simplicial sets, $s\mathbf{Strat}_P^{\text{Joyal-Kan}}$ [Hai23]. It can not be transported along the adjunction $(|-|_P, \mathrm{Sing}_P)$ to a model structure on $\mathbf{Strat}_P$. Indeed, supposing this is possible, this hypothetical model structure would satisfy (1), since the cofibrations in $s\mathbf{Strat}_P^{\text{Joyal-Kan}}$ are the monomorphisms. Furthermore, it would satisfy (2), since Sing_P preserves stratified homotopy equivalences, and those are weak-equivalences in $s\mathbf{Strat}_P^{\text{Joyal-Kan}}$. But then, it can not satisfy (3), by Proposition 3.A.0.1. This implies that if such a model structure existed, the classical homotopy links would not be invariants of the stratified homotopy type.

However, we can show that **some** homotopy links would have to be preserved, preventing the existence of this model structure altogether, by the strengthened version of Proposition 3.A.0.1 (see Remark 3.A.0.2). Indeed, let us first assume that P admits a successor pair, that is $p < q \in P$ such that there exists no $m \in P$ satisfying $p < m < q$. Then, by Lemma 3.A.0.8, and recalling that

$$\mathrm{Sing}(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X)) = \mathrm{HoLink}_{\mathcal{I}}(\mathrm{Sing}_P(X)),$$

we see that $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}$ sends weak equivalences to weak equivalences of spaces, for $\mathcal{I} = [p < q]$. The argument is slightly more subtle if P does not admit successor pairs. Let $p < q \in P$, and consider stratified spaces of the form $X \to P$ such that the m stratum of X is empty for all $p < m < q$. Then, $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}$ sends weak equivalences between such stratified spaces to weak equivalences of topological spaces, for $\mathcal{I} = [p < q]$. But this also leads to a contradiction when considering Example 3.A.0.6 as stratified over $\{p < q\} \subset P$.

Remark 3.A.0.5. Similarly, let Strat^N be the category of such stratified spaces, which have no empty strata, as defined in [Nan19]. In [Nan19] the author asked the question, whether the structure on the model category for quasi categories $\mathbf{sSet}^{Joyal}$ can be transferred to Strat^N along an adjunction defined using the functors $(|-|, \mathrm{Sing}_{\mathbf{Strat}})$. Arguing as in the proof of Proposition 3.A.0.1, and using Example 3.A.0.6, we may show that it is indeed not possible.

Assume that the transported model structure exists. Then, since cofibrations in $\mathbf{sSet}^{\text{Joyal}}$ are monomorphisms, the model category would satisfy the analogue of (1). Furthermore, assume that $H\colon X \times [0,1] \to Y$ is a stratified homotopy, between two stratified maps $f, g\colon X \to Y$, and consider the following composition

$$\mathrm{Sing}_{\mathrm{Strat}}(X) \times \mathrm{Sing}([0,1]) \to \mathrm{Sing}_{\mathrm{Strat}}(X \times [0,1])) \to \mathrm{Sing}_{\mathrm{Strat}}(Y). \tag{3.18}$$

This composition is a homotopy (in the Joyal model structure) between $\mathrm{Sing}_{\mathrm{Strat}}(f)$ and $\mathrm{Sing}_{\mathrm{Strat}}(g)$. This follows from the fact that $\mathrm{Sing}([0,1])$ is a cylinder in the Joyal model structure. In particular, in the transported model structure on $\mathbf{Strat}^N$, stratum-preserving homotopy equivalences are weak-equivalences, i.e. the model category satisfies the analoge of (2). Now, assume that $f\colon X \to Y$, is a stratum-preserving map over the poset $P = \{0 < 1\}$ such that $\mathrm{Sing}_{\mathrm{Strat}}(f)$ is a Joyal equivalence in $\mathbf{sSet}$. Then, the map $\mathrm{Sing}_P(f)\colon \mathrm{Sing}_P(X) \to \mathrm{Sing}_P(Y)$ is a map in $s\mathbf{Strat}_P$, and is a weak equivalence in the structure $s\mathbf{Strat}_P^{\text{Joyal}}$ (that is, the slice model category). But since $s\mathbf{Strat}_P^{\text{Joyal-Kan}}$ is a Bousfield localization of the former, $\mathrm{Sing}_P(f)$ is also a weak equivalence in $s\mathbf{Strat}_P^{\text{Joyal-Kan}}$. Finally, note that for $P = \{0 < 1\}$, the model structure on $s\mathbf{Strat}_P^{\text{Joyal-Kan}}$ coincides with the one studied in this paper, giving that $\mathrm{Sing}_P(f)$ must be a weak equivalence in $s\mathbf{Strat}_P$, and hence, must induce weak-equivalences

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y).$$

Applying the above to the map of Example 3.A.0.6 gives a contradiction, hence the transported model structure on $\mathbf{Strat}^N$ may not exist.

Proposition 3.A.0.1 is a direct consequence of the following example.

Example 3.A.0.6. Consider the simplicial complex Y with vertices a, b, c, d, e, f generated by the simplices $\{a, b, c, d\}$, $\{a, c, e\}$ and $\{a, d, f\}$ and with the stratification over $P = \{0 < 1\}$ defined as follows. The entire segment $\{a, b\}$ is sent to 0, then, consider two segments between

a and the middle of $\{d, f\}$ and between a and the middle of $\{e, c\}$, which we will promptly identify with two copies of the interval $[0, 1]$ (with a at 0). In those intervals, send all the points of the sequence $\frac{1}{2^{n+1}}$, for $n \geq 0$, to 0 (any decreasing sequence converging to 0 will give an isomorphic result). The remaining points are mapped to 1.

Note that the stratified space obtained in this way is not isomorphic in $\mathbf{Strat}_P$ to the realization of a stratified simplicial set. In particular, Y does not come from a strongly stratified space. Nevertheless one can compute its homotopy link with respect to the flag $\mathcal{I} = [0 < 1]$. We will also consider the subspace $X \subset Y$ obtained by deleting the maximal simplex $\{a, b, c, d\}$ and its face $\{a, c, d\}$, see Fig. 3.9.

We will only be interested in $\pi_0(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X))$, and $\pi_0(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y))$ or in other words, in exit-paths up to stratum-preserving homotopies. Note that any exit path starting from one of the isolated points is entirely determined - up to stratum-preserving homotopy - by its starting point. Similarly, any point starting in the interval $(a, b]$ is equivalent to the exit-path spanning the segment $[b, d]$.

On the other hand, there are a vast number of inequivalent classes of exit-paths starting from a. A particular set of exit-paths that can be easily described is those that get away from a steadily (as in Fig. 3.9). For such an exit-path in X, it is enough to specify which face ($\{a, d, f\}$ or $\{a, c, e\}$) it lies on, and then for each isolated singular point if the path passes under, or over. This can be summarized as a binary sequence (where the first few terms, corresponding to points away from a might be ill-defined). One then checks that two such exit-paths in X are equivalent if and only if they lie on the same face and their associated sequence differ in only finitely many places. This implies in particular that $\pi_0(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X))$ is uncountable.

Now consider $\pi_0(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y))$. The paths we just described in X also exist as exit-paths in Y, but there also exist paths going back on forth between the faces $\{a, c, e\}$ and $\{a, d, f\}$. If one restricts to those paths that are moving away from a, it is still possible to parametrize those by ternary sequences, where the n-th entry indicates if the path passes to the left, right or middle of the n-th pair of isolated singular points. One can show again that two such paths are equivalent up to stratum-preserving homotopies if and only if their associated sequences differ only at finitely many places. But now, note that an exit-path in Y zig-zaging infinitely many times between the left side of $\{a, c, e\}$ and the right side of $\{a, d, f\}$ can not be in the image of $\pi_0(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X)) \to \pi_0(\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y))$. This implies that $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(Y)$ is not a weak-equivalence, and in turn that $X \to Y$ is not a weak-equivalence in $\mathbf{Strat}_P$.

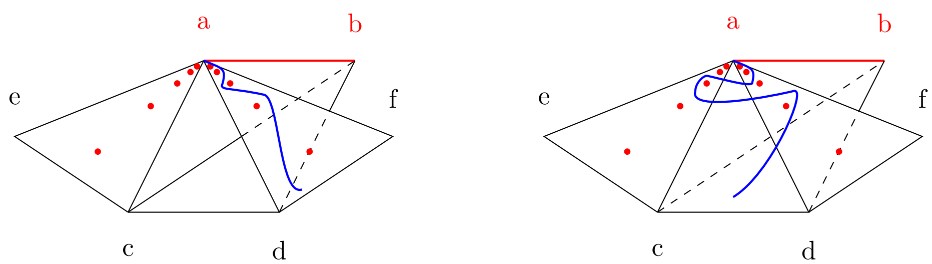


Figure 3.9: the simplicial complexes X and Y with their pathological stratifications over $P = \{0 < 1\}$. The isolated singular points admit a as an accumulation point. Two exit-paths starting from a have been represented in blue.

Proof of Proposition 3.A.0.1. Assume that there exists a model structure on $\mathbf{Strat}_P$ satisfying (1) and (2). This implies that $|\Lambda^{\mathcal{J}}_k|_P \to |\Delta^{\mathcal{J}}|_P$ is a trivial cofibration in this model structure for any admissible horn inclusion $\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}}$ since it is simultaneously the realization of a cofibration and a stratified homotopy equivalence (see [Dou21a, Proposition 1.13]). Now note that the inclusion $X \hookrightarrow Y$ from Example 3.A.0.6 can be obtained as the following pushout

$$\begin{tikzcd} |\Lambda^{\mathcal{J}}_k|_P \arrow[r] \arrow[d] & X \arrow[d] \\ |\Delta^{\mathcal{J}}|_P \arrow[r] & Y \end{tikzcd}$$

where $\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}}$ is an admissible horn inclusion. In particular, $X \to Y$ is the pushout of a trivial cofibration, so it must be a trivial cofibration. But as demonstrated in Example 3.A.0.6, the inclusion $X \to Y$ does not induce weak-equivalences on all classical homotopy links. In particular, the model structure can not satisfy (3), which concludes the proof. □

Remark 3.A.0.7. One might object, that Example 3.A.0.6 is pathological insofar, as the 0-stratum considered as a subspace in the usual subspace topology is not Δ-generated. It may be possible to pass to an appropriate tamer subcategory of $\mathbf{Strat}_P$, which would then allow for the structure of a model category fulfilling all of the requirements of Proposition 3.A.0.1. The main point of this article however, is that even without attempting to change the underlying point-set topological framework and defining such a model structure, many of the consequences of its existence may nevertheless be obtained.

Lemma 3.A.0.8. *Let $\mathcal{I} = \{p < q\}$ be a flag in P and let $X \in \mathrm{s}\mathbf{Strat}_P^{Joyal-Kan}$ be such that X_m is empty, for all $p < m < q$. Then, for any weak equivalence $X \to Y$ in $\mathrm{s}\mathbf{Strat}_P^{Joyal-Kan}$, the induced map on simplicial homotopy links,* $\mathrm{HoLink}_{\mathcal{I}}(X) \to \mathrm{HoLink}_{\mathcal{I}}(Y)$*, is a weak equivalence in the Kan model structure on simplicial sets.*

Proof. Recall that the fibrant objects in $\mathbf{sSet}_P^{\text{Joyal-Kan}}$ are those that admit the right lifting property against all stratified horn inclusions $\Lambda^{\mathcal{J}}_k$ which are either inner horn inclusions, or admissible horn inclusions. Call those horn inclusions weakly admissible. Now, given a weak-equivalence in $\mathbf{sSet}_P^{\text{Joyal-Kan}}$, $f\colon X \to Y$, where the m-stratum of X is empty for all $p < m < q$, consider fibrant replacements for X and Y obtained by the small object argument applied to the set of weakly admissible horn inclusions. We get a commutative diagram of weak equivalences

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \\ X^{\mathrm{fib}} \arrow[r, "f^{\mathrm{fib}}"] & Y^{\mathrm{fib}} \end{tikzcd}$$

Now, since $\mathbf{sSet}_P^{\text{Joyal-Kan}}$ is a left Bousfield localization of $\mathrm{s}\mathbf{Strat}_P$ (see Remark 3.2.9.4), and since the map f^{fib} is a weak equivalence between fibrant object in $\mathbf{sSet}_P^{\text{Joyal-Kan}}$, it must also be a weak equivalence in $\mathrm{s}\mathbf{Strat}_P$. But then, by Corollary 3.6.0.2, f^{fib} must induce a weak equivalence $\mathrm{HoLink}_{\mathcal{I}}(X^{\mathrm{fib}}) \to \mathrm{HoLink}_{\mathcal{I}}(Y^{\mathrm{fib}})$. By two out of three, it is thus enough to show Lemma 3.A.0.8 for maps of the form $X \hookrightarrow X^{\mathrm{fib}}$ (noting that the m-strata of Y must also be empty for $p < m < q$, since weak equivalences in $\mathbf{sSet}_P^{\text{Joyal-Kan}}$ preserve the homotopy type of strata). Now, by Theorem 3.1.0.4, together with the (pseudo)-naturality of $\mathrm{Link}_{\mathcal{I}}$ with respect to monomorphisms (see Proposition 3.4.1.1), we can reduce to comparing $\mathrm{Link}_{\mathcal{I}}(X)$ and $\mathrm{Link}_{\mathcal{I}}(X^{\mathrm{Fib}})$. Finally, since the map $X \hookrightarrow X^{\mathrm{fib}}$ is a transfinite composition of pushouts along weakly admissible horn inclusions, and since $\mathrm{Link}_{\mathcal{I}}$ preserves colimits, it is enough to prove Lemma 3.A.0.8 for maps $X \hookrightarrow Y$ obtained by pushing out along a single weakly admissible horn inclusion, $\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}}$. Now, note that since X has empty m strata for all $p < m < q$, so must have $\Lambda^{\mathcal{J}}_k$, and $\Delta^{\mathcal{J}}$. Finally, it is enough to show that

$$\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k) \hookrightarrow \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$$

is a weak equivalence of simplicial sets, whenever $\Delta^{\mathcal{J}}$ has empty m-strata, for $p < m < q$. Let $\mathcal{J} = [p_0 \le \cdots \le p_n]$, and consider the following cases.

- if $\Lambda^{\mathcal{J}}_k$ is an admissible horn, then this follows from Lemma 3.4.3.2.
- else, $\Lambda^{\mathcal{J}}_k$ is an inner horn, and thus $n \geq 2$. But then, the only vertices that can possibly be in $\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$ but not in $\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k)$ are those corresponding to the simplex $\Delta^{\mathcal{J}}$ and to its face, $\Delta^{\mathcal{J}'} = d_k(\Delta^{\mathcal{J}})$. Those would be in $\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$ if and only if they degenerate from $\Delta^{\mathcal{I}}$. If the former degenerates from $\Delta^{\mathcal{I}}$, $\Lambda^{\mathcal{J}}_k$ is an admissible horn. If the latter degenerates from $\Delta^{\mathcal{I}}$, then $\Delta^{\mathcal{J}'} = [p_0, \dots, \widehat{p_k}, \dots, p_n]$ with $p_i = p$ or q for all $i \neq k$. But since $p_{k-1} \leq p_k \leq p_{k+1}$, and since $\Delta^{\mathcal{J}}$ has empty m-strata for all $p < m < q$, then $p_k = p$ or q, and the horn is also admissible. Finally, in all the remaining cases, we have $\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k) = \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$.

$\square$

3.B Relating labellings, vertical stratifications and diagrams.

In Remark 3.6.2.8 we have already hinted at the fact that P-labelled simplicial sets can be thought of as a particularly concise description of certain cofibrant diagrams in $\mathbf{Diag}_P$. Let us now expand on this and make the relationship between labelled objects, diagrams and vertical objects precise. Before we do so, let us quickly remark on the topological counterpart of $\mathbf{Diag}_P$. We denote $\mathbf{Diag}_P^{\mathbf{Top}} := \mathrm{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$.

Remark 3.B.0.1. Just like its simplicial counterpart $\mathbf{Diag}_P^{\mathbf{Top}}$ can be equipped with the projective model structure (use for example [Hir03, Thrm. 11.6.1]). Assigning to a P-stratified space X the diagram given by its (topological) homotopy links $\mathcal{I} \mapsto \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(X)$, and conversely sending a diagram F to $\int^{\mathcal{I}} F(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_P$ then defines a Quillen adjunction

$$D'_P : \mathbf{Diag}_P^{\mathbf{Top}} \leftrightarrow \mathbf{Strat}_P : C'_P.$$

We obtain a diagram of Quillen functors

$$\begin{tikzcd} & \mathbf{Diag}_P^{\mathbf{Top}} \arrow[dr, shift left, "C'_P"] \arrow[dl, shift left] & \\ \mathbf{Diag}_P \arrow[ur, shift left] \arrow[rr, shift left, "C_P"] & & \mathbf{Strat}_P \arrow[ll, shift left, "D_P"] \arrow[ul, shift left, "D'_P"] \end{tikzcd} \tag{3.19}$$

where the left diagonals are induced by the adjunction $|-| \dashv \mathrm{Sing}$. Both the left adjoint, as well as the right adjoint part of this diagram commute up to natural isomorphism. Hence, it follows from the two out of three property for Quillen equivalences that $C'_P \dashv D'_P$ is also a Quillen equivalence.

Next, let us describe the relationship between labelled objects and diagrams defined on $\mathrm{sd}(P)^{\mathrm{op}}$. Both P-labeled simplicial sets, as well as CW-complexes can readily be equipped with a functor

$$U : P\text{-}\mathbf{sSet} \to \mathbf{Diag}_P,$$
$$U^{\mathbf{Top}} : P\text{-}\mathbf{CW} \to \mathbf{Diag}_P^{\mathbf{Top}}$$

respectively. In case of a P-labeled CW-complex (T, λ_T), the diagram $U^{\mathbf{Top}}(T, \lambda_T) \in \mathbf{Diag}_P^{\mathbf{Top}}$, at a regular flag $\mathcal{I}$, is given by the unions of cells

$$U^{\mathbf{Top}}(T, \lambda_T)(\mathcal{I}) = \bigcup_{e_\alpha,\ \Delta^{\mathcal{I}} \subset \lambda_T(e_\alpha)} e_\alpha,$$

and structure maps are given by inclusions. One extends this construction to morphisms in the obvious way. The definition for simplicial sets is analogous, replacing open cells by non-degenerate simplices. The precise behavior of the U functors is then described in the following proposition:

Proposition 3.B.0.2. *The functors U and $U^{\mathbf{Top}}$ are fully faithful and fit into a commutative diagram*

$$\begin{tikzcd} P\text{-}\mathbf{sSet} \arrow[r, "U"] \arrow[d] & \mathbf{Diag}_P \arrow[d] \\ P\text{-}\mathbf{CW} \arrow[r, "U^{\mathbf{Top}}"] & \mathbf{Diag}_P^{\mathbf{Top}} \end{tikzcd}$$

with verticals induced by realization. Both U and $U^{\mathbf{Top}}$ factor through the respective subcategories of cofibrant diagrams. Furthermore, U induces an equivalence of categories P-$\mathbf{sSet} \xrightarrow{\sim} \mathbf{Diag}_P^{\mathrm{Cof}}$.

Proof. Commutativity is easily verified from the constructions, while fully-faithfulness is an immediate consequence of the definition of labelled maps. That both U functors have image in the subcategories of cofibrant objects follows from the characterization of generating cofibrations in a projective model structure in [Hir03, Thrm. 11.6.1]. Finally, that every cofibrant diagram in $\mathbf{Diag}_P$ is in fact (up to natural isomorphism) of the form $U(S, \lambda_S)$, for $(S, \lambda_S) \in P$-$\mathbf{sSet}$, follows from the characterization of cofibrant diagrams in Proposition 3.2.7.4 (see also Remark 3.6.2.8). ☐

In this sense, P-labeled simplicial sets are a particularly concise, but equivalent, description of cofibrant diagrams. Since not every absolute cell complex is a CW-complex, the analogous essential surjectivity fails for P-labelled CW-complexes. However, as a consequence of Theorem 3.6.2.18, essential surjectivity is restored when passing to homotopy categories.

Next, let us study the precise relationship of diagrams with verticalization. One easily verifies that the two diagrams

$$\begin{tikzcd} P\text{-}\mathbf{sSet} \arrow[rr, "U"] \arrow[dr, "V"'] & & \mathbf{Diag}_P \arrow[dl, "C_P"] \\ & \mathrm{s}\mathbf{Strat}_P & \end{tikzcd}, \quad \begin{tikzcd} P\text{-}\mathbf{CW} \arrow[rr, "U^{\mathbf{Top}}"] \arrow[dr, "V"'] & & \mathbf{Diag}_P^{\mathbf{Top}} \arrow[dl, "C'_P"] \\ & \mathbf{Strat}_P & \end{tikzcd},$$

commute up to natural isomorphism. In this sense, we can think of verticalization as an alternative description of the C_P functors. Now, clearly the verticalization functors are not full. We can amend this by passing to the vertical setting. Denote by $\mathrm{Vs}\mathbf{Strat}_P$ and $\mathrm{V}\mathbf{CW}_P$ respectively the categories of vertically stratified simplicial sets and CW-spaces, with vertical maps. Using the natural pre-verticalizations which the verticalization functors come with, we obtain lifts

$$\begin{aligned} V\colon & P\text{-}\mathbf{sSet} \to \mathrm{Vs}\mathbf{Strat}_P \\ & (S, \lambda_S) \to \{V(S, \lambda_S) \hookrightarrow S \times N(P)\}, \\ V\colon & P\text{-}\mathbf{CW} \to \mathrm{V}\mathbf{CW}_P \\ & (T, \lambda_T) \to \{V(T, \lambda_T) \hookrightarrow T \times |N(P)|_P\}, \end{aligned}$$

denoted the same by abuse of notation. It turns out that both these functors induce equivalences of categories. One may immediately verify from the definition of label preserving maps that these two functors are fully faithful. They are essentially surjective, by construction of the categories of vertical objects.

Hence, verticalization defines equivalences between P-labeled and vertical objects. In this sense, using Proposition 3.B.0.2, the vertical categories give a description of (certain) cofibrant diagrams, which is more intrinsic to $\mathrm{s}\mathbf{Strat}_P$ or $\mathbf{Strat}_P$. Together with Proposition 3.B.0.2 we may summarize the whole situation in the following proposition.

Proposition 3.B.0.3. *The following diagram of functors commutes up to natural isomorphism.*

$$\begin{tikzcd}
s\mathbf{Strat}_P \arrow[d] & \mathrm{V}s\mathbf{Strat}_P \arrow[l] \arrow[d] & P\text{-}\mathbf{sSet} \arrow[l, "V"', "\sim"] \arrow[r, "U", "f.f."'] \arrow[d] & \mathbf{Diag}_P^{\mathrm{Cof}} \arrow[d] \arrow[lll, bend right, "C_P"'] \\
\mathbf{Strat}_P & \mathrm{V}\mathbf{CW}_P \arrow[l] & P\text{-}\mathbf{CW} \arrow[l, "V"', "\sim"] \arrow[r, "U^{\mathbf{Top}}", "f.f"'] & \mathbf{Diag}_P^{\mathbf{Top},\mathrm{Cof}} \arrow[lll, bend left, "C'_P"]
\end{tikzcd} \tag{3.20}$$

Here, all verticals are induced by realization, and the left two horizontals are given by the forgetful functors forgetting the pre-verticalizations. Both V functors define equivalences of categories. The U functors are both fully faithful, and the upper one is even an equivalence of categories.

The fact that P-labeled CW-complexes embed fully faithfully into $\mathbf{Diag}_P^{\mathbf{Top},\mathrm{Cof}}$ (Proposition 3.B.0.2), allows one to model the homotopy category $\mathrm{ho}\mathbf{Strat}_P$ through these objects. To see this, we first need a definition of label preserving homotopy.

Definition 3.B.0.4. Let $f, g\colon (T, \lambda_T) \to (T', \lambda_{T'})$ be two label preserving maps between P-labelled CW-complexes. A *label preserving homotopy* from f to g is a map,

$$H\colon T \times [0,1] \to T',$$

such that for all cells $e_\alpha \in T$ and $e_\beta \in T'$, if $H(e_\alpha \times [0,1]) \cap e_\beta \neq \varnothing$, then $\lambda_T(e_\alpha) \subset \lambda_{T'}(e_\beta)$. Equivalently, H is a label-preserving map for the induced labelling on $T \times [0,1]$. If such a homotopy exists, we say f and g are label preserving homotopic, writing $f \simeq_{\mathrm{lab}} g$. We write $P\text{-}\mathbf{CW}/\simeq_{\mathrm{lab}}$ for the category whose objects are P-labelled CW-complex and whose sets of morphisms between (T, λ_T) and $(T', \lambda_{T'})$ is $P\text{-}\mathbf{CW}((T, \lambda_T), (T', \lambda_{T'}))/\simeq_{\mathrm{lab}}$.

Remark 3.B.0.5. Note, that under the fully faithful functor $V\colon P\text{-}\mathbf{CW} \to \mathrm{V}\mathbf{CW}_P$, P-labelled homotopies correspond one to one to vertical homotopies. Furthermore, under the fully faithful functor $U^{\mathbf{Top}}\colon P\text{-}\mathbf{CW} \to \mathbf{Diag}_P^{\mathbf{Top},\mathrm{Cof}}$ they correspond one to one to those homotopies in $\mathbf{Diag}_P^{\mathbf{Top}}$ which are defined through the cylinder given by $(F \times [0,1])(\mathcal{I}) = F(\mathcal{I}) \times [0,1]$.

We may use Proposition 3.B.0.3 to obtain a more high-level proof of part of Proposition 3.6.2.16. We focus on the $\mathbf{Strat}_P$ part here. A proof for $\mathbf{Top}_{N(P)}$ works analogously.

Alternative proof of Proposition 3.6.2.16. Let us first focus on the absolute case. By Remark 3.B.0.5 and Proposition 3.B.0.3, we may equivalently show that in the following following commutative diagram of categories

$$\begin{tikzcd}
P\text{-}\mathbf{CW} \arrow[r, "U^{\mathbf{Top}}", "f.f."'] \arrow[d] \arrow[rr, bend left, "V"] & \mathbf{Diag}_P^{\mathbf{Top},\mathrm{Cof}} \arrow[r, "C'_P"] \arrow[d] & \mathbf{Top}_P^{\mathrm{Cof}} \arrow[d] \\
P\text{-}\mathbf{CW}/\simeq_{\mathrm{lab}} \arrow[r] & \mathbf{Diag}_P^{\mathbf{Top},\mathrm{Cof}}/_{\sim} \arrow[r] & \mathrm{ho}\mathbf{Strat}_P,
\end{tikzcd}$$

the lower horizontal composition is fully faithful. Here, by $\mathbf{Diag}_P^{\mathbf{Top},\mathrm{Cof}}/_{\sim}$ we denote the category obtained by identifying morphisms which are homotopic through the cylinder given by $(F \times [0,1])(\mathcal{I}) = F(\mathcal{I}) \times [0,1]$. By Proposition 3.B.0.3, the left upper horizontal is fully faithful. Using the second part of Remark 3.B.0.5, we also obtain that the left lower horizontal is fully faithful. Since every object in $\mathbf{Diag}_P^{\mathbf{Top}}$ is fibrant, we have a natural equivalence $\mathbf{Diag}_P^{\mathbf{Top},\mathrm{Cof}}/_{\sim} \cong \mathrm{ho}\mathbf{Diag}_P^{\mathbf{Top}}$. Now, finally note that $C_P'^{\mathbf{Top}}\colon \mathbf{Diag}_P^{\mathbf{Top}} \to \mathbf{Strat}_P$ is the left part of a Quillen equivalence (see Remark 3.B.0.1). It follows, that the last remaining lower horizontal arrow is also fully faithful, finishing the proof of the absolute case. The relative case follows analogously, after verifying that $U^{\mathbf{Top}}$ maps inclusion $(A, \lambda_A) \hookrightarrow (T, \lambda_T)$ (where λ_A is the induced labelling) to a cofibration in the projective model structure on $\mathbf{Diag}_P^{\mathbf{Top}}$. □

Finally, we may also use Proposition 3.B.0.3 together with Remark 3.B.0.5 to obtain the following alternative version of Theorem 3.6.2.18, which allows one to perform stratified homotopy theory in the labelled setting.

Corollary 3.B.0.6. *Verticalization induces an equivalence of categories*

$$P\text{-}\mathbf{CW}/\simeq_{\text{lab}} \cong \text{ho}\mathbf{Strat}_P.$$

Chapter 4

From samples to persistent stratified homotopy types

Note to the reader: The following chapter presents the article [MW24], which was written in joint work with Tim Mäder and appeared in the *Journal of Applied and Computational Topology*. We have slightly adapted some notation in order to be consistent with Chapter 1. The only major notation difference is that stratified spaces are referred to with calligraphic notation, in Chapter 1, while we will usually just use a different letter for the stratified space and its underlying topological space in this chapter.

The natural occurrence of singular spaces in applications has led to recent investigations on performing topological data analysis (TDA) in a stratified framework. In many applications, there is no a priori information on what points should be regarded as singular or regular. For this purpose we describe a fully implementable process that provably approximates the stratification for a large class of two-strata Whitney stratified spaces from sufficiently close non-stratified samples.
Additionally, in this work, we establish a notion of persistent stratified homotopy type obtained from a sample with two strata. In analogy to the non-stratified applications in TDA which rely on a series of convenient properties of (persistent) homotopy types of sufficiently regular spaces, we show that our persistent stratified homotopy type behaves much like its non-stratified counterpart and exhibits many properties (such as stability, and inference results) necessary for an application in TDA.
In total, our results combine to a sampling theorem guaranteeing the (approximate) inference of (persistent) stratified homotopy types of sufficiently regular two-strata Whitney stratified spaces.

4.1 Introduction

Topological data analysis has proven itself to be a source of qualitative and quantitative data features that were not readily accessible by other means. Arguably, the most important concept for the development of this field is persistent homology ([ELZ00; ZC05; CEH07; Ghr08; NSW08; Car09; Oud15]). Both in practice, as well as abstractly speaking, persistent homology usually is divided up into a two-step process. First, one assigns to a data set $\mathbb{X}$ a filtration of topological or combinatorial objects $(\mathbb{X}_\alpha)_{\alpha \geq 0}$. Most prominently, this is done for $\mathbb{X} \subset \mathbb{R}^N$, by taking $\mathbb{X}_\alpha$ to be an α-thickening of $\mathbb{X}$, which is the case we will consider in the following. Then, from this filtered object, a persistence module is computed, essentially given by computing homology in each filtration degree while keeping track of the functoriality of homology on the inclusions. As homology is a homotopy invariant what is relevant to this computation is only what one may call the *persistent homotopy type of* $\mathbb{X}$. More precisely, if we think of $(\mathbb{X}_\alpha)_{\alpha\geq 0}$ as a functor from the non-negative reals $\mathbb{R}_+$ into the category of topological spaces **Top** then the persistent homotopy type is the isomorphism class of $(\mathbb{X}_\alpha)_{\alpha\geq 0}$ in the homotopy category ho$\mathbf{Top}^{\mathbb{R}_+}$ obtained by inverting pointwise (weak) homotopy equivalences. From this perspective, persistent homology is the composition

$$\mathrm{PH}_i\colon \mathbf{Sam} \xrightarrow{\mathcal{P}} \mathrm{ho}\mathbf{Top}^{\mathbb{R}_+} \xrightarrow{\mathrm{H}_i^{\mathbb{R}_+}} \mathbf{Vec}_k^{\mathbb{R}_+}. \tag{4.1}$$

Here **Sam** is the category of subspaces of some fixed $\mathbb{R}^N$, $\mathcal{P}$ assigns an object in the persistent homotopy category (for example, through thickening spaces or possibly using combinatorial models thereof) and $\mathrm{H}_i^{\mathbb{R}_+}$ computes homology degree-wise. The composition produces an object in the category of persistence modules over some field k, denoted $\mathbf{Vec}_k^{\mathbb{R}_+}$. Many of the advantages of persistent homology turn out to not be properties of the right-hand side of this composition but of the left-hand side $\mathcal{P}$. That is, they are properties of the persistent homotopy type. Such properties include, for example:

P(1): The fact that persistent homology defined through thickenings is computable at all; (This is a consequence of the nerve theorem (see e.g. [Hat02, Prop. 4G.3] or [Bor48]), which states that for $\mathbb{X} \subset \mathbb{R}^N$ the persistent stratified homotopy type $\mathcal{P}(\mathbb{X})$ may equivalently be represented by a filtered Čech complex.)

P(2): The stability of persistent homology with respect to Hausdorff and interleaving type distances (see [CEH07; Cha+09; BL15]);

P(3): The possibility to infer information from the sampling source by using persistent homology. (This is usually justified by stability together with the result that $\mathcal{P}(T)_\alpha \xleftarrow{\simeq} \mathcal{P}(T)_0 = T$ for α sufficiently small and T a sufficiently regular space such as a compact smooth submanifold of Euclidean space (compare to [NSW08]).)

At the same time, many of the limitations of persistent homology also stem from the factorization in (4.1). Consider, for example, the two subspaces of $\mathbb{R}^2$ depicted in Fig. 4.1. It is not hard to see that (up to a rescaling) they have the same persistent homotopy type and thus have the same persistent homology.

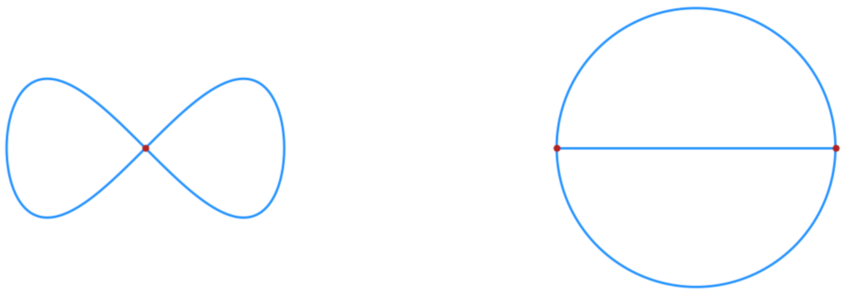


Figure 4.1: The lemniscate $V = \{x \in \mathbb{R}^2 \mid x_1^4 - x_1^2 + x_2^2 = 0\}$ on the left and a circle with a diameter filament on the right

Of course, the spaces themselves are topologically quite different, the lemniscate shown in Fig. 4.1 and the circle with a filament having two singularities. Depending on the application, one may be interested in an invariant capable of distinguishing the two. For example, one may consider the two spaces in Fig. 4.1 as so-called stratified spaces, taking care to mark their singularities.

The topological data analysis of stratified objects has recently received increased interest (see, for example, [Mil21; Sto+20; Nan20; SW14; BWM12; FW16]). However, as suggested by the properties of the non-stratified scenario described in P(1) to P(3), to successfully establish persistent methods in a stratified framework a notion of persistent stratified homotopy type is needed. No such thing was available so far, at least not to our knowledge and not in a way that satisfies analogs to the properties P(1) to P(3). This is because stratified homotopy theory has only recently received a wave of renewed attention from the theoretical perspective [Woo09; Lur17; Mil13; Hai23; Dou21c; Dou21b; DW22]. A series of new results in this field now lay the foundation for stratified investigations in topological data analysis.
Establishing such a notion of persistent stratified homotopy type and showing that it fulfills properties much like the non-stratified persistent homotopy type is precisely what this work is concerned with. Thus, the focus lies entirely on the left-hand side of the factorization in (4.1), leaving investigating algebraic invariants of the latter (for example, intersection homology, as in [BH11]) for future work. Note, however, that whatever invariants they may be, they automatically inherit many of the convenient properties of persistent homology.

4.1.1 Persistent stratified homotopy types

Let us illustrate our methods and results by following the example of the lemniscate V shown in Fig. 4.1. We may treat the lemniscate as a so-called Whitney stratified space (see Recollection 4.2.1.10) W, with two strata given by $W_p = \{0\}$, the singularity, and $W_q = V \setminus W_p$ given by the regular part. It follows from results in [Dou19b; DW22] (see Theorem 4.2.2.10 and Recollection 4.2.3.5) that, for a Whitney stratified space with two strata, the so-called *stratified homotopy type* (the analogue of the classical homotopy type, obtained by considering a stratum-preserving notion of map and homotopy) may equivalently be thought of as (the homotopy type of) a diagram of spaces of the form

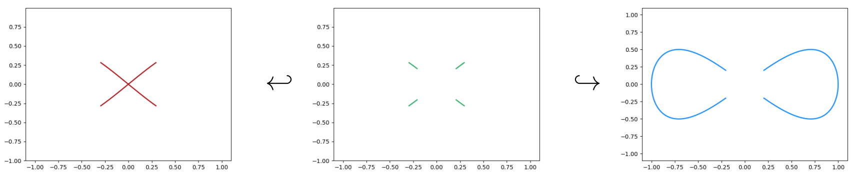
$$D(W)_p \longleftarrow D(W)_{\{p,q\}} \longrightarrow D(W)_q. \tag{4.2}$$

Here, the spaces $D(W)_p$ and $D(W)_q$ correspond respectively to the strata of the stratified space W, while $D(W)_{\{p,q\}}$ corresponds to the homotopy type of the space connecting the two strata, the so-called *(homotopy)link* (see Definition 4.2.2.4). More explicitly, it follows from Proposition 4.3.1.14 that for sufficiently small positive real numbers $v_l < v_h$ the stratified homotopy type of W is encoded in the diagram

$$\mathrm{d}_{W_p}^{-1}[0, v_h] \hookleftarrow \mathrm{d}_{W_p}^{-1}[v_l, v_h] \hookrightarrow \mathrm{d}_{W_p}^{-1}[v_l, \infty], \tag{4.3}$$

where d_{W_p} is the function assigning to a point its distance to the singular stratum W_p.

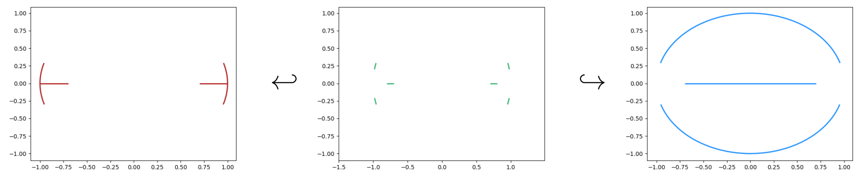
Example 4.1.1.1. For the lemniscate, as shown in Fig. 4.1, the Diagram (4.3) with parameter $v = (0.2, 0.3)$ can be visualized as follows:



This diagram is (weakly) equivalent to the simpler diagram of discrete spaces

$$\{c\} \longleftarrow \{a, b\} \times \{x, y\} \xrightarrow{\pi_2} \{x, y\}. \tag{4.4}$$

In the case of the space shown in Fig. 4.1 on the right, with the singular stratum given by the singularities, Diagram (4.3) is given by:



From the perspective of homotopy theory, this diagram simplifies to the diagram of discrete spaces

$$\{a, b\} \xleftarrow{\pi_1} \{a, b\} \times \{x, y, z\} \xrightarrow{\pi_2} \{x, y, z\}. \tag{4.5}$$

A simple point count already shows that Diagrams (4.4) and (4.5) distinguish the lemniscate from the circle with a filament shown in Fig. 4.1.

This example illustrates how the stratified homotopy type provides a significantly finer invariant than the classical homotopy type.
In order to perform topological data analysis with such diagram representations of stratified homotopy types, we need a persistent analogue, i.e. a *persistent stratified homotopy type*. (In this paper, we focus on the two strata case, for reasons elaborated in more detail in the end of the introduction.)
In Section 4.3, we construct such an object by separately thickening the pieces of Diagram (4.3) in a surrounding Euclidean space. In this fashion, after having chosen parameters $v = (v_l, v_h)$, we may associate to a stratified subset $S \subset \mathbb{R}^N$ a persistent stratified homotopy type, $\mathcal{P}_v(S)$. By construction, for each S the persistent stratified homotopy type $\mathcal{P}_v(S)$ is represented by a space valued functor

$$\{p \leftarrow \{p, q\} \rightarrow q\} \times \mathbb{R}_+ \rightarrow \mathbf{Top},$$

where $\{p \leftarrow \{p, q\} \rightarrow q\}$ is the indexing category for diagrams such as Diagram (4.2). For each fixed $\varepsilon \in \mathbb{R}_+$ we recover a diagram indexed over $\{p \leftarrow \{p, q\} \rightarrow q\}$ (see Example 4.1.1.2 for a visualization), which under Recollection 4.2.3.5 corresponds to a (weak) stratified homotopy type. Our main results pertaining to this construction may be summarized as follows:

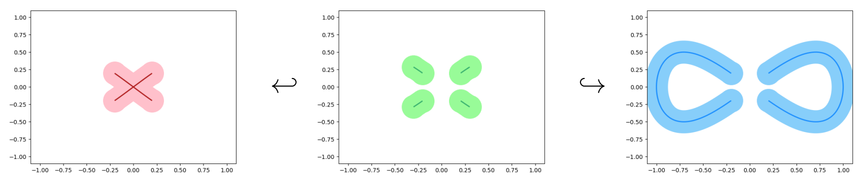
Main Result D. *The assignment*

$$S \mapsto \mathcal{P}_v(S),$$

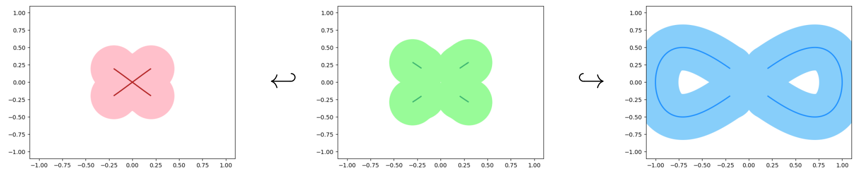
sending a stratified subset S of $\mathbb{R}^N$ with two strata to its persistent stratified homotopy type (depending on a choice of parameters v) fulfills stratified analogues of P(1) to P(3).

More specifically, the analogue of P(1) is guaranteed by the fact, that for finite stratified point clouds, we may encode diagrams of the form Diagram (4.3) in terms of diagrams of Čech complexes (see Remark 4.3.1.17). The invariance under thickenings part of P(3) is guaranteed by Propositions 4.3.1.19 and 4.3.1.20. Proposition 4.3.1.19 roughly states that for sufficiently constructible two strata subspaces $S \subset \mathbb{R}^N$, and sufficiently small choices of v, the (weak) stratified homotopy type given by $\mathcal{P}_v(S)_\varepsilon$ agrees with the one of S for sufficiently small $\varepsilon > 0$. In other words, the stratified homotopy type does not change under small thickenings.

Example 4.1.1.2. Again, consider the example of the lemniscate, using parameter values $v = (0.2, 0.3)$. For $\varepsilon = 0.12$, the homotopy type of the thickened diagram



agrees with the unthickened diagram in Example 4.1.1.1. If instead we take $\varepsilon = 0.24$, then we obtain:



The latter diagram is (weakly) equivalent to the diagram

$$\star \longleftarrow \star \longrightarrow S^1 \cup_\star S^1, \tag{4.6}$$

which is not (weakly) equivalent to Diagram (4.4).

Finally, a version of stability - and hence an analogue of P(2) - is guaranteed by Theorems 4.3.3.11 and 4.3.4.8. It follows from these results that for a two strata Whitney stratified $W \subset \mathbb{R}^N$ and a sequence of stratified space $\mathbb{S}^i \subset \mathbb{R}^N$ converging to W (in a stratified version of the Hausdorff distance, see Definition 4.3.1.4) the associated persistent stratified homotopy types $\mathcal{P}_v(\mathbb{S}^i)$ converge to $\mathcal{P}_v(W)$, and for small v this convergence is even of Lipschitz type.

Example 4.1.1.3. Convergence in the stratified version of the Hausdorff distance that we used is equivalent to convergence in both the underlying spaces, as well as in the singular strata. For example, consider the family of real algebraic varieties

$$V^s := \{x \in \mathbb{R}^2 \mid f_s(x) = x_1^4 - x_1^2 + x_2^2 = s\}, \text{ for } s \in \mathbb{R},$$

equipped with the stratification given by singular loci (see Fig. 4.2). In other words, the singular stratum of V^s is given by the intersection of V^s with the vanishing set of the Jacobian of f_s.

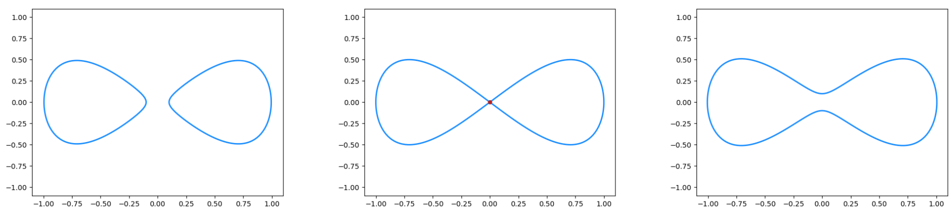


Figure 4.2: The real algebraic variety V^s, for $s \in \{-0.1, 0, 0.1\}$, with singularities marked in red

Note that V^0 is the lemniscate as in Fig. 4.1. We may not expect convergence

$$\mathcal{P}_v(V^s) \xrightarrow{s \to 0} \mathcal{P}_v(V^0)$$

in stratified Hausdorff distance, since for $s \neq 0$ the variety has no singular points, and hence the singular stratum is empty. In particular, the stratified Hausdorff distance between V^s and V^0 is in fact infinite for $s \neq 0$.
However, instead of classifying points as singular by the vanishing of the Jacobian of f_s, we could, for example, use a more quantitative measure of singularity and let $V_0^s = \{x \in V^s \mid \|Jf_x\| \leq 3\sqrt{s}\}$ (see Fig. 4.3 for an illustration). With these alternative stratifications V^s converges to V^0 in stratified Hausdorff distance, as indicated in Fig. 4.3. Hence, convergence of persistent stratified homotopy types also holds. The general approach of using a more quantitative measure of singularity is also central in Section 4.4, which deals with stratifying point clouds.

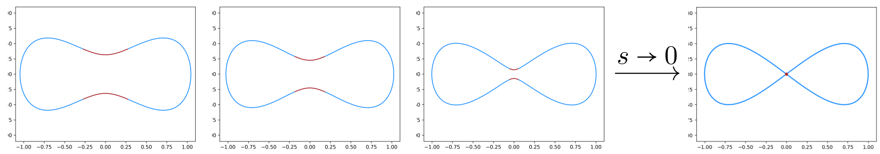


Figure 4.3: V^s for $s = 0.1, 0.5, 0.05$ with bottom stratum in red given by $\{x \in V^s \mid \|Jf_x\| \leq 3\sqrt{s}\}$.

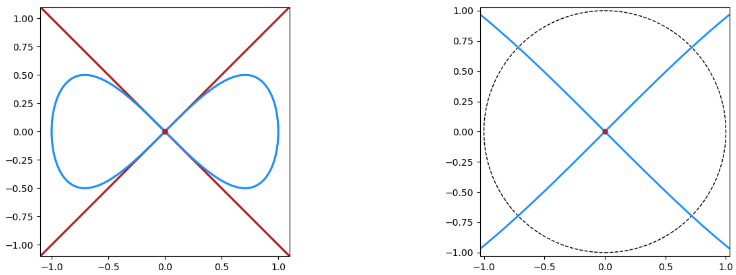


Figure 4.4: Tangent cone (red) at origin of the lemniscate and magnification at the origin for $\zeta = 3$

Together, the statements of Main Result D allow for the computational inference of stratified homotopy theoretic information about a two strata Whitney stratified space W from a sufficiently close (potentially noisy), finite, stratified sample.

4.1.2 Stratification learning through tangent cones

In an applied scenario we would generally not expect a point cloud to already be equipped with an appropriate stratification. Much attention has been paid to how such stratifications can be obtained [Mil21; Sto+20; Nan20; SW14; BWM12; FW16]. In Section 4.4, we provide a method as well as theoretical guarantees (Theorem 4.4.5.8) for the scenario of approximating a stratified space from a non-stratified sample, in the stratified Hausdorff distance. Our approach is inspired by the approaches to persistent local homology in [BWM12; SW14; Nan20; Mil21], which detects singularities using local data of the form $\mathrm{B}_{\frac{1}{\zeta}}(x) \cap W$, for large $\zeta > 0$ and $x \in W \subset \mathbb{R}^N$. From a scale independent point of view, one may equivalently consider the so-called *magnifications*

$$\mathcal{M}_x^{\zeta}(W) := \zeta(W - x) \cap \mathrm{B}_1(0).$$

If W is a sufficiently constructible Whitney stratified space, then it is a classical result (see [Hir69; BL07]) that as ζ converges to ∞ these magnifications converge to the (unit ball in the) so-called *extrinsic tangent cone* at $x \in W$ (see Definition 4.4.1.5) - a generalization of tangent spaces in the singular setting. Thus, the class of stratified spaces we investigate are the *tangentially stratified* (see Definition 4.4.1.8) spaces, for which singularities may be detected by their extrinsic tangent cones.

Example 4.1.2.1. In the case of the lemniscate, for example, the tangent cone at the origin is given by the algebraic variety

$$\{x \in \mathbb{R}^2 \mid (x_1 - x_2)(x_1 + x_2) = 0\}$$

which identifies the origin as singular (see Fig. 4.4). Hence, with respect to the standard stratification, the lemniscate is tangentially stratified.

For the purpose of topological data analysis, we needed a global sampling version of the convergence results of [Hir69]. More specifically, we needed a version which at the same time allows for the replacement of W by a sufficiently close sample, and furthermore holds with respect a global notion of convergence, where a single tangent cone is replaced by a bundle of tangent cones. We prove such convergence results in Propositions 4.4.2.7 and 4.4.4.6.

In Section 4.4.5, we leverage the convergence guaranteed by Proposition 4.4.4.6 to provide theoretical guarantees for learning stratifications with two strata from a non-stratified sample $\mathbb{X} \subset \mathbb{R}^N$ close to a tangentially Whitney stratified space W. To decide which points of $\mathbb{X}$ should be considered singular, we consider specific functions (see Definition 4.4.1.10)

$$\Phi\colon \{\text{Local Data}\} \to [0,1],$$

which quantifiy the degree of degeneracy of $\mathcal{M}^{\zeta}_x(\mathbb{X})$ (with 1 being regular and 0 highly singular). Examples of such functions Φ come from minimizing truncated Hausdorff distances to q-dimensional linear subspaces (where q is the dimension of W, see Example 4.4.1.11) and local persistent homology (see Example 4.4.1.12).
A Whitney stratified space $W \subset \mathbb{R}^N$, with two strata, for which the singular stratum W_p is precisely given by such points $x \in W$, for which $\Phi(\mathrm{T}^{\mathrm{ex}}_x(W)) < 1$, is called *(tangentially) Φ-stratified*. In other words, Φ-stratifications are precisely the stratifications which we may hope to learn through the lens of Φ. Tangentially stratified spaces, for example, are precisely the Φ-stratified spaces, with respect to the function of Example 4.4.1.11.
After a choice of cutoff parameter $u \in (0,1)$, one may then turn a non-stratified point cloud $\mathbb{X} \subset \mathbb{R}^N$ into a stratified point cloud with two strata, denoted $\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X})$, by taking the singular stratum to be

$$\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X})_p = \{x \in \mathbb{X} \mid \Phi(\mathcal{M}^{\zeta}_x(\mathbb{X})) \leq u\}.$$

This construction depends, of course, on the magnification parameter $\zeta > 0$. Our main result on stratification learning through tangent cones then describes the convergence behavior of $\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X})$ in ζ and $\mathbb{X}$. In the following $\mathrm{d}_{\mathrm{HD}(W)}(-,-)$ denotes non-stratified Hausdorff distance:

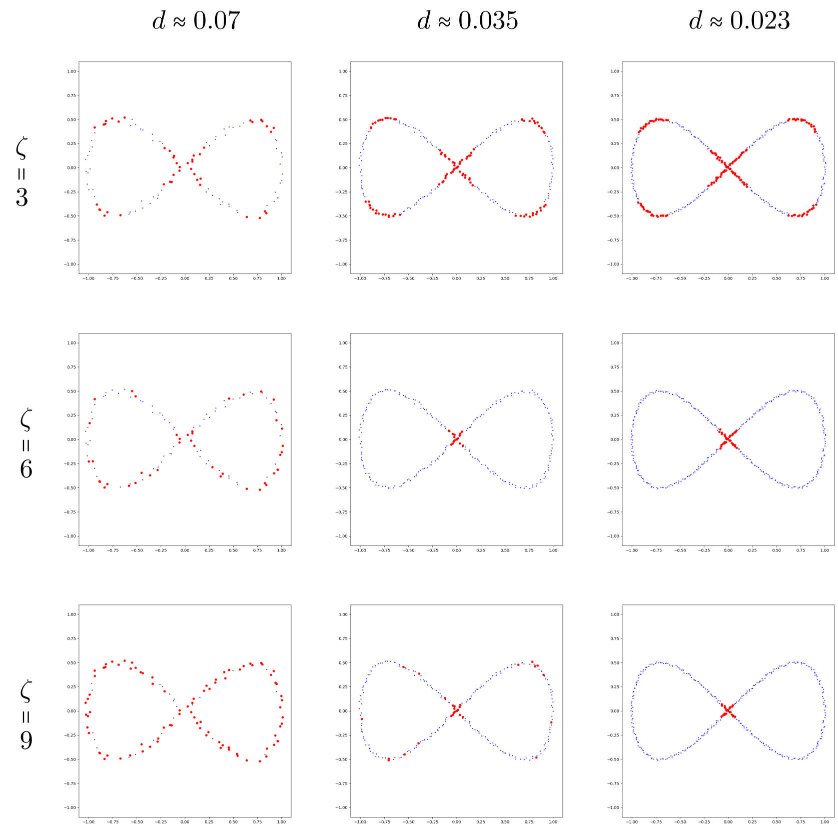
Main Result E (Theorem 4.4.5.8)**.** *Let $W \subset \mathbb{R}^N$ be a compact (sufficiently constructible, see Definition 4.4.2.2) Whitney stratified space, which is Φ-stratified with respect to a function Φ as in Definition 4.4.1.10. Denote by X the underlying space of W. Then there exists $u_0 \in (0,1)$ such that for all $u \in [u_0, 1)$, the convergence in stratified Hausdorff distance*

$$\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X}) \to W,$$

holds, for $\zeta \to \infty$ and $\mathbb{X} \to X$ such that $\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X}, X) \to 0$.

In other words, we may approximate the singular stratum W_p of W by $\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X})_p$ if ζ is sufficiently large and given that $\mathbb{X}$ is a sufficiently fine approximation of X, where the sufficient degree of fineness depends on ζ. This dependence on ζ is not surprising at all, in fact, it is simply a rigorous restatement of the following principle: To recover local information from a sample, the quality of the sample needs to be finer by some magnitude than the locality scale we work at.

Example 4.1.2.2. Below, we depict a family of point clouds converging to the lemniscate V^0 in Hausdorff distance. These samples have been stratified using Φ as in Example 4.4.1.11 with $u = 0.6$, and we marked points in $\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X})_p$ with thickened red dots. In horizontal direction from left to right the non-stratified Hausdorff distance, denoted d, to V^0 decreases. In vertical direction going downwards we increase the magnification parameter ζ.



In practice, these choices of Φ and u often tend to be too sensitive, especially when it comes to detecting curvature (see also Example 4.4.1.11). In this example, however, they serve well to illustrate the convergence behavior in magnification parameter and Hausdorff distance. First, note that for $\zeta = 3$, points in regions with high curvature are generally classified as belonging to $\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X})_p$. This is not surprising as for such comparatively small values, we may not expect magnifications at regular points to be close to the tangent spaces yet. By doubling ζ, we may correctly classify these regions, provided that the sample quality is good enough to also approximate things at a local level (see the middle row). In this case, only an area around the singularity is classified as belonging to $\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X})_p$. If we want to further shrink this area, and thus obtain a better approximation in stratified Hausdorff distance, we may again decrease ζ (see third row, in particular the most right picture). In case our sample quality is not sufficient, this may also classify several points far away from the singularity as singular though (see the second picture of the third row).
For less sensitive choices of parameter u and alternative choice of Φ, samples of lesser quality and smaller ζ may lead to good approximations in stratified Hausdorff distance. Consider, for example, Fig. 4.5 which was obtained with the same Φ and $u = 0.4$.

Finally, combining our two methods, i.e. persistent stratified homotopy types and stratification learning through tangent cones, we obtain a pipeline which associates to a non-stratified sample a persistent stratified homotopy type. The combination of Theorem 4.3.4.8 and Theorem 4.4.5.8, which is Corollary 4.4.5.9, guarantees that we may indeed approximate persistent stratified homotopy types of sufficiently regular Whitney stratified spaces from nearby samples, under the assumptions of Theorem 4.4.5.8. Using Proposition 4.3.1.19 we may then infer from these approximations information about the stratified homotopy type of W.

Example 4.1.2.3. As an illustration of the convergence of persistent stratified homotopy types obtained from non-stratified samples, consider Fig. 4.6. It shows the barcode of the 0-homology of the link part of the persistent stratified homotopy types associated to the stratified point clouds of Example 4.1.2.2, going in a zigzag above the diagonal from the upper left to the lower

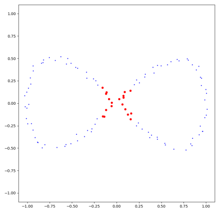


Figure 4.5: Approximated stratification of a sample around the lemniscate with $\zeta = 3$ and $u = 0.4$.

right corner. Note that for $\zeta = 3$, five disjoint regions are detected as singular and the link

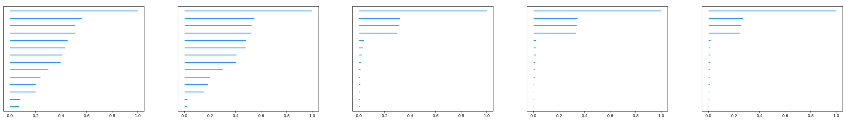


Figure 4.6: Barcodes (showing the 14 longest bars) of the 0-th homology of the link part of $\mathcal{P}_v(\mathcal{S}^{\zeta}_{\Phi,u}(\mathbb{X}))$, with varying $\mathbb{X}$ and ζ. Here $\mathbb{X}$ is a point cloud of Hausdorff distance to V^0 approximately d.

ends up having twelve path components. For $\zeta = 6$ and $\zeta = 9$, however, the stratified samples are close to V^0 in stratified Hausdorff distance and we detect four path components in the link. This is the expected number of path components from Example 4.1.1.1.

4.1.3 The case of more than two strata

Let us end this introduction with some remarks on the case of multiple strata. In fact, many of the constructions throughout this paper (such as the construction of diagrams representing stratified homotopy types) can be generalized to the case of more than two strata through a serious amount of inductive and technical effort. We have chosen to focus on the two strata case for the following reasons:

1. The main goal of this paper is to establish a pipeline going all the way from a (non-stratified) sample of a stratified space to a persistent stratified homotopy type and investigate the properties of such a construction. In this sense, it is partially intended as a proof of concept, leaving room for many improvements and generalizations at several steps of the pipeline for future works. While some of the steps are fairly easy to replicate in a multi-strata scenario, this is not the case for all of them. In particular, learning stratifications from non-stratified data gets significantly harder when the underlying stratification poset is more complicated (this is due to examples such as the Whitney Umbrella, see for example [HN22] or [Ban07, p. 128-129]). Note that our results on convergence of magnifications to tangent cones are proven in the setting of more than two strata (Proposition 4.4.2.7), and thus may allow generalizations for appropriate families of functions Φ, for suitably nice spaces, avoiding examples such as the Whitney Umbrella. We are aware that these types of stratification learning questions are currently the research focus of several other groups.

2. While it is certainly possible to generalize the definition of the persistent stratified homotopy type to more complicated posets (note that a lot of the abstract homotopy theory is already in place [Dou19b; DW22; Hai23; AFR19]), this significantly increases the

technical complexity of definitions and proofs involved, adding an inductive component. This adds another technical difficulty to an already somewhat lengthy paper, which we wanted to avoid at this point.

3. In the case of multiple strata, there are several slightly different approaches to what the homotopy category of stratified spaces should be (compare [Hai23] and [DW22]). While all of these approaches agree in the two strata case and on the class of Whitney stratified spaces, the investigation and comparison of the multi-strata case is still the content of ongoing research. New results and insights on the theoretical side may still greatly influence and simplify the transfer to topological data analysis, making it worthwhile to save the development of the multi-strata case for future projects. In particular, having inductive interpretations of the stratified homotopy theories defined in [Dou19b; DW22; Hai23; AFR19] would significantly streamline the transfer to the setting of topological data analysis.

4.2 Stratified homotopy theory

In this section, we summarize material concerning stratified spaces and their homotopy theory, as far as it is relevant to our investigations (Sections 4.2.1 to 4.2.3). The exposition on stratified homotopy theory should be accessible for a reader familiar with basic notions in algebraic topology and category theory. Both for details and the complete model categorical picture we refer the reader to [DW22] (or Chapter 3 in this text), which contains a comprehensive overview. For more details on stratified spaces and their invariants consider, for example, [Ban07].

4.2.1 Stratified spaces

We begin by recalling some of the basic notions relevant to the theory of stratified space. Recall that the Alexandrov topology on a poset P, is the topology in which the closed sets are the sets which are closed below, under the relation on P.

Definition 4.2.1.1. A *stratified space* (over a poset P) is a pair $S = (T, s : T \to P)$ where T is a topological space and s is continuous with respect to the Alexandrov topology on P. The map s is called the *stratification of* S. The fiber of the stratification over $p \in P$

$$S_p := s^{-1}\{p\}$$

is called the *p-stratum of* S.

Example 4.2.1.2. From an abstract point of view, any filtration $(T_{\leq 0} \subset ... \subset T_{\leq n} = X)$ by closed subsets of a topological space T induces a stratification over the poset $[n] = \{0 < ... < n\}$. However, stratified spaces also arise quite naturally in different fields of mathematics, and are often assumed to have manifold strata.

- Let $(M, \partial M)$ be a compact manifold with boundary and let T be the space obtained by coning off the boundary of M, i.e. $T = M \cup_{\partial M} C(\partial M)$, where

$$C(Y) = {Y \times [0,1]} \big/ {(y,0) \sim (y',0)}$$

denotes the cone on a space Y. One obtains a stratification of T by the map

$$s : X \to \{p, q\}; \qquad \begin{cases} x \mapsto q, & \text{for } x \in X \setminus \{\text{cone point}\}, \\ x \mapsto p, & \text{for } x = \text{cone point}. \end{cases}$$

The resulting stratified space is locally Euclidean away from one isolated singularity, at which arbitrarily small neighborhoods are homeomorphic to the open cone $\mathring{C}(\partial M) = C(\partial M) \setminus \{1\} \times \partial M$.

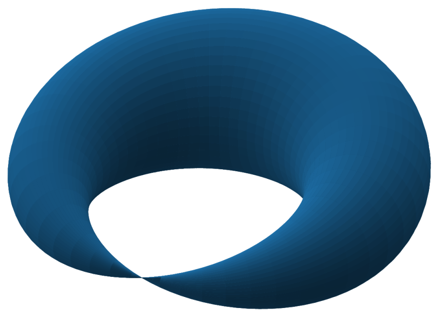


Figure 4.7: Pinched torus

- Given a smooth manifold M with a compact Lie group G acting smoothly and properly on M. The orbit space M/G can then be stratified by orbit types (see, e.g., [Pfl01, Chapter 4] for more details).
- Any n-dimensional complex algebraic variety T can be equipped with the structure of a stratified space. A filtration by closed subsets is given by iteratively taking singular loci.

Example 4.2.1.3. The so-called pinched torus PT^2 can be described as the quotient space of the torus $T^2 = S^1 \times S^1$ by collapsing one circle $* \times S^1$ to a point, see Fig. 4.7. The image of this circle is the singular point, denoted s, of the pinched torus. The filtration $\{s\} \subset PT^2$ induces a stratification over the poset $\{0 < 2\}$. The pinched torus is an example of a so-called *pseudomanifold*, an important class of stratified spaces that have been the subject of research to recover a form of generalized Poincaré duality for singular spaces ([GM80; GM83]).

Remark 4.2.1.4. It is common to abuse notation insofar as one usually refers to the stratified space by its underlying topological space. Thus, we will freely use notation such as $x \in S$, when we mean $x \in T$. However, as the second half of this paper is particularly concerned with learning stratifications, we will take care to differentiate rigorously between stratified and non-stratified objects then.

Definition 4.2.1.5. A *stratum-preserving map* between two P-stratified spaces $T \to P$ and $Y \to P$ is a continuous map $f : T \to Y$, making the diagram

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & P & \end{tikzcd}$$

commute.

Notation 4.2.1.6. Stratified spaces over a poset P together with stratum-preserving maps define a category which we denote $\mathbf{Strat}_P$. Isomorphisms in $\mathbf{Strat}_P$ - i.e. stratum-preserving homeomorphisms - will be denoted by $\cong_P$.

Remark 4.2.1.7. There is a slight technical issue here insofar, as the homotopy theoretical perspective needs assumptions on the underlying topological spaces used. We assure the reader unfamiliar with the following technicalities that they can safely ignore them. We generally denote by **Top** the category of Δ-generated spaces, i.e. spaces which have the final topology with respect to maps coming from simplices (see [Dug03] for details). We generally assume all topological spaces involved to have this property. At times, this will mean that the topology on a space has to be slightly modified and replaced by a Δ-generated one (for example $\mathbb{Q} \subset \mathbb{R}$ is not Δ-generated, its Δ-ification is given by a discrete countable space). However, since this operation does not change weak homotopy types, it is mostly irrelevant to our investigations of homotopy theory (see also [DW22, Rem 2.10] which is Remark 3.2.2.1 in this text).

Notation 4.2.1.8. Given a stratified space $S = (T, s : T \to P)$ and $p \in P$ we write

$$\begin{aligned} S_{\leq p} &:= s^{-1}(\{q \leq p\}), \\ S_{<p} &:= s^{-1}(\{q < p\}), \\ S^{\geq p} &:= s^{-1}(\{q \geq p\}), \\ S^{>p} &:= s^{-1}(\{q > p\}). \end{aligned}$$

For many theoretical as well as for our more applied investigations of stratified spaces, it is fruitful to impose additional regularity assumptions on the strata (such as manifold assumptions) and the way they interact. The notion central to this paper is the notion of a Whitney stratified space. These are characterized by the convergence behavior of secant lines around singularities.

Notation 4.2.1.9. Given two distinct vectors $v, u \in \mathbb{R}^{\mathbb{N}}$, with $v \neq u$, we denote by $l(v, u)$ the 1-dimensional subspace of $\mathbb{R}^N$ spanned by $v - u$.

Recollection 4.2.1.10. A stratified space $W = (T, s{:}T \to P)$ with $T \subset \mathbb{R}^N$ locally closed is called *Whitney stratified*, if it fulfills the following properties.

1. *Local finiteness*: Every point $x \in T$ has a neighborhood intersecting only finitely many of the strata of W.
2. *Frontier condition:* W_p is dense in $W_{\leq p}$, for all $p \in P$.
3. *Manifold condition:* W_p is a smooth submanifold of $\mathbb{R}^N$, for all $p \in P$.
4. *Whitney's condition (b):* Let $p, q \in P$ such that $p < q$ and let x_n, y_n be sequences in W_q and W_p respectively, both convergent to some $y \in W_p$. Furthermore, assume that the secant lines $l(x_n, y_n)$ converge to a 1-dimensional space $l \subset \mathbb{R}^N$ and that the tangent spaces $\mathrm{T}_{x_n}(W_q)$ converge to a linear subspace $\tau \subset \mathbb{R}^N$. Then $l \subset \tau$. (By convergence of vector spaces we mean convergence in the respective Grassmannians.)

Example 4.2.1.11. Whitney's work ([Whi65a], [Whi65b]) states that every algebraic and analytic variety admits a Whitney stratification. More general, Whitney stratifications can even be given to spaces such as semianalytic sets (see e.g. [Łoj65]) or o-minimally definable sets (see e.g. [Loi98]). Finally, if T is such that it has only isolated singularities and admits a Whitney stratification, then any stratification of T, fulfilling frontier and boundary condition, with smooth strata is automatically a Whitney stratification. In particular, any definable set with isolated singularities and a dense open submanifold is canonically Whitney stratified with two strata. Another class of Whitney stratified spaces arises from G-manifolds, already noted in Example 4.2.1.2. For a proof, see [Pfl01, Theorem 4.3.7].

Whitney's condition (b) has a series of immanent topological consequences, which ultimately led to the more general notion of a conically stratified space. The latter are (with some additional assumptions) one of the main objects of interest in the algebro-topological study of stratified spaces [Sie72; GM80; GM83; Qui88; Lur17]. In addition to the Whitney stratification assumption, we will frequently need additional control over how pathological the subsets of Euclidean space we allow for can be. To obtain such additional control, we use the notion of a set $T \subset \mathbb{R}^N$, definable with respect to some o-minimal structure (see [Dri98] for a definition). For the reader entirely unfamiliar with these notions it suffices to know that all semialgebraic or compact subanalytic sets have this property. On the one hand, definability assumptions guarantee the existence of certain mapping cylinder neighborhoods (see Example 4.2.3.16) that allow thickenings that do not change the homotopy type (see Lemma 4.B.1.1). At the same time, asserting additional control over the functions defining a set (polynomially bounded), has several consequences for the convergence behavior of tangent cones, already noted in [Hir69; BL07]. We will use these to recover stratifications from samples in Section 4.4.

Definition 4.2.1.12. We say that a stratified space $S = (T, s\colon T \to P)$, with $T \subset \mathbb{R}^N$ and P finite, is *definable* (or *definably stratified*) if all of its strata are definable with respect to some fixed o-minimal structure.

4.2.2 Homotopy categories of stratified spaces

Many of the algebraic invariants of stratified spaces - most prominently intersection homology - are invariant under a stratified notion of homotopy equivalence.

Definition 4.2.2.1. Let $f, f'\colon S \to S'$ be stratum-preserving maps. We call f and f' *stratified homotopic*, if there exists a stratum-preserving

$$\mathrm{H}\colon (T \times [0,1], T \times [0,1] \to T \xrightarrow{s} P) \to S'$$

such that $\mathrm{H}_{|T\times\{0\}} = f$ and $\mathrm{H}_{|T\times\{1\}} = f'$. Furthermore, f is called a *stratified homotopy equivalence*, if there exists another stratum-preserving map $g\colon S' \to S$ such that $f \circ g$ and $g \circ f$ are stratified homotopic to $\mathrm{id}_{S'}$ and id_S respectively.

Remark 4.2.2.2. Since we use different notions of equivalences of stratified spaces in this paper, we use the convention of speaking of *strict stratified homotopy equivalences* instead of stratified homotopy equivalences, to avoid any possibility of confusion. The class of all stratified spaces *strictly stratified homotopy equivalent* to a stratified space S is called the *strict stratified homotopy type of S*.

The use of strict stratified homotopy equivalence for topological data analysis faces one apparent issue. Many of the justifications for the use of persistent approaches to the analysis of geometrical data rely on the fact that homotopy types of (sufficiently regular) spaces do not change under small thickenings (see for example [NSW08]). Unlike classical homotopy equivalence, however, stratified homotopy equivalence is a rather rigid notion.

Example 4.2.2.3. Consider the space $T = S^1 \vee S^1$ embedded in $\mathbb{R}^2$ as a curve, shown in Fig. 4.8 on the left. It features a singular point at the self-crossing. Denote the resulting stratified space over $P = \{0 < 1\}$ with the singularity sent to 0 and the remainder to 1 by S. While there generally seems to be no canonical way to thicken such a space, one possibility is to thicken both the total space as well as the singularity as in Fig. 4.8 on the right. The resulting thickened space S'' is strictly stratified homotopy equivalent to the original curve with the singular stratum extended from a point to the crossing, denoted S', see Fig. 4.8. However, S and S' (and hence S'') are not strictly stratified homotopy equivalent. To see this, note that a stratified homotopy equivalence between S and S' would also have to be a homotopy equivalence of the underlying spaces. Such a map has to send a circle S^1 with degree ± 1 onto another circle. But the image of any stratum-preserving map between S and S' is (non-stratifiedly) contractible.

In some sense, the failure of stratified homotopy equivalence in Example 4.2.2.3 is due to the fact that the two thickenings are not sufficiently regular (i.e. Whitney stratified, or more generally conically stratified in the sense of [Lur17]) spaces anymore (this will become more apparent later on from Theorem 4.2.2.10 and Fig. 4.11). Here, we already encounter the issue that to perform topological data analysis on *nicely* stratified spaces, one generally needs to

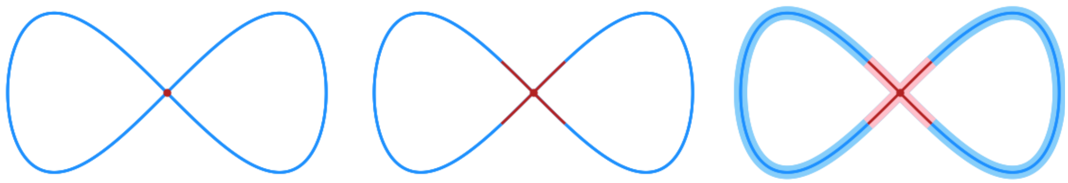


Figure 4.8: From left to right: A stratified singular curve S; an alternative stratification S'; and a stratified thickening S''

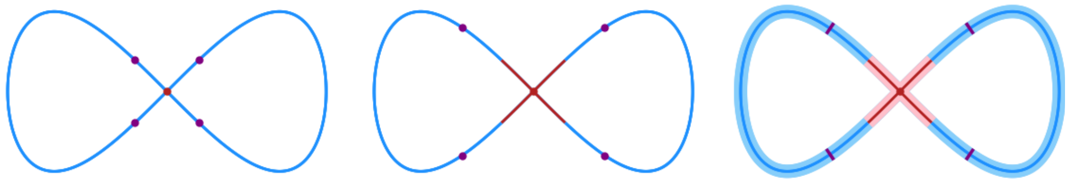


Figure 4.9: Geometric models of homotopy links marked in purple

leave the *nice* category. To make the intuition of why this phenomenon leads to the failure of stratified homotopy equivalence in Example 4.2.2.3 more rigorous, we need the notion of a homotopy link. These were first introduced in [Qui88] and can be thought of as a homotopy theoretical analog of the boundary of a regular neighborhood in the piecewise linear scenario. See also [DW22] for more geometrical intuitions.

Definition 4.2.2.4. Let S be a stratified space and $p, q \in P$ with $p < q$. The *homotopy link* of the p-stratum in the q-stratum is the space of so-called *exit paths*

$$\mathcal{H}o\mathrm{Link}_{\{p<q\}}(S) = \{\gamma\colon [0,1] \to T \mid \gamma(0) \in S_p, \gamma(t) \in S_q, \forall t > 0\}$$

with its topology induced by $\mathcal{H}o\mathrm{Link}_{\{p<q\}}(S) \subset C^0([0,1], T)$, where the latter denotes the space of continuous functions equipped with the compact open topology. The induced functors

$$\mathbf{Strat}_P \to \mathbf{Top}$$

come with natural transformations

$$S_p \leftarrow \mathcal{H}o\mathrm{Link}_{\{p<q\}}(S) \to S_q,$$

given by the starting point and end point evaluation map.

Example 4.2.2.5. Let us return to Example 4.2.2.3 to give an illustration of the homotopy link. For the original singular curve and both thickenings, the homotopy links are all homotopy equivalent to four isolated points (see Fig. 4.9). This can be seen from Construction 4.2.3.19, which states that the homotopy links are homotopy equivalent to the boundary of a cylinder neighborhood of the singular stratum.

In [Mil13, Theorem 6.3], it was first shown that a stratum-preserving between sufficiently regular stratified spaces is a stratified homotopy equivalence, if and only if it induces homotopy equivalences on all homotopy links and strata. This behavior is akin to the one described by the classical Whitehead theorem (see [Whi49a], [Whi49b]) or more generally the behavior of cofibrant, fibrant objects in a model category. It is a general paradigm in abstract homotopy theory that to study a class of in some sense regular objects within a larger class of objects, up to a notion of equivalence, it can be useful to weaken that notion in a way, that it becomes less rigid on the whole class, but still agrees with the original notion on the class of regular objects. This is also the perspective on stratified homotopy theory that we take here that also allows us to circumvent the issue alluded to in Example 4.2.2.3.

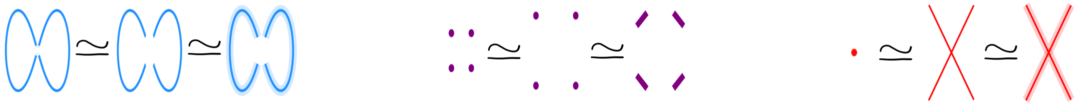


Figure 4.10: Regular strata, homotopy links and singular strata of the spaces in Example 4.2.2.3

Recollection 4.2.2.6. The definition of a homotopy link for pairs $\{p < q\}$ generalizes to the case where $\{p < q\}$ is replaced by a regular, i.e. strictly increasing, flag $\mathcal{I} = \{p_0 < \ldots < p_n\}$. The resulting spaces are denoted

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(S).$$

One then needs to replace the stratified interval $[0,1]$ by a stratified simplex corresponding to $\mathcal{I}$. In the case of $\mathcal{I} = \{p\}$ a singleton, this definition comes down to

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(S) = S_p.$$

Since we are mainly concerned with the two strata case here, we refer the interested reader to [DW22] (Chapter 3 in this text) for rigorous definitions.

Definition 4.2.2.7. A stratum-preserving map $f\colon S \to S'$ in $\mathbf{Strat}_P$ is called a *weak equivalence of stratified spaces*, if it induces weak equivalences of topological spaces

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(S) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(S'),$$

for all regular flags $\mathcal{I} \subset P$.

Notation 4.2.2.8. We denote by $\mathrm{ho}\mathbf{Strat}_P$ the category obtained by localizing $\mathbf{Strat}_P$ at the class of weak equivalences. The isomorphism class of $S \in \mathrm{ho}\mathbf{Strat}_P$ is called the *stratified homotopy type of* S. Isomorphisms in $\mathrm{ho}\mathbf{Top}_P$ will be denoted by $\simeq_P$.

It is an immediate consequence of the fact that homotopy links map stratified homotopy equivalences to homotopy equivalences that any strict stratified homotopy equivalence is also a weak equivalence of stratified spaces. The converse is generally false.

Example 4.2.2.9. Let us illustrate these concepts for the spaces from Example 4.2.2.3 where we already discussed that there is no strict stratified homotopy equivalence between the original curve and any of the described thickenings. However, all the spaces are weakly stratified homotopy equivalent. Indeed, this is already hinted at by the fact that we may find a homotopy equivalence between the respective regular and singular parts as well as the homotopy links as described in Example 4.2.2.5. Consider Fig. 4.10 for an illustration.

Miller's result ([Mil13, Thm. 6.3]) can in fact be strengthened to a fully faithful embedding of homotopy categories. Roughly speaking, a stratified space is called triangulable if it admits a triangulation compatible with the stratification (for details see [DW22]). For the purpose of this paper, it suffices to know that Whitney stratified and (locally compact) definably stratified spaces even admit a PL-structure compatible with the stratification and are thus triangulable, see [Gor78], [Shi05], [Cza12]. As a consequence of [DW22, Theorem 1.2] (Theorem 3.1.0.2 in this text), one then obtains the following result:

Theorem 4.2.2.10. *[DW22, Theorem 1.2] Let $\mathbf{Whit}_P \subset \mathbf{Strat}_P$ be the full subcategory of Whitney stratified spaces over P, and $\simeq$ be the relation of stratified homotopy. Denote by $\mathbf{Whit}_P/_{\simeq}$ the category obtained by identifiying stratified homotopic morphisms in $\mathbf{Whit}_P$. Then the induced functor*

$$\mathbf{Whit}_P/_{\simeq} \to \mathrm{ho}\mathbf{Strat}_P$$

is a fully faithful embedding.

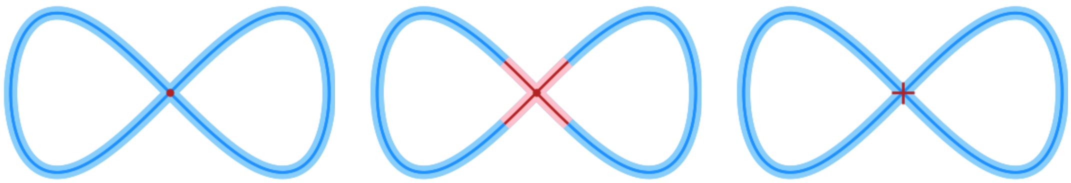


Figure 4.11: Three possible thickenings

For our purpose, this result entails that for the study of stratified homotopy invariants of sufficiently regular stratified spaces through topological data analysis, one may as well work in the category ho$\mathbf{Strat}_P$. As long as the spaces we investigate have these regularity properties, no information is lost by considering the stratified homotopy type instead of the strict stratified homotopy type. At the same time, Proposition 4.3.1.19 and Theorems 4.3.4.8 and 4.4.5.8 point towards the fact that stratified homotopy types are well suited for applications in topological data analysis, ultimately fulfilling many of the relevant properties of the classical homotopy type.

4.2.3 Stratification Diagrams

As noted in the previous section, for the passage to a persistent scenario, some notion of thickening of a stratified space is needed. In analogy to the classical scenario, this should assign to a stratified space $S \subset \mathbb{R}^N$, a functor from the category given by the (positive) reals with the usual order $\mathbb{R}_+$ into some category representing stratified homotopy types $\mathcal{C}$. In the classical scenario, $\mathcal{C}$ is often taken to be the category of simplicial complexes (sets) using constructions such as the Čech or Vietoris-Rips complex. For now, let us refer to the image under such a functor $\mathcal{P}(S)$ as the *persistent stratified homotopy type* of S, and similarly to the non-stratified construction using thickenings or Čech complexes as the *persistent homotopy type*.

This leaves us with the following question: How does one thicken a stratified subspace $S \subset \mathbb{R}^N$ while fulfilling a series of stability and invariance properties that justify the use for topological data analysis (compare with P(1) to P(3)). We explain and show a series of such properties in Section 4.3.

Example 4.2.3.1. In Fig. 4.11 we exhibit three different thickenings of the original space from Example 4.2.2.3. The first thickening is neither weakly nor strictly stratified homotopy equivalent to the original curve (as can be seen by comparing homotopy links). The second thickening, being only weakly equivalent to the unthickened space, was discussed in Example 4.2.2.9. However, note that the inclusion of the original curve into it is not stratum-preserving. Hence, this notion of thickening does not allow for a persistent approach. For the third thickening, the inclusion of the original curve is even a strict stratified homotopy equivalence. However, it seems unclear how to systematically achieve such a thickening, particularly when working with samples.

As illustrated in detail in Section 4.3, thickenings can be done successfully by representing stratified homotopy types by so-called stratification diagrams.

Definition 4.2.3.2. We denote by $\mathrm{R}(P)$ the category with objects given by regular (i.e. strictly increasing) flags $\mathcal{I} = \{p_0 < \cdots < p_k\}$ in P and morphisms given by inclusion relations of flags. We denote by

$$\mathbf{Diag}_P \coloneqq \mathrm{Fun}(\mathrm{R}(P)^{\mathrm{op}}, \mathbf{Top})$$

the category of $\mathrm{R}(P)^{\mathrm{op}}$ indexed diagrams of topological spaces. We call elements of $\mathbf{Diag}_P$ *(stratification) diagrams*.

Definition 4.2.3.3. A morphism $f\colon D \to D'$ in $\mathbf{Diag}_P$, for which $f_{\mathcal{I}}$ is a weak equivalence at all $\mathcal{I} \in \mathrm{R}(P)$ is called a *weak equivalence of (stratification) diagrams*.

Notation 4.2.3.4. We denote by $\mathrm{ho}\mathbf{Diag}_P$ the category obtained by localizing $\mathbf{Diag}_P$ at weak equivalences of diagrams.

For our purposes, the important result on stratification diagrams is that they can equivalently be used to describe stratified homotopy types. This is due to the following result.

Recollection 4.2.3.5. (For details see [Dou19b; DW22]). (Generalized) homotopy links induce a functor

$$\begin{aligned} \mathrm{D}_P\colon \mathbf{Strat}_P &\to \mathbf{Diag}_P \\ S &\mapsto \{\mathcal{I} \mapsto \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(S)\}. \end{aligned}$$

By definition, a stratum-preserving map is a weak equivalence, if and only if its image under D_P is a weak equivalence. In particular, one obtains an induced functor

$$\mathrm{D}_P\colon \mathrm{ho}\mathbf{Strat}_P \to \mathrm{ho}\mathbf{Diag}_P$$

which turns out to be an equivalence of categories. In this sense, the stratification diagram encodes the same homotopy theoretic information as the original space. We will use this equivalence to identify these two homotopy categories and often not distinguish between a stratified space and its stratification diagram.

Homotopy links (and thus also stratification diagrams) defined as subspaces of mapping spaces are, at first glance, objects unsuited to a computational or algorithmic approach. To obtain more geometrical and combinatorially interpretable models of the latter, we will also use another equivalent description of stratified homotopy types, which occur naturally, particularly when trying to quantitatively recover stratifications from non-stratified data in Section 4.4. Since our TDA investigation is mainly concerned with the two strata case, we will only consider $P = \{p < q\}$ for the remainder of this section and only give definitions in this scenario. The relevant observation (see [Dou19b]) is that instead of considering the poset P as a space with Alexandrov topology, we may instead consider it as a simplicial complex via its nerve $\mathrm{N}(P)$ (with vertices the elements of P and simplices given by flags) and then consider its realization. In the particular case $\{p < q\}$, the resulting space is canonically homeomorphic to $[0, 1]$, with p corresponding to 0 and q corresponding to $(0, 1]$. This leads to the following definition:

Definition 4.2.3.6. A *strongly stratified space* (over $P = \{p < q\}$) is a pair

$$S = (T, s : T \to [0, 1])$$

where T is a topological space and s is continuous. A *strongly stratum-preserving map* $f : S = (T, s) \to (T', s') = S'$ is a map of topological spaces $f : T \to T'$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "s"'] & & X' \arrow[dl, "s'"] \\ & {[0,1]} & \end{tikzcd}$$

commute.

Remark 4.2.3.7. The name strongly stratified space $S = (T, s : T \to [0, 1])$ relates to the fact that we may recover a stratified space by postcomposing with the stratification of $[0, 1]$ given by

$$\begin{aligned} [0, 1] &\to \{p < q\} \\ t &\mapsto \begin{cases} p & t = 0; \\ q & t > 0 \end{cases}. \end{aligned}$$

In this sense, a strong stratification is a stronger notion than a stratification, which is obtained by storing the additional information of a parametrization of a neighborhood around the singular stratum.

Notation 4.2.3.8. We denote by $\mathbf{Top}_{\mathrm{N}(P)}$ the category with objects given by strongly stratified spaces and morphisms given by strongly stratum-preserving maps. Isomorphisms in this category - i.e. strongly stratum-preserving homeomorphisms - will be denoted by $\cong_{\mathrm{N}(P)}$.

In a TDA scenario, where one usually works with metric spaces, strong stratifications arise naturally from stratifications.

Example 4.2.3.9. Let $S = (T, s)$ be a stratified space equipped with a metric $\mathrm{d}(-,-)$ on T. Then, S can be equipped with the structure of a strongly stratified space, compatible with the original stratification. The strong stratification map is given by the minimum of the distance to the singular stratum function and 1, i.e. by

$$\begin{aligned} \mathrm{d}_{S_p} : T &\to [0,1] \\ x &\mapsto \min\{\mathrm{d}(x, S_p), 1\}. \end{aligned}$$

The central examples of particularly well-behaved strongly stratified spaces are those that have the structure of a mapping cylinder close to the singular stratum (see Definition 4.2.3.13). The structure of such spaces near the singular stratum is specified by the following example.

Example 4.2.3.10. Given a map of topological spaces $r: L \to T$, we can consider the mapping cylinder of r

$$M_r := L \times [0,1] \cup_{L \times 0, r} T$$

equipped with the teardrop topology [Qui88, Definition 2.1] as a strongly stratified space via

$$\begin{aligned} \pi_{[0,1]} : M_r &\to [0,1] \\ [(x,t)] &\mapsto t. \end{aligned}$$

Note that if the above r is a proper map between locally compact Hausdorff spaces, then the usual quotient space topology agrees with the teardrop topology on the mapping cylinder [Hug99a]. When working with metric spaces, there is the following criterion for a map

$$f : M_r \to Z$$

into a metric space Z to be continuous. The map f is continuous, if and only if its restrictions to $L \times (0,1]$ and T are continuous, and the family of maps $f(-,t) : L \to Z$ with $t > 0$, converges uniformly to $f|_X \circ r$, as $t \to 0$ (consider [Qui88, Definition 2.1]).

Similar to the relation between diagrams and stratified spaces, strongly stratified spaces can also be used to describe stratified homotopy types, as explained in the following recollection.

Recollection 4.2.3.11. We have only described the construction of $\mathbf{Top}_{\mathrm{N}(P)}$ in the case of $P = \{p < q\}$ here. For the more general case see [Dou19b; DW22]. Similarly to the stratified case, the strongly stratified category can be equipped with a notion of weak equivalence, leading to a homotopy category $\mathrm{ho}\mathbf{Top}_{\mathrm{N}(P)}$. The forgetful functor

$$\mathbf{Top}_{\mathrm{N}(P)} \to \mathbf{Strat}_P,$$

obtained by post composing the strong stratification with the stratification of the interval

$$[0,1] \to \{p < q\}$$

given by taking 0 as the p-stratum, then (by passing to derived functors with respect to the model structures explained in [Dou21c]) induces an equivalence of homotopy categories

$$\mathrm{ho}\mathbf{Top}_{\mathrm{N}(P)} \to \mathrm{ho}\mathbf{Strat}_P.$$

We will often treat strongly stratified spaces as stratified spaces under this forgetful functor. The equivalence of homotopy categories guarantees that no homotopy theoretical information is lost.

We will not be making mathematical use of this result here. Nevertheless, it conceptually explains the multiple occurrences of strongly stratified spaces in our investigations of strongly stratified homotopy types.

As in the stratified scenario we make frequent use of some short notation to access the analogs of strata in the strongly stratified case.

Notation 4.2.3.12. Let S be a strongly stratified space and $v' \leq v \in [0,1]$. We use the following notation:

$$\begin{aligned} S_v &:= s^{-1}\{v\}, \\ S_{\leq v} &:= s^{-1}[0,v], \\ S^{\geq v'} &:= s^{-1}[v',1], \\ S^{v'}_v &:= s^{-1}[v',v]. \end{aligned}$$

For values of v, v' outside of $[0,1]$ we define these as above, using the closest allowable value.

It turns out that for particularly nice strongly stratified spaces, these sub- and superlevel sets can be used to recover the stratification diagram, cf. [Qui88; Mil94; DW22]. However, for the sake of completeness, we include details of this behavior with Example 4.2.3.16. As already alluded to above, these are stratified spaces for which the strata have cylinder neighborhoods.

Definition 4.2.3.13. We say a stratified space S over $P = \{p < q\}$ is *cylindrically stratified*, if there exists a neighborhood N of T_p and a space L and a map of spaces $r\colon L \to T_p$, such that

$$N \cong_P M_r,$$

where M_r denotes the stratified mapping cylinder of r from Example 4.2.3.10. We say a strongly stratified space $S = (T, s\colon T \to [0,1)])$ is *cylindrically stratified*, if it is cylindrically stratified as a stratified space and there is a homeomorphism $f\colon s^{-1}(0,1) \xrightarrow{\sim} S_{\frac{1}{2}} \times (0,1)$, making the diagram

$$\begin{tikzcd} s^{-1}(0,1) \arrow[rr, "f", "\sim"'] \arrow[dr, "s|_{s^{-1}(0,1)}"'] & & S_{\frac{1}{2}} \times (0,1) \arrow[dl, "\pi_{(0,1)}"] \\ & (0,1) & \end{tikzcd} \tag{4.7}$$

commute (i.e. a strongly stratum-preserving homeomorphism, with respect to the strong stratifications induced by s and $\pi_{[0,1]}$.)

Remark 4.2.3.14. Note, that the definition of a cylindrically stratified space in the strong case is slightly weaker than assuming a strongly stratified mapping cylinder neighborhood. We choose this definition for our purposes as it has precisely the same consequences and is much easier to verify. Nevertheless, for compact S, it follows by an application of the two-out-of-six property, as in Lemma 4.B.1.1, that the inclusions

$$S_p \hookrightarrow S_{\leq v}$$

for $0 \leq v < 1$, are homotopy equivalences.

Definition 4.2.3.15. A *cylindrically stratified metric space* S over $P = \{p < q\}$ is a stratified space equipped with a metric $\mathrm{d}(-,-)$, which is cylindrically stratified when considered as a strongly stratified space, with respect to the strong stratification induced by the metric (Example 4.2.3.9).

It turns out that many of the stratified spaces, that we are interested in, are cylindrically stratified.

Example 4.2.3.16. Whitney stratified spaces, equipped with the metric induced by the inclusion into $\mathbb{R}^N$, are cylindrically stratified up to a rescaling. They even admit neighborhoods that are strongly stratum-preserving homeomorphic to a strongly stratified mapping cylinder of a fiber bundle (in particular, they are conically stratified). This is a classical result, found for example already in [Tho69; Mat12]. We sketch a proof here for the sake of completeness. Let $S = (T, T \to \{p < q\})$ be a Whitney stratified space with S_p compact and $T \subset \mathbb{R}^N$. By passing to a sufficiently small neighborhood of S_p we may assume X to lie in a (standard) tubular neighborhood N of S_p in $\mathbb{R}^N$, such that the retraction map

$$\begin{aligned} r: N &\to S_p \\ x &\mapsto y_m, \end{aligned}$$

where y_m minimizes $\mathrm{d}(x,y)$, is well defined and smooth. Next, consider the distance to S_p map

$$\begin{aligned} \mathrm{d}_{S_p}: T &\to \mathbb{R} \\ x &\mapsto \mathrm{d}(x, S_p). \end{aligned}$$

It is then a consequence of Thom's first isotopy lemma (which in this two strata case amounts to Ehresmann's Lemma [Ehr51], see e.g.[Tho69] and [Ban07, Thm. 6.7] for a modern source), that the map

$$\begin{aligned} T \cap N &\to S_p \times \mathbb{R} \\ x &\mapsto (r(x), \mathrm{d}_{S_p}(x)) \end{aligned}$$

restricts to a fiber bundle over $S_p \times (0, \varepsilon]$, for ε small enough. If we denote by $\mathrm{N}_\varepsilon(S_p)$ a closed ε-neighborhood of S_p in X and set $L = \mathrm{d}_{S_p}^{-1}(\varepsilon)$ this means that there is a homeomorphism

$$f: \mathrm{N}_\varepsilon(S_p) \setminus S_p \xrightarrow{\sim} L \times (0, \varepsilon]$$

such that the diagram

$$\begin{tikzcd} \mathrm{N}_\varepsilon(S_p) \setminus S_p \arrow[rr, "f"] \arrow[dr, "r \times \mathrm{d}_{S_p}"'] & & L \times (0,\varepsilon] \arrow[dl, "r \times \pi_{(0,\varepsilon]}"] \\ & S_p \times (0,\varepsilon] & \end{tikzcd} \tag{4.8}$$

commutes. By rescaling, we may assume without loss of generality that $\varepsilon = 1$ and let $N = \mathrm{N}_1(S_p)$, the closed neighborhood of points with distance ≤ 1 to S_p.
Now, consider the map

$$g: M_r \to N; \qquad \begin{cases} (x,t) \mapsto f(x,t), & \text{for } t > 0, \\ [(x,0)] = [y] \mapsto r(x) = y, & \text{for } t = 0. \end{cases}$$

g is clearly bijective and continuous on S_p and $L \times (0,1]$. Furthermore, by the commutativity of Diagram 4.8, for $t \to 0$, $f(-,t): L \to N$ converges uniformly to $r\mid_L$. By the alternative characterization of the mapping cylinder topology in Example 4.2.3.10, it follows that g is a continuous bijection, from a compactum to a Hausdorff space, and thus a homeomorphism.

Example 4.2.3.17. Compact definably stratified spaces $S = (T, s: T \to \{p < q\})$, are (up to a rescaling) cylindrically stratified. Indeed, note first that they are cylindrically stratified as topological spaces. This follows from the fact that they are triangulable in a way that is compatible with the strata (see [Dri98]). In particular, S_p always admits a mapping cylinder neighborhood given by a regular neighborhood in the piecewise linear sense. Furthermore, note that the map

$$\mathrm{d}_{S_p}: T \to \mathbb{R}$$

again is definable. Thus, by Hardt's Theorem for definable sets (see [Dri98]), it restricts to a trivial fiber bundle over $(0,\varepsilon]$, for ε sufficiently small. In other words, after rescaling, we indeed have a homeomorphism

$$\mathrm{d}_{S_p}^{-1}(0,1) \to S_{\frac{1}{2}} \times (0,1).$$

over $(0,1)$.

Remark 4.2.3.18. We will generally consider all compact definably or Whitney stratified spaces to be appropriately rescaled, such that they are cylindrically stratified. Similar assumptions will be made for definably stratified spaces when using Lemma 4.B.1.1.

Finally, the following construction, together with Proposition 4.2.3.20, tells us that stratification diagrams of cylindrically stratified spaces have more interpretable geometric models, usable for TDA.

Construction 4.2.3.19. Given a stratified mapping cylinder M_r for $r : L \to Y$ a map of metrizable spaces, we may consider the map

$$\begin{aligned} \alpha\colon L &\to \mathcal{H}o\mathrm{Link}_{\{p<q\}} M_r \\ x &\mapsto \{t \mapsto [(x,t)]\}, \end{aligned}$$

mapping a point x to the corresponding line segment in M_r. A homotopy inverse to this map is given by

$$\begin{aligned} \beta\colon \mathcal{H}o\mathrm{Link}_{\{p<q\}} M_r &\to L \\ \gamma &\mapsto \pi_L(\gamma(1)). \end{aligned}$$

Clearly, $\beta \circ \alpha = 1_L$. A homotopy $\alpha \circ \beta \simeq 1_{\mathcal{H}o\mathrm{Link}_{\{p<q\}} M_r r}$ is given by

$$\begin{aligned} \mathcal{H}o\mathrm{Link}_{\{p<q\}} M_r \times [0,1] &\to \mathcal{H}o\mathrm{Link}_{\{p<q\}} M_r \\ (\gamma, s) &\mapsto \{t \mapsto (\pi_L(\gamma(s + (1-s)t), t). \end{aligned}$$

Compare [DW22], [Fri03] and [Qui88] for similar, more detailed arguments covering the continuity of such maps. Now, if S is a metrizable, cylindrically stratified space over $P = \{p < q\}$ and $N \cong M_r$ is a stratified mapping cylinder neighborhood of S_p with boundary L, then the inclusion

$$\mathcal{H}o\mathrm{Link}_{\{p<q\}} N \hookrightarrow \mathcal{H}o\mathrm{Link}_{\{p<q\}} S$$

is a (weak) homotopy equivalence. Essentially, the idea of the proof is to continuously retract paths in S into N (see [Fri03, Appendix] for details under slightly stronger assumptions). In particular, we then have a commutative diagram

$$\begin{tikzcd} S_q \arrow[rr, equal] & & S_q \\ L \times \{v\} \arrow[u, hook] \arrow[r, hook, "\simeq"] \arrow[d, "r"] & \mathcal{H}o\mathrm{Link}_{\{p<q\}}(N) \arrow[r, hook, "\simeq"] & \mathcal{H}o\mathrm{Link}_{\{p<q\}}(S), \arrow[u] \arrow[d] \\ S_p \arrow[rr, equal] & & S_p\,, \end{tikzcd}$$

for $v \in (0,1]$

Proposition 4.2.3.20. *Let S be a compact, cylindrically stratified metric space and (v_l, v_h), such that $0 < v_l \leq v_h < 1$. Then there is an isomorphism in* $\mathrm{ho}\mathbf{Diag}_P$

$$\{S_{\leq v_h} \hookleftarrow S^{v_l}_{v_h} \hookrightarrow S^{\geq v_l}\} \simeq \mathrm{D}_P(S).$$

Proof. Let s be the strong stratification induced by the metric on S. By assumption, S admits a mapping cylinder neighborhood $N \cong M_r$, for some map $r\colon L \to S_p$. Denote $\tilde{s}\colon N \to [0,1]$, the alternative strong stratification induced by this choice of mapping cylinder neighborhood. Since we assume that S_p is compact, we may assume, without loss of generality, that $N \subset s^{-1}[0,1)$. By Construction 4.2.3.19 (using the same notation), it suffices to expose a (canonical) zigzag of weak equivalence to the diagram

$$\{S_p \leftarrow L \times \{v\} \hookrightarrow S_q\},$$

for some $v \in (0,1]$. Such a zigzag between diagrams is given as follows:

$$\begin{tikzcd}
\{S_{\leq v_h} & S_{v_h}^{v_l} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\simeq"] & S^{\geq v_l}\} \\
\{s^{-1}[0,1) & s^{-1}(0,1) \arrow[l, hook'] \arrow[r, hook] & s^{-1}(0,1]\} \\
\{s^{-1}[0,1) & L \times \{v\} \arrow[l, hook'] \arrow[r, hook] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & s^{-1}(0,1] = S_q\} \\
\{S_p & L \arrow[l, "r"] \arrow[r, hook] & S_q\}.
\end{tikzcd} \tag{4.9}$$

We describe the morphisms of diagrams from top to bottom, and show that they are weak equivalences. The first morphism is given by inclusions. Since we have assumed that $s^{-1}(0,1)$ has the shape of an open cylinder $L' \times (0,1)$, this morphism is clearly given by weak equivalences at $\{p\},\{q\}$ and $\{p<q\}$. The next morphism is again given by inclusions. To see that it is a weak equivalence, we need to show that

$$L \times \{v\} \hookrightarrow \tilde{s}^{-1}(0,1] \hookrightarrow s^{-1}(0,1) \cong L' \times (0,1)$$

is a weak equivalence. Since the first of these maps is a weak equivalence, it suffices to show that the second is one too. Since S_p is compact we find $\varepsilon, \varepsilon' > 0$ such that

$$\tilde{s}^{-1}(0,\varepsilon') \subset s^{-1}(0,\varepsilon) \subset \tilde{s}^{-1}(0,1) \subset s^{-1}(0,1).$$

Now, note that since all sets involved are given by open cylinders (on L and L' respectively), these inclusions fulfill the requirements for the two-out-of-six property of homotopy equivalences. In particular, all maps involved are weak equivalences (even homotopy equivalences). Finally, the last morphism is constructed as follows. Both at q and $\{p<q\}$ it is given by the identity. Assume that $v \in (0,1]$ was taken such that

$$L \times \{v\} \cong \tilde{s}^{-1}\{v\} \subset s^{-1}(0,\varepsilon] \subset \tilde{s}^{-1}(0,1),$$

for some $\varepsilon > 0$ sufficiently small. Note that this is indeed possible by the compactness of S_p. Then, at $\{p\}$ the morphism is given by the composition

$$s^{-1}[0,1) \to s^{-1}[0,\varepsilon] \hookrightarrow N \cong M_r \to S_p$$

where the first of these maps is given by

$$(x,t) \mapsto (x, \min\{t,\varepsilon\})$$

under the identification $s^{-1}(0,1) \cong L' \times (0,1)$. By the assumption that $L \times \{v\}$ maps into $s^{-1}[0,\varepsilon]$, this map induces a morphism of diagrams. It remains to show that it is a weak equivalence. By the cylinder structure of $s^{-1}(0,1)$, the first map in this composition is a homotopy equivalence. The same holds true for the second map by a two-out-of-six argument, completely analogous to the one performed when comparing L and $L' \times (0,1)$. Finally, the last map is the retraction of a mapping cylinder and thus also a homotopy equivalence. Combining this, we have shown that the final morphism is also a weak equivalence of diagrams. □

4.3 Persistent stratified homotopy types

In this section, we introduce the notion of persistent stratified homotopy type (Section 4.3.1) and investigate its stability properties (Section 4.3.3), in the particular case of Whitney stratified spaces with two strata (Section 4.3.4). Before we focus on the specific case of persistent *stratified* homotopy types, let us first make a little more precise what we mean by persistent homotopy types and their role in topological data analysis. As already alluded to, for the case of persistent homology in the introduction, the perspective we take here on persistent approaches to TDA is that they can usually be decomposed into a two-step process. Conceptually, it can be described as follows:

Let $\mathbf{S}$ denote some categories of objects representing datasets which contain geometrical information. For example, we can take the category given by all subsets of $\mathbb{R}^N$, with morphisms given by inclusions. Let $\mathbf{T}$ denote some category of geometrical and or combinatorial objects, for example the categories $\mathbf{Top}$, $\mathbf{sSet}$ (the category of simplicial sets), $\mathbf{Strat}_P$, $\mathbf{Diag}_P$, equipped with a class of morphisms called weak equivalences. Let $\mathbf{A}$ denote some category of algebraic objects, for example, the category of vector spaces over some fixed field. Finally, let

$$H\colon \mathbf{T} \to \mathbf{A}$$

be a functor that sends weak equivalences to isomorphisms, for example for $\mathbf{T} = \mathbf{Top}$ this could be a homology functor. We denote $\mathbb{R}_+$ the category given by the poset of nonnegative reals.

Notation 4.3.0.1. Given any category $\mathcal{C}$ and another category U (most prominently $\mathbb{R}_+$). We denote by $\mathcal{C}^U$ the category of functors from U to $\mathcal{C}$, with morphisms given by natural transformations.

Then, a persistent version of H is constructed by taking a composition

$$\mathbf{S} \xrightarrow{\mathcal{P}} \mathbf{T}^{\mathbb{R}_+} \xrightarrow{H^{\mathbb{R}_+}} \mathbf{A}^{\mathbb{R}_+},$$

for some functor $\mathbf{S} \xrightarrow{\mathcal{P}} \mathbf{T}^{\mathbb{R}_+}$ turning objects in $\mathbf{S}$ into persistent objects in $\mathbf{T}$, i.e. elements of $\mathbf{T}^{\mathbb{R}_+}$. Examples of such functors include sending a subspace of $\mathbb{R}^N$ to the family of its ε thickenings, filtered Čech- or Vietoris-Rips complexes. Since H sends weak equivalences into isomorphisms, we obtain a factorization

$$\begin{tikzcd}
\mathbf{S} \arrow[r] \arrow[rr, bend left, "PH"] \arrow[dr, "\mathcal{P}"'] & \mathbf{T}^{\mathbb{R}_+} \arrow[r, "H^{\mathbb{R}_+}"] \arrow[d] & \mathbf{A}^{\mathbb{R}_+} \\
& \mathrm{ho}\mathbf{T}^{\mathbb{R}_+} \arrow[ur, dashed] &
\end{tikzcd} \tag{4.10}$$

Notation 4.3.0.2. All throughout this paper, $\mathrm{ho}\mathbf{T}^{\mathbb{R}_+}$ denotes the category obtained by localizing $\mathbf{T}^{\mathbb{R}_+}$ at such natural transformations, which are given by a weak equivalence at each $\alpha \in \mathbb{R}_+$. Such natural transformations will be called weak equivalences of functors. We will also refer to isomorphisms in the homotopy category $\mathrm{ho}\mathbf{T}^{\mathbb{R}_+}$ (which are always represented by zigzags of weak equivalences in $\mathbf{T}^{\mathbb{R}_+}$) as weak equivalences. The same notation and language is used when $\mathbb{R}_+$ is replaced by a more general indexing category. (The reader wondering about the relation of this construction to taking the homotopy category first and then passing to persistent objects is referred to Remark 4.A.0.1.)

The functor $\mathcal{P}$ can then be understood as assigning to an object in $\mathbf{S}$ a persistent homotopy type. By a slight abuse of language, we thus refer to $\mathcal{P}(\mathbb{X})$ as the *persistent homotopy type of* $\mathbb{X} \in \mathbf{S}$, even though it depends on a choice of functor $\mathbf{S} \to \mathbf{T}^{\mathbb{R}_+}$. (Note, however, that this abuse of language is no more incorrect than speaking of *the persistent homology*, which also depends on choices such as Čech- or Vietoris-Rips complexes.) Then, as we illustrated in the introduction to the scenario of classical persistent homology, many properties of the functor PH may already be understood by studying the functor $\mathcal{P}$. At the same time, the advantage of such a modular approach is that it quickly allows obtaining results for all sorts of choices of H.

4.3.1 Definition of the persistent stratified homotopy type

Let us now move to the specific case of persistent stratified homotopy types. The goal is to expose a functor $\mathcal{P}$ with values in the category $\mathrm{ho}\mathbf{Strat}_P^{\mathbb{R}_+}$. Furthermore, for the values of such functor to deserve the name, *persistent stratified homotopy types*, we need to show it fulfills properties analogous to the properties P(1), P(2) and P(3) of the non-stratified persistent homotopy type. Let us summarize the pipeline suggested by Sections 4.2.2 and 4.2.3.
We start with a Whitney stratified or definably stratified space $S \subset \mathbb{R}^N$ (or later on a sample of any of the latter) aiming to obtain a persistent version of its stratified homotopy type. By Recollection 4.2.3.5 the stratified homotopy type of S is equivalently described by its stratification diagram $\mathrm{D}_P(S)$. It is an immediate consequence of [Dou21c, Thm. 2.12] (which is a stronger version of Recollection 4.2.3.5) that the homotopy link functor from Recollection 4.2.3.5 sending a stratified space to a stratification diagram, induces an equivalence of categories

$$\mathrm{ho}\mathbf{Strat}_P^{\mathbb{R}_+} \xrightarrow{\sim} \mathrm{ho}\mathbf{Diag}_P^{\mathbb{R}_+}. \tag{4.11}$$

Hence, we may equivalently expose a functor $\mathcal{P}$ valued in $\mathrm{ho}\mathbf{Diag}_P^{\mathbb{R}_+}$. To do so, we need to obtain a geometric description of the stratification diagram $\mathrm{D}_P(S)$, which admits thickenings. By Examples 4.2.3.16 and 4.2.3.17, S naturally admits the structure of a cylindrically stratified space (up to a rescaling). Then, by Proposition 4.2.3.20, up to a weak equivalence we may recover the stratification diagram $\mathrm{D}_P(S)$ of S by the diagram of sub- and superlevel sets

$$S_{\leq v_h} \hookleftarrow S_{v_h}^{v_l} \hookrightarrow S^{\geq v_l}.$$

The diagram obtained in this fashion has the advantage that it admits an obvious notion of thickening. One simply thickens the parts of the diagram contained in $\mathbb{R}^N$ separately.

Let us now set up the framework to analyze the stability properties of these constructions and their interactions with sampling. For the remainder of this subsection let $P = \{p < q\}$ be a poset with two elements. Next, we define a series of spaces of subspaces of $\mathbb{R}^N$. One should mainly think of elements of these spaces as samples, sampled nearby a continuous space, whose convergence behavior we are investigating. Nevertheless, in the generality below all sorts of complicated, non-finite sets are permitted. We will follow the naming convention of using blackboard bold letters like $\mathbb{X}$, when suggesting an object conceptually takes the role of a sample. We use usual letters, like T, when an objects takes the role of a space nearby which samples are taken. Of course, this convention does not apply to the fields $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$.

Remark 4.3.1.1. Throughout the remainder of this paper, we will be defining several distances on categories and spaces whose elements are themselves some version of spaces. In this context, the term metric will mean *symmetric Lawvere metric*, that is, we allow for the value ∞, and do not require nonidentical elements to have positive distance.

Notation 4.3.1.2. Throughout the remainder of this paper, $||-||$ always denotes the Euclidean norm on $\mathbb{R}^N$. Given a subset $\mathbb{X} \subset \mathbb{R}^N$, and some non-negative real number $\alpha \geq 0$, we denote by $\mathbb{X}_\alpha$ the α thickening of $\mathbb{X}$, given by

$$\{y \in \mathbb{R}^N \mid \exists x \in \mathbb{X} : ||x - y|| \leq \alpha\}.$$

We will take care, to always use greek letters for thickenings, as to avoid any possible confusion with the indexing letters denoting strata.

Definition 4.3.1.3. Denote by **Sam** the space of subspaces of $\mathbb{R}^N$, $\{\mathbb{X} \subset \mathbb{R}^N\}$, equipped with the (extended pseudo) metric given by the Hausdorff-distance. That is, for $\mathbb{X}, \mathbb{X}' \in \mathbf{Sam}$, we set

$$\mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X}, \mathbb{X}') = \inf\{\alpha > 0 \mid \mathbb{X} \subset \mathbb{X}'_\alpha, \mathbb{X}' \subset \mathbb{X}_\alpha\}.$$

We refer to **Sam** as the *space of samples* (of $\mathbb{R}^N$).

Next, we define a metric on the set of stratified samples.

Definition 4.3.1.4. Denote by $\mathbf{Sam}_P$ the space of pairs in $\mathbb{R}^N$,

$$\{(\mathbb{X}, \mathbb{X}_p) \mid \mathbb{X}_p \subset \mathbb{X}\} \subset \mathbf{Sam}^2,$$

equipped with the (extended pseudo) metric induced by the inclusion.
That is, for $\mathbb{X}, \mathbb{X}' \in \mathbf{Sam}_P$, we set

$$\mathrm{d}_{\mathbf{Sam}_P}((\mathbb{X}, \mathbb{X}_p), (\mathbb{X}', \mathbb{X}'_p)) := \max\{\mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X}, \mathbb{X}'), \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X}_p, \mathbb{X}'_p)\}.$$

We also refer to $\mathbf{Sam}_P$ as the *space of (P-)stratified samples* (of $\mathbb{R}^N$).

We can think of $\mathbf{Sam}_P$ as a metricized, sample version of the category $\mathbf{Strat}_P$. Next, we need an analogous construction for the category $\mathbf{Diag}_P$.

Definition 4.3.1.5. Denote by $\mathbf{D}_P\mathbf{Sam}$ the space of triples of subspaces of $\mathbb{R}^N$

$$\{(\mathbb{D}_p, \mathbb{D}_{\{p<q\}}, \mathbb{D}_q) \mid \mathbb{D}_q \supset \mathbb{D}_{\{p<q\}} \subset \mathbb{D}_p, \} \subset \mathbf{Sam}^3,$$

equipped with the subspace metric. That is, for $\mathbb{D}, \mathbb{D}' \in \mathbf{D}_P\mathbf{Sam}$ we set

$$\mathrm{d}_{\mathbf{D}_P\mathbf{Sam}}(\mathbb{D}, \mathbb{D}') := \max_{\mathcal{I} \in \mathrm{R}(P)} \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{D}_{\mathcal{I}}, \mathbb{D}'_{\mathcal{I}}).$$

We also refer to $\mathbf{D}_P\mathbf{Sam}$ as the *space of stratification diagram samples* (of $\mathbb{R}^N$).

Finally, we repeat a similar process for $\mathbf{Top}_{\mathrm{N}(P)}$.

Definition 4.3.1.6. Denote by $\mathbf{Sam}_{\mathrm{N}(P)}$ the space

$$\{(\mathbb{X}, s\colon \mathbb{X} \to [0,1]) \mid \mathbb{X} \subset \mathbb{R}^N\},$$

equipped with the (extended pseudo) metric given as follows. Embed $\mathbf{Sam}_{\mathrm{N}(P)}$ into the space of subspaces of $\mathbb{R}^N \times [0,1]$, equipped with the Hausdorff distance on the product metric, by sending s to its graph. The metric on $\mathbf{Sam}_{\mathrm{N}(P)}$ is then given by the subspace metric under this embedding.
Equivalently, this comes down to the following. For $(\mathbb{X}, s), (\mathbb{X}', s') \in \mathbf{Sam}_{\mathrm{N}(P)}$, we set

$$\mathrm{d}_{\mathbf{Sam}_{\mathrm{N}(P)}}((\mathbb{X}, s), (\mathbb{X}', s')) := \max_{\mathbb{X}_0, \mathbb{X}_1 \in \{\mathbb{X}, \mathbb{X}'\}^2} \{\inf\{\varepsilon > 0 \mid \forall x \in \mathbb{X}_0 \exists y \in \mathbb{X}_1 : \\ \|x - y\|, |s_0(x) - s_1(y)| \le \varepsilon\}\}.$$

We also refer to $\mathbf{Sam}_{\mathrm{N}(P)}$ as the *space of strongly (P-)stratified samples* (of $\mathbb{R}^N$).

Notation 4.3.1.7. For the remainder of this work, we denote $v = (v_l, v_h) \in (0,1)^2$ with $v_l < v_h$, and $u \in (0,1)$. Furthermore, we equip $(0,1)^2$ with the usual order in the second, and the opposite order in the first component, that is we write

$$v \le v' \iff v'_l \le v_l \quad \text{and} \quad v_h \le v'_h.$$

We denote by $\Omega \subset (0,1)^2$ the subposet of $(0,1)^2$ with this order, consisting of (v_l, v_h) with $0 < v_l < v_h < 1$.

Finally, let us model the several constructions connecting stratified spaces, strongly stratified spaces and stratification diagrams we described in Section 4.2 in this framework.

Construction 4.3.1.8. Consider the following three maps.

$$\begin{tikzcd} \mathbf{Sam}_P \arrow[r, shift left, "\mathcal{N}"] & \mathbf{Sam}_{\mathrm{N}(P)} \arrow[l, shift left, "\mathcal{F}_u"] \arrow[r, "\mathcal{D}_v"] & \mathbf{D}_P\mathbf{Sam} \end{tikzcd}$$

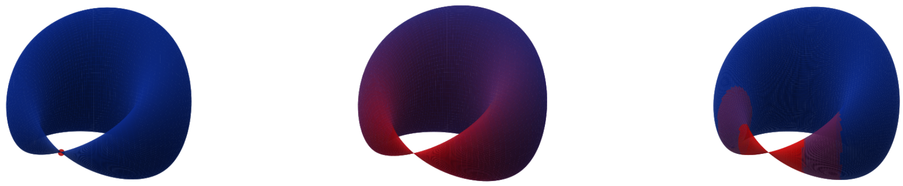


Figure 4.12: From left to right, illustrations of: The pinched torus PT as an element of PT; the strongly stratified space $\mathcal{N}(PT)$; the associated diagram $\mathcal{D}_v(\mathcal{N}(PT))$

These are defined via:

$$\begin{aligned}(\mathbb{X}, \mathbb{X}_p) &\overset{\mathcal{N}}{\gtrsim\!\!\to} (\mathbb{X}, \min\{\mathrm{d}_{\mathbb{X}_p}, 1\}),\\ \mathbb{S} = (\mathbb{X}, s) &\overset{\mathcal{F}_u}{\gtrsim\!\!\to} (\mathbb{X}, \mathbb{S}_{\leq u}),\\ \mathbb{S} = (\mathbb{X}, s) &\overset{\mathcal{D}_v}{\gtrsim\!\!\to} (\mathbb{S}_{\leq v_h}, \mathbb{S}^{v_l}_{v_h}, \mathbb{S}^{\geq v_l}) .\end{aligned}$$

The map $\mathcal{N}$ corresponds to the assignment of a strong stratification to a stratified metric space (see Example 4.2.3.9). $\mathcal{F}_u$ gives a family of models for the forgetful functor, $\mathbf{Top}_{\mathrm{N}(P)} \to \mathbf{Strat}_P$, described in Recollection 4.2.3.11. Finally, by Proposition 4.2.3.20, $\mathcal{D}_v$ (composed with $\mathcal{N}$) provides a model for the functor assigning to a stratified space its stratification diagram, $\mathrm{D}_P \colon \mathbf{Strat}_P \to \mathbf{Diag}_P$ (see Recollection 4.2.3.5).

Example 4.3.1.9. Consider the three pictures in Fig. 4.12. The first shows the pinched torus PT as a stratified subspace of $\mathbb{R}^3$, with the singularity marked in red. The second shows $\mathcal{N}(PT)$, where the color coding indicates the strong stratification. Finally, the third shows the image under $\mathcal{D}_v$ for $v = (0.2, 0.4)$. Specifically, the union of the red and purple part give the p-part of the diagram, the purple part the $\{p < q\}$-part, and the union of the purple and the blue one the q-part.

We will later make use of the following immediate elementary relation between $\mathcal{D}_v$ and $\mathcal{F}_u$.

Lemma 4.3.1.10. *Let $v = (v_l, u) \in \Omega$. Then,*

$$\mathcal{F}_u(\mathbb{S}) = (\mathcal{D}_v(\mathbb{S})_p \cup \mathcal{D}_v(\mathbb{S})_{\{p<q\}} \cup \mathcal{D}_v(\mathbb{S})_q, \mathcal{D}_v(\mathbb{S})_p)$$

for all $\mathbb{S} \in \mathbf{Sam}_{\mathrm{N}(P)}$.

Remark 4.3.1.11. Note, that all of the described sample spaces naturally admit the structure of a poset. In the case of $\mathbf{Sam}$, $\mathbf{Sam}_P$ and $\mathbf{D}_P\mathbf{Sam}$ it is simply given by inclusions. In case of $\mathbf{Sam}_{\mathrm{N}(P)}$, it is obtained by treating elements of $\mathbf{Sam}_{\mathrm{N}(P)}$ as their graph, i.e. as a subset of $\mathbb{R}^N \times [0, 1]$ and then using the inclusion relation. Equivalently, this means

$$(\mathbb{X}, s) \leq (\mathbb{X}', s') \iff (x \in \mathbb{X} \implies (x \in \mathbb{X}' \ \& \ s(x) = s'(x))).$$

In this fashion, the spaces of samples may also be treated as categories and the maps of Construction 4.3.1.8 are functors. Furthermore, from this perspective we can treat $\mathbf{Sam}_P$ as a subcategory of $\mathbf{Strat}_P$ treating the equivalence in Proposition 4.3.1.14 as a natural equivalence. We will not make much use of this perspective here. However, it allows for notation such as $\mathbf{Sam}^I$, where I is some indexing category to make sense, and we will use this freely. Furthermore, from this perspective the metrics on $\mathbf{Sam}_P$ and $\mathbf{D}_P\mathbf{Sam}$ are induced by the flow given by componentwise thickening (see Example 4.3.2.3 for details).

Notation 4.3.1.12. In the remainder of the section, we will frequently state that certain functors are *homotopically constant*, which means the following: If $\mathbf{T}$ is a category with weak equivalences and U some indexing category, then we say $F \in \mathrm{ho}\mathbf{T}^U$ is homotopically constant with value $T \in \mathbf{T}$, if there is an isomorphism in $\mathrm{ho}\mathbf{T}^U$

$$F \simeq T,$$

where we treat T as as object of the functor category $\mathbf{T}^U$ by sending it to the constant functor of value T. Note that this implies in particular that all structure maps of F are weak equivalences.

The categorical perspective on the sampling spaces can be used to define a parameter independent version of $\mathcal{D}_v$.

Construction 4.3.1.13. Note that for $v \leq v'$ we have natural inclusions

$$\mathcal{D}_v \hookrightarrow \mathcal{D}_{v'}.$$

These induce a map

$$\mathcal{D}\colon \mathbf{Sam}_{\mathrm{N}(P)} \to \mathbf{D}_P\mathbf{Sam}^{\Omega}.$$

Proposition 4.2.3.20 may then be rephrased as follows.

Proposition 4.3.1.14. *Let $S \in \mathbf{Sam}_P$ be cylindrically stratified. Then, for all $v \in \Omega$ we have a weak equivalence*

$$\mathrm{D}_P(S) \simeq \mathcal{D}_v \circ \mathcal{N}(S).$$

In fact, even more, there is an isomorphism in the homotopy category $\mathrm{ho}\mathbf{Diag}_P^{\Omega}$

$$\mathrm{D}_P(S) \simeq \mathcal{D} \circ \mathcal{N}(S).$$

Proof. Note, that in the proof of Proposition 4.2.3.20 we in fact first constructed a weak equivalence of $\mathcal{D}_v \circ \mathcal{N}(S)$ with a diagram independent of v. It is immediate from the construction there, that this weak equivalences induces a weak equivalence from $\mathcal{D} \circ \mathcal{N}(S)$ to a constant functor. The second part of the proof then shows that this constant functor is weakly equivalent to the constant functor with value $\mathrm{D}_P(S)$. ☐

In particular, under the equivalence of homotopy categories $\mathrm{ho}\mathbf{Strat}_P \cong \mathrm{ho}\mathbf{Diag}_P$, S and $\mathcal{D}_v \circ \mathcal{N}(S)$ represent the same stratified homotopy type. As a consequence, to define persistent stratified homotopy types, we can thicken stratification diagrams instead of stratified spaces.

Construction 4.3.1.15. Define *the thickening of* $\mathbb{D}$ *by* $\alpha \geq 0$ via:

$$\mathbb{D}_\alpha := ((\mathbb{D}_q)_\alpha, (\mathbb{D}_{\{p<q\}})_\alpha, (\mathbb{D}_q)_\alpha).$$

For $\alpha \leq \alpha'$ there are the obvious inclusions of diagrams

$$\mathbb{D}_\alpha \hookrightarrow \mathbb{D}_{\alpha'}$$

We thus obtain a map (functor from the categorical perspective)

$$\begin{aligned} \mathbf{D}_P\mathbf{Sam} &\to \mathbf{D}_P\mathbf{Sam}^{\mathbb{R}_+} \\ \mathbb{D} &\mapsto \{\alpha \mapsto \mathbb{D}_\alpha\} \end{aligned}$$

with the structure maps given by inclusions. We may then treat the sample diagrams as elements of $\mathbf{Diag}_P$, ultimately obtaining the composition:

$$\mathcal{DP}\colon \mathbf{D}_P\mathbf{Sam} \to \mathbf{D}_P\mathbf{Sam}^{\mathbb{R}_+} \to \mathrm{ho}\mathbf{Diag}_P^{\mathbb{R}_+} \simeq \mathrm{ho}\mathbf{Strat}_P^{\mathbb{R}_+}.$$

We now have everything in place to define persistent stratified homotopy types.

Definition 4.3.1.16. The *persistent stratified homotopy type* of a stratified sample $\mathbb{S} \in \mathbf{Sam}_P$ (depending on the parameter v) is defined as the image of $\mathbb{S}$ under the composition

$$\mathcal{P}_v\colon \mathbf{Sam}_P \xrightarrow{\mathcal{N}} \mathbf{Sam}_{\mathrm{N}(P)} \xrightarrow{\mathcal{D}_v} \mathbf{D}_P\mathbf{Sam} \xrightarrow{\mathcal{DP}} \mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+},$$

where the final map is the one defined in Construction 4.3.1.15.

Remark 4.3.1.17. Note, that by construction, $\mathcal{P}_v$ fulfills an analog of Property P(1), i.e. it admits a combinatorial interpretation which, for finite samples, can be stored on a computer. Indeed, by construction the persistent stratified homotopy type of $\mathbb{S} \in \mathbf{Sam}_P$ is equivalently represented by the image of $\mathbb{S}$ under

$$\mathcal{P}_v\colon \mathbf{Sam}_P \xrightarrow{\mathcal{N}} \mathbf{Sam}_{\mathrm{N}(P)} \xrightarrow{\mathcal{D}_v} \mathbf{D}_P\mathbf{Sam} \xrightarrow{\mathcal{DP}} \mathrm{ho}\mathbf{Diag}_P^{\mathbb{R}_+},$$

named the same by abuse of notation. Then, it is a consequence of the classical nerve theorem (see e.g. [Hat02, Prop. 4G.3] or [Bor48]) that for $\mathbb{S} \in \mathbf{Sam}_P$, $\mathcal{P}_v(\mathbb{S})$ is equivalently represented by the diagram of Čech complexes

$$\alpha \mapsto \{\mathcal{I} \mapsto \check{\mathrm{C}}_\alpha(\mathcal{D}_v(\mathcal{N}(\mathbb{S}))_{\mathcal{I}})\}$$

where $\check{\mathrm{C}}_\alpha$ denotes the Čech complex with respect to α. When $\mathbb{S}$ is finite, this data can be stored on a computer and algorithmically evaluated.

Definition 4.3.1.18. The (parameter-free) *persistent stratified homotopy type* of a stratified sample $\mathbb{S} \in \mathbf{Sam}_P$ is defined as the image of $\mathbb{S}$ under the composition

$$\mathcal{P}\colon \mathbf{Sam}_P \xrightarrow{\mathcal{N}} \mathbf{Sam}_{\mathrm{N}(P)} \xrightarrow{\mathcal{D}} \mathbf{D}_P\mathbf{Sam}^\Omega \xrightarrow{\mathcal{DP}^\Omega} (\mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+})^\Omega \cong \mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+\times\Omega}.$$

Then, the following two results guarantee that for sufficiently regular stratified spaces the homotopy type does not change under small thickenings (this is Property P(3), see [NSW08] for the analogous result in the non-stratified smooth setting). This justifies the use of persistent stratified homotopy types as a means to infer stratified homotopic information. Recall that the weak feature size of a subspace T of $\mathbb{R}^N$ ([CL05]), is a non-negative real number ε associated to T, which has the property that the natural inclusion $T_\alpha \hookrightarrow T_{\alpha'}$ is a homotopy equivalences, for $0 < \alpha \leq \alpha' < \varepsilon$.

Proposition 4.3.1.19. *Let $S \in \mathbf{Sam}_P$ be a compact, definable stratified metric space. Then, for any $v \in \Omega$, there exists an $\varepsilon > 0$, such that the structure map*

$$\mathcal{P}_v(S)(\alpha) \to \mathcal{P}_v(S)(\alpha')$$

is a weak equivalence, for all $0 \leq \alpha \leq \alpha' < \varepsilon$. In particular,

$$\mathcal{P}_v(S)\,|_{[0,\varepsilon)} \simeq S.$$

In other words, the persistent stratified homotopy type of S at v restricted to $[0,\varepsilon)$, is weakly equivalent to the constant functor with value S. Furthermore, ε can be taken to be the minimum of the weak feature size of the entries of $\mathcal{D}_v(T)$ (see [CL05]), and the latter is positive.

Proof. Note that by definition of a weak equivalence in the category of stratification diagrams, this statement really just says there exists an $\varepsilon > 0$, such that for each flag $\mathcal{I}$ in P the inclusions

$$\mathcal{D}_v(S)_{\mathcal{I}} \hookrightarrow (\mathcal{D}_v(S)_{\mathcal{I}})_\alpha$$

are weak equivalences, for $\alpha \leq \varepsilon$. Note, however, that by the definability assumption $\mathcal{D}_v(S)_{\mathcal{I}}$ is again definable. Hence, this follows from the fact that the homotopy type of compact definable sets is invariant under slight thickenings (see Lemma 4.B.1.1 for the precise statement and the fact that ε can be taken as the minimum of the weak feature size of the entries of $\mathcal{D}_v(T)$). For $\alpha = 0$, we have

$$\mathcal{P}_v(S)(0) \simeq S$$

by Proposition 4.3.1.14. $\square$

Proposition 4.3.1.20. *Let $S \in \mathbf{Sam}_P$ be a compact, definably stratified space. Then (up to a linear rescaling), the persistent stratified homotopy type*

$$\mathcal{P}(S) : \Omega \times \mathbb{R}_+ \to \mathrm{ho}\mathbf{Strat}_P$$

is homotopically constant with value S on an open neighborhood of $\Omega \times \{0\}$.

Proof. That the functor is homotopically constant with value S on $\Omega \times \{0\}$ is the content of Proposition 4.3.1.14. Let ω_v denote the minimum of the weak feature sizes (compare to [CL05]) of the entries of $\mathcal{D}_v(S)$. An elementary argument shows that ω_v varies continuously in v. By Lemma 4.B.1.1 all weak feature sizes involved are positive. We take

$$U := \{(v, \alpha) \mid \alpha \text{ is smaller than } \omega_v\}.$$

From Proposition 4.3.1.19 it follows, that all structure maps of $\mathcal{P}(S)$ on U in direction $\mathbb{R}_+$ are weak equivalences. From this, it already follows that all structure maps of $\mathcal{P}(S)\mid_U$ are weak equivalences. For the slightly stronger result that this already implies that $\mathcal{P}(S)\mid_U$ is homotopically constant, see Lemma 4.A.0.3. □

4.3.2 Metrics on categories of persistent objects

One of the central requirements for the use of persistent homology in practice is the fact that it is stable with respect to Hausdorff and interleaving distance (P(2), first shown in [Cha+09]). Investigating the use of metrics in persistent scenarios and the stability of functors with respect to them has since been the content of ongoing research ([BW20; Hof+17; Les15; BL21; BSS20]). The stability of persistent homology with respect to the interleaving distance may, however, already be phrased at the level of persistent homotopy types (even on the level of persistent spaces), as we explain in the remainder of this subsection. The goal of Section 4.3.3 is to investigate the stability behavior of the persistent stratified homotopy type. To do so, we make us of the notion of a flow introduced [SMS18]. For the sake of conciseness, we recall a slightly less general definition here.

Recollection 4.3.2.1 ([SMS18])**.** A *strict flow* on a category $\mathcal{C}$ is a strict monoidal functor $(-)_- : \mathbb{R}_+ \to \mathrm{End}(\mathcal{C})$. In other words, to each $\varepsilon \in \mathbb{R}_+$ we assign an endofunctor $(-)_\varepsilon$ and whenever $\varepsilon \leq \varepsilon'$ we assign (functorially) a natural transformation $s_{\varepsilon \to \varepsilon'} : (-)_\varepsilon \to (-)_{\varepsilon'}$. Being strict monoidal means that $(-)_0 = 1_{\mathcal{C}}$, $(-)_{\varepsilon'} \circ (-)_\varepsilon = (-)_{\varepsilon + \varepsilon'}$ and $(s_{\varepsilon \leq \varepsilon'})_\delta = s_{\varepsilon + \delta \leq \varepsilon' + \delta}$. Generally, one should think of flows as a notion of shift on $\mathcal{C}$. Then, just as in the scenario of the interleaving distance for persistence modules [Cha+09], one says that $X, Y \in \mathcal{C}$ are *ε-interleaved* if there are morphisms $f : X \to Y_\varepsilon$ and $g : Y \to X_\varepsilon$ and such that the diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr] & Y \arrow[d, "g"] \arrow[dl] \\
Y_\varepsilon \arrow[d, "g_\varepsilon"'] \arrow[dr] & X_\varepsilon \arrow[d, "f_\varepsilon"] \arrow[dl] \\
X_{2\varepsilon} & Y_{2\varepsilon}
\end{tikzcd} \tag{4.12}$$

commutes (all unlabelled morphisms are given by the flow). One then obtains a (symmetric Lawvere) metric space by setting

$$\mathrm{d}_{\mathrm{In}}(X, Y) := \inf\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-interleaved}\}. \tag{4.13}$$

An immediate consequence of this definition is that any functor $F : \mathcal{C} \to \mathcal{D}$ between categories with a (strict) flow that fulfills $(-)_\varepsilon \circ F = F \circ (-)_\varepsilon$ for all $\varepsilon \in \mathbb{R}_+$ is necessary 1-Lipschitz with respect to the interleaving distances, that is it fulfills

$$\mathrm{d}_{\mathrm{In}}(F(X), F(Y)) \leq \mathrm{d}_{\mathrm{In}}(X, Y),$$

for $X, Y \in \mathcal{C}$.

Example 4.3.2.2. Consider $\mathbb{R}^k$ equipped with the product partial ordering, given by $u \leq v :\iff \forall i \in \{1, \dots, n\}: u_i \leq v_i$. Let $U \subset \mathbb{R}^k$ be a generalized interval (i.e. a subset of $\mathbb{R}^k$, such that $u, v \in U, u \leq w \leq v$ implies $w \in U$). Furthermore, let $\mathbf{T}$ be a category with a terminal object $\star$. Then any functor category $\mathbf{T}^U$ is naturally equipped with a shift type flow, given by

$$X_\varepsilon(u) := \begin{cases} X(u + \varepsilon(1, \dots, 1)) & \text{, for } u + \varepsilon(1, \dots, 1) \in U; \\ \star & \text{, for } u + \varepsilon(1, \dots, 1) \notin U. \end{cases}$$

If $\mathbf{T}$ is equipped with a notion of weak equivalence (which includes all isomorphisms), then by construction the flow respects weak equivalences in $\mathbf{T}^U$. Thus, it descends to a flow on the homotopy category $\mathrm{ho}\mathbf{T}^U$ obtained by localizing weak equivalences of functors. In particular, this construction equips the persistent homotopy category $\mathrm{ho}\mathbf{Top}^{\mathbb{R}_+}$ with a symmetric Lawvere metric, such that the functor

$$\mathbf{Top}^{\mathbb{R}_+} \to \mathrm{ho}\mathbf{Top}^{\mathbb{R}_+}$$

is 1-Lipschitz. More generally, the same construction works for the cases $\mathbf{T} = \mathbf{Strat}_P, \mathbf{Diag}_P$. We call distances of this type *interleaving distances*. Furthermore, for any functor between two such categories $\mathbf{T}, \mathbf{T}'$, which descends to the homotopy category, the induced functors

$$\mathbf{T}^{\mathbb{R}_+} \to \mathbf{T}'^{\mathbb{R}_+},$$
$$\mathrm{ho}\mathbf{T}^{\mathbb{R}_+} \to \mathrm{ho}\mathbf{T}'^{\mathbb{R}_+}$$

commute with shifting and are thus 1-Lipschitz. Whenever we refer to a metric on such a functor category (or its homotopy category), we will be referring to the interleaving distance.

Example 4.3.2.3. Denote by **Sam** the category of subsets of $\mathbb{R}^N$, with morphisms given by inclusions. If we take $\mathbb{X}_\varepsilon$ to be given by an ε thickening, for $\varepsilon \in \mathbb{R}_+$ then this construction defines a strict flow on **Sam**. The distance induced by the flow is the Hausdorff distance (compare [SMS18]). Clearly, the functor

$$\begin{aligned} \mathcal{P}: \mathbf{Sam} &\to \mathbf{Top}^{\mathbb{R}_+} \\ \mathbb{X} &\mapsto \{\varepsilon \mapsto \mathbb{X}_\varepsilon\} \end{aligned}$$

commutes with flows. In particular, this gives an, albeit somewhat unnecessarily abstract, proof of the fact that persistent homotopy types are stable with respect to Hausdorff and interleaving distance, which immediately implies the stability of persistent homology ([Cha+09]). More generally, if we define thickening componentwise, then the Hausdorff style distances on $\mathbf{Sam}_P$ and $\mathbf{D}_P\mathbf{Sam}$ (Definitions 4.3.1.4 and 4.3.1.5) are also induced by the thickening flow.

In the next subsection, we are going to make frequent use of *tautological* stability results as in Examples 4.3.2.2 and 4.3.2.3. For example, from the flow perspective, just as in the non-stratified scenario, one immediately obtains:

Lemma 4.3.2.4. *$\mathcal{DP}: \mathbf{D}_P\mathbf{Sam} \to \mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+}$ is 1-Lipschitz.*

Example 4.3.2.5. We may identify Ω with the subset

$$U = \{(x, y) \in (-1, 0) \times (0, 1) \mid -x < y\}$$

by mapping $(v_l, v_h) \mapsto (-v_l, v_h)$. Note that this defines an order preserving map. Then the construction in Example 4.3.2.2 defines a flow distance on $\mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+ \times \Omega}$.

4.3.3 Stability of persistent stratified homotopy types

As we have seen in Section 4.3.2, especially Example 4.3.2.3, the stability of persistent homotopy type with respect to the Hausdorff distance is almost tautological. When beginning with a stratification diagram of samples, the situation is similarly easy for the stratified scenario (see

Lemma 4.3.2.4). The situation for persistent stratified homotopy types when starting from a stratified sample is a little more subtle. This stems from the fact that taking sublevel sets with respect to some function (or equivalently intersecting with a closed subset) is generally not even a continuous operation with respect to the Hausdorff distance.

Example 4.3.3.1. Let $N = 1$, i.e. $\mathbb{R}^N = \mathbb{R}$ and $A = (-\infty, 0]$. Let $\mathbb{X} = \{0\}$ and $\mathbb{X}_n = \{\frac{1}{n}\}$. $\mathbb{X}_n$ converges to $\mathbb{X}$ in the Hausdorff distance. However, the intersections $A \cap \mathbb{X}_n$ are empty, while $A \cap \mathbb{X} = \mathbb{X}$. In particular, we have

$$\mathrm{d}_{\mathrm{HD}(W)}(A \cap \mathbb{X}_n, A \cap \mathbb{X}) = \infty,$$

for all $n > 0$.

The problem in Example 4.3.3.1 is that $\mathbb{X}$ is lacking a certain amount of homogeneity while passing into the interior of A. Indeed, if we replaced $\mathbb{X}$ by $[-\varepsilon, 0]$ then no such phenomena occur and one can easily show that $A \cap -$ is continuous in $[-\varepsilon, 0]$. As a more involved incarnation of this phenomenon, we now show stability of persistent stratified homotopy types, when sampling around a compact cylindrically stratified metric space.
A peculiarity of the stratified setting is that stability of persistent stratified homotopy types does generally not hold globally, as it does in the non-stratified case, but only at sufficiently regular elements of $\mathbf{Sam}_P$. To capture this notion of stability, we need the following notion of local Lipschitz continuity.

Definition 4.3.3.2. Let $K \in [0, \infty)$. We say that a function of (symmetric Lawvere) metric spaces $f : M \to M'$ is *K-Lipschitz (continuous) at $x \in M$*, if there is a $\delta > 0$, such that

$$\mathrm{d}(f(x), f(y)) \leq K\mathrm{d}(x, y),$$

for all $y \in M$ with $\mathrm{d}(x, y) \leq \delta$. We say f *is K-Lipschitz (continuous)*, if it is K-Lipschitz for $\delta = \infty$, at every $x \in M$.

Let us begin by gathering some of the more obvious elementary results. For the remainder of this subsection let $P = \{p < q\}$ be a poset with two elements and let $v = (v_l, v_h) \in \Omega$ and $u \in [0, 1]$.

Proposition 4.3.3.3. *The map*

$$\mathcal{N}\colon \mathbf{Sam}_P \to \mathbf{Sam}_{\mathrm{N}(P)}$$

is 2-Lipschitz.

Proof. This is immediate from the triangle inequality. □

As an immediate consequence of the definition of the metric for strongly stratified samples, one obtains.

Lemma 4.3.3.4. *Let $\mathbb{S}, \mathbb{S}' \in \mathbf{Sam}_{\mathrm{N}(P)}$ and $v' \leq v \in [0, 1]$. Then*

$$(\mathbb{S}')_v^{v'} \subset (\mathbb{S}_{v+\delta}^{v'-\delta})_\delta,$$

for any $\delta > \mathrm{d}_{\mathbf{Sam}_{\mathrm{N}(P)}}(\mathbb{S}, \mathbb{S}')$.

As a corollary of Lemma 4.3.3.4 and Proposition 4.3.3.3 together with the definition of the flow distance on $\mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+ \times \Omega}$ (Example 4.3.2.5), we obtain:

Corollary 4.3.3.5. *The map*

$$\mathcal{P}\colon \mathbf{Sam}_P \to \mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+ \times \Omega}$$

is 2-Lipschitz.

The case of the persistent stratified homotopy type depending on v is more subtle, since one lacks the possibility of diagonal interleavings. The following technical lemma is the decisive argument in showing stability nevertheless.

Lemma 4.3.3.6. *Let $\mathbb{S}, \mathbb{S}' \in \mathbf{Sam}_{\mathrm{N}(P)}$. Let $\delta > \mathrm{d}_{\mathbf{Sam}_{\mathrm{N}(P)}}(\mathbb{S}, \mathbb{S}')$ and suppose $v + \delta := (v_l - \delta, v_h + \delta) \in \Omega$ and $v - \delta := (v_l + \delta, v_h - \delta) \in \Omega$. Then*

$$\mathrm{d}_{\mathbf{D}_P\mathbf{Sam}}(\mathcal{D}_v(\mathbb{S}), \mathcal{D}_v(\mathbb{S}')) \leq \delta + \max\{\mathrm{d}_{\mathbf{D}_P\mathbf{Sam}}(\mathcal{D}_v(\mathbb{S}), \mathcal{D}_{v\pm\delta}(\mathbb{S}))\}.$$

Similarly, if $u \pm \delta \in (0,1)$, then

$$\mathrm{d}_{\mathbf{Sam}_P}(\mathcal{F}_u(\mathbb{S}), \mathcal{F}_u(\mathbb{S}')) \leq \delta + \max\{\mathrm{d}_{\mathbf{Sam}_P}(\mathcal{F}_u(\mathbb{S}), \mathcal{F}_{u\pm\delta}(\mathbb{S}))\}.$$

Proof. We prove the diagram case, the other one can be shown completely analogously. Let $\alpha > \mathrm{d}_{\mathbf{D}_P\mathbf{Sam}}(\mathcal{D}_v(\mathbb{S}), \mathcal{D}_{v\pm\delta}(\mathbb{S}))$. We then have inclusions

$$\mathcal{D}_v(\mathbb{S}) \hookrightarrow \mathcal{D}_{v-\delta}(\mathbb{S})_\alpha \hookrightarrow \mathcal{D}_v(\mathbb{S}')_{\delta+\alpha},$$

$$\mathcal{D}_v(\mathbb{S}') \hookrightarrow \mathcal{D}_{v+\delta}(\mathbb{S})_\delta \hookrightarrow \mathcal{D}_v(\mathbb{S})_{\delta+\alpha}.$$

The upper left and lower right inclusion follow by the assumption on α. The lower left and upper right inclusions follow by Lemma 4.3.3.4. Hence, the result follows by considering the diagram distance as coming from a thickening flow as in Example 4.3.2.3. ☐

Morally speaking, the way we should think of Lemma 4.3.3.6, is that the continuity of $\mathcal{D}_v$ in a strongly stratified sample $\mathbb{S}$ depends on the continuity of $\mathcal{D}_v(\mathbb{S})$ in the parameter v. As an immediate corollary of the second part of Lemma 4.3.3.6 we obtain the following result, which will come in handy in Section 4.4.5.

Corollary 4.3.3.7. *Let $\delta > 0$ such that $u \pm \delta \in (0,1)$. Let $\mathbb{S} = (\mathbb{X}, s) \in \mathbf{Sam}_{\mathrm{N}(P)}$ be such that $\mathbb{S}_{\leq u} = \mathbb{S}_{\leq u\pm\delta}$. Then*

$$\mathcal{F}_u \colon \mathbf{Sam}_{\mathrm{N}(P)} \to \mathbf{Sam}_P$$

is 1-Lipschitz at $\mathbb{S}$ (on an open ball with radius δ).

The continuity of $\mathcal{D}_v(\mathbb{S})$ in v can furthermore be reduced to the continuity of the $\{p < q\}$ parts of diagrams, by the following lemma.

Lemma 4.3.3.8. *Let $\mathbb{S} \in \mathbf{Sam}_{\mathrm{N}(P)}$ and $v, v' \in \Omega$ and set $a = \min\{v_l, v_l'\}, b = \max\{v_h, v_h'\}$, then*

$$\mathrm{d}_{\mathbf{D}_P\mathbf{Sam}}(\mathcal{D}_v(\mathbb{S}), \mathcal{D}_{v'}(\mathbb{S})) \leq \max\{\mathrm{d}_{\mathrm{HD}(W)}(\mathbb{S}^{v_l}_{v_h}, \mathbb{S}^{v_l'}_{v_h'}), \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{S}^a_{v_h'}, \mathbb{S}^a_{v_h}), \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{S}^{v_l}_b, \mathbb{S}^{v_l'}_b)\}.$$

Proof. This is an immediate consequence of the fact that

$$\mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X}, \mathbb{Y}) \leq \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X} \setminus \mathbb{A}, \mathbb{Y} \setminus \mathbb{A}),$$

for $\mathbb{A} \subset \mathbb{X}, \mathbb{Y} \in \mathbf{Sam}$. ☐

In case of compact cylindrically stratified spaces, $\mathcal{D}_v(\mathbb{S})$ does indeed vary continuously in v.

Proposition 4.3.3.9. *Let $S \in \mathbf{Sam}_{\mathrm{N}(P)}$ be compact and cylindrically stratified. Then*

$$\begin{aligned}(0,1) &\to \mathbf{Sam}_P\\ u &\mapsto \mathcal{F}_u(S)\end{aligned}$$

and

$$\begin{aligned}\Omega &\to \mathbf{D}_P\mathbf{Sam}\\ v &\mapsto \mathcal{D}_v(S)\end{aligned}$$

are continuous.

Proof. Note that it suffices to show the case of $\mathcal{D}_v$, since the nontrivial part of the continuity for $\mathcal{F}_u$ is given by the $(\mathcal{F}_u)_p$ component, and the latter is defined identically to the p-component of $\mathcal{D}_v(S)$. By Lemma 4.3.3.8 it suffices to show that for $v \to v^0$, we also have

$$\mathrm{d}_{\mathrm{HD}(W)}(S^{v_l}_{v_h}, S^{v^0_l}_{v^0_h}) \to 0.$$

Next, note that the topology of the Hausdorff distance on the space of compact subspaces of a space only depends on the topology of the latter. Set $L := S^{\frac{1}{2}}_{\frac{1}{2}}$. Then by the cylinder assumption we may without loss of generality compute d in $L \times (0,1)$ equipped with the product metric. In particular, $S^{v_l}_{v_h} = L \times [v_l, v_h]$. We then have

$$\begin{aligned}\mathrm{d}_{\mathrm{HD}(W)}(S^{v_l}_{v_h}, S^{v^0_l}_{v^0_h}) &= \mathrm{d}_{\mathrm{HD}(W)}(L \times [v_l, v_h], L \times [v^0_l, v^0_h]) \\ &\leq \max\{|v_l - v^0_l|, |v_h - v^0_h|\} \xrightarrow{v \to v^0} 0.\end{aligned}$$

□

From Proposition 4.3.3.9 and Lemma 4.3.3.6 we obtain the following result. Here Ω is equipped with the metric induced by the maximum norm.

Corollary 4.3.3.10. *Let $S \in \mathbf{Sam}_{\mathrm{N}(P)}$ be compact and cylindrically stratified. Then*

$$\mathcal{F}_u \colon \mathbf{Sam}_{\mathrm{N}(P)} \to \mathbf{Sam}_P,$$
$$\mathcal{D}_v \colon \mathbf{Sam}_{\mathrm{N}(P)} \to \mathbf{D}_P\mathbf{Sam}$$

are continuous at S.
Even more, if $S_{\le -} \colon (0,1) \to \mathbf{Sam}$ is K-Lipschitz in a neighborhood of u (respectively S^-_- in a neighborhood of v), then $\mathcal{F}_u$ ($\mathcal{D}_v$) is $(K+1)$-Lipschitz at S.

In total, we finally obtain the following stability result for persistent stratified homotopy types, which can be seen as a (slightly weaker) version of the classical, non-stratified Property P(1). In the next subsection (specifically in Theorem 4.3.4.8), we strengthen this general stability result significantly for the case of Whitney stratified spaces.

Theorem 4.3.3.11. *Let $S \in \mathbf{Sam}_P$ be compact and cylindrically stratified. Then*

$$\mathcal{P}_v \colon \mathbf{Sam}_P \to \mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+}$$

is continuous at S. Even more, if $S^-_- \colon \Omega \to \mathbf{Sam}$ is K-Lipschitz in a neighborhood of v, then $\mathcal{P}_v$ is $2(K+1)$-Lipschitz at S.

Proof. Recall that $\mathcal{P}_v = \mathcal{DP} \circ \mathcal{D}_v \circ \mathcal{N}$. $\mathcal{N}$ is 2-Lipschitz by Proposition 4.3.3.3. Furthermore, by Corollary 4.3.3.10, $\mathcal{D}_v$ is continuous in $\mathcal{N}(S)$. Finally, $\mathcal{DP}$ is 1-Lipschitz by Lemma 4.3.2.4. The second statement follows similarly. □

4.3.4 Stability at Whitney stratified spaces

One way to think of Whitney's condition (b) is that it gives additional control over the derivatives of the rays of the mapping cylinder neighborhood of a stratified space. This additional control can be used to improve the stability result in Theorem 4.3.3.11 to Lipschitz continuity (Proposition 4.3.4.7). To show this, we need to first consider an asymmetric version of the Hausdorff distance for subspaces of $\mathbb{R}^N$. For the remainder of this subsection, P is not restricted to the case of two elements.

Definition 4.3.4.1. Let $V, U \subset \mathbb{R}^N$ be linear subspaces. The (asymmetric) *distance of V to U* is given by

$$\vec{d}(V,U) = \sup_{v \in V, \|v\|=1} \inf\{\|v - u\| \mid u \in U\} = \sup_{v \in V, \|v\|=1} \{\|\pi_{U^\perp}(v)\|\},$$

where $\pi_{U^\perp}$ denotes the orthonormal projection to the orthogonal complement of U.

Whitney's condition (b) can be expressed in terms of a function, which measures the failure of secants being contained in the tangent space, as follows (compare [Hir69]).

Construction 4.3.4.2. Let $S = (T, s: T \to P)$ be a stratified space with smooth strata, contained in $\mathbb{R}^N$. Consider the function

$$\beta: T \times T \to \mathbb{R}; \qquad \begin{cases} (x,y) & \mapsto \vec{d}(l(x,y), \mathrm{T}_x(X_{s(x)})) \text{ if } x \neq y, \\ (x,x) & \mapsto 0 \text{ else} \end{cases}$$

where we consider all tangent spaces involved as linear subspaces of $\mathbb{R}^N$.

Condition (b) can be formulated as β restricting to a continuous function on certain subspaces of $X \times X$.

Proposition 4.3.4.3. *Let $S = (T, s: T \to P)$, be as in the assumption of Construction 4.3.4.2 and further so that the frontier and local finiteness condition are fulfilled. Then, S is a Whitney stratified space if and only if*

$$\beta\mid_{(T_q \times T_p) \cup \Delta_{T_p}} : (T_q \times T_p) \cup \Delta_{T_p} \to \mathbb{R},$$

is continuous, for all pairs $q \geq p \in P$. Here Δ_{T_p} denotes the diagonal of $T_p \times T_p$,

Proof. This statement is somewhat folklore. For the sake of completeness, we provide a proof in Section 4.B.2. ☐

Next, we need the notion of integral curves, as defined for example in [Hir69].

Proposition 4.3.4.4. *[Hir69, Lemma 4.1.1] Let $W = (T, s: T \to P)$ be a Whitney stratified space and $y \in W_p$, for some $p \in P$. Let $B = \mathrm{B}_d(y) \subset \mathbb{R}^N$ be a ball of radius d around y, such that $\beta(-, y)$ is bounded uniformly by some $\delta < 1$, on $W_{\geq p} \cap B$. Then, for any $x \in W_{\geq p} \cap B$, $x \neq y$, there exists a unique curve $\phi: [0, d] \to W \cap B$, fulfilling*

1. *$\phi(0) = y$ and $\phi(||y - x||) = x$,*
2. *ϕ is almost everywhere differentiable. At differentiable points, $t \neq 0$, the differential is given by*
$$\phi'(t) = \frac{||\phi(t) - y||}{||\pi_{\phi(t)}(\phi(t) - y)||^2} \pi_{\phi(t)}(\phi(t) - y),$$
where $\pi_{\phi(t)}$ denotes the projection to $\mathrm{T}_{\phi(t)}(W_{s(\phi(t))})$.

Definition 4.3.4.5. A curve as in Proposition 4.3.4.4 is called the *integral curve associated to the pair x, y.*

The existence of integral curves allows for additional control over the mapping cylinder neighborhoods defined in Example 4.2.3.16. This is essentially due to the following result.

Proposition 4.3.4.6. *[Hir69, Proof of 4.1.1] Let W be a Whitney stratified space over P and let $\phi: [0, d] \to W$ be the integral curve associated to $x \in W_q$, $y \in W_p$, $q \geq p \in P$, with notation as in Proposition 4.3.4.4. Then ϕ has the following properties.*

1. *$||\phi(t) - y|| = t$, for $t \in [0, d]$.*
2. *$||\phi(t) - \phi(t')|| \leq \frac{1}{\sqrt{1-\delta^2}} |t - t'|$, for $t, t' \in [0, d]$.*

As a consequence of this result, the continuity result of Theorem 4.3.3.11 can be improved to Lipschitz continuity.

Proposition 4.3.4.7. *Let $P = \{p < q\}$ and let $W \in \mathbf{Sam}_P$ be a Whitney stratified space with compact singular stratum W_p. Then, for any $C > 1$, there exists an $R > 0$, such that the function*

$$\begin{aligned} \Omega \cap (0,R)^2 &\to \mathbf{D}_P\mathbf{Sam} \\ v &\mapsto \mathcal{D}_v(\mathcal{N}(W)) \end{aligned}$$

is C-Lipschitz continuous.

Proof. We omit the $\mathcal{N}$, to keep notation concise. By Lemma 4.3.3.8, it again suffices to consider the link part of the diagrams given by $W^{v_l}_{v_h}$. Choose $\delta < 1$ such that $\frac{1}{\sqrt{1-\delta^2}} < C$. Next, take R small enough such that $\mathrm{N}_R(W_p)$, with retraction $r{:}\,\mathrm{N}_R(W_p) \to W_p$ is a standard tubular neighborhood of W_p. By [NV23, Lemma 2.1], for R small enough the spaces $W^y = r^{-1}(y) \cap W$ of y are given by Whitney stratified spaces with singular stratum given by a point. Then, using Construction 4.B.3.1, we may also choose R so small, that

$$\beta(x,y) \leq \delta,$$

for the respective β on the fiber W^y. Now, let $v, v' \in \Omega \cap [0,R]$. Let $x \in W^{v_l}_{v_h}$ and assume that $v_h > v'_h$ (the other cases work similarly). Now, consider the integral curve ϕ from $y := r(x) \in W_p$ to x in $r^{-1}(y) \cap W$. By Proposition 4.3.4.6 we have,

$$|x - \phi(v'_h)| = |\phi(|x|) - \phi(v'_h)| \leq C||x| - v'_h| \leq C|v_h - v'_h| \leq C|v - v'|.$$

Since $\phi(v'_h) \in W^{v'_l}_{v'_h}$, going through all the cases, we obtain

$$W^{v_l}_{v_h} \subset (W^{v'_l}_{v'_h})_{C|v-v'|}.$$

Thus, the result follows by symmetry. □

We thus obtain, as a corollary of Theorem 4.3.3.11, that for v sufficiently small the persistent stratified homotopy type $\mathcal{P}_v$ is even Lipschitz continuous at a Whitney stratified space.

Theorem 4.3.4.8. *Let $P = \{p < q\}$ and $W \in \mathbf{Sam}_P$ be Whitney stratified with W_p compact. Then, for any $C > 1$, there exists some $R > 0$, such that the map*

$$\mathcal{P}_v : \mathbf{Sam}_P \to \mathrm{ho}\mathbf{Strat}^{\mathbb{R}_+}$$

is $2(C+1)$-Lipschitz continuous at W, for all $v \in \Omega \cap (0,R)^2$.

4.4 Learning stratifications

In practice, we can generally not expect that sample data is already equipped with a stratification. This requires for notions of stratification which are intrinsic to the geometry of a space. One such example are homology stratifications, as used by Goresky and MacPherson in [GM83].

Example 4.4.0.1. For the sake of simplicity, we describe the case of two strata. Suppose $S = (s{:}\,T \to \{p < q\})$ is stratified conically as follows:
S_q is locally Euclidean of dimension q, and S_p of dimension p, and $x \in S_p$ admits a neighborhood

$$U \cong_P \mathbb{R}^p \times C(L)$$

for some $q - (p+1)$ dimensional compact manifold L, called the link of x. Here $C(L)$ is the stratified cone on L, stratified over $\{p < q\}$, by sending only the cone point to p. This holds, for example, if S is a Whitney stratified space. Suppose further that L is not a homology sphere, and that the strata are connected. Then, the stratification of S can be recovered from the underlying space as follows. For each $x \in T$, we can compute the local homology of X at x

$$\mathrm{H}_\bullet(T;x) \coloneqq \mathrm{H}_\bullet(T, T \setminus \{x\}) = \varinjlim \mathrm{H}_\bullet(T, T \setminus U),$$

where the colimit ranges over the open subsets of T containing x. By the assumption on the local geometry of T, for any $x \in T$ there exists a small open neighborhood U_x such that the natural map

$$\mathrm{H}_\bullet(T, T \setminus U_x) \to \mathrm{H}_\bullet(T;x)$$

is an isomorphism. In particular, for each $x \in T$ one obtains natural maps

$$\mathrm{H}_\bullet(T;x) \cong \mathrm{H}_\bullet(T, T \setminus U_x) \to \mathrm{H}_\bullet(T;y)$$

for $y \in U_x$. If x, y are contained in the same stratum, then all of these maps are given by isomorphisms. By the path connectedness assumption any two points in the same strata are connected by such a sequence of isomorphisms. Conversely, since we assumed that L is not a homology sphere, we have

$$\mathrm{H}_\bullet(T;x) \cong \mathrm{H}_\bullet(U_x;x) \cong \mathrm{H}_\bullet(\mathbb{R}^p \times C(L);x) \cong \tilde{\mathrm{H}}_{\bullet-(p+1)}(L) \neq \mathrm{H}_\bullet(T;y),$$

whenever $x \in S_p$ and $y \in S_q$. Thus, we can reobtain the stratification of S, by assigning to points the same stratum, if and only if they are connected through such a sequence of isomorphisms. Stratifications with the property that all the induced maps of local homologies on a stratum are isomorphisms are called *homology stratifications*.

Local homology as a means to obtain stratifications of point clouds (or combinatorial objects) have recently been investigated in several works ([BWM12; SW14; FW16; Nan20; Sto+20; Mil21]). Both [BWM12] and [Nan20] make use of the structure maps $\mathrm{H}_\bullet(T;x) \to \mathrm{H}_\bullet(T;y)$ to determine the strata. Note, however, that in the case of two strata it suffices to study the isomorphism type at each point, and there is no need to study the maps themselves, as stated by the following lemma.

Lemma 4.4.0.2. *Let $S = (T, s{:}T \to \{p < q\})$ be a Whitney stratified space (more generally conically stratified space) with manifold strata of dimension q and p respectively. Then s is a homology stratification.*

Furthermore, if the local homology of T is different from $\mathrm{H}_\bullet(\mathbb{R}^q;0)$, at each $y \in S_p$, then s is the only homology stratification of T with two strata.

Conversely, one always obtains a homology stratification $\tilde{s} : T \to \{p < q\}$ defined by:

$$\tilde{s}(x) = q \iff \mathrm{H}_\bullet(T;x) \cong \mathrm{H}_\bullet(\mathbb{R}^q;0),$$

for $x \in T$.

Proof. See Section 4.B.6. □

Now, let us consider the scenario of working with a (potentially noisy) sample $\mathbb{X}$ instead of considering the whole space T. Even when working persistently, to obtain non-trivial information, one can not pass all the way to the limit, when computing local homology. Indeed, for any thickening $\mathbb{X}_\alpha$, $\alpha > 0$, $\mathrm{H}_\bullet(\mathbb{X}_\alpha;x) = \mathrm{H}_\bullet(\mathbb{R}^N;0)$. Instead, one considers persistent local homology of the sample, with respect to a parameter $\frac{1}{\zeta}$, specifying the radius of the ball representing U_x (see [BWM12], [SW14]). In other words, one computes persistent local homology using the spaces

$$(\mathbb{X}_\alpha, \mathbb{X}_\alpha \setminus \mathring{\mathrm{B}}_{\frac{1}{2\zeta}}(x)).$$

For computational reasons, it is beneficial to use the intrinsically local notion of this structure. By the excision theorem, one may equivalently work with:

$$((\mathbb{X} \cap \mathrm{B}_{\frac{1}{\zeta}}(x))_\alpha, (\mathbb{X} \cap \mathrm{B}_{\frac{1}{\zeta}}(x))_\alpha \setminus \mathring{\mathrm{B}}_{\frac{1}{2\zeta}}(x)).$$

If one does not want the resulting barcodes to become shorter as $\zeta \to \infty$ and instead wants a measure of singularity that is comparable for different scales, then this needs to be normalized, and one may instead compute persistent homology via the stretched pair

$$((\zeta\mathbb{X} \cap \mathrm{B}_1(\zeta x))_\alpha, (\zeta\mathbb{X} \cap \mathrm{B}_1(\zeta x))_\alpha \setminus \mathring{\mathrm{B}}_{\frac{1}{2}}(\zeta x)).$$

Let us take a bit more of a conceptual look on this procedure in the following remark.

Remark 4.4.0.3. The procedure we just described may abstractly be rephrased as follows. We want to obtain a stratification of $\mathbb{X}$ using local data. Hence, we only consider sets of the form

$$\zeta\mathbb{X} \cap \mathrm{B}_1(\zeta x).$$

By shifting into the origin, we may equivalently investigate the space

$$\mathcal{M}^\zeta_x(\mathbb{X}) \coloneqq \zeta(\mathbb{X} - x) \cap \mathrm{B}_1(0) \subset \mathbb{R}^N,$$

with $\mathbb{X} - x = \{y - x \mid y \in \mathbb{X}\}$. We can think of $\mathcal{M}^\zeta_x(\mathbb{X})$ as zooming into $\mathbb{X}$ at x by a magnification parameter ζ. We then want to determine how far from a q-dimensional euclidean unit disk $D^q \subset \mathbb{R}^q \hookrightarrow \mathbb{R}^N$ the space $\mathcal{M}^\zeta_x(\mathbb{X})$ is. In the particular case of persistent local homology, we apply the map

$$\mathcal{P}\mathrm{L}_\bullet : M \mapsto \{\alpha \mapsto \mathrm{H}_\bullet(M_\alpha, M_\alpha \setminus \mathring{\mathrm{B}}_{\frac{1}{2}}(0))\}$$

to obtain a persistence module indexed over $[0, \frac{1}{2})$ and thus a quantitative invariant. The interleaving distance to $\mathcal{P}\mathrm{L}_\bullet(D^q) \cong \mathcal{P}\mathrm{L}_\bullet(\mathbb{R}^q)$ then gives a quantitative measure of singularity.

4.4.1 Magnifications and Φ-stratifications

Let us now put our observations on persistent local homology made in the beginning of this section and especially in Remark 4.4.0.3 into a more abstract framework.

Definition 4.4.1.1. Denote by $\mathbf{Sam}_\star$ the (symmetric Lawvere) metric space

$$\mathbf{Sam}_\star \coloneqq \{\mathbb{X} \mid \mathbb{X} \subset \mathbb{R}^N\},$$

equipped with the following truncated version of the Hausdorff distance: We pull back the metric on $\mathbf{Sam}$ along

$$\begin{aligned} \mathbf{Sam}_\star &\to \mathbf{Sam} \\ \mathbb{B} &\mapsto \mathrm{B}_1(0) \cap \mathbb{B}. \end{aligned}$$

We call $\mathbf{Sam}_\star$ the *space of local samples* (of $\mathbb{R}^N$), and denote its metric by $\mathrm{d}_{\mathbf{Sam}_\star}(-,-)$.

Remark 4.4.1.2. Note that the way the metric on $\mathbf{Sam}_\star$ is defined, it automatically identifies a local sample with its intersection with a unit ball around the origin. Indeed, $\mathbf{Sam}_\star$ is by definition isometric to the space of subspaces of $\mathrm{B}_1(0) \subset \mathbb{R}^N$. One may as well have used the latter, however, that involves a series of inconvenient truncations, so the above perspective is notationally preferable. In particular, in this context it makes sense to write $V \in \mathbf{Sam}_\star$, for $V \subset \mathbb{R}^N$ a linear subspace.

Next, we define the magnified spaces which showed up in our analysis of local homology in Remark 4.4.0.3.

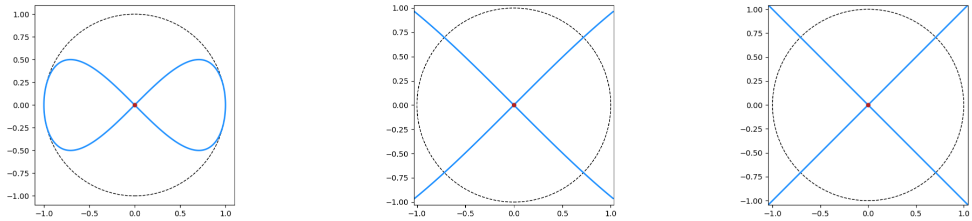


Figure 4.13: Three magnifications of X at the origin

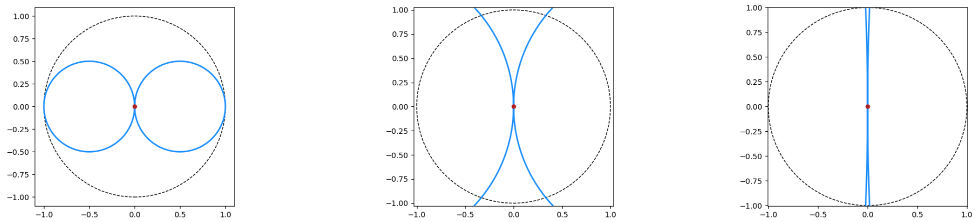


Figure 4.14: Three magnifications of Y at the origin.

Definition 4.4.1.3. Let $\mathbb{X} \in \mathbf{Sam}$, $x \in \mathbb{R}^N$ and $\zeta > 0$. We denote by

$$\mathcal{M}_x^\zeta(\mathbb{X}) \coloneqq \zeta(\mathbb{X} - x) \cap \mathrm{B}_1(0) \in \mathbf{Sam}_\star,$$

with $\mathbb{X} = \{y - x \mid y \in \mathbb{X}\}$, the *$\zeta$-magnification of $\mathbb{X}$ at x.*

Let us assume for a second that $\mathbb{X} = T$ and the latter admits a locally conelike stratification (as in Example 4.4.0.1), such that we need not worry about zooming in too far. Then, theoretically speaking, to make sure we identify every locally Euclidean region as such, we want the information obtained to be as local as possible, i.e. we want to consider the case $\zeta \to \infty$. Local homology, as described in Remark 4.4.0.3, defines a continuous map on $\mathbf{Sam}_\star$. Hence, to understand the behavior of local persistent homology for $\zeta \to \infty$ it suffices to understand the behavior of $\mathcal{M}_x^\zeta(T)$, for $\zeta \to \infty$. The following example illustrates when this limit can be used to determine local singularity.

Example 4.4.1.4. Consider the two real algebraic varieties

$$X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 - x_1^2 + x_2^2 = 0\}$$

and

$$Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid ((x_1 + 0.5)^2 + x_2^2 - 0.25)((x_1 - 0.5)^2 + (x_2)^2 - 0.25) = 0\}.$$

These varieties are Whitney stratified spaces with the singular set containing only the origin. In Fig. 4.13, we show magnifications of X at the origin $x = (0,0)$, i.e. $\mathcal{M}_x^\zeta(X)$ for three different $\zeta \in \{1, 3, 45\}$. We can observe that the homeomorphism type of the magnifications stabilizes as we increase ζ. In the limit the spaces $\mathcal{M}_x^\zeta(X)$ converge (in Hausdorff distance for $\zeta \to \infty$), to a space of the same homeomorphism type. In contrast, Y shows a different convergence behavior. Although the spaces $\mathcal{M}_x^\zeta(Y)$ share the same homeomorphism type with the magnifications of X at the origin, for ζ large enough, Fig. 4.14 illustrates that the homeomorphism type changes when passing to the limit (see also Fig. 4.15).

If T admits a (subanalytic) Whitney stratification, then limit spaces of magnifications (in $\mathbf{Sam}_\star$) exist and are known as the (extrinsic) tangent cones of T at x. For a more detailed investigation of metric tangent cones see [Lyt04; BL07]. For our purpose, the following definition will suffice.

Definition 4.4.1.5. Let $T \subset \mathbb{R}^N$. The *(extrinsic) tangent cone of T at $x \in T$* is defined as

$$\mathrm{T}^{\mathrm{ex}}_x(T) := \{v \in \mathbb{R}^N \mid \forall \varepsilon > 0 \exists y \in \mathrm{B}_\varepsilon(x) \cap T : v \in (\mathbb{R}_{\geq 0}(y - x))_\varepsilon\}.$$

The extrinsic tangent cones define a map

$$\begin{aligned} \mathrm{T}^{\mathrm{ex}}(T) : T &\to \mathbf{Sam}_\star \\ x &\mapsto \mathrm{T}^{\mathrm{ex}}_x(T). \end{aligned}$$

Example 4.4.1.6. By Taylor's expansion theorem one has

$$\mathrm{T}^{\mathrm{ex}}_x(T) = \mathrm{T}^{\mathrm{ex}}_x(U) = \mathrm{T}_x(U)$$

where $U \subset T \subset \mathbb{R}^N$ is a neighborhood of x in X and furthermore U is a smooth submanifold of $\mathbb{R}^N$.

Example 4.4.1.7. For an (affine) complex algebraic variety T the tangent cone at the origin coincides with the algebraic tangent cone, i.e. the set of common zeroes of all polynomials in the ideal generated by the homogeneous elements of lowest degree of all polynomials that vanish identically on T.

It is a classical result (see e.g. [Hir69], [BL07]) that when T admits a subanalytic Whitney stratification, then

$$\mathcal{M}^\zeta_x(T) \xrightarrow{\zeta \to \infty} \mathrm{T}^{\mathrm{ex}}_x(T)$$

in $\mathbf{Sam}_\star$. Since we are mostly interested in the case of what happens when one replaces T by samples and needs uniform versions of this result, we will recover this result as a special case of Section 4.4.2. However, it already points at what kind of information one may expect to obtain when one uses local features such as local persistent homology obtained from magnifications to stratify a data set. In the limit $\zeta \to \infty$ one can only expect to extract information that is contained in the extrinsic tangent cones. This leads to the following definition.

Definition 4.4.1.8. Let $P = \{p < q\}$. Let $W = (T, s) \in \mathbf{Sam}_P$ be a q-dimensional Whitney stratified space. We say that W is *tangentially stratified* if

$$d_{\mathbf{Sam}_\star}(\mathrm{T}^{\mathrm{ex}}_x(W), V) > 0,$$

for all $x \in W_p$ and for all $V \subset \mathbb{R}^N$ q-dimensional linear subspaces of $\mathbb{R}^N$.

Tangentially stratified spaces are precisely the type of Whitney stratified spaces for which we may expect to recover stratifications by using magnifications with large $\zeta > 0$. That this holds true rigorously is essentially the content of Section 4.4.5.

Example 4.4.1.9. Not every Whitney stratified space is tangentially stratified. Consider again $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid ((x_1+0.5)^2+x_2^2-0.25)((x_1-0.5)^2+(x_2)^2-0.25) = 0\}$ from Example 4.4.1.4. In this case, the above condition specifies to $d_{\mathbf{Sam}_\star}(\mathrm{T}^{\mathrm{ex}}_{(0,0)}(Y), V) > 0$, for all 1-dimensional linear subspaces $V \subset \mathbb{R}^N$. The tangent cone of Y at the origin is a 1-dimensional linear space given by

$$\mathrm{T}^{\mathrm{ex}}_{(0,0)}(Y) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\},$$

see Fig. 4.15 on the right, which already serves as a linear subspace $V \subset \mathbb{R}^2$ such that $d_{\mathbf{Sam}_\star}(\mathrm{T}^{\mathrm{ex}}_{(0,0)}(Y), V) = 0$. For the space

$$T = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 - x_1^2 + x_2^2 = 0\}$$

on the other hand we find that the tangent cone at the origin is given by

$$\mathrm{T}^{\mathrm{ex}}_{(0,0)}(T)\{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 + x_2)(x_2 - x_1) = 0\},$$

see Fig. 4.15 on the left. Clearly, there is no 1-dimensional linear subspace $V \subset \mathbb{R}^2$ such that $d_{\mathbf{Sam}_\star}(\mathrm{T}^{\mathrm{ex}}_{(0,0)}(T), V) = 0$.

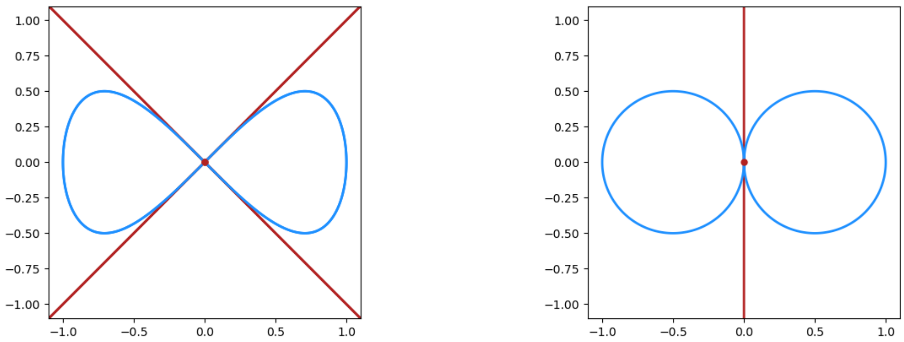


Figure 4.15: Two curves with their respective tangent cones (red) at their singular stratum

In practice, we may want to use other local invariants, such as local persistent homology in Remark 4.4.0.3, to identify singular points as in Lemma 4.4.0.2. This leads to the following definition. Again, for the remainder of this subsection, let $P = \{p < q\}$.

Definition 4.4.1.10. Let $\Phi : \mathbf{Sam}_\star \to [0,1]$ be a continuous function, such that $\Phi(V,0) = 1$, whenever V is a q-dimensional linear subspace of $\mathbb{R}^N$. Let $W \in \mathbf{Sam}_P$ be q-dimensional Whitney stratified space. We say that W is *(tangentially) Φ-stratified* if

$$\Phi(\mathrm{T}^{\mathrm{ex}}_x(W)) < 1,$$

for all $x \in W_p$.

Let us begin with some examples of functions $\Phi : \mathbf{Sam}_\star \to [0,1]$ which may be used to detect singularities.

Example 4.4.1.11. Consider the continuous map

$$\begin{aligned} \Phi_q \colon \mathbf{Sam}_\star &\to [0,1] \\ \mathbb{B} &\mapsto 1 - \inf\{\mathrm{d}_{\mathbf{Sam}_\star}(\mathbb{B}, V)\}, \end{aligned}$$

where V ranges over the q-dimensional linear subspaces of $\mathbb{R}^N$. A q-dimensional Whitney stratified space $W \in \mathbf{Sam}_P$ is tangentially stratified if and only if it is Φ_q- stratified. Φ_q is thus universal in the sense that if W is Φ-stratified for some Φ as in Definition 4.4.1.10, then it is Φ_q-stratified.

Example 4.4.1.12. Persistent local homology can be used as a function Φ, as was done similarly in [BWM12; Sto+20; Nan20; Mil21]. Precisely, we use $\mathcal{P}\mathrm{L}_i : \mathbf{Sam}_\star \to \mathbf{Vec}_k^{[0,\frac{1}{2})}$ as defined in Remark 4.4.0.3. Consider a linear embedding $\mathbb{R}^q \subset \mathbb{R}^N$, allowing us to write $\mathbb{R}^q \in \mathbf{Sam}_\star$ and set

$$\begin{aligned} \Phi \colon \mathbf{Sam}_\star &\to [0,1] \\ \mathbb{B} &\mapsto 1 - 2\max_{i \le q} \mathrm{d}_{\mathrm{In}}(\mathcal{P}\mathrm{L}_i(\mathbb{B}), \mathcal{P}\mathrm{L}_i(\mathbb{R}^q)). \end{aligned}$$

Indeed, as no bar in $\mathcal{P}\mathrm{L}_i$ can be longer than $\frac{1}{2}$, this function is well defined. Let $W \in \mathbf{Sam}_P$ be a (definable) q-dimensional Whitney stratified such that for all $x \in W_p$ we have $\mathcal{P}\mathrm{L}_\bullet(\mathrm{T}^{\mathrm{ex}}_x(W)) \neq \mathcal{P}\mathrm{L}_\bullet(\mathbb{R}^q)$, i.e. the two persistence modules are not isomorphic. Then, W is a Φ-stratified space.

One of the advantages of allowing for different Φ than just the universal one is that in practice one may use a series of rougher invariants which may be easier to compute.

Example 4.4.1.13. Instead of using

$$\mathbb{B} \mapsto 1 - \inf\{\mathrm{d}(\mathbb{B}, V)\},$$

as in Example 4.4.1.11, one can only use half of the numbers used to compute Hausdorff distance, i.e. only consider

$$\mathbb{B} \mapsto \inf\{\varepsilon \mid \mathrm{B}_1(0) \cap \mathbb{B} \subset \mathrm{B}_1(0) \cap V_\varepsilon\}.$$

Note that this definition of Φ will identify points in the boundary of a smooth manifold as regular. While this decreases the class of Φ-stratified spaces, this Φ can generally be easier to compute when using optimization techniques to find an optimal V. Similarly, instead of computing $\mathcal{P}\mathrm{L}_\bullet(\mathbb{B})$ as in Example 4.4.1.12 one may use a Vietoris-Rips version of the latter as described in [SW14] or only consider certain dimensions.

4.4.2 Lojasiewicz-Whitney stratified spaces

The previous section illustrates that in order to reconstruct stratifications from sample data we have to obtain a better understanding of the convergence properties of the magnification spaces to tangent cones. Such results are the content of Section 4.4.4. Before we investigate these, we need a series of results on Whitney stratified spaces which are definable with respect to particularly well behaved o-minimal structures. Our methods heavily rely on the work of [Hir69] and [BL07]. However, note that the results there are local, while ours are more global in nature, and that we consider the case of magnifications of samples as well. We use the following result due to Hironaka, which gives us additional control over integral curves.

Lemma 4.4.2.1. *Let $W \to P$ be a Whitney stratified space, $p \in P$ and $y \in W_p$. Suppose there exists $d_0 > 0$ such that there exists $\alpha > 0$, with*

$$\beta(x, y) \leq \|y - x\|^\alpha$$

for all $x \in W_{\geq p} \cap \mathrm{B}_{d_0}(y)$. Then, for any $C > 0$, there exists $d > 0$ only depending on d_0, α (and the dimension of W), such that for any integral curve $\phi : [0, d] \to W$ starting in y and ending in $\mathrm{B}_d(y)$ the inequality

$$\left\|\frac{1}{t}(\phi(t) - \phi(0)) - \frac{1}{s}(\phi(s) - \phi(0))\right\| \leq C|t - s|^\alpha$$

holds for all $t, s \in [0, d]$. In particular, all integral curves starting at y are differentiable at 0.

Proof. A complete proof of this statement can be found in [Hir69]. □

Spaces fulfilling a local version of the above condition were investigated in [Hir69]. It was called the strict Whitney condition there.

Definition 4.4.2.2. A Whitney stratified space fulfilling the requirements of Lemma 4.4.2.1 on any compactum K contained in some pure stratum W_p of W, is called a *Lojasiewicz-Whitney stratified space*. That is, W is called Lojasiewicz-Whitney stratified, if the following condition holds. Let $K \subset W_p$ be a compact, definable subset of some stratum W_p of W. Then there exist $\alpha > 0, d_0 > 0$ such that

$$\beta(x, y) \leq \|y - x\|^\alpha,$$

for all $y \in K$ and $x \in W \cap \mathrm{B}_{d_0}(y)$.

In other words, Lojasiewicz-Whitney stratified spaces are Whitney stratified spaces for which the speed at which secant lines diverge from the tangent spaces is bounded by some root. It turns out that most of the definably stratified spaces one is interested in i.e. compact subanalytic or semialgebraic are Lojasiewicz-Whitney stratified (Proposition 4.4.2.4).

Recollection 4.4.2.3. Recall that an o-minimal structure is called polynomially bounded, if for all $f\colon\mathbb{R}\to\mathbb{R}$ definable with respect to the structure, there exists an $n\in\mathbb{N}$ such that

$$|f(t)|\leq|t|^n,$$

for t sufficiently large. Polynomially bounded structures include the structure of semialgebraic sets and finitely subanalytic sets (see [Dri86] and [Mil94]). In particular, any compact subanalytically definable stratified space is definable with respect to a polynomially bounded o-minimal structure.

A proof of the following statement can be found in Section 4.B.5.

Proposition 4.4.2.4. *Let W be a Whitney stratified space which is definable with respect to some polynomially bounded o-minimal structure. Then, W is Lojasiewicz-Whitney stratified.*

Remark 4.4.2.5. In this section and in the Sections 4.4.3 and 4.4.4, there is no need to restrict to the two strata case. The results hold for general P.

As an almost immediate consequence of Lemma 4.4.2.1 and Proposition 4.4.2.4, we obtain:

Proposition 4.4.2.6. *Let W be a Lojasiewicz-Whitney stratified space. Then, for any $x\in W$, every integral curve starting at x is differentiable in 0. Furthermore, we have*

$$\mathrm{T}^{\mathrm{ex}}_x(W)\cap\partial\mathrm{B}_1(x)=\overline{\{\phi'(0)\mid\phi\text{ is an integral curve starting at }x\}}.$$

Hence,

$$\mathrm{T}^{\mathrm{ex}}_x(W)=\overline{\{\alpha\phi'(0)\mid\phi\text{ is an integral curve starting at }x,\alpha\geq0\}}.$$

Proof. First, note that $\mathrm{T}^{\mathrm{ex}}_x(W)$ is closed by definition. The containment of the right hand side in the left hand side is immediate by definition of the derivative (compare to Proposition 4.3.4.4). For the converse inclusion, let $v\in\mathrm{T}^{\mathrm{ex}}_x(W)\cap\partial\mathrm{B}_1(x)$. For $\varepsilon>0$ small enough, we have $y\in W_{\geq p}\cap\mathrm{B}_\varepsilon(x)$, with $p=s(x)$, such that

$$||v-\lambda(y-x)||<\varepsilon,$$

for some $\lambda\geq0$. In particular, we also obtain

$$\big|1-\lambda||y-x||\big|<\varepsilon.$$

Now, $t=||y-x||$ and let $\phi:[0,d]\to W$ be the integral curve starting at x and passing through y. We then have

$$\begin{aligned}||v-\phi'(0)||&\leq||v-\lambda(y-x)||+||\lambda(y-x)-\frac{y-x}{||y-x||}||+||\frac{y-x}{||y-x||}-\phi'(0)||\\&=||v-\lambda(y-x)||+\big|1-\lambda||y-x||\big|+||\frac{\phi(t)-x}{t}-\phi'(0)||\\&\leq\varepsilon+\varepsilon+C\varepsilon^\alpha,\end{aligned}$$

for some $C,\alpha>0$ independent of the choices above. In particular, we can choose ϕ such that $\phi'(0)$ is arbitrarily close to v. ☐

We can now obtain the following key technical result, to investigate the convergence behavior of magnifications of samples.

Proposition 4.4.2.7. *Let W be a Lojasiewicz-Whitney stratified space over P, with underlying space $X\subset\mathbb{R}^N$. Let $p\in P$ and $K\subset W_p$ be a compact subset. Then, there exist $d,C,\alpha>0$ such that the following holds.*
For all ζ such that $\frac{1}{\zeta}\in[0,d]$ there exists $\varepsilon_0>0$, such that

$$\mathrm{d}_{\mathbf{Sam}_\star}(\mathrm{T}^{\mathrm{ex}}_x(W),\mathcal{M}^\zeta_w(\mathbb{X}))\leq C(\zeta^{-\alpha}+\zeta\varepsilon),$$

for $\mathbb{X}\in\mathbf{Sam}$ with $\mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},X)=\varepsilon\leq\varepsilon_0$, $w\in\mathbb{R}^N$ and $x\in K$ with $|x-w|\leq\varepsilon$.

Proof. Denote $r := \frac{1}{\zeta}$. We work with the non-normalized spaces instead, that is instead of working in the unit ball of $\mathcal{M}^{\zeta}_x(\mathbb{X})$, we work in the ball of radius r in $\mathbb{X}$. Furthermore, without loss of generality let $x = 0$. Again, choose d, C', α as in Lemma 4.4.2.1, possibly slightly decreasing d, such that the requirements on r still hold for $r + 2\varepsilon$. Let $c \in \mathrm{T}^{\mathrm{ex}}_x(W)$ with $|c| \leq r$. Let $\tilde{c} := \frac{r-2\varepsilon}{r}c$. We have

$$|c - \tilde{c}| \leq 2\varepsilon.$$

Next, using Proposition 4.4.2.6, consider the integral curve starting in 0 with initial direction $\frac{c}{|c|}$, $\phi : [0, d] \to W$ (or, by passing to the limit if necessary a curve with initial direction arbitrarily close to $\frac{c}{\|c\|}$). We then have

$$\|\tilde{c} - \phi(\|\tilde{c}\|)\| \leq C' r^{\alpha+1}$$

and

$$|\phi(|\tilde{c}|) - w| \leq |\tilde{c}| + \varepsilon \leq r - \varepsilon. \tag{4.14}$$

Choose $w' \in \mathbb{X}$ with $|w' - \phi(\|\tilde{c}\|)| \leq \varepsilon$. Then, by Eq. (4.14) $w' \in \mathrm{B}_r(w) \cap \mathbb{X}$. Summarizing, we have

$$|c + w - w'| \leq 2\varepsilon + \varepsilon + C' r^{\alpha+1} + \varepsilon \leq C(r^{\alpha+1} + \varepsilon),$$

for appropriate $C > 0$.

Conversely, let $w' \in \mathbb{X}$ with $|w - w'| \leq r$. By assumption, we find $y \in W$ with $|y - w'| \leq \varepsilon$ and have $|y| \leq r + 2\varepsilon$. Thus, for ϕ_y the integral curve starting in 0 through y we have

$$||y|\phi_y'(0) - y| \leq C'(r + 2\varepsilon)^{\alpha+1}.$$

Take $c = (r - \varepsilon)\frac{|y|}{r+2\varepsilon}\phi_y'(0) \in \mathrm{T}^{\mathrm{ex}}_x(W) \cap \mathrm{B}_{r-\varepsilon}(x)$. Note, that $|c + w| \leq |w| + |c| \leq r$ i.e. $c + w \in \mathrm{B}_r(w) \cap (\mathrm{T}^{\mathrm{ex}}_x(W) + w)$. We further have

$$|c - |y|\phi_y'(0)| \leq |y|(1 - \frac{r - \varepsilon}{r + 2\varepsilon}) \leq 3\varepsilon.$$

Summarizing, we have

$$\begin{aligned} |c + w - w'| &\leq \varepsilon + |c - w'| \leq \varepsilon + |c - |y|\phi_y'(0)| + ||y|\phi_y'(0) - y| + |y - w'| \\ &\leq \varepsilon + 4\varepsilon + C'(r + 2\varepsilon)^{\alpha+1} \\ &\leq C(r^{\alpha+1} + \varepsilon) \end{aligned}$$

for appropriate $C > 0$ and $\varepsilon < r/2$. We obtain the result by multiplying with ζ to pass to the magnification. ☐

As a first corollary of Proposition 4.4.2.7, we obtain that the tangent cones of a Lojasiewicz-Whitney stratified space vary continuously on each stratum.

Proposition 4.4.2.8. *Let W be a Lojasiewicz-Whitney stratified space over P and $p \in P$. Then, the map*

$$\begin{aligned} \mathrm{T}^{\mathrm{ex}}_{-}(W) : W_p &\to \mathbf{Sam}_{\star} \\ x &\mapsto \mathrm{T}^{\mathrm{ex}}_x(W) \end{aligned}$$

is continuous.

Proof. To see this, note that by Proposition 4.4.2.7, restricted to any compactum, $\mathrm{T}^{\mathrm{ex}}_{-}(X)$ is the uniform limit of the family of maps given by $f_\zeta : x \mapsto \mathcal{M}^{\zeta}_x(W)$. By exhausting W_p by compacta it suffices to see that the f_ζ are continuous for ζ large enough. Again, set $r = \frac{1}{\zeta}$. So, let $K \subset W_p$ be a compactum and let r be small enough, such that $\mathrm{N}_r(K) \cap W \subset W_{\geq p}$. In other words, we may assume without loss of generality that W_p is the minimal stratum of W. Next, note that

$$\mathrm{d}_{\mathbf{Sam}_\star}(\mathcal{M}_x^\zeta(W), \mathcal{M}_{x'}^\zeta(W)) \leq r\mathrm{d}_{\mathrm{HD}(W)}(\mathrm{B}_r(x) \cap W, \mathrm{B}_r(x') \cap W) + \|x - x'\|, \tag{4.15}$$

for $x, x' \in W_p$. By an application of Thom's isotopy lemma, the map

$$\begin{aligned} \hat{g} : W \times W_p &\to [0, \infty) \times W_p \\ (x, y) &\mapsto (\|x - y\|, y) \end{aligned}$$

restricts to a fiber bundle over $(0, r] \times W_p$ for r small enough. In particular, it follows that if we set $X = \{(x, y) \in W \times W_p \mid \|x - y\| \leq r\}$, we obtain an induced fiber bundle

$$\begin{aligned} g : X &\to W_p \\ (x, y) &\mapsto y \end{aligned}$$

with fiber $\mathrm{B}_r(y) \cap W$ over y. Again, locally using the independence of the Hausdorff-distance topology of the choice of metric, we obtain that $\mathrm{B}_r(y) \cap W$ varies continuously in y. Hence, by Eq. (4.15) so does $\mathcal{M}_y^\zeta(W)$. □

4.4.3 Pointwise convergence of magnifications of a sample

As an immediate consequence of Proposition 4.4.2.7, we obtain that for a Lojasiewicz-Whitney stratified space W we have

$$\mathcal{M}_x^\zeta(W) \xrightarrow{\zeta \to \infty} \mathrm{T}_x^{\mathrm{ex}}(W),$$

for all $x \in W$. This result can already be found in similar form in [Hir69]. What we want to do, however, is to describe the case occurring in application. That is, we aim to analyse the convergence behavior of magnifications for samples of T, as $\zeta \to \infty$. At first glance, this is a nonsensical question. For a fixed sample $\mathbb{X}$, $\mathcal{M}_x^\zeta(\mathbb{X})$ has distance 0 to a one point (or empty) space, when ζ is large enough. Instead, the correct notion of convergence is already suggested by the inequality in Proposition 4.4.2.7. What needs to be described is a convergence behavior where the quality of the sample is allowed to improve at the same time as $\zeta \to \infty$.

Notation 4.4.3.1. Given a function $f\colon M \times (0, \infty) \to T$, where M is a metric space and T a topological space, we write

$$f(\mathbb{X}, \zeta) \xrightarrow[\zeta \mathrm{d}(\mathbb{X}, T) \to 0]{\zeta \to \infty} Y$$

for $T \in M$ and $Y \in T$ to state that for any pair of sequences $\zeta_n \in (0, \infty)$ converging to ∞, and $\mathbb{X}_n \in M$, such that $\zeta_n \mathrm{d}(\mathbb{X}_n, T)$ converges to 0, the sequence $f(\mathbb{X}_n, \zeta_n)$ converges to Y.

Remark 4.4.3.2. We may think of the type of convergence in Notation 4.4.3.1, as convergence of $f(\mathbb{X}, \zeta)$ to Y, for $\mathbb{X} \to X$ and $\zeta \to \infty$, under the additional condition that the convergence in the $\mathbb{X}$ variable is faster than the convergence in the ζ variable. This corresponds to the idea that if we want to zoom in further by a magnitude of k, and investigate some point locally, the quality of the sample also needs to improve by more than this magnitude k, so that we do not zoom in too far and end up only considering a single point. We can think of this as a notion of convergence in ζ, while improving the quality of the sample. Hence, we will also speak of *convergence while sampling.*

Now, we can interpret Proposition 4.4.2.7 with $\mathbb{X} = X$, $x = w$ and $K = \{x\}$ as the following convergence while sampling result.

Corollary 4.4.3.3. *Let $T \in$ **Sam** be a Lojasiewicz-Whitney stratifiable space. Let $x \in T$. Then,*

$$\mathcal{M}_x^\zeta(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X}, T) \to 0]{\zeta \to \infty} \mathrm{T}_x^{\mathrm{ex}}(T).$$

Furthermore, this convergence is uniform on any compactum K contained in a stratum.

4.4.4 Convergence of tangent bundles

To prove a global recovery of stratifications result, we need to obtain a more global version of Corollary 4.4.3.3. For this we need to treat tangent cones not as separate spaces but as a (stratified) bundle of cones. To describe the resulting convergence result, we need a space of samples of bundles.

Definition 4.4.4.1. Denote by **BSam** the set

$$\{(\mathbb{X}, F : \mathbb{X} \to \mathbf{Sam}_\star) \mid \mathbb{X} \in \mathbf{Sam}\},$$

equipped with the (extended pseudo) metric given by regarding F as a subset of $\mathbb{R}^N \times \mathbf{Sam}_\star$, equipping the latter with the product metric, and then using the resulting Hausdorff distance. That is, for $(\mathbb{X}, F), (\mathbb{X}', F') \in \mathbf{BSam}$, we define

$$\begin{aligned} \mathrm{d}_{\mathbf{BSam}}((\mathbb{X}, F), (\mathbb{X}', F')) := \max_{(\mathbb{X}_0, \mathbb{X}_1) \in \{\mathbb{X}, \mathbb{X}'\}^2} \inf\{\varepsilon > 0 \mid &\forall x \in \mathbb{X}_0 \exists y \in \mathbb{X}_1 : \|x - y\|, \\ &\mathrm{d}_{\mathbb{B}}(F_0(x), F_1(y)) \leq \varepsilon\}. \end{aligned}$$

We also refer to **BSam** as the *space of bundle samples* (of $\mathbb{R}^N$).

Definition 4.4.4.2. The *ζ-magnification bundle of* $\mathbb{X} \in \mathbf{Sam}$ is defined as the image of $\mathbb{X}$ under the map

$$\begin{aligned} \mathcal{M}^\zeta : \mathbf{Sam} &\to \mathbf{BSam} \\ \mathbb{X} &\mapsto (\mathbb{X}, \{x \mapsto \mathcal{M}^\zeta_x(\mathbb{X})\}). \end{aligned}$$

The *tangent cone bundle of* $T \in \mathbf{Sam}$ is defined as the image of T under the map

$$\begin{aligned} \mathrm{T}^{\mathrm{ex}} : \mathbf{Sam} &\to \mathbf{BSam} \\ T &\mapsto (T, \{x \mapsto \mathrm{T}^{\mathrm{ex}}_x(T)\}). \end{aligned}$$

Remark 4.4.4.3. Note that the nomenclature warrants some care, as for an arbitrary space $\mathbb{X}$, neither $\mathcal{M}^\zeta$ nor $\mathrm{T}^{\mathrm{ex}}(\mathbb{X})$ are anything close to a fiber bundle and even for a Lojasiewicz-Whitney stratified space they are stratified fiber bundles at best.

Note that Proposition 4.4.2.7 does not imply convergence of magnification bundles in the metric on **BSam**, as the convergence is only uniform on compacta contained in pure strata. However, we may equip the spaces **BSam** with alternative topologies, allowing us to formulate notions of convergence on a compactum. Again, for the remainder of this subsection let $P = \{p < q\}$.

Construction 4.4.4.4. Let $K \in \mathbf{Sam}$ and let T be any of the spaces $\mathbf{Sam}_{\mathrm{N}(P)}$, **BSam**. Let $\varepsilon : \mathbf{Sam} \to \mathbb{R}_+$ be some continuous map. Define a map

$$\begin{aligned} g^K_\varepsilon &: T \to T \\ (\mathbb{X}, f) &\mapsto (\mathbb{X} \cap K_{\varepsilon(\mathbb{X})}, f\,|_{K_{\varepsilon(\mathbb{X})}}). \end{aligned}$$

If $\mathcal{K} = (E, \varepsilon)$ is a pair consisting of a set $E \subset \mathbf{Sam}$, together with a continuous map $\varepsilon : \mathbf{Sam} \to \mathbb{R}_+$, we denote by $T^{\mathcal{K}}$, the space with the same underlying set as T, but equipped with the initial topology with respect to the maps g^K_ε and $T \to \mathbf{Sam} \xrightarrow{\varepsilon} \mathbb{R}_+$. In particular, with respect to this topology, a sequence $\mathbb{B}_n = (\mathbb{X}_n, F_n)$ in T converges to $\mathbb{B} = (\mathbb{X}, F) \in T$, if and only if

$$g^K_\varepsilon(\mathbb{B}_n) \xrightarrow{n \to \infty} g^K_\varepsilon(\mathbb{B}),$$

for all $K \in E$ and

$$\varepsilon(\mathbb{X}_n) \xrightarrow{n \to \infty} \varepsilon(\mathbb{X}).$$

Remark 4.4.4.5. In the case where E is a countable set, the topology on $T^{\mathcal{K}}$ is still first countable. All cases we consider here can be reduce to this scenario. Alternatively, all of the proofs using sequences below work identically when using nets instead of sequences.

We can now rephrase Proposition 4.4.2.7 as a global convergence result, which is essential for the stratification learning theorems of Section 4.4.5.

Proposition 4.4.4.6. *Let $T \in$ **Sam** be equipped with a Lojasiewicz-Whitney stratification $W = (T, T \to P)$. Denote $\varepsilon := \mathrm{d}_{\mathrm{HD}(W)}(T, -)$. Let $E \subset$ **Sam** be such that for all $K \in E$ there exist a decomposition into compacta $K = K_p \sqcup K_q$ such that $K_p \subset W_p$, $K_q \subset W_q$. Denote $\mathcal{K} = (E, \varepsilon)$. Then,*

$$\mathcal{M}^{\zeta}(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T)\to 0]{\zeta\to\infty} \mathrm{T}^{\mathrm{ex}}(T) \text{ in } \mathbf{BSam}^{\mathcal{K}}.$$

Proof. Let $K \in E$. We need to show

$$g_{\varepsilon}^{K}(\mathcal{M}^{\zeta}(\mathbb{X})) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T)\to 0]{\zeta\to\infty} g_{\varepsilon}^{K}(\mathrm{T}^{\mathrm{ex}}(T)).$$

Note that since $K \subset W$, $g_{\varepsilon}^{K}(\mathrm{T}^{\mathrm{ex}}(T)) = \mathrm{T}^{\mathrm{ex}}(T)|_K$. The result is now an immediate consequence of Proposition 4.4.2.7. ☐

4.4.5 The stratification learning theorem

We now have all the tools in place to recover stratifications from samples. We have seen in Theorem 4.3.4.8 that the persistent stratified homotopy type is (Lipschitz) continuous in compact Whitney stratified spaces W over $P = \{p < q\}$. In particular, we can approximate the persistent stratified homotopy type of W from a stratified sample $\mathbb{W}$ close to W in the metric on $\mathbf{Sam}_P$. In practice, we can generally only expect to be given non-stratified samples. Even naively, if one had a means to decide when a point has ended up precisely in the singular stratum, one should expect the latter to be a 0-set with respect to the used density, and hence usually end up with non-stratified sets. Nevertheless, our investigations of magnifications and Φ-stratifications already suggest that local tangent cones may be used to recover stratifications which approximate the original one. Let us first illustrate how the procedure works in case one is given a perfect sample, i.e. one can work with the whole of W. Again, for the remainder of this section let $P = \{p < q\}$.

Construction 4.4.5.1. Let $W \in \mathbf{Sam}_P$ be a compact Lojasiewicz-Whitney Φ-stratified space, with respect to a function Φ as in Definition 4.4.1.10. Suppose we forget the stratification of $W = (T, s)$, and only have the data given by T. We can then associate to T its tangent cone bundle $\mathrm{T}^{\mathrm{ex}}T \in \mathbf{BSam}$. Next, we use the function Φ to decide which regions should be considered singular. We can do so by applying Φ to $\mathrm{T}^{\mathrm{ex}}T$ fiberwise. As a result we obtain a strong stratification $\tilde{s}$ of T, given by

$$x \mapsto \mathrm{T}_x^{\mathrm{ex}}(T) \mapsto \Phi(\mathrm{T}_x^{\mathrm{ex}}(T)).$$

By Proposition 4.4.2.8, this map is continuous on all strata. In particular, by assumption, it takes a maximum value $m < 1$ on W_p. Since W_q is a manifold, we have

$$\mathrm{T}_x^{\mathrm{ex}}(T) = \mathrm{T}_x^{\mathrm{ex}}(W_q) = \mathrm{T}_x(W_q) = \mathbb{R}^q$$

for $x \in W_q$, and thus the strong stratification has constant value 1 on W_q. Therefore, we may recover the stratification of s by choosing $u > m$ and applying $\mathcal{F}_u$:

$$\mathcal{F}_u(T, \tilde{s}) = W.$$

We now replicate the procedure described in Construction 4.4.5.1 in case of working with samples and investigate its convergence behavior.

Lemma 4.4.5.2. *Let $\Phi : \mathbf{Sam}_\star \to [0,1]$ be a continuous map. Then, the induced map*

$$\begin{aligned}\Phi_* : \mathbf{BSam} &\to \mathbf{Sam}_{\mathrm{N}(P)}\\ (\mathbb{X}, F) &\mapsto (\mathbb{X}, \Phi \circ F)\end{aligned}$$

is continuous. Even more, if Φ is C-Lipschitz, then so is Φ_*.

Proof. Since $\mathbf{Sam}_\star$ is isometric to the space of compact subspaces of $\mathrm{B}_1(0) \subset \mathbb{R}^N$ and thus compact, Φ is a uniformly continuous map. Hence, the result follows immediately by definition of the metrics on $\mathbf{BSam}$ and $\mathbf{Sam}_{\mathrm{N}(P)}$. □

It turns out Φ_* also descends to a continuous map on the alternative topologies of Construction 4.4.4.4.

Lemma 4.4.5.3. *Let $\Phi : \mathbf{Sam}_\star \to [0,1]$ be a continuous map. Let $E \subset \mathbf{Sam}$, $\varepsilon : \mathbf{Sam} \to \mathbb{R}_+$ be some continuous function and $\mathcal{K} = (E, \varepsilon)$. Then, the map*

$$\begin{aligned}\Phi_* : \mathbf{BSam}^{\mathcal{K}} &\to \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)}\\ (\mathbb{X}, F) &\mapsto (\mathbb{X}, \Phi \circ F)\end{aligned}$$

is continuous.

Proof. By definition of the topologies on $\mathbf{BSam}^{\mathcal{K}} \to \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)}$, it suffices to show the result for the case where $E = \{K\}$ is a singleton. Continuity of $\mathbf{BSam}^{\mathcal{K}} \to \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)} \to \mathbf{Sam} \xrightarrow{\varepsilon} \mathbb{R}_+$ holds trivially. Next, note that the diagram

$$\begin{tikzcd}
\mathbf{BSam}^{\mathcal{K}} \arrow[r, "\Phi_*"] \arrow[d, "g^K_\varepsilon"] & \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)} \arrow[d, "g^K_\varepsilon"] \\
\mathbf{BSam} \arrow[r, "\Phi_*"] & \mathbf{Sam}_{\mathrm{N}(P)}
\end{tikzcd}$$

trivially commutes, since the g are given by restricting, i.e. precomposition and Φ_* by postcomposition.

Then, for a sequence $\mathbb{B}_n \in \mathbf{BSam}^{\mathcal{K}}$ and $\mathbb{B} \in \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)}$ we have:

$$\begin{aligned}
\mathbb{B}_n \xrightarrow{n\to\infty} \mathbb{B} \text{ in } \mathbf{BSam}^{\mathcal{K}} &\iff g^K_\varepsilon(\mathbb{B}_n) \xrightarrow{n\to\infty} g^K_\varepsilon(\mathbb{B}) \text{ in } \mathbf{BSam}\\
&\implies \Phi_*(g^K_\varepsilon(\mathbb{B}_n)) \xrightarrow{n\to\infty} \Phi_*(g^K_\varepsilon(\mathbb{B})) \text{ in } \mathbf{Sam}_{\mathrm{N}(P)}\\
&\iff g^K_\varepsilon(\Phi_*(\mathbb{B}_n)) \xrightarrow{n\to\infty} g^K_\varepsilon(\Phi_*(\mathbb{B})) \text{ in } \mathbf{Sam}_{\mathrm{N}(P)}\\
&\iff \Phi_*(\mathbb{B}_n) \xrightarrow{n\to\infty} \Phi_*(\mathbb{B}) \text{ in } \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)},
\end{aligned}$$

where the implication in the second line follows by Lemma 4.4.5.2. □

We have already seen, that with respect to the alternative topologies the magnification bundles do indeed converge uniformly to the tangent cone bundle. This is however not the case with the usual topologies. Hence, to approximate stratifications using a magnification version of Construction 4.4.5.1, we need to show that $\mathcal{F}_u$ is continuous in the respective tangent cone bundles with respect to the alternative topology.

Proposition 4.4.5.4. *Let $S = (T, s) \in \mathbf{Sam}_{\mathrm{N}(P)}$, T compact. Let $u \in [0,1]$ be such that $S_{\leq u}$ is closed and such that $S_{\leq u \pm \delta} = S_{\leq u}$ for δ sufficiently small. Let $\varepsilon = \mathrm{d}_{\mathrm{HD}(W)}(T, -)$. Finally, let*

$$\mathcal{K} = (\{K \in \mathbf{Sam} \mid K = K_p \sqcup K_q, K_p, K_q \text{ compact}, K_p \subset S_{\leq u}, K_q \subset s^{-1}(u, 1]\}, \varepsilon).$$

Then

$$\mathcal{F}_u \colon \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)} \to \mathbf{Sam}_P$$

is continuous at S.

Proof. Let $\mathbb{S} = (\mathbb{X}, s') \in \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)}$. Note that convergence in $\mathbf{Sam}_P$ may be verified componentwise. Since convergence in $\mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)}$ also implies $\varepsilon(\mathbb{X}) = \mathrm{d}_{\mathrm{HD}(W)}(T, \mathbb{X}) \to 0$, we only need to verify convergence in the component $\mathbb{S}_{\leq u}$. We have

$$\mathrm{d}_{\mathrm{HD}(W)}(S_{\leq u}, \mathbb{S}_{\leq u}) \leq \mathrm{d}_{\mathrm{HD}(W)}(S_{\leq u}, \mathbb{S}_{\leq u} \cap K_{\varepsilon(\mathbb{X})}) + \mathrm{d}_{\mathrm{HD}(W)}(S_{\leq u}, \mathbb{S}_{\leq u} \cap (S_{\leq u})_{\gamma}),$$

whenever $K = \overline{(T - (S_{\leq u})_{\frac{\gamma}{2}})} \sqcup S_{\leq u}$ and $\gamma > 0$ such that, $\mathbb{X} \subset K_{\varepsilon(\mathbb{X})} \cup (S_{\leq u})_{\gamma}$. Note, that for this to hold, it suffices that $\varepsilon(\mathbb{X}) \leq \frac{\gamma}{2}$. For the left summand we obtain,

$$\mathrm{d}_{\mathrm{HD}(W)}(S_{\leq u}, \mathbb{S}_{\leq u} \cap K_{\varepsilon(\mathbb{X})}) = \mathrm{d}_{\mathrm{HD}(W)}(\mathcal{F}_u(g^K_{\varepsilon}(S)), \mathcal{F}_u(g^K_{\varepsilon}(\mathbb{S}))) \leq \mathrm{d}_{\mathbf{Sam}_{\mathrm{N}(P)}}(g^K_{\varepsilon}(S), g^K_{\varepsilon}(\mathbb{S})),$$

by Corollary 4.3.3.7, for $\varepsilon(\mathbb{X})$ sufficiently small and $g^K_{\varepsilon}(\mathbb{S})$ close to $g^K_{\varepsilon}(S)$. For the other summand we first split the Hausdorff distance into the directed distances

$$\mathrm{d}_{\mathrm{HD}(W)}(S_{\leq u}, \mathbb{S}_{\leq u} \cap (S_{\leq u})_{\gamma}) \leq d_L(S_{\leq u}, \mathbb{S}_{\leq u} \cap (S_{\leq u})_{\gamma}) + d_L(\mathbb{S}_{\leq u} \cap (S_{\leq u})_{\gamma}, S_{\leq u})$$

where $d_L(A, B) = \inf\{\delta \geq 0 \mid A \subset B_{\delta}\}$. Then, the second summand is bounded by γ and for the first summand we observe that

$$d_L(S_{\leq u}, \mathbb{S}_{\leq u} \cap (S_{\leq u})_{\gamma}) \leq d_L(S_{\leq u}, \mathbb{S}_{\leq u} \cap (S_{\leq u})_{\varepsilon(\mathbb{X})}).$$

This is due to the fact that $\varepsilon(\mathbb{X}) < \gamma$ and $\mathbb{S}_{\leq u} \cap (S_{\leq u})_{\varepsilon(\mathbb{X})} \subset \mathbb{S}_{\leq u} \cap (S_{\leq u})_{\gamma}$. If we set $K' = S_{\leq u}$ and invoke Corollary 4.3.3.7 again we obtain

$$\begin{aligned} d_L(S_{\leq u}, \mathbb{S}_{\leq u} \cap (S_{\leq u})_{\varepsilon(\mathbb{X})}) &= d_L(\mathcal{F}_u(g^{K'}_{\varepsilon}(S)), \mathcal{F}_u(g^{K'}_{\varepsilon}(\mathbb{S}))) \\ &\leq \mathrm{d}_{\mathbf{Sam}_{\mathrm{N}(P)}}(g^{K'}_{\varepsilon}(S), g^{K'}_{\varepsilon}(\mathbb{S})), \end{aligned}$$

for $g^{K'}_{\varepsilon}(\mathbb{S})$ close to $g^{K'}_{\varepsilon}(S)$. Summarizing, we have:

$$\mathrm{d}_{\mathrm{HD}(W)}(S_{\leq u}, \mathbb{S}_{\leq u}) \leq \mathrm{d}_{\mathbf{Sam}_{\mathrm{N}(P)}}(g^K_{\varepsilon}(S), g^K_{\varepsilon}(\mathbb{S})) + \mathrm{d}_{\mathbf{Sam}_{\mathrm{N}(P)}}(g^{K'}_{\varepsilon}(S), g^{K'}_{\varepsilon}(\mathbb{S})) + \gamma.$$

In particular, we may first fix some γ while the other terms converge to 0 for $\mathbb{S} \to S$ in $\mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)}$ by assumption. Since γ can be taken arbitrarily small, the result follows. ☐

We are now finally in shape to define a map which equips samples with stratifications, depending on their approximate tangential structure.

Definition 4.4.5.5. Let $\Phi : \mathbf{Sam}_{\star} \to [0, 1]$ be a continuous map and $u \in [0, 1)$, $\zeta \in \mathbb{R}_{+}$. Let $\mathbb{X} \in \mathbf{Sam}_{\star}$. We call the image of $\mathbb{X}$ under the composition

$$\mathcal{S}^{\zeta}_{\Phi,u} : \mathbf{Sam} \xrightarrow{\mathcal{M}^{\zeta}} \mathbf{BSam} \xrightarrow{\Phi_*} \mathbf{Sam}_{\mathrm{N}(P)} \xrightarrow{\mathcal{F}_u} \mathbf{Sam}_P$$

the *ζ-th Φ-stratification of $\mathbb{X}$ (with respect to u).* In the case where $\zeta = \infty$, replace $\mathcal{M}^{\zeta}$ by T^{ex}.

Example 4.4.5.6. To illustrate the concepts in Definition 4.4.5.5 let us walk through every component of the composition defining $\mathcal{S}^{\zeta}_{\Phi,u}$ for a specific sample. Let T denote the algebraic variety given by

$$\{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2 + z^2 + 1.44)^2 - 7.84x^2 + 1.44y^2 = 0\}. \tag{4.16}$$

In the bottom left of Fig. 4.16, a visual representation of T can be found. A finite sample from this variety, denoted $\mathbb{X}$, was obtained by randomly picking points from an enclosing rectangular cuboid and only keeping points that satisfy (4.16) up to a small error. Choosing a magnification parameter $\zeta = 5$ we obtain the magnification bundle $\mathcal{M}^{\zeta}(\mathbb{X})$ for $\mathbb{X}$, depicted in the top middle of Fig. 4.16. Φ was chosen as in Example 4.4.1.12. Evaluating the fibers of $\mathcal{M}^{\zeta}(\mathbb{X})$ we obtain a strongly stratified sample $\Phi_*(\mathcal{M}^{\zeta}(\mathbb{X}))$, shown on the left of Fig. 4.16. Next, picking the threshold value $u \in [0, 1)$ to be 0.83 induces a stratified sample via $\mathcal{F}_u$. A visual comparison indicates that the resulting stratified sample is close to the Whitney stratified space given by T with two isolated singularities. This already points at the convergence behavior predicted by Theorem 4.4.5.8.

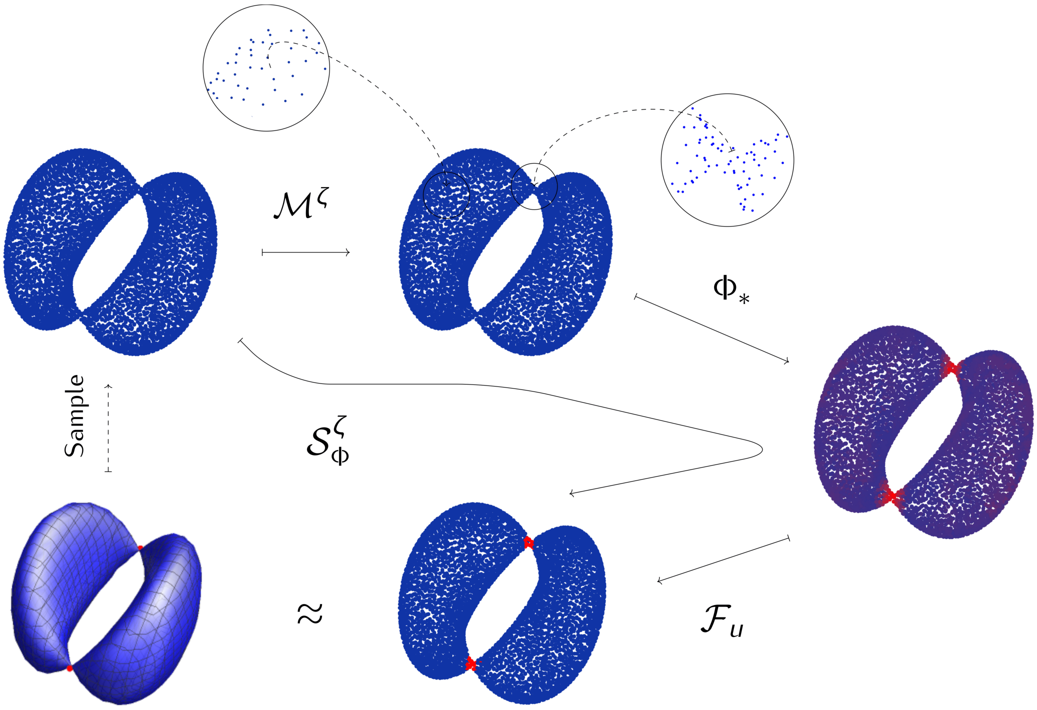


Figure 4.16: Illustration of $\mathcal{S}^\zeta_{\Phi,u}$ for a sample from a 2-dimensional real algebraic variety

Using Definition 4.4.5.5, we can restate the content of Construction 4.4.5.1 as follows.

Proposition 4.4.5.7. *Let $W \in \mathbf{Sam}_P$ be a Lojasiewicz-Whitney stratified space, Φ-stratified with respect to $\Phi : \mathbf{Sam}_\star \to [0,1]$ as in Definition 4.4.1.10. Then,*

$$\sup\{\Phi(\mathrm{T}^{\mathrm{ex}}_x(T)) \mid x \in W_p\} < 1.$$

In particular,

$$\mathcal{S}^\zeta_{\Phi,u}(T) = W,$$

for $\sup\{\Phi(\mathrm{T}^{\mathrm{ex}}_x(T)) \mid x \in W_p\} < u < 1$.

Proof. This was already covered in Construction 4.4.5.1. □

We can now finally state the main theorem about approximating the stratification of a Lojasiewicz-Whitney Φ-stratified space W. In practice, it guarantees that for ζ large enough and given a sufficiently good sample one can use the ζ-th Φ-stratification to approximate the stratified space W. In particular, this result can be applied to all compact, subanalytically Whitney stratified spaces.

Theorem 4.4.5.8. *Let $P = \{p < q\}$ and let $W = (T, T \to P) \in \mathbf{Sam}_P$ be a compact Lojasiewicz-Whitney stratified space, Φ-stratified with respect to $\Phi : \mathbf{Sam}_\star \to [0,1]$. Then there exists $u_0 \in (0,1)$ such that*

$$\mathcal{S}^\zeta_{\Phi,u}(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T) \to 0]{\zeta \to \infty} W,$$

for $u \in [u_0, 1)$.

Proof. Let $\mathcal{K}$ be as in Proposition 4.4.4.6. It is the content of the latter proposition that

$$\mathcal{M}^\zeta(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T)\to 0]{\zeta\to\infty} \mathrm{T}^{\mathrm{ex}}(T) \text{ in } \mathbf{BSam}^{\mathcal{K}}.$$

Applying Φ_* to this and using Lemma 4.4.5.3, we obtain

$$\Phi_* \circ \mathcal{M}^\zeta(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T)\to 0]{\zeta\to\infty} \Phi_* \circ \mathrm{T}^{\mathrm{ex}}(W) \text{ in } \mathbf{Sam}^{\mathcal{K}}_{\mathrm{N}(P)}.$$

Now, note that $\Phi_* \circ \mathrm{T}^{\mathrm{ex}}(T)$ fulfills the requirements of Proposition 4.4.5.4, if we take $1 > u > \max\{\Phi_* \circ \mathrm{T}^{\mathrm{ex}}(T)(x) \mid x \in T\}$. Hence,

$$\mathcal{S}^\zeta_{\Phi,u}(\mathbb{X}) = \mathcal{F}_u \circ \Phi_* \circ \mathcal{M}^\zeta(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T)\to 0]{\zeta\to\infty} \mathcal{F}_u \circ \Phi_* \circ \mathrm{T}^{\mathrm{ex}}(T) = W,$$

where the equality follows by Proposition 4.4.5.7. □

Finally, we can now combine this result with Corollary 4.3.3.5 and Theorem 4.3.3.11 which guarantees that $\mathcal{S}^\zeta_{\Phi,u}$ may be used to infer stratified homotopy types from non-stratified samples. Note that in the following we again assume W to be linearly rescaled in such a way that it is cylindrically stratified. Equivalently, this does not need to be assumed if Ω is reparametrized by the scaling factor.

Corollary 4.4.5.9. *Let $P = \{p < q\}$ and let $W = (T, T \to P) \in \mathbf{Sam}_P$ be a compact Lojasiewicz-Whitney stratified space, Φ-stratified with respect to $\Phi : \mathbf{Sam}_\star \to [0,1]$. Then there exists $u_0 \in (0,1)$ such that*

$$\mathcal{P} \circ \mathcal{S}^\zeta_{\Phi,u}(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T)\to 0]{\zeta\to\infty} \mathcal{P}(W),$$

for $u \in [u_0, 1)$. Furthermore,

$$\mathcal{P}_v \circ \mathcal{S}^\zeta_{\Phi,u}(\mathbb{X}) \xrightarrow[\zeta \mathrm{d}_{\mathrm{HD}(W)}(\mathbb{X},T)\to 0]{\zeta\to\infty} \mathcal{P}_v(W),$$

for $u \in [u_0, 1)$ and $v \in \Omega$.

4.5 Conclusion

The central advantage of the approach to stratified TDA we have described in this work is that it is highly modular. In summary, it can be decomposed into three steps.

1. From non-stratified data obtain stratified data (Section 4.4.5).

2. From stratified data obtain a persistent stratified homotopy type (Section 4.3.1).

3. From a stratified homotopy type compute algebraic invariants.

The goal of this work was to show the feasibility of the first two steps in the restricted case of two strata. Our results in Sections 4.3 and 4.4 show that the resulting notion of persistent stratified homotopy type fulfills many of the properties required in application (P(1), P(2) and P(3)), which are fulfilled by the classical persistent homotopy type, such as stability (Theorem 4.3.4.8), computability (Remark 4.3.1.17) and the availability of inference results (Proposition 4.3.1.20, Theorem 4.4.5.8, and Corollary 4.4.5.9). There are a series of promising avenues arising from this first step in persistent stratified homotopy theory.

1. So far, our constructions are mostly developed for the case of two strata. In the introduction, we have already described in some detail why we decided to restrict to this scenario. Nevertheless, for possible applications, the case of multiple strata seems of great interest. We are aware that there is currently ongoing research concerning how to recover stratifications in the case of arbitrary posets, which could greatly increase the possible realm of application. At the same time, such an approach would also require a generalization of the inference and stability results of Sections 4.3.1, 4.3.3 and 4.3.4 to persistent stratified homotopy types with more than two strata. Proofs of such are expected to be inductive in nature, which suggests an inductive approach to stratified homotopy theory on the theoretical side. This has yet to be established in detail.

2. While our results in this work are mostly theoretical, we are currently working on implementing the stratification learning method and persistent stratified homotopy types on a computer. One possible next step is then to apply these methods to inherently singular data sets such as retinal artery photos, and investigate the stability and expressiveness of our approach in practice. This also requires a more detailed study and evaluation of choices of functions Φ, for the construction of Φ-stratifications (see Definition 4.4.1.10) in an applied scenario.

3. The application of persistent stratified homotopy types to real-world data also requires a further investigation of the last step - i.e. passing to algebraic invariants such as persistent homology. While there are some expressive and well-understood algebraic invariants at hand - for example the persistent homology of the links and strata - there is a series of more intricate invariants to consider. These include a persistent version of intersection homology, as well as an interpretation of the persistent stratified homotopy type as a multi-parameter persistence module. Studying the properties of such invariants, ranging from computability to expressiveness, leaves much room for future research projects both in theory as well as in application.

4.A Some details on abstract homotopy theory

Remark 4.A.0.1. There are some subtleties to be considered, which come down to the order in which one passes to the persistent and homotopical perspective. We emphasize that by $\mathrm{ho}\mathbf{T}^I$, for some indexing category I, we mean the localization of the functor category at pointwise weak equivalences, and not the functor category $(\mathrm{ho}\mathbf{T})^I$, obtained by localizing at weak equivalences first. The universal property of the localization induces a canonical functor

$$\mathrm{ho}\mathbf{T}^I \to (\mathrm{ho}\mathbf{T})^I.$$

This functor is essentially never an equivalence of categories. For example, for $\mathbf{T} = \mathbf{Top}$ with the usual class of weak equivalences, the notion of isomorphism on the left-hand side is fine enough to compute homotopy limits and colimits. This is not the case on the right-hand side (see for example [Hir03], for an introduction to the theory). Generally, the functor will be neither essentially surjective nor fully faithful. Essential surjectivity, for example, comes down to whether or not a homotopy commutative diagram is equivalent to an actual commutative diagram (see [DK84] for a detailed discussion.)
To see that faithfulness is generally not the case, consider replacing $\mathbb{R}_+$ by $I = \{0 < 1\}$, and taking $\mathbf{T} = \mathbf{Top}$, $D = \{* \to S^1\}$ and $D' = \{* \to X\}$, for some pointed space X. Both objects may be considered as pointed spaces. Then, the hom-objects from D to D' in $\mathrm{ho}\mathbf{T}^I$ are the homotopy groups of X. In $(\mathrm{ho}\mathbf{T})^I$, however, the hom-object is given by free homotopy classes from S^1 to X, i.e. by the abelianization of the homotopy group of X.
In the special case where $I = \mathbb{R}_+$ and $\mathbf{T} = \mathbf{Top}$ this leaves, a priori, an ambivalence by what one means by a persistent homotopy type. Given a persistent space, i.e. an object in $\mathbf{Top}^{\mathbb{R}_+}$, one can either consider its isomorphism class in $\mathrm{ho}(\mathbf{Top}^{\mathbb{R}_+})$ or in $(\mathrm{ho}\mathbf{Top})^{\mathbb{R}_+}$. We argue that the former is the conceptually better notion since properties P(1) to Item P(3) may already

be stated on this level. At the same time, due to the comparison functor between the two categories, results obtained in $\mathrm{ho}(\mathbf{Top}^{\mathbb{R}_+})$ are generally stronger than results in $(\mathrm{ho}\mathbf{Top})^{\mathbb{R}_+}$. However, one should note that when passing to the algebraic world by applying homology index-wise, both perspectives agree. Finally, we may add that for most applications the difference is negligible. This is a consequence of Lemma 4.A.0.2 which, among other things, implies, as long as one restricts to persistent objects which are tame in the sense that their homotopy type only changes at finitely many points, then the functor

$$\mathrm{ho}\mathbf{T}^{\mathbb{R}_+} \to (\mathrm{ho}\mathbf{T})^{\mathbb{R}_+} \tag{4.17}$$

induces a bijection on isomorphism classes. In particular, there is no difference in the resulting notion of persistent homotopy type.

Lemma 4.A.0.2. *Let $\mathcal{M}$ be a (simplicial) model category and I be any small indexing category. Then,*

$$F : \mathrm{ho}\mathcal{M}^I \to (\mathrm{ho}\mathcal{M})^I$$

reflects isomorphisms. Furthermore, let I be a finite, totally ordered poset. Then F is essentially surjective and full. In particular, two objects in $\mathrm{ho}\mathcal{M}^I$ are isomorphic, if and only if their images under F are isomorphic.

Proof. Ultimately, this comes down to the fact that homotopy coherent diagrams of the particularly simple shapes involved are easy to understand. The proof requires a series of standard arguments in the theory of model categories. See [Hir03] for a comprehensive overview. To see that the functor characterizes isomorphisms, note that a morphism in a functor category is an isomorphism, if and only if is so pointwise. Since a morphism descends to an isomorphism in the homotopy category, if and only if it is a weak equivalence, and weak equivalences in $\mathcal{M}^I$ (equipped with any of the usual model structures) are defined pointwise, this shows that a morphism in $\mathcal{M}^I$ is an isomorphism in $\mathrm{ho}(\mathcal{M}^I)$ if and only if its image under F is an isomorphism.
The next statement holds in general. However, we show only the case of a simplicial model category, since all model categories relevant in this paper fulfill this property and the availability of a canonical cylinder object makes the proof somewhat more digestible. Essential surjectivity is immediate, as functors D defined on a totally ordered set $I \cong [n] = \{0, ..., n\}$ are entirely determined by their values valued on $D(i \leq i+1)$ and conversely any sequence of morphisms $X_i \to X_{i+1}$, uniquely determines a functor. Thus, (up to an isomorphism in the right-hand side category) being a functor with values in the homotopy category $\mathrm{ho}\mathcal{M}^I$ is equivalent to being a functor with values in $\mathcal{M}^I$.
Now, to see fullness, consider objects D and D' on the left-hand side. Without loss of generality, we may assume that D is a cofibrant and D' a fibrant object with respect to the injective model structure (which exists since I is a Reedy category in the obvious fashion. See [Hir03] for an introduction to Reedy model structures.) In particular, morphisms in the homotopy categories between D and D' and between D_i, D'_i are given by (simplicial) homotopy classes. We now proceed to show fullness by induction over n. The case $n = 0$ is trivial. Now let $f : D \to D'$ be a morphism in $(\mathrm{ho}\mathcal{M})^I$, i.e. we are given a homotopy commutative diagram

$$\begin{tikzcd} D_0 \arrow[r] \arrow[d, "f_0"] & \dots \arrow[r] & D_n \arrow[r, "i_n"] \arrow[d, "f_n"] & D_{n+1} \arrow[d, "f_{n+1}"] \\ D'_0 \arrow[r] & \dots \arrow[r] & D'_n \arrow[r, "i'_n"] & D'_{n+1} \end{tikzcd} \tag{4.18}$$

by inductive assumption, we can assume that up to D_n the diagram is actually commutative. It remains to show, that there exists a morphism g_{n+1}, simplicially homotopic to f_{n+1}, such that the right-hand square commutes on the nose. Let $H : D_n \to D'^{\Delta^1}_{n+1}$ be the adjoint to the simplicial homotopy from the right down to the down right composition. We may instead solve the induced lifting problem

$$\begin{tikzcd} D_n \arrow[r, "H"] \arrow[d, "i_n"] & {D'_{n+1}}^{\Delta^1} \arrow[d, "ev_1"] \\ D_{n+1} \arrow[r, "f_{n+1}"'] \arrow[ur, dashed] & D'_{n+1} \end{tikzcd} \tag{4.19}$$

such a lift exists, since, by assumption, the left-hand side map is a cofibration and the right-hand side map is a fibration. Thus, we have shown fullness. ☐

Lemma 4.A.0.3. *Let $\mathcal{M}$ be a relative category (i.e. a category equipped with a notion of weak equivalence). Let $U \subset \Omega \times \mathbb{R}_+$ be a subset containing $\Omega \times \{0\}$. Let $D \in \mathcal{M}^U$, be such that*

$$D(v,0) \to D(v,\alpha)$$

is a weak equivalence, for all $(v,\alpha) \in U$. Let $D\mid_{\Omega\times\{0\}}$ be homotopically constant of value $M \in \mathcal{M}$. Then D is also homotopically constant of value M.

Proof. This follows from the fact that Ω is initial in U. Let $i{:}\,\Omega \times \{0\} \hookrightarrow U$ be the inclusion. Note that, in this specific scenario

$$i_*(D')(v,\alpha) = D'_{(v,0)}$$

since any slice involved in the right Kan-extension have a terminal object of the form $(v,0)$. In particular, this means that i_* preserves weak equivalences between all objects. Furthermore, by assumption, this equality implies that the natural transformation

$$i_*D\mid_{(\Omega,0)} \to D$$

is a weak equivalence. Now, by assumption, $D\mid_{(\Omega,0)}$ is weakly equivalent to some constant functor C in $\mathcal{M}^{\Omega\times\{0\}}$. In particular, this implies that there is a zigzag of weak equivalences

$$C \longleftrightarrow D' \longleftrightarrow \ldots \longleftrightarrow D\mid_{\Omega\times 0} \tag{4.20}$$

applying i_* and using the fact that it preserves weak equivalences between all objects, we thus obtain a zigzag of weak equivalences

$$i_*C \longleftrightarrow i_*D' \longleftrightarrow \ldots \longleftrightarrow i_*D\mid_{(\Omega,0)} \longrightarrow D \tag{4.21}$$

which induces an isomorphism in $\mathrm{ho}\mathcal{M}^U$. Finally, note that if C is constant of value $M \in \mathcal{M}$, then so is i_*C. ☐

4.B Results on definable and Whitney stratified spaces

4.B.1 Definable sets can be thickened

The following lemma seems folklore knowledge to some degree. We provide it here for the sake of completeness. It seems to us that, with some extra technical effort, methods used in [CL05] may even be used to obtain strongly stratified mapping cylinder neighborhoods. However, the following result suffices for our purposes.

Lemma 4.B.1.1. *Let $T \subset Y \subset \mathbb{R}^N$ be definable with respect to some o-minimal structure and X compact. Then, there exists a $\varepsilon > 0$ such for $0 < \alpha < \varepsilon$ the following holds:*

1. *$T \hookrightarrow T_\alpha \cap Y$ is a strong deformation retract.*

2. *There is a homeomorphism* $(T_\alpha \cap Y) \setminus X \cong \mathrm{d}_T^{-1}(\frac{\alpha}{2}) \times (0, \alpha]$, *such that the diagram*

$$\begin{tikzcd} (T_\alpha \cap Y) \setminus X \arrow[rr, "\sim"] \arrow[dr, "\mathrm{d}_X"'] & & \mathrm{d}_T^{-1}(\frac{\alpha}{2}) \times (0,\alpha] \arrow[dl, "\pi_{(0,\alpha]}"] \\ & (0,\alpha] & \end{tikzcd}$$

commutes.

Furthermore, if $Y = \mathbb{R}^N$, *then* ε *may be taken to be the weak feature size of* T *as in [CL05, Definition 3.1].*

Proof. The statement on the homemomorphism type of the complements is an immediate application of Hardt's theorem for definable sets together with the fact that d_X is definable (see e.g. [Dri98]). One may then use the isotopies induced by flows used for example in [CL05] to extend this homeomorphism to the case where $Y = \mathbb{R}^N$ and ε is the weak feature size. To see that the latter is positive, note that the argument for positivity of weak feature sizes of semialgebraic sets in [Fu85, Remark 5.3] also applies to the definable case. Finally, we need to see that the inclusion is a strong deformation retraction. Note that by the triangulability of definable sets (see for example [Dri98, Theorem 2.9]), $\mathbb{R}^N$ may be equipped with a triangulation compatible with X and Y. In particular, by subdividing if necessary, X has arbitrarily small mapping cylinder neighborhoods in Y, given by piecewise linear regular neighborhoods. Furthermore, this means that $X \hookrightarrow X_\alpha \cap Y$ is a cofibration. Thus, it suffices to show that $X \hookrightarrow X_\alpha \cap Y$ is a homotopy equivalence. Now, for $\alpha < \alpha' < \varepsilon$, with ε such that 2 holds. Then, we have inclusions

$$X \hookrightarrow X_\alpha \cap Y \hookrightarrow N \hookrightarrow X_{\alpha'} \cap Y,$$

where N and N' are regular neighborhoods with respect to the piecewise linear structure induced by the triangulation. By the open cylinder structure (assumption 2) of the set $(X_{\alpha'} \cap Y) \setminus X$, the inclusion $X_\alpha \cap Y \hookrightarrow X_{\alpha'} \cap Y$ is a homotopy equivalence. The same holds for the inclusion $X \hookrightarrow N$. It follows by the two-out-of-six property of homotopy equivalences, that all maps are homotopy equivalences. ☐

4.B.2 Proof of Proposition 4.3.4.3

Proof of Proposition 4.3.4.3. The map β is clearly continuous on $S_q \times S_p$. The condition on β is thus equivalent to the extension by 0 to Δ_{S_p} being continuous. Indeed, by continuity of $\vec{d}(-,-)$, this extension condition immediately implies condition (b). For the converse, as $\beta \geq 0$, it suffices to show upper semi-continuity. This is the content of Proposition 4.B.2.1. ☐

Proposition 4.B.2.1. *Let* $W = (X, s : X \to P)$ *be a Whitney stratified space. Then, the restriction of* β *to* $W_{\geq p} \times W_p \to \mathbb{R}$ *is upper semi-continuous.*

Proof. β is clearly continuous on the strata of $W \times W$. Now, suppose $(x_n, y_n) \in W_{\geq p} \times W_p$ is a sequence converging to a point $(x, y) \in W_{p'} \times W_p$, for some $p' \geq p$. Then, for sufficiently large $n \in \mathbb{N}$, we have $s(x_n) \geq p'$. To show upper semi-continuity, we may thus without loss of generality assume that x_n lies in the same stratum W_q. We show that any subsequence of (x_n, y_n) has a further subsequence (all named the same by abuse of notation), for which $\beta(x_n, y_n)$ converges to a value lesser or equal then $\beta(x, y)$. By compactness of Grassmannians, we may first restrict to a subsequence such that $\mathrm{T}_{x_n}(W_q)$ and $l(x_n, y_n)$ converge to τ and l respectively. By Whitney's condition (a) ([Whi65a], [Whi65b]) - which by [Mat12] follows from condition (b) - we have $\mathrm{T}_x(W_{p'}) \subset \tau$. Summarizing, this gives:

$$\lim \beta(x_n, y_n) = \vec{d}(l, \tau) \leq \vec{d}(l, \mathrm{T}_x(W_{p'})).$$

Now, in case when $x \neq y$, the last expression equals $\beta(x,y)$ by definition. In the case when $x = y$ then, by condition (b), $l \subset \tau$. Thus, again, we have

$$\lim \beta(x_n, y_n) = \vec{d}(l, \tau) = 0 = \beta(y,y)$$

finishing the proof. □

4.B.3 A normal bundle version of β

Furthermore, we are going to make use of the following fiberwise version of β.

Construction 4.B.3.1. Again, in the framework of Construction 4.3.4.2, assume that $W = (T, s{:}T \to P)$ is a Whitney stratified space, with W_p compact. Take N to be a standard tubular neighborhood of W_p in $\mathbb{R}^N$ with retraction $r : N \to W_p$. Note that by Whitney's condition (a), for N sufficiently small, $r_{|W_q}$ is a submersion for $q \geq p$. In particular, by [NV23, Lemma 2.1] the fiber of

$$W^y := (r)^{-1}_{|N \cap W^{\geq p}}(y)$$

is a Whitney stratified space over $\{q \in P \mid q \geq p\}$ with the p-stratum given by $\{y\}$. Furthermore, we have

$$\mathrm{T}_x(W_q) \cap \nu_{r(x)}(W_p) = \mathrm{T}_x(W_q^{r(x)}),$$

where $\nu_{r(x)}(W_p)$ denotes the normal space of W_p at $r(x)$. In particular, the dimension of these spaces is constant, and they vary continuously in x. Then, consider the following function:

$$\tilde{\beta}_p(-) : N \cap W^{\geq p} \to \mathbb{R} \qquad \begin{cases} x & \mapsto \vec{d}(l(x, r(x)), \mathrm{T}_x(W^{r(x)}_{s(x)})), \text{ for } s(x) > p \\ x & \mapsto 0, \text{ for } s(x) = p. \end{cases}$$

Noting that $l(x, r(x)) \in \nu_{r(x)}(W_p)$, by an analogous argument to the proof of Proposition 4.3.4.3, one obtains that $\tilde{\beta}_p(-)$ is continuous on $W_q \cup W_p$. Note that if we restrict $\tilde{\beta}_p(-)$ to W^y, then we obtain the function $\beta(-, y)$ associated to W^y. Let us denote this β_y. In particular, by compactness of W_p, we obtain that the functions β_y can be globally bounded by any $\delta > 0$, for N sufficiently small.

4.B.4 Definability of β

Proposition 4.B.4.1. *Let $S = (T, s{:}T \to P)$ be as in Construction 4.3.4.2. Then, if $T \subset \mathbb{R}^N$ is definable, then so is β.*

Proof. As all the strata of $T \times T$ are again definable, it suffices to show that β is definable on the strata of $T \times T$. Furthermore, as β is 0 along Δ_T, it suffices to show definability away from the diagonal. Here β is equivalently given by

$$\beta(x,y) = \inf_{v \in \mathrm{T}_x(T_{s(x)})} \left|\left| \frac{x-y}{||x-y||} - v \right|\right|.$$

It follows from the fact that for $q \in P$, $\mathrm{T}(T_q) \subset \mathbb{R}^N \times \mathbb{R}^N$ is definable (see [Cos00] and Lemma 4.B.5.1) that this defines a definable function $T_q \times T_p \to \mathbb{R}$. □

4.B.5 Proof of Proposition 4.4.2.4

We begin by proving a series of technical lemmas.

Lemma 4.B.5.1. *Consider two definable maps $f : X \to \mathbb{R}$, $\pi : X \to Y$ such that f is bounded from above on every fiber of π. Then the map*

$$\begin{aligned} g : Y &\to \mathbb{R} \\ y &\mapsto \sup_{x \in \pi^{-1}(y)} f(x) \end{aligned}$$

is again definable.

Proof. This is immediate, if one interprets the graph of g in terms of a formula being expressible with respect to the o-minimal structure. □

Lemma 4.B.5.2. *Let $X \to \{p < q\}$ be a stratified metric space and Y a first countable, locally compact Hausdorff space. Let $\pi : X \to Y$ be a proper map, such that both the fibers of π, as well as the fibers of $\pi_{|X_p}$ vary continuously in the Hausdorff distance. Let $f : X \to \mathbb{R}$ be upper semi-continuous and continuous on the strata. Then,*

$$\begin{aligned} g : Y &\to \mathbb{R} \\ y &\mapsto \sup_{x \in \pi^{-1}(y)} f(x) \end{aligned}$$

is continuous.

Proof. Note first that as the fibers of π are compact and f is upper semi continuous, it takes its maximum on every fiber. Now, let $y_n \to y$ be a convergent sequence in Y. We show that any of its subsequences y'_n, has a further subsequence $\tilde{y}_n \to y$, with

$$\sup_{x \in \pi^{-1}(\tilde{y}_n)} f(x) \to \sup_{x \in \pi^{-1}(y)} f(x).$$

Let $x'_n \in \pi^{-1}(y_n)$ for all n such that $f(x'_n) = \sup_{x \in \pi^{-1}(y'_n)} f(x)$. As Y is locally compact and π is proper, x'_n has a convergent subsequence $\tilde{x}_n \to \tilde{x}$. Define $\tilde{y}_n := \pi(\tilde{x}_n)$. Since the fibers of π vary continuously and $\tilde{y}_n \to y$, we also have $\tilde{x} \in \pi^{-1}(y)$. Thus, we have

$$\limsup \sup_{x \in \pi^{-1}(\tilde{y}_n)} f(x) = \limsup f(\tilde{x}_n) \leq f(\tilde{x}) \leq g(y).$$

It remains to see the converse inequality for a subsequence of $\tilde{y}_n$. Let $\hat{x} \in \pi^{-1}(y)$ be such that $f(\hat{x}) = \sup_{x \in \pi^{-1}(y)} f(x)$. By assumption we can find a sequence x''_n with $x''_n \in \pi^{-1}(\tilde{y}_n)$ converging to $\hat{x}$. If $\hat{x} \in X_p$, then x''_n can be taken to be in X_p, as $\pi^{-1}(\tilde{y}_n) \cap X_p$ converges to $\pi^{-1}(y) \cap X_p$. If $\hat{x} \in X_q$, then, as the latter is open, x''_n ultimately lies in X_q. Hence, by continuity of f on the strata, we have

$$g(y) = f(\hat{x}) = \lim f(x''_n) = \liminf f(x''_n) \leq \liminf \sup_{x \in \pi^{-1}(\tilde{y}_n)} f(x).$$

□

As a consequence of the prior two lemmas we obtain:

Lemma 4.B.5.3. *If W is a definably Whitney stratified over $P = \{p < q\}$. Then the map*

$$\begin{aligned} \hat{\beta} : W_p \times \mathbb{R}_{\geq 0} &\to \mathbb{R} \\ (y, d) &\mapsto \sup_{||x - y|| = d, x \in W} \beta(x, y) \end{aligned}$$

is continuous in a neighborhood of $W_p \times \{0\}$, definable and vanishes on $W_p \times \{0\}$.

Proof. Definability follows immediately from Lemma 4.B.5.1. consider the map

$$\begin{aligned} B : W \times W_p &\to W_p \times \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto (y, ||x - y||). \end{aligned}$$

Over $W_p \times \mathbb{R}_{>0}$ it is given by submersion on each stratum of $W \times W_p$. In particular, by Thom's first isotopy lemma [Mat12, Proposition 11.1] it is a fiber bundle with fibers $\partial \mathrm{B}_d(y)$ at (y, d) over $\mathbb{R}_{>0}$. In particular, the fibers of B vary continuously over $W_p \times \mathbb{R}_{>0}$. Additionally, for $(y_n, d_n) \to (y, 0)$ the fiber converges to the point y. Hence, B fulfills the requirements of Lemma 4.B.5.2. Furthermore, $\beta : W \times W_p \to \mathbb{R}$ also fulfills the requirements of Lemma 4.B.5.2, showing the continuity of $\hat{\beta}$. Lastly, $\hat{\beta}$ vanishes on $W_p \times \mathbb{R}_{\leq 0}$ by definition of β. □

We now have all the tools available to obtain a proof of Proposition 4.4.2.4,

Proof of Proposition 4.4.2.4. We conduct this proof for the case of $P = \{p < q\}$ and $K = W_p$ (with notation as in Definition 4.4.2.2). The general case follows analogously by working strata-wise and then passing to maxima. By Lemma 4.B.5.3 for d small enough, the function $\hat{\beta} : W_p \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ fulfills the requirements of Lojasiewicz' theorem for (polynomially bounded) o-minimal structures [Loi16]. Hence, we find $\hat{\phi} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ to be a definable and monotonous bijection such that on $W_p \times [0, d]$ we have

$$\hat{\phi}(\hat{\beta}(y, t)) \leq t.$$

If the relevant o-minimal structure is polynomially bounded, then there exist $n > 0$, such that

$$t^n \leq \hat{\phi}(t)$$

for $t \in [0, d']$. Hence, we obtain

$$\begin{aligned} \hat{\beta}(y, t)^n &\leq \hat{\phi}(\hat{\beta}(y, t)) \leq t. \\ \implies \hat{\beta}(y, t) &\leq t^\alpha \end{aligned}$$

for $t \in [0, d]$, $\alpha = \frac{1}{n}$ and $d := \phi^{-1}(d')$. □

4.B.6 Proof of Lemma 4.4.0.2

Proof of Lemma 4.4.0.2. The first result is immediate from the local conical structure of T. The second is immediate from the definition of a homology stratification, as clearly $T - S_p$ is a homology manifold. For the final result, note first that by the local conical structure, having local homology isomorphic to $\mathrm{H}_\bullet(\mathbb{R}^q; 0)$ is an open condition on S_p. In particular, since this condition holds on all of $T - S_p$ it is an open condition on all of T. Thus, $s : T \to \{p < q\}$ as defined in the statement is actually a stratification of T. To see that this is indeed a homology stratification we need to see that the local isomorphism condition is fulfilled. By construction, we have $T - S_p \subset \tilde{s}^{-1}\{q\}$. Within $S_p \cap \tilde{s}^{-1}\{p\}$ the local isomorphism condition again holds by the local conical structure of T. Thus, it remains to consider the case where $x \in S_p$, and $\mathrm{H}_\bullet(T; x) \cong \mathrm{H}_\bullet(\mathbb{R}^q; 0)$. We need to show that, for $U_x \cong \mathbb{R}^{q-p-1} \times \mathring{C}(L_x)$, an open neighborhood of x, the natural map

$$\mathrm{H}_\bullet(T; x) \cong \mathrm{H}_\bullet(W, W - U_x) \to \mathrm{H}_\bullet(T; y)$$

is an isomorphism, for all $y \in U_x$. The only nontrivial degree in this case is $q = \dim W$. By an application of the Künneth formula L_x is again an orientable manifold. Hence, up to suspension, from this perspective, the claim reduces to the fact that if L_x is an orientable, closed manifold. Then, under the natural isomorphism

$$\mathrm{H}_\bullet(CL_x, L_x) \cong \tilde{\mathrm{H}}_{\bullet-1}(L_x)$$

the fundamental class of L_x induces a fundamental class of CL_x. □

Chapter 5

Combinatorial models for stratified homotopy theory

Note to the reader: The following chapter was structured as an independent article, in order to allow for easier accessibility. A preliminary version was made publicly available on the arXiv (see [Waa24a]). Notation in this chapter is entirely consistent with Chapter 1. There may be minor notation differences compared to Chapter 3. However, as all notation is introduced separately in this chapter, this should not pose an issue.

This paper is part of a series of three articles with the objective of investigating a stratified version of the homotopy hypothesis in terms of semimodel structures that interact well with classical examples of stratified spaces, such as Whitney stratified spaces. To this end, we prove the existence of several combinatorial simplicial model structures in the combinatorial setting of stratified simplicial sets. One of these we show to be Quillen equivalent to the left Bousfield localization of the Joyal model structure that presents the $(\infty, 1)$-category of small layered $(\infty, 1)$-categories, i.e., such small $(\infty, 1)$-categories in which every endomorphism is an isomorphism.

5.1 Introduction

Stratified spaces were originally introduced by Whitney, Thom and Mather as a tool to investigate spaces with singularities (see [Whi65b; Mat12; Mat73; Tho69]). In the broadest sense, a stratified space consists of the data of a topological space together with a decomposition into disjoint pieces, with additional varying assumptions on the properties of these pieces - the so-called strata - and their interactions. In more recent years, the investigation of such objects has shifted from being primarily concerned with studying a single object to studying classes of stratified spaces and the stratified maps between them (such maps that map strata into strata). Even more, instead of focusing on this 1-categorical perspective, the focus has been on the $(\infty, 1)$-categorical point of view: Studying homotopy theories of (certain classes of) stratified spaces, induced by stratified notions of homotopy (see [Qui88; Hug99b; Mil13; AFT17; AFR19; Dou21c; DW22; Hai23; Nan19]). Starting with Quinn's theory of homotopically stratified spaces (named homotopically stratified sets in [Qui88]), several homotopy theories of stratified spaces were introduced and studied, for example, in [DW22; Hai23; Nan19]. This paper is part of a three-part series of articles concerned with these homotopy theories of stratified spaces, the goal of which is to develop (semi-)model structures for stratified homotopy theory which interact well with classical geometric and topological examples of stratified spaces, and ultimately lead to a tractable and interpretable version of the so-called *topological stratified homotopy hypothesis*:

> The homotopy theory of stratified topological spaces is the same as the homotopy theory of such $(\infty, 1)$-categories in which every endomorphism is an isomorphism.

It is a general paradigm in homotopy theory (most prominently realized in [Qui67]) that homotopy theoretic phenomena are often easier understood after being translated into the world of combinatorics. Thus, our approach to constructing model structures for stratified homotopy theory consists of developing the theory in a combinatorial framework first and then transferring it to the world of stratified topological spaces.
The goal of this paper is to cover the purely combinatorial part of this program. To this end, we survey several model structures for stratified simplicial sets over a fixed poset already on the market, exposing the precise connections between them. We then extend these model categories to model categories of stratified simplicial sets with varying poset, and connect the latter to the Joyal model category for $(\infty, 1)$-categories. Let us explicitly state that our goal here is not to obtain results which are new from a purely conceptual (∞-categorical) point of view, but rather to produce an overview of combinatorial models for stratified homotopy theory and mirror several results and structures already known on the ∞-categorical level from [BGH18; Hai23] in the language of model categories. This has the advantage that it will ultimately allow us to transfer these structures and results to the topological stratified framework, in which the additional structure of a (semi)model category is necessary to connect the homotopy theory with the geometry and topology of stratified spaces.

In more detail, the content of this paper is as follows. First, in Section 5.2.1, we recall the Douteau-Henriques model structure (defined by Douteau in [Dou21a], and independently defined by Henriques in [Hen]), as well as the Joyal-Kan model structure defined by Haine in [Hai23], which are both defined on categories of simplicial sets stratified over a fixed poset. The latter of these presents ∞-categories with a conservative functor into a poset, so-called *abstract stratified homotopy types*. We show that the Joyal-Kan model structure is the left Bousfield localization of the Douteau-Henriques model structure at the class of inner stratified horn inclusions (Proposition 5.2.1.3). This provides a useful approach to investigating the categorical homotopy theories of stratified spaces defined by Haine, and the one defined by Nand-Lal in [Nan19]: One can often obtain results about the categorical theories from results about the Douteau-Henriques theories, which often turn out to be significantly easier to handle, due to the explicit description of weak equivalences in the latter (see, for example, the proof of [Hai23, Thm 0.1.1]). To illustrate this method, in Section 5.2.2, we provide combinatorial simplicial model structure for the homotopy theory of décollages described in [Hai23] - roughly space valued presheaves indexed over the finite increasing sequences over a poset fulfilling a Segal style fibrancy condition - and prove a Quillen equivalence between Haine's model structure for abstract stratified homotopy types and the model structure for décollages (Theorem 5.2.2.20). This Quillen equivalence presents an equivalence of ∞-categories already proven in [Hai23], without appealing to the theory of complete Segal spaces. Our proof works by constructing a new left Quillen functor model for the functors of homotopy links studied in detail in [DW22] (Construction 5.2.2.4 and Proposition 5.2.2.11).
Then, in Section 5.3.1 we move from the case of a fixed poset to the case of flexible posets by gluing the model structures described in [Dou21a; Hai23] using a method of [CM20], already employed in [Dou21c]. These model structures provide combinatorial simplicial models for the homotopy theories of stratified spaces with varying posets investigated in [Dou21c; Hai23] (see Proposition 5.3.1.8).
Both of the homotopy theories of stratified simplicial sets constructed in Section 5.3.1 have the property that morphisms in them are not entirely determined by the underlying map of spaces but include the additional data of a map of posets, in opposition to the classical scenario (see, for example, [Wei94]) where stratification was purely a property of a map (see the beginning of Section 5.3.2 and particularly Remark 5.3.2.1). If one is looking to get closer to the classical scenario, one can instead work with so-called *refined stratified simplicial sets* (called 0-connected in the case of abstract stratified homotopy types in [BGH18]), which are, roughly speaking, the class of stratified simplicial sets for which the underlying poset is

entirely encoded in the closure relations of the strata (see Definition 5.3.2.12). To account for this, in Section 5.3.2, we provide right Bousfield localizations of the global model structures in which maps between bifibrant objects have the property that maps are defined entirely on the space (simplicial set) level. Our main results in this subsection may be summarized as follows. The category of stratified simplicial sets s**Strat** admits the structures of two combinatorial simplicial model categories, s$\mathbf{Strat}^{\mathfrak{d}}$ and s$\mathbf{Strat}^{\mathfrak{c}}$ which are respectively right Bousfield localizations of the global versions of the Douteau-Henriques and the Joyal-Kan model structures (Theorem 5.3.2.19). In both model structures, the cofibrant objects are precisely the refined stratified simplicial sets. s$\mathbf{Strat}^{\mathfrak{c}}$ is the left Bousfield localization of s$\mathbf{Strat}^{\mathfrak{d}}$ at inner stratified horn inclusions and presents the ∞-category of refined (0-connected) abstract stratified homotopy types (Propositions 5.3.1.8 and 5.3.2.24). We furthermore show that weak equivalences in both s$\mathbf{Strat}^{\mathfrak{c}}$ and s$\mathbf{Strat}^{\mathfrak{d}}$ are stable under filtered colimits, which is one of the key ingredients to transferring these model structures to the topological realm (Proposition 5.3.1.5).
In the next subsection (Section 5.3.3), we then show that one of these model structures is Quillen equivalent to the left Bousfield localization of the Joyal model structure on simplicial sets that presents the ∞-category of small ∞-categories in which every endomorphism is an isomorphism (Theorem 5.3.3.6). This lifts a result proven on the ∞-categorical level in [BGH18, p. 2.3.8] to the level of model categories, and provides one necessary core result for our version of the topological stratified homotopy hypothesis proven in [Waa24c]. Finally, we prove that all of the model structures on s**Strat** defined in this paper are cartesian closed (Theorem 5.3.5.4), which allows us to recover a result of Bruce Hughes (see [Hug99b, Main Result]) on homotopically stratified spaces in our purely combinatorial setting (Construction 5.3.5.5).

5.1.1 Language and notation

Let us begin by introducing some of the relevant categories and recalling some notation. We will follow the convention of denoting 1-categories in bold letters, simplicial categories in bold underlined letters, and $(\infty, 1)$-categories (modeled by quasi-categories) by writing their first capital letter in caligraphic script. If we wish to denote the underlying 1-category of a simplicial category, we do so by simply omitting the line under the name. We use the same notation for model categories mutatis mutandis.

Notation 5.1.1.1. We are going to use the following terminology and notation for partially ordered sets, drawn partially from [Dou21a] and [Hai23]:

- We denote by **Pos** the category of partially ordered sets, with morphisms given by order-preserving maps.
- We denote by Δ the full subcategory of **Pos** given by the finite, nonempty, linearly ordered posets of the form $[n] := \{0, \cdots, n\}$, for $n \in \mathbb{N}$.
- Given $P \in \mathbf{Pos}$, we denote by Δ_P the slice category $\Delta_{/P}$. That is, objects are given by arrows $[n] \to P$ in **Pos**, $n \in \mathbb{N}$, and morphisms are given by commutative triangles.
- We denote by $\mathrm{sd}(P)$ the *subdivision of* P, given by the full subcategory of Δ_P of such arrows $[n] \to P$, which are injective.
- The objects of Δ_P are called *flags of* P. We represent them by strings $[p_0 \leq \cdots \leq p_n]$, of $p_i \in P$. We refer to n as the *length* of the flag $[p_0 \leq \cdots \leq p_n]$.
- Objects of $\mathrm{sd}(P)$ are called regular *flags of* P. We represent them by strings $[p_0 < \cdots < p_n]$, of $p_i \in P$.

Notation 5.1.1.2. We use the following terminology and notation for (stratified) simplicial sets, drawn partially from [Dou21a] and [Hai23]:

- We denote by $\underline{\mathbf{sSet}}$ the simplicial category of simplicial sets, i.e. the category of set valued presheaves on Δ^{op}, equipped with the canonical simplicial structure induced by the product (see [Lur09] for all of the standard notation used for simplicial sets).
- When we treat $\underline{\mathbf{sSet}}$ as a model category this will generally be with respect to the Kan-Quillen model structure (see [Qui67]), unless otherwise noted. When we use Joyal's model structure for quasi-categories ([JT08]) instead, we will denote this model category by $\mathbf{sSet}^{\mathfrak{J}}$.
- We think of $\mathbf{Pos}$ as being fully faithfully embedded in $\mathbf{sSet}$, via the nerve functor (compare [Hai23]). By abuse of notation, we just write P for the simplicial set given by the nerve of $P \in \mathbf{Pos}$.
- For $P \in \mathbf{Pos}$, we denote by $\mathrm{s}\mathbf{Strat}_P$ the slice category $\mathbf{sSet}_{/P}$, which is equivalently given by the category of set-valued presheaves on Δ_P. We treat $\underline{\mathrm{s}\mathbf{Strat}}_P$ as a simplicial category, denoted $\underline{\mathrm{s}\mathbf{Strat}}_P$, with the structure inherited from $\underline{\mathbf{sSet}}$ (see [DW22, Recol. 2.21.], which is Recollection 3.2.3.3 in this text).
- Objects of $\underline{\mathrm{s}\mathbf{Strat}}_P$ are called *P-stratified simplicial sets.* They are given by a tuple $\mathcal{X} = (X, s_{\mathcal{X}}\colon X \to P)$. In the literature, a P-stratified simplicial set $\mathcal{X} = (X, s_{\mathcal{X}}\colon X \to P)$ is often simply referred to by its underlying simplicial set X, omitting the so-called *stratification* $s_{\mathcal{X}}\colon X \to P$. We are not going to adopt this notation here, as we will frequently consider the same simplicial set with changing stratifications. We are always going to use calligraphic letters for stratified simplicial sets and their non-calligraphic counterparts for the underlying simplicial set.
- Morphisms in $\underline{\mathrm{s}\mathbf{Strat}}_P$ are called *stratum-preserving simplicial maps.* Simplicial homotopies in $\underline{\mathrm{s}\mathbf{Strat}}_P$ are called *stratified simplicial homotopies.* Simplicial homotopy equivalences in $\underline{\mathrm{s}\mathbf{Strat}}_P$ are called *stratum-preserving simplicial homotopy equivalences.*
- Given a map of posets $f\colon Q \to P$ and $\mathcal{X} \in \underline{\mathrm{s}\mathbf{Strat}}_P$, we denote by $f^*\mathcal{X} \in \underline{\mathrm{s}\mathbf{Strat}}_Q$ the stratified simplicial set $X \times_P Q \to Q$. We are mostly concerned with the case where f is given by the inclusion of a singleton $\{p\}$, of a subset $\{q \sim p \mid q \in P\}$, for $p \in P$ and $\sim$ some relation on the partially ordered set P (such as $\leq$), or more generally, a subposet $Q \subset P$. We then write $\mathcal{X}_p$ (or, respectively, $\mathcal{X}_{\sim p}$, $\mathcal{X}_Q$) instead of $f^*\mathcal{X}$. The simplicial sets $\mathcal{X}_p$, for $p \in P$ are called the *strata of* $\mathcal{X}$.
- For $f\colon Q \to P$ in $\mathbf{Pos}$, we denote by $f_!$ the left adjoint to the simplicial functor $f^*\colon \underline{\mathrm{s}\mathbf{Strat}}_P \to \underline{\mathrm{s}\mathbf{Strat}}_Q$, given on objects by $(s_{\mathcal{X}}\colon X \to Q) \mapsto (f \circ s_{\mathcal{X}}\colon X \to Q \to P)$.
- Let $\mathbf{sSet}^{[1]}$ be the category of arrows of simplicial sets. We denote by $\mathrm{s}\mathbf{Strat}$ the category of all stratified simplicial sets, given by the full sub-category of $\mathbf{sSet}^{[1]}$ of such arrows $X \to P$, where $X \in \mathbf{sSet}$ and $P \in \mathbf{Pos}$ is (the nerve of) a poset. In particular, every object of $\mathrm{s}\mathbf{Strat}$ is given by a P-stratified simplicial set, for some $P \in \mathbf{Pos}$, and a morphism $(X \to P) \to (Y \to Q)$ is given by a pair of morphisms $f\colon X \to Y$ and $g\colon P \to Q$, where f is a simplicial map and g can be seen as a map of posets, making the obvious square commute (see also [DW22, Def. 2.19], which is Definition 3.2.3.1 in this text). Morphisms are called *stratified simplicial maps.*
- Given $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$, we are going to use the notational convention $\mathcal{X} = (X, s_{\mathcal{X}}, P_{\mathcal{X}})$ to refer, respectively, to the underlying simplicial set, the stratification and the poset and proceed analogously for morphisms.
- We equip $\mathrm{s}\mathbf{Strat}$ with the structure of a simplicial category, tensored and cotensored over $\underline{\mathbf{sSet}}$, denoted $\underline{\mathrm{s}\mathbf{Strat}}$, with the tensoring induced by setting

  $$\mathcal{X} \otimes \Delta^n = (X \times \Delta^n \to X \to P_{\mathcal{X}})\,.$$

  Simplicial homotopies in $\underline{\mathrm{s}\mathbf{Strat}}$ are called *stratified simplicial homotopies.* Simplicial homotopy equivalences in $\underline{\mathrm{s}\mathbf{Strat}}$ are called *stratified simplicial homotopy equivalences.*

- The forgetful functor $s\mathbf{Strat} \to \mathbf{sSet}$, $\mathcal{X} \mapsto X$ will be denoted $\mathcal{F}$, and has a right adjoint and a left adjoint. The left adjoint is given by left Kan extending the functor on simplices: $\Delta^n \mapsto \{\Delta^n \xrightarrow{1_{\Delta^n}} \Delta^n = [n]\}$. We denote it by $\mathcal{L}\colon \mathbf{sSet} \to s\mathbf{Strat}$. The right adjoint is given by mapping $K \in \mathbf{sSet}$ to the trivially stratified simplicial set $\{K \to [0]\}$. By abuse of notation, we will often write K to refer to the trivially stratified simplicial set associated to a simplicial set K.

Remark 5.1.1.3. There is a canonical forgetful functor, $s\mathbf{Strat} \to \mathbf{Pos}$ given by $\mathcal{X} \mapsto P_{\mathcal{X}}$ and we may identify its fiber at $P \in \mathbf{Pos}$ with $s\mathbf{Strat}_P$. This functor is easily seen to be a Grothendieck bifibration, with right action given by $f \mapsto f^*$ and left action given by $f \mapsto f_!$. It follows that we may use the results in [CM20] to glue local model structures on the fibers to global model structures.

Remark 5.1.1.4. Both $s\mathbf{Strat}$ and $s\mathbf{Strat}_P$, for $P \in \mathbf{Pos}$, are bicomplete categories (see, for example, [Dou21a]). Limits and colimits in $s\mathbf{Strat}_P$ are simply given by the limits and colimits in a slice category. Both limits and colimits in $s\mathbf{Strat}$ are computed by taking, respectively, the limit or colimit both on the simplicial set and on the poset level.

Notation 5.1.1.5. We are going to need some additional notation for flags and stratified simplices.

- For a flag $\mathcal{J} = [p_0 \le \dots \le p_n] \in \Delta_P$, we write $\Delta^{\mathcal{J}}$ for the image of $\mathcal{J}$ in $s\mathbf{Strat}_P$ under the Yoneda embedding $\Delta_P \hookrightarrow s\mathbf{Strat}_P$. Equivalently, $\Delta^{\mathcal{J}}$ is given by the unique simplicial map $\Delta^n \to P$ mapping $i \mapsto p_i$. $\Delta^{\mathcal{J}}$ is called the *stratified simplex* associated to $\mathcal{J}$.

- Given a stratified simplex $\Delta^{\mathcal{J}}$, for $\mathcal{J} = [p_0 \le \dots \le p_n]$, we write $\partial\Delta^{\mathcal{J}}$ for its *stratified boundary*, given by the composition $\partial\Delta^n \to \Delta^n \to P$.

- Furthermore, for $0 \le k \le n$, we write $\Lambda^{\mathcal{J}}_k \subset \Delta^{\mathcal{J}}$ for the stratified subsimplicial set given by the composition $\Lambda^n_k \to \Delta^n \to P$ (we use the horn notation as in [Lur09]). The stratum-preserving map $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{I}}$ is called the *stratified horn inclusion associated to* $\mathcal{J}$ and k. The inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{I}}$ is called *admissible*, if $p_k = p_{k+1}$ or $p_k = p_{k-1}$. The inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{I}}$ is called *inner* if $0 < k < n$.

- Using the fully faithful (and continuous) embedding $\Delta_P \hookrightarrow s\mathbf{Strat}_P$, we extend the base-change notation for stratified simplicial sets to flags. That is, for $f\colon Q \to P$ we write $f^*\mathcal{J}$ for the unique flag of Q corresponding to $f^*(\Delta^{\mathcal{J}})$. We use the same shorthand notation for subsets $Q \subset P$. For example, $\mathcal{J}_{\le p}$ is the flag obtained from $\mathcal{J}$, by removing all entries not less than or equal to p.

- It will also be convenient to have a concise notation for the images of simplices, horns, and boundaries under $\mathcal{L}\colon \mathbf{sSet} \to s\mathbf{Strat}$. These are denoted by replacing the exponent $n \in \mathbb{N}$, by the poset $[n]$. That is, we write $\Delta^{[n]} := \mathcal{L}(\Delta^n)$, $\partial\Delta^{[n]} := \mathcal{L}(\partial\Delta^n)$, $\Lambda^{[n]}_k := \mathcal{L}(\Lambda^n_k)$, for $0 \le k \le n$.

5.2 Combinatorial models over a fixed poset

Before we begin with the construction of model structures for the category of stratified simplicial sets over varying posets $s\mathbf{Strat}$, we first cover the case of categories of stratum-preserving maps. Later, in Section 5.3, we will piece together the model structures defined in this section for one fixed poset, to obtain model structures on $s\mathbf{Strat}$. For the remainder of this subsection, fix some poset P.

5.2.1 The minimalist- and the Joyal-Kan approach

In this subsection, we recall the model structures on P-stratified simplicial sets defined in [Dou21a] and [Hai23] and point out the precise relationship between them. Since $\mathrm{s}\mathbf{Strat}_P$ is isomorphic to the category of set-valued presheaves on Δ_P, we may use the methods of [Cis06] to construct model structures on it.

Recollection 5.2.1.1 ([DW22])**.** The *Douteau-Henriques model structure* on $\underline{\mathrm{s}\mathbf{Strat}}_P$, defined first in [Dou21a], is the Cisinski model structure (see [Cis19, Thm. 2.4.19]) induced by the simplicial cylinder $X \mapsto X \otimes \Delta^1$, with the empty set of anodyne extensions. This defines a combinatorial, cofibrant, simplicial model structure on $\underline{\mathrm{s}\mathbf{Strat}}_P$ whose defining classes may be characterized as follows (see [DW22], specifically Theorem 3.1.0.3 in this text) for this characterization, which is stronger than the one provided in [Dou21a]):

1. Cofibrations are precisely the monomorphisms in $\underline{\mathrm{s}\mathbf{Strat}}_P$.
2. Weak equivalences are precisely such stratum-preserving simplicial maps $\mathcal{X} \to \mathcal{Y}$ for which the induced map of simplicial sets
$$\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathcal{X}) \to \underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathcal{Y})$$
is a weak homotopy equivalence, for all $\mathcal{I} \in \mathrm{sd}(P)$. We call such a map a *diagrammatic equivalence*.
3. Fibrations are precisely the simplicial maps which have the right lifting property with respect to all admissible horn inclusions.

We denote the resulting simplicial model category by $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{d}}$. It carries the minimal model structure (with respect to weak equivalences) in which the cofibrations are the monomorphisms, and stratified simplicial homotopy equivalences are weak equivalences. $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{d}}$ is cofibrantly generated by the classes of stratified boundary inclusions and admissible horn inclusions.

Since $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{d}}$ is in some sense minimal among model structures on $\underline{\mathrm{s}\mathbf{Strat}}_P$, it is not surprising that alternative theories arise as a localization of the homotopy theory presented by $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{d}}$. In particular, this is the case for the model structure defined in [Hai23].

Recollection 5.2.1.2 ([Hai23])**.** The *Joyal-Kan model structure* on $\underline{\mathrm{s}\mathbf{Strat}}_P$ is the one obtained by localizing the model structure inherited from the Joyal-model structure on $\mathbf{sSet}$ at the cylinder $- \otimes \Delta^1$. The Joyal-Kan model structure is simplicial, cofibrantly generated, and its defining classes have the following descriptions:

1. Cofibrations are precisely the monomorphisms in $\mathrm{s}\mathbf{Strat}_P$.
2. Fibrant objects are precisely the stratified simplicial sets $\mathcal{X}$ for which the underlying simplicial set X is a quasi-category and $s_{\mathcal{X}}: X \to P_{\mathcal{X}}$ is a conservative functor. Fibrations between fibrant objects are precisely the stratum-preserving simplicial maps that have the right lifting property with respect to all inner and admissible stratified horn inclusions.
3. Weak equivalences between fibrant objects are equivalently characterized as the class of
   (a) stratified homotopy equivalences;
   (b) Joyal equivalences (over P);
   (c) stratum-preserving maps that induce weak equivalences on $\underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, -)$, for all regular flags $\mathcal{I}$ of length lesser or equal to 1.

We denote the model category (uniquely determined by these classes) by $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{c}}$. Weak equivalences in this model structure will be called *Joyal-Kan equivalences*. It follows by the characterization of weak equivalences and fibrant objects above that $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{c}}$ presents the ∞-category of conservative functors from a quasi-category into P, also called *abstract stratified homotopy types* over P.

In view of the minimality of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$, the following is not surprising:

Proposition 5.2.1.3. *The simplicial model category $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ is the left Bousfield localization of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ at the class of stratified inner horn inclusions.*

Proof. It suffices to see that the localization described above has the same fibrant objects as $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$. Let $\mathcal{X}$ be fibrant in the localization. In particular, $\mathcal{X}$ has the filler property for all admissible and inner stratified horn inclusions. It follows that X is a quasi-category. Furthermore, as $X \to P$ has the right lifting property with respect to every admissible horn inclusion (which includes horn inclusions entirely contained in one stratum) for every $p \in P$ the stratum $\mathcal{X}_p$ is a Kan complex. In particular, $s_{\mathcal{X}}\colon X \to P$ is conservative. Now, conversely, suppose that $\mathcal{X}$ is such that $s_{\mathcal{X}}\colon X \to P_{\mathcal{X}}$ is a conservative functor of quasi-categories. Then, since P is the nerve of a 1-category, $s_{\mathcal{X}}$ is also an inner fibration, which shows that $\mathcal{X}$ admits fillers for all inner horn inclusions. Now, consider a horn inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$, with $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, which is not inner, but admissible. We cover the case $k = n$, as the other is analogous. Hence, we may assume that $p_{n-1} = p_n$. Since $s_{\mathcal{X}}$ is conservative, it follows that the edge of Λ^n_k from $n-1$ to n maps to an isomorphism f in X. In particular, f is cartesian ([Lur09, Prop. 2.4.1.5]) and a lift with respect to $\Lambda^{\mathcal{I}}_k \hookrightarrow \Delta^{\mathcal{J}}$ exists by [Lur09, Rem. 2.4.1.4]. Therefore, $\mathcal{X}$ admits a filler for all inner and all admissible horn inclusions. The latter shows that it is fibrant in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$. To see that $\mathcal{X}$ is local with respect to inner horn inclusions $\Lambda^{\mathcal{I}}_k \hookrightarrow \Delta^{\mathcal{I}}$, we may equivalently show that $\mathcal{X} \to P$ has the right lifting property with respect to the maps

$$\Lambda^{\mathcal{I}}_k \otimes \Delta^n \cup_{\Lambda^{\mathcal{I}}_k \otimes \partial\Delta^n} \Delta^{\mathcal{I}} \otimes \partial\Delta^n \hookrightarrow \Delta^{\mathcal{I}} \otimes \Delta^n.$$

It is a standard argument that these may be decomposed into a composition of pushouts of inner horn inclusions (see, for example, [Cis19, Cor 3.2.4]). ☐

Again, using [Cis19, Cor 3.2.4], we obtain:

Corollary 5.2.1.4. *Fibrant objects in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ are precisely such stratified simplicial sets that have the horn filling property with respect to all admissible and inner stratified horn inclusions.*

Proposition 5.2.1.3 is particularly useful, because it provides a criterion to check for weak equivalences $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$. Generally, in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$, the lack of an explicit criterion for weak equivalences can make such verifications challenging. In many cases, however, we may already verify the relevant property in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ and then use the fact that they are preserved under left Bousfield localization.[1] In this sense, the homotopy theories defined by $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ are really not in a competing, but in a mutually supportive relationship. Let us finish this subsection with a general remark and a proposition which we use to transfer model structures from the simplicial to the topological world in [Waa24c].

Remark 5.2.1.5. Every stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$ (or in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$) is the homotopy colimit of its stratified simplices. In fact, by [Cis06, Ex. 8.2.5, Prop. 8.2.9], this holds for any Cisinski model structure on $\mathrm{s}\mathbf{Strat}_P$.

Proposition 5.2.1.6. *Weak equivalences in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ are stable under filtered colimits.*

Proof. For $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ this is [Hai23, p. 2.5.9]. For $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ this follows from the fact that weak equivalences are detected by a finite set of functors with values in simplicial sets, that preserve filtered colimits. That weak equivalences of simplicial sets are stable under filtered colimits follows, for example, as an application of Kan's Ex^{∞} functor, which preserves all filtered colimits (see [Kan57]). ☐

[1] See for example the approach to proving the existence of semi-models tructures on the topological side we take in [Waa24c]. Similarly, the proof of a version of a stratified homotopy hypothesis in [Hai23] was built on [Dou21c].

5.2.2 Homotopy links and a model structure for décollages

As is apparent from the characterization of weak equivalences in $\underline{\mathbf{sStrat}}_P^{\mathfrak{d}}$ (Recollection 5.2.1.1), the simplicial mapping spaces $\underline{\mathbf{sStrat}}_P(\Delta^{\mathcal{I}}, \mathcal{X})$, for $\mathcal{X} \in \underline{\mathbf{sStrat}}_P$, play a central role in understanding the homotopy theory of $\underline{\mathbf{sStrat}}_P^{\mathfrak{d}}$.

Recollection 5.2.2.1. For $\mathcal{I} \in \mathrm{sd}(P)$ and $\mathcal{X} \in \underline{\mathbf{sStrat}}_P$, the simplicial set $\underline{\mathbf{sStrat}}_P(\Delta^{\mathcal{I}}, \mathcal{X})$ is called the *$\mathcal{I}$-th (simplicial) homotopy link of $\mathcal{X}$*. It is denoted $\mathrm{HoLink}_{\mathcal{I}}(\mathcal{X})$ (see also [DW22, Def. 2.31], which is Definition 3.2.5.1 in this text). The simplicial sets $\mathrm{HoLink}_{\mathcal{I}}(\mathcal{X})$ are organized in the structure of a simplicial presheaf on $\mathrm{sd}(P)$, denoted $\mathrm{HoLink}(\mathcal{X})$. Denote by $\underline{\mathbf{Diag}}_P$ the simplicial category of simplicial presheaves on $\mathrm{sd}P$. Homotopy links induce a nerve-style functor

$$\mathrm{HoLink}\colon \underline{\mathbf{sStrat}}_P \to \underline{\mathbf{Diag}}_P$$

that admits a left adjoint, given by mapping $D \in \underline{\mathbf{Diag}}_P$ to the coend $\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes D_{\mathcal{I}}$. This left adjoint functor preserves all monomorphisms. Furthermore, it preserves all weak equivalences in both directions, by [DW22, Thm. 1.3] (which is Theorem 3.1.0.3 in this text). Hence, we obtain a pair of (simplicial) Quillen adjoint functors

$$\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -_{\mathcal{I}} : \underline{\mathbf{Diag}}_P^{\mathrm{inj}} \rightleftharpoons \underline{\mathbf{sStrat}}_P^{\mathfrak{d}} \colon \mathrm{HoLink}$$

between $\underline{\mathbf{sStrat}}_P^{\mathfrak{d}}$ and $\underline{\mathbf{Diag}}_P$ equipped with the injective model structure.

As an immediate corollary of [DW22, Thm. 1.3] (Theorem 3.1.0.3), one obtains the following. Recall that a functor F between categories with weak equivalences is said to *create weak equivalences*, if it has the property that $F(w)$ is a weak equivalence, if and only if w is a weak equivalence, for every morphism w in the source category.

Corollary 5.2.2.2. *The simplicial Quillen adjunction*

$$\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -_{\mathcal{I}} : \underline{\mathbf{Diag}}_P^{\mathrm{inj}} \rightleftharpoons \underline{\mathbf{sStrat}}_P^{\mathfrak{d}} \colon \mathrm{HoLink}$$

is a Quillen equivalence that creates weak equivalences in both directions.

Remark 5.2.2.3. Note that the condition for an adjunction between model categories (more generally, categories with weak equivalences) to create weak equivalences in both directions is equivalent to both functors preserving weak equivalences and the unit and counit being given by weak equivalences.

We may interpret Corollary 5.2.2.2 as follows. If one takes the perspective that inclusions of stratified simplicial sets should be the cofibrations, and that at least the stratified (simplicial) homotopy equivalences should be weak equivalences, then the minimal homotopy theory one ends up with is the one of simplicial presheaves on $\mathrm{sd}(P)$. Consequently, one would also expect to be able to interpret the homotopy theory of $\underline{\mathbf{sStrat}}_P^{\mathfrak{c}}$ in terms of a category of (certain) presheaves on $\mathrm{sd}(P)$. Such a result was first shown in [BGH18, Thm. 2.7.4] and in [Hai23]. Here, we are going to give a version of this result in the language of model categories. This serves to illustrate a method of proof, which we are also going to employ when we show the existence of convenient model structures for topological stratified spaces in [Waa24c]:
Up to weak equivalence, the functor $\mathrm{HoLink} : \underline{\mathbf{sStrat}}_P \to \underline{\mathbf{Diag}}_P$ is both a right and a left Quillen adjoint.

Construction 5.2.2.4. Homotopy links admit a more geometric model, which is constructed as follows. Let $\mathcal{I} = [p_0 < \dots < p_n]$ be a regular flag of P. We then obtain a functor

$$\mathrm{Link}_{\mathcal{I}}\colon \Delta_P \to \mathbf{sSet}$$

by mapping

$$\mathcal{J} \mapsto \prod_{p_i \in \mathcal{I}} \Delta^{\mathcal{J}_{p_i}}.$$

If $\mathcal{I}_0 \subset \mathcal{I}_1$, then the projections of the product induce a natural transformation

$$\mathrm{Link}_{\mathcal{I}_1} \to \mathrm{Link}_{\mathcal{I}_0},$$

Under left Kan extension, we therefore obtain a functor

$$\mathrm{Link} \colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P.$$

Let us explicitly compute $\mathrm{Link}_{\mathcal{I}}$ for a stratified version of Joyal's join functor.

Construction 5.2.2.5. Suppose that $\mathcal{I} \in \mathrm{sd}(P)$ is a non-degenerate flag such that $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, with $\mathcal{I}_0$ and $\mathcal{I}_1$ disjoint (and non-empty). Given two flags $\mathcal{J}_0$ an $\mathcal{J}_1$ degenerating from a subflag of $\mathcal{I}_0$ and $\mathcal{I}_1$ respectively, the associated object $\mathcal{J}_0, \mathcal{J}_1 \in \Delta_P$ admits a coproduct, denoted $\mathcal{J}_0 \sqcup \mathcal{J}_1$. It is given by the (appropriately ordered) union of the sequences defining $\mathcal{J}_0$ and $\mathcal{J}_1$. In particular, whenever $\mathcal{J}$ degenerates from a subflag of $\mathcal{I}$ that intersects $\mathcal{I}_0$ and $\mathcal{I}_1$ non-trivially, then $\mathcal{J} = \mathcal{J}_0 \sqcup \mathcal{J}_1$, where $\mathcal{J}_0$ and $\mathcal{J}_1$ denote the respective restrictions of $\mathcal{J}$ to $\mathcal{I}_0$ and $\mathcal{I}_1$. Let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_{\mathcal{I}_0}$ and $\mathcal{Y} \in \mathrm{s}\mathbf{Strat}_{\mathcal{I}_1}$. We denote by $\mathcal{X} *_P \mathcal{Y}$ the stratified simplicial set given by the presheaf on Δ_P mapping

$$\mathcal{J} \mapsto \begin{cases} \varnothing & \text{, if } \mathcal{J} \text{ does not degenerate from a subflag of } \mathcal{I} \\ \mathcal{X}(\mathcal{J}) & \text{, if } \mathcal{J} \text{ degenerates from a subflag of } \mathcal{I}_0 \\ \mathcal{Y}(\mathcal{J}) & \text{, if } \mathcal{J} \text{ degenerates from a subflag of } \mathcal{I}_1 \\ \mathcal{X}(\mathcal{J}_0) \times \mathcal{Y}(\mathcal{J}_1) & \text{, if } \mathcal{J} = \mathcal{J}_0 \sqcup \mathcal{J}_1 \text{ for } \mathcal{J}_0 \text{ and } \mathcal{J}_1 \text{ as above} \end{cases}$$

with all face and degeneracy maps induced by the ones on $\mathcal{X}$ and $\mathcal{Y}$, the functoriality of restriction to $\mathcal{I}_0$ and $\mathcal{I}_1$ and the universal property of the product. This construction induces a functor

$$- *_P - \colon \mathrm{s}\mathbf{Strat}_{\mathcal{I}_0} \times \mathrm{s}\mathbf{Strat}_{\mathcal{I}_1} \to \mathrm{s}\mathbf{Strat}_P,$$

functorial in morphisms in the obvious way. It comes together with a natural transformation

$$\mathcal{X} \sqcup \mathcal{Y} \hookrightarrow \mathcal{X} *_P \mathcal{Y}.$$

where we treat $\mathcal{X} \sqcup \mathcal{Y}$ as a stratified simplicial set over P. We call this construction the *P-stratified join functor.* Indeed, if we restrict to stratified simplices, then there is a canonical natural isomorphism

$$\Delta^{\mathcal{J}_0} *_P \Delta^{\mathcal{J}_1} \cong \Delta^{\mathcal{J}_0 \sqcup \mathcal{J}_1}$$

If we fix any of the two arguments (say the first, which suffices, since the construction is symmetric), then we may use this natural transformation to obtain a lift of the stratified join functor.

$$\begin{aligned} \mathcal{X} *_P - \colon \mathrm{s}\mathbf{Strat}_{\mathcal{I}_1} &\to (\mathrm{s}\mathbf{Strat}_P)_{\mathcal{X}/} \\ \mathcal{Y} &\mapsto (\mathcal{X} \hookrightarrow \mathcal{X} \sqcup \mathcal{Y} \hookrightarrow \mathcal{X} *_P \mathcal{Y}). \end{aligned}$$

It follows immediately from the definition of $\mathcal{X} *_P -$ and the elementary laws for computing colimits in presheaf and under-categories that $\mathcal{X} *_P -$ is cocontinuous as a functor with image in $(\mathrm{s}\mathbf{Strat}_P)_{\mathcal{X}/}$.

Let us now take a look at the interaction of the stratified join functor with the link functors.

Lemma 5.2.2.6. *Using the notation of Construction 5.2.2.5, there is a natural isomorphism of bifunctors*

$$\mathrm{Link}_{\mathcal{I}}(- *_P -) \cong \mathrm{Link}_{\mathcal{I}_0}(-) \times \mathrm{Link}_{\mathcal{I}_1}(-).$$

Proof. We use the notation of Construction 5.2.2.5. Observe that that there is a canonical isomorphism

$$\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}_0 \sqcup \mathcal{J}_1}) \cong \mathrm{Link}_{\mathcal{I}_0}(\Delta^{\mathcal{J}_0}) \times \mathrm{Link}_{\mathcal{I}_1}(\Delta^{\mathcal{J}_1}).$$

We have already seen that there is an isomorphism

$$\Delta^{\mathcal{J}_0} *_P \Delta^{\mathcal{J}_0} \cong \Delta^{\mathcal{J}_0 \sqcup \mathcal{J}_1}$$

natural in $\mathcal{J}_0$ and $\mathcal{J}_1$. Hence, after restricting to $\Delta_{\mathcal{I}_0} \times \Delta_{\mathcal{I}_1}$, there is a natural isomorphism of bivariate functors

$$\mathrm{Link}_{\mathcal{I}}(- *_P -) \cong \mathrm{Link}_{\mathcal{I}_0} \times \mathrm{Link}_{\mathcal{I}_1}.$$

We now want to extend this isomorphism to a natural isomorphism of functors on all of $\mathrm{s}\mathbf{Strat}_{\mathcal{I}_0} \times \mathrm{s}\mathbf{Strat}_{\mathcal{I}_1}$. To see this, via left Kan extension in both arguments, it suffices to show that $\mathrm{Link}_{\mathcal{I}}(- *_P -)$ is cocontinuous in both arguments. Note that $\mathcal{X} *_P -$ is only cocontinuous as a functor into the under-category, hence an additional argument is required. To this end, observe that colimits in the under category $(\mathrm{s}\mathbf{Strat}_P)_{\mathcal{X}/}$ of a diagram of arrows $i \mapsto (\mathcal{X} \xrightarrow{f_i} \mathcal{Y}_i)$ can be computed as the lower horizontal arrow in the following pushout

$$\begin{tikzcd} \varinjlim \mathcal{X} \arrow[r, "\varinjlim f_i"] \arrow[d] & \varinjlim \mathcal{Y}_i \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Z}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{5.1}$$

In particular, given a colimit of a diagram of stratified $i \mapsto \mathcal{Y}_i \in \mathrm{s}\mathbf{Strat}_{\mathcal{I}_1}$ and $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_{\mathcal{I}_0}$ there is a pushout square

$$\begin{tikzcd} \varinjlim \mathcal{X} \arrow[r, "\varinjlim f_i"] \arrow[d] & \varinjlim (\mathcal{X} *_P \mathcal{Y}_i) \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} *_P \varinjlim \mathcal{Y}_i. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{5.2}$$

in $\mathrm{s}\mathbf{Strat}_P$. If we apply the colimit preserving functor $\mathrm{Link}_{\mathcal{I}}$ to this square, we obtain a pushout square of simplicial sets

$$\begin{tikzcd} \varinjlim \mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \arrow[r, "\varinjlim f_i"] \arrow[d] & \varinjlim \mathrm{Link}_{\mathcal{I}}(\mathcal{X} *_P \mathcal{Y}_i) \arrow[d] \\ \mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \arrow[r] & \mathrm{Link}_{\mathcal{I}}(\mathcal{X} *_P \varinjlim \mathcal{Y}_i). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{5.3}$$

Observe that since $\mathcal{I}_1$ is non-empty, it follows that $\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) = \varnothing$. Hence, the left hand vertical in the last pushout square is an isomorphism, showing that the right hand vertical is also an isomorphism. This shows that $\mathrm{Link}_{\mathcal{I}}(- *_P -)$ preserves colimits in the right argument. The case of the left argument follows by symmetry. ☐

Example 5.2.2.7. We may use the stratified join to compute the links of stratified horns. Let $\mathcal{I}$ be a regular flag, $\mathcal{J} = [p_0 \leq \cdots \leq p_l]$ be some arbitrary flag of P and $k \in [l]$. Furthermore, denote by $\overline{\mathcal{J}}$ the unique regular flag from which $\mathcal{J}$ degenerates. Then, the horn inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$ has the following image under $\mathrm{Link}_{\mathcal{I}}$:

1. If $\mathcal{I}$ is not a subflag of $\overline{\mathcal{J}}$, it is immediate from the definition of $\mathrm{Link}_{\mathcal{I}}$ that

$$\mathrm{Link}_{\mathcal{I}}\Lambda^{\mathcal{J}}_k = \varnothing = \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}}.$$

2. If $\mathcal{I} = \overline{\mathcal{J}} \setminus \{p_k\}$ and $\mathcal{J}_{p_k}$ has length 0, then $\Lambda^{\mathcal{J}}_k \cong \partial\Delta^{\mathcal{J}_{\mathcal{I}}} *_P \Delta^{[p_k]}$ and it follows that the image is given by the inclusion
$$\mathrm{Link}_{\mathcal{I}}\Lambda^{\mathcal{J}}_k = \mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k)_{\mathcal{I}} = \mathrm{Link}_{\mathcal{I}}\partial\Delta^{\mathcal{J}_{\mathcal{I}}} \subset \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}_{\mathcal{I}}} = \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}}.$$
$\mathrm{Link}_{\mathcal{I}}(\Lambda^{\mathcal{J}}_k)_{\mathcal{I}}$ is precisely given by the boundary of the polygon $\mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}} = \prod_{p_i \in \mathcal{I}} \Delta^{\mathcal{J}_{p_i}}$.

3. If $\mathcal{I} \subsetneq \overline{\mathcal{J}}$, and furthermore $\mathcal{I} \neq \overline{\mathcal{J}} \setminus \{p_k\}$ or the length of $\mathcal{J}_{p_k}$ is not 0, then we obtain the identity
$$\mathrm{Link}_{\mathcal{I}}\Lambda^{\mathcal{J}}_k = \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}_{\mathcal{I}}} = \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}}.$$

4. If $\mathcal{I} = \overline{\mathcal{J}}$, then we may represent $\Lambda^{\mathcal{J}}_k$ as a join as follows. Denote $\mathcal{I}_0 = \mathcal{I} \setminus \{p_k\}$, $\mathcal{I}_1 = \{p_k\}$, and by $\mathcal{J}_0$ the restriction of $\mathcal{J}$ to $\mathcal{I}_0$. Let k_0 be minimal with the property that $p_{k_0} = p_k$. Denote by F_k the $(k-k_0)$-th face of $\Delta^{\mathcal{J}_{p_k}}$. Then
$$\Lambda^{\mathcal{J}}_k = (\Delta^{\mathcal{J}_0} *_P F_k) \cup_{\partial\Delta^{\mathcal{J}_0} *_P F_k} (\partial\Delta^{\mathcal{J}_0} *_P \Delta^{\mathcal{J}_{p_k}}).$$
Therefore, if we apply $\mathrm{Link}_{\mathcal{I}}$ and use the interaction with stratified joins, we obtain
$$\begin{aligned}\mathrm{Link}_{\mathcal{I}}\Lambda^{\mathcal{J}}_k &= (\mathrm{Link}_{\mathcal{I}_0}\Delta^{\mathcal{J}_0} \times F_k) \cup_{\mathrm{Link}_{\mathcal{I}_0}(\partial\Delta^{\mathcal{J}_0}) \times F_k} (\mathrm{Link}_{\mathcal{I}_0}(\partial\Delta^{\mathcal{J}_0}) \times \Delta^{\mathcal{J}_{p_k}}) \\ &\subset \mathrm{Link}_{\mathcal{I}_0}\Delta^{\mathcal{J}_0} \times \Delta^{\mathcal{J}_{p_k}} = \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}}.\end{aligned}$$

As a consequence of our computations in Example 5.2.2.7, we obtain the following corollary, characterizing admissible horn inclusions:

Corollary 5.2.2.8. *A stratified horn inclusion $j\colon \Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$ is admissible if and only if $\mathrm{Link}_{\mathcal{I}} j$ is a weak homotopy equivalence for all regular flags $\mathcal{I}$.*

Proof. Example 5.2.2.7 covers all possible examples of combinations of $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{J}$ and k be such that j is admissible. Then, in the first and third cases, the induced map $\mathrm{Link}_{\mathcal{I}} j$ is an isomorphism. The second case cannot occur, as it is assumed that $\mathcal{J}_{p_k}$ has a length greater than or equal to 1, by the definition of admissibility. Therefore, the only remaining case is the fourth. Note that $F_k \hookrightarrow \Delta^{\mathcal{J}_{p_k}}$ is an acyclic cofibration in the Quillen model structure. Hence, in the fourth case it follows from the description of $\mathrm{Link}_{\mathcal{I}} j\colon \mathrm{Link}_{\mathcal{I}}\Lambda^{\mathcal{J}}_k \hookrightarrow \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}}$ in Example 5.2.2.7 that $\mathrm{Link}_{\mathcal{I}} j$ is given by the box product of a cofibration and an acyclic cofibration, and hence is also an acyclic cofibration of simplicial sets. Conversely, suppose that $\mathrm{Link}_{\mathcal{I}} j$ is an acyclic cofibration for all $\mathcal{I}$. Then, in particular, $\mathcal{J}_{p_k}$ cannot have length 0, as this would imply that for $\mathcal{I} = \overline{\mathcal{J}} \setminus \{p_k\}$ the second case of Example 5.2.2.7 applies. In this case, $\mathrm{Link}_{\mathcal{I}}$ is given by the boundary inclusion of a polygon, which is not a weak homotopy equivalence. ☐

Proposition 5.2.2.9. *The functor*
$$\mathrm{Link}\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$$
is the left part of a Quillen adjunction between $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$ *and* $\mathbf{Diag}^{\mathrm{inj}}_P$.

Proof. That Link admits a right adjoint is immediate from its construction via Kan extension on a category of presheaves. Furthermore, one may easily see that, for any regular flag $\mathcal{I}$, $\mathrm{Link}_{\mathcal{I}}$ sends monomorphisms to pointwise monomorphisms, and hence preserves all cofibrations. A generating set of acyclic cofibrations in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$ is given by the admissible horn inclusions ([Dou21a, Thm. 2.14]). Therefore, we only need to show that, for any regular flag $\mathcal{I}$ and any admissible horn inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}$, the induced simplicial map
$$\mathrm{Link}_{\mathcal{I}}\Lambda^{\mathcal{J}}_k \hookrightarrow \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}}$$
is a weak homotopy equivalence. This is the content of Corollary 5.2.2.8. ☐

Let us now compare $\mathrm{Link}_{\mathcal{I}}$ with the simplicial homotopy link.

Construction 5.2.2.10. A natural transformation

$$\mathrm{Link} \to \mathrm{HoLink}$$

is constructed as follows: Given a flag $\mathcal{J}$ of length m of P, which contains a regular flag $\mathcal{I} = [p_0 < \cdots < p_n]$, a k-simplex τ of $\mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}})$ is given by an $n+1$-tuple $(\tau_0, \tau_1, \cdots, \tau_n)$, with $\tau_i \colon \Delta^k \to \Delta^{\mathcal{J}_{p_i}}$. Under the inclusions $\Delta^{\mathcal{J}_{p_i}} \hookrightarrow \Delta^{\mathcal{J}}$, we may equivalently interpret these data as a $[m]$ valued matrix $(i_{lj})_{l \in [n], j \in [k]}$, with the properties:

- $q_{i_{lj}} = p_j$ for all $l \in [n], j \in [k]$;
- $i_{lj} \leq i_{l(j+1)}$, for all $l \in [n], j \in [k-1]$.

As a consequence of the first property, any such matrix also fulfills

- $i_{lj} < i_{l+1j}$ for all $l \in [n-1]$, $j \in [k]$.

Together, the second and the third property imply that

- $i_{lj} \leq i_{l'j'}$, for $l \leq l' \in [n]$ and $j \leq j' \in [k]$.

Equivalently, such a matrix is precisely the data of a stratum-preserving simplicial map

$$\hat{\tau} \colon \Delta^{\mathcal{I}} \times \Delta^k \to \Delta^{\mathcal{J}}$$

given by uniquely extending the map of vertices

$$(l, j) \mapsto (i_{lj}).$$

One may easily check that this construction is compatible with face and degeneracy maps. Thus, we obtain an induced isomorphism of simplicial sets

$$\begin{aligned} \mathrm{Link}_{\mathcal{I}}(\Delta^{\mathcal{J}}) &\to \mathrm{HoLink}_{\mathcal{I}}(\Delta^{\mathcal{J}}) \\ \tau &\mapsto \hat{\tau} \end{aligned}$$

natural in $\mathcal{J}$ and $\mathcal{I}$ (when $\mathcal{I}$ is not a subflag of $\mathcal{J}$, both simplicial sets are empty by definition). Therefore, again by left Kan extension, we obtain a natural transformation

$$\mathrm{Link} \to \mathrm{HoLink}.$$

Proposition 5.2.2.11. *The natural transformation $\tau \colon \mathrm{Link} \to \mathrm{HoLink}$ is given by weak equivalences in $\mathbf{Diag}_P^{\mathrm{inj}}$.*

We are going to give a purely abstract proof here. Before we do so, let us, however, give a geometrical intuition for why the statement holds.

Example 5.2.2.12. Suppose $P = \{p < q\}$ is a poset with two strata. For a stratified simplex $\Delta^{\mathcal{J}}$, the image of $\Delta^{\mathcal{J}}$ under Link is the diagram

$$D = \{\Delta^{\mathcal{J}}_p \leftarrow \Delta^{\mathcal{J}}_p \times \Delta^{\mathcal{J}}_q \to \Delta^{\mathcal{J}}_q\}.$$

If we apply $\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -_{\mathcal{I}}$ to this diagram, we obtain the quotient of the stratified simplicial set

$$\Delta^{\mathcal{J}}_p \times \Delta^{\mathcal{J}}_q \times \Delta^{[p<q]}$$

obtained by collapsing $\Delta^{\mathcal{J}}_p \times \Delta^{\mathcal{J}}_q$ to $\Delta^{\mathcal{J}}_p$ and $\Delta^{\mathcal{J}}_q$, respectively, at the ends of the interval $\Delta^{[p<q]}$. Note that this construction is just a stratified version of Joyal's alternative join (see, for example, [Cis19, p. 4.2.1]). We obtain a natural comparison map

$$\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes D_{\mathcal{I}} \to \Delta^{\mathcal{J}}_p *_P \Delta^{\mathcal{J}}_q = \Delta^{\mathcal{J}}.$$

This comparison is natural in $\mathcal{J}$, and we thus obtain a natural transformation

$$\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes \mathrm{Link}_{\mathcal{I}}(-) \to 1_{\mathrm{s}\mathbf{Strat}_P}.$$

This map is not an isomorphism. However, it is stratified homotopic to a stratified homeomorphism after passing to the topological stratified world. In this sense, Link can be thought of as an actual (left) inverse to $\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -_{\mathcal{I}}$ up to passing from combinatorics to topology. We may just think of this as the statement that a piecewise linear space may be decomposed into a double mapping cylinder along the boundary of some regular neighborhood.

Proof of Proposition 5.2.2.11. As a consequence of Corollary 5.2.2.2, $\mathrm{HoLink}_{\mathcal{I}}$ preserves homotopy colimits. Since $\mathrm{Link}_{\mathcal{I}}$ is the left part of a Quillen adjunction (with source a cofibrant model category), the same holds for $\mathrm{Link}_{\mathcal{I}}$. As every stratified simplicial set is the homotopy colimit of its stratified simplices (Remark 5.2.1.5), it hence suffices to show that τ is a weak equivalence on the latter. However, on stratified simplices, τ is even an isomorphism of simplicial sets. □

The fact that, up to weak equivalence, this makes $\mathrm{HoLink}_{\mathcal{I}}$ both the left part and the right part of a Quillen equivalence turns out to be quite useful in practice. Let us illustrate this by providing some model structures for décollages, as defined in [Hai23]. In particular, this gives an example of how results on abstract stratified homotopy types can be deduced from a deeper understanding of the Douteau-Henriques model structure.

Recollection 5.2.2.13. A diagram $D \in \mathbf{Diag}_P$ is called a *décollage* over **Pos**, if for every regular flag $\mathcal{I} = [p_0 < \dots < p_n]$ in $\mathrm{sd}(P)$ the induced simplicial map from $D_{\mathcal{I}}$ into the homotopy limit of

$$D_{p_0} \leftarrow D_{[p_0,p_1]} \to \dots \leftarrow D_{[p_{n-1},p_n]} \to D_{p_n}$$

is a weak homotopy equivalence. In [Hai23, Thm. 1.1.7] the author shows that the homotopy link construction induces an equivalence of ∞-categories between abstract stratified homotopy types and décollages (using a homotopy coherent model of décollages).

Let us construct a model structure presenting the ∞-category of décollages. We will need the following observation.

Observation 5.2.2.14. Observe that, for a subcomplex $K \subset \mathrm{N}(P)$, the associated simplicial homotopy link diagram $\mathrm{HoLink}(K) \in \mathbf{Diag}_P$ is given by $\emptyset$, at $\mathcal{I}$ with $\Delta^{\mathcal{I}} \not\subset K$ and by the terminal simplicial set Δ^0 otherwise. Consequently, for any simplicial set S, a morphism $\mathrm{HoLink}(K) \otimes \Delta^n \to D$ specifies the same data as a morphism from the constant simplicial presheaf on $\mathrm{sd}(K)^{\mathrm{op}} \subset \mathrm{sd}(P)^{\mathrm{op}}$ with value Δ^n into $D|_{\mathrm{sd}(K)^{\mathrm{op}}}$. It follows that, for $D \in \mathbf{Diag}_P$, there is a canonical isomorphism

$$\underline{\mathbf{Diag}}_P(\mathrm{HoLink}(K), D) \cong \varprojlim_{\mathcal{I} \in \mathrm{sd}(K)^{\mathrm{op}}} D_{\mathcal{I}},$$

where $\mathrm{sd}(K)$ denotes the subcategory of $\mathrm{sd}(P)$ given by the simplices of K.

Notation 5.2.2.15. Given $K \subset \mathrm{N}(P)$, and $S \in \mathbf{sSet}$, we denote

$$K \otimes_D S \coloneqq \mathrm{HoLink}(K) \otimes S \in \mathbf{Diag}_P.$$

This construction defines a functor from the product of the category of subobjects of $\mathrm{N}(P)$ with the category **sSet** into $\mathbf{Diag}_P$.

Observation 5.2.2.16. By Observation 5.2.2.14 and the simplicial adjunction $\mathrm{HoLink}(K) \otimes - \dashv \underline{\mathbf{Diag}}_P(\mathrm{HoLink}(K), -)$, it follows that morphisms

$$K \otimes_D S \to D$$

are in natural bijection with arrows

$$S \to \varprojlim_{\mathcal{I} \in \mathrm{sd}(K)^{\mathrm{op}}} D_{\mathcal{I}}.$$

In the special case where $K = \Delta^{\mathcal{I}}$, the category $\mathrm{sd}(K)$ has the terminal object $\Delta^{\mathcal{I}}$, and we obtain a canonical isomorphism

$$\varprojlim_{\mathcal{I}' \in \mathrm{sd}(K)^{\mathrm{op}}} D_{\mathcal{I}'} = D_{\mathcal{I}}.$$

Notation 5.2.2.17. Given a flag $\mathcal{J} = [p_0 \leq \cdots \leq p_n] \in \Delta_P$, we denote by $\mathrm{Sp}(\mathcal{J}) \subset \Delta^{\mathcal{J}}$ the stratified subsimplicial set whose underlying simplicial set is the spine of Δ^n, i.e. the union of all 1-simplices of the form $\Delta^{\{k,k+1\}}$, for $0 \leq k \leq n-1$.

Observe that the diagrams that one takes a homotopy limit over in Recollection 5.2.2.13 are precisely the restriction of D to $\mathrm{sd}(\mathrm{Sp}(\mathcal{J}))^{\mathrm{op}}$.

Construction 5.2.2.18. As $\underline{\mathrm{sSet}}$ (with the Kan-Quillen model structure) is a left proper, combinatorial, simplicial model category, so is $\underline{\mathbf{Diag}}_P^{\mathrm{inj}}$ ([Lur09, Rem. 2.8.4, A.3.3.2]). By [Bar10, Thm. 4.7], we can therefore localize $\underline{\mathbf{Diag}}_P^{\mathrm{inj}}$ with respect to either of the following sets of morphisms:

$$\{\Lambda_k^{\mathcal{I}} \otimes_D \Delta^0 \hookrightarrow \Delta^{\mathcal{I}} \otimes_D \Delta^0 \mid \mathcal{I} \in \mathrm{sd}(P), \Lambda_k^{\mathcal{I}} \hookrightarrow \Delta^{\mathcal{I}} \text{ is inner}\};$$
$$\{\mathrm{Sp}(\mathcal{I}) \otimes_D \Delta^0 \hookrightarrow \Delta^{\mathcal{I}} \otimes_D \Delta^0 \mid \mathcal{I} \in \mathrm{sd}(P)\}.$$

It turns out that these two localizers result in the same left Bousfield localization (see the proof below). An injectively fibrant diagram D is then local with respect to these inclusions, if and only if the induced maps

$$D_{\mathcal{I}} \cong \underline{\mathbf{Diag}}_P(\Delta^{\mathcal{I}} \otimes_D \Delta^0, D) \to \underline{\mathbf{Diag}}_P(\mathrm{Sp}(\mathcal{I}) \otimes_D \Delta^0, D) \cong \varprojlim_{\mathcal{I}' \in \mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}}} D_{\mathcal{I}'},$$

for $\mathcal{I} \in \mathrm{sd}(P)$, are weak equivalences. The resulting simplicial model category is called the model category of décollages and denoted $\underline{\mathbf{Diag}}_P^{\mathrm{dé}}$.

Let us show that these two localizers do indeed produce the same localizations:

Proof. We denote the first localizer by L_0 and the second by L_1. It suffices to see that each of the two localizers is contained, respectively, in the set of acyclic cofibrations generated by the other. To this end, observe that within the class of cofibrations, acyclic cofibrations in a model-category are closed under the operations

1. pushouts along monomorphisms[2];
2. right cancellation;
3. composition.

Hence, it suffices to see that each element of L_0 is generated under these operations by the elements of L_1, and vice versa. Next, observe that the functor $K \mapsto K \otimes_D \Delta^0$ (from the category of subobjects of $\mathrm{N}(P)$) maps such squares that define pushouts in $\mathrm{s}\mathbf{Strat}_P$ into pushouts. Hence, it suffices to see that the class of inner horn inclusions and spine inclusions of subobjects of $\mathrm{N}(P)$ generate the same class under the three operations

1. pushouts in $\mathrm{s}\mathbf{Strat}_P$ along arrows in the category of subobjects of $\mathrm{N}(P)$;
2. right cancellation;

[2] Of course, they are also closed under more general pushouts, but this will suffice here.

3. composition.

Indeed, any spine inclusion $\mathrm{Sp}(\mathcal{I}) \hookrightarrow \Delta^{\mathcal{I}}$ can be written as a composition of pushouts of inner horn inclusions, along inclusions (see, for example, the proof of [Lan21, Prop. 1.3.22]). For the converse inclusion, consider the proof of [JT07, Lem. 3.5]. □

Let us verify that the bifibrant objects of $\underline{\mathbf{Diag}}_P^{\text{dé}}$ are indeed precisely such injectively fibrant diagrams that fulfill the décollage condition.

Proposition 5.2.2.19. *A bifibrant object $D \in \underline{\mathbf{Diag}}_P^{\mathrm{inj}}$ is a décollage if and only if it is a bifibrant object in $\underline{\mathbf{Diag}}_P^{\text{dé}}$.*

Proof. Observe that all objects in $\underline{\mathbf{Diag}}_P^{\mathrm{inj}}$ are cofibrant, and thus that bifibrancy is equivalent to fibrancy. Under both conditions D is a fibrant object in $\underline{\mathbf{Diag}}_P^{\mathrm{inj}}$. By definition, D is fibrant in $\underline{\mathbf{Diag}}_P^{\text{dé}}$ if and only if

$$D_{\mathcal{I}} \to \varprojlim_{\mathcal{I}' \in \mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}}} D_{\mathcal{I}'}$$

is a weak equivalence, for each $\mathcal{I} \in \mathrm{sd}(P)$. To show that this is equivalent to being a décollage, it suffices to show that the right-hand expression computes the homotopy limit of the restriction of D to $\mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}}$. Observe that

$$E \mapsto \varprojlim_{\mathcal{I}' \in \mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}}} E_{\mathcal{I}} = \underline{\mathbf{Diag}}_P(\mathrm{Sp}(\mathcal{I}) \otimes_D \Delta^0, E)$$

defines a right Quillen functor (since $\underline{\mathbf{Diag}}_P^{\mathrm{inj}}$ is a cofibrant simplicial model category). Let us denote this functor by F. Equivalently, we may write F as the composition of the right Quillen functor

$$\varprojlim \colon \mathbf{Fun}(\mathrm{sd}(\mathrm{Sp})(\mathcal{I})^{\mathrm{op}}, \mathbf{sSet}) \to \mathbf{sSet}$$

with the restriction functor along

$$j \colon \mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}} \to \mathrm{sd}(P)^{\mathrm{op}},$$

denoted j^*. That is, we have $F = \varprojlim \circ j^*$. Observe that j^* is also a right Quillen functor. To see this, we may treat $\mathrm{sd}(P)^{\mathrm{op}}$ as a Reedy category, with all morphisms being degree decreasing and apply [Bar07, Thm 2.7], from which the claim follows. In the following, given a right Quillen functor G, we denote by RG its right derived functor. As D was assumed to be fibrant, it follows that $\varprojlim_{\mathcal{I}' \in \mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}}} D_{\mathcal{I}'}$ computes the right derived functor of F. Hence, we have

$$\varprojlim_{\mathcal{I}' \in \mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}}} D_{\mathcal{I}'} = R(\varprojlim \circ j^*)(D) = (R\varprojlim) \circ (Rj^*)(D) = (R\varprojlim) j^* D \simeq \mathrm{ho}\varprojlim(j^* D),$$

and we have shown that $\varprojlim_{\mathcal{I}' \in \mathrm{sd}(\mathrm{Sp}(\mathcal{I}))^{\mathrm{op}}} D_{\mathcal{I}'}$ computes precisely the homotopy limit in the defining property of a décollage. □

Theorem 5.2.2.20. *The adjunction*

$$\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -_{\mathcal{I}} \colon \underline{\mathbf{Diag}}_P \rightleftharpoons \underline{\mathbf{sStrat}}_P \colon \mathrm{HoLink}$$

defines a simplicial Quillen equivalence between $\underline{\mathbf{sStrat}}_P^{\mathfrak{c}}$ and $\underline{\mathbf{Diag}}_P^{\text{dé}}$, creating weak equivalences in both directions.

Proof. We are first going to show that $\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -_{\mathcal{I}}$ sends the localizer defining $\underline{\mathbf{Diag}}_P^{\text{dé}}$ to weak equivalences in $\underline{\mathbf{sStrat}}_P^{\mathfrak{c}}$. It then follows by the universal property of Bousfield localization that the adjunction in the statement of the theorem is a Quillen adjunction. We then show that HoLink also preserves all weak equivalences. Since the ordinary unit of adjunction is given by weak equivalences, it follows from $\mathrm{HoLink}_{\mathcal{I}}$ preserving weak equivalences that the derived unit is also a weak equivalence. Consequently, the induced Quillen adjunction is a Quillen equivalence, with (ordinary) unit and counit given by weak equivalences (Remark 5.2.2.3). To see the statement about $\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes -_{\mathcal{I}}$, note that for $K \subset N(P)$ a subcomplex, we have

$$\int^{\mathcal{I}} \Delta^{\mathcal{I}} \otimes (K \otimes_D \Delta^0)_{\mathcal{I}} = K.$$

Hence, the elements of the localizer defining $\underline{\mathbf{Diag}}_P^{\text{dé}}$ are mapped to the stratified inner horn inclusions

$$\Lambda_k^{\mathcal{I}} \hookrightarrow \Delta^{\mathcal{I}},$$

which are acyclic cofibrations by definition of the model structure on $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{c}}$. To show the statement about $\mathrm{HoLink}_{\mathcal{I}}$, observe that by Proposition 5.2.2.11 we may equivalently show that Link preserves weak equivalences. As every object in $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{c}}$ is cofibrant, this follows if we can show that Link defines a left Quillen functor with respect to the localizations. Again, by the universal property of the left Bousfield localization, it suffices to show that, for $\Lambda_k^{\mathcal{J}} \to \Delta^{\mathcal{I}}$ an inner horn inclusion ($\mathcal{J} = [q_0 \leq \cdots \leq q_n]$) that is not also admissible, the induced morphism

$$\mathrm{Link}(\Lambda_k^{\mathcal{J}}) \hookrightarrow \mathrm{Link}(\Delta^{\mathcal{J}})$$

is a weak equivalence. Let $\overline{\mathcal{J}}$ be the unique non-degenerate flag which $\mathcal{J}$ degenerates from. Denote $\mathcal{J}_0 := \mathcal{J} \setminus \{p_k\}$ and $\overline{\mathcal{J}}_0 = \overline{\mathcal{J}} \setminus \{p_k\}$. Since $\Lambda_k^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}$ is not admissible, we have that $\mathcal{J}_{p_k}$ has length 0. If we apply Example 5.2.2.7, we obtain the following computations of $\mathrm{Link}(\Lambda_k^{\mathcal{J}}) \hookrightarrow \mathrm{Link}(\Delta^{\mathcal{J}})$ at $\mathcal{I} \in \mathrm{sd}(P)$:

(i) If $\mathcal{I}$ is not a subflag of $\overline{\mathcal{J}}$:

$$\varnothing \hookrightarrow \varnothing;$$

(ii) If $\mathcal{I} \subset \overline{\mathcal{J}}$, $\mathcal{I} \neq \overline{\mathcal{J}}$ and $\mathcal{I} \neq \overline{\mathcal{J}}_0$:

$$\mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}} \to \mathrm{Link}_{\mathcal{I}}\Delta^{\mathcal{J}};$$

(iii) If $\mathcal{I} = \overline{\mathcal{J}}, \overline{\mathcal{J}}_0$:

$$\mathrm{Link}_{\overline{\mathcal{J}}_0}(\partial\Delta^{\mathcal{J}_0}) \hookrightarrow \mathrm{Link}_{\overline{\mathcal{J}}_0}(\Delta^{\mathcal{J}_0}).$$

Let us denote the inclusion of Description (iii) by $S \hookrightarrow D$. Consider the canonical morphisms (adjoint to the identities on S and D)

$$\Delta^{\overline{\mathcal{J}}} \otimes_D S \to \mathrm{Link}(\Lambda_k^{\mathcal{J}});$$
$$\Delta^{\overline{\mathcal{J}}} \otimes_D D \to \mathrm{Link}(\Delta^{\mathcal{J}}).$$

These morphisms induce a commutative diagram

$$\begin{tikzcd} \Delta^{\overline{\mathcal{J}}} \otimes_D S \cup_{\Lambda_l^{\overline{\mathcal{J}}} \otimes_D S} \Lambda_l^{\overline{\mathcal{J}}} \otimes_D D \arrow[r] \arrow[d] & \mathrm{Link}(\Lambda_k^{\mathcal{J}}) \arrow[d] \\ \Delta^{\overline{\mathcal{J}}} \otimes_D D \arrow[r] & \mathrm{Link}(\Delta^{\mathcal{J}}), \end{tikzcd} \tag{5.4}$$

where l is uniquely determined by $\overline{\mathcal{J}} = [q_0 < \cdots < q_m]$ fulfilling, $q_l = p_k$. We claim that this diagram is pushout. Proving this finishes the proof, since the left vertical is given by a box product of a localizer defining $\underline{\mathbf{Diag}}_P^{\text{dé}}$ (namely $\Lambda_l^{\overline{\mathcal{J}}} \otimes_D \Delta^0 \to \Delta^{\overline{\mathcal{J}}} \otimes_D \Delta^0$) with a cofibration of simplicial sets (namely $S \hookrightarrow D$). Let us verify the cocartesianity of this diagram at each $\mathcal{I} \in \mathrm{sd}(P)$. If $\mathcal{I}$ is not a subflag of $\mathcal{J}$, then Diagram (5.4) is empty by Description (i). For $\mathcal{I} \subset \overline{\mathcal{J}}$, $\mathcal{I} \neq \overline{\mathcal{J}}, \overline{\mathcal{J}}_0$, by Description (ii), both verticals are isomorphisms, which makes the diagram cocartesian. Finally, by Description (iii), if $\mathcal{I} = \overline{\mathcal{J}}, \overline{\mathcal{J}_0}$, then both horizontals are isomorphisms. □

5.3 Combinatorial models over varying posets

In this section, we define global analogues of the Douteau-Henriques and the Joyal-Kan model structures described in the previous section.

5.3.1 From local to global model structures

Now, let us piece the model structures on $\underline{\mathrm{s}\mathbf{Strat}}_P$, where P varies over all posets, together to model structures on $\underline{\mathrm{s}\mathbf{Strat}}$. To do so, we make use of the following general principle, which is the special case of the characterization of bifibrations over a model category in [CM20], where the base category carries the trivial model structure (we use the notation of [CM20]). This approach was first used in [Dou21c].

Lemma 5.3.1.1 ([CM20, Thm. 4.4])**.** *Suppose that we are given a Grothendieck bifibration $P:\mathbf{M} \to \mathbf{B}$. Suppose further that , for every $A \in \mathbf{B}$, the fiber $\mathbf{M}_A$ is equipped with the structure of a model category and that, for every morphism $u : A \to B$ in $\mathbf{B}$, the induced functor $u_!:\mathbf{M}_A \to \mathbf{M}_B$ is a left Quillen functor. Then $\mathbf{M}$ carries the structure of a model category with the following defining classes. Let $f : X \to Y$ be a morphism in $\mathbf{M}$:*

1. *f is a weak equivalence, if and only if $P(f)$ is an isomorphism and $f^{\triangleleft}$ is a weak equivalence in $\mathbf{M}_{P(Y)}$ (or equivalently $f_{\triangleright}$ is a weak equivalence in $\mathbf{M}_{P(X)}$).*
2. *f is a cofibration, if and only if $f_{\triangleright}$ is a cofibration in $\mathbf{M}_{P(Y)}$.*
3. *f is a fibration, if and only if $f^{\triangleleft}$ is a fibration in $\mathbf{M}_{P(X)}$.*

Furthermore, assume that $\mathbf{M}$ is a simplicial category and P a simplicial functor (with respect to the discrete structure on $\mathbf{B}$) such that $u_! \dashv u^*$ is a simplicial adjunction, for all $u \in \mathbf{B}$. If for each $A \in \mathcal{B}$, the category $\mathbf{M}_A$ is a simplicial model category with respect to the simplicial structure inherited from $\mathbf{M}$, then so is $\mathbf{M}$.

To apply Lemma 5.3.1.1 to glue the fiberwise model structures on $\underline{\mathrm{s}\mathbf{Strat}}_P$, we need the following lemma.

Proposition 5.3.1.2. *For any morphism of posets $u:P \to P'$, the induced adjunction*

$$u_!:\underline{\mathrm{s}\mathbf{Strat}}_P \rightleftharpoons \underline{\mathrm{s}\mathbf{Strat}}_{P'}:u^*$$

- given by postcomposition and pulling back along u - is a simplicial Quillen adjunction, with respect to the Douteau-Henriques and the Joyal-Kan model structures (taken the same on both sides, respectively). Furthermore, again in both scenarios, $u_!$ reflects fibrations and creates acyclic fibrations.

Proof. Simpliciality is immediate by definition. Clearly, $u_!$ preserves all cofibrations. Furthermore, $u_!$ preserves admissible horn inclusions, which shows the case of the Douteau-Henriques model structure, as the latter generate the acyclic cofibrations. For the case of the Joyal-Kan model structure, by Proposition 5.2.1.3, it suffices to show that $u_!$ sends stratified inner horn inclusions to acyclic cofibrations. Clearly, the image of every stratified inner horn inclusion under $u_!$ remains an inner horn inclusion. This shows that $u_!$ is left Quillen. Now to see that $u_!$ reflects (acyclic) fibrations, note that for any lifting diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{5.5}$$

any dashed solution to

$$\begin{tikzcd} u_!\mathcal{A} \arrow[r] \arrow[d] & u_!\mathcal{X} \arrow[d] \\ u_!\mathcal{B} \arrow[r] \arrow[ur, dashed] & u_!\mathcal{Y} \end{tikzcd} \tag{5.6}$$

already provides a solution to Diagram (5.5). Indeed, commutativity at the level of simplicial sets already implies that $\mathcal{B} \to \mathcal{X}$ is stratum-preserving, as all these diagrams can be considered in the slice category over $\mathcal{Y}$, which is independent from the stratifications. Hence, the reflection property follows from $u_!$ being left Quillen. That $u_!$ preserves acyclic fibrations follows similarly, if we note that every lifting diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, hook] & u_!\mathcal{X} \arrow[d] \\ \mathcal{B} \arrow[r] \arrow[ur, dashed] & u_!\mathcal{Y} \end{tikzcd} \tag{5.7}$$

lies in the image of $u_!$ and that $u_!$ creates cofibrations. □

Definition 5.3.1.3. We denote by $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ and $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$ the simplicial model categories with underlying category $\underline{\mathbf{sStrat}}$, defined by applying Lemma 5.3.1.1 to the forgetful functor

$$\underline{\mathbf{sStrat}} \to \mathbf{Pos},$$

with the fiberwise model structures given by $\underline{\mathbf{sStrat}}^{\mathfrak{d}}_P$ and $\underline{\mathbf{sStrat}}^{\mathfrak{c}}_P$, for $P \in \mathbf{Pos}$, respectively. The model structure on $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ is called the *Douteau-Henriques model structure* on $\underline{\mathbf{sStrat}}$. The model structure on $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$, is called the *Joyal-Kan model structure* on $\underline{\mathbf{sStrat}}$. Weak equivalences in these model categories are called *poset-preserving diagrammatic equivalences* and *poset-preserving Joyal-Kan equivalences*, respectively.

Let us begin our investigation of these model structures with the following observation:

Lemma 5.3.1.4. *Let $\mathcal{X} \in \mathbf{sStrat}$ and let $f{:}\, Q \to P_{\mathcal{X}}, g{:}\, P_{\mathcal{X}} \to Q' \in \mathbf{Pos}$. Then the induced natural map $f^*\mathcal{X} \to \mathcal{X}$ is a fibration and the natural map $\mathcal{X} \to g_!\mathcal{X}$ is a cofibration, in $\mathbf{sStrat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathbf{sStrat}^{\mathfrak{d},\mathfrak{p}}$.*

Proof. This is immediate from the simple observation that $f^{\triangleleft}$ and $g_{\triangleright}$ are both given by isomorphisms (the identity even). □

Proposition 5.3.1.5. *Weak equivalences in $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$ and $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ are stable under filtered colimits.*

Proof. Note that as every weak equivalence is given on posets by an isomorphism, and filtered diagrams lack monodromy, it follows that the colimit of all posets involved is canonically isomorphic to any of the posets in the filtered diagram, and we may easily reduce the statement to such diagrams of weak equivalences, which are given by the identity on the poset level. Now, the result follows from Proposition 5.2.1.6. □

Remark 5.3.1.6. Note that the acyclic fibrations in $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ and $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$ are precisely the stratified maps that induce an isomorphism on posets and an acyclic fibration (in the Joyal or Kan model structure) on simplicial sets. Indeed, this follows by applying Proposition 5.3.1.2 to $u{:}\, P \to [0]$.

We may then state the following global version of [Hai23, Cor. 2.5.11].

Recollection 5.3.1.7. Recall by [Lur09] that the quasi-category of all (small) quasi-categories $\mathcal{C}\mathbf{at}_\infty$ is given by the homotopy coherent nerve of the simplicial category $\underline{\mathbf{Cat}}_\infty$, whose objects are small quasi-categories X, and whose mapping spaces are given by the Kan complexes $\underline{\widetilde{\mathbf{sSet}}}(X,Y)^{\simeq}$, given by the maximal Kan complex in $\underline{\widetilde{\mathbf{sSet}}}(X,Y)$. The infinity category of *abstract stratified homotopy types* (see [Hai23]), denoted $\mathcal{AS}\mathbf{trat}$, is the full subcategory of the arrow quasi-category $\mathcal{C}\mathbf{at}_\infty^{\Delta^1}$ of conservative functors $F{:}\, X \to P$, where P is a poset. (The tilde over $\underline{\mathbf{sSet}}$ indicates that, in order to avoid set-theoretic issues, $\underline{\widetilde{\mathbf{sSet}}}$ is modeled on a larger Grothendieck universe than $\underline{\mathbf{sSet}}$.)

Proposition 5.3.1.8. $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$ *presents the ∞-category of abstract stratified homotopy types.*

Proof. A stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ is fibrant, if and only if it is fibrant as an element of $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$, with $P = P_{\mathcal{X}}$. It follows that the bifibrant objects of $\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ are precisely the abstract stratified homotopy types. Next, consider the simplicial functor category $\underline{\mathbf{Cat}}_\infty^{[1]}$, i.e. the simplicial category of arrows in $\underline{\mathbf{Cat}}_\infty$. Given two such fibrant objects $\mathcal{X}$ and $\mathcal{Y}$, $\underline{\mathbf{Cat}}_\infty^{[1]}(\mathcal{X},\mathcal{Y})[n]$ is given by the set of such morphisms

$$F\colon (X \times \Delta^n \to P_{\mathcal{X}} \times \Delta^n) \to (Y \to P_{\mathcal{Y}})$$

which fulfill

$$F_0(\{x\} \times \Delta^n) \subset Y^{\simeq} \text{ and } F_1(\{p\} \times \Delta^n) \subset P_{\mathcal{Y}}^{\simeq} \text{ for } x \in X, p \in P.$$

Note that since $s_{\mathcal{Y}}$ is a conservative functor, the condition that $F(\{x\} \times \Delta^n) \subset Y^{\simeq}$, for $x \in X$, is redundant. Furthermore, $P^{\simeq}$ is discrete (every isomorphism in a poset is the identity). Therefore, the condition $F_1(\{p\} \times \Delta^n) \subset P_{\mathcal{Y}}^{\simeq}$ is equivalent to saying that $F_1\colon P \times \Delta^n \to P_{\mathcal{Y}}$ is of the form $P \times \Delta^n \to P \xrightarrow{u} P_{\mathcal{Y}}$. Hence, $\underline{\mathbf{Cat}}_\infty^{[1]}(\mathcal{X},\mathcal{Y})[n]$ is equivalently the set of stratified maps

$$\mathcal{X} \otimes \Delta^n \to \mathcal{Y}$$

which is precisely $\underline{\mathrm{s}\mathbf{Strat}}(\mathcal{X},\mathcal{Y})[n]$. To summarize, we have shown that if we denote by $\underline{\mathrm{s}\mathbf{Strat}}^o$ the simplicial category of bifibrant objects in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$, then $\underline{\mathrm{s}\mathbf{Strat}}^o$ is even isomorphic to the full subcategory of $\underline{\mathbf{Cat}}_\infty^{[1]}$ given by conservative functors into a poset. Making use of this, we treat $\underline{\mathrm{s}\mathbf{Strat}}^o$ as a full subcategory of $\underline{\mathbf{Cat}}_\infty^{[1]}$. Denote by $(\underline{\mathbf{Cat}}_\infty^{[1]})^o$ the full subcategory of $\underline{\mathbf{Cat}}_\infty^{[1]}$ given by such functors $f\colon X \to Y$ that are an iso-fibration, i.e. the full simplicial subcategory of bifibrant objects in the injective model structure. $(\underline{\mathbf{Cat}}_\infty^{[1]})^o$ is a model for the category of arrows in $\mathcal{C}\mathbf{at}_\infty$ in terms of simplicial categories. More precisely, if we denote by $\mathcal{I}\mathbf{sofib}$ the full subcategory of $\mathcal{C}\mathbf{at}_\infty^{\Delta^1}$ of isofibrations, then there is a natural zig-zag of Joyal-equivalences

$$\mathcal{C}\mathbf{at}_\infty^{\Delta^1} \xhookleftarrow{\simeq} \mathcal{I}\mathbf{sofib} \xrightarrow{\simeq} \mathrm{N}\big((\underline{\mathbf{Cat}}_\infty^{[1]})^o\big).$$

The left-hand side equivalence follows from fibrant replacement in the Joyal model structure. The right-hand equivalence is induced by the natural transformations $\mathcal{S}(\Delta^n \times \Delta^1) \to \mathcal{S}(\Delta^n) \times [1]$, where $\mathcal{S}$ is the left-adjoint of the homotopy coherent nerve, and is a weak equivalence by [Lur09, A.3.4.13.] applied to the model structure of marked simplicial sets presenting $(\infty,1)$-categories. Note that every conservative functor from a quasi-category into a poset is necessarily an isofibration. Indeed, every functor with target the nerve of a 1-category is an inner fibration, and every isomorphism in a poset is the identity, which clearly admits a lift. It follows that $\mathcal{AS}\mathbf{trat} \subset \mathcal{I}\mathbf{sofib}$ as a full subcategory and that $\underline{\mathrm{s}\mathbf{Strat}}^o \subset (\underline{\mathbf{Cat}}_\infty^{[1]})^o$. We thus obtain a commutative square

$$\begin{tikzcd}
\mathcal{I}\mathbf{sofib} \arrow[r, "\simeq"] & \mathrm{N}\big((\underline{\mathbf{Cat}}_\infty^{[1]})^o\big) \\
\mathcal{AS}\mathbf{trat} \arrow[u, hook, "f.f."] \arrow[r, dashed] & \mathrm{N}(\underline{\mathrm{s}\mathbf{Strat}}^o)\,, \arrow[u, hook, "f.f."']
\end{tikzcd} \tag{5.8}$$

with the lower horizontal induced by the fact that the composition of the left vertical and right horizontal has image in $\mathrm{N}(\underline{\mathrm{s}\mathbf{Strat}}^o)$. This dashed functor even is a bijection on objects, as we have already noted in the beginning of this proof. Furthermore, by commutativity of the diagram, it is fully faithful. Hence, we have constructed an equivalence of quasi-categories $\mathcal{AS}\mathbf{trat} \simeq \mathrm{N}(\underline{\mathrm{s}\mathbf{Strat}}^o)$ as claimed. $\square$

Next, we gather some general properties of the model categories $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$. We begin with the following general lemma.

Lemma 5.3.1.9. *In the situation of Lemma 5.3.1.1, assume that $P: \mathbf{M} \to \mathbf{B}$ admits a left adjoint $L\colon \mathbf{B} \to \mathbf{M}$. Furthermore, let S be a set of morphisms in $\mathbf{B}$, such that a morphism u in $\mathbf{B}$ is an isomorphism if and only if it has the right lifting property with respect to S. Let I be a set of (acyclic) cofibrations in $\mathbf{M}$ such that:*

1. *Each $i \in I$ is contained in some fiber $\mathbf{M}_A$, for some $A \in \mathbf{B}$.*

2. *For each $A \in \mathbf{B}$, the set*
$$\{u_! i \mid i \in I, u\colon B \to A, B \in \mathbf{B}\}$$
is a set of generating (acyclic) cofibrations for $\mathbf{M}_A$.

Then $L(S) \cup I$ (I) is a set of generating (acyclic) cofibrations for $\mathbf{M}$.

Proof. We prove the case of cofibrations. Consider a morphism $f\colon X \to Y$ in $\mathcal{M}$. We need to show that f is an acyclic fibration (i.e. $P(f)$ is an isomorphism and $f^{\triangleleft}$ is an acyclic fibration in $\mathcal{M}_{P(X)}$), if and only if f has the right lifting property with respect to $L(S) \cup I$. Note that by the adjunction $L \dashv P$, the map $P(f)$ is an isomorphism, if and only if f has the right lifting property with respect to $L(S)$. Hence, in the following we may assume without loss of generality that $P(f) = 1_X$. Next, note that since P is a Grothendieck left fibration, any lifting problem

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "i"] & X \arrow[d, "f"] \\ X_1 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{5.9}$$

with $P(i)$ an identity is equivalent to a unique lifting problem

$$\begin{tikzcd} P(g)_! X_0 \arrow[r] \arrow[d, "P(g)_! i"'] & X \arrow[d, "f"] \\ P(g)_! X_1 \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd} \tag{5.10}$$

Hence, as $\{u_! i \mid i \in I, u\colon B \to A, B \in \mathcal{A}\}$ is a set of generating cofibrations for $\mathcal{M}_{P(X)}$, follows that $f^{\triangleleft}$ is an acyclic fibration, if and only if f has the right lifting property with respect to I. ☐

Corollary 5.3.1.10. *The model category $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ is cofibrantly generated. A generating set of cofibrations is given by the set of stratified boundary inclusions $\{\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]} \mid n \in \mathbb{N}\}$, together with the two morphisms*

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ \varnothing \arrow[r] & {[0]}, \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ {[0] \sqcup [0]} \arrow[r, hook] & {[1]}. \end{tikzcd} \tag{5.11}$$

A generating set of acyclic cofibrations for $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ is given by the set of admissible horn inclusions

$$\begin{tikzcd} \Lambda^n_k \arrow[rr] \arrow[dr] & & \Delta^n \arrow[dl] \\ & {[m]} & \end{tikzcd} \quad , \tag{5.12}$$

for $n, m \in \mathbb{N}$.

Proof. This is a consequence of Lemma 5.3.1.9. Note that $i\colon \mathcal{L}(\partial\Delta^1 \hookrightarrow \Delta^1) = (\partial\Delta^{[1]} \hookrightarrow \Delta^{[n]})$ while being a cofibration, is not contained in a fiber of $\underline{\mathbf{sStrat}} \to \mathbf{Pos}$. However, we may replace i by the stratified simplicial map obtained by pushing out along the stratified simplicial map

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "1"] \arrow[d, "s_{\mathcal{L}(\partial\Delta^1)}"] & \partial\Delta^1 \arrow[d, hook] \\ {[0] \sqcup [0]} \arrow[r, hook] & {[1]}. \end{tikzcd} \tag{5.13}$$

Denote by I the class of cofibrations in $\underline{\mathrm{s}\mathbf{Strat}}$ obtained in this manner. Now, let us verify the requirements of Lemma 5.3.1.9. First, note that the forgetful functor $\underline{\mathrm{s}\mathbf{Strat}} \to \mathbf{Pos}$ admits a left adjoint given by mapping P to $\varnothing \to P$. The two morphisms of posets $\varnothing \to [0]$ and $[0] \sqcup [0] \hookrightarrow [1]$ detect all isomorphisms of posets. Indeed, the former detects surjectivity. The latter detects surjectivity on relations. Any morphism of posets which is surjective on points and relations is necessarily an isomorphism. For any $P \in \mathbf{Pos}$, the stratified boundary inclusions over P, together with the admissible horn inclusions, form sets of (acyclic) cofibrant generators (see Recollection 5.2.1.1). Clearly, the elements of these sets are respectively of the form $u_!(i)$, for $i \in I$ or i an admissible horn inclusion as in the statement of the corollary, where u is an appropriate map of posets with target P. ☐

Next, we show that $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ is cofibrantly generated. However, since we lack an explicit set of acyclic generators for $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$, some additional work needs to be done to show that there is a set of acyclic generators for $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$. We are going to take a slight detour to see this. As a corollary of Proposition 5.2.1.3, we have:

Proposition 5.3.1.11. $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ *is the left Bousfield localization of* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ *at the class of inner stratified horn inclusions*

$$\Lambda^{[n]}_k \hookrightarrow \Delta^{[n]}$$

for $0 < k < n$.

We may then conclude:

Proposition 5.3.1.12. *The simplicial model categories* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ *and* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ *are cofibrant and combinatorial.*

Proof. Cofibrancy is obvious. It is not hard to see that $\underline{\mathrm{s}\mathbf{Strat}}$ is generated by the sources and targets of the generating cofibrations in Corollary 5.3.1.10, under filtered colimits. It follows that $\underline{\mathrm{s}\mathbf{Strat}}$ is finitely locally presentable. Since $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ is cofibrantly generated, we may hence conclude that $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ is combinatorial. As $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ is a left Bousfield localization of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ at a set of morphisms, it follows by [Bar10, Thm. 4.7] that $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ is also combinatorial. ☐

5.3.2 Model structures of refined stratified simplicial sets

For classical examples of stratified spaces the stratification poset is usually strongly related to the topology of the underlying space. In fact, originally, the poset structure arises from the closure containment relation on a partition of a space into disjoint subsets ([Mat12]). For general stratified simplicial sets, $\mathcal{X}$, the only relationship between the underlying object and P is that the existence of an edge $x \to y$ implies a relation $s_{\mathcal{X}}(x) \leq s_{\mathcal{X}}(y)$. This degree of generality is, of course, necessary when we are working over a fixed poset, at least if we want to have access to all stratified simplices over $\mathbf{Pos}$. If we allow for flexible posets, however, then this amount of generality has some peculiar side effects. In fact, we may take it to the extreme as follows:

Remark 5.3.2.1. Denote by $L\colon \mathbf{Pos} \to \mathrm{s}\mathbf{Strat}$ the left adjoint to the forgetful functor $\mathrm{s}\mathbf{Strat} \to \mathbf{Pos}$, given by $P \mapsto (\varnothing \to P)$. Clearly, L is fully faithful. If we equip $\mathbf{Pos}$ with the trivial model structure (in which weak equivalences are precisely the isomorphisms, and all maps are cofibrations and fibrations), then L becomes a left Quillen functor with target $\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ ($\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$). One may then verify that L induces a fully faithful embedding $\mathbf{Pos} \hookrightarrow \mathrm{hos}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ ($\mathrm{hos}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$). In other words, the homotopy category $\mathrm{hos}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ contains a complete copy of $\mathbf{Pos}$, consisting of empty stratified simplicial sets.

We may aim for a notion of stratified simplicial sets for which the poset structure is minimal in some sense, which at least should imply that maps are uniquely determined on the level of simplicial sets. To do so, let us first consider the following functor:

Construction 5.3.2.2. Consider the fully faithful inclusion

$$\mathbf{Pos} \hookrightarrow \mathbf{sSet}$$

given by taking the nerve of a poset. It admits a left adjoint, which we denote P, explicitly constructed by sending K to the poset generated from K_0 by adding the relation

$$x \leq y \iff \exists f\colon x \to y \text{ with } f \in \tau(X).$$

where $\tau(X)$ is the homotopy category of X. In particular, this means that whenever there are arrows $f\colon x \to y$, $g\colon y \to x$ in $\tau(X)$, then $x = y$ in $P(X)$.

Remark 5.3.2.3. One should be careful to note that there is a certain overload of notation here. Namely, there are two ways of associating to a stratified simplicial set $\mathcal{X}$ a poset. We may either associate to it the poset $P_{\mathcal{X}}$, or the poset $P(X)$. These two posets will generally be different, as is evident from the fact that $P(X)$ does not depend on the stratification of $\mathcal{X}$.

It follows immediately from the definition in Construction 5.3.2.2 that the construction factors through taking homotopy categories and one obtains:

Lemma 5.3.2.4. *Let $f\colon X \to Y$ in* **sSet** *be a categorical equivalence. Then $P(f)$ is an isomorphism.*

Remark 5.3.2.5. Now, to remove redundancies in the stratification poset, at first glance, one may try to invert the stratified maps

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d] & X \arrow[d, "s_{\mathcal{X}}"] \\ P(X) \arrow[r] & P_{\mathcal{X}}. \end{tikzcd} \tag{5.14}$$

This does, however, not lead to a meaningful homotopy theory of stratified spaces. Denote the functor $\mathcal{X} \mapsto (X \to P(X))$ by $(-)^{\mathrm{red}}$. Consider a stratified simplicial set $\mathcal{X}$, and consider X as a trivially stratified simplicial set. Then, if we invert $\mathcal{X}^{\mathrm{red}} \to \mathcal{X}$ and $X^{\mathrm{red}} \to X$, we obtain weak equivalences

$$X \simeq X^{\mathrm{red}} = \mathcal{X}^{\mathrm{red}} \simeq \mathcal{X},$$

in other words: We forget all stratifications and simply recover classical homotopy theory. Instead, we need to work with a derived version of the functor $\mathcal{X} \mapsto P(X)$, which remembers which paths are within a stratum and should be considered invertible.

Proposition 5.3.2.6. *Let $\mathcal{X}, \mathcal{Y} \in \underline{\mathbf{sStrat}}$ such that all strata of $\mathcal{X}$ and $\mathcal{Y}$ are Kan complexes. Then, for any weak equivalence $f\colon \mathcal{X} \to \mathcal{Y}$ in $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$ the induced morphism of posets $P(X) \to P(Y)$ is an isomorphism.*

Proof. Without loss of generality, we may assume that f is the identity on posets $P_{\mathcal{X}} = P_{\mathcal{Y}} = P$, i.e., that f is a weak equivalence in $\underline{\mathbf{sStrat}}^{\mathfrak{c}}_P$. $\underline{\mathbf{sStrat}}^{\mathfrak{c}}_P$ is equivalently constructed by localizing the model structure on the overcategory $\underline{\mathbf{sSet}}_{/P}$, coming from the Joyal model structure on $\underline{\mathbf{sSet}}$, at inclusions $\Delta^{[p \leq p]} \hookrightarrow \Delta^{[p \leq p \leq p]} \cup_{\Delta^{\{0,2\}}} \Delta^{[p]}$, for $p \in P$. Indeed, being local with respect to these inclusions precisely means that every morphism in the fibers is an isomorphism, that is, that $X \to P$ induces a conservative functor of infinity categories (after fibrantly replacing X). It follows that $\mathcal{X}$ and $\mathcal{Y}$ are local with respect to these inclusions. Hence, f is a Joyal-Kan equivalence in $\underline{\mathbf{sStrat}}_P$, if and only if the underlying simplicial map $X \to Y$ is a categorical equivalence ([nLa24i, Prop. 6.3]). Consequently, f induces an equivalence of homotopy categories $\tau(X) \to \tau(Y)$. It follows by construction of $P\colon \mathbf{sSet} \to \mathbf{Pos}$ that the induced morphism $P(f)$ is an isomorphism. □

In particular, the right derived functors of $P \circ \mathcal{F}\colon \underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}, \underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}} \to \mathbf{Pos}$ agree and may be computed by only replacing strata by Kan complexes.

Notation 5.3.2.7. We denote the right derived functor (with respect to $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$ or $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$) of the composition

$$\underline{\mathbf{sStrat}} \xrightarrow{\mathcal{F}} \underline{\mathbf{sSet}} \xrightarrow{P(-)} \mathbf{Pos}$$

by $P_{-^{\mathfrak{r}}}$. For $\mathcal{X} \in \underline{\mathbf{sStrat}}$, we call $P_{\mathcal{X}^{\mathfrak{r}}}$ the *refined poset* associated to $\mathcal{X}$.

As an immediate corollary of Proposition 5.3.2.6 we have:

Corollary 5.3.2.8. *Let $\mathcal{X} \in \underline{\mathbf{sStrat}}$ such that all strata of $\mathcal{X}$ are Kan complexes. Then, the canonical map*

$$P(X) \to P_{\mathcal{X}^{\mathfrak{r}}},$$

is an isomorphism.

We obtain the following explicit description of $P_{\mathcal{X}^{\mathfrak{r}}}$.

Proposition 5.3.2.9. *Let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$. Then the underlying set of $P_{\mathcal{X}^{\mathfrak{r}}}$ is the set of path components of non-empty strata of $\mathcal{X}$. Furthermore, for any two such components $[x]$ and $[y]$, for $x, y \in X$, there is a relation $x \leq y$, if and only if there is a path of 1-simplices*

$$x = x_0 \leftrightarrow x_1 \leftrightarrow x_2 \leftrightarrow \cdots \leftrightarrow x_n = y$$

where only simplices that are contained within a stratum of $\mathcal{X}$ are allowed to point in direction of x.

The following lemma follows from the explicit description Proposition 5.3.2.9.

Lemma 5.3.2.10. *The functor $P_{-^{\mathfrak{r}}}\colon \mathrm{s}\mathbf{Strat} \to \mathbf{Pos}$ preserves filtered colimits.*

Construction 5.3.2.11. For $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$ we denote by $\mathcal{X}^{\mathfrak{r}}$, its so-called *refinement*, is given by the canonical simplicial map $X \to P_{\mathcal{X}^{\mathfrak{r}}}$ that maps a vertex to the path component of its stratum (using the explicit construction of $P_{\mathcal{X}^{\mathfrak{r}}}$ as in Proposition 5.3.2.9). This construction induces an idempotent functor

$$(-)^{\mathfrak{r}}\colon \underline{\mathbf{sStrat}} \to \underline{\mathbf{sStrat}}$$

together with a natural transformation $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$, given by

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d] & X \arrow[d] \\ P_{\mathcal{X}^{\mathfrak{r}}} \arrow[r] & P_{\mathcal{X}} \end{tikzcd} \tag{5.15}$$

where the lower map maps a path component to the stratum it is contained in.

Definition 5.3.2.12. A stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$ is called *refined* if the natural stratified map $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$ is an isomorphism.

Being refined may be interpreted as being stratified in a way that uses the minimal poset (in the sense of minimal amounts of elements and relations) capable of reflecting the same stratified topology (see [Waa24c], or specifically Section 7.5.3 in this text, for topological characterizations).

Remark 5.3.2.13. Note that by Proposition 5.3.2.9, it follows that a stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$ is refined if and only if $s_{\mathcal{X}}\colon X \to P_{\mathcal{X}}$ does not have empty strata, and whenever there is a relation $s_{\mathcal{X}}(x) \leq s_{\mathcal{X}}(y)$, for $x, y \in X([0])$ there is a sequence

$$x = x_0 \leftrightarrow x_1 \leftrightarrow x_2 \leftrightarrow \cdots \leftrightarrow x_n = y$$

of 1-simplices in X, with $s_{\mathcal{X}}(x) = p$ and $s_{\mathcal{X}}(y) = q$, and such that only simplices that are contained in one stratum are allowed to point in the direction of x. In particular, all strata are path connected.

Remark 5.3.2.14. If $\mathcal{X} \in \underline{\mathbf{Strat}}$ is fibrant in $\underline{\text{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$, that is, given by a quasi-category X together with a conservative functor $X \to P_{\mathcal{X}}$ then being refined is equivalent to being 0-connected, in the sense of [BGH18, Def. 2.3.6]. See also Remark 5.3.2.21.

One may now easily verify the following:

Proposition 5.3.2.15. *The refinement functor $\mathcal{X} \mapsto \mathcal{X}^{\mathfrak{r}}$ has image in the full simplicial subcategory of refined stratified simplicial sets $\mathcal{X}^{\mathfrak{r}}$. It induces the right adjoint to the inclusion of refined stratified simplicial sets into all stratified simplicial sets. The counit of adjunction is given by the refinement morphisms $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$.*

By Lemma 5.3.2.10 we have:

Lemma 5.3.2.16. *The functor $(-)^{\mathfrak{r}}\colon \underline{\text{s}\mathbf{Strat}} \to \underline{\text{s}\mathbf{Strat}}$ preserves filtered colimits.*

Let us begin by investigating how $(-)^{\mathfrak{r}}$ interacts with the model structures on $\underline{\text{s}\mathbf{Strat}}$.

Construction 5.3.2.17. We will make use of the stratified Ex^{∞} functors of [DW22, Def. 3.7] (these were referred to with a "*naiv*" exponent in Definition 3.3.1.7). Denote by $\mathrm{sd}\colon \mathbf{sSet} \to \mathbf{sSet}$ the barycentric subdivision functor and by Ex its right-adjoint (see [Kan57]). These constructions are extended to stratified spaces as follows: For $\mathcal{X} \in \text{s}\mathbf{Strat}$, we denote by $\mathrm{Ex}\mathcal{X}$ the stratified simplicial set obtained by the left vertical in the pullback square

$$\begin{tikzcd} \mathcal{F}(\mathrm{Ex}\mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Ex}X \arrow[d] \\ P_{\mathcal{X}} \arrow[r, hook] & \mathrm{Ex}P_{\mathcal{X}}\,. \end{tikzcd} \tag{5.16}$$

This construction induces a right adjoint to the stratified subdivision functor $\mathcal{X} \mapsto (\mathrm{sd}X \to X \to P_{\mathcal{X}})$. There is a natural inclusion $\mathcal{X} \hookrightarrow \mathrm{Ex}\mathcal{X}$ adjoint to the stratified last vertex map $\mathrm{sd}\mathcal{X} \to \mathcal{X}$. We denote by $\mathrm{Ex}^{\infty}\mathcal{X}$ the colimit of the diagram

$$\mathcal{X} \hookrightarrow \mathrm{Ex}\mathcal{X} \hookrightarrow \mathrm{Ex}^2\mathcal{X} \hookrightarrow \cdots.$$

One may easily verify that Ex^{∞} is compatible with taking strata, in the sense that $(\mathrm{Ex}^{\infty}\mathcal{X})_p = \mathrm{Ex}^{\infty}(X_p)$, for $p \in P_{\mathcal{X}}$. It follows from the classical results of [Kan57] that $\mathrm{Ex}^{\infty}\mathcal{X}$ has strata given by Kan complexes. We have shown in [DW22, Prop. 3.9] (see Proposition 3.3.1.9 in this text) that the natural inclusion $\mathcal{X} \hookrightarrow \mathrm{Ex}^{\infty}\mathcal{X}$ is an acyclic cofibration in $\underline{\text{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$. In particular, we can compute

$$P_{\mathcal{X}^{\mathfrak{r}}} = P(\mathcal{F}(\mathrm{Ex}^{\infty}\mathcal{X})),$$

for all $\mathcal{X} \in \underline{\text{s}\mathbf{Strat}}$.

Proposition 5.3.2.18. *The functors*

$$(-)^{\mathfrak{r}}\colon \underline{\text{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}} \to \underline{\text{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}};$$
$$(-)^{\mathfrak{r}}\colon \underline{\text{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}} \to \underline{\text{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$$

preserve cofibrations, acyclic fibrations, and acyclic cofibrations. In particular, they preserve weak equivalences. Furthermore, a cofibration j that induces an isomorphism on posets is acyclic if and only if $j^{\mathfrak{r}}$ is an acyclic cofibration.

Proof. To see the statement concerning acyclic fibrations, note that both model categories we are concerned with have the same acyclic fibrations, and by Remark 5.3.1.6 these are precisely given by such morphisms which induce isomorphisms on posets, and acyclic fibrations in the Joyal model structure on the underlying simplicial sets. Hence, we only need to show that for an acyclic fibration $\mathcal{X} \to \mathcal{Y}$, without loss of generality over the same poset P, the induced map $P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{Y}^{\mathfrak{r}}}$ is an isomorphism. Now, just as in the classical scenario [Kan57], one may show that the functor $\mathrm{Ex}^{\infty}\colon \text{s}\mathbf{Strat} \to \text{s}\mathbf{Strat}$ preserves acyclic fibrations. Therefore, we may assume without loss of generality that $\mathcal{X}$ and $\mathcal{Y}$ have strata given by Kan complexes, and hence that $P_{\mathcal{X}^{\mathfrak{r}}} = P(X)$ and $P_{\mathcal{Y}^{\mathfrak{r}}} = P(Y)$. As any acyclic fibration is a categorical equivalence in **sSet**, it follows that f induces an isomorphism $P(X) = P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{Y}^{\mathfrak{r}}} = P(Y)$, as was to be shown. It remains to show the statement on acyclic cofibrations. Clearly, $(-)^{\mathfrak{r}}$ creates cofibrations, as these are defined only in terms of the underlying simplicial sets. Thus, it suffices to show that a cofibration $j\colon \mathcal{A} \hookrightarrow \mathcal{B}$ that is an isomorphism on posets is a weak equivalence, if and only if $j^{\mathfrak{r}}$ is a weak equivalence. Without loss of generality, we may assume that j is the identity on posets. Furthermore, since $(-)^{\mathfrak{r}}$ is given by $P_{-^{\mathfrak{r}}}$ on the posets level, which is a derived functor, it follows that $j^{\mathfrak{r}}$ also is given by an isomorphism on posets. Hence, we also assume that $j^{\mathfrak{r}}$ is given by the identity on the latter. Denote by $Q = P_{\mathcal{B}^{\mathfrak{r}}}$ and by $\mathcal{Q}$ the stratified simplicial set given by $Q \to P$. We may thus consider j as an object of the slice category $(\underline{\mathrm{s}\mathbf{Strat}}_P)_{/\mathcal{Q}}$. We may then instead show the following stronger claim: The isomorphism of simplicial categories

$$(\underline{\mathrm{s}\mathbf{Strat}}_P)_{/\mathcal{Q}} \to \underline{\mathrm{s}\mathbf{Strat}}_Q$$

is an isomorphism of model categories, where on the left-hand side we use the slice model structure (with respect to $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{j}}_P$ or $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$).
Note that as this is an isomorphism of simplicial categories, and the cofibrations in all categories involved are given by monomorphisms, it suffices to show that the isomorphism identifies the classes of fibrant objects. On the left-hand side, these are given by fibrations $\mathcal{X} \to \mathcal{Q}$ (respectively in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{j}}_P$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$). By Proposition 5.3.2.9, the map $f\colon Q \to P$ has fibers which contain no relations, but the identity. In other words, the functor $f\colon Q \to P$ is conservative. It follows from this (using [Lur09, Prop. 2.4.1.5]), that f has the right lifting property with respect to all inner and admissible horn inclusions. Hence, $\mathcal{Q}$ is a fibrant object of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{j}}_P$ and of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$. Consequently, we only need to show that $\mathcal{X}$ being fibrant implies $\mathcal{X} \to \mathcal{Q}$ being a fibration, in both scenarios. Since $\mathcal{Q}$ is fibrant in both scenarios, fibrancy of $\mathcal{X} \to \mathcal{Q}$ can be checked by having the right lifting property with respect to admissible, and inner and admissible horn inclusions, respectively. Now, consider a lifting diagram

$$\begin{tikzcd}
\Lambda^{\mathcal{J}}_k \arrow[r] \arrow[dd, hook] & \mathcal{X} \arrow[d] \\
& \mathcal{Q} \arrow[d] \\
\Delta^{\mathcal{J}} \arrow[r] \arrow[uur, dashed] \arrow[ur] & P
\end{tikzcd} \tag{5.17}$$

where the solid part of the diagram commutes, and the dashed map makes the upper and lower triangle of the outer rectangle diagram commute. Furthermore, assume that the left vertical horn inclusion is either admissible or inner. To finish the proof, it suffices to prove that this also implies that the middle triangle commutes. Since Q is a simplicial complex, it suffices to verify commutativity on vertices. If a vertex $x \in \Delta^{\mathcal{J}}$ lies in $\Lambda^{\mathcal{J}}_k$, then, by commutativity of the upper left triangle, there is nothing to show. Hence, we may restrict to the case where $\Lambda^{\mathcal{J}}_k$ is admissible and $\Delta^{\mathcal{J}}$ of dimension 1, i.e. $\mathcal{J} = [p \leq p]$ and $k = 0$ or $k = 1$. Then, however, we may without loss of generality assume that $P = \{p\}$ is a singleton. Since $f\colon Q \to P$ has discrete fibers, this means that Q is discrete. In this case, commutativity of the middle triangle follows immediately from commutativity of the upper left triangle, using path connectedness of $\Delta^{\mathcal{J}}$. □

We may now use the refinement functor to obtain model structures which will take care of the pathologies we explain in Remark 5.3.2.1. The model structure derived from the Joyal-Kan model structure on $\underline{\mathrm{s}\mathbf{Strat}}$ will allow us to think of stratified spaces as fully faithfully embedded into ∞-categories (Theorem 5.3.3.6). We now define model categories presenting homotopy theories of (certain) refined stratified simplicial sets. These are constructed by forcing $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$ to be a weak equivalence, and hence turn out to be right Bousfield localizations (and thus coreflective localizations).

Theorem 5.3.2.19. *Let S be the class of refinement morphisms $\{\mathcal{X}^{\mathfrak{r}} \to \mathcal{X} \mid \mathcal{X} \in \underline{\mathrm{s}\mathbf{Strat}}\}$. Then the right Bousfield localization of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$) at S exists and is again combinatorial and simplicial. Its defining classes can be characterized as follows:*

(i) The cofibrations are generated by the set of stratified boundary inclusions $\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]}$ together with the boundary inclusion $\partial\Delta^{[1]} \hookrightarrow \Delta^1$ into the trivially stratified simplex.

Equivalently, cofibrations are precisely those morphisms $j\colon \mathcal{A} \to \mathcal{B}$ that induce a monomorphism on simplicial sets (i.e. are a cofibration in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ or $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$) and are furthermore such that the diagram

$$\begin{array}{ccc} \mathcal{A}^{\mathfrak{r}} & \longrightarrow & \mathcal{B}^{\mathfrak{r}} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array} \tag{5.18}$$

is pushout. In particular, the cofibrant objects are precisely the refined stratified simplicial sets.

(ii) Weak equivalences are precisely those morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ for which $f^{\mathfrak{r}}$ is a weak equivalence in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$).

(iii) Acyclic fibrations are precisely those morphisms $f\colon \mathcal{X} \to \mathcal{Y}$, for which $f^{\mathfrak{r}}$ is an acyclic fibration in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ (or equivalently in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$). In other words, f induces an isomorphism on refined posets and an acyclic fibration on the underlying simplicial sets.

(iv) Fibrations and acyclic cofibrations are the same as in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$).

Proof. We denote by I the set of generating cofibrations described in (i) and by W the class of weak equivalences described in (ii). Furthermore, we denote by $\mathrm{inj}(I)$ the class of morphisms that have the right lifting property with respect to I, and denote $\mathrm{cof}(I)$ the class of morphisms that have the left lifting property with respect to $\mathrm{inj}(I)$. Finally, denote by AC and F the classes of acyclic cofibrations and fibrations in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$). To prove the existence of the localization above, it suffices to show the following:

(a) $\mathrm{inj}(I)$ is precisely the class of morphisms described in (iii).

(b) $\mathrm{cof}(I) \cap W = AC$.

To see this, note first that it follows from the small object argument that $\mathrm{cof}(I)$ and $\mathrm{inj}(I)$ form a weak factorization system. AC and F form a weak factorization system, by the respective property of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$). Hence, it only remains to show $F \cap W = \mathrm{inj}(I)$. That $\mathrm{inj}(I) \subset W$ follows by the characterization in (iii). That $\mathrm{inj}(I) \subset F$ follows from $F = \mathrm{inj}(AC)$ and $AC \subset \mathrm{cof}(I)$. Finally, to see that $F \cap W = \mathrm{inj}(I)$, consider $f\colon \mathcal{X} \to \mathcal{Y} \in F \cap W$ as well as factorization

$$\mathcal{X} \xhookrightarrow{i} \hat{\mathcal{X}} \xrightarrow{\hat{f}} \mathcal{Y}$$

of f into $i \in \mathrm{cof}(I)$ and $\hat{f} \in \mathrm{inj}(I)$. Since $f, \hat{f} \in W$, it follows by two-out-of-three, that the same holds for i. It follows from Claim (b) that $i \in AC$. In particular, i has the left lifting property with respect to f from which it follows that f is a retract of $\hat{f}$, and hence an element of $\mathrm{inj}(I)$. Let us assume that we have shown Claims (a) and (b) as well as the equivalence in (i) for now. Note that the thus defined model category is again combinatorial. Indeed, we have provided a set of generators for cofibrations in (i) and a set of generators for acyclic cofibrations is given by the ones for $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$). Next, let us verify simpliciality. Suppose that $i\colon \mathcal{A} \to \mathcal{B}$ lies in $\mathrm{cof}(I)$ and that $j\colon A \to B$ is a cofibration of simplicial sets. We need to show that

$$f\colon \mathcal{C} := \mathcal{B} \otimes A \cup_{\mathcal{A} \otimes A} \mathcal{A} \otimes B \to \mathcal{B} \otimes B =: \mathcal{D}$$

again lies in $\mathrm{cof}(I)$. That the induced map of simplicial sets is a monomorphism is immediate from the corresponding statement on simplicial sets. Note that, by the equivalent characterization of $\mathrm{cof}(I)$ in (i), $\mathrm{cof}(I)$ has the property that for any morphism $g \in \mathrm{s}\mathbf{Strat}$ that induces a monomorphism on simplicial sets and any $i' \in \mathrm{cof}(I)$ with target the source of g, it holds that

$$g \in \mathrm{cof}(I) \iff g \circ i' \in \mathrm{cof}(I).$$

Thus, it suffices to show that

$$\mathcal{B} \otimes A \to \mathcal{B} \otimes B$$

and

$$\mathcal{B} \otimes A \to \mathcal{B} \otimes A \cup_{\mathcal{A} \otimes A} \mathcal{A} \otimes B$$

are in $\mathrm{cof}(I)$. Using the stability of $\mathrm{cof}(I)$ under pushouts, we may thus reduce to the cases where either $\mathcal{A}$ or A is empty, i.e. $\mathcal{C}$ is of the form $\mathcal{B} \otimes A$ or $\mathcal{A} \otimes B$. Now, again using (i), and the fact that pushout diagrams in $\mathrm{s}\mathbf{Strat}$ are detected on the poset and simplicial set level, it suffices to show that

$$\begin{tikzcd} P_{\mathcal{C}^{\mathfrak{r}}} \arrow[r] \arrow[d] & P_{\mathcal{D}^{\mathfrak{r}}} \arrow[d] \\ P_{\mathcal{C}} \arrow[r] & P_{\mathcal{D}} \end{tikzcd} \tag{5.19}$$

is a pushout diagram in $\mathbf{Pos}$. Finally, note that applying $- \otimes K$ acts as $- \times \pi_0(A)$ (with $\pi_0(A)$ equipped with the discrete poset structure) both on the level of posets as well as on the level of refined posets. If $\mathcal{A} = \varnothing$, then by assumption $P_{\mathcal{B}^{\mathfrak{r}}} \to P_{\mathcal{B}}$ is an isomorphism and it follows that Diagram (5.19) is of the form

$$\begin{tikzcd} P_{\mathcal{B}^{\mathfrak{r}}} \times \pi_0(A) \arrow[r] \arrow[d, "\cong"] & P_{\mathcal{B}^{\mathfrak{r}}} \times \pi_0(B) \arrow[d, "\cong"] \\ P_{\mathcal{B}} \times \pi_0(A) \arrow[r] & P_{\mathcal{B}} \times \pi_0(B)\,, \end{tikzcd} \tag{5.20}$$

with horizontals induced by j. Since both verticals are isomorphisms, this diagram is pushout. If A is empty, then Diagram (5.19) is of the form

$$\begin{tikzcd} P_{\mathcal{A}^{\mathfrak{r}}} \times \pi_0(B) \arrow[r] \arrow[d] & P_{\mathcal{B}^{\mathfrak{r}}} \times \pi_0(B) \arrow[d] \\ P_{\mathcal{A}} \times \pi_0(B) \arrow[r] & P_{\mathcal{B}} \times \pi_0(B)\,, \end{tikzcd} \tag{5.21}$$

with horizontal induced by i. Consequently, it follows from

$$\begin{tikzcd} P_{\mathcal{A}^{\mathfrak{r}}} \arrow[r] \arrow[d] & P_{\mathcal{B}^{\mathfrak{r}}} \arrow[d] \\ P_{\mathcal{A}} \arrow[r] & P_{\mathcal{B}} \end{tikzcd} \tag{5.22}$$

being pushout by assumption, that Diagram (5.19) is also pushout in this case.
Finally, if either i or j is an acyclic cofibration, then it follows by the simpliciality of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$) and Claim (b) that f is also an acyclic cofibration.
To finish the proof, it remains to show Claims (a) and (b) as well as the equivalence in (i). This is the content of Lemmas 5.3.2.28 to 5.3.2.30. ☐

Definition 5.3.2.20. We denote by $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$, respectively, the simplicial right Bousfield localizations in Theorem 5.3.2.19. They are, respectively, called the *diagrammatic* and the *categorical* model structure on $\underline{\mathrm{s}\mathbf{Strat}}$. Weak equivalences in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}$ are called *diagrammatic equivalences.* Weak equivalences in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$ are called *Joyal-Kan equivalences.* We call the homotopy theory presented by $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$ the $(\infty, 1)$-category of refined abstract homotopy types and denote it by $\mathcal{AS}\mathbf{trat}^{\mathfrak{r}}$.

Remark 5.3.2.21. It follows from Remark 5.3.2.14 that the refined abstract stratified homotopy types are precisely what [BGH18] calls 0-connected stratified spaces. In this sense, the part of Theorem 5.3.2.19 that is concerned with the Joyal-Kan model structure can be taken to be the construction of a model structure presenting 0-connected stratified spaces.

Let us also make the following observation, which is immediate from the characterization of the defining classes in Theorem 5.3.2.19.

Lemma 5.3.2.22. *For any $\mathcal{X} \in \underline{\text{s}\mathbf{Strat}}$, the natural transformation $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$ is an acyclic fibration in $\underline{\text{s}\mathbf{Strat}}^{\mathfrak{d}}$ ($\underline{\text{s}\mathbf{Strat}}^{\mathfrak{c}}$). It defines a cofibrant replacement of $\mathcal{X} \in \underline{\text{s}\mathbf{Strat}}^{\mathfrak{d}}$ ($\underline{\text{s}\mathbf{Strat}}^{\mathfrak{c}}$).*

Furthermore, we are going to need the following property of the refined model structures, which follows from Lemma 5.3.2.16 and Proposition 5.3.1.5.

Lemma 5.3.2.23. *Weak equivalences in $\underline{\text{s}\mathbf{Strat}}^{\mathfrak{d}}$ and $\underline{\text{s}\mathbf{Strat}}^{\mathfrak{c}}$ are stable under filtered colimits.*

Also note that it follows from Theorem 5.3.2.19 together with Proposition 5.3.1.11 that:

Proposition 5.3.2.24. *$\underline{\text{s}\mathbf{Strat}}^{\mathfrak{c}}$ is the left Bousfield localization of $\underline{\text{s}\mathbf{Strat}}^{\mathfrak{d}}$ at the set of stratified inner horn inclusions $\{\Lambda^{[n]}_k \hookrightarrow \Delta^{[n]} \hookrightarrow \Delta^n \mid 0 < k < n\}$.*

Finally, the following observation will be useful when passing to the topological scenario:

Proposition 5.3.2.25. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a stratified simplicial map between stratified simplicial sets $\mathcal{X}, \mathcal{Y}$ whose strata are Kan complexes. Then f is a Joyal-Kan equivalence if and only if the underlying map of simplicial sets $\mathcal{F}(f)$ is a categorical equivalence (also called Joyal equivalences).*

Proof. By definition, f is a Joyal-Kan equivalence if and only if $f^{\mathfrak{r}}$ is a categorical equivalence. By [Hai23, Thm. 0.2.2.2] this is, in turn, equivalent to the following two conditions being fulfilled.

1. The underlying simplicial map of f, $\mathcal{F}(f)$, is a categorical equivalence.

2. f induces an isomorphism on refined posets.

However, to compute the map on refined posets, by Corollary 5.3.2.8, there is no need to derive at all, and it is given by $P(\mathcal{F}(f))$. Since $\mathcal{F}(f)$ is a categorical equivalence, it follows from Lemma 5.3.2.4 that the second condition is redundant, as was to be shown. ☐

We may summarize the whole situation as follows.

Proposition 5.3.2.26. *The simplicial, combinatorial model structures on $\underline{\text{s}\mathbf{Strat}}$ fit into a diagram of Bousfield localizations*

$$\begin{tikzcd} \underline{\text{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}} \arrow[r] \arrow[d] & \underline{\text{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}} \arrow[d] \\ \underline{\text{s}\mathbf{Strat}}^{\mathfrak{d}} \arrow[r] & \underline{\text{s}\mathbf{Strat}}^{\mathfrak{c}} \end{tikzcd} \tag{5.23}$$

with the verticals right Bousfield and the horizontals left Bousfield. The verticals are obtained by localizing the stratified inner horn inclusions. The horizontals are obtained by localizing the refinement morphisms $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$.

Furthermore, consider the following result which - retroactively - justifies the naming conventions for the different notions of equivalences of stratified simplicial sets:

Proposition 5.3.2.27. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a stratified simplicial map. Then f is a poset-preserving Joyal-Kan equivalence if and only if f is a Joyal-Kan equivalence and the underlying map of posets, $P(f) \colon P_{\mathcal{X}} \to P_{\mathcal{Y}}$, is an isomorphism. The analogous result for diagrammatic equivalences holds.*

Proof. Both in the diagrammatic and categorical scenario, the only if case is immediate by Proposition 5.3.2.18 together with the idempotency of the refinement functor, and the characterization of weak equivalences in Theorem 5.3.2.19. Next, let us show the if case in the case of Joyal-Kan equivalences. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] \arrow[ddd, bend right=60, "\simeq"'] & \mathcal{Y} \arrow[d, "\simeq"] \arrow[ddd, bend left=60, "\simeq"] \\ \mathrm{Ex}^\infty \mathcal{X} \arrow[r] \arrow[d] & \mathrm{Ex}^\infty \mathcal{Y} \arrow[d] \\ \mathrm{Ex}^\infty \mathcal{X}^{\mathfrak{r}} \arrow[r] & \mathrm{Ex}^\infty \mathcal{Y}^{\mathfrak{r}} \\ \mathcal{X}^{\mathfrak{r}} \arrow[r] \arrow[u, "\simeq"] & \mathcal{Y}^{\mathfrak{r}} \arrow[u, "\simeq"'] , \end{tikzcd} \tag{5.24}$$

and note that all but the upper most row has the property that all stratified simplicial sets involved have strata given by Kan complexes. We have marked all maps which are known to be Joyal-Kan equivalences from previous results in this article with a $\simeq$ symbol. That these maps are weak equivalences follows either by assumption, or from the natural map $1 \to \mathrm{Ex}^\infty$ even being a poset-preserving diagrammatic equivalence (see [DW22, Prop. 3.9], which is Proposition 3.3.1.9 in this text). A quick diagram chase using the two-out-of-three property shows that all morphisms in the diagram are Joyal-Kan equivalences. It follows from Proposition 5.3.2.25 that the underlying simplicial map of $\mathrm{Ex}^\infty \mathcal{X} \to \mathrm{Ex}^\infty \mathcal{Y}$ is a categorical equivalence. Hence, by [Hai23, Thm. 0.2.2.2], using the assumption that $\mathcal{X} \to \mathcal{Y}$ induces an isomorphism on posets, it follows that $\mathrm{Ex}^\infty \mathcal{X} \to \mathrm{Ex}^\infty \mathcal{Y}$ is a poset-preserving Joyal-Kan equivalence. Finally, the upper two verticals are also poset-preserving Joyal-Kan equivalences (diagrammatic even), from which, again by two-out-of-three, it follows that $\mathcal{X} \to \mathcal{Y}$ is a poset-preserving Joyal-Kan equivalence. It remains to show that a stratified simplicial map that induces isomorphisms on the poset level and is a diagrammatic equivalence is a poset-preserving diagrammatic equivalence. Let $\mathcal{I}$ be a non-degenerate flag of $P_{\mathcal{X}}$. We obtain an induced commutative diagram of simplicial sets

$$\begin{tikzcd} \bigsqcup_{\mathcal{I}' \mapsto \mathcal{I}} \mathrm{HoLink}_{\mathcal{I}'}(\mathcal{X}^{\mathfrak{r}}) \arrow[r] \arrow[d] & \bigsqcup_{\mathcal{I}' \mapsto P(f)(\mathcal{I})} \mathrm{HoLink}_{\mathcal{I}'}(\mathcal{Y}^{\mathfrak{r}}) \arrow[d] \\ \mathrm{HoLink}_{\mathcal{I}}(\mathcal{X}) \arrow[r] & \mathrm{HoLink}_{P(f)(\mathcal{I})}(\mathcal{Y}) , \end{tikzcd} \tag{5.25}$$

where the coproducts are indexed over regular flags mapping to $\mathcal{I}$ (respectively $P(f)(\mathcal{I})$) under $P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{X}}$ ($P_{\mathcal{Y}^{\mathfrak{r}}} \to P_{\mathcal{Y}}$). Since $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X}^{\mathfrak{r}} \to \mathcal{Y}^{\mathfrak{r}}$ are assumed to be injective on the poset level, the two horizontals are well-defined. By assumption, the upper horizontal is a weak homotopy equivalence of simplicial sets. Furthermore, it follows from an application of Proposition 5.3.2.15 that the two verticals are isomorphisms of simplicial sets. Hence, the lower vertical is a weak homotopy equivalence. Since $\mathcal{X} \to \mathcal{Y}$ is assumed to induce an isomorphism on the poset level, it follows that it is a poset-preserving diagrammatic equivalence. □

Now, let us prove the remaining open statements.

Lemma 5.3.2.28. *In the framework of Theorem 5.3.2.19 and its proof,* $\mathrm{inj}(I)$ *is the class of stratified maps* $f\colon \mathcal{X} \to \mathcal{Y}$ *such that* $f^{\mathfrak{r}}$ *is an isomorphism on posets and the underlying simplicial map of* f *is a trivial fibration (with respect to any of the model structures on presheaves on* $\underline{\mathbf{sSet}}$*). In other words,* $f \in \mathrm{inj}(I)$ *if and only if* $f^{\mathfrak{r}}$ *is an acyclic fibration in* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ *(or equivalently in* $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$*).*

Proof. Denote by S the set of boundary inclusions in $\underline{\mathbf{sSet}}$ and by $i \in \underline{\mathrm{s}\mathbf{Strat}}$ the remaining cofibration $\partial\Delta^{[1]} \to \Delta^1$ specified in the statement of the theorem. It follows from the adjunction of $\mathcal{L}\colon \underline{\mathbf{sSet}} \to \underline{\mathbf{sStrat}}$ with the forgetful functor $\underline{\mathbf{sStrat}} \to \underline{\mathbf{sSet}}$, that $f \in \mathrm{inj}(\mathcal{L}(S))$ if and only if the underlying simplicial map is an acyclic fibration. Let us now assume that $f \in \mathcal{L}(S)$. Under this assumption, we show that $f \in \mathrm{inj}(i)$ is equivalent to the induced map $\tilde{f}\colon P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{Y}^{\mathfrak{r}}}$ being an isomorphism. $\tilde{f}$ is an isomorphism, if and only if $\tilde{f}$ is surjective on elements and relations. Assume that $f \in \mathrm{inj}(i \cup \mathcal{L}(S))$. Surjectivity on elements follows from the fact that the underlying simplicial map of f is surjective (as it is an acyclic fibration). Now, by Proposition 5.3.2.9, it suffices to show that any zigzag as in Proposition 5.3.2.9 lifts. For 1-simplices that point in direction of y, this follows from $f \in \mathrm{inj}(\{\partial\Delta^{[1]} \to \Delta^{[1]}\})$. For 1-simplices pointing in the direction of x, this follows from $f \in \mathrm{inj}(i)$. Conversely, let $\tilde{f}$ be an isomorphism. Given a lifting problem as the right square in

$$\begin{tikzcd} \partial\Delta^{[1]} \arrow[r, "1"] \arrow[d, hook] & \partial\Delta^{[1]} \arrow[r] \arrow[d, "g'"'] & \mathcal{X} \arrow[d] \\ \Delta^{[1]} \arrow[r] \arrow[urr, dashed] & \Delta^1 \arrow[r] \arrow[ur, dashed, "g"'] & \mathcal{Y}, \end{tikzcd} \tag{5.26}$$

by the assumption that $f \in \mathrm{inj}(\mathcal{L}(S))$ it follows that a solution g' of the outer rectangle exists. Since $\tilde{f}$ is an isomorphism, any two points in $\mathcal{X}$ that are mapped into the same stratum of $\mathcal{Y}$ and are connected by a path in the latter, already lie in the same stratum of $\mathcal{X}$. It follows by commutativity of the outer rectangle that g' factors through $\partial\Delta^{[1]} \to \Delta^1$ into a stratified map $g\colon \Delta^1 \to \mathcal{X}$. Since both left horizontals are epimorphisms, g is a solution for the right lifting square. ☐

Lemma 5.3.2.29. *In the framework of Theorem 5.3.2.19 and its proof,* $\mathrm{cof}(I)$ *is precisely the class of stratified maps* $f\colon \mathcal{A} \to \mathcal{B}$ *such that the diagram*

$$\begin{tikzcd} \mathcal{A}^{\mathfrak{r}} \arrow[r] \arrow[d] & \mathcal{B}^{\mathfrak{r}} \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{B} \end{tikzcd} \tag{5.27}$$

is pushout and such that f *is a cofibration in* $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ *(or equivalently* $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$*) (i.e.* f *induces a monomorphism on the simplicial set level).*

Proof. First, let us show that any $\mathcal{A} \hookrightarrow \mathcal{B}$ in $\mathrm{cof}(I)$ has the pushout property (that it is a monomorphism on simplicial sets is immediate). We only need to show that

$$\begin{tikzcd} P_{\mathcal{A}^{\mathfrak{r}}} \arrow[r] \arrow[d] & P_{\mathcal{B}^{\mathfrak{r}}} \arrow[d] \\ P_{\mathcal{A}} \arrow[r] & P_{\mathcal{B}} \end{tikzcd} \tag{5.28}$$

is pushout. Using the small object argument, we may reduce to showing that this is true for any $j \in I$, and $\mathcal{A} \to \mathcal{B}$ a pushout of j. Let us compute explicitly the maps

$$P_{\mathcal{A}^{\mathfrak{r}}} \to P_{\mathcal{B}^{\mathfrak{r}}}$$
$$P_{\mathcal{A}} \to P_{\mathcal{B}}$$

in terms of generators and relations (using the explicit description in Proposition 5.3.2.9). If the codomain of j is a simplex of dimension greater than 1, then both maps on the poset level are isomorphisms. It remains to consider the three cases:

$$\varnothing \hookrightarrow \Delta^{[0]};$$
$$\partial\Delta^{[1]} \hookrightarrow \Delta^{[1]};$$
$$\partial\Delta^{[1]} \hookrightarrow \Delta^1.$$

The first of these adds an extra element to $P_{\mathcal{A}^{\mathfrak{r}}}$ and $P_{\mathcal{A}}$. For the second and third case let $p_0, p_1 \in P_{\mathcal{A}}$ be the strata corresponding to the images of the points in $\partial\Delta^{[1]}$. For the second case, we obtain $P_{\mathcal{B}}$ by adding precisely one generating relation $r : p_0 \leq p_1$ to $P_{\mathcal{A}}$. This identifies all elements of $P_{\mathcal{A}}$ that are now contained in a finite ordered cycle of relations. For the third case, two generating relations $r{:}\, p_0 \leq p_1$ and $r^{-1}{:}\, p_1 \leq p_0$ are added to $P_{\mathcal{A}}$. In both the second and the third case, $P_{\mathcal{B}^{\mathfrak{r}}}$ is obtained from $P_{\mathcal{A}^{\mathfrak{r}}}$ by adding one additional generating relation $\hat{r}$, added from $[x_0]$ to $[x_1]$, where x_0 and x_1 are the respective boundary vertices of the glued in 1-simplex, and furthermore, one generating relation (pointing in the opposite direction) is added for every 1-simplex, whose strata become identical in $P_{\mathcal{X}}$ after adding r, (r, r^{-1}). We may now check by hand that Diagram (5.28) is pushout. To do this, note that pushouts in partially ordered sets are computed from elements and relations by taking a pushout of the elements in sets and taking the generating relations coming from $P_{\mathcal{B}^{\mathfrak{r}}}$ and $P_{\mathcal{A}}$. If none of the three cases above apply, then all horizontals are isomorphisms and there is nothing to be shown. In the first case, Diagram (5.28) is of the form

$$\begin{tikzcd} P_{\mathcal{A}^{\mathfrak{r}}} \arrow[r] \arrow[d] & P_{\mathcal{A}^{\mathfrak{r}}} \sqcup [0] \arrow[d] \\ P_{\mathcal{A}} \arrow[r] & P_{\mathcal{A}} \sqcup [0] \end{tikzcd} \tag{5.29}$$

and therefore pushout. In the second and third case, the upper horizontal is surjective on elements. It follows by the explicit construction above that the pushout $P_{\mathcal{A}} \cup_{P_{\mathcal{A}^{\mathfrak{r}}}} \cup P_{\mathcal{B}^{\mathfrak{r}}}$ may simply be constructed by adding to $P_{\mathcal{A}}$, all relations of the form $s_{\mathcal{A}}(x) \leq s_{\mathcal{A}}(y)$, where $[y] \leq [x]$ is a generating relation in $P_{\mathcal{B}^{\mathfrak{r}}}$, not already present in $P_{\mathcal{A}^{\mathfrak{r}}}$. Hence, by our explicit description above, in these cases the pushout is computed by adding the relation $r : s_{\mathcal{A}}(x_0) \leq s_{\mathcal{A}}(x_1)$, as well as one additional relation $s_{\mathcal{A}}(y_0) \leq s_{\mathcal{A}}(y_1)$, for all edges $y_0 \to y_1$, whose endpoint strata are identified after adding r, (r, r^{-1}). Note how in both cases the additional relations $s_{\mathcal{A}}(y_0) \leq s_{\mathcal{A}}(y_1)$ are redundant, by their definition. To summarize, we have presented the pushout $P_{\mathcal{A}} \cup_{P_{\mathcal{A}^{\mathfrak{r}}}} \cup P_{\mathcal{B}^{\mathfrak{r}}}$ in terms of the same generators and relations as $P_{\mathcal{B}}$, which finishes this part of the proof.

Let us now, conversely, show that any map $f{:}\, \mathcal{A} \to \mathcal{B}$ that induces a monomorphism of the underlying simplicial sets, and a pushout square as in the claim, lies in $\mathrm{cof}(I)$. Suppose that we are given a lifting diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{Y}, \end{tikzcd} \tag{5.30}$$

with $\mathcal{X} \to \mathcal{Y}$ in $\mathrm{inj}(I)$. Note that the induced diagram

$$\begin{tikzcd} \mathcal{A}^{\mathfrak{r}} \arrow[r] \arrow[d] & \mathcal{X}^{\mathfrak{r}} \arrow[d] \\ \mathcal{B}^{\mathfrak{r}} \arrow[r] & \mathcal{Y}^{\mathfrak{r}}. \end{tikzcd} \tag{5.31}$$

admits a solution. Indeed, the left vertical is a cofibration in $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ and, by Lemma 5.3.2.28, the right vertical is an acyclic fibration $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$. In particular, we have a solution to the composed diagram

$$\begin{tikzcd} \mathcal{A}^{\mathfrak{r}} \arrow[r] \arrow[d] & \mathcal{X}^{\mathfrak{r}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{B}^{\mathfrak{r}} \arrow[r] \arrow[urr, dashed] & \mathcal{Y}^{\mathfrak{r}} \arrow[r] & \mathcal{Y}. \end{tikzcd} \tag{5.32}$$

Now, consider the solid commutative diagram

$$\begin{tikzcd}
\mathcal{A}^{\mathfrak{r}} \arrow[rr, tail] \arrow[dd] \arrow[dr] & & \mathcal{X} \arrow[dd] \\
& \mathcal{A} \arrow[ur] \arrow[dd] & \\
\mathcal{B}^{\mathfrak{r}} \arrow[rr] \arrow[dr] \arrow[uurr, bend left] & & \mathcal{Y} \\
& \mathcal{B} \arrow[ur] \arrow[uuur, dashed] &
\end{tikzcd} \quad . \tag{5.33}$$

A diagram chase shows that the universal property of the pushout induced a dashed solution to our original lifting problem. ☐

Lemma 5.3.2.30. *In the framework of Theorem 5.3.2.19 and its proof,* $\mathrm{cof}(I) \cap W$ *is precisely the class of acyclic cofibrations in* $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ *(*$\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$*).*

Proof. Suppose that $j{:}\,\mathcal{A} \to \mathcal{B}$ is an acyclic cofibration in $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$). By Proposition 5.3.2.18, it follows that $j^{\mathfrak{r}}$ is an isomorphism on posets. Consequently, the diagram

$$\begin{tikzcd}
\mathcal{A}^{\mathfrak{r}} \arrow[r] \arrow[d] & \mathcal{B}^{\mathfrak{r}} \arrow[d] \\
\mathcal{A} \arrow[r] & \mathcal{B}
\end{tikzcd} \tag{5.34}$$

is pushout, which by Lemma 5.3.2.29 implies that j lies in $\mathrm{cof}(I)$. Furthermore, again by Proposition 5.3.2.18, we also have that $j^{\mathfrak{r}}$ is a weak equivalence, i.e., that $j \in W$. Now, conversely, assume that $j \in \mathrm{cof}(I) \cap W$. By the definition of W, it follows that $j^{\mathfrak{r}}$ is a weak equivalence in $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$). As j is given by a monomorphism on simplicial sets, it thus follows that $j^{\mathfrak{r}}$ is an acyclic cofibration in $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$). Since, by Lemma 5.3.2.29, the diagram

$$\begin{tikzcd}
\mathcal{A}^{\mathfrak{r}} \arrow[r] \arrow[d] & \mathcal{B}^{\mathfrak{r}} \arrow[d] \\
\mathcal{A} \arrow[r] & \mathcal{B}
\end{tikzcd} \tag{5.35}$$

is pushout, it follows that j is also an acyclic cofibration in $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$). ☐

5.3.3 Refined abstract stratified homotopy types and layered ∞-categories

Let us give an alternative description of the homotopy theory defined by categorical model structure on $\underline{\mathbf{sStrat}}$. It turns out that it is a fully faithful subcategory of the infinity category of all small infinity categories $\mathcal{C}\mathbf{at}_\infty$.

Definition 5.3.3.1. [BGH18] Let $X \in \mathbf{sSet}$ be a quasi-category. We say X is *layered*, if the natural functor

$$X \to P(X)$$

is conservative. More generally, we say that an arbitrary $Y \in \mathbf{sSet}$ is *layered*, if this holds for any fibrant replacement of Y in the Joyal model structure, $\mathbf{sSet}^{\mathfrak{J}}$.

Remark 5.3.3.2. In other words, a quasi-category $X \in \mathbf{sSet}$ is layered if and only if each endomorphism in X is an isomorphism. This has the effect that the isomorphism classes naturally carry the structure of a poset, with a relation $[x] \le [y]$ if and only if there is a morphism $x \to y$. This poset then agrees with $P(X)$.

Notation 5.3.3.3. We denote by $\mathcal{L}\mathbf{ay}_\infty$ the full subcategory of $\mathcal{C}\mathbf{at}_\infty$ given by the layered quasi-categories.

Let us construct a model structure on **sSet** corresponding to $\mathcal{L}\mathbf{ay}_\infty$.

Construction 5.3.3.4. Denote by E the quotient of Δ^2 obtained by collapsing the edge $[0,2]$ and identifying the vertices $[0]$ and $[1]$. In other words, we have specified the generating data for a free endomorphism, which has a right inverse. Furthermore, denote by S^1 the quotient of Δ^1 by $\partial\Delta^1$. The inclusion $\Delta^1 \hookrightarrow \Delta^2$, mapping to the $[0,1]$ face, induces an inclusion of simplicial sets

$$l\colon S^1 \hookrightarrow E.$$

We denote by $\mathbf{sSet}^{\mathfrak{D}}$ the left Bousfield localization of $\mathbf{sSet}^{\mathfrak{J}}$ at l, which exists by [Bar10, Thm. 4.7].

Proposition 5.3.3.5. $\mathbf{sSet}^{\mathfrak{D}}$ *is a model for the* ∞*-category* $\mathcal{L}\mathbf{ay}_\infty$.

Proof. Since, $\mathcal{L}\mathbf{ay}_\infty$ is a full subcategory of $\mathcal{C}\mathbf{at}_\infty$ and $\mathbf{sSet}^{\mathfrak{D}}$ is a left Bousfield localization of $\mathbf{sSet}^{\mathfrak{J}}$, which models $\mathcal{C}\mathbf{at}_\infty$, we only need to show that the fibrant objects of $\mathbf{sSet}^{\mathfrak{D}}$ are precisely the layered quasi-categories. Now, note that a quasi-category X is l-local, if and only if the induced simplicial map

$$\underline{\mathbf{sSet}}(E,X)^{\simeq} \to \underline{\mathbf{sSet}}(S^1,X)^{\simeq},$$

where $(-)^{\simeq}$ denotes the maximal Kan complex contained in these quasi categories, is a weak homotopy equivalence (indeed this follows from the fact that the latter Kan complexes define derived mapping spaces for $\mathbf{sSet}^{\mathfrak{J}}$). The path components of the left-hand side correspond to (isomorphism classes of) morphisms which have a right inverse. The path components on the right-hand side correspond to (isomorphism classes of) endomorphisms. Hence, this map being a weak equivalence implies that every endomorphism in X has a right inverse. Since this also holds for the respective right inverses, it follows that every endomorphism in X is an isomorphism, i.e. that X is layered (Remark 5.3.3.2). Conversely, if every endomorphism of X is an isomorphism, then every simplicial map from $A = E, S^1$ to X has image in $X^{\simeq}$. Hence, it follows (by [Cis19, Cor. 3.5.12.]) that

$$\underline{\mathbf{sSet}}(A,X)^{\simeq} = \underline{\mathbf{sSet}}(A,X^{\simeq})^{\simeq} = \underline{\mathbf{sSet}}(A,X^{\simeq}),$$

as the middle term is already a Kan complex. It is not hard to see that $S^1 \hookrightarrow E$ is a weak homotopy equivalence of simplicial sets (see the proof of Theorem 5.3.3.6 below), which implies that

$$\underline{\mathbf{sSet}}(E,X)^{\simeq} = \underline{\mathbf{sSet}}(E,X) \to \underline{\mathbf{sSet}}(S^1,X) = \underline{\mathbf{sSet}}(S^1,X)^{\simeq},$$

is also a weak homotopy equivalence. $\square$

We may now expose $\underline{\mathbf{sStrat}}^{\mathfrak{c}}$ as a different model for the homotopy theory of layered ∞-categories. The ∞-categorical version of this statement was already proven in [BGH18, 2.3.8]. Here is the model-categorical version of this statement:

Theorem 5.3.3.6. *The adjunction*

$$\mathcal{L}\colon \mathbf{sSet} \rightleftharpoons \mathrm{s}\mathbf{Strat}\colon \mathcal{F}$$

induces a Quillen equivalence between $\mathbf{sSet}^{\mathfrak{D}}$ *and* $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$.

Proof. We begin by showing that $\mathcal{L}$ is left Quillen. It follows immediately from the construction of $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$ that $\mathcal{L}$ is left Quillen as a functor with domain $\mathbf{sSet}^{\mathfrak{J}}$. For cofibrations, this follows by construction. For acyclic cofibrations, note that any categorical equivalence $X \to Y$ induces a morphism $\mathcal{L}(X) \to \mathcal{L}(Y)$ that is an isomorphism on posets $P(X) \to P(Y)$. In particular, by definition of the model structure on $\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$, $\mathcal{L}(X) \to \mathcal{L}(Y)$ is a weak equivalence. Now, consider $\mathcal{L}(l) \in \mathrm{s}\mathbf{Strat}$. On the poset level, $\mathcal{L}(l)$ is given by the identity on the poset $[0]$. It follows, by construction of $\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ that $\mathcal{L}(l)$ is a weak equivalence if and only if it is a weak homotopy equivalence of (trivially stratified) simplicial sets. Indeed, l is given by the pushout

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & (\Delta^2)/\Delta^{[0,2]} \arrow[d] \\ S^1 \arrow[r] & E. \end{tikzcd} \tag{5.36}$$

The upper horizontal is an acyclic cofibration in the Kan-Quillen model structure on simplicial sets, hence so is l. By the universal property of the left Bousfield localization, it follows that $\mathcal{L}\colon \mathbf{sSet}^{\mathfrak{D}} \to \mathrm{s}\mathbf{Strat}$ is indeed left Quillen. It remains to show that $\mathcal{L} \dashv \mathcal{F}$ is a Quillen equivalence. Let $\mathcal{Y} \in \mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$ be fibrant (i.e. $s_{\mathcal{Y}}\colon Y \to P_{\mathcal{Y}}$ a conservative functor of quasi-categories) and $X \in \mathbf{sSet}$. Consider a morphism

$$f\colon \mathcal{L}(X) \to \mathcal{Y}$$

and its adjoint

$$g\colon X \to Y.$$

We need to show that f is a weak equivalence, if and only if g is a weak equivalence. By replacing X and $\mathcal{Y}$ fibrantly and cofibrantly, respectively, we may without loss of generality assume that X and $\mathcal{Y}$ are bifibrant. Note that by definition of the model structure on $\mathbf{sSet}^{\mathfrak{D}}$, $\mathcal{L}$ sends fibrant objects in $\mathbf{sSet}^{\mathfrak{D}}$ (i.e. layered infinity categories) to fibrant objects in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$. Similarly, as every object in $\mathbf{sSet}^{\mathfrak{D}}$ is cofibrant, $\mathcal{F}$ preserves cofibrant objects. It follows by the construction of $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$ as a right Bousfield localization (and the Whitehead theorem for Bousfield localizations [Hir03, Thm. 3.2.13]), that f is a weak equivalence in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$, if and only if it is a weak equivalence in $\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$. Hence, we may without loss of generality, assume that f is the identity on posets, for $P = P_{\mathcal{Y}}$. Thus, from Recollection 5.2.1.2 it follows that f is a weak equivalence, if and only if the underlying map of simplicial sets (which is g) is a categorical equivalence. Finally, again using the local Whitehead theorem, g is a categorical equivalence if and only if it is a weak equivalence in $\mathbf{sSet}^{\mathfrak{D}}$. □

Remark 5.3.3.7. Denote by $\mathcal{AS}\mathbf{trat}^{\mathfrak{r}}$ the full coreflective subcategory of $\mathcal{AS}\mathbf{trat}$ given by refined abstract stratified homotopy types. Then it follows from Theorem 5.3.3.6 that the forgetful functor $\mathrm{s}\mathbf{Strat} \to \mathbf{sSet}$ induces a fully faithful reflective embedding

$$\mathcal{AS}\mathbf{trat}^{\mathfrak{r}} \hookrightarrow \mathcal{C}\mathbf{at}_\infty$$

with essential image the full subcategory of layered ∞-categories $\mathcal{L}\mathbf{ay}_\infty$.

5.3.4 Homotopy links for the global stratified setting

Weak equivalences in $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{d}}$ can be detected entirely in terms of generalized homotopy links. The question arises whether we can make a similar argument in the global scenario. To do so, we first need a global version of the homotopy link.

Definition 5.3.4.1. For $n \in \mathbb{N}$ and $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$, we denote

$$\hat{\mathrm{H}}\mathrm{oLink}_n(\mathcal{X}) \coloneqq \underline{\mathrm{s}\mathbf{Strat}}(\Delta^{[n]}, \mathcal{X})$$

and call this simplicial set the n-th extended homotopy link of $\mathcal{X}$.

Remark 5.3.4.2. Note that we may decompose the extended homotopy link into two parts:

$$\hat{\mathrm{H}}\mathrm{oLink}_n(\mathcal{X}) = \bigsqcup_{\mathcal{I} \in (NP)_n, \mathcal{I} \text{ n.d.}} \mathrm{HoLink}_{\mathcal{I}}(\mathcal{X}) \sqcup \bigsqcup_{\mathcal{J} \in (\mathrm{N}P)_n, \mathcal{J} \text{ d.}} \underline{\mathrm{s}\mathbf{Strat}}_{P_{\mathcal{X}}}(\Delta^{\mathcal{J}}, \mathcal{X}),$$

where the left-hand union ranges over regular flags, and the right-hand union over degenerate flags. Now if $\mathcal{J}$ degenerates from a regular flag $\mathcal{I}$ of $P_{\mathcal{X}}$, then $\Delta^{\mathcal{J}}$ and $\Delta^{\mathcal{I}}$ are stratum-preserving homotopy equivalent. It follows that $\underline{\mathrm{s}\mathbf{Strat}}_{P_{\mathcal{X}}}(\Delta^{\mathcal{J}}, \mathcal{X})$ is naturally homotopy equivalent to $\underline{\mathbf{sStrat}}_{P_{\mathcal{X}}}(\Delta^{\mathcal{I}}, \mathcal{X}) = \mathrm{HoLink}_{\mathcal{I}}(\mathcal{X})$. In other words, $\hat{\mathrm{H}}\mathrm{oLink}_n$ carries a lot of homotopy-theoretically redundant data, which is already contained in links of lower dimension. This extra data is only necessary to make $\hat{\mathrm{H}}\mathrm{oLink}_n$ functorial in morphisms that do not induce injections on the poset level.

Extended homotopy links turn out to create weak equivalences in $\underline{\mathbf{sStrat}}^{\mathfrak{d}}$. To see this, note first that:

Proposition 5.3.4.3. *A stratified simplicial map $f: \mathcal{X} \to \mathcal{Y} \in$ s***Strat** *that induces isomorphisms on $\pi_0\hat{\mathrm{H}}\mathrm{oLink}_0$ and $\pi_0\hat{\mathrm{H}}\mathrm{oLink}_1$ induces an isomorphism $P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{Y}^{\mathfrak{r}}}$.*

Proof. Consider the explicit construction of $P_{\mathcal{X}^{\mathfrak{r}}}$ in terms of elements and relations in Proposition 5.3.2.9. The elements correspond precisely to the elements of $\pi_0\hat{\mathrm{H}}\mathrm{oLink}_0(\mathcal{X})$. The generating relations correspond precisely to the elements of $\pi_0\hat{\mathrm{H}}\mathrm{oLink}_1(\mathcal{X})$ (with components of degenerate flags corresponding to equalities). Hence, the result follows. □

We may then show:

Proposition 5.3.4.4. *A stratified simplicial map $f: \mathcal{X} \to \mathcal{Y}$ is a diagrammatic equivalence if and only if it induces weak homotopy equivalences on all extended homotopy links $\hat{\mathrm{H}}\mathrm{oLink}_n$, for $n \geq 0$.*

Proof. It follows from Proposition 5.3.2.15, that $\hat{\mathrm{H}}\mathrm{oLink}_n$ sends the refinement morphisms $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$ into isomorphisms. Consequently, by the characterization of refined diagrammatic equivalences in Theorem 5.3.2.19, we may, without loss of generality, assume that $\mathcal{X}$ and $\mathcal{Y}$ are refined. By Proposition 5.3.4.3, under both assumptions that we want to show are equivalent the induced map $P_{\mathcal{X}} = P_{\mathcal{X}^{\mathfrak{r}}} \to P_{\mathcal{Y}^{\mathfrak{r}}} = P_{\mathcal{Y}}$ is an isomorphism. Hence, we may, without loss of generality, assume that it is given by the identity. Since $\mathcal{X}$ and $\mathcal{Y}$ are colocal objects with respect to the right Bousfield localization defining $\underline{\mathbf{sStrat}}^{\mathfrak{d}}$, f is a diagrammatic equivalence if and only if it is a poset-preserving diagrammatic equivalence. In particular, this is the case if and only if f induces weak equivalences on all homotopy links. Since the extended homotopy links are given as coproducts of all homotopy links and spaces naturally weakly equivalent to the latter, it follows that f is poset-preserving diagrammatic equivalence if and only if it induces an equivalence on extended homotopy links. □

Remark 5.3.4.5. We may rephrase Proposition 5.3.4.4 in the sense that the weak equivalences on $\underline{\mathbf{sStrat}}^{\mathfrak{d}}$ are transported from weak equivalences of bisimplicial sets (interpreted as simplicial presheaves on Δ), under the functor into bisimplicial sets

$$\hat{\mathrm{H}}\mathrm{oLink}: \mathrm{s}\mathbf{Strat} \to \mathbf{ssSet}$$

induced by the functoriality of $\hat{\mathrm{H}}\mathrm{oLink}_n$ in n. This justifies the name diagrammatic equivalences, as these equivalences are created by the diagram of extended homotopy links. It seems plausible that $\underline{\mathbf{sStrat}}^{\mathfrak{d}}$ is Quillen equivalent to a localization of **ssSet** equipped with the Reedy model structure. This is also supported by the fact that $\underline{\mathbf{sStrat}}^{\mathfrak{c}}$ is a left Bousfield localization of $\underline{\mathbf{sStrat}}^{\mathfrak{d}}$, which is equivalent to $\mathbf{sSet}^{\mathfrak{D}}$, whose homotopy theory may in turn be presented as a left Bousfield localization of complete Segal spaces.

5.3.5 Stratified mapping spaces

Give two layered ∞-categories X and Y, the ∞-category of functors Y^X is itself layered. Indeed, it follows from the fact that isomorphisms of functors are detected pointwise that it even suffices for Y to be layered. Similarly, in the world of topological stratified spaces (more specifically homotopically stratified spaces) [Hug99b] equipped the space of stratified maps with a natural decomposition (which generally may not be a stratification) and investigated the lifting properties of such mapping spaces. In [Nan19], the author refined the topology on these mapping spaces in order to obtain internal mapping spaces, at least for stratified spaces with non-empty strata. Hence, it is not surprising that the homotopy theories on stratified simplicial sets defined in this paper admit a notion of stratified mapping space. In other words, in this subsection we prove that all of the model structures on s**Strat** presented in this paper are cartesian closed (Theorem 5.3.5.4). Let us begin with the corresponding statement on 1-categories.

Proposition 5.3.5.1. s**Strat** *is a cartesian closed category.*

Proof. Recall first that the category **Pos** is also cartesian closed. Given two posets P, Q, the inner hom P^Q is obtained by equipping $\mathbf{Pos}(P,Q)$ with the poset structure given by

$$f \leq g \iff \forall p \in P \colon f(p) \leq g(p),$$

for $f, g \in \mathbf{Pos}(P,Q)$. Next, note that the adjunction $P(-) \dashv \mathrm{N}(-)$ between simplicial sets and posets has the property that the left adjoint preserves finite products. It follows by an easy application of the Yoneda lemma that there is a natural isomorphism

$$\mathrm{N}(P)^{\mathrm{N}(Q)} \to \mathrm{N}(P^Q).$$

On vertices, it is simply given by the identification $\mathbf{sSet}(\mathrm{N}(P), \mathrm{N}(Q)) \cong \mathbf{Pos}(P,Q)$, which under the adjunction $P \dashv \mathrm{N}$ entirely describes the map. Now, let $\mathcal{X}, \mathcal{Y} \in \mathrm{s}\mathbf{Strat}$. We construct the exponential object $\mathcal{Y}^{\mathcal{X}}$, that is, we construct a stratified simplicial set $\mathcal{Y}^{\mathcal{X}}$ together with a natural isomorphism $\mathrm{s}\mathbf{Strat}(- \times \mathcal{X}, \mathcal{Y}) \cong \mathrm{s}\mathbf{Strat}(-, \mathcal{Y}^{\mathcal{X}})$. Consider the pullback diagram of simplicial sets

$$\begin{tikzcd} Y^X \times_{\mathrm{N}(P_{\mathcal{Y}})^X} \mathrm{N}(P_{\mathcal{Y}})^{\mathrm{N}(P_{\mathcal{X}})} \arrow[r] \arrow[d] & Y^X \arrow[d] \\ \mathrm{N}(P_{\mathcal{Y}}^{P_{\mathcal{X}}}) \cong \mathrm{N}(P_{\mathcal{Y}})^{\mathrm{N}(P_{\mathcal{X}})} \arrow[r] & \mathrm{N}(P_{\mathcal{Y}})^X. \end{tikzcd} \tag{5.37}$$

The left-hand side defines a stratified simplicial set over the poset $P_{\mathcal{Y}}^{P_{\mathcal{X}}}$. Let us denote the latter by $\mathcal{Y}^{\mathcal{X}}$. A morphism from a stratified simplicial set $\mathcal{Z}$ into this stratified simplicial set corresponds to the data of a morphism $\mathrm{N}(P_{\mathcal{Z}}) \to \mathrm{N}(P_{\mathcal{Y}}^{P_{\mathcal{X}}})$, together with a morphism $Z \to Y^X$ making the induced diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[d] & & Y^X \arrow[d] \\ \mathrm{N}(P_{\mathcal{Z}}) \arrow[r] & \mathrm{N}(P_{\mathcal{Y}}^{P_{\mathcal{X}}}) \cong \mathrm{N}(P_{\mathcal{Y}})^{\mathrm{N}(P_{\mathcal{X}})} \arrow[r] & \mathrm{N}(P_{\mathcal{Y}})^X. \end{tikzcd} \tag{5.38}$$

commute. Using the cartesian structure of **sSet**, this in turn specifies the same data as a commutative diagram

$$\begin{tikzcd} Z \times X \arrow[r] \arrow[d, "s_{\mathcal{Z}} \times s_{\mathcal{X}}"] & Y \arrow[d, "s_{\mathcal{Y}}"] \\ \mathrm{N}(P_{\mathcal{Z}} \times P_{\mathcal{X}}) \cong \mathrm{N}(P_{\mathcal{Z}}) \times \mathrm{N}(P_{\mathcal{X}}) \arrow[r] & P_{\mathcal{Y}} \end{tikzcd} \tag{5.39}$$

that is of a morphism $\mathcal{Z} \times \mathcal{X} \to \mathcal{Y}$. The naturality of the thus constructed bijection

$$\mathrm{s}\mathbf{Strat}(\mathcal{Z} \times \mathcal{X}, \mathcal{Y}) \cong \mathrm{s}\mathbf{Strat}(\mathcal{Z}, \mathcal{Y}^{\mathcal{X}})$$

shows that $\mathcal{Y}^{\mathcal{X}}$ defines the required exponential object. ☐

Lemma 5.3.5.2. *The functor* $- \times - \colon \mathrm{s}\mathbf{Strat} \to \mathrm{s}\mathbf{Strat}$ *preserves (poset-preserving) diagrammatic and (poset-preserving) Joyal-Kan equivalences.*

Proof. Clearly, if $f\colon \mathcal{X} \to \mathcal{Y}$ induces an isomorphism on posets, then so does every product $f \times 1_{\mathcal{Z}}$, for $\mathcal{Z} \in s\mathbf{Strat}$. Thus, it follows from Proposition 5.3.2.27 that it suffices to show the diagrammatic and the Joyal-Kan case, and the poset-preserving versions follow from the latter. The case of diagrammatic equivalences is immediate from Proposition 5.3.4.4, which states that the functor $\hat{\mathrm{H}}\mathrm{oLink}\colon \mathbf{ssSet}$ creates weak equivalences and the fact that the latter commutes with products. Finally, for the Joyal-Kan case, suppose that $f\colon \mathcal{X} \to \mathcal{Y}$ is a Joyal-Kan equivalence and consider the following induced commutative diagram:

$$\begin{array}{ccc} \mathcal{X} \times \mathcal{Z} & \longrightarrow & \mathrm{Ex}^{\infty}\mathcal{X} \times \mathrm{Ex}^{\infty}\mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{Y} \times \mathcal{Z} & \longrightarrow & \mathrm{Ex}^{\infty}\mathcal{Y} \times \mathrm{Ex}^{\infty}\mathcal{Z}\,. \end{array} \tag{5.40}$$

Since the natural transformation $1 \to \mathrm{Ex}^{\infty}$ is a poset-preserving diagrammatic equivalence, it follows from the diagrammatic case that both horizontals are diagrammatic and thus also Joyal-Kan equivalences. Hence, by two-out-of-three, we only need to show that the right vertical is a Joyal-Kan equivalence. By Proposition 5.3.2.25, using the fact that a product of Kan complexes is a Kan complex, it follows that the right vertical is a Joyal-Kan equivalence if and only if the underlying simplicial map is a categorical equivalence. This map is given by the product of the underlying simplicial maps of $1_{\mathrm{Ex}^{\infty}\mathcal{Z}}$ and $\mathrm{Ex}^{\infty}(f)$. By assumption, and since Ex^{∞} preserves Joyal-Kan equivalences, $\mathrm{Ex}^{\infty}(f)$ is a Joyal-Kan equivalence. We may again apply Proposition 5.3.2.25, from which it follows that the underlying simplicial map of $\mathrm{Ex}^{\infty}(f)$ is a categorical equivalence. Thus, the claim follows from the fact that products in **sSet** preserve categorical equivalences. ☐

Lemma 5.3.5.3. *Given a model category* $\mathbf{M}$*, suppose that the product functor* $-\times-\colon \mathbf{M}\times\mathbf{M} \to \mathbf{M}$ *preserves colimits and weak equivalences in both arguments, and also is such that for any pair of cofibrations* $i\colon A \to B$ *and* $j\colon A' \to B'$ *the induced morphism*

$$i \boxtimes j\colon A \times B' \cup_{A\times A'} B \times A' \to B \times B'$$

is a cofibration. Then $-\times-$ *is a Quillen bifunctor.*

Proof. We need to show that, given two cofibrations as in the statement of the lemma, if (without loss of generality) i is additionally a weak equivalence, then so is $i \boxtimes j$. Consider the diagram

$$\begin{array}{ccccc} A \times A' & \longrightarrow & B \times A' & & \\ \downarrow & & \downarrow & & \\ A \times B' & \longrightarrow & A \times B' \cup_{A\times A'} B \times A' & \longrightarrow & B \times B'. \end{array} \tag{5.41}$$

By assumption, the upper horizontal and lower horizontal compositions are weak equivalences. Since, in addition to this, the upper horizontal is a cofibration and the square is pushout, it follows that its parallel is also a weak equivalence. Hence, by two out of three, so is the right lower horizontal. ☐

Theorem 5.3.5.4. *Let* $s\mathbf{Strat}$ *be equipped with any of the model structures of Sections 5.3.1 and 5.3.2. Then* $s\mathbf{Strat}$ *is a cartesian closed model category.*

Proof. We need to show that the map from the initial object $\varnothing \in s\mathbf{Strat}$ to the terminal objects $\star \in s\mathbf{Strat}$ is a cofibration and, furthermore, that $-\times-\colon s\mathbf{Strat} \times s\mathbf{Strat} \to s\mathbf{Strat}$ is a Quillen bifunctor. The former statement holds, since $(\varnothing \to \star) \cong (\partial\Delta^{[0]} \hookrightarrow \Delta^{[0]})$, which is a generator for the cofibrations in any of the model structures (by Corollary 5.3.1.10 and Theorem 5.3.2.19). For the second statement, we make use of Proposition 5.3.5.1 and Lemmas 5.3.5.2 and 5.3.5.3 and it remains to show that for every pair of cofibrations $i\colon \mathcal{A} \hookrightarrow \mathcal{B}$, $j\colon \mathcal{A}' \hookrightarrow \mathcal{B}'$ the induced stratified simplicial map

$$i \boxtimes j\colon \mathcal{A} \times \mathcal{B}' \cup_{\mathcal{A}\times\mathcal{A}'} \mathcal{B} \times \mathcal{A}' \to \mathcal{B} \times \mathcal{B}'$$

is a cofibration. Both in $s\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ and in $s\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ a map is a cofibration, if and only if the underlying simplicial map is a cofibration (i.e. a monomorphism) hence in these cases the $i \boxtimes j$ is a cofibration, since the underlying simplicial map $\mathcal{F}(i \boxtimes j) \cong \mathcal{F}(i) \boxtimes \mathcal{F}(j)$ is a cofibration. Even more, by [Hov07, Cor. 4.2.5] we only need to consider the cases where i and j are generating cofibrations. For the cases $s\mathbf{Strat}^{\mathfrak{c}}$ and $s\mathbf{Strat}^{\mathfrak{d}}$, it follows that we only need to show that

$$\partial\Delta^{[n]} \times \Delta^{[m]} \cup_{\partial\Delta^{[n]}\times\partial\Delta^{[m]}} \Delta^{[n]} \times \partial\Delta^{[m]} \to \Delta^{[n]} \times \Delta^{[m]}$$

is a cofibration. Since both the source and target of this cofibration are refined, it follows from the cases $s\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ and $s\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$, together with the characterization of cofibrations between cofibrant objects in a right-bousfield localization ([Hir03, p. 3.3.16]) that $i \boxtimes j$ is a cofibration. □

One of the main results of [Hug99b] was that for certain particularly convenient stratified spaces $\mathcal{X}$ - so-called homotopically stratified spaces - for any closed union of strata $\mathcal{A} \hookrightarrow \mathcal{X}$ the starting point evaluation map from the space of stratified paths $|\Delta^{[1]}|_s \to \mathcal{X}$ starting in $\mathcal{A}$, $\text{Path}_{nsp}(\mathcal{A}, \mathcal{X})$, is a stratified fibration (i.e., has the right lifting property with respect to inclusions into the stratified cylinder). Homotopically stratified spaces have the property that they are mapped into fibrant objects in $s\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$, and being a fibration in $s\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ is even a stronger property than just lifting (simplicial) stratified homotopies. Thus, we may interpret the following result as a combinatorial analogue of [Hug99b, Main Result].

Construction 5.3.5.5. Let $s\mathbf{Strat}$ be equipped with one of the model structures of Section 5.3.2. Let $\mathcal{X} \in s\mathbf{Strat}$ and let $A \subset X$ be a simplicial subset. Denote by $\mathcal{A}$ the stratified simplicial set $A \to X \to P_{\mathcal{X}}$. Now, consider the pullback diagram in $\mathbf{Strat}$

$$\begin{tikzcd} \text{Path}_{nsp}(\mathcal{X}, \mathcal{A}) \coloneqq \mathcal{X}^{\Delta^{[1]}} \times_{\text{ev}_0} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{X}^{\Delta^{[1]}} \arrow[d, "\text{ev}_0"] \\ \mathcal{A} \arrow[r] & \mathcal{X}. \end{tikzcd} \tag{5.42}$$

$\text{Path}_{nsp}(\mathcal{X}, \mathcal{A})$ is a stratified space over $\{(p,q) \in P_{\mathcal{X}} \times P_{\mathcal{X}} \mid p \leq q\}$. Its vertices are precisely the 1-simplices (i.e. paths) in $\mathcal{X}$, starting in $\mathcal{A}$. Now, if $\mathcal{X}$ is fibrant, then it follows from the cartesian closedness of $\mathbf{Strat}$ that the right-hand vertical is a fibration. Consequently, so is the starting point evaluation map $\text{Path}_{nsp}(\mathcal{X}, \mathcal{A}) \to \mathcal{A}$. In particular, this map has the right lifting property with respect to the acyclic cofibrations $\mathcal{B} \hookrightarrow \mathcal{B} \otimes \Delta^1$.

Chapter 6

On the homotopy links of stratified cell complexes

Note to the reader: The following chapter was structured as an independent article, in order to allow for easier accessibility. A preliminary version was made publicly available on the arXiv (see [Waa24b]). Notation in this chapter is entirely consistent with Chapters 1 and 5. There may be minor notation differences compared to Chapter 3. However, as all notation is introduced separately in this chapter, this should not pose an issue.

Homotopy links have proven to be one of the most powerful tools of stratified homotopy theory. In previous work, we described combinatorial models for the generalized homotopy links of a stratified simplicial set. For many purposes, in particular to investigate the stratified homotopy hypothesis, a more general version of this result pertaining to stratified cell complexes is needed. Here we prove that, given a stratified cell complex X, the generalized homotopy links can be computed in terms of a certain subcomplex of a subdivision of X. As a consequence, it follows that generalized homotopy links map certain pushout diagrams of stratified cell complexes into homotopy pushout diagrams. This result is crucial to the development of (semi-)model structures for stratified homotopy theory in which geometric examples of stratified spaces, such as Whitney stratified spaces, are bifibrant.

6.1 Introduction

Stratified spaces were first introduced by Whitney, Thom and Mather to investigate spaces with singularities (see [Whi65b; Mat12; Mat73; Tho69]). One of the central insights of [Mat73] was that a key ingredient in the study of stratified spaces with smooth manifold strata was having a theory of tubular neighborhoods of strata available. These made it possible to study stratified spaces in terms of their strata and the so-called link bundles, connecting the latter. In a less geometric scenario, such tubular (or regular) neighborhoods may generally not be available. To avoid this difficulty, [Qui88] introduced the notion of a homotopy link - a homotopy-theoretic proxy for the boundary of a regular neighborhood. Given two strata X_p and X_q in a poset-stratified space $s_{\mathcal{X}}\colon X \to P$, with $p < q \in P$, the associated homotopy link is the space of paths starting in X_p and immediately exiting into X_q.

It turns out that much of the homotopy theory of stratified spaces may be understood in terms of the homotopy types of homotopy links and strata (and the structure maps between them). For example, [Mil13] proved that a stratum-preserving map between two sufficiently regular stratified spaces is a stratum-preserving homotopy equivalence, if and only if it induces equivalences on strata and (pairwise) homotopy links. [Dou21c; Hen] built on this insight, and developed a homotopy theory of stratified spaces in which weak equivalences are defined as such

stratified maps that induce weak equivalences on all generalized homotopy links, which replace exit-paths by more general stratified singular simplices. We call this theory the Douteau-Henriques homotopy theory, henceforth. It turns out that the Douteau-Henriques homotopy theory is in some sense minimal among many stratified homotopy theories (see [Dou21a]). Thus, it is not surprising that many other approaches to stratified homotopy theory turn out to be localizations, global versions, or subtheories of the latter (see [Waa24a]). It follows from this that much about stratified homotopy theory (not only about the Douteau-Henriques one) can be understood in terms of generalized homotopy links. For example, in [DW22], we obtained explicit combinatorial models - in terms of a subobject of a subdivision - for the homotopy link of a stratified simplicial set. We used this to prove a stratified ∞-categorical analogue of the Kan-Quillen equivalence between topological spaces and simplicial sets ([DW22, Thm. 5.1, Rem. 5.4], which are Theorem 3.5.1.1 and Remark 3.5.1.4 here). The latter can be used to prove a Quillen equivalence version of the topological stratified homotopy hypothesis (see [Waa24c], which is Chapter 7 here).
The applications in [DW22] show the strength of a general paradigm: Homotopy links as a mathematical tool become most powerful when geometric or combinatorial, as well as homotopy-theoretic models are available.

The main goal of this article is to extend the availability of such models to the case of so-called *stratified cell complexes*. Roughly speaking, a stratified cell complex is a stratified space obtained by inductively gluing in stratified simplices along stratum-preserving maps defined on the boundaries of the simplices. From a technical point of view, constructing and verifying models for generalized homotopy links in terms of subcomplexes of a subdivision of a stratified cell complex turns out to be significantly more involved than the simplicial set case. This is mainly due to the fact that the case of cell complexes allows arbitrarily complicated gluing maps, rather than only allowing for (piecewise) linear ones (see Example 6.4.2.6). Nevertheless, here we show the following.
We describe a construction that, given a stratified cell complex $\mathcal{X}$, stratified over a poset P, and $\mathcal{I} = \{p_0 < \dots < p_n\} \subset P$ a finite increasing sequence, produces a stratified subspace $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ of $\mathcal{X}$. This construction relies on a choice of (appropriate topological barycentric) subdivision Ψ of the cell structure on $\mathcal{X}$, and is such that $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I}) \subset \mathcal{X}$ defines a subcomplex of this subdivision that contains the stratum X_{p_0}. It can be seen as a generalization of regular neighborhoods in the PL setting, both to the case of more than two strata, as well as to the case of more general stratified cell complexes. Crucially, the construction has the following property:

Theorem HA (Theorems 6.2.4.14 and 6.4.2.7 and Proposition 6.3.2.11)**.** *Let $\mathcal{X}$ be a stratified cell complex, stratified over a poset P, and $\mathcal{I} = \{p_0 < \dots < p_n\} \subset P$ a finite increasing sequence. Denote by $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X}$ the $\mathcal{I}$-th generalized homotopy link of $\mathcal{X}$ (see [Dou21c; DW22]). There exists a subdivision Ψ of the cell structure on $\mathcal{X}$, such that $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ defines a well-defined subcomplex with respect to this subdivision. Given such a Ψ, there is a canonical weak homotopy equivalence*

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X} \simeq \mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})_{p_n},$$

between the $\mathcal{I}$-th generalized homotopy link and the p_n-stratum of $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$. Furthermore, subdivisions can be chosen such that the construction of $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ is compatible with stratum-preserving maps and pushouts along inclusions of subcomplexes.

In fact, we show that the subcomplexes $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ can even be used to model the whole homotopy link diagram of [Dou21c]. Theorem HA has the following corollary, which is central to the construction of semimodel categories of stratified spaces in [Waa24c].

Theorem HB (Corollary 6.4.2.8)**.** *Let P be a poset and $\mathcal{I} = \{p_0 < \dots < p_n\} \subset P$ a finite*

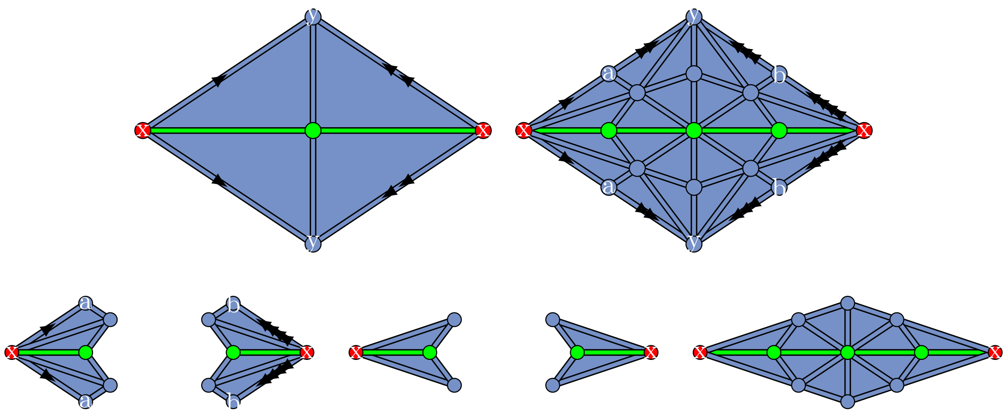


Figure 6.1: The upper left corner shows a stratified cell structure for the pinched torus, stratified over the poset $\{0 < 1 < 2\}$. Vertices with the same name, and edges with the same markings are being identified and the stratification is indicated by the coloring. To its right, a barycentric subdivision Ψ of this cell structure is shown. In the following row there are illustrations of the subcomplexes $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ for $\mathcal{I} = \{0 < 2\}, \{0 < 1 < 2\}, \{1 < 2\}$.

increasing sequence. Consider a pushout diagram of P-stratified cell complexes

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "c"] \arrow[d, "f"] & \mathcal{B} \arrow[d] \\ \mathcal{X} \arrow[r, hook] & \mathcal{Y}, \end{tikzcd} \tag{6.1}$$

with c an inclusion of a stratified subcomplex. Then the induced diagram of spaces

$$\begin{tikzcd} \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{A} \arrow[r] \arrow[d] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{B} \arrow[d] \\ \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X} \arrow[r] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{Y}, \end{tikzcd} \tag{6.2}$$

obtained by taking generalized homotopy links, is homotopy cocartesian.

In [Waa24c] we use the latter result to construct new (semi-)model structures for stratified homotopy theory in which classical examples of stratified spaces such as Whitney stratified spaces are bifibrant. Furthermore, we derive from this semimodel structure a version of the stratified homotopy hypothesis pertaining to a conjecture of [AFR19] ([Waa24c, Thm. B]).

6.1.1 Overview of the article

It is a well-known classical result that the (pairwise) homotopy link of a piecewise linear two strata stratified space may be computed in terms of the boundary of a regular neighborhood of the lower stratum. Equivalently, one may take the homotopy type of the complement of the lower stratum in the regular neighborhood. In [Qui88] the author generalized this result to more general topological notions of regular neighborhood $X_p \subset N \subset X$ of a stratum X_p, which admit a so-called *nearly stratum-preserving deformation retraction* (this nomenclature was first used in [Mil13], Quinn speaks of *tame inclusions of strata*). These are given by homotopies

$$R \colon N \times [0,1] \to X$$

such that $R \times (0,1]$ is stratum-preserving, R_1 is the inclusion $N \hookrightarrow X$, R is constant on X_p and such that R_0 has image entirely in X_p. In the case of the realization of a stratified-simplicial set, such regular neighborhoods can be obtained by first taking a barycentric subdivision, and then taking the union of closed simplices intersecting the lower stratum (see [Qui88]). The goal here is a two-fold generalization. Firstly, we aim to replace pairwise homotopy links with generalized homotopy links, obtained by replacing the stratified interval with stratified simplices. Secondly, we generalize from stratified simplicial sets to stratified cell complexes. The case of generalized homotopy links of stratified simplicial sets was studied in detail in [DW22] (Chapter 3). To further generalize these results to stratified cell complexes, we need to generalize the notions of neighborhood and nearly stratum-preserving deformation retraction to the n-strata case. We proceed to do so in the following steps.

1. In Section 6.2.4, we introduce the notion of a *system of strata-neighborhoods* of a stratified space $\mathcal{X}$. These are defined in a way that they allow for a computation of homotopy links close to a singularity. The ultimate goal of the theory is to identify a class of such neighborhood systems for which the topology of the neighborhoods themselves may be used to compute the homotopy type of homotopy links, and there is no need to pass to path-spaces. Such neighborhood systems are called *homotopy link models*. Following from this, our strategy of proof is then to show that every stratified cell complex may be equipped with a homotopy link model, and furthermore that this can be done in a way that is compatible with pushout diagrams.

2. A first step lies in constructing the so-called *standard neighborhood-systems* for realizations of stratified simplicial sets (Section 6.4.1). We show that these standard neighborhoods turn out to be universal in some sense (Proposition 6.3.1.19). This result is crucial to obtain homotopy link models that are compatible with given maps of stratified cell complexes. Furthermore, it provides a new proof of a locality principle for homotopy links of strata-neighborhoods, which ultimately provides a more conceptual proof of [DW22, Thm. 4.8] (Theorem 3.4.4.1 in this text).

3. In Section 6.3.2, we then use the results on strata-neighborhoods of stratified simplicial sets to generalize the construction of standard neighborhood systems to stratified cell complexes.

4. Having strengthened our understanding of strata-neighborhood systems, we return to the notion of a homotopy link model. At this point, we have all the results necessary available to prove that the *regular complement diagram* (Construction 6.2.4.11) associated to a homotopy link model is weakly equivalent to the diagram of homotopy links of the associated stratified space (Theorem 6.2.4.14).

5. Finally, it remains to prove that the standard neighborhood systems we have constructed for stratified cell complexes are homotopy link models. To accomplish this, we first define an adaptation to the n-strata scenario of the notion of nearly stratum-preserving neighborhood retracts (as they were defined in [Qui88]) a so-called ASP$\mathcal{I}$R (see Definition 6.4.0.1). In Section 6.4, we show that the existence of these ASP$\mathcal{I}$Rs provides a way to guarantee that a neighborhood system is a homotopy link model (Proposition 6.4.0.7).

6. In Section 6.4.1, we then return to the standard neighborhood systems of realization of stratified simplicial sets and show that these can be equipped with ASP$\mathcal{I}$Rs (Proposition 6.4.1.6). In particular, this result has [DW22, Thm. 4.8] (Theorem 3.4.4.1 in this article) as a corollary.

7. Finally, in Section 6.4.2, we generalize the results of the previous section to standard neighborhood systems of stratified cell complexes. To do so, we first provide a technical gluing lemma that ultimately allows a cell-by-cell construction of ASP$\mathcal{I}$Rs (Lemma 6.4.2.2). It then remains to investigate the case of a single cell (Lemma 6.4.2.3), to finish the proof that standard-neighborhood systems provide homotopy link models (Proposition 6.4.2.4).

6.2 Basic notions: From homotopy links to their models

The goal of this section is to introduce the basic objects and notions under investigation. We begin by recalling the necessary language and notation from stratified homotopy theory Section 6.2.1 in particular the notion of homotopy link Section 6.2.2. We then introduce the central objects of study to this paper: Stratified cell complexes (Section 6.2.3). Our goal is to find convenient models for the homotopy links of such stratified cell complexes. We make this idea rigorous with the notions of strata-neighborhood systems and homotopy link models in Section 6.2.4.

6.2.1 Language and notation

Let us first fix some language and notation pertaining to stratified homotopy theory, mostly lifted from [Dou21a; Dou21c; DW22; Hai23].

Notation 6.2.1.1. We will denote simplicial categories in the form $\underline{\mathbf{S}}$ and their underlying categories in the form $\mathbf{S}$. Given two categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\mathbf{Fun}(\mathbf{C},\mathbf{D})$ the category of functors between the two categories.

Notation 6.2.1.2. We use the following notation for categories of simplicial sets.

- We denote by Δ the category of finite, linearly ordered posets of the form $[n] \coloneqq \{0,\cdots,n\}$, for $n \in \mathbb{N}$.
- We denote by $\underline{\mathbf{sSet}}$ the simplicial category of simplicial sets, i.e., the category of set-valued presheaves on Δ^{op}, equipped with the canonical simplicial structure induced by the product (see [Lur09] for all of the standard notation used for simplicial sets).
- When we treat $\underline{\mathbf{sSet}}$ as a (simplicial) model category, this will generally be with respect to the Kan-Quillen model structure (see [Qui67]), unless otherwise noted. When we use Joyal's model structure for quasi-categories ([JT08]) instead, we will denote this model category by $\mathbf{sSet}^{\mathfrak{J}}$.

Notation 6.2.1.3. **Top** is going to denote either of the following three categories of topological spaces.

1. The category of *all topological spaces*, which we will also refer to as *general* topological spaces.
2. The category of compactly generated topological spaces, i.e., such spaces that have the final topology with respect to compact Hausdorff spaces (see, for example, [Rez17]).
3. The category of Δ-generated topological spaces, i.e., such spaces which have the final topology with respect to realizations of simplices, or equivalently just with respect to the unit interval (compare [Dug03; Gau21]).

We denote by $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ the realization functor of simplicial sets and by $\mathrm{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$ its right adjoint. **Top** naturally carries the structure of a simplicial category, tensored and powered over $\underline{\mathbf{sSet}}$, induced by left Kan extension of the construction

$$T \otimes \Delta^n \coloneqq T \times |\Delta^n|.$$

We denote the resulting simplicial category by $\underline{\mathbf{Top}}$. Furthermore, we will always consider $\underline{\mathbf{Top}}$ to be equipped with the Quillen-model structure [Qui67], which makes $|-| \dashv \mathrm{Sing}$ a simplicial Quillen equivalence, between $\underline{\mathbf{Top}}$ and $\underline{\mathbf{sSet}}$, which creates weak equivalences in both directions.

Remark 6.2.1.4. Note that one commonly only defines the simplicial structure for compactly or Δ-generated spaces. This is, however, mostly due to the fact that, for general topological spaces T and infinite simplicial sets K, the tensoring $T \otimes K$ does not agree with the inner product $T \times |K|$. Instead, it is given by a colimit of products of T with the simplices of K. Similarly, the power T^K is not given by an internal mapping space (which does not necessarily exist for arbitrary K) but by the limit of the mapping spaces of simplices of K, equipped with the compact open topology. The reason we do not take the regular approach of restricting to one fixed convenient category of topological spaces is that much of the literature has been formulated for the Δ-generated case, while we will make several arguments in the category of general topological spaces later on, which seem to lack an internal analogue in the category of Δ-generated spaces. Note that from a homotopy-theoretic perspective these choices in set-theoretic-topological foundations are usually not relevant, as any space is canonically weakly equivalent to its Δ-fication (compactly generated replacement). Furthermore, all our results concern spaces in the Δ-generated category (which is included in the other two categories) and the choice of larger framework is thus mostly inessential.

For the remainder of this section, we fix some category of topological spaces **Top** as in Notation 6.2.1.3.

Notation 6.2.1.5. We are going to use the following terminology and notation for partially ordered sets, drawn partially from [Dou21a] and [Hai23]:

- We denote by **Pos** the category of partially ordered sets, with morphisms given by order-preserving maps.
- We consider Δ as a subcategory of **Pos** in the obvious fashion. Given $P \in \mathbf{Pos}$, we denote by Δ_P the slice category $\Delta_{/P}$. That is, objects are given by arrows $[n] \to P$ in **Pos**, $n \in \mathbb{N}$, and morphisms are given by commutative triangles.
- We denote by $\mathrm{sd}(P)$ the *subdivision of* P, given by the full subcategory of Δ_P of such arrows $[n] \to P$, which are injective.
- The objects of Δ_P are called *flags of* P. We represent them by strings $[p_0 \leq \cdots \leq p_n]$, of $p_i \in P$.
- Objects of $\mathrm{sd}(P)$ are called regular *flags of* P. We represent them by strings $[p_0 < \cdots < p_n]$, of $p_i \in P$.

Notation 6.2.1.6. Having fixed a category of topological spaces **Top**, we then use the following notation for stratified topological spaces (all of these constructions can be found in [Dou21c] among other places).

- We think of the 1-category **Pos** as naturally embedded in **Top**, via the Alexandrov topology functor, equipping a poset P with the topology where the closed sets are given by the downward closed sets. By abuse of notation, we just write P, for the Alexandrov space corresponding to it (compare [DW22, Def. 2.2], which is Definition 3.2.1.2 in this article).
- For $P \in \mathbf{Pos}$, we denote by $\mathbf{Strat}_P$ the slice category $\mathbf{Top}_{/P}$.
- Objects of $\mathbf{Strat}_P$ are called P*-stratified spaces*. They are given by a tuple $(T, s{:}T \to P)$. We will usually use calligraphic letters for stratified spaces and stick to the notational convention $\mathcal{X} = (X, s_{\mathcal{X}})$ to refer to the underlying space and the stratification.
- Morphisms in $\mathbf{Strat}_P$ are called *stratum-preserving maps.*
- Given a map of posets $f{:}Q \to P$ and $\mathcal{X} \in \mathbf{Strat}_P$, we denote by $f^*\mathcal{X} \in \mathbf{Strat}_Q$ the stratified space $X \times_P Q \to Q$. We are mostly concerned with the case where f is given by the inclusion of a singleton $\{p\}$, of a subset $\{q \sim p \mid q \in P\}$, for $p \in P$ and $\sim$ some relation

on the partially ordered set P (such as $\leq$), or more generally a subposet $Q \subset P$. We then write $\mathcal{X}_p$ (or, respectively, $\mathcal{X}_{\sim p}$, $\mathcal{X}_Q$) instead of $f^*\mathcal{X}$. The spaces $\mathcal{X}_p$, for $p \in P$ are called the *strata of* $\mathcal{X}$.

Notation 6.2.1.7. Throughout this article, we will consider a series of subspaces of stratified spaces in the category **Top**, that is, use maps on them that are not stratum-preserving. We keep following the convention (Notation 6.2.1.3), which is that calligraphic letters indicate the stratified context, and regular letters the non-stratified one. For example, for $\mathcal{X} \in \mathbf{Strat}_P$ and $p \in P$, $\mathcal{X}_{\leq p}$ is the stratified space over $\{q \in P \mid q \leq p\}$, given by restricting $\mathcal{X}$ and $X_{\leq p}$ is its underlying topological space. Note that in the case of the strata there is no notational conflict with using both $\mathcal{X}_p$ and X_p, if we identify stratified spaces over a poset with one element with topological spaces. This type of notational convention reaches its syntactic limits when applied to expressions such as $|\Delta^{\mathcal{I}}|_s$, which do not have calligraphic counterparts. In this case, we will simply write $|\Delta^{\mathcal{I}}|$ to indicate the underlying topological space.

Notation 6.2.1.8. We use the following terminology and notation for (stratified) simplicial sets, drawn partially from [Dou21a] and [Hai23]:

- We think of **Pos** as being naturally embedded in **sSet**, via the nerve functor (compare [Hai23]). By abuse of notation, we just write P, for the simplicial set given by the nerve of $P \in \mathbf{Pos}$.

- For $P \in \mathbf{Pos}$, we denote by $\mathrm{s}\mathbf{Strat}_P$ the slice category $\mathbf{sSet}_{/P}$, which is equivalently given by the category of set valued presheaves on Δ_P.

- Objects of $\mathrm{s}\mathbf{Strat}_P$ are called *P-stratified simplicial sets.* They are given by a tuple $\mathcal{X} = (X, s_{\mathcal{X}}: X \to P)$.

- Morphisms in $\mathrm{s}\mathbf{Strat}_P$ are called *stratum-preserving simplicial maps.* Simplicial homotopies in $\mathrm{s}\mathbf{Strat}_P$ are called *stratified simplicial homotopies.* Simplicial homotopy equivalences in $\mathrm{s}\mathbf{Strat}_P$ are called *stratum-preserving simplicial homotopy equivalences*.

- Given a map of posets $f: Q \to P$ and $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, we denote by $f^*\mathcal{X} \in \mathrm{s}\mathbf{Strat}_Q$ the stratified simplicial set $X \times_P Q \to Q$. We are mostly concerned with the case where f is given by the inclusion of a singleton $\{p\}$, of a subset $\{q \sim p \mid q \in P\}$, for $p \in P$ and $\sim$ some relation on the partially ordered set P (such as $\leq$), or more generally, a subposet $Q \subset P$. We then write $\mathcal{X}_p$ (or, respectively, $\mathcal{X}_{\sim p}$, $\mathcal{X}_Q$) instead of $f^*\mathcal{X}$. The simplicial sets $\mathcal{X}_p$, for $p \in P$ are called the *strata of* $\mathcal{X}$.

- For a flag $\mathcal{J} = [p_0 \leq \cdots \leq p_n] \in \Delta_P$, we write $\Delta^{\mathcal{J}}$ for the image of $\mathcal{J}$ in $\mathrm{s}\mathbf{Strat}_P$ under the Yoneda embedding $\Delta_P \hookrightarrow \mathrm{s}\mathbf{Strat}_P$. Equivalently, $\Delta^{\mathcal{J}}$ is given by the unique simplicial map $\Delta^n \to P$ mapping $i \mapsto p_i$. $\Delta^{\mathcal{J}}$ is called the *stratified simplex associated to* $\mathcal{J}$.

- Using the fully faithful (and continuous) embedding $\Delta_P \hookrightarrow \mathrm{s}\mathbf{Strat}_P$, we extend the base change notation for stratified simplicial sets to flags. That is, for $f: Q \to P$ we write $f^*\mathcal{J}$ for the unique flag of Q corresponding to $f^*(\Delta^{\mathcal{J}})$. We use the same shorthand notation for subsets $Q \subset P$. For example $\mathcal{J}_{\leq p}$ is the flag obtained from $\mathcal{J}$ by removing all entries not lesser equal to p.

- Given a stratified simplex $\Delta^{\mathcal{J}}$, for $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, we write $\partial\Delta^{\mathcal{J}}$ for its *stratified boundary*, given by the composition $\partial\Delta^n \to \Delta^n \to P$.

Recollection 6.2.1.9 ([Dou21a])**.** For fixed $P \in \mathbf{Pos}$, the two categories $\mathbf{Strat}_P$ and $\mathrm{s}\mathbf{Strat}_P$ are connected through a singular simplicial set, realization style adjunction, denoted

$$|-|_s : \mathrm{s}\mathbf{Strat}_P \rightleftharpoons \mathbf{Strat}_P : \mathrm{Sing}_s.$$

The left adjoint is constructed by mapping a stratified simplex $\Delta^{\mathcal{J}} \to P$, with $\mathcal{J} = [p_0 \leq \dots \leq p_n]$, to the stratified space

$$\begin{aligned} |\Delta^n| &\to P \\ x &\mapsto \sup\{p_i \in \mathcal{J} \mid x_i > 0\}, \end{aligned}$$

where we consider $|\Delta^n|$ as embedded in $\mathbb{R}^{n+1} \cong \mathbb{R}^{\mathcal{J}}$. If we consider $\mathrm{s}\mathbf{Strat}_P$ as the category of set-valued presheaves on Δ_P, then by the logic of a nerve and realization functor, $\mathrm{Sing}_s\mathcal{X}$ is hence given the stratified simplicial set

$$\mathrm{Sing}_s\mathcal{X}(\mathcal{J}) = \mathbf{Strat}_P(|\Delta^{\mathcal{J}}|_s, \mathcal{X})$$

with the obvious structure morphisms.

6.2.2 Generalized homotopy links

Homotopy links were originally introduced in [Qui88] order to obtain a homotopy-theoretic replacement for the boundary of a regular neighborhood in the piecewise-linear scenario. As functors, they may be understood as the right adjoint to taking products with stratified simplices.

Notation 6.2.2.1. In the following subsections, we will make frequent use of the action of **Top** on $\mathbf{Strat}_P$, given by

$$\begin{aligned} \mathbf{Top} \times \mathbf{Strat}_P &\to \mathbf{Strat}_P \\ (T, \mathcal{X}) &\mapsto T \times \mathcal{X} := (T \times X \xrightarrow{\pi_X} X \xrightarrow{s_{\mathcal{X}}} P). \end{aligned}$$

In case there is any possibility of confusion, the stratification always arises from the second component.

Recollection 6.2.2.2 (See [Dou21c])**.** Given a (locally compact in the case of general topological spaces) stratified space $\mathcal{S}$, the functor

$$- \times \mathcal{S} \colon \mathbf{Top} \to \mathbf{Strat}_P$$

admits a right adjoint. It is given by equipping $\mathbf{Strat}_P(\mathcal{S}, \mathcal{X})$ with the respective subspace topology (depending on the choice of category **Top**) of the space of all continuous maps, equipped with the compact open topology. We are particularly interested in the case $\mathcal{S} = |\Delta^{\mathcal{I}}|_s$, for $\mathcal{I} \in \mathrm{sd}(P)$ a regular flag. For $\mathcal{X} \in \mathbf{Strat}_P$, the image under the right adjoint to $- \times |\Delta^{\mathcal{I}}|_s$ is called the *$\mathcal{I}$-th (generalized) homotopy link of $\mathcal{X}$*. Explicitly, it is given by topologizing the set of stratum-preserving maps

$$\{|\Delta^{\mathcal{I}}|_s \to \mathcal{X}\}$$

as described above. We then can summarize the homotopy links in a global functor

$$\mathcal{H}o\mathrm{Link} \colon \mathbf{Strat}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top}),$$

with structure maps of the diagram $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$, $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}'}(\mathcal{X})$ given by restricting along the inclusion $|\Delta^{\mathcal{I}'}|_s \subset |\Delta^{\mathcal{I}}|_s$, for $\mathcal{I}' \subset \mathcal{I}$.

Example 6.2.2.3. If $\mathcal{I} = [p]$ is a singleton, then $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X}$ is naturally homeomorphic to the stratum $\mathcal{X}_p$. For $\mathcal{I} = [p_0 < p_1]$ a pair, the homotopy link $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X}$ is the space of paths starting in $\mathcal{X}_p$ and immediately exiting into $\mathcal{X}_q$, so-called *exit paths* defined in [Qui88].

Example 6.2.2.4. Let $\mathcal{I} \in \mathrm{sd}(P)$ and $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ be a stratified simplicial set. We can consider the first barycentric subdivision of the underlying simplicial set $\mathrm{sd}X$. The vertices of $\mathrm{sd}(P)$ correspond to pairs $(\Delta^{\mathcal{J}} \to \mathcal{X}, \mathcal{J}')$ with $\mathcal{J}' \subset \mathcal{J} \in \Delta_P$ and $\Delta^{\mathcal{J}} \to \mathcal{X}$ non-degenerate. We may then consider the full subsimplicial set of $\mathrm{sd}X$ spanned by such vertices for which

$\mathcal{J}'$ degenerates from $\mathcal{I}$. The latter is called the *simplicial* link, denoted $\mathrm{Link}_{\mathcal{I}}\mathcal{X}$. In the case of two strata, for $\mathcal{I} = [p < q]$, $|\mathrm{Link}_{\mathcal{I}}|$ is precisely the boundary of a regular neighborhood of $(|\mathcal{X}|_s)_p$. In particular, in this case, it is weakly equivalent to the homotopy link of $|\mathcal{X}|_s$. In [DW22], we proved the case of general $\mathcal{I}$ of this result obtaining weak homotopy equivalences

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}|\mathcal{X}|_s \simeq |\mathrm{Link}_{\mathcal{I}}\mathcal{X}|.$$

6.2.3 Stratified cell complexes

Let us now move on to the case of more general stratified cell complexes. We perform a general investigation of structured cell complexes in an abstract categorical scenario in Chapter 8. For the purposes of this chapter, the following will suffices.

Definition 6.2.3.1. Let $\mathcal{X} \in \mathbf{Strat}_P$. A *stratified cell structure on* $\mathcal{X}$ is a family of stratum-preserving maps $(\sigma_i : |\Delta^{\mathcal{J}_i}|_s \to \mathcal{X})_{i \in I}$ such that the following properties hold.

1. X has the final topology with respect to the maps σ_i.
2. For each $i \in I$, σ_i induces a homeomorphism from $|\Delta^{\mathcal{J}_i}|_s \setminus |\partial\Delta^{\mathcal{J}_i}|_s$ onto its image. Denote these images by e_i. Furthermore, denote the image of $|\partial\Delta^{\mathcal{J}}_i|_s$ under σ by ∂e_i.
3. X is given by the (set-theoretic) disjoint union of the cells e_i.
4. Denote by $\prec$ the relation on I, which is generated under transitivity by
$$i \prec j \iff e_i \cap \partial e_j \neq \varnothing.$$
Then $\prec$ is irreflexive, and every element of I only has finitely many precursors with respect to $\prec$.

A stratified Hausdorff space $\mathcal{X}$ together with a stratified cell structure $(\sigma_i)_{i\in I}$ on it is called a *stratified cell complex*. A *stratified subcomplex* of $(\mathcal{X}, (\sigma_i)_{i\in I})$, is a stratified subspace $\mathcal{A} \subset \mathcal{X}$, together with a subset $I' \subset I$ such that I', $\mathcal{A} = \bigcup_{i \in I'} e_i$, and such that I' is closed below under $\prec$.

Remark 6.2.3.2. In many respects, stratified cell complexes behave much like their unstratified counterparts. In particular, it is not hard to see that every stratified subcomplex is closed, and itself a stratified cell complex, with the induced cell structure. We will usually refer to a stratified cell complex just by its underlying stratified space, and keep the cell structure implicit. At times, we will say $\mathcal{X}$ is a stratified cell complex, to refer to the fact that it is Hausdorff and admits a stratified cell structure.

Example 6.2.3.3. Every realization of a stratified simplicial set $\mathcal{X} \in s\mathbf{Strat}_P$ naturally inherits the structure of a stratified cell complexes, with cells given by the realizations of non-degenerate simplices $\Delta^{\mathcal{J}} \to \mathcal{X}$.

If we forget about the cell structure, stratified cell complexes are simply the spaces that arise as absolute cell complexes (in the sense of [Hir03]) from the set of stratified boundary inclusions $\{|\partial\Delta^{\mathcal{J}}|_s \hookrightarrow |\Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P\}$.

Proposition 6.2.3.4. *Let $\mathcal{X} \in \mathbf{Strat}_P$. Then the following are equivalent:*

1. *$\mathcal{X}$ is an absolute cell complex with respect to the set $\{|\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P\}$.*
2. *$\mathcal{X}$ is Hausdorff and admits a stratified cell structure.*

Furthermore, the following relative version of this result holds. Suppose that $\mathcal{A}$ is a stratified cell complex. Then, for a stratified map $i : \mathcal{A} \to \mathcal{X}$, the following are equivalent.

1. *i is a relative cell complex with respect to the set $\{|\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P\}$.*

2. *$\mathcal{X}$ is Hausdorff and admits a stratified cell structure, which makes $\mathcal{A}$ a stratified subcomplex.*

Proof. Essentially, this argument is identical with the non-stratified case. For some reason, we were unable to find a reference for this in the literature. First, note that being an absolute cell complex as above implies that $\mathcal{X}$ is Hausdorff. Indeed, this already holds for absolute cell complexes in the Quillen model structure, and may be seen by extending disjoint opens cell by cell, via a transfinite inductive argument. The remaining translation of structures is handled by Construction 6.2.3.5 below. □

Construction 6.2.3.5. Let $\mathcal{X} \in \mathbf{Strat}_P$. If we write $\varnothing \to \mathcal{X}$ as a transfinite compositions

$$\mathcal{X}^0 \to \mathcal{X}^1 \to \dots \to \mathcal{X}^\alpha = \mathcal{X}$$

with pushout diagrams

$$\begin{tikzcd} |\partial\Delta^{\mathcal{J}_\beta}|_s \arrow[r, hook] \arrow[d] & |\Delta^{\mathcal{J}_\beta}|_s \arrow[d] \\ \mathcal{X}^\beta \arrow[r, hook] & \mathcal{X}^{\beta+1}, \end{tikzcd} \tag{6.3}$$

then the compositions

$$|\Delta^{\mathcal{J}_\beta}|_s \to \mathcal{X}^\beta \to \mathcal{X}$$

define a cell structure on $\mathcal{X}$. The finite precursor conditions follows from the fact that every compactum in an absolute cell complex is contained in a finite subcomplex (see [Hir03, Prop. 10.7.4], for the topological case). Conversely, if $\mathcal{X}$ admits a stratified cell structure, $(\sigma_i)_{i\in I}$, then we may extend the precursor order on I to a well-order as follows: By well-founded induction, we obtain an order-preserving map $\nu: I \mapsto \mathbb{N}$, inductively defined via $\tau \mapsto \sup\{\nu(\tau') \mid \tau' \prec \tau\}$. Then, for every fiber $\nu^{-1}(n)$, choose a well order $\prec_n$, and finally equip

$$I = \bigsqcup_{n\in\mathbb{N}} \nu^{-1}(n)$$

with the lexicographic order

$$i < p \iff i \prec j \vee (\nu(i) = \nu(j) = n \wedge i \prec_n j).$$

Under this construction, we can identify I as an ordinal α_I . Then, for $\beta \leq \alpha_I$ denote $\mathcal{X}^\beta = \bigcup_{j<i} e_\beta \subset \mathcal{X}$. By construction, $|\partial\Delta^{\mathcal{J}_\beta}|_s \hookrightarrow |\Delta^{\mathcal{J}_\beta}|_s \xrightarrow{\sigma_\beta} \mathcal{X}$ factors through $\mathcal{X}^\beta$, and one may check that the diagrams

$$\begin{tikzcd} |\partial\Delta^{\mathcal{J}_\beta}|_s \arrow[r, hook] \arrow[d] & |\Delta^{\mathcal{J}_\beta}|_s \arrow[d] \\ \mathcal{X}^\beta \arrow[r, hook] & \mathcal{X}^{\beta+1}, \end{tikzcd} \tag{6.4}$$

are pushout and that whenever β is a limit element, then $\mathcal{X}^\beta = \varinjlim_{\beta'<\beta} \mathcal{X}^{\beta'}$. The relative case is essentially analogous.

As a consequence of [Hir03, Prop. 10.7.6], one obtains the following.

Lemma 6.2.3.6. *If $\mathcal{X} \in \mathbf{Strat}_P$ is a stratified cell complex, then every compactum $K \subset \mathcal{X}$ is contained in a finite subcomplex $\mathcal{A}$ of $\mathcal{X}$.*

6.2.4 Systems of strata-neighborhoods

One of the central properties of pairwise homotopy links is that they may be computed locally using neighborhoods of a stratum (compare [Qui88]). This type of argument was also central in the proof of [DW22, Thm 4.8] (Theorem 3.4.4.1). In this subsection, we introduce the notion of a strata-neighborhood system of a stratified space $\mathcal{X}$ which provide a general framework for these type of local computations. We then move on to the question of when the homotopy links may instead be computed entirely in terms of these strata-neighborhood systems, in which case we speak of a homotopy link model.

Definition 6.2.4.1. Let $\mathcal{X} \in \mathbf{Strat}_P$ and $S \subset \mathcal{X}$. A *Δ_P-neighborhood of S in $\mathcal{X}$* is a subset $U \subset X$ such that for any flag $\mathcal{J}$ of P and any stratum-preserving map $\sigma: |\Delta^{\mathcal{J}}|_s \to \mathcal{X}$ the set $\sigma^{-1}(U)$ is a neighborhood of $\sigma^{-1}(S)$ in $|\Delta^{\mathcal{J}}|_s$.

The following elementary property follows immediately from the definition of a Δ_P-neighborhood.

Lemma 6.2.4.2. *Let $f: \mathcal{X} \to \mathcal{Y}$ in $\mathbf{Strat}_P$. If $U \subset \mathcal{Y}$ is a Δ_P-neighborhood of $S \subset \mathcal{Y}$, then $f^{-1}(U)$ is a Δ_P-neighborhood of $f^{-1}(S)$ in $\mathcal{Y}$.*

Roughly speaking, we should think of Δ_P-neighborhoods as subsets of $\mathcal{X}$ that look like a neighborhood if one takes the perspective of a finite stratified cell complex. This heuristic is made rigorous by the following lemma.

Lemma 6.2.4.3. *If $\mathcal{X} \in \mathbf{Strat}_P$ is equipped with the structure of a finite stratified cell complex and $S \subset U \subset X$, then the following are equivalent:*

1. *U is a Δ_P-neighborhood of S.*
2. *For every cell $\sigma: |\Delta^{\mathcal{J}}|_s \to \mathcal{X}$, the set $\sigma^{-1}(U)$ is a neighborhood of $\sigma^{-1}(S)$.*
3. *U is a neighborhood of S.*

Proof. That the first condition implies the second is trivial. Clearly, also the third implies the first. It remains to show that the second condition implies the third. This is the content of Lemma 6.A.0.1. □

For infinite stratified cell complexes we may still state the following lemma.

Lemma 6.2.4.4. *If $\mathcal{X}$ admits the structure of a cell complex, then $U \subset \mathcal{X}$ is a Δ_P-neighborhood of $S \subset U \subset \mathcal{X}$, if and only if the inverse image under every cell $\sigma: |\Delta^{\mathcal{J}}|_s \to \mathcal{X}$ of U is a neighborhood of $\sigma^{-1}(S)$ in $|\Delta^{\mathcal{J}}|$.*

Proof. The only if case is immediate by the definition of a Δ_P-neighborhood. Now, for the converse, note that any continuous map $|\Delta^{\mathcal{J}}|_s \to \mathcal{X}$ factors through a finite subcomplex $\mathcal{A}$ of $\mathcal{X}$. It follows from this that it suffices to show that $U \cap A$ is a Δ_P-neighborhood of $S \cap A$, for any finite subcomplex $\mathcal{A}$ of $\mathcal{X}$. Therefore, the result follows from Lemma 6.2.4.3. □

Next, let us define the notion of a strata-neighborhood system.

Definition 6.2.4.5. A *strata-neighborhood system* of a stratified space $\mathcal{X} \in \mathbf{Strat}_P$ is a family of subspaces $\mathfrak{U}_{\mathcal{X}} = (U_{\mathcal{X}}(p))_{p \in P}$ such that $U_{\mathcal{X}}(p)$ is a Δ_P-neighborhood of X_p in $\mathcal{X}$ that further fulfills $X_{\leq p} \subset U_{\mathcal{X}}(p)$.

Remark 6.2.4.6. The additional requirement that $X_{\leq p} \subset U_{\mathcal{X}}(p)$ is only there for the sake of notational convenience, when passing to flags in Notation 6.2.4.9. Aside from this, this is immaterial for the constructions in this section. We decided to add this condition, as it is automatically fulfilled for the strata-neighborhood systems we construct in this article, and can always be achieved by replacing $U_{\mathcal{X}}(p)$ with $U_{\mathcal{X}}(p) \cup X_{<p}$.

The goal of a strata-neighborhood system is ultimately to provide a geometric model for the homotopy link diagram of a stratified space. However, we do not only need to model the homotopy links, but also the functoriality of the latter. This requires the following definition.

Definition 6.2.4.7. Let $\mathcal{X}, \mathcal{Y} \in \mathbf{Strat}_P$ be equipped with strata-neighborhood systems $\mathfrak{U}_{\mathcal{X}}$ and $\mathfrak{U}_{\mathcal{Y}}$, respectively. We say $f: \mathcal{X} \to \mathcal{Y}$ *lifts to a map of strata-neighborhood systems* $\mathfrak{U}_{\mathcal{X}} \to \mathfrak{U}_{\mathcal{Y}}$, if $f(U_{\mathcal{X}}(p)) \subset U_{\mathcal{Y}}(p)$, for all $p \in P$.

Notation 6.2.4.8. Denote by **SNS** the category which is defined as follows. Objects are given by pairs $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}})$ with $\mathcal{X} \in \mathbf{Strat}_P$ and $\mathfrak{U}_{\mathcal{X}}$ a strata-neighborhood system of $\mathcal{X}$. The set of morphisms from $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}})$ to $(\mathcal{Y}, \mathfrak{U}_{\mathcal{Y}})$ is given by such stratum-preserving maps $f: \mathcal{X} \to \mathcal{Y}$ which lift to a map of neighborhood systems $\mathfrak{U}_{\mathcal{X}} \to \mathfrak{U}_{\mathcal{Y}}$. We will usually just refer to a pair $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}})$ just by $\mathfrak{U}_{\mathcal{X}}$.

To extract an object of $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ from a strata-neighborhood system, we need the following construction.

Notation 6.2.4.9. For $\mathfrak{U}_{\mathcal{X}}$ a strata-neighborhood system of $\mathcal{X} \in \mathbf{Strat}_P$ and $\mathcal{I} \in \mathrm{sd}(P)$ a regular flag, we write $U_{\mathcal{X}}(\mathcal{I}) := \bigcap_{p \in \mathcal{I}} U_{\mathcal{X}}(p)$.

Notation 6.2.4.10. Given a strata-neighborhood system $\mathfrak{U}_{\mathcal{X}}$ a strata-neighborhood system of $\mathcal{X} \in \mathbf{Strat}_P$, and $\mathcal{I} \in \mathrm{sd}(P)$, we are going to follow our conventions on stratified and non-stratified objects (see Notation 6.2.1.6), and write $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$, for the P-stratified space obtained by equipping $U_{\mathcal{X}}(\mathcal{I})$ with the stratification inherited from $\mathcal{X}$. In particular, for $p \in P$, it makes sense to use expressions such as $U_{\mathcal{X}}(\mathcal{I})_{\leq p}$, which in this case refers to the subspace of $U_{\mathcal{X}}(\mathcal{I})$, given by the strata of index lesser or equal to p.

Construction 6.2.4.11. Let $\mathfrak{U}_{\mathcal{X}}$ be a strata-neighborhood system for $\mathcal{X} \in \mathbf{Strat}_P$. We denote by $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}}) \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ the diagram

$$\mathcal{I} = [p_0 < \cdots < p_n] \mapsto U_{\mathcal{X}}(\mathcal{I})_{\geq p_n}$$

with structure maps given by inclusions. The resulting object $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}}) \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ is called the *regular complement diagram* of $\mathcal{X} \in \mathbf{Strat}_P$. This construction defines a functor

$$\mathrm{D}^T: \mathbf{SNS} \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top}).$$

We may now ask the question under which conditions on a neighborhood system $\mathfrak{U}_{\mathcal{X}}$, on a stratified space $\mathcal{X}$, there is a canonical weak equivalence of diagrams between $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ and $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})$.

Definition 6.2.4.12. Let $\mathcal{X} \in \mathbf{Strat}_P$. We say that a strata-neighborhood system $\mathfrak{U}_{\mathcal{X}}$ of $\mathcal{X}$ *is a model for the homotopy links of* $\mathcal{X}$ (is a homotopy link model) if the natural maps

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{U}_{\mathcal{X}}(\mathcal{I})) \xrightarrow{\mathrm{ev}} \mathcal{H}o\mathrm{Link}_{p_n}(\mathcal{U}_{\mathcal{X}}(\mathcal{I})) = U_{\mathcal{X}}(\mathcal{I})_{p_n},$$

and

$$U_{\mathcal{X}}(\mathcal{I})_{p_n} \hookrightarrow U_{\mathcal{X}}(\mathcal{I})_{\geq p_n}$$

are weak equivalences of topological spaces, for each regular flag $\mathcal{I} = [p_0 < \cdots < p_n] \subset P$.
We denote by **HLMod** the full subcategory of **SNS** given by pairs $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}})$ with $\mathfrak{U}_{\mathcal{X}}$ a homotopy link model for $\mathcal{X}$.

Remark 6.2.4.13. We should note that the second condition in the definition of a homotopy link model is necessary since (using the notation of Construction 6.2.4.11) $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})(\mathcal{I})$ is defined as $U_{\mathcal{X}}(\mathcal{I})_{\geq p_n}$ rather than just using $U_{\mathcal{X}}(\mathcal{I})_{p_n}$. This in turn is necessary if we want $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})$ to provide a model for the whole diagram $\mathcal{H}o\mathrm{Link}(\mathcal{X})$, not just its pointwise values. To obtain the structure maps of $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})$ we need to have inclusions $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})(\mathcal{I}_1) \subset \mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})(\mathcal{I}_0)$, whenever $\mathcal{I}_0 \subset \mathcal{I}_1$. In particular, if we also want this inclusion to hold when $\mathcal{I}_0$ and $\mathcal{I}_1$ do not share a maximal element, then we need to define D^T as in Construction 6.2.4.11. In any case, the second condition for being a homotopy link model will usually turn out to be the easy one to verify, as it can be verified entirely in the language of classical homotopy theory, unlike the first one which requires stratified considerations.

Let us now finish this subsection by stating the result that legitimizes the nomenclature of homotopy link models.

Theorem 6.2.4.14. *The diagram*

$$\begin{tikzcd} \mathbf{HLMod} \arrow[r, hook] \arrow[d] & \mathbf{SNS} \arrow[d, "\mathrm{D}^T"] \\ \mathbf{Strat}_P \arrow[r, "\mathcal{H}o\mathrm{Link}"] & \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top}) \end{tikzcd} \tag{6.5}$$

commutes up to weak equivalence.

At this point, we do not yet have the tools necessary for a proof of Theorem 6.2.4.14. The proof follows in Section 6.3.3.

6.3 Strata-neighborhood systems for stratified simplicial sets and cell complexes

From the perspective of Theorem 6.2.4.14, the goal of this paper is to construct homotopy link models for stratified cell complexes. In this section, we provide a general construction for strata-neighborhood systems, first for the simplicial case (Section 6.3.1) and later the case of stratified cell complexes (Section 6.3.2). Importantly, we prove that this construction can be made compatible with stratum-preserving maps, which ultimately leads to a proof of Theorem 6.2.4.14.

6.3.1 Standard strata-neighborhood systems for stratified simplicial sets

Before we move on to investigating strata-neighborhood systems on stratified cell complexes, let us first consider the simpler case of the realization of a stratified simplicial set. To do so, we are going to make heavy use of the following coordinates.

Construction 6.3.1.1. Let $\mathcal{J}$ be a flag in P. For $p \in P$, we denote $\mathcal{J}_p$ the unique maximal subflag of $\mathcal{J}$ that degenerates from the regular flag $[p]$ (see Notation 6.2.1.8 for an overview). The inclusion $\mathcal{J}_p \subset \mathcal{J}$ induces a natural projection

$$\mathbb{R}^{\mathcal{J}} \to \mathbb{R}^{\mathcal{J}_p},$$

where $\mathbb{R}^{\mathcal{J}}$ denotes the vector space spanned by the elements of $\mathcal{J}$ (counted with repetition). Next, consider the canonical embedding $|\Delta|_s{}^{\mathcal{J}} \hookrightarrow \mathbb{R}^{\mathcal{J}}$, which embeds $|\Delta^{\mathcal{J}}|_s$ as the affine hull of the standard basis vectors. For $x \in |\Delta^{\mathcal{J}}|_s$, we write x_p for the image of x under the composition

$$|\Delta^{\mathcal{J}}|_s \hookrightarrow \mathbb{R}^{\mathcal{J}} \to \mathbb{R}^{\mathcal{J}_p}.$$

Now, if $\mathcal{I} = [p_0 < \dots < p_n]$ is the regular flag such that all elements of $\mathcal{J}$ are contained in $\mathcal{I}$, then $\mathcal{J}$ is equivalently given by the concatenation

$$\mathcal{J}_{p_0} \cup \dots \cup \mathcal{J}_{p_n}.$$

(Note that we may indeed allow p_i that are not in $\mathcal{J}$, as then $\mathcal{J}_{p_i}$ is the empty flag.) In particular, we have a canonical isomorphism

$$\mathbb{R}^{\mathcal{J}} \cong \Pi_{i \in [n]} \mathbb{R}^{\mathcal{J}_{p_i}},$$

which allows us to make sense of the expression

$$x = (x_{p_0}, \cdots, x_{p_n}).$$

Further, recall from Notation 6.2.1.8 that $\mathcal{J}_{\leq p}$ denotes the maximal subflag given by such entries of $\mathcal{J}$, of value lesser or equal to p. Analogously, we denote by $\mathcal{J}_{\nleq p}$ the maximal subflag given by such entries of $\mathcal{J}$, of value not lesser equal to p, and so on. Just as in the case of a singleton, we denote by $x_{\leq p}$ the image of $x \in |\Delta^{\mathcal{J}}|_s$ under

$$|\Delta^{\mathcal{J}}|_s \hookrightarrow \mathbb{R}^{\mathcal{J}} \to \mathbb{R}^{\mathcal{J}_{\leq p}},$$

and similarly proceed with $x_{\nleq p}$, $x_{<p}$ and so on.

Next, we need a normalized version of the coordinates defined in Construction 6.3.1.1, so-called join coordinates. The notation here will be somewhat sloppy, in the sense that we are often going to write expressions like y_p, when we formally should be writing $y_p(x)$.

Construction 6.3.1.2. We again use the setup of Construction 6.3.1.1. At the level of underlying simplicial sets, we may then identify $\Delta^{\mathcal{J}}$ as the join

$$\Delta^{\mathcal{J}} = \Delta^{\mathcal{J}_{p_0}} * \cdots * \Delta^{\mathcal{J}_{p_n}}.$$

Doing so, we can equip $|\Delta^{\mathcal{J}}|_s \subset \mathbb{R}^{\mathcal{J}}$ with n-fold join coordinates. Explicitly, the topological join

$$|\Delta^{\mathcal{J}_{p_0}}| * \cdots * |\Delta^{\mathcal{J}_{p_n}}|$$

can be described as follows. For $I \subset [n]$, denote by $|\Delta^I| \subset |\Delta^n|$ the face corresponding to I. Furthermore, for $I' \subset I$, denote by $\pi_{I,I'}$ the obvious projection $\Pi_{i \in I}|\Delta^{\mathcal{J}_{p_i}}| \to \Pi_{i \in I'}|\Delta^{\mathcal{J}_{p_i}}|$. Then, the n-fold join can be described as the quotient space of

$$\bigsqcup_{I \subset [n]} (\Pi_{i \in I}|\Delta^{\mathcal{J}_{p_i}}|) \times |\Delta^I|$$

where we mod out by the equivalence relation generated by

$$(y, s) \sim (\pi_{I,I'}(y), s)$$

whenever $y \in \Pi_{i \in I}|\Delta^{\mathcal{J}_{p_i}}|$ and $s \in |\Delta^{I'}| \subset |\Delta^I|$. The homeomorphism to $|\Delta^{\mathcal{J}}|$ is then given by mapping

$$|\Delta^{\mathcal{J}_{p_0}}| * \cdots * |\Delta^{\mathcal{J}_{p_n}}| \ni [(y, s)] \mapsto \sum_{i \in [n]} s_i y_i \in |\Delta^{\mathcal{J}}|.$$

Note that if y_i is not defined, then $s_i = 0$ and this expression makes sense. Conversely, an inverse is obtained by

$$|\Delta^{\mathcal{J}}| \ni x \mapsto \big((\frac{x_{p_0}}{|x_{p_0}|}, \cdots, \frac{x_{p_n}}{|x_{p_n}|}), (|x_{p_0}|, \cdots, |x_{p_n}|) \big) \in |\Delta^{\mathcal{J}_{p_0}}| * \cdots * |\Delta^{\mathcal{J}_{p_n}}|.$$

Again, note that this expression makes sense, even if $|x_p| = 0$. This leads us to the following change to join coordinates, which we are going to use frequently in this section. For $x \in |\Delta^{\mathcal{J}}|_s$ we denote

$$y_p := \frac{x_p}{|x_p|};$$
$$s_p := |x_p|.$$

Finally, we will need another set of coordinates using the decomposition $x = (x_{\leq p}, x_{\nleq p})$, for $x \in |\Delta^{\mathcal{J}}|_s$ and $p \in P$.

Construction 6.3.1.3. Again, in the setup of Construction 6.3.1.1, we may - just as in Construction 6.3.1.2 - identify

$$\begin{aligned} \Delta^{\mathcal{I}} &= \Delta^{\mathcal{I}_{<p}} * \Delta^{\mathcal{I}_{\nless p}}, \\ \Delta^{\mathcal{I}} &= \Delta^{\mathcal{I}_{\leq p}} * \Delta^{\mathcal{I}_{\nleq p}}, \\ \cdots &= \cdots \end{aligned}$$

etc. This leads to a change of coordinates

$$\begin{aligned} s_{<p} &:= |x_{<p}|; \quad s_{\nless p} := |x_{\nless p}|; \quad \cdots = \cdots \\ y_{<p} &:= \frac{x_{<p}}{|x_{<p}|}; \quad y_{\nless p} := \frac{x_{\nless p}}{|x_{\nless p}|}; \quad \cdots = \cdots \end{aligned}$$

Remark 6.3.1.4. Let us remark on some immediate facts on the s-coordinates of Construction 6.3.1.2 and Construction 6.3.1.3. First, note that they are indeed invariant under stratified face and degeneracy maps and therefore extend to any realization of a stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$. Then, the s-coordinates interact with the stratification of $|\mathcal{X}|_s$ as follows. In the following, as the notation $s_{|\mathcal{X}|_s}(x)$, for $x \in |X|_s$, is rather cumbersome, we will use the shortened notation $s(x) := s_{|\mathcal{X}|_s}(x) \in P$ to refer to the stratum of x. Let $x \in |\mathcal{X}|_s$. Then, we have equivalences

- $s(x) < p \iff s_{<p} = 1 \iff s_{\nless p} = 0.$
- $s(x) \leq p \iff s_{\leq p} = 1 \iff s_{\nleq p} = 0.$
- $\cdots.$

It immediately follows that

- $s(x) = p \iff s_{\leq p} = 1 \wedge s_p > 0 \iff s_{\nleq p} = 0 \wedge s_p > 0 \iff \cdots.$

We are now going to use these coordinates to define strata-neighborhoods for realizations of stratified simplicial sets.

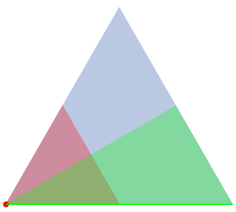
Construction 6.3.1.5. Suppose $p \in P$ and $\varphi_p : [0,1] \to [0,1]$ is a continuous function such that $\varphi_p(s) > 0$, whenever $s > 0$. Let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$. We may then consider the following subspaces of $|\mathcal{X}|_s$. We set

$$U^{\varphi_p}_{\mathcal{X}}(p) := \{x \in |\mathcal{X}|_s \mid s_{\nleq p} \leq \varphi_p(s_{\nleq p}) s_p\}$$

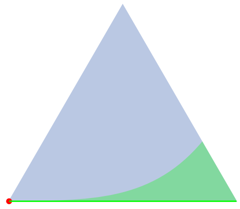
and call it the *φ_p-standard neighborhood associated to $\mathcal{X}$*. Note that, since all s-coordinates are invariant under realizations of maps of stratified simplicial sets, this construction extends to a functor

$$\mathcal{U}^{\varphi_p}_{-}(p) \colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_P.$$

Example 6.3.1.6. Consider the stratified simplex $|\Delta^{[0<1<2]}|_s$, pictured below with the strata colored in red, green and blue, in ascending order. If we set $\varphi_p = 1$ for $p = 0, 1$ we obtain the following standard neighborhoods shaded in red and green respectively for $p = 0, 1$.



For a smaller choice of φ_1, here with $\varphi_1(0) = 0$, we obtain a φ_1-standard neighborhood whose boundary is tangential to the 1-stratum at the 0-stratum:



Proposition 6.3.1.7. *Let φ_p be as in Construction 6.3.1.5. For any stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ the space $U^{\varphi_p}_{\mathcal{X}}(p) \subset |\mathcal{X}|_s$ defines a p-stratum neighborhood.*

Proof. Consider the open subset of $O \subset U^{\varphi_p}_{\mathcal{X}}(p) \subset (|\mathcal{X}|_s)_{\nleq p}$ defined by the condition

$$s_{\nleq p} < \varphi_p(s_{\nleq p}) s_p.$$

By Remark 6.3.1.4, for any $x \in (|\mathcal{X}|_s)_p$ the value of $s_{\nleq p}$ is 0, $s_p > 0$ and $s_{\nleq p} \geq s_p > 0$. As $\varphi_p(s) > 0$ for $s > 0$, it follows that

$$s_{\nleq p} = 0 < \varphi_p(s_{\nleq p}) s_p.$$

Consequently, $(|\mathcal{X}|_s)_p \subset O \subset U^{\varphi_p}_{\mathcal{X}}(p)$ and $U^{\varphi_p}_{\mathcal{X}}(p)$ is even a neighborhood of the p-stratum in the strong sense. ☐

In particular, standard neighborhoods allow us to factor stratified realization through the category of strata-neighborhood systems:

Corollary 6.3.1.8. *Let $\varphi = (\varphi_p)_{p \in P}$ be a family of functions as in Construction 6.3.1.5. Then, $\mathcal{X} \mapsto \mathfrak{U}^{\varphi}_{\mathcal{X}} = (|\mathcal{X}|_s, (U^{\varphi_p}_{\mathcal{X}}(p))_{p \in P})$ defines a factorization*

$$\begin{tikzcd} \mathrm{s}\mathbf{Strat}_P \arrow[r, "\mathfrak{U}^{\varphi}_{-}"] \arrow[dr, "|-|_s"'] & \mathbf{SNS} \arrow[d] \\ & \mathbf{Strat}_P . \end{tikzcd} \tag{6.6}$$

Proof. By Proposition 6.3.1.7, it suffices to verify that $(|\mathcal{X}|_s)_{\leq p} \subset U^{\varphi_p}_{\mathcal{X}}(p)$, for $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ and $p \in P$. This is immediate from Remark 6.3.1.4. ☐

Example 6.3.1.9. The most important case to consider is the case where all functions φ_p are given by the constant function with value 1. In this case, the condition for a point x to lie in $U^{\varphi_p}_{\mathcal{X}}(p)$ is simply that

$$s_{\nleq p} \leq s_p.$$

In this case, we denote the resulting neighborhood system by $\mathfrak{U}_{\mathcal{X}}$ and call it the *standard neighborhood system*.

Let us also remark on some of the more degenerate examples of standard neighborhoods:

Lemma 6.3.1.10. *Let φ_p be as in Construction 6.3.1.5. If $p \notin \mathcal{J}$, then $\mathcal{U}^{\varphi_p}_{\Delta^{\mathcal{J}}}(p) = |\Delta^{\mathcal{J}_{<p}}|_s$.*

Proof. Indeed, when $p \notin \mathcal{J}$, then for any $x \in |\Delta^{\mathcal{J}}|_s$ it holds that $s_p = 0$. Hence, the defining condition for $U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p)$ is fulfilled if and only if $s_{\nleq p} = 0$, that is, when $x \in (|\Delta^{\mathcal{J}}|_s)_{\leq p} = |\Delta^{\mathcal{J}_{<p}}|_s$. ☐

Next, we give a purely combinatorial description of the standard neighborhood in the special case where $\mathcal{X}$ is a stratified simplicial complex, by making use of barycentric subdivisions. By a *stratified simplicial complex*, we mean a stratified simplicial set, such that its underlying simplicial set has the property that every non-degenerate simplex is uniquely determined by its set of vertices. In particular, this means that every face of every non-degenerate simplex is non-degenerate.

Construction 6.3.1.11. For $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, consider the first barycentric subdivision of $\mathcal{X}$, equipped with the stratification induced by the last vertex map $\mathrm{sd}X \to X$, and denote it $\mathrm{sd}\mathcal{X}$ (see [DW22, Def. 3.7], which is Definition 3.3.1.7 in this text). If $\mathcal{X} = \Delta^{\mathcal{J}}$, for some flag $\mathcal{J}$ of P, then we denote by $N_{\Delta^{\mathcal{J}}}(p) \subset \mathrm{sd}\Delta^{\mathcal{J}}$ the subcomplex, spanned by those vertices that correspond to the subflags $\mathcal{J}'$ of $\mathcal{J}$ such that

$$p \in \mathcal{J}' \quad \vee \quad \forall q \in \mathcal{J}' : q < p.$$

We denote by $\mathcal{N}_{\Delta^{\mathcal{J}}}(p)$ the stratified simplicial set obtained by equipping $N_{\Delta^{\mathcal{J}}}(p)$, with the stratification inherited from $\mathrm{sd}\Delta^{\mathcal{J}}$. This construction is functorial with respect to maps of stratified simplices and hence extends to a functor

$$\mathcal{N}_{-}(p) \colon \mathrm{s}\mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P,$$

via left Kan extension, together with a natural transformation $\mathcal{N}_{-}(p) \hookrightarrow \mathrm{sd}$, identifying $\mathcal{N}_{\mathcal{X}}(p)$ with a stratified subsimplicial set of $\mathrm{sd}\mathcal{X}$, for $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$.

Construction 6.3.1.12. If $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ is a stratified simplicial complex (i.e., every non-degenerate simplex in $\mathcal{X}$ is uniquely determined by its set of vertices), then we may identify $\mathcal{U}_{\mathcal{X}}(p)$ with the realization of $\mathcal{N}_{\mathcal{X}}(p)$[1]. A stratum-preserving homeomorphism $|\mathcal{N}_{\mathcal{X}}(p)|_s \to \mathcal{U}_{\mathcal{X}}(p)$ is constructed as follows. On each stratified simplex $|\Delta^{\mathcal{J}}|_s$, where $\mathcal{J}$ degenerates from a regular flag $[p_0 < \dots < p_n]$, consider the weighted barycenter $b_{\mathcal{J}}$, given in join coordinates by

$$b_{\mathcal{J}} := [(b_{p_0}, \dots, b_{p_n}), (\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \frac{1}{2^n})],$$

where b_{p_i} is the barycenter of $|\Delta^{\mathcal{J}_{p_i}}|_s$. We denote

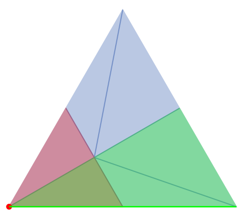
$$\Psi : |\mathrm{sd}\mathcal{X}|_s \to |\mathcal{X}|_s$$

the stratum-preserving homeomorphism which is affinely extended from the map on vertices

$$v \mapsto |\sigma_v|_s(b_{\mathcal{J}}),$$

where v corresponds to a non-degenerate simplex $\sigma_v \colon \Delta^{\mathcal{J}} \to \mathcal{X}$ with flag $\mathcal{J}$. Then, the content of Proposition 6.3.1.15 is that Ψ restricts to a homeomorphism from $|\mathcal{N}_{\mathcal{X}}(p)|_s$ to $\mathcal{U}_{\mathcal{X}}(p)$.

Example 6.3.1.13. In the context of Example 6.3.1.14, the following figure shows the weighted barycentric subdivision given by Construction 6.3.1.12. As indicated in the following picture, the standard neighborhoods of the 0 and 1-stratum are precisely spanned by such vertices in the subdivision fulfilling the condition of Construction 6.3.1.11.



Example 6.3.1.14. For a non-degenerate flag $\mathcal{I} = [p_0 < \dots < p_n]$ and $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, we denote by $\mathcal{N}_{\mathcal{X}}(\mathcal{I})$ the intersection $\bigcap_{p \in \mathcal{I}} \mathcal{N}_{\mathcal{X}}(p)$. If $\mathcal{X} = \Delta^{\mathcal{I}}$, then $\mathcal{N}_{\mathcal{X}}(\mathcal{I})$ is given by the image of the unique embedding $\Delta^{\mathcal{I}} \hookrightarrow \mathrm{sd}\Delta^{\mathcal{I}}$.

The following proposition then shows that the construction in Construction 6.3.1.12 does indeed provide a combinatorial model for standard neighborhoods.

[1] This also works in the general simplicial set case. However, the constructions become significantly more involved, and we have no need for this case here.

Proposition 6.3.1.15. *Let $p \in P$ and $\mathcal{X} \in s\mathbf{Strat}_P$ be a stratified simplicial complex. Then $\Psi \colon |\mathrm{sd}\mathcal{X}|_s \to |\mathcal{X}|_s$ - as defined in Construction 6.3.1.12 - restricts to a stratum-preserving homeomorphism $|\mathcal{N}_{\mathcal{X}}(p)|_s \xrightarrow{\sim} \mathcal{U}_{\mathcal{X}}(p)$.*

Proof. The statement is easily reduced to the case where $\mathcal{X} = \Delta^{\mathcal{J}}$, for $\mathcal{J}$ a flag of P that degenerates from a non-degenerate flag $[p_0 < \cdots < p_n]$. Let us begin by computing the value of $s_{\nleq p}(b_{\mathcal{J}})$ and $s_p(b_{\mathcal{J}})$: If $p = p_n$, then

$$s_{\nleq p}(b_{\mathcal{J}}) = 0 < 2^{-n} = s_p(b_{\mathcal{J}}). \tag{6.7}$$

If $p = p_k$, for some $k \in [n-1]$, then $b_{\mathcal{J}}$ fulfills

$$s_{\nleq p}(b_{\mathcal{J}}) = 2^{-(k+1)} = s_p(b_{\mathcal{J}}). \tag{6.8}$$

If $p_k < p$, for all $k \in [n]$, then

$$s_{\nleq p}(b_{\mathcal{J}}) = 0 \leq 0 = s_p(b_{\mathcal{J}}). \tag{6.9}$$

Finally, if $p \notin \mathcal{J}$ and $k \in [n]$ is minimal with the property that $p_k \nleq p$, then

$$s_{\nleq p}(b_{\mathcal{J}}) = 2^{-k} > 0 = s_p(b_{\mathcal{J}}). \tag{6.10}$$

It follows, from the inequalities (6.7) to (6.9) that Ψ does indeed restrict to an embedding $|\mathcal{N}_{\mathcal{X}}(p)|_s \to \mathcal{U}_{\mathcal{X}}(p)$. It remains to show surjectivity of this restriction. So, let $x \in U_{\Delta^{\mathcal{J}}}(p) \subset |\Delta^{\mathcal{J}}|_s$ be a point in $|\Delta^{\mathcal{J}}|_s$. Let $\{\mathcal{J}_0 \subset \cdots \subset \mathcal{J}_m\}$, be the minimal set of subflags of $\mathcal{J}$ such that x lies in the affine span of $(b_{\mathcal{J}_i})_{i \in m}$, i.e., we have

$$x = t_0 b_{\mathcal{J}_0} + \cdots + t_m b_{\mathcal{J}_m}$$

with $t_i > 0$, for all $i \in [m]$. We need to show that for each $i \in [m]$ either $p \in \mathcal{J}_i$ or $q < p$, for all $q \in \mathcal{J}_i$. In other words, we need to show that the set

$$S = \{ i \in [m] \mid p \notin \mathcal{J}_i \wedge \big(\exists q \in \mathcal{J}_i \colon q \nleq p\big) \}$$

is empty. Since $x \in U_{\Delta^{\mathcal{J}}}(p)$, we have

$$s_{\nleq p}(x) \leq s_p(x)$$

and thus

$$t_0 s_{\nleq p}(b_{\mathcal{J}_0}) + \cdots + t_m s_{\nleq p}(b_{\mathcal{J}_m}) = s_{\nleq p}(x) \leq s_p(x) = t_0 s_p(b_{\mathcal{J}_0}) + \cdots + t_m s_p(b_{\mathcal{J}_m})$$

Using Eqs. (6.8) and (6.9) it follows that

$$\sum_{i \in S} t_i s_{\nleq p}(b_{\mathcal{J}_i}) \leq \sum_{i \in S} t_i s_p(b_{\mathcal{J}_i}). \tag{6.11}$$

By Eq. (6.10), the right-hand side of this equality is 0 and the left-hand side is a (possibly empty) sum of strictly positive numbers. It follows that $S = \varnothing$, as was to be shown. ☐

We will need the following technical lemma. Roughly speaking, it states that, for a finite simplicial set $\mathcal{X}$, the strata-neighborhood system $\mathfrak{U}^{\varphi}_{\mathcal{X}}$ of Construction 6.3.1.5 are universal.

Lemma 6.3.1.16. *Let $\mathcal{X} \in s\mathbf{Strat}_P$ be a finite stratified simplicial set. Let $\mathfrak{U}_{|\mathcal{X}|_s}$ be any neighborhood system on $|\mathcal{X}|_s$. Then there exists a family of functions φ as in Corollary 6.3.1.8 such that, for any $p \in P$, we have*

$$U^{\varphi_p}_{\mathcal{X}}(p) \subset U_{|\mathcal{X}|_s}(p).$$

In other words, the identity on $|\mathcal{X}|_s$ lifts to a map of neighborhood systems $\mathfrak{U}^{\varphi}_{\mathcal{X}} \to \mathfrak{U}_{|\mathcal{X}|_s}$.

Proof. Note first that it suffices to solve the problem on each simplex and then define φ_p by passing to minima. Hence, without loss of generality we may assume that $\mathcal{X} = \Delta^{\mathcal{J}}$, for some flag $\mathcal{J}$. Fix some $p \in P$ and denote $U = U_{|\Delta^{\mathcal{J}}|_s}(p)$. By Lemma 6.2.4.3, we may therefore assume that U is actually a neighborhood of the p-stratum. Next, note that for any $0 < \alpha \leq 1$ the set

$$S_\alpha := \{x \in (|\Delta^{\mathcal{J}}|_s)_p \mid s_{\nleq p} \geq \alpha\}$$

is compact. A neighborhood basis for S_α in $|\Delta^{\mathcal{J}}|_s$ is given by the sets

$$S_{\alpha,\beta} := \{x \in |\Delta^{\mathcal{J}}|_s \mid s_{\nless p} \geq \alpha - \beta \wedge s_{\nleq p} \leq \beta\},$$

where $\beta > 0$. Hence, for any $n \in \mathbb{N}$ there exists some $\beta_n > 0$ such that for any $x \in |\mathcal{X}|_s$, the implication

$$s_{\nless p} \geq \frac{1}{n} \wedge s_{\nleq p} \leq \beta_n \implies x \in U$$

holds. Without loss of generality, we may assume that the sequence β_n is decreasing. Choosing a partition of unity σ_n on the family $((\frac{1}{n}, 1])_{n \in \mathbb{N}}$ covering $(0, 1]$, we set

$$\varphi^p(s) = \sum_{n \in \mathbb{N}} \sigma_n(s)\beta_n.$$

In this fashion, we obtain a continuous function $\varphi^p \colon [0,1] \to [0,1]$, which is positive outside of 0. Now, let $x \in |\mathcal{X}|_s$ be such that $s_{\nless p} \in (\frac{1}{m}, \frac{1}{m-1}]$, for some $m > 1$, and suppose that $s_{\nleq p} \leq \varphi_p(s_{\nless p})$. Then, since $\sigma_n(s) = 0$ for $s < \frac{1}{n}$, we obtain

$$\begin{aligned} s_{\nleq p} \leq \varphi_p(s_{\nless p}) &= \sum_{n \in \mathbb{N}} \sigma_n(s_{\nless p})\beta_n \\ &= \sum_{n \geq m} \sigma_n(s_{\nless p})\beta_n \\ &\leq \beta_m , \end{aligned}$$

where the last inequality follows as $\sum_{n \geq m} \sigma_n(s_{\nless p})\beta_n$ is a convex combination and $(\beta_n)_{n \in \mathbb{N}}$ is a decreasing sequence. Thus, it follows that

$$s_{\nless p} > 0 \wedge s_{\nleq p} \leq \varphi_p(s_{\nless p}) \implies x \in U,$$

and in particular also

$$s_{\nless p} > 0 \wedge s_{\nleq p} \leq \varphi^p(s_{\nless p})s_p \implies x \in U,$$

for any $x \in |\Delta^{\mathcal{J}}|_s$. We deduce that

$$(U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p))_{\nless p} \subset U.$$

Since U contains $(|\mathcal{X}|_s)_{\leq p}$ by assumption, it follows that

$$U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p) \subset U$$

as was to be shown. ☐

In the next step, we show that (up to a stratum-preserving homeomorphism) we may really replace $\mathfrak{U}^{\varphi}_{\mathcal{X}}$ by $\mathfrak{U}_{\mathcal{X}}$, making the latter universal among strata-neighborhood systems in this sense. Before we do so, let us introduce another set of coordinates, to simplify notation.

Construction 6.3.1.17. In the framework of Construction 6.3.1.3 we may repeat the procedure described there with $\Delta^{\mathcal{J}_{\nless p}}$, and decompose

$$\Delta^{\mathcal{J}_{\nless p}} = \Delta^{\mathcal{J}_p} \star \Delta^{\mathcal{J}_{\nleq p}}.$$

We denote the resulting coordinates by

$$t_p := |y_p|; \quad t_{\nleq p} := |y_{\nleq p}|; \quad z_p := \frac{y_p}{|y_p|}; \quad z_{\nleq p} := \frac{y_{\nleq p}}{|y_{\nleq p}|}.$$

Using the affine relations between the several variables (such as $1 = s_{<p} + s_{\nless p}$), we may then express x entirely in terms of $y_{<p}$, z_p, $z_{\nleq p}$ and $s_{\nless p}$, $t_{\nleq p}$. Explicitly, we have

$$x = (1 - s_{\nless p})y_{<p} + s_{\nless p}(t_{\nleq p} z_{\nleq p} + (1 - t_{\nleq p})z_p).$$

Remark 6.3.1.18. By construction, whenever $t_{\nleq p}$ is defined, we have an equality

$$t_{\nleq p} = \frac{s_{\nleq p}}{s_{\nless p}}.$$

Using this, the condition for $x \in |\mathcal{X}|_s$ to lie in $U^{\varphi_p}_{\mathcal{X}}(p)$ may equivalently be rewritten as

$$s_{\nless p} = 0 \quad \vee \quad t_{\nleq p} \leq \frac{\varphi_p(s_{\nless p})}{1 + \varphi_p(s_{\nless p})}.$$

Proposition 6.3.1.19. *Let φ be a system of functions as in Construction 6.3.1.5. Then there exists a natural stratum-preserving automorphism*

$$\Phi \colon |-|_s \to |-|_s$$

that, for each $\mathcal{X} \in s\mathbf{Strat}_P$, lifts to a map $\mathfrak{U}_{\mathcal{X}} \to \mathfrak{U}^{\varphi}_{\mathcal{X}}$. In particular, Φ induces a natural transformation

$$\mathfrak{U}_{-} \to \mathfrak{U}^{\varphi}_{-}$$

in **SNS**. *Furthermore, Φ can be taken naturally stratum-preserving homotopic to the identity, through a family of natural homeomorphisms which lift to maps $\mathfrak{U}_{\mathcal{X}} \to \mathfrak{U}_{\mathcal{X}}$.*

Proof. We use the coordinates as in Construction 6.3.1.17. We first define separate homeomorphisms for each $p \in P$, Φ_p. Note that by left Kan extension it suffices to construct the natural transformation Φ_p for stratified simplices. On $|\Delta^{\mathcal{J}}|_s$, we define Φ_p via

$$[(y_{<p}, z_p, z_{\nleq p}), (s_{\nless p}, t_{\nleq p})] \mapsto [(y_{<p}, z_p, z_{\nleq p}), (s_{\nless p}, \hat{t}_{\nleq p})]$$

where

$$\hat{t}_{\nleq p} := \begin{cases} 2t_{\nleq p} - 1 + (2 - 2t_{\nleq p})\frac{\varphi_p(s_{\nless p})}{1+\varphi_p(s_{\nless p})} & \text{, for } t_{\nleq p} \geq \frac{1}{2} \\ 2t_{\nleq p}\frac{\varphi_p(s_{\nless p})}{1+\varphi_p(s_{\nless p})} & \text{, for } t_{\nleq p} \leq \frac{1}{2}. \end{cases}$$

One may easily verify that this assignment is well defined (under the identifications in the join), using the fact that the only coordinate that changes is $t_{\nleq p}$ that $t_{\nleq p} = 1 \iff \hat{t}_{\nleq p} = 1$ and that $t_{\nleq p} = 0 \iff \hat{t}_{\nleq p} = 0$. Similarly, one can easily verify that the resulting map

$$\Phi_p : |\Delta^{\mathcal{J}}|_s \to |\Delta^{\mathcal{J}}|_s$$

is stratum-preserving. Naturality follows from the fact that both $s_{\nless p}$ and $t_{\nleq p}$ are invariant under stratified face inclusions and degeneracy maps. Let us assume for a second that $p \in \mathcal{J}$. Then, if $s_{\nless p} > 0$ and all coordinates except $t_{\nleq p}$ remain fixed, the $\hat{t}_{\nleq p}$ component of Φ_p is given by gluing affine homeomorphisms $[0, \frac{1}{2}] \xrightarrow{\sim} [0, \frac{\varphi_p(s_{\nless p})}{1+\varphi_p(s_{\nless p})}]$ and $[\frac{1}{2}, 1] \xrightarrow{\sim} [\frac{\varphi_p(s_{\nless p})}{1+\varphi_p(s_{\nless p})}, 1]$. It follows from this fiberwise decomposition that Φ_p does indeed define a bijection (which is clearly continuous). Since source and target are compact Hausdorff spaces, this already shows that Φ_p defines a stratum-preserving homeomorphism. Next, let us agglomerate some more observations about Φ_p, the first of which verifies that Φ_p is a homeomorphism if $p \notin \mathcal{J}$.

(i)$_\Phi$ Whenever $p \notin \mathcal{J}$, then $t_{\nleq p} = 1$ for all $x \in |\Delta^{\mathcal{J}}|_s$. Hence, then Φ_p is given by the identity.

(ii)$_\Phi$ For any $p \in P$, we have $\Phi_p(U_{\Delta^{\mathcal{J}}}(p)) \subset U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p)$.

(iii)$_\Phi$ For $q \leq p$, $\Phi_p(U^{\varphi_q}_{\Delta^{\mathcal{J}}}(q))) \subset \Phi_p(U^{\varphi_q}_{\Delta^{\mathcal{J}}}(q)))$.

(iv)$_\Phi$ For any $q \in P$, $\Phi_p(U_{\Delta^{\mathcal{J}}}(q))) \subset \Phi_p(U_{\Delta^{\mathcal{J}}}(q))$.

Properties (i)$_\Phi$ and (ii)$_\Phi$ are immediate from the construction of Φ_p. Let us verify Properties (iii)$_\Phi$ and (iv)$_\Phi$. Note first that by Property (i)$_\Phi$, we may assume that $p \in \mathcal{J}$. Furthermore, by Lemma 6.3.1.10 we may assume that $q \in \mathcal{J}$. Therefore, since $\mathcal{J}$ is a flag, we may proceed with the remaining cases $q < p$, $q = p$, and $q > p$. Furthermore, since any of the relevant q-strata neighborhoods contains $(|\Delta^{\mathcal{J}}|_s)_{<q}$ and Φ_p is stratum-preserving, we may always assume $s_{\nless q} > 0$. For $q < p$, note that $\Phi_p(x)_{\leq q} = x_{\leq q}$, for all $x \in |\Delta^{\mathcal{J}}|_s$ (this follows from the computation of x in Construction 6.3.1.17). From this it follows that

$$s_{\nless q}(\Phi_p(x)) = 1 - s_{<p}(\Phi_p(x)) = 1 - s_{<p}(x) = s_{\nless q}(x)$$

and similarly

$$t_{\nleq q}(\Phi_p(x)) = t_{\nleq q}(x)$$

(whenever the latter is defined). In particular, this immediately implies Properties (iii)$_\Phi$ and (iv)$_\Phi$. If $q = p$, then by Property (ii)$_\Phi$ and since $U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p) \subset U_{\Delta^{\mathcal{J}}}(p)$, we have

$$\Phi_p(U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p)) \subset \Phi_p(U_{\Delta^{\mathcal{J}}}(p)) \subset U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p) \subset U_{\Delta^{\mathcal{J}}}(p).$$

It remains to consider the case where $q > p$ for Property (iv)$_\Phi$. In this case, one can compute from the description of $x \in |\Delta^{\mathcal{J}}|_s$ in Construction 6.3.1.17 the equalities

$$s_{\nless q}(\Phi_p(x)) = \frac{t_{\nleq p}(\Phi_p(x))}{t_{\nleq p}(x)} s_{\nless q}(x) \tag{6.12}$$

$$s_{\nleq q}(\Phi_p(x)) = \frac{t_{\nleq p}(\Phi_p(x))}{t_{\nleq p}(x)} s_{\nleq q}(x) \tag{6.13}$$

whenever these expressions are defined. An elementary verification shows that this is indeed the case whenever $t_{\nleq q}(\Phi_p(x)) > 0$. It follows that in this case

$$t_{\nleq q}(\Phi_p(x)) = \frac{s_{\nleq q}(\Phi_p(x))}{s_{\nless q}(\Phi_p(x))} = \frac{s_{\nleq q}(x)}{s_{\nless q}(x)} = t_{\nleq q}(x).$$

In particular, Property (iv)$_\Phi$ holds for the remaining case $q > p$. This finishes the verification of the properties of the natural transformation Φ_p. Next, for a flag $\mathcal{J}$ degenerating from a regular flag $[p_0 < \dots < p_n]$, we set

$$\Phi = \Phi_{p_n} \circ \dots \circ \Phi_{p_0} \colon |\Delta^{\mathcal{J}}|_s \to |\Delta^{\mathcal{J}}|_s.$$

Note that this still defines a natural transformation of the realization functor (on the stratified simplex category). Indeed, whenever there is a stratum-preserving simplicial map

$$\Delta^{\mathcal{J}} \to \Delta^{\mathcal{J}'}$$

and $\mathcal{J}'$ degenerates from $[q_0 < \dots < q_m]$, then $[p_0 < \dots < p_n] \subset [q_0 < \dots < q_m]$. Thus, it follows from Property (i)$_\Phi$ that on $|\Delta^{\mathcal{J}}|_s$

$$\Phi_{p_n} \circ \dots \circ \Phi_{p_0} = \Phi_{q_m} \circ \dots \circ \Phi_{q_0}.$$

Using this equation, the naturality of Φ follows from the naturality of the Φ_{q_n}. By left Kan extension Φ extends to a natural automorphism of $| - |_s$. It remains to verify that $\Phi_{\mathcal{X}}$ lifts to a map $\mathfrak{U}_{\mathcal{X}} \to \mathfrak{U}^{\varphi}_{\mathcal{X}}$. By construction of these neighborhood systems, it suffices to verify this on stratified simplices $\Delta^{\mathcal{J}}$, where $\mathcal{J}$ degenerates from the regular flag $[p_0 < \dots < p_n]$. As before, we may assume that $p \in \mathcal{J}$, i.e., $p = p_k$, for some $k \in [n]$. Then, we have

$$\begin{aligned}\Phi(U_{\Delta^{\mathcal{J}}}(p)) &= \Phi_{p_n} \circ \dots \circ \Phi_{p_0}(U_{\Delta^{\mathcal{J}}}(p)) \\ &\subset \Phi_{p_n} \circ \dots \circ \Phi_{p_k}(U_{\Delta^{\mathcal{J}}}(p)) \\ &\subset \Phi_{p_n} \circ \dots \circ \Phi_{p_{k+1}}(U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p)) \\ &\subset U^{\varphi_p}_{\Delta^{\mathcal{J}}}(p),\end{aligned}$$

where the first inclusion follows by Property (iv)$_\Phi$, the second inclusions follows by Property (ii)$_\Phi$, and the final inclusion follows by Property (iii)$_\Phi$.
It remains to see that Φ is stratum-preserving homotopic to the identity. Given any family of functions φ as in the assumption, we write $\Phi(\varphi)$ for the corresponding Φ, constructed as above. Note that Φ varies continuously in each φ_p (with respect to the supremum distance on $C^0([0,1],[0,1])$ and that $\Phi(\varphi) = 1$, if φ is given by the constant functions of value 1. For $t \in [0,1]$, define φ^t via $\varphi^t_p(s) = (1-t) + t\varphi_p$, for $p \in P$. Then, $t \mapsto \Phi(\varphi^t)$ defines the required natural homotopy. □

We may combine Lemma 6.3.1.16 and Proposition 6.3.1.19 as the following result, which will be central to generalizing our construction of strata-neighborhood systems to stratified cell complexes.

Proposition 6.3.1.20. *Let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ be a finite stratified simplicial set. Let $\mathcal{U}_{\mathcal{Y}}$ be a strata-neighborhood system on $\mathcal{Y} \in \mathbf{Strat}_P$ and $f\colon |\mathcal{X}|_s \to \mathcal{Y}$ be any stratum-preserving map. Then, there exists a natural stratum-preserving automorphism Φ of $|-|_s$, naturally stratified homotopic to the identity through a family of automorphisms which lift to maps $\mathfrak{U}_{\mathcal{X}} \to \mathfrak{U}_{\mathcal{X}}$ such that $f \circ \Phi_{\mathcal{X}} : |\mathcal{X}|_s \to \mathcal{Y}$ lifts to a map $\mathfrak{U}_{\mathcal{X}} \to \mathcal{U}_{\mathcal{Y}}$.*

A first consequence of Proposition 6.3.1.20 is that strata-neighborhood systems may be used to compute homotopy links using only data close to a stratum. Such an argument was essentially the decisive step in the proof of [DW22, Theorem 4.8] (see Theorem 3.4.4.1 in this text), where we gave an elementary proof of a special case of the following more general statement.

Proposition 6.3.1.21. *Let $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}}) \in \mathbf{SNS}$ and $\mathcal{I} \subset P$ be a regular flag. Then the inclusion $\mathcal{U}_{\mathcal{X}}(\mathcal{I}) \hookrightarrow \mathcal{X}$ induces a weak equivalence*

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{U}_{\mathcal{X}}(\mathcal{I})) \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}).$$

Proof. We use Lemma 6.B.0.1. Under the adjunction

$$- \times |\Delta^{\mathcal{I}}|_s \colon \mathbf{Top} \rightleftharpoons \mathbf{Strat}_P \colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}},$$

we may thus equivalently show that every stratum-preserving map $g_0 : D^{n+1} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ which maps $S^n \times |\Delta^{\mathcal{I}}|_s$ to $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$ is stratum-preserving homotopic to a map $g_1 : D^{n+1} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ with image in $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$ through a homotopy mapping $S^n \times |\Delta^{\mathcal{I}}|_s$ to $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$. Now, fix some identification $|\Delta^{n+1}| \cong D^{n+1}$. Under this identification, we obtain a canonical isomorphism

$$|\Delta^{n+1} \times \Delta^{\mathcal{I}}|_s \cong |\Delta^{n+1}| \times |\Delta^{\mathcal{I}}|_s \cong D^{n+1} \times |\Delta^{\mathcal{I}}|_s,$$

identifying $|\partial\Delta^{n+1} \times \Delta^{\mathcal{I}}|_s$ with $S^n \times |\Delta^{\mathcal{I}}|_s$. We can now apply Proposition 6.3.1.20 to

$$g_0\colon |\Delta^{n+1} \times \Delta^{\mathcal{I}}|_s \cong D^{n+1} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X},$$

from which it follows that g_0 is stratum-preserving homotopic to a stratum-preserving map $g_0'\colon |\Delta^{n+1} \times \Delta^{\mathcal{I}}|_s \cong D^{n+1} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ that maps $U_{\Delta^{n+1} \times \Delta^{\mathcal{I}}}(\mathcal{I})$ into $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$. Furthermore, by naturality of the homotopy in Proposition 6.3.1.20, it follows that the homotopy between g_0 and

g'_0 maps $S^n \times |\Delta^{\mathcal{I}}|_s$ into $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$. Next, note that the identification $|\Delta^{n+1} \times \Delta^{\mathcal{I}}|_s \cong D^{n+1} \times |\Delta^{\mathcal{I}}|_s$ restricts to an identification

$$U_{\Delta^{n+1}\times\Delta^{\mathcal{I}}}(\mathcal{I}) \cong D^{n+1} \times U_{\Delta^{\mathcal{I}}}(\mathcal{I}).$$

Hence, we may assume without loss of generality that g_0 maps $D^{n+1} \times U_{\Delta^{\mathcal{I}}}(\mathcal{I})$ into $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$. Finally, note that, by Example 6.3.1.14, the inclusion $U_{\Delta^{\mathcal{I}}}(\mathcal{I}) \hookrightarrow |\Delta^{\mathcal{I}}|_s$, equivalently given by

$$|\Delta^{\mathcal{I}}|_s \hookrightarrow |\mathrm{sd}\Delta^{\mathcal{I}}|_s \cong |\Delta^{\mathcal{I}}|_s.$$

Choose any strong (stratum-preserving) deformation retraction $R : |\Delta^{\mathcal{I}}|_s \times [0,1] \to |\Delta^{\mathcal{I}}|_s$ of the inclusion $|\Delta^{\mathcal{I}}|_s \hookrightarrow |\mathrm{sd}\Delta^{\mathcal{I}}|_s \cong |\Delta^{\mathcal{I}}|_s$ (given, for example, by affine interpolation between the identity and the last vertex map). Then the homotopy

$$g_0 \circ (1_{D^{n+1}} \times R) \colon D^{n+1} \times |\Delta^{\mathcal{I}}|_s \times [0,1] \to D^{n+1} \times |\Delta^{\mathcal{I}}|_s$$

has the required properties. □

6.3.2 Strata-neighborhood systems for stratified cell complexes

Next, let us generalize the construction of standard neighborhood systems to stratified cell complexes. The obvious issue at hand is that we may generally not expect the choices of standard neighborhood on cells to be compatible with attaching maps. To amend this difficulty, we first need a notion of subdivision of a stratified cell complex. For the remainder of this section, by a stratified cell complex we will always mean a P-stratified space, together with a fixed choice of cell structure $(\sigma_i \colon |\Delta^{\mathcal{J}_i}|_s \to \mathcal{X})_{i\in I}$. By a slight abuse of notation, we will often just refer to the underlying space.

Definition 6.3.2.1. Let $\mathcal{X}$ be a stratified cell complex, defined by cells $(\sigma_i \colon |\Delta^{\mathcal{J}_i}|_s \to \mathcal{X})_{i\in I}$. By a barycentric subdivision of $\mathcal{X}$ we mean a family of stratum-preserving homeomorphisms $\Psi_i \colon |\mathrm{sd}\Delta^{\mathcal{J}_i}|_s \xrightarrow{\sim} |\Delta^{\mathcal{J}_i}|_s$, for $i \in I$, which fulfill $\Psi_i(|\mathrm{sd}\Delta^{\mathcal{J}}|_s) \subset |\Delta^{\mathcal{J}}|_s$, for $\mathcal{J} \subset \mathcal{J}_i$.

Remark 6.3.2.2. Note that any choice of barycentric subdivision $(\Psi_i)_{i\in I}$ on a stratified cell complex $\mathcal{X}$ naturally induces a new cell structure on $\mathcal{X}$ which is indexed over

$$\{(i,\tau) \mid i \in I, \tau \text{ simplex of } \mathrm{sd}\Delta^{\mathcal{J}_i} \text{ s.t. } \mathcal{J}_i \in \tau\}.$$

We write $\mathrm{sd}_\Psi\mathcal{X}$, for the underlying space of $\mathcal{X}$ equipped with this new cell structure.

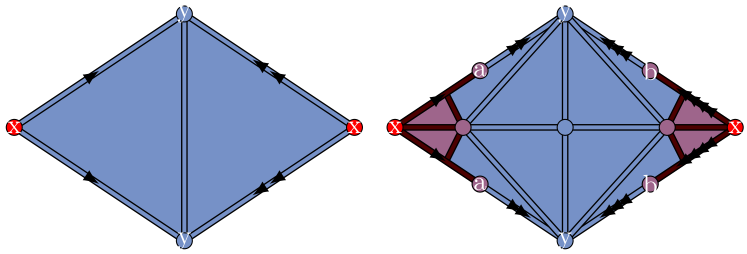
Example 6.3.2.3. If $\mathcal{X} \in \mathbf{Strat}_P$ is the realization of a stratified simplicial complex $\mathcal{K}$, equipped with the induced cell structure, and we take $\Psi_i \colon |\mathrm{sd}\Delta^{\mathcal{J}_i}|_s \to |\Delta^{\mathcal{J}_i}|_s$ to be the barycentric subdivision homeomorphism, then the cell structure induced by the subdivision Ψ is the one coming from the barycentric subdivision homeomorphism $|\mathrm{sd}\mathcal{K}|_s \cong |\mathcal{K}|_s = \mathcal{X}$.

Definition 6.3.2.4. Let $\mathcal{X}$ be a stratified cell complex, and Ψ a choice of subdivision of $\mathcal{X}$. We say that Ψ *defines a standard neighborhood system on* $\mathcal{X}$ if for every $p \in P$ and every $i \in I$ the inclusion

$$\sigma_i \circ \Psi_i(|\mathcal{N}_{\partial\Delta^{\mathcal{J}_i}}(p)|_s) \subset \bigcup_{j<i} \sigma_j \circ \Psi_j(|\mathcal{N}_{\Delta^{\mathcal{J}_j}}(p)|_s),$$

holds (here $<$ denotes the precursor order of Definition 6.2.3.1).

Example 6.3.2.5. Consider the pinched torus $S^1 \times S^1/(S^1 \times x_0)$, stratified over $\{0 < 2\}$ taking the equivalence class $S^1 \times x_0$ as the 0-stratum. To the left, a stratified cell structure induced by a simplicial model is shown, with the stratification indicated by the coloring. To the right, we show a subdivision of this cell structure, which defines a standard neighborhood system on the pinched torus. The standard-neighborhood of the p-stratum induced by this subdivision is shown shaded in red.



Note that the definition of a stratified cell complex allows for the attachment of n-cells to n-cells, as long as this does not lead to cycles in attachment.

Remark 6.3.2.6. Note that the condition in Definition 6.3.2.4 can be equivalently defined by replacing the precursor order by any ordinal structure on I, which exposes $\mathcal{X}$ as an absolute cell complex with respect to stratified boundary inclusions of simplices (see Construction 6.2.3.5).

Construction 6.3.2.7. The condition in Definition 6.3.2.4 precisely guarantees that when a subdivision Ψ of a cell complex $\mathcal{X}$ defines a standard neighborhood system on $\mathcal{X}$, then the indexing set

$$\{(i,\tau) \mid i \in I, \mathcal{J}_i \in \tau; \tau \subset \mathcal{N}_{\Delta^{\mathcal{J}_i}}(p)\}$$

corresponding to such cells in $\mathrm{sd}\mathcal{X}$ that lie in standard neighborhoods in the stratified cells define a subcomplex of $\mathrm{sd}_\Psi\mathcal{X}$. As a stratified space, it is given by the union

$$\bigcup_{i \in I} \sigma_i \circ \Psi_i(|\mathcal{N}_{\Delta^{\mathcal{J}_i}}(p)|_s) \subset \mathcal{X}.$$

We denote this subcomplex by $U^\Psi_\mathcal{X}(\mathcal{I})$. We call $U^\Psi_\mathcal{X}(\mathcal{I})$ the *p-th standard neighborhood associated to the subdivision* Ψ. Furthermore, we denote

$$\mathfrak{U}^\Psi_\mathcal{X} \coloneqq (U^\Psi_\mathcal{X}(p))_{p \in P}$$

and call this family the *standard neighborhood system of $\mathcal{X}$ associated to the subdivision* Ψ.

Let us verify that the nomenclature of Construction 6.3.2.7 does make sense, that is, that we have indeed defined a strata-neighborhood system. Before we do so, note the following remark.

Remark 6.3.2.8. For any stratified simplicial complex $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ we may use Proposition 6.3.1.15 to identify $|\mathcal{N}_\mathcal{X}(p)|_s$ with $\mathcal{U}^\Psi_{|\mathcal{X}|_s}(\mathcal{I})$, where Ψ is the barycentric subdivision of Construction 6.3.1.12.

Proposition 6.3.2.9. *In the setting of Construction 6.3.2.7, the family $\mathfrak{U}^\Psi_\mathcal{X}$ defines a strata-neighborhood system on $\mathcal{X}$. Furthermore, $\mathfrak{U}^\Psi_\mathcal{X}$ has the property that $\mathcal{U}^\Psi_\mathcal{X}(\mathcal{I}) \subset \mathcal{X}$ is a subcomplex of $\mathrm{sd}_\Psi\mathcal{X}$, for every regular flag $\mathcal{I} \subset P$.*

Proof. That, for any $p \in P$, it holds that $X_{\leq p} \subset U^\Psi_\mathcal{X}(p)$, is immediate from Remark 6.3.2.8, Proposition 6.3.1.15 and Corollary 6.3.1.8. Next, let us verify that $U^\Psi_\mathcal{X}(\mathcal{I})$ does indeed define a p-stratum neighborhood. By Lemma 6.2.4.4, it suffices to show that, for every $p \in P$ and every cell $\sigma\colon |\Delta^{\mathcal{J}_i}|_s \to \mathcal{X}$ of $\mathcal{X}$, the set $\sigma^{-1}(U^\Psi_\mathcal{X}(\mathcal{I}))$ defines a neighborhood of the p-stratum. Since Ψ_i is a stratum-preserving homeomorphism, we may equivalently show that $(\sigma \circ \Psi_i)^{-1}(U^\Psi_\mathcal{X}(\mathcal{I}))$ has this property. Note that by construction we have

$$|\mathcal{N}_{\Delta^{\mathcal{J}_i}}(p)| \subset (\sigma \circ \Psi_i)^{-1}(U^\Psi_\mathcal{X}(\mathcal{I})).$$

By Proposition 6.3.1.15, up to a stratum-preserving homeomorphism of $|\Delta^\mathcal{J}_i|_s$ we have

$$|\mathcal{N}_{\Delta^{\mathcal{J}_i}}(p)|_s = \mathcal{U}_{\Delta^{\mathcal{J}_i}}(p),$$

which shows that both $|\mathcal{N}_{\Delta^{\mathcal{J}_i}}(p)|$ and thus also $(\Psi \circ \sigma)^{-1}(U^\Psi_\mathcal{X}(\mathcal{I}))$ is a neighborhood of the p-stratum. The statement on subcomplexes is immediate from the fact that the intersection of subcomplexes is again a subcomplex. $\square$

Example 6.3.2.10. If $\mathcal{X} \in \mathbf{Strat}_P$ is the realization of a stratified simplicial complex $\mathcal{K}$, equipped with the induced cell structure, and we take $\Psi_i \colon |\mathrm{sd}\Delta^{\mathcal{J}_i}|_s \to |\Delta^{\mathcal{J}_i}|_s$ to be the subdivision homeomorphism of Construction 6.3.1.12, then $U^{\Psi}_{\mathcal{X}}(\mathcal{I}) = U_{\mathcal{K}}(\mathcal{I})$.

Next, let us show that there always exists a subdivision which defines a standard neighborhood system on a stratified cell complex $\mathcal{X}$. This is ultimately a consequence of Proposition 6.3.1.19, and provides a first step towards Theorem HA.

Proposition 6.3.2.11. *For every stratified cell complex $\mathcal{X}$, there exists a subdivision Ψ of $\mathcal{X}$ such that Ψ defines a standard neighborhood system on $\mathcal{X}$. Additionally, for any P-stratified space $\mathcal{Y}$ equipped with a strata-neighborhood system $\mathfrak{U}_{\mathcal{Y}}$ and any stratum-preserving map $f \colon \mathcal{X} \to \mathcal{Y}$, Ψ may be chosen such that f lifts to a map $\mathfrak{U}^{\Psi}_{\mathcal{X}} \to \mathfrak{U}_{\mathcal{Y}}$. Furthermore, if such a subdivision $\Psi_{\mathcal{A}}$ has already been chosen on a subcomplex of $\mathcal{A} \subset \mathcal{X}$, then Ψ may be taken such that*

$$\Psi_{\mathcal{A},i} = \Psi_i$$

whenever $i \in I$ defines a cell of $\mathcal{A}$.

Proof. Via transfinite induction, it suffices to show the following. For any commutative diagram

$$\begin{tikzcd} |\partial\Delta^{\mathcal{J}_i}|_s \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ |\Delta^{\mathcal{J}_i}|_s \arrow[r] & \mathcal{Y}, \end{tikzcd} \tag{6.14}$$

where $\mathcal{A} \to \mathcal{Y}$ lifts to a map of neighborhood systems $\mathfrak{U}_{\mathcal{A}} \to \mathfrak{U}_{\mathcal{Y}}$, there exists a stratum-preserving homeomorphism $\Psi_i \colon |\mathrm{sd}\Delta^{\mathcal{J}_i}|_s \to |\Delta^{\mathcal{J}_i}|_s$ (which is compatible with faces) with the following properties.

1. The composition $|\mathrm{sd}\partial\Delta^{\mathcal{J}_i}|_s \xrightarrow{\Psi_i|_{|\mathrm{sd}\partial\Delta^{\mathcal{J}_i}|_s}} |\partial\Delta^{\mathcal{J}_i}|_s \to \mathcal{A}$ lifts to a map of neighborhood systems
$$(|\mathcal{N}_{\partial\Delta^{\mathcal{J}_i}}(p)|_s)_{p \in P} \to \mathfrak{U}_{\mathcal{A}}.$$

2. The composition $|\mathrm{sd}\Delta^{\mathcal{J}_i}|_s \xrightarrow{\Psi_i} |\Delta^{\mathcal{J}_i}|_s \to \mathcal{Y}$ lifts to a map of neighborhood systems
$$(|\mathcal{N}_{\Delta^{\mathcal{J}_i}}(p)|_s)_{p \in P} \to \mathfrak{U}_{\mathcal{Y}}.$$

We may first apply Proposition 6.3.1.15 and instead show the analogous statement with $\mathrm{sd}\Delta^{\mathcal{J}_i}$ replaced by $\Delta^{\mathcal{J}_i}$ and $(|\mathcal{N}_{\Delta^{\mathcal{J}_i}}(p)|_s)_{p \in P}$ replaced by $\mathfrak{U}_{\Delta^{\mathcal{J}_i}}$. Next, apply Proposition 6.3.1.20 twice, first to $|\Delta^{\mathcal{J}_i}|_s \to \mathcal{Y}$, obtaining a natural stratum-preserving automorphism $\Phi_{\mathcal{Y}}$ of $|-|_s$, and then to the composition $|\partial\Delta^{\mathcal{J}_i}|_s \xrightarrow{\Phi_{\mathcal{Y}}} |\partial\Delta^{\mathcal{J}_i}|_s \to \mathcal{A}$, obtaining a natural stratum-preserving automorphism $\Phi_{\mathcal{A}}$ of $|-|_s$. By construction, these have the following properties:

1. The composition $|\partial\Delta^{\mathcal{J}_i}|_s \xrightarrow{\Phi_{\mathcal{Y}} \circ \Phi_{\mathcal{A}}} |\partial\Delta^{\mathcal{J}_i}|_s \to \mathcal{A}$ lifts to a map $\mathfrak{U}_{\partial\Delta^{\mathcal{J}_i}} \to \mathfrak{U}_{\mathcal{A}}$.

2. $|\Delta^{\mathcal{J}_i}|_s \xrightarrow{\Phi_{\mathcal{A}}} |\Delta^{\mathcal{J}_i}|_s$ lifts to a map $\mathfrak{U}_{\Delta^{\mathcal{J}_i}} \to \mathfrak{U}_{\Delta^{\mathcal{J}_i}}$.

3. The composition $|\Delta^{\mathcal{J}_i}|_s \xrightarrow{\Phi_{\mathcal{Y}}} |\Delta^{\mathcal{J}_i}|_s \to \mathcal{Y}$ lifts to a map $\mathfrak{U}_{\Delta^{\mathcal{J}_i}} \to \mathfrak{U}_{\mathcal{Y}}$.

In particular, by the composability of morphisms of strata-neighborhood systems, it follows that the composition $\Psi_i := \Phi_{\mathcal{Y}} \circ \Phi_{\mathcal{A}}$ also has the property that $|\Delta^{\mathcal{J}_i}|_s \xrightarrow{\Psi_i} |\Delta^{\mathcal{J}_i}|_s \to \mathcal{Y}$ also lifts to a map $\mathfrak{U}_{\Delta^{\mathcal{J}_i}} \to \mathfrak{U}_{\mathcal{Y}}$. ☐

Furthermore, we can now prove a first result towards Theorem HB.

Corollary 6.3.2.12. *Suppose, we are given a pushout square in* $\mathbf{Strat}_P$

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "c"] \arrow[d, "f"] & \mathcal{B} \arrow[d] \\ \mathcal{X} \arrow[r, hook] & \mathcal{Y}, \end{tikzcd} \tag{6.15}$$

with $\mathcal{A}, \mathcal{X}$ *stratified cell complexes and* c *an inclusion of a stratified subcomplex. Let* $(\sigma_i)_{i \in I}$ *be the cell structure on* $\mathcal{X}$*,* $(\sigma_j)_{j \in J}$ *be the cell structure on* $\mathcal{A}$ *and* $(\sigma_j)_{j \in J \sqcup J'}$ *be the cell structure on* $\mathcal{B}$*, extending the one on* $\mathcal{A}$ *along* c*. Then, there exist barycentric subdivisions* Ψ *of* $\mathcal{X}$ *and* $\hat{\Phi}$ *of* $\mathcal{B}$ *such that the following holds:*
Denote by Φ *the restriction of* $\hat{\Phi}$ *to* $\mathcal{A}$*. Denote by* $\hat{\Psi}$ *the subdivision of the induced cell structure on* $\mathcal{Y}$*, given by* $(\Psi_i)_{i \in I} \cup (\Phi_j)_{j \in J'}$*. Then, Diagram* (6.15) *lifts to a diagram of strata-neighborhood systems*

$$\begin{tikzcd} \mathfrak{U}^{\Phi}_{\mathcal{A}} \arrow[r, "\tilde{c}"] \arrow[d] & \mathfrak{U}^{\hat{\Phi}}_{\mathcal{B}} \arrow[d] \\ \mathfrak{U}^{\Psi}_{\mathcal{X}} \arrow[r] & \mathfrak{U}^{\hat{\Psi}}_{Y}\,. \end{tikzcd} \tag{6.16}$$

Proof. By Proposition 6.3.2.11, we obtain subdivisions $\hat{\Phi}$ and Ψ as in the claim such that $\mathcal{A} \to \mathcal{X}$ lift to maps of strata-neighborhood systems $\mathfrak{U}^{\Phi}_{\mathcal{A}} \to \mathfrak{U}^{\Psi}_{\mathcal{X}}$. Now, define $\hat{\Psi}$ as above. Let us show that $\hat{\Psi}$ does define a strata-neighborhood system on the induced cell structure on $\mathcal{Y}$. Via transfinite induction, we may without loss of generality assume that $\mathcal{B}$ is given by gluing a single cell $\sigma_m \colon |\Delta^{\mathcal{J}}|_s \to \mathcal{B}$ to $\mathcal{A}$. Then, $\mathcal{Y}$ is given by gluing $|\Delta^{\mathcal{J}}|_s$ to $\mathcal{X}$ along $|\partial\Delta^{\mathcal{J}}|_s \to \mathcal{A} \to \mathcal{X}$. Denote the resulting cell of $\mathcal{Y}$, by $\hat{\sigma}_m \colon |\Delta^{\mathcal{J}}|_s \to \mathcal{Y}$. Then, by construction, we have

$$\begin{aligned} \hat{\sigma}_m \circ \hat{\Psi}_m(|\mathcal{N}_{\partial\Delta^J}(p)|_s) &= f \circ \sigma_m \circ \Phi_m(|\mathcal{N}_{\partial\Delta^J}(p)|_s) \\ &\subset f(U^{\Phi}_{\mathcal{A}}(p)) \\ &\subset U^{\Psi}_{\mathcal{X}}(p) \\ &= \bigcup_{i \in I} \sigma_j \circ \Psi_j(|\mathcal{N}_{\Delta^{\mathcal{J}_j}}(p)|_s) \end{aligned}$$

for all $p \in P$, which (by Remark 6.3.2.6) was to be shown. It is then immediate by construction that $\mathcal{B} \to \mathcal{Y}$ also lifts to a map of strata-neighborhood systems. □

Next, let us verify that the functor $\mathrm{D}^T \colon \mathbf{SNS} \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ sends Diagram (6.16) to a homotopy pushout square.

Lemma 6.3.2.13. *In the situation of Corollary 6.3.2.12, the image of Diagram* (6.16) *under* D^T *has the following property. For each* $\mathcal{I} \in \mathrm{sd}(P)$*, the resulting square*

$$\begin{tikzcd} \mathrm{D}^T(\mathfrak{U}^{\Phi}_{\mathcal{A}})(\mathcal{I}) \arrow[r, hook] \arrow[d] & \mathrm{D}^T(\mathfrak{U}^{\hat{\Phi}}_{\mathcal{B}}(\mathcal{I})) \arrow[d] \\ \mathrm{D}^T(\mathfrak{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})) \arrow[r, hook] & \mathrm{D}^T(\mathfrak{U}^{\hat{\Psi}}_{\mathcal{Y}}(\mathcal{I})) \end{tikzcd} \tag{6.17}$$

is such that:

1. *All objects of the square are cell complexes in* **Top***;*

2. *The square is a pushout in* **Top***;*

3. *The horizontals are relative cell complexes in* **Top**.

In particular, Diagram (6.17) *is a homotopy pushout diagram in* **Top**.

Proof. It is immediate from the construction of the standard neighborhoods of a stratified cell complex that the diagrams

$$\begin{tikzcd} \mathcal{U}^{\Phi}_{\mathcal{A}}(\mathcal{I}) \arrow[r, hook] \arrow[d] & \mathcal{U}^{\hat{\Phi}}_{\mathcal{B}}(\mathcal{I}) \arrow[d] \\ \mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I}) \arrow[r, hook] & \mathcal{U}^{\hat{\Psi}}_{\mathcal{Y}}(\mathcal{I}) \end{tikzcd} \tag{6.18}$$

are pushout diagrams of stratified cell complexes (see also Proposition 6.3.2.9), with the upper vertical a relative (stratified) cell complex, for every $\mathcal{I} \in \mathrm{sd}(P)$. Indeed, note that the cells missing in $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ from $\mathcal{U}^{\hat{\Psi}}_{\mathcal{Y}}(\mathcal{I})$ correspond precisely to the respective cells missing in $\mathcal{U}^{\Phi_0}_{\mathcal{A}}(\mathcal{I})$ from $\mathcal{U}^{\Phi}_{\mathcal{B}}(\mathcal{I})$. What remains to be shown is that these properties are preserved under applying the functor $(-)_{\geq p}\colon \mathbf{Strat}_P \to \mathbf{Top}$. This is an immediate consequence of Lemma 6.A.0.4 and Lemma 6.A.0.3. We can thus summarize that Diagram (6.17) is a pushout diagram of cell complexes where the upper horizontal is given by a relative cell complex, in particular a cofibration. It follows from [Lur09, A.2.4.4] that the diagram is homotopy cocartesian. □

6.3.3 The proof of Theorem 6.2.4.14

As a consequence of Proposition 6.3.1.21 we are now ready to give a proof of Theorem 6.2.4.14, which tells us that we may indeed use homotopy link models to compute homotopy links. Precisely, Proposition 6.3.1.21 guarantees us that for $\mathcal{X} \in \mathbf{Strat}_P$ the diagram $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ may equivalently computed via the diagram given by $\mathcal{I} \mapsto \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{U}_{\mathcal{X}}(\mathcal{I}))$, where $\mathfrak{U}_{\mathcal{X}}$ is any strata-neighborhood system of $\mathcal{X}$.

Notation 6.3.3.1. Let $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}}) \in \mathbf{SNS}$. We denote by $\mathrm{D}^H(\mathfrak{U}_{\mathcal{X}})$ the element of $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$ given by

$$\mathcal{I} \mapsto \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{U}_{\mathcal{X}}(\mathcal{I}))$$

with the obvious structure maps induced by the ones on $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$. We denote

$$\mathrm{D}^H\colon \mathbf{SNS} \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})$$

the functor induced by this construction.

We may then rephrase Proposition 6.3.1.21 as follows.

Corollary 6.3.3.2. *The inclusions $\mathcal{U}_{\mathcal{X}}(\mathcal{I}) \hookrightarrow \mathcal{X}$, for $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}}) \in \mathbf{SNS}$ and $\mathcal{I} \in \mathrm{sd}(P)$, induce a natural weak equivalence of functors*

$$\mathrm{D}^H \xrightarrow{\simeq} \mathcal{H}o\mathrm{Link}.$$

The obvious next step to prove Theorem 6.2.4.14 is to show that D^H is in turn weakly equivalent to D^T. The definition of a homotopy link model suggests to use the maximal vertex evaluation maps

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{U}_{\mathcal{X}}(\mathcal{I})) \xrightarrow{\mathrm{ev}_{p_n}} U_{\mathcal{X}}(\mathcal{I})_{p_n} \hookrightarrow U_{\mathcal{X}}(\mathcal{I})_{\geq p_n}$$

However, there is a technical difficulty to overcome first. In fact, these maps do not induce a morphisms of diagrams. Already in the case where $\mathcal{I} = [p_0 < p_1]$ the diagram

$$\begin{tikzcd} \mathcal{H}o\mathrm{Link}_{p_0<p_1}(\mathcal{U}_{\mathcal{X}}(p_0 < p_1)) \arrow[r, "\mathrm{ev}_{p_1}"] \arrow[d] & U_{\mathcal{X}}(p_0 < p_1)_{\geq p_1} \arrow[d, hook] \\ \mathcal{H}o\mathrm{Link}_{p_0}(\mathcal{U}_{\mathcal{X}}(p_0)) = (U_{\mathcal{X}}(p_0))_{p_0} \arrow[r, hook] & U_{\mathcal{X}}(p_0)_{\geq p_0} \end{tikzcd} \tag{6.19}$$

is only commutative up to homotopy. What we may do instead is to construct a natural transformation $\mathrm{D}^H \to \mathrm{D}^T$ only up to homotopy coherence. We may then use rigidification results such as [Lur09, Prop. A.3.4.12] to obtain a weak equivalence of functors.

Remark 6.3.3.3. There will occur a slight set-theoretical difficulty when using [Lur09, Prop. A.3.4.12]. Namely, we will want to consider the homotopy coherent nerve of $\underline{\mathbf{Top}}$ as an element of $\underline{\mathbf{sSet}}$. Size issues require us to pass to a larger Grothendieck universe. To make this rigorous, we need to assume large cardinals $\kappa < \kappa'$, and denote by $\underline{\mathbf{sSet}}$ the category of simplicial sets of size smaller than κ some fixed large cardinal, and by $\widetilde{\underline{\mathbf{sSet}}}$ the category of simplicial sets of cardinality smaller than κ'.

Definition 6.3.3.4. In the case where **Top** denotes either Δ-generated or topologically generated spaces (i.e., $\mathbf{Strat}_P$ is cartesian closed). We denote by X^Y the internal mapping space of $Y, X \in \mathbf{Top}$. For any $X \in \underline{\mathbf{Top}}$, this constructions defines a simplicial functor

$$X^{-}\colon \underline{\mathbf{Top}}^{\mathrm{op}} \to \underline{\mathbf{Top}},$$

by mapping an n-simplex

$$\sigma\colon |\Delta^n| \times Z \to Y$$

of $\underline{\mathbf{Top}}(Z, Y)$ to the adjoint map of

$$X^Y \times |\Delta^n| \times Z \xrightarrow{1\times\sigma} X^Y \times Y \xrightarrow{\mathrm{ev}} X,$$

which indeed defines an n-simplex of $\underline{\mathbf{Top}}(X^Y, X^Z)$.

Construction 6.3.3.5. We only construct the weak equivalence for a fixed $(\mathcal{X}, \mathfrak{U}_{\mathcal{X}}) \in \mathrm{D}^T$. Generalizing to the case of a whole natural transformation essentially just comes down to an increase in notation. Furthermore, we only prove the case where $\underline{\mathbf{Top}}$ is cartesian closed. The general case follows from this using that every space is weakly equivalent to its kelleyfication with respect to Δ-generated spaecs. We denote by $\mathcal{N}$ the homotopy coherent nerve functor from the category of (κ') small simplicial categories $\widetilde{\mathrm{s}\mathbf{Cat}}$ to $\widetilde{\mathbf{sSet}}$. We denote its leftadjoint by $\mathcal{S}$.

Denote by $\underline{\mathbf{Pos}}$ the category of all (κ small) posets, with its simplicial structure inherited from $\underline{\mathbf{sSet}}$. Furthermore, consider the posets $Q = \mathrm{sd}(P) \times [1]^{\mathrm{op}}$ as a category. Now, consider the assigment $E\colon Q \to \mathbf{Pos}$ by mapping

$$\begin{aligned}(\mathcal{I}, 1) &\mapsto [0]\\ (\mathcal{I}, 0) &\mapsto \mathcal{I}\end{aligned}$$

and

$$\begin{aligned}(\mathcal{I}, 1) \le (\mathcal{I}', 1) &\mapsto ([0] \to [0]])\\ (\mathcal{I}, 1) \le (\mathcal{I}', 0) &\mapsto \{0 \mapsto \max \mathcal{I}'\}\\ (\mathcal{I}, 0) \le (\mathcal{I}', 0) &\mapsto (\mathcal{I} \hookrightarrow \mathcal{I}').\end{aligned}$$

This assignment does not define a functor! However, we can turn it into a homotopy coherent functor. This is due to the fact that E has the property

$$E(f \circ g)(p) \ge E(f) \circ E(g)(p), \tag{6.20}$$

for composable $f, g \in Q$ and p in the source of $E(g)$. For $\alpha_1, \alpha_0 \in Q$, denote by $Q_{\alpha_1,\alpha_0} \subset Q$ the poset of all regular flags $S \subset Q$, with $\min S = \alpha_1$ and $\max S = \alpha_0$ ordered by reverse inclusion. Next, consider map

$$\begin{aligned}\mathcal{E}\colon Q_{\alpha_1,\alpha_0} \times E(\alpha_1) &\to E(\alpha_0)\\ ([S_0 < \cdots < S_n], p) &\mapsto E(S_{n-1} \le S_n) \circ \cdots \circ E(S_0 \le S_1)(p).\end{aligned}$$

It follows by Eq. (6.20) that $\mathcal{E}$ defines a map of posets. Thus, equivalently $\mathcal{E}$ specifies a simplicial map

$$\mathrm{N}(Q_{\alpha_1,\alpha_0}) \to \underline{\mathbf{sSet}}(\mathrm{N}(E(\alpha_1)), \mathrm{N}(E(\alpha_0))).$$

In this manner, we have defined a simplicial functor

$$\mathcal{E}: \mathcal{S}(\mathrm{sd}(P) \times [1]^{\mathrm{op}}) \to \underline{\mathbf{Pos}},$$

where $\mathcal{S}$ is the left adjoint to the homotopy coherent nerve (see for example [Lur09, Sec. 1.1.5]) and where the simplicial structure on the right hand side is inherited from the one on $\underline{\mathbf{sSet}}$. Next, consider the composition of simplicial functors

$$\mathcal{S}(\mathrm{sd}(P)^{\mathrm{op}} \times [1]) \xrightarrow{\mathcal{E}} \underline{\mathbf{Pos}}^{\mathrm{op}} \xrightarrow{\mathrm{N}} \underline{\mathbf{sSet}}^{\mathrm{op}} \xrightarrow{|-|} \underline{\mathbf{Top}}^{\mathrm{op}} \xrightarrow{X^{-}} \underline{\mathbf{Top}}.$$

It specifies a homotopy coherent diagram D in $\underline{\mathbf{Top}}$, indexed over $\mathrm{sd}(P)^{\mathrm{op}} \times [1]$, which restricts to the constant diagram of value X at 1 and to the diagram D_0 given by $\mathcal{I} \mapsto X^{|\Delta^{\mathcal{I}}|}$ at 0. We may then consider $\mathrm{D}^H(\mathfrak{U}_{\mathcal{X}})$ as a subdiagram of D_0 and $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})$ as a subdiagram of D_1. For $\alpha_0 = (\mathcal{I}_0, 0)$ and $\alpha_1 = (\mathcal{I}_1, 1)$, and $p \in \mathcal{I}_1$, $\mathcal{E}_{\alpha_1,\alpha_0}$ has the property that $\mathcal{E}(-,p)$ has image in $\{q \in \mathcal{I}_0 \mid q \geq \max \mathcal{I}_0\}$. It follows from this that restricting to $\mathrm{D}^H(\mathfrak{U}_{\mathcal{X}})$ at 0 and $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})$ at 1 defines a homotopy coherent subdiagram of D. To summarize, we have constructed a simplicial functor

$$\mathrm{ev} \in \widetilde{\mathrm{s}\mathbf{Cat}}(\mathcal{S}(\mathrm{sd}(P)^{\mathrm{op}} \times [1]), \ \underline{\mathbf{Top}}) \cong \widetilde{\mathbf{sSet}}(\mathrm{sd}(P)^{\mathrm{op}} \times [1], \mathcal{N}(\ \underline{\mathbf{Top}}))$$

which restricts to D^H at 0 and D^T at 1, or in other words by the identity

$$\widetilde{\mathbf{sSet}}(\mathrm{sd}(P)^{\mathrm{op}} \times [1], \mathcal{N}(\ \underline{\mathbf{Top}})) \cong \mathrm{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{N}(\ \underline{\mathbf{Top}}))_1.$$

a natural transformation of functors of quasi categories between

$$\mathcal{N}(\mathrm{sd}(P)^{\mathrm{op}} \xrightarrow{\mathrm{D}^H(\mathfrak{U}_{\mathcal{X}})} \underline{\mathbf{Top}})$$

and

$$\mathcal{N}(\mathrm{sd}(P)^{\mathrm{op}} \xrightarrow{\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})} \underline{\mathbf{Top}}).$$

For any fixed flag $\mathcal{I} = [p_0 < \cdots < p_n]$ this natural transformation is given by

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(U_{\mathcal{X}}(\mathcal{I})) \xrightarrow{\mathrm{ev}_{p_n}} U_{\mathcal{X}}(\mathcal{I})_{\geq p_n}.$$

Now, if $\mathfrak{U}_{\mathcal{X}}$ is a homotopy link model for $\mathcal{X}$, then the latter map is a weak equivalence. Hence, if we pass to Kan-complex ($\underline{\mathbf{sSet}}^o$) by applying singular simplicial sets, then this natural transformation is given pointwise by an isomorphism in the quasi-category $\mathcal{N}(\underline{\mathbf{sSet}}^o)$. We have thus defined an isomorphism between the functors of quasi-categories

$$\mathcal{N}(\mathrm{sd}(P)^{\mathrm{op}} \xrightarrow{\mathrm{D}^H(\mathfrak{U}_{\mathcal{X}})} \underline{\mathbf{Top}} \to \underline{\mathbf{sSet}}^o),$$
$$\mathcal{N}(\mathrm{sd}(P)^{\mathrm{op}} \xrightarrow{\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})} \underline{\mathbf{Top}} \to \underline{\mathbf{sSet}}^o).$$

We may now finish the proof of Theorem 6.2.4.14.

Proof of Theorem 6.2.4.14. We only provide a weak equivalence for some fixed homotopy link model $\mathfrak{U}_{\mathcal{X}}$. The global case is essentially analogous. We consider $\widetilde{\mathrm{s}\mathbf{Cat}}$ as equipped with the model structure for simplicial categories (see [Ber07a]) making the adjunction $\mathcal{S} \dashv \mathcal{N}$ a Quillen equivalence between $\widetilde{\mathrm{s}\mathbf{Cat}}$ and simplicial sets equipped with the Joyal model structure, $\widetilde{\mathbf{sSet}}^{\mathfrak{J}}$ ([Joy, Thm. 1.21]). If not indicated otherwise by an superscript $\mathfrak{J}$, we consider $\underline{\mathbf{sSet}}$ to be equipped with the Kan-Quillen model structure. Let $\mathfrak{U}_{\mathcal{X}}$ be a homotopy link model for a stratified space $\mathcal{X} \in \mathbf{Strat}_P$. We need to show that $\mathcal{H}o\mathrm{Link}\mathcal{X}$ and $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})$ are weakly equivalent. By Corollary 6.3.3.2, we may instead show that $\mathrm{D}^T(\mathfrak{U}_{\mathcal{X}})$ and $\mathrm{D}^H(\mathfrak{U}_{\mathcal{X}})$ are weakly equivalent. Using the Quillen equivalence between $\underline{\mathbf{Top}}$ and $\underline{\mathbf{sSet}}$, we may equivalently show that $\mathrm{Sing} \circ \mathrm{D}^T(\mathfrak{U}_{\mathcal{X}}): \mathrm{sd}(P)^{\mathrm{op}} \to \underline{\mathbf{sSet}}$ and $\mathrm{Sing} \circ \mathrm{D}^H(\mathfrak{U}_{\mathcal{X}}): \mathrm{sd}(P)^{\mathrm{op}} \to \underline{\mathbf{sSet}}$ are weakly equivalent.

In other words, we need to show that these two functors present the same path component in $\pi_0(\mathbf{Fun}\big((\mathrm{sd}P)^{\mathrm{op}}, \underline{\mathbf{sSet}}\big))$ (using the notation of [Lur09, Prop. A.3.4.12].) By [Lur09, Prop. A.3.4.12] there is a canonical bijection:

$$\pi_0(\mathbf{Fun}\big((\mathrm{sd}P)^{\mathrm{op}}, \underline{\mathbf{sSet}}\big)) = \mathrm{ho}\widetilde{\mathrm{s}\mathbf{Cat}}(\mathrm{sd}(P)^{\mathrm{op}}, \underline{\mathbf{sSet}}^o).$$

Furthermore, under the Quillen equivalence between simplicial and quasi-categories [Joy, Thm. 1.21], this bijection extends to

$$\pi_0(\mathbf{Fun}\big((\mathrm{sd}P)^{\mathrm{op}}, \underline{\mathbf{sSet}}\big)) = \mathrm{ho}\widetilde{\mathrm{s}\mathbf{Cat}}(\mathrm{sd}(P)^{\mathrm{op}}, \underline{\mathbf{sSet}}^o) = \mathrm{ho}\widetilde{\mathbf{sSet}}^{\mathfrak{J}}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{N}(\underline{\mathbf{sSet}}^o)).$$

$\mathrm{ho}\widetilde{\mathbf{sSet}}^{\mathfrak{J}}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{N}(\underline{\mathbf{sSet}}^o))$ is the set of isomorphism classes of functors of quasi-categories $\mathrm{sd}(P)^{\mathrm{op}} \to \mathcal{N}(\underline{\mathbf{sSet}}^o)$. We have constructed such an isomorphism between $\mathcal{N}(\mathrm{Sing} \circ \mathrm{D}^T(\mathfrak{U}_{\mathcal{X}}))$ and $\mathcal{N}(\mathrm{Sing} \circ \mathrm{D}^H(\mathfrak{U}_{\mathcal{X}}))$ in Construction 6.3.3.5. □

6.4 Regular neighborhoods and homotopy link models

In the previous section, we have constructed strata-neighborhood systems for stratified simplicial sets and stratified cell complexes. For a proof of Theorem HA, in light of Theorem 6.2.4.14, it remains to show that these strata-neighborhood systems are homotopy link models. To do so, we develop a generalized notion of regular neighborhoods for stratified spaces, which also applies to flags $\mathcal{I} \in \mathrm{sd}(P)$ of length greater equal to two. Recall from [Fri03, A] the notion of a nearly stratum-preserving deformation retraction (introduced in similar form in [Qui88]). The following generalizes this notion to the case of more than two strata. In the following sections, we will generally omit the index from the stratification maps $s_{\mathcal{X}}: \mathcal{X} \to P$ and just write $s(x) \in P$, for $x \in \mathcal{X}$.

Definition 6.4.0.1. Let $\mathcal{X} \in \mathbf{Strat}_P$ be a stratified space and let $\mathcal{I} = [p_0 < \cdots < p_n]$ be a regular flag in P. We say that $\mathcal{X}$ admits *an almost*[2] *stratum-preserving $\mathcal{I}$-retraction* - ASP$\mathcal{I}$R for short - if the following holds: There exists a stratum-preserving map $R: X_{p_n} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ such that, for each $p \in \mathcal{I}$, the map of (general topological) spaces

$$(X_{p_n} \cup X_{\leq p}) \times |\Delta^{\mathcal{I}_{\geq p}}|_s \to X$$
$$(x, u) \mapsto \begin{cases} R(x, u) & s(x) = p_n \\ x & s(x) \leq p \end{cases}$$

is well defined and continuous.

Remark 6.4.0.2. To get a first intuition for Definition 6.4.0.1, let us decode what the requirements in Definition 6.4.0.1 mean in the case where $\mathcal{I} = [p_0 < p_1] = P$. Then, we may identify $|\Delta^{\mathcal{I}}|_s$ with the (stratified) interval $[0, 1]$. Suppose a (stratified) neighborhood $\mathcal{N} \subset \mathcal{X}$ of X_{p_0} admits an ASP$\mathcal{I}$R R. Then, equivalently R is a stratum-preserving map

$$R: N_{p_1} \times [0, 1] \cong N_{p_1} \times |\Delta^{\mathcal{I}}|_s \to N$$

which extends to

$$N \times [0, 1] \to N$$

by taking the constant homotopy of the inclusion on $X_{p_0} \hookrightarrow N$, and furthermore R restricted to $N_{p_n} \times \{1\}$ is given by the inclusion $N_{p_n} \hookrightarrow N$.
We may summarize this information as R defining a strong deformation retraction from N to X_{p_0}, which is stratum-preserving, except at time 0 when all of N is mapped into X_{p_0}. Note that this is (up to a slight but inessential variation in target space) the definition of a *nearly stratum-preserving strong deformation retraction* given in [Fri03, A] (adapted from [Qui88]).

[2]The usage of 'almost' instead of 'nearly' is purely for the sake of having a phonetically pleasant acronym.

It is a consequence of [Fri03, Prop. A.1] that (under some additional conditions on $\mathcal{X}$) the existence of such a deformation retraction guarantees that the map

$$\mathcal{H}o\mathrm{Link}_{p_0<p_1}(N) \xrightarrow{\mathrm{ev}_{p_1}} N_{p_1}$$

is a homotopy equivalence. Note that this is half the condition required for N to be part of a homotopy link model for $\mathcal{X}$. This already makes it plausible that ASP$\mathcal{I}$Rs may be used to verify that certain strata-neighborhood systems are homotopy link models.

Another technical remark on questions of set theoretic topology is in order.

Remark 6.4.0.3. In Definition 6.4.0.1, we required the map $(X_{p_n} \cup X_{\leq p}) \times |\Delta^{\mathcal{I}_{\geq p}}|_s \to X$ to be a continuous map of *general* topological spaces. In particular, we take $(X_{p_n} \cup X_{\leq p}) \subset X$ to have the classical relative topology, not the Δ-generated or compactly generated one. Indeed, since $(X_{p_n} \cup X_{\leq p})$ is not open in X, this is generally a stronger requirement. For example, this subtlety will be important in the proof of Proposition 6.4.0.9.

Remark 6.4.0.4. We are often going to treat an ASP$\mathcal{I}$R $R: X_{p_n} \times |\Delta^{\mathcal{I}}|_s \to X$ as a (not-necessarily continuous) map

$$\mathcal{X}_{p_n} \times |\Delta^{\mathcal{I}}|_s \cup \bigcup_{p \in \mathcal{I}} X_{\leq p} \times |\Delta^{\mathcal{I}_{\geq p}}|_s \to X.$$

In this sense, we also write

$$R(x,u) \coloneqq x$$

for $x \in X_p$ and $u \in |\Delta^{\mathcal{I}_{\geq p}}|_s$. Furthermore, under the adjunction $- \times |\Delta^{\mathcal{I}}|_s \dashv \mathcal{H}o\mathrm{Link}_{\mathcal{I}}$ it can be useful to treat an ASP$\mathcal{I}$R R as a map $X_{p_n} \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$, the value of which at $x \in X$ we denote by R_x.

Finally, let us give another characterization of ASP$\mathcal{I}$Rs in the case where X is a metric space, which may be somewhat more intuitive.

Remark 6.4.0.5. When X is metrizable, we may equivalently require the stratum-preserving map R as in Definition 6.4.0.1 to have the following property. Whenever a sequence $x_m \in X_{p_n}$ converges to $x \in X_p$, then the sequence of stratum-preserving simplices

$$R_{x_m}|_{|\Delta^{\mathcal{I}_{\geq p}}|_s}: |\Delta^{\mathcal{I}_{\geq p}}|_s \to \mathcal{X},$$

converges uniformly to the constant map

$$c_x: |\Delta^{\mathcal{I}_{\geq p}}| \to X$$

of value x. In particular, if we denote by v_n the maximal vertex of $|\Delta^{\mathcal{I}}|_s$, then $R(x, v_n) = x$.

Remark 6.4.0.6. The question may arise, why we have chosen to use the more technical condition, to only require ASP$\mathcal{I}$Rs to extend continuously to certain subspaces of $X_{\leq p_n} \times |\Delta^{\mathcal{I}}|_s$, and not to the whole space. For realizations of standard neighborhoods of stratified simplicial sets one can indeed produce ASP$\mathcal{I}$Rs which extends to the whole space (see Proposition 6.4.1.6). For stratified cell complexes, however, this is not the case (see Example 6.4.2.6). This is ultimately due to the fact that stratified cell complexes allow for vastly pathological gluing maps, which are generally far from being piecewise linear. Nevertheless, the more general definition of ASP$\mathcal{I}$Rs we have chosen here also applies to stratified cell complexes.

Remark 6.4.0.2 already suggests the following proposition.

Proposition 6.4.0.7. *Let $\mathcal{X} \in \mathbf{Strat}_P$ and let $\mathcal{I} = [p_0 < \cdots < p_n] \subset P$ be a regular flag. If $\mathcal{X}$ admits an ASP$\mathcal{I}$R, then*

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \xrightarrow{\mathrm{ev}} X_{p_n}$$

is a weak homotopy equivalence in **Top**.

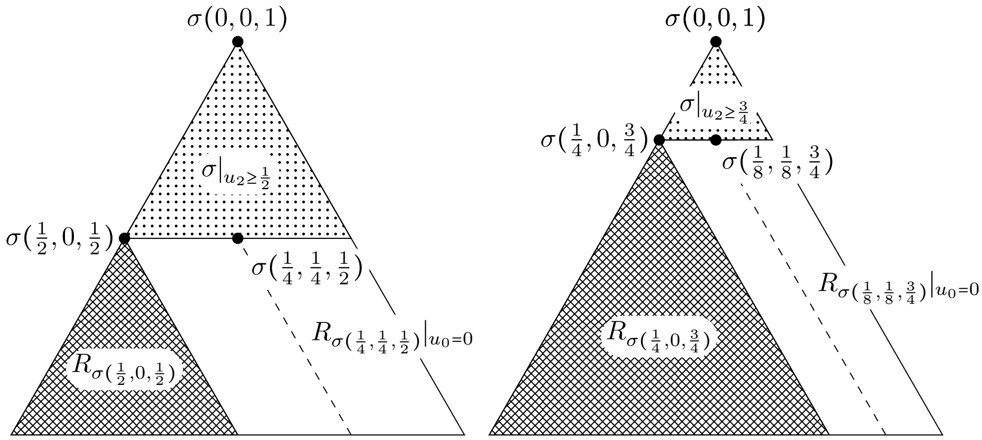


Figure 6.2: Illustration in barycentric coordinates of the maps $\sigma \star_t R$ for $t = \frac{1}{2}$, $t = \frac{3}{4}$ and $\mathcal{I} = [0 < 1 < 2]$. As t increases from 0 to 1, the stratified simplex σ is gradually replaced by simplices of the form $R_{\sigma(u)}$, ending in $R_{\sigma(v_n)}$, for $t = 1$.

To prove this proposition, we need the following construction:

Construction 6.4.0.8. Let $\mathcal{X} \in \mathbf{Strat}_P$ and let $\mathcal{I} = [p_0 < \cdots < p_n] \subset P$ be a regular flag. Let $R\colon A_{p_n} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{A}$ define an ASP$\mathcal{I}$R on a closed subspace $\mathcal{A}$ of $\mathcal{X}$. Then, for any $p \in P$, it follows from $\mathcal{A} \subset \mathcal{X}$ being closed that the restriction of R to $A_{p_n} \times |\Delta^{\mathcal{I}_{\geq p}}|_s$ extends continuously to a map

$$(A_{p_n} \cup X_{\leq p}) \times |\Delta^{\mathcal{I}_{\geq p}}|_s \to X$$

by mapping (x,u) to x, whenever $s(x) \leq p$. For notational simplicity, we consider R as a (not necessarily continuous) map

$$R\colon \bigcup_{p \in \mathcal{I}} (A_{p_n} \cup X_{\leq p}) \times |\Delta^{\mathcal{I}_{\geq p}}|_s \to X$$

in this fashion. We may identify $|\Delta^{\mathcal{I}}|_s \times [0,1] = |\Delta^{\mathcal{I}} \times \Delta^1|_s$. Having done so, we can consider the natural embedding

$$|\Delta^{\mathcal{I}} \times \Delta^1|_s \hookrightarrow |\Delta^{\mathcal{I}} \star \Delta^{\mathcal{I}}|$$

under which $|\Delta^{\mathcal{I}} \times \Delta^1|_s$ corresponds to the union of joins $|\Delta^{\mathcal{I}_{\leq p}} \star \Delta^{\mathcal{I}_{\geq p}}|$, $p \in \mathcal{I}$. This embedding induces join coordinates $(u,t) \mathrel{\hat{=}} [y_0, y_1, t]$ on $|\Delta^{\mathcal{I}} \times \Delta^1|_s$.
Now, denote by v_n the maximal vertex of $|\Delta^{\mathcal{I}}|$. Let $\sigma\colon |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ be a stratum-preserving map, and $t \in [0,1]$ such that $\sigma(u) \in A \cup X_{<p_n}$, if $u_{p_n} = t$. We define

$$\begin{aligned} \sigma \star_t R\colon |\Delta^{\mathcal{I}}|_s &\to \mathcal{X} \\ u &\mapsto R(\sigma((1-t)y_0 + tv_n), y_1)\,. \end{aligned}$$

See also the illustration of $\sigma \star_t R$ in Fig. 6.2. If $\sigma(u) \in A$, for all $u \in |\Delta^{\mathcal{I}}|_s$, then this construction extends to a homotopy

$$\begin{aligned} \sigma \star R\colon |\Delta^{\mathcal{I}}|_s \times [0,1] &\to \mathcal{X} \\ (u,t) &\mapsto \sigma \star_t R(u). \end{aligned}$$

Proposition 6.4.0.9. *Using the notation of Construction 6.4.0.8, $\sigma \star R$ is well defined and has the following properties:*

1. *$\sigma \star_t R$ is stratum-preserving.*

2. *$(\sigma \star R)_0 = \sigma$ and $(\sigma \star R)_1 = R(\sigma(v_n), -)$.*

3. *Consider $X_{\leq p_0}$ as a subspace of the space of continuous maps from $|\Delta^{\mathcal{I}}|$ to X, $C^0(|\Delta^{\mathcal{I}}|, X)$, equipped with the compact open topology, by mapping x to the constant map c_x with value x. Furthermore, let S denote the union of*

$$\{(\sigma, t) \mid \sigma \in \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \wedge \forall u \in |\Delta^{\mathcal{I}}|_s : u_{p_n} = t \implies \sigma(u) \in A\}$$

with $X_{\leq p_0} \times [0,1]$ in $C^0(|\Delta^{\mathcal{I}}|, X) \times [0,1]$. We equip S with the compactly generated topology, that is, the Kelleyfication of the subspace topology in $C^0(|\Delta^{\mathcal{I}}|, X) \times [0,1]$ with respect to compact Hausdorff spaces.
Then, the map

$$\begin{aligned} - \star R \colon S &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \cup X_{\leq p_0} \\ (\sigma, t) &\mapsto \sigma \star_t R. \\ (x, t) &\mapsto x \end{aligned}$$

is continuous.

In particular, if $\mathcal{X} = \mathcal{A}$, then we obtain a homotopy

$$\begin{aligned} - \star R \colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \times [0,1] &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \\ (\sigma, t) &\mapsto \sigma \star_t R. \end{aligned}$$

(with respect to the Kelleyfication topology) between the identity and $\sigma \mapsto R_{\sigma(v_n)}$.

Proof. Let us first verify that $\sigma \star_t R$ is indeed well defined on each join $|\Delta^{\mathcal{I}_{\leq p}} * \Delta^{\mathcal{I}_{\geq p}}|$. For $p = p_n$, and $u \in |\Delta^{\mathcal{I}_{\leq p}} * \Delta^{\mathcal{I}_{\geq p}}|$ the coordinate $y_1(u)$ is given by v_n. It follows that $R(-, y_1)$ is given by the identity on $\mathcal{X}$ and there is nothing to show.
For any $t > 0$ and $p < p_n$ the point $(1-t)y_0 + tv_n$ satisfies $((1-t)y_0 + tv_n))_{p_n} = t$. As σ is stratum-preserving, this also implies $\sigma((1-t)y_0 + tv_n) \in A_{p_n}$, making $R(\sigma((1-t)y_0 + tv_n), y_1)$ a well-defined expression, as long as we show independence from a choice of representatives in join coordinates. If $t = 0$, then $(1-t)y_0 + tv_n = y_0 = u$, and hence $\sigma((1-t)y_0 + tv_n)) \in X_{p_k}$, for some $p_k \leq p$. As $y_1 \in |\Delta^{\mathcal{I}_{\geq p}}|_s \subset |\Delta^{\mathcal{I}_{\geq p_k}}|_s$, it follows that then the expression $R(\sigma((1-t)y_0 + tv_n), y_1)$ is independent of y_1, hence well defined in join coordinates. Precisely, we have

$$R(\sigma((1-t)y_0 + tv_n), y_1) = \sigma(u).$$

Conversely, if $t = 1$, $R(\sigma((1-t)y_0 + tv_n), y_1)$ is clearly independent of y_0 and given by

$$R(\sigma((1-t)y_0 + tv_n), y_1) = R(\sigma(v_n), y_1).$$

Next, note that $\sigma \star_t R$ is stratum-preserving. We only need to check the case $t > 0$. Then, $(1-t)x + tv_n \in (|\Delta^{\mathcal{I}}|_s)_{p_n}$. Hence, as σ was assumed to be stratum-preserving, it also follows that $\sigma((1-t)x + tv_n) \in X_{p_n}$. Now, the stratum of $[y_0, y_1, t]$ (in join coordinates) is given by $s(y_1)$, whenever $t > 0$. Hence, it follows from the assumption that R is stratum-preserving that we indeed have

$$s(R(\sigma((1-t)y_0 + tv_n), y_1)) = s(y_1) = s(y_0, y_1, t),$$

as was to be shown. It remains to verify the continuity of

$$\begin{aligned} - \star R \colon S &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \cup X_{\leq p_0} \\ (\sigma, t) &\mapsto \sigma \star_t R \\ (x, t) &\mapsto x. \end{aligned}$$

Using mapping space adjunctions, it suffices to verify the following statement: Let D be a compact Hausdorff space and let $f \colon D \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ and $\tau : D \to [0,1]$ be a pair of maps such that for all $a \in D$ either $f(a,-)$ is stratum-preserving and $f(a,u) \in A$ whenever $u_{p_n} = \tau(a)$, or f is constant with value in $X_{\leq p_0}$. Then the map

$$\begin{aligned} f' \colon D \times |\Delta^{\mathcal{I}}|_s &\to \mathcal{X} \\ (a,u) &\mapsto R\Big(f\big(a, (1-\tau(a))y_0(u,\tau(a)) + \tau(a)v_n, \tau(a)\big), y_1(u,\tau(a))\Big) \end{aligned}$$

is continuous.

For $p \in \mathcal{I}$, denote by T^p the pushout of general topological spaces

$$(X_{\leq p} \cup A_{p_n}) \times |\Delta^{\mathcal{I}_{\geq p}}|_s \cup_{X_{\leq p} \times |\Delta^{\mathcal{I}_{\geq p}}|_s} X_{\leq p}.$$

Note that R induces a continuous maps

$$R^p \colon T^p \to X\,.$$

We obtain a closed covering of $D \times |\Delta^{\mathcal{I}}|_s$ by the sets D^p, for $p \in \mathcal{I}$, where

$$D^p = \{(a,u) \in D \times |\Delta^{\mathcal{I}}|_s \mid (u, \tau(a)) \in |\Delta^{\mathcal{I}_{\leq p}} * \Delta^{\mathcal{I}_{\geq p}}|\},$$

and verify continuity of f' separately on these pieces. Now, on each D^p, f' is given by a composition

$$D^p \to D^p \times_{\tau,t} |\Delta^{\mathcal{I}_{\leq p}} * \Delta^{\mathcal{I}_{\geq p}}| \to T^p \xrightarrow{R^p} \mathcal{X}$$

with the respective maps defined by

$$\begin{aligned} (a,u) &\mapsto ((a,u), [y_0(a,u), y_1(a,u), \tau(a)]) \\ ((a,u), [y_0, y_1, t]) &\mapsto [f(a, (1-t)y_0 + tv_n), y_1] \\ [z,y] &\mapsto R^p[z,y]. \end{aligned}$$

To verify the continuity of the first of these maps, one needs to treat the set $D^p \times_{\tau,t} |\Delta^{\mathcal{I}_{\leq p}} * \Delta^{\mathcal{I}_{\geq p}}|$ as a pullback, while for the second, one needs to use the topology given by taking the pushout of

$$\begin{tikzcd} D^p \times_{\tau,\pi_{\{0,1\}}} (|\Delta^{\mathcal{I}_{\leq p}}| \times |\Delta^{\mathcal{I}_{\geq p}}| \times \{0,1\}) \arrow[r] \arrow[d] & D^p \times_{\tau,\pi_{[0,1]}} (|\Delta^{\mathcal{I}_{\leq p}}| \times |\Delta^{\mathcal{I}_{\geq p}}| \times [0,1]) \\ D^p \times_{\tau,\pi_{\{0\}}} |\Delta^{\mathcal{I}_{\leq p}}| \times \{0\} \sqcup D^p \times_{\tau,\pi_{\{1\}}} |\Delta^{\mathcal{I}_{\geq p}}| \times \{1\} & . \end{tikzcd} \tag{6.21}$$

The latter is, a priori, finer than the former. Since $D^p \times_{\tau,t} |\Delta^{\mathcal{I}_{\leq p}} * \Delta^{\mathcal{I}_{\geq p}}|$ is Hausdorff, with respect to the former topology, and compact, with respect to the latter, the two topologies do in fact agree. Summarizing, we have shown continuity of f' on each D^p, and hence continuity of f'.

□

We can now prove Proposition 6.4.0.7.

Proof of Proposition 6.4.0.7. We are going to show that ev is a homotopy equivalence, if we pass to the Δ-generated topology. Note that since the Δ-generated topology on S, as in Proposition 6.4.0.9, is finer than the Kelleyfication with respect to compact Hausdorff spaces, $- \star R$ is also continuous with respect to the Δ-generated topology. Since any space is naturally weakly equivalent to its Δ-ification, this shows the result also for the case of compactly generated and general topological spaces. Let $R \colon X_{p_n} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$ define an ASP$\mathcal{I}$R on $\mathcal{X}$. Consider the map

$$\begin{aligned} \iota : X_{p_n} &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X} \\ x &\mapsto \{u \mapsto R(x,u)\}. \end{aligned}$$

Since R is stratum-preserving, this map is indeed well defined. Furthermore, since $R(x, v_n) = x$, we have

$$\mathrm{ev} \circ \iota = 1.$$

By Proposition 6.4.0.9, the map

$$\begin{aligned} - \star R \colon \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \times [0,1] &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \\ (\sigma, t) &\mapsto \sigma \star_t R. \end{aligned}$$

defines a homotopy between the identity and $\iota \circ \mathrm{ev}$. □

6.4.1 ASP$\mathcal{I}$Rs of standard neighborhoods

Now, let us construct ASP$\mathcal{I}$Rs for the standard neighborhoods of stratified simplicial sets of Construction 6.3.1.5. To accomplish this, let us first describe a class of retracts of the inclusions $X_{\leq p} \hookrightarrow U_{\mathcal{X}}(p)$.

Construction 6.4.1.1. Let $p \in P$ and $\mathcal{J} \subset P$ be a flag. We use coordinates $y_{\leq p}$, $y_{\nleq p}$ and $s_{\leq p}$ (as in Construction 6.3.1.17) on $|\Delta^{\mathcal{J}}|_s$. Consider the map

$$\begin{aligned} \rho^p \colon U_{\Delta^{\mathcal{J}}}(p) &\to (|\Delta^{\mathcal{J}}|_s)_{\leq p} \\ [y_{\leq p}, y_{\nleq p}, s_{\leq p}] &\mapsto [y_{\leq p}, y_{\nleq p}, 1] = [y_{\leq p}]. \end{aligned}$$

Note that since $s_{\leq p} \geq \frac{1}{2}$, for $x \in U_{\Delta^{\mathcal{J}}}(p)$, this map is indeed well defined. Under left Kan extension, ρ^p extends to a natural transformation

$$\rho^p \colon U_{\mathcal{X}}(p) \to (|\mathcal{X}|_s)_{\leq p}$$

which defines a retract to the natural inclusion

$$(|\mathcal{X}|_s)_{\leq p} \hookrightarrow U_{\mathcal{X}}(p)$$

of functors $\mathrm{s}\mathbf{Strat}_P \to \mathbf{Top}$. In fact, ρ^p extends to a strong deformation through the natural homotopy defined simplexwise by

$$([y_{\leq p}, y_{\nleq p}, s_{\leq p}], t) \mapsto [y_{\leq p}, y_{\nleq p}, (1-t)s_{\leq p} + t].$$

If we consider $U_{\mathcal{X}}(p)$ as stratified over $P_{\leq p}$ via

$$x \mapsto \begin{cases} s(x) & s(x) \leq p \\ p & s(x) \nleq p \end{cases}$$

then this construction, in fact, defines a natural stratum-preserving strong deformation retraction of functors $\mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_{P_{\leq p}}$.

Next, we verify that the retractions ρ^p are compatible with intersections of p-standard neighborhoods.

Lemma 6.4.1.2. *Let $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$. Then, for any $q \leq p$, the inclusion*

$$\rho^p(U_{\mathcal{X}}(q) \cap U_{\mathcal{X}}(p)) \subset U_{\mathcal{X}}(q)$$

holds.

Proof. Similarly to the proof of Lemma 6.3.1.16 one may easily verify that

$$s_q(\rho^p(x)) = \frac{s_q(x)}{s_{\leq p}(x)}, \tag{6.22}$$

$$s_{\nleq q}(\rho^p(x)) = 1 - s_{\leq q}(\rho^p(x)) = 1 - \frac{s_{\leq q}(x)}{s_{\leq p}(x)}. \tag{6.23}$$

Let $x \in U_{\mathcal{X}}(q) \cap U_{\mathcal{X}}(p)$. Then by Eqs. (6.22) and (6.23)

$$\begin{aligned} s_{\not\leq q}(\rho^p(x)) &= 1 - \frac{s_{\leq q}(x)}{s_{\leq p}(x)} \\ &= \frac{1}{s_{\leq p}(x)}(s_{\leq p(x)} - s_{\leq q}(x)) \\ &\leq \frac{1}{s_{\leq p}(x)}(1 - s_{\leq q}(x)) \\ &\leq \frac{s_q(x)}{s_{\leq p}(x)} \\ &= s_q(\rho^p(x)), \end{aligned}$$

that is, $\rho^p(x) \in U_{\mathcal{X}}(q)$, as was to be shown. □

Using the simplex-wise convexity of the standard neighborhoods, we immediately obtain.

Corollary 6.4.1.3. *For any regular flag $\mathcal{I} = [p_0 < \cdots < p_n] \subset P$ and any $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ the natural transformation $\rho^{p_n} \colon U_{\mathcal{X}}(p_n) \to (|\mathcal{X}|_s)_{\leq p_n}$ restricts to a natural transformation*

$$\rho^{\mathcal{I}} \colon U_{\mathcal{X}}(\mathcal{I}) \to U_{\mathcal{X}}(\mathcal{I})_{\leq p_n}.$$

Even more, $\rho^{\mathcal{I}}$ is part of a natural strong deformation retraction (over $P_{\leq p_n}$) of the inclusion

$$U_{\mathcal{X}}(\mathcal{I})_{\leq p_n} \hookrightarrow U_{\mathcal{X}}(\mathcal{I}).$$

As a first consequence of Corollary 6.4.1.3 we obtain that the standard neighborhood systems $\mathfrak{U}_{\mathcal{X}}$, for $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, fulfill the second requirement of being a homotopy link model:

Corollary 6.4.1.4. *For any $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$ and $\mathcal{I} = [p_0 < \cdots < p_n] \in \mathrm{sd}(P)$ the inclusion*

$$U_{\mathcal{X}}(\mathcal{I})_{p_n} \hookrightarrow U_{\mathcal{X}}(\mathcal{I})_{\geq p_n}$$

is a homotopy equivalence of topological spaces.

Proof. By Corollary 6.4.1.3, the inclusion $U_{\mathcal{X}}(\mathcal{I})_{\leq p_n} \hookrightarrow U_{\mathcal{X}}(\mathcal{I})$ is a stratum-preserving homotopy equivalence over $P_{\leq p_n}$. Consequently, the restriction of this inclusion to the p_n-stratum is a homotopy equivalence, as was to be shown. □

Next, we use the retractions ρ^p to define ASP$\mathcal{I}$Rs for standard neighborhoods.

Construction 6.4.1.5. Let $\mathcal{I} = [p_0 < \cdots < p_n]$ be a regular flag in P and let $\mathcal{J}$ be some other flag. It follows from Lemma 6.4.1.2 and the convexity of standard neighborhoods that the map

$$\begin{aligned} U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s &\to U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \\ (x, t) &\mapsto \sum_{p \in \mathcal{I}} t_p \rho^p(x) \end{aligned}$$

is well defined. One may easily verify that this construction is natural in $\mathcal{J}$, and thus induces a natural transformation

$$R_{\mathcal{I}} \colon U_{-}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s \to U_{-}(\mathcal{I}),$$

of functors $\mathrm{s}\mathbf{Strat}_P \to \mathbf{Top}$.

Proposition 6.4.1.6. *For any $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, the natural transformation $R_{\mathcal{I}} \colon U_{\mathcal{X}}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s \to U_{\mathcal{X}}(\mathcal{I})$ restricts to an ASP$\mathcal{I}$R on $\mathcal{U}_{\mathcal{X}}(\mathcal{I})$.*

Proof. Denote $U := U_{\mathcal{X}}(\mathcal{I})$. Note that by construction $R_{\mathcal{I}}$ may even be defined continuously on all of $U_{\mathcal{X}}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s$. Let us first verify that the restriction $R_{\mathcal{I}}|_{U_{p_n} \times |\Delta^{\mathcal{I}}|_s}$ is stratum-preserving. First, note that for $x \in U_{p_n}$, and $p < p_n$ we have

$$s_p \geq s_{\nleq p} \geq s_{p_n} > 0.$$

It follows that $\rho^p(x) \in U_p$, for all $p \in \mathcal{I}$. It follows from this that $\sum_{p \in \mathcal{I}} t_p \rho^p(x)$ lies in the stratum corresponding to the maximal p with $t_p > 0$, as was to be shown. Furthermore, whenever $x \in U_{\leq p}$ and $t \in |\Delta^{\mathcal{I}_{\geq p}}|_s$, then

$$R_{\mathcal{I}}(x,t) = \sum_{q \in \mathcal{I}} t_q \rho^q(x) = \sum_{q \in \mathcal{I}_{\geq p}} t_q \rho^q(x) = \sum_{q \in \mathcal{I}_{\geq p}} t_q x = x$$

as required. □

We may now summarize Proposition 6.4.0.7, Section 6.4.1, and Corollary 6.4.1.4 as:

Corollary 6.4.1.7. *For any $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, the standard neighborhood system $\mathfrak{U}_{\mathcal{X}}$ is a homotopy link model for $|\mathcal{X}|_s$.*

6.4.2 ASP$\mathcal{I}$Rs for stratified cell complexes

The problem with extending the construction of an ASP$\mathcal{I}$R as in Construction 6.4.1.5 to stratified cell complexes is of course that ASP$\mathcal{I}$Rs may generally not be compatible with gluing. This is circumvented by the following construction.

Construction 6.4.2.1. Let $\mathcal{I} = [p_0 < \cdots < p_n]$ be a regular flag in P and suppose we are given a pushout diagram of finite stratified cell complexes in $\mathbf{Strat}_P$

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, "f"] & \mathcal{B} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, hook, "i"] & \mathcal{X} \cup_{\mathcal{A}} \mathcal{B} = \mathcal{Y}\,. \end{tikzcd} \tag{6.24}$$

where $\mathcal{A} \hookrightarrow \mathcal{B}$ is the inclusion of a subcomplex. Furthermore, suppose we are given the following data:

1. A function $\psi \colon B_{p_n} \to [0,1]$ such that $\psi^{-1}(1) = A_{p_n}$, which we consider as extended by 1 to Y_{p_n};
2. An ASP$\mathcal{I}$R $R_{\mathcal{X}} \colon X_{p_n} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{X}$;
3. An ASP$\mathcal{I}$R $R_{\mathcal{B}} \colon B_{p_n} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{B}$ such that $R_{\mathcal{B}}(x,u) \in A$, whenever $u_{p_n} = \psi(x)$.

Then, we denote by $R_{\mathcal{Y}}$ the map

$$\begin{aligned} R_{\mathcal{Y}} \colon Y_{p_n} \times |\Delta^{\mathcal{I}}|_s &\to Y \\ ([x],u) &\mapsto \big((g \circ R_{\mathcal{B},x}) \star_{\psi(x)} R_{\mathcal{X}}\big)(u) && \text{, for } x \in B \\ ([x],u) &\mapsto R_{\mathcal{X}}(x,u) && \text{, for } x \in X. \end{aligned}$$

Lemma 6.4.2.2. *$R_{\mathcal{Y}}$ as in Construction 6.4.2.1 defines an ASP$\mathcal{I}$R on $\mathcal{Y}$, which extends $R_{\mathcal{X}}$.*

Proof. Note first that since all the spaces involved are finite cell complexes, we need not distinguish between the Δ-generated topology and the relative topology on subspaces of the form $T_{p_n} \cup T_{\leq p_i}$ (see Proposition 6.A.0.5). In particular, both of these topologies also agree with the compactly generated topology. $R_{\mathcal{Y}}$ may then equivalently be constructed as follows.

Consider the set $S \subset C^0(|\Delta^{\mathcal{I}}|, Y) \times [0,1]$, defined as in Proposition 6.4.0.9 with respect to the closed inclusion $\mathcal{X} \hookrightarrow \mathcal{Y}$. Furthermore, consider the map

$$\begin{aligned} R'^0 \colon B_{p_n} &\to S && \xrightarrow{- \star R_{\mathcal{X}}} \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y}) \cup Y_{\leq p_0} \\ b &\mapsto (g \circ R_{\mathcal{B},b}, \psi(b)) && \mapsto (g \circ R_{\mathcal{B},b}) \star_{\psi(b)} R_{\mathcal{X}} \end{aligned}$$

which is continuous by Proposition 6.4.0.9. Note that, for $a \in A$, this map is given by

$$a \mapsto (g \circ R_{\mathcal{B},a}, 1) \mapsto R_{\mathcal{X}}(g(a)) = i \circ R_{\mathcal{X},f(a)}.$$

We claim that R'^0 extends to a continuous map

$$R^0 \colon B_{p_n} \cup B_{\leq p_0} \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y}) \cup Y_{\leq p_0}$$

by mapping $b \mapsto g(b)$, for $b \in B_{\leq p_0}$. Since $B_{p_n} \cup B_{\leq p_0}$ is metrizable, it suffices to see that, for any sequence $b_m \in B_{p_n}$ converging to $b \in B_{\leq p_0}$, it also holds that $R'^0(b_m)$ converges to $R^0(b) = g(b)$ (with respect to the topology of uniform convergence on $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y}) \cup Y_{\leq p_0}$). Furthermore, we may without loss of generality assume that the sequence $\psi(b_m)$ converges. Indeed, this follows from the standard argument that a sequence y_n converges to y, if and only if each of its subsequences has, in turn, a subsequence converging to y, together with compactness of $[0,1]$. For ease of notation, denote $\psi(b) := \lim_{m \to \infty} \psi(b_m)$. Let D denote the subspace of $B_{p_n} \cup B_{\leq p_0}$, given by the elements of the sequence b_m and b. Since b_m converges to b, D is a compact Hausdorff space. It follows that the map

$$\begin{aligned} D &\to S \\ b_m &\mapsto (g \circ R_{\mathcal{B},b_m}, \psi(b_m)) \\ b &\mapsto (g \circ R_{\mathcal{B},b}, \psi(b)) \end{aligned}$$

is continuous, both with respect to the subspace topology on S as well as with respect to the compactly generated topology. In particular, the composition of the last map with $- \star R_{\mathcal{X}}$ is also continuous. It follows that

$$\lim_{m \to \infty} R'_0(b_m) = g \circ R_{\mathcal{B},b} \star_{\psi(b)} R_{\mathcal{X}} = g(b) \star_{\psi(b)} R_{\mathcal{X}} = g(b)$$

as was to be shown.
It follows from the assumption on i and g being closed and Lemma 6.A.0.2 that the square

$$\begin{tikzcd} A_{\leq p_0} \cup A_{p_n} \arrow[r, hook] \arrow[d] & B_{\leq p_0} \cup B_{p_n} \arrow[d] \\ X_{\leq p_0} \cup X_{p_n} \arrow[r, hook] & Y_{\leq p_0} \cup Y_{p_n} \end{tikzcd} \tag{6.25}$$

remains a pushout square. Hence, together with

$$\begin{aligned} R^1 \colon X_{p_n} \cup X_{\leq p_0} &\to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y}) \cup Y_{\leq p_0} \\ x &\mapsto i \circ R_{\mathcal{X},x}\,, \end{aligned}$$

R^0 glues to a map

$$R \colon Y_{p_n} \cup Y_{\leq p_0} \to \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y}) \cup Y_{\leq p_0}$$

whose adjoint is the extension of $R_{\mathcal{Y}}$ to $(Y_{p_n} \cup Y_{\leq p_0}) \times |\Delta^{\mathcal{I}}|_s$ as defined in the proposition. This shows that $R_{\mathcal{Y}}$ is indeed well-defined and stratum-preserving. It remains to verify that $R_{\mathcal{Y}}$ interacts with lower strata, as required in the definition of an ASP$\mathcal{I}$R. We have already covered the case $p = p_0$. All other cases can be reduced to this one, by replacing $\mathcal{I}$ by $\mathcal{I}_{\geq p}$ and restricting $R_{\mathcal{B}}$ and $R_{\mathcal{X}}$ accordingly. That $R_{\mathcal{Y}}$ extends $R_{\mathcal{X}}$ is immediate by definition. □

Suppose now, for a second, that we have already shown the following lemma.

Lemma 6.4.2.3. *Let $\mathcal{I} = [p_0 < \cdots < p_n]$ be a regular flag in P. Then, for any flag $\mathcal{J}$, there exists a function $\psi: U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \to [0,1]$, together with an ASP$\mathcal{I}$R $\overline{R}: U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{U}_{\Delta^{\mathcal{J}}}(\mathcal{I})$ such that*

1. *$\psi^{-1}(1) = U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$;*
2. *$\overline{R}$ restricts to the standard ASP$\mathcal{I}$R on $\mathcal{U}_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$ (see Construction 6.4.1.5);*
3. *For $u \in |\Delta^{\mathcal{I}}|_s$, and $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$ such that $u_{p_n} = \psi(x)$, we have $\overline{R}(x,u) \in U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$.*

Then we may proceed to show the following statement.

Proposition 6.4.2.4. *Let $\mathcal{X}$ be a finite stratified cell complex and Ψ a barycentric subdivision of $\mathcal{X}$ that defines a standard neighborhood system of $\mathcal{X}$. Then, for any regular flag $\mathcal{I} \subset P$, $U^{\Psi}_{\mathcal{X}}(\mathcal{I})$ admits an ASP$\mathcal{I}$Rthat is compatible with subcomplexes, i.e. whenever $\mathcal{B} \subset \mathcal{X}$ is a subcomplex of $\mathcal{X}$, then the ASP$\mathcal{I}$R on $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ restricts to one on $\mathcal{U}^{\Psi|_{\mathcal{B}}}_{\mathcal{B}}(\mathcal{I})$. Furthermore, if $\mathcal{A} \subset \mathcal{X}$ is a subcomplex, then for any such ASP$\mathcal{I}$R $R_{\mathcal{A}}$ on $\mathcal{U}^{\Psi|_{\mathcal{A}}}_{\mathcal{A}}(\mathcal{I})$, the ASP$\mathcal{I}$R on $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$ may be taken to extend $R_{\mathcal{A}}$.*

Proof. Via induction over the number of cells, it suffices to consider the case where $\mathcal{A} \subset \mathcal{X}$ differ only in one cell. Then, using Proposition 6.3.1.15, we have a pushout diagram

$$\begin{tikzcd}
\mathcal{U}_{\partial\Delta^{\mathcal{J}_i}}(\mathcal{I}) \arrow[r, hook] \arrow[d] & \mathcal{U}_{\Delta^{\mathcal{J}_i}}(\mathcal{I}) \arrow[d] \\
\mathcal{U}^{\Psi|_{\mathcal{A}}}_{\mathcal{A}}(\mathcal{I}) \arrow[r, hook] & \mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})
\end{tikzcd} \tag{6.26}$$

of finite stratified cell complexes. We may then use Construction 6.4.2.1 together with Lemma 6.4.2.3 to extend the ASP$\mathcal{I}$R on $\mathcal{U}^{\Psi|_{\mathcal{A}}}_{\mathcal{A}}(\mathcal{I})$ to one on $\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})$. One may verify directly from the construction in Construction 6.4.2.1 that the ASP$\mathcal{I}$Rs defined inductively in this fashion are compatible with subcomplexes. □

Remark 6.4.2.5. One may hope that the construction in Proposition 6.4.2.4 generalizes to arbitrary cell complexes via transfinite composition. While it is true that the construction goes through, note that Construction 6.4.2.1 requires X to be a finite cell complex. This assumption was needed to circumvent the subtle differences between Δ-generated topology and relative topology described in Example 6.A.0.6. Note, however that the main purpose of ASP$\mathcal{I}$Rs in this work is to compute homotopy links. For this, existence of ASP$\mathcal{I}$Rs on finite subcomplexes is sufficient.

The analogue of Proposition 6.4.2.4 fails, if one instead changes the definition of ASP$\mathcal{I}$Rs such that they are required to extend continuously to $U_{\mathcal{I}} \times |\Delta^{\mathcal{I}}|_s \to \mathcal{U}$. Let us give an example to illustrate this:

Example 6.4.2.6. Let $\mathcal{I} = P = \{p < p_1 < p_2\}$. Consider the flag $\mathcal{J} = [p_0 \le p_1 \le p_1 \le p_2]$. Now, we may glue $|\Delta^{\mathcal{I}}|_s$ to $|\Delta^{\mathcal{J}}|_s$, along any stratum-preserving map $\xi: |\Delta^{[p_0<p_1]}|_s \to |\Delta^{\mathcal{J}}|_s$. Denote the resulting stratified cell complex by $\mathcal{X}$ and let $\mathcal{A}$ be the subcomplex defined by $|\Delta^{\mathcal{J}}|_s$. Next, fix any barycentric subdivision Ψ of the stratified cell complex $\mathcal{X}$, and denote by Φ the induced subdivision of $|\Delta^{\mathcal{I}}|_s$ (by treating the latter as a cell of $\mathcal{X}$). Furthermore, denote $U := U^{\Psi}_{\mathcal{X}}(\mathcal{I})$ and $V := U^{\Psi_{\mathcal{A}}}_{\mathcal{A}}(\mathcal{I}) \subset U$, and by V' the image of $\mathcal{U}^{\Phi}_{|\Delta^{\mathcal{I}}|_s}$ in $\mathcal{X}$. Suppose we are given a map

$$R: U \times |\Delta^{\mathcal{I}}|_s \to U,$$

which is stratum-preserving when restricted to $(U \cap \mathcal{X}_{p_n}) \times |\Delta^{\mathcal{I}}|_s$, and fulfills $R(x, v_i) = x$, for $i \in [2]$, v_i the vertex of $|\Delta^{\mathcal{I}}|_s$ corresponding to $p_i \in \mathcal{I}$ and $x \in \mathcal{X}_{p_i}$. In particular, all of these properties are consequences of the altered definition of ASP$\mathcal{I}$Rs we are investigating. For connectivity reasons, using the fact that the p_2 stratum consists of two disjoint cells, R

must also fulfill $R(x,u) \in V$, for any $x \in V$, as well as $R(x,u) \in V'$, for any $x \in V'$ and all $u \in |\Delta^{\mathcal{I}}|_s$. Consequently, it follows that $R(V \cap V' \times |\Delta^{\mathcal{I}}|_s) \subset V \cap V'$. In particular, it follows that R restricts to a map

$$R' \colon (V \cap V') \times |\Delta^{[p_0 < p_1]}|_s \to V \cap V'.$$

By identifying $|\Delta^{[p_0<p_1]}|_s$ with the interval $[0,1]$, it follows that R' defines a homotopy between the identity and the constant map with value the unique point y in the p_0 stratum of $\mathcal{X}$. Note that $V \cap V'$ is of the form $\xi([0,a])$, for some $a > 0$. Since ξ was allowed to be arbitrarily complicated, there is no reason to assume that the image $\xi([0,a])$ is contractible, for any choice of a (think of a spiral converging to y which intersects itself infinitely often, as it does so). Note that if ξ was piecewise linear, we could indeed assume contractibility for sufficiently small a.

We may now finally prove the following theorem:

Theorem 6.4.2.7. *Let $\mathcal{X}$ be a P-stratified cell complex and let Ψ be any subdivision of $\mathcal{X}$ that induces a strata-neighborhood system. Then $\mathfrak{U}^{\Psi}_{\mathcal{X}}$ defines a homotopy link model for $\mathcal{X}$.*

Proof. By the standard compactness arguments, it suffices to show the case where $\mathcal{X}$ is a finite stratified cell complex. First, let us show that the maps $U^{\Psi}_{\mathcal{X}}(\mathcal{I})_{p_n} \hookrightarrow U^{\Psi}_{\mathcal{X}}(\mathcal{I})_{\geq p_n}$ are weak equivalences. Let us first note that the result holds when $\mathcal{X}$ is the realization of a stratified simplicial complex $\mathcal{K} = \partial\Delta^{\mathcal{J}}, \Delta^{\mathcal{J}}$, for some flag $\mathcal{J}$ in P and Ψ is the subdivision given by Construction 6.3.1.12. Indeed, then we have $\mathcal{U}^{\Psi}_{|\mathcal{K}|_s}(\mathcal{I}) = \mathcal{U}_{\mathcal{K}}(\mathcal{I})$, for which the result holds by Proposition 6.3.1.15. Next, let us proceed to show the result for $\mathcal{X}$ a finite cell complex, via induction over the number of cells. Suppose $\mathcal{X}$ is obtained by gluing a cell $\sigma \colon |\Delta^{\mathcal{J}}|_s \to \mathcal{X}$ along $|\partial\Delta^{\mathcal{J}}|_s \to \mathcal{A}$, for some finite complex $\mathcal{A}$. Using Proposition 6.3.1.15 it follows that there is a pushout diagram of P-stratified spaces

$$\begin{tikzcd}
\mathcal{U}_{\partial\Delta^{\mathcal{J}}}(\mathcal{I}) \arrow[r, hook] \arrow[d] & \mathcal{U}_{\Delta^{\mathcal{J}}}(\mathcal{I}) \arrow[d] \\
\mathcal{U}^{\Psi|_{\mathcal{A}}}_{\mathcal{A}}(\mathcal{I}) \arrow[r] & \mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I}).
\end{tikzcd} \tag{6.27}$$

From this, we obtain the following commutative cube.

$$\begin{tikzcd}
 & U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})_{\geq p_n} \arrow[rr, hook] \arrow[dd] & & U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{\geq p_n} \arrow[dd] \\
U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \arrow[rr, hook] \arrow[ur, "\simeq"] \arrow[dd] & & U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \arrow[ur, "\simeq"] \arrow[dd] & \\
 & U^{\Psi|_{\mathcal{A}}}_{\mathcal{A}}(\mathcal{I})_{\geq p_n} \arrow[rr, hook] & & U^{\Psi}_{\mathcal{X}}(\mathcal{I})_{\geq p_n} \\
U^{\Psi|_{\mathcal{A}}}_{\mathcal{A}}(\mathcal{I})_{p_n} \arrow[rr, hook] \arrow[ur, "\simeq"] & & U^{\Psi}_{\mathcal{X}}(\mathcal{I})_{p_n} \arrow[ur] &
\end{tikzcd} \tag{6.28}$$

By inductive assumption all the diagonal maps but the lower vertical one are known to be weak homotopy equivalences in **Top**. By the standard properties of homotopy pushouts (see for example [Hir03, Prop 13.5.4]) it suffices to show that the front and the back face of this cube are homotopy cocartesian. This follows from Lemma 6.A.0.4 together with Lemma 6.A.0.2 and the characterization of homotopy cocartesian squares in a model category in [Lur09, Prop. A.2.4.4].

Next, we need to show that for any regular flag $\mathcal{I} = [p_0 < \dots < p_n] \subset P$, the natural map

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{U}^{\Psi}_{\mathcal{X}}(\mathcal{I})) \xrightarrow{\mathrm{ev}} U^{\Psi}_{\mathcal{X}}(\mathcal{I})_{p_n}$$

is a weak equivalence. This follows directly from Proposition 6.4.2.4 together with Proposition 6.4.0.7. □

As a corollary of Theorem 6.4.2.7, we obtain the following result, which is central to our investigation of the stratified homotopy hypothesis in [Waa24c].

Corollary 6.4.2.8. *Let $\mathcal{I} = [p_0 < \cdots < p_n]$ be a regular flag in P and suppose we are given a pushout diagram of stratified cell complexes in* $\mathbf{Strat}_P$

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, "f"] & \mathcal{B} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, hook, "i"] & \mathcal{X} \cup_{\mathcal{A}} \mathcal{B} = \mathcal{Y} \end{tikzcd} \tag{6.29}$$

where $\mathcal{A} \hookrightarrow \mathcal{B}$ is the inclusion of a subcomplex. Then the image of this square under $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}$

$$\begin{tikzcd} \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{A} \arrow[r] \arrow[d] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{B} \arrow[d] \\ \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X} \arrow[r] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{Y} \end{tikzcd} \tag{6.30}$$

is homotopy cocartesian in $\mathbf{Top}$.

Proof. By Corollary 6.3.2.12, Diagram (6.29) lifts to a diagram of strata-neighborhood systems

$$\begin{tikzcd} \mathfrak{U}^{\Phi}_{\mathcal{A}} \arrow[r] \arrow[d] & \mathfrak{U}^{\hat{\Phi}}_{\mathcal{B}} \arrow[d] \\ \mathfrak{U}^{\Psi}_{\mathcal{X}} \arrow[r] & \mathfrak{U}^{\hat{\Psi}}_{\mathcal{Y}}, \end{tikzcd} \tag{6.31}$$

for appropriate choice of subdivisions $\Phi, \hat{\Phi}, \Psi, \hat{\Psi}$. By Theorem 6.4.2.7, Diagram (6.31) is a diagram of homotopy link models. Thus, by Theorem 6.2.4.14, Diagram (6.30) is weakly equivalent to the image of Diagram (6.31) under D^T at $\mathcal{I}$. That the latter is homotopy cocartesian is the content of Lemma 6.3.2.13. □

Finally, to finish this section, we still need to provide a proof of Lemma 6.4.2.3.

Proof of Lemma 6.4.2.3. Using Proposition 6.3.1.15 we may identify $U_{\Delta^{\mathcal{J}}}(p)$ with the realization of $N_{\Delta^{\mathcal{J}}}(\mathcal{I}) = \bigcap_{p\in\mathcal{I}} N_{\Delta^{\mathcal{J}}}(p)$ as defined in Construction 6.3.1.12 and proceed analogously with $\partial\Delta^{\mathcal{J}}$. As full subcomplexes of $\mathrm{sd}\Delta^{\mathcal{J}}$ the two complexes $N_{\Delta^{\mathcal{J}}}(\mathcal{I})$ and $N_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$ only differ in the vertex corresponding to the maximal simplex of $\Delta^{\mathcal{J}}$, $x_{\mathcal{J}}$. If $x_{\mathcal{J}} \in S_{\Delta^{\mathcal{J}}}(\mathcal{I})$, then either $\mathcal{I} \subset \mathcal{J}$ or $\max \mathcal{J} < p_k$, for some $k \in [n]$. If $\max \mathcal{J} < p_k < p_n$, then $U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} = \varnothing$ and there is nothing to show. Hence, we may assume that $\max \mathcal{J} = p_n$ and $\mathcal{I} \subset \mathcal{J}$.
Step 1: Let $x_0 \in |\Delta^{\mathcal{J}}|_s \setminus \bigcup_{p\in\mathcal{I}, p<p_n} U_{\Delta^{\mathcal{J}}}(p) \cup |\partial\Delta^{\mathcal{J}}|_s$. That such a point exists is a consequence of the inclusion $\mathcal{I} \subset \mathcal{J}$. Indeed, any point x in the interior of $|\Delta^{\mathcal{J}}|_s$, with $s_{p_n}(x) > \frac{1}{2}$ will do. Next, consider the straight line projection through x_0

$$r\colon |\Delta^{\mathcal{J}}|_s \setminus \{x_0\} \to |\partial\Delta^{\mathcal{J}}|_s.$$

Let us show that r maps $U_{\Delta^{\mathcal{J}}}(\mathcal{I})$ to $U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$. By definition of $U_{-}(\mathcal{I})$, and using that $\max \mathcal{J} = p_n$, we may instead show that r maps $U_{\Delta^{\mathcal{J}}}(p)$ to $U_{\partial\Delta^{\mathcal{J}}}(p)$ for all $p \in \mathcal{I}_{<p_n}$. The map r maps x to the intersection point of the ray

$$\{x + \alpha(x - x_0) \mid \alpha \geq 0\}$$

with $U_{\partial\Delta^{\mathcal{J}}}(p)$. Let α_x be the unique value in $[0,1]$, specifying this intersection point. In particular, for any $p \in \mathcal{I}_{<p_n}$, we may compute

$$\begin{aligned} s_{\nleq p}(r(x)) &= s_{\nleq p}(x + \alpha_x(x - x_0)) &&= (1+\alpha_x)s_{\nleq p}(x) - \alpha_x s_{\nleq p}(x_0) \\ s_p(r(x)) &= \cdots &&= (1+\alpha_x)s_p(x) - \alpha_x s_p(x_0). \end{aligned}$$

By assumption, $s_{\nleq p}(x_0) > s_p(x_0)$ and $s_{\nleq p}(x) \leq s_p(x)$. Since, $\alpha_x \geq 0$, it follows that

$$s_{\nleq p}(r(x)) = (1+\alpha_x)s_{\nleq p}(x) - \alpha_x s_{\nleq p}(x_0) \leq (1+\alpha_x)s_p(x) - \alpha_x s_p(x_0) = s_p(r(x)),$$

as was to be shown.

Step 2: We need to verify an additional property of r, namely that it is close to being stratum-preserving. We show that for $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$

$$s(r(x)) \geq p_{n-1}. \tag{6.32}$$

Assume, to the contrary that $\alpha_x > 0$ and $s_q(r(x)) = 0$, for all $q \geq p_{n-1}$. Then, for such q, we have

$$s_q(x) = \frac{\alpha_x}{1+\alpha_x} s_q(x_0)$$

and

$$s_{\nleq q}(x) = \frac{\alpha_x}{1+\alpha_x} s_{\nleq q}(x_0)$$

and obtain

$$s_{p_{n-1}}(x) = \frac{\alpha_x}{1+\alpha_x} s_{p_{n-1}}(x_0) < \frac{\alpha_x}{1+\alpha_x} s_{\nleq p_{n-1}}(x_0) = s_{\nleq p_{n-1}}(x)$$

in contradiction to the assumption that $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})$.

Step 3: Denote by R the standard ASP$\mathcal{I}$R on $U_{\Delta^{\mathcal{J}}}(\mathcal{I})$ (see Construction 6.4.1.5). Furthermore, denote by $\hat{R}\colon U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s$ the (extended) ASP$\mathcal{I}$R obtained by affinely extending

$$\begin{aligned} (x, v_k) &\mapsto x && \text{, for } k = n \\ (x, v_k) &\mapsto \rho^{p_k}(r(x)) && \text{, for } k < n \end{aligned}$$

where v_k denotes the k-th vertex of $|\Delta^{\mathcal{I}}|_s$ and ρ^p are as in Construction 6.4.1.1. $\hat{R}$ is well defined by Lemma 6.4.1.2 and the fact that r maps into $U_{\Delta^{\mathcal{J}}}(\mathcal{I})$. If $x \in U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$, then $r(x) = x$ and hence $\hat{R}_x$ agrees with R. Furthermore, it follows from the inclusion $U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{<p_n} \subset U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$ that $\hat{R}$ agrees with R on $U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{\leq p} \times |\Delta^{\mathcal{I}_{\geq p}}|_s$, for any $p \in \mathcal{I}$, $p < p_n$. Clearly, also $\hat{R}(x, v_n) = x$. Hence, to see that $\hat{R}$ does indeed define an ASP$\mathcal{I}$R, we only need to verify that $\hat{R}_x$ is stratum-preserving for $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$. Just as for the proof of the analogous statement in Proposition 6.4.1.6, one shows that $\hat{R}_x$ being stratum-preserving is equivalent to showing that $r(x) \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{\geq p_{n-1}}$ which is the content of Eq. (6.32).

Step 4: The idea of the remainder of the proof is to now combine $\hat{R}$ and R. To do so, we will make use of a (continuous) function $\psi\colon U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \to [0,1]$, with the properties that

1. $\psi^{-1}(0) = r^{-1}(U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})_{<p_n}) \cap U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$;
2. $\psi^{-1}(1) = U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$.

Notice that both of these sets are closed subsets of $U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$ and that since $r(x) = x$, for $x \in U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$, they are also disjoint. Hence, such a function ψ exists. Furthermore, we are going to need another interpolation function

$$\begin{aligned} H\colon U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s \times [0,1] &\to U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \\ (x,u,t) &\mapsto (1-t)\hat{R}(x,u) + tR(r(x),u). \end{aligned}$$

Then one may verify the following properties of H:

(i) $H_0 = \hat{R}$ and H_1 has value in $U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$.

(ii) If $x \in U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$, then $H(x,-,-)$ is the constant homotopy with value R_x.

(iii) If $s(x) = p_n$, then for any $t < 1$, $H(x,-,t)$ is stratum-preserving.

(iv) If $\psi(x) > 0$ and $s(x) = p_n$, then $H(x,-,-)$ is stratum-preserving.

(v) Restricted to $U_{\Delta^{\mathcal{J}}}(\mathcal{I})\times|\Delta^{\mathcal{I}_{<p_n}}|_s$, H is given by the constant homotopy of value $R_{r(x)}|_{|\Delta^{\mathcal{I}_{<p_n}}|_s}$.

(vi) If $s(x) = p_n$ and $\psi(x) = 1$, then $H(x,-,-)$ is the constant homotopy with value $R_{r(x)} = R_x$.

Step 5: We may now finally define the ASP$\mathcal{I}$R promised in the statement of the proposition. Consider the map

$$
\begin{aligned}
\overline{R}\colon U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s &\to U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \\
(x,u) &\mapsto H(x,u,1) && \text{, for } x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{<p_n} \\
(x,u) &\mapsto H(x,u,1) && \text{, for } x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \text{ and } u_{p_n} \le \psi(x) \\
(x,u) &\mapsto H(x,u,\frac{1-u_{p_n}}{1-\psi(x)}) && \text{, for } x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \text{ and } u_{p_n} > \psi(x).
\end{aligned}
$$

Let us verify the continuity of $\overline{R}$. Notice that the first two conditions on (x,t) define a closed subspace $D \subset U_{\Delta^{\mathcal{J}}}(\mathcal{I}) \times |\Delta^{\mathcal{I}}|_s$. Hence, the only thing to check is that for any sequence (x_i,u_i), $i \in \mathbb{N}$, with $x_i \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$ and $(u_i)_{p_n} > \psi(x_i)$, converging to $(x,u) \in D$, it also follows that $H(x_i,u_i,\frac{1-(u_i)_{p_n}}{1-\psi(x_i)})$ converges to $H(x,u,1)$. In the following, by convergence of functions we will always mean uniform convergence. There are two cases to consider. If $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{<p_n} \subset U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$, then by Property (ii) $H(x_i,-,-)$ converges to a constant homotopy and hence $H(x_i,u_i,\frac{1-(u_i)_{p_n}}{1-\psi(x_i)})$ converges to $H(x,u,1)$. If $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$, then, by assumption, $u_{p_n} = \psi(x)$ and thus if $\psi(x) < 1$ continuity is immediate from the definition. It remains to consider the case $u_{p_n} = \psi(x) = 1$. In this case, it follows from Property (vi) that $H(x_i,-,-)$ converges to the constant homotopy with value R_x. Hence, again it follows that $H(x_i,u_i,\frac{1-(u_i)_{p_n}}{1-\psi(x_i)})$ converges to $H(x,u,1)$.

Step 6: Let us now verify that $\overline{R}$ restricts to an ASP$\mathcal{I}$R. If $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n}$ and $\psi(x) > 0$, then $\overline{R}_x\colon|\Delta^{\mathcal{I}}|_s \to U_{\Delta^{\mathcal{J}}}(\mathcal{I})$ is stratum-preserving by Property (iv). If $\psi(x) = 0$, then $\overline{R}_x(s) = H(x,u,1-u_{p_n})$. Hence, by Property (iii), we obtain preservation of strata for $u_{p_n} > 0$. For $u_{p_n} = 0$ and $\psi(x) = 0$, it follows by Property (v) that then $H(x,u,1)) = R(r(x),u)$. Since $s(r(x)) \ge p_{n-1}$, by Eq. (6.32), it follows that $R_{r(x)}|_{|\Delta^{\mathcal{I}_{<p_{n-1}}}|_s}$ is stratum-preserving. This shows that the restriction of $\overline{R}$

$$R\colon U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{p_n} \times |\Delta^{\mathcal{I}}|_s \to U_{\Delta^{\mathcal{J}}}(\mathcal{I})$$

is stratum-preserving. Finally, let $p \in \mathcal{I}$, $u \in |\Delta^{\mathcal{I}_{\ge p}}|_s$ and $x \in U_{\Delta^{\mathcal{J}}}(\mathcal{I})_{\le p}$. Note first that whenever $s(x) < p_n$ or $\psi(x) = 1$, then $x \in U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$ and thus

$$\overline{R}(x,u) = H(x,u,1) = R(x,u) = x$$

by Property (ii). Furthermore, If $s(x) = p_n$ and $\psi(x) < 1$ then, by the assumption that $s(x) \le p$ and $u \in |\Delta^{\mathcal{I}_{\ge p}}|_s$, it follows that $u_{p_n} = 1$ and hence

$$\overline{R}(x,u) = H(x,u,0) = \hat{R}(x,u) = x$$

by Property (i). To summarize, we have shown that $\overline{R}$ is an ASP$\mathcal{I}$R. By Property (ii), R defines an extension of the standard ASP$\mathcal{I}$R on $U_{\partial\Delta^{\mathcal{I}}}(\mathcal{I})$. Finally, if $\psi(x) = u_{p_n}$, for $(x,u) \in U_{\Delta^{\mathcal{I}}}(\mathcal{I})_{p_n} \times |\Delta^{\mathcal{I}}|_s$, then $R(x,u) = H(x,u,1) \in U_{\partial\Delta^{\mathcal{J}}}(\mathcal{I})$, by Property (i), which finishes the proof. $\square$

6.A A series of tools from point-set topology

In this section, we list a series of elementary results in point-set topology.

Lemma 6.A.0.1. *Consider a pushout diagram of compact Hausdorff spaces*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f'"] \\ X \arrow[r, "g'"] & Y . \end{tikzcd} \tag{6.33}$$

Let $Z \subset U \subset Y$. If both $g'^{-1}(U)$ is a neighborhood of $g'^{-1}(Z)$ in X and $f'^{-1}(U)$ is a neighborhood of $f'^{-1}(Z)$ in B, then U is a neighborhood of Z in Y.

Proof. It is generally true that for any quotient map $\pi\colon T \to \tilde{Y}$ of compact Hausdorff spaces the image of any neighborhood $\tilde{V}$ of $\pi^{-1}(\tilde{Z})$ is a neighborhood of $\tilde{Z}$. Indeed, for any open $O \subset \tilde{V}$, containing $\pi^{-1}(Z)$, the set $T \setminus \pi^{-1}(\pi(T \setminus O))$ is a saturated open set (this uses that $T \setminus O$ is closed, due to the compact-Hausdorff assumptions), which contains $\pi^{-1}(\tilde{Z})$ and is contained in $\tilde{V}$. Since the map $X \sqcup B \to Y$ is such a quotient map and by assumption $g'^{-1}(U) \sqcup f'^{-1}(U)$ is a neighborhood of $g'^{-1}(Z) \sqcup f'^{-1}(Z)$, it follows that $U = g'(g'^{-1}(U)) \cup f'(f'^{-1}(U))$ is a neighborhood of Z. ☐

Lemma 6.A.0.2. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{6.34}$$

be a cocartesian square in $\mathbf{Strat}_P$ such that all arrows pointing into Y are closed maps (or open maps). Then, for any subset $Q \subset P$, the square of general topological spaces

$$\begin{tikzcd} A_Q \arrow[r] \arrow[d] & B_Q \arrow[d] \\ X_Q \arrow[r] & Y_Q \end{tikzcd} \tag{6.35}$$

remains cocartesian. Furthermore, if Q is open or closed in P, then the square remains cocartesian without any assumptions on the maps.

Proof. We cover the closed cases. We need to verify that the bijection

$$\phi\colon X_Q \cup_{A_Q} B_Q \to Y_Q$$

is a closed map. Now, any closed set $Z \subset X_Q \cup_{A_Q} B_Q$ is given by the image of some closed set $Z_X \sqcup Z_B \subset X_Q \sqcup B_Q$. The latter is given by the restriction to Q, of some closed set $\tilde{Z}_X \sqcup \tilde{Z}_B \subset X \sqcup B$. Denote by $\tilde{Z}$ the image of $\tilde{Z}_X \sqcup \tilde{Z}_B$ in $Y = X \cup_A B$. By assumption, $\tilde{Z}$ is again closed, and by construction we have $\phi(Z) = \tilde{Z}_Q$. The second statement follows similarly, using the fact that then $Y_Q \hookrightarrow Y$ is given by a closed inclusion. ☐

Lemma 6.A.0.3. *Let $p \in P$. Furthermore, let* **Top** *be any of the categories of topological spaces in Notation 6.2.1.3. The functor*

$$(-)_{\geq p}\colon \mathbf{Strat}_P \to \mathbf{Top}$$

preserves all colimits.

Proof. First, let us show that $(-)_{\geq p}$ preserves all colimits of general topological spaces. It is an immediate consequence of the more general statement on over-categories of topological spaces, which one may easily verify using the elementary construction of colimits via final topologies. For any space $T \in \mathbf{Top}$ and any open subspace $U \subset T$, the restriction functor

$$\mathbf{Top}_{/T} \to \mathbf{Top}_{/U}$$

preserves all colimits. Since $\{q \geq p\} \subset \mathbf{Pos}$ is an open subset in the Alexandrow topology and the forgetful functor $\mathbf{Top}_{/U} \to \mathbf{Top}$ admits a right adjoint, the result follows. Next, let us cover the case of Δ-generated spaces. The one of compactly generated spaces is analogous. If $D\colon I \to \mathbf{Strat}_P$ is a diagram of Δ-generated stratified spaces, then since the inclusion into general topological spaces is left adjoint (see, for example, [Gau21]), the colimit of D in Δ-generated spaces also defines the colimit in general topological spaces. Hence, we may apply the previous case, to see that $\varinjlim D_{\geq p} = (\varinjlim D)_{\geq p}$ in general topological spaces. Now, since $(T)_{\geq p} \subset T$ always defines an open subspace, and open subspaces of Δ-generated subspaces are again Δ-generated (see [Gau21, Sec. 2]), the diagram $D_{\geq p}$ lives in the category of Δ-generated spaces. Since the inclusion into general spaces preserves all colimits, we also have $\varinjlim D_{\geq p} = (\varinjlim D)_{\geq p}$ in Δ-generated spaces, as was to be shown. □

Lemma 6.A.0.4. *Let $p \in P$. The functors*

$$(-)_{\geq p}, (-)_p\colon \mathbf{Strat}_P \to \mathbf{Top}$$

and

$$(-)_{\leq p}\colon \mathbf{Strat}_P \to \mathbf{Strat}_P$$

send relative stratified cell complexes into relative (stratified) cell complexes.

Proof. Let us begin with $(-)_{\geq p}$. By Lemma 6.A.0.3, it suffices to show that $(-)_{\geq p}$ sends stratified boundary inclusions $|\partial\Delta^{\mathcal{J}}|_s \hookrightarrow |\Delta^{\mathcal{J}}|_s$ into a relative cell complex of topological spaces. Let us assume that not all elements of $\mathcal{J}$ are smaller then p, otherwise both spaces are empty after applying $(-)_{\geq p}$, and there is nothing to be shown. Then, applying $(-)_{\geq p}$ corresponds to removing a face of the simplex $|\Delta^{\mathcal{J}}|_s$ from both spaces. In particular, we may reduce to the following general statement: Let $T \hookrightarrow T'$ be an inclusion of a piecewise linear closed subspace into a piecewise linear space T. Let $A \subset T$ be a further inclusion of a piecewise linear closed subspace. Then, $T \setminus A \hookrightarrow T' \setminus A$ also admits the structure of a closed inclusion of a piecewise linear subspace (this is ultimately a consequence of the existence of piecewise linear regular neighborhoods). In particular, there is a compatible triangulation of $T \setminus A$ and $T' \setminus A$, which makes $T \setminus A \hookrightarrow T' \setminus A$ a relative cell complex. The case of $(-)_{\leq p}$ follows similarly by the natural isomorphisms $(|\mathcal{X}|_s)_{\leq p} \cong |\mathcal{X}_{\leq p}|_s$, for $\mathcal{X} \in s\mathbf{Strat}_P$. Finally, the case of $(-)_p$ follows from the equality $(-)_p = (-)_{\geq p} \circ (-)_{\leq p}$. □

Proposition 6.A.0.5. *Let $\mathcal{X} \in \mathbf{Strat}_P$ be a finite stratified cell complex. Then for any $Q \subset P$ the relative topology on $X_Q \subset X$ makes X_Q a Δ-generated space.*

Proof. First, let us show that for any flag $\mathcal{J}$ of P, the space $(|\Delta^{\mathcal{J}}|_s)_Q$ with the relative topology is Δ-generated. $(|\Delta^{\mathcal{J}}|_s)_Q \subset |\Delta^{\mathcal{J}}|_s \subset \mathbb{R}^{\mathcal{J}}$ may equivalently described by

$$\{s \in |\Delta^{\mathcal{J}}|_s \mid \forall p \in P \setminus Q\colon (\exists q \in Q\colon q > p \wedge s_q > 0) \vee s_p = 0\}.$$

It follows from this description that $(|\Delta^{\mathcal{J}}|_s)_Q \subset |\Delta^{\mathcal{J}}|_s \subset \mathbb{R}^{\mathcal{J}}$ is a convex set. It turns out that every convex subset C of $\mathbb{R}^n$ is Δ-generated. Indeed, let $A \subset C$ be such that $\sigma^{-1}(A)$ is closed, for every continuous map $\sigma\colon |\Delta^1| \to C$. Let x_n, $n \in \mathbb{N}$, be a sequence in A which converges to $c \in C$. Since C is convex, we may use affine interpolation to define a continuous map $\sigma\colon [0,1] \to C$ with $\sigma(2^{-n}) = x_n$. In particular, the inverse image of A under σ contains $\{2^{-n} \mid n \in \mathbb{N}\}$. As $\sigma^{-1}(A)$ is closed, it follows that $0 \in \sigma^{-1}(A)$. By continuity of σ, we hence have $c = \sigma(0) \in A$, showing that A is closed.

We now proceed to show the case of a general complex $\mathcal{X}$ via induction over the number of cells. The case $n = 0$ is trivial, so let $\mathcal{X}$ admit the structure of a stratified cell complex with $n+1$ cells. In other words $\mathcal{X}$ fits into a pushout diagram

$$\begin{tikzcd} |\partial\Delta^{\mathcal{J}}|_s \arrow[r, hook] \arrow[d] & |\Delta^{\mathcal{J}}|_s \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{X}, \end{tikzcd} \tag{6.36}$$

with $\mathcal{A}$ a stratified space admitting a cell structure with n cells. By Lemma 6.A.0.2, the diagram

$$\begin{tikzcd} (|\partial\Delta^{\mathcal{J}}|)_Q \arrow[r, hook] \arrow[d] & (|\Delta^{\mathcal{J}}|)_Q \arrow[d] \\ A_Q \arrow[r] & X_Q \end{tikzcd} \tag{6.37}$$

is a pushout diagram of general topological spaces. In particular X_Q is a quotient of $A_Q \sqcup (|\Delta^{\mathcal{J}}|_s)_Q$. We have already seen that $(|\Delta^{\mathcal{J}}|_s)_Q$ is Δ-generated. By the inductive assumption, the same holds for A_Q. Thus X_Q is Δ-generated as a quotient of Δ-generated spaces. □

Example 6.A.0.6. The statement of Proposition 6.A.0.5 is generally not true for infinite stratified cell complexes, even if X is given by the realization of a stratified simplicial set. Let $P = \{p_0 < p_1 < p_2\}$ and $Q = \{p_0 < p_2\}$. Consider the realization of the stratified simplicial set given by gluing countably many Δ^P along $\Delta^{\{p_0<p_1\}}$, i.e., there is a pushout diagram

$$\begin{tikzcd} \bigsqcup_{n\in\mathbb{N}} |\Delta^{\{p_0<p_1\}}|_s \arrow[r] \arrow[d] & \bigsqcup_{n\in\mathbb{N}} |\Delta^P|_s \arrow[d] \\ |\Delta^{\{p_0<p_1\}}|_s \arrow[r, "i"] & \mathcal{X}\,. \end{tikzcd} \tag{6.38}$$

Then, X_Q is not Δ-generated. In the following, we denote closures in the form $\overline{A}$. To see that X_Q is not Δ-generated, consider the subset S of X given by

$$\bigcup_{n\in\mathbb{N}} S_n$$

where

$$S_n = \{s \in |\Delta^P|_s \mid s_{p_0} \leq 1 - \frac{1}{n}\}$$

lies in the n-th copy of $|\Delta^P|_s$ in X. The set $S_Q \subset X_Q$ is not closed in the relative topology on X_Q. To see this, let $A \subset X$ be any closed set containing S_Q. Then, as $(S_n)_Q$ is dense in S_n, it follows that A contains S. Observe that S contains the image of the p_1-stratum of $|\Delta^{\{p_0<p_1\}}|_s$ under $i\colon |\Delta^{\{p_0<p_1\}}|_s \to \mathcal{X}$, but it does not contain any point in the p_0-stratum of X. Denote by x_0 the unique element in the p_0-stratum of $|\Delta^{\{p_0<p_1\}}|_s$. It lies in the closure of $(|\Delta^{\{p_0<p_1\}}|_s)_{p_1}$. As A is closed and contains S, it follows that we have

$$i(x_0) \in i(\overline{(|\Delta^{\{p_0<p_1\}}|_s)_{p_1}}) = \overline{i((|\Delta^{\{p_0<p_1\}}|_s)_{p_1})} \subset \overline{S} \subset A.$$

To summarize, we have $i(x_0) \in A \cap X_Q$, for any closed subset $A \subset X$, which shows that the closure of S_Q in X_Q contains $i(x_0)$. As $i(x_0) \notin S_Q$, S_Q is not closed in X_Q.
However, S_Q is Δ-closed in X_Q. To see this, denote by X_n the union of the first n copies of $|\Delta^P|_s$ in X. Note that since $|\Delta^1|$ is compact, it follows that any map continuous $f\colon |\Delta^1| \to X_Q$ factors through some $(X_n)_Q$, for n sufficiently large. Furthermore, we have

$$S_Q \cap (X_n)_Q = (\bigsqcup_{m\leq n} S_m)_Q$$

which shows that $S_Q \cap (X_n)_Q$ is a closed subset of X_Q. Consequently, $f^{-1}(S_Q) = f^{-1}(S_Q \cap (X_n)_Q)$ is closed in $|\Delta^1|$, which proves that S_Q is Δ-closed.

6.B A characterization of weak equivalences of topological spaces

The following characterization of weak equivalences is certainly well known. For a lack of convenient reference, we nevertheless give a proof here.

Lemma 6.B.0.1. *A map $f\colon T \to T'$ is a weak homotopy equivalence in* **Top***, if and only if for every solid commutative diagram*

$$\begin{tikzcd} S^n \arrow[r, "g^1"] \arrow[d] & T \arrow[d, "f"] \\ D^{n+1} \arrow[r, "g^0"] \arrow[ur, dashed, "l"] & T' , \end{tikzcd} \tag{6.39}$$

with $n \geq -1$, there exists a dashed arrow l such that $(1_T, f) \circ (l|_{S^n}, l)$ is homotopic to (g^1, g^0) as a map of arrows (i.e. homotopic in the presheaf category $\mathbf{Fun}(([1])^{\mathrm{op}}, \mathbf{Top})$ with respect to the cylinder given by the pointwise product with $[0,1]$).

Proof. We use the classical characterization of weak equivalences found for example in [May99, Ch. 9.6]. Indeed, the classical characterization of weak equivalences even guarantees a lift, where the homotopy may be taken constant on S^1. Conversely, if we are given such a lift l, together with a homotopy $(H^1, H^0) : (g^1, g^0) \implies (1_T, f) \circ (l|_{S^n}, l)$, then any extension

$$\begin{tikzcd} S^n \times [0,1] \times D^{n+1} \times \{1\} \arrow[r, "H^1 \cup l"] \arrow[d, hook] & T \\ D^{n+1} \times [0,1] \arrow[ur, dashed, "\hat{L}"'] & \end{tikzcd} \tag{6.40}$$

will provide $\hat{l} = \hat{L}_0$ such that the upper left triangle in

$$\begin{tikzcd} S^n \arrow[r, "g^1"] \arrow[d] & T \arrow[d, "f"] \\ D^{n+1} \arrow[r, "g^0"] \arrow[ur, dashed, "\hat{l}"] & T' \end{tikzcd} \tag{6.41}$$

commutes on the nose. Furthermore, then $f \circ \hat{L}$ and H^0 both provide extensions

$$\begin{tikzcd} S^n \times [0,1] \times D^{n+1} \times \{1\} \arrow[r, "f \circ (H^1 \cup l)"] \arrow[d, hook] & T' \\ D^{n+1} \times [0,1] \arrow[ur, dashed] & \end{tikzcd} . \tag{6.42}$$

Since the left hand vertical of the last diagram is an acyclic cofibration, any two such extensions are homotopic relative to $S^n \times [0,1] \times D^{n+1} \times \{1\}$. It follows that $f \circ \hat{l} = (f \circ \hat{L})_0$ and $g^0 = (H^0)_0$ are homotopic relative to S^n. $\square$

Chapter 7

Presenting the stratified homotopy hypothesis

Note to the reader: The following chapter was structured as an independent article, in order to allow for easier accessibility. A preliminary version was made publicly available on the arXiv (see [Waa24c]). Notation in this chapter is entirely consistent with Chapters 1, 5 and 6. There may be minor notation differences compared to Chapter 3. However, as all notation is introduced separately in this chapter, this should not pose an issue.

This article is concerned with three different homotopy theories of stratified spaces: The one defined by Douteau and Henriques, the one defined by Haine, and the one defined by Nand-Lal. One of the central questions concerning these theories has been how precisely they connect with geometric and topological examples of stratified spaces, such as piecewise linear pseudomanifolds, Whitney stratified spaces, or more recently Ayala, Francis and Tanaka's conically smooth stratified spaces. More precisely, so far, it has been an open question whether there exist (semi-)model structures on stratified topological spaces that present these theories, in which such relevant examples of stratified spaces are bifibrant. Here, we prove an affirmative answer to this question. As a consequence, we obtain a model categorical interpretation of a stratified homotopy hypothesis. Specifically, we show that Lurie's stratified singular simplicial set functor induces a Quillen equivalence between the semi-model category of stratified topological spaces presenting Nand-Lal's homotopy theory of stratified spaces and the left Bousfield localization of the Joyal model structure that corresponds to such ∞-categories in which every endomorphism is an isomorphism. We then perform a detailed investigation of bifibrant objects in these model structures of stratified spaces, proving a series of detection criteria and illuminating the relationship to Quinn's homotopically stratified spaces.

7.1 Introduction

Conceptually speaking, Grothendieck's homotopy hypothesis refers to the following statement:

> Assigning to a topological space its ∞-groupoid of paths induces an equivalence between the *homotopy theory of spaces* (more precisely, CW-complexes) and the *homotopy theory of ∞-groupoids.*

Whether this statement is regarded as a theorem or a conjecture, of course, strongly relies on the precise model of ∞-categories - and consequently of ∞-groupoids - one has in mind. Nowadays, the following result due to Kan and Quillen is often taken as a formal interpretation of the homotopy hypothesis:

Theorem HH. *[Qui67] The geometric realization and singular simplicial set adjunction*

$$|-|\colon \mathbf{sSet} \rightleftharpoons \mathbf{Top} : \mathrm{Sing}$$

induces a Quillen equivalence between the Quillen model structure on topological spaces and the Kan-Quillen model structure on simplicial sets.

That this is a formal interpretation of the homotopy hypothesis can be argued as follows.

1. The Quillen equivalence induces an equivalence

$$\mathbf{Top}[W^{-1}] \simeq \mathbf{Kan}[H_k^{-1}]$$

between topological spaces localized at weak homotopy equivalences and Kan complexes localized at homotopy equivalences of Kan complexes.

2. Quasi-categories (as introduced by Joyal and popularized by Lurie in [Lur09]) have proven to be a powerful and versatile model for the theory of $(\infty,1)$-categories (∞-categories henceforth). Kan complexes are the ∞-groupoid within the framework of quasi-categories, and it follows from the existence of the Kan-Quillen and the Joyal model structure that the right-hand side thus defines the homotopy theory of ∞-groupoids $\mathcal{G}\mathbf{rpd}_\infty$. The singular simplicial set then provides a model for the ∞-groupoid of paths in this interpretation of ∞-categories. This justifies the usage of Kan complexes as a model for ∞-groupoids.

3. It follows from the existence of the Quillen model structure on **Top**, or more classically an argument involving Whitehead's theorem, that the inclusion of CW complexes into topological spaces, $\mathbf{CW} \hookrightarrow \mathbf{Top}$ induces an equivalence $\mathbf{CW}[H^{-1}] \simeq \mathbf{Top}[W^{-1}]$, where H is the class of homotopy equivalences. Since most spaces of geometric interest at least have the homotopy type of a CW complex (see, for example, [Mil59]), it follows that if one is interested in studying the homotopy types of such classical spaces, one may perform such an investigation in $\mathbf{Top}[W^{-1}]$. Thus, the latter can rightfully be called the homotopy theory of spaces.

Combining these insights, one obtains an equivalence

$$\mathcal{S}\mathbf{paces} = \mathbf{CW}[H^{-1}] \simeq \mathbf{Top}[W^{-1}] \simeq \mathbf{Kan}[H_k^{-1}] = \mathcal{G}\mathbf{rpd}_\infty,$$

as asserted in the homotopy hypothesis.
One of the central assertions of [AFR19] is a smooth stratified version of the homotopy hypothesis. Recall that, roughly speaking and in the broadest sense, a stratified space is a topological space together with a decomposition into disjoint pieces, the so-called strata. A stratified map is, again roughly speaking, a continuous map between such objects that has the property that the image of each stratum in the source is completely contained in a stratum in the target. Stratifications of topological spaces often arise naturally when investigating spaces with singularities, by decomposing a singular space into manifold pieces (see, for example, [Whi65b; Mat12; Mat73; Tho69]) and in these scenarios the set of strata tends to naturally inherit the structure of a poset from the topological closure relation. The homotopy theory of such stratified spaces, using homotopies that also preserve the strata, was first investigated in detail by Quinn in [Qui88]. Quinn focused on a class of stratified spaces with excellent homotopical properties, the so-called homotopically stratified spaces (called homotopically stratified sets in [Qui88]), proving, among other results, a stratified version of the s-cobordism theorem for certain homotopically stratified spaces with manifold strata. Following a more differential topological framework, which they established in [AFT17], in [AFR19] Ayala, Francis, and Rozenblyum developed a homotopy theory of stratified spaces within the context of differential topology. One of their central assertions was the following statement:

Let $\mathbf{Strat}_{C^\infty}$ denote the category of conically smooth stratified spaces (with conically smooth stratified maps), and H_s the class of stratified homotopy equivalences (in the conically smooth sense). Then there is a fully faithful embedding

$$\mathbf{Strat}_{C_\infty}[H_s^{-1}] \hookrightarrow \mathcal{C}\mathbf{at}_\infty$$

into the homotopy theory of ∞-categories[1].

Furthermore, they conjectured a topological analogue of this result. This relies on the exit path construction of McPherson, Treuman, Woolf, and Lurie ([Woo09; Tre09; Lur17]). Roughly speaking, this construction associates to a stratified space an ∞-category in which morphisms are given by paths that either remain within one stratum, or start in one stratum and immediately exit into a higher one.

Conjecture 7.1.0.1. [AFR19] Topological exit paths define a fully faithful functor

$$\mathbf{Exit}\colon \mathcal{S}\mathbf{trat} \hookrightarrow \mathcal{C}\mathbf{at}_\infty$$

from a homotopy theory of topological stratified spaces $\mathcal{S}\mathbf{trat}$ into ∞-categories.

Again, just as in the case of the classical homotopy hypothesis, any answer to this conjecture must first provide a formal interpretation of the homotopy theories in question. In this case, the difficulty lies with the left-hand side, specifically the question after a homotopy theory of topological stratified spaces. In recent years, three of such theories have been independently proposed (not all with the intent to tackle the topological stratified homotopy hypothesis). In the following **Strat** will always denote the category of all poset-stratified spaces, with stratified maps between them (see, for example, [DW22], which is Chapter 3 here, for an overview). Similarly, we denote by $\mathbf{Strat}_P$, for a partially ordered set P, the category of poset-stratified spaces over a fixed poset, with morphisms given by stratified maps that descend to the identity on P, so-called *stratum-preserving* maps. One way of defining a homotopy theory of stratified spaces is, of course, to specify a category of topologically stratified spaces, in these cases, the category of all poset-stratified spaces with stratified maps, and then to localize the latter at an appropriate class of *stratified weak equivalences.*

1. In [Dou21c; Hen] Douteau and Henriques independently built on a result of Miller's in [Mil13], concerning Quinn's homotopically stratified spaces. In [Mil13], it was shown that stratum-preserving homotopy equivalences between homotopically stratified spaces are precisely such maps that induce homotopy equivalence on the strata and on the so-called *$[p < q]$-homotopy links* (where $p < q$ are elements of the stratifying poset). Recall, that these are given by the spaces of exit-paths, starting in the p-stratum and immediately exiting into the q-stratum. Since for general poset-stratified spaces pairwise homotopy links are insufficient to even guarantee that the underlying map is a weak equivalence of topological spaces, [Dou21c; Hen] additionally considered so-called generalized homotopy links, obtained by replacing the stratified interval by a stratified simplex. Then a weak equivalence of stratified spaces is defined to be a stratified map that induces an isomorphism on stratifying posets and weak equivalences on all generalized homotopy links (which include the strata). We call such stratified maps *poset-preserving diagrammatic equivalences* and denote the resulting ∞-category obtained by localizing poset-preserving diagrammatic equivalences by $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$. Analogously, given a fixed poset P, we denote the localization of $\mathbf{Strat}_P$ at stratum-preserving (poset-preserving)

[1] At the current point in time, it appears that [AFR19] is missing a definition of what exactly the category of conically smooth stratified spaces they consider is. Note that one cannot simply use the category of conically smooth stratified spaces of [AFT17], since the latter does not contain the stratified simplices. Without a definition of the stratified smooth category accommodating such spaces with corners (and the development of the necessary theory), it is currently not possible to verify the truth of the smooth stratified homotopy hypothesis.

diagrammatic equivalences over P by $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$. Then taking generalized homotopy links induces an equivalence of ∞-categories

$$\mathcal{H}o\mathrm{Link}\colon \mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \simeq \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$$

where the right hand side denotes the category of $\mathcal{S}\mathbf{paces}$ valued presheaves on the subdivision on the poset P. Explicitly, $\mathrm{sd}(P)$ is the category whose objects are finite increasing sequences $[p_0 < \dots < p_n]$ in P (so-called regular flags) and whose morphisms are inclusions of subsequences.

2. In [Hai23], Haine followed the insight that the categories of exit paths associated to a classical stratified space (for example, conically stratified space or homotopically stratified space) always come with a conservative functor into the stratification poset. This is simply due to the fact that restricted to each stratum the construction produces the classical ∞-groupoid of paths. Furthermore, the fact that exit-paths define an ∞-category at all implies that, for such stratified spaces, the natural maps from the generalized homotopy links into the homotopy pullbacks of pairwise homotopy links

$$\mathcal{H}o\mathrm{Link}_{[p_0 < \dots < p_n]}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}_{p_0 < p_1}(\mathcal{X}) \times^H_{X_{p_1}} \dots \times^H_{X_{p_{n-1}}} \mathcal{H}o\mathrm{Link}_{p_{n-1} < p_n}(\mathcal{X})$$

are weak homotopy equivalences. Hence, only certain diagrams $D \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ — those were the corresponding morphism is an isomorphism in the ∞-category of spaces — may arise as the homotopy-link diagram of classical examples of stratified spaces. Diagrams that satisfy this property are called *décollages* (see [Hai23; BGH18]). The full ∞-subcategory of décollages, $\mathcal{D}\mathbf{\acute{e}c}_P$, turns out to be a (left) localization of $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ (i.e. a reflective subcategory). [Hai23] then defines his class of weak equivalences of stratified spaces over a fixed poset P as the class of such stratified maps that map into isomorphisms under the composition

$$\mathbf{Strat}_P \xrightarrow{\mathcal{H}o\mathrm{Link}} \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces}) \to \mathcal{D}\mathbf{\acute{e}c}_P.$$

We call such weak equivalences (stratum-preserving) categorical equivalences, and denote the resulting homotopy theory by $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$. This makes $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ a left localization of Douteau and Henriques' theory $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$, that is canonically equivalent to $\mathcal{D}\mathbf{\acute{e}c}_P$. One may then extend this class of weak equivalences to the case of varying poset $\mathbf{Strat}$, by requiring a weak equivalence to induce isomorphisms on stratifying posets and a stratum-preserving categorical equivalence after identifying the posets along the isomorphism. We call such stratified maps *poset-preserving categorical equivalences* (for reasons which will become apparent later) and denote the resulting homotopy theory by $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$.

3. Finally, in [Nan19], Nand-Lal took the approach of transferring the weak equivalences along Lurie's functor of stratified singular simplices, which provides a model for the ∞-category of exit paths associated to a stratified space[2]:

$$\mathrm{Sing}_s\colon \mathbf{Strat} \to \mathbf{sSet}.$$

More precisely, a stratified map $\mathcal{X} \to \mathcal{Y}$ is called a *categorical equivalence* if the induced simplicial map $\mathrm{Sing}_s(\mathcal{X}) \to \mathrm{Sing}_s(\mathcal{Y})$ is a Joyal equivalence (also called categorical equivalences; see [Lur09]). One major difference compared to the previous two theories is that weak equivalences in this setting are not required to preserve the stratification poset. In this sense, the stratification posets are only essential to a stratified homotopy type insofar as they determine what paths are exit paths (what simplices are exit-simplices). We denote the induced homotopy theory by $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$.

[2]Strictly speaking, Nand Lal works with a full subcategory of $\mathbf{Strat}$, but from a homotopy theoretic perspective this difference turns out to be irrelevant (see Proposition 7.A.0.2)

The first answer to Conjecture 7.1.0.1 was given in [Hai23], where Haine used a result of Douteau ([Dou21c, Thm. 3]) to prove an equivalence of ∞-categories

$$\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P \simeq \mathcal{AS}\mathbf{trat}_P,$$

where $\mathcal{AS}\mathbf{trat}_P$ is the ∞-category of conservative functors of ∞-categories with target P, so-called abstract stratified homotopy types. Thus, the equivalence provides an answer to Conjecture 7.1.0.1 for the setting of a fixed poset. The equivalence is constructed by passing through the equivalence of $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ with $\mathcal{D}\mathbf{\acute{e}c}_P$, and then, in turn, constructing an equivalence of the latter with a model for $\mathcal{AS}\mathbf{trat}_P$ given in the language of complete Segal spaces. In this sense, the equivalence is not constructed directly in terms of Lurie's Sing_s construction (from a 1-categorical perspective at least). A priori, it is unclear whether the weak equivalences defining $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ can be characterized through Sing_s.

Here, we aim to provide an answer to the topological stratified homotopy hypothesis that is similarly tractable to the incarnation of the classical homotopy hypothesis in terms of Theorem HH. To begin with, this means we want to obtain tractable versions of the comparison functors between stratified spaces and ∞-categories, in terms of a presentation through the stratified singular simplicial set functor and its left adjoint given by stratified realization. Secondly, we want to obtain a better understanding of the homotopy theories of stratified spaces – $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$,$\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$, and their fixed poset counterparts – and how they interact with the 1-category **Strat** as well as classical approaches to stratified homotopy theory, such as the one pursued in [Qui88].

Question 7.1.0.2. More specifically, we aim to answer the following questions.

Q(1) Can the equivalence $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P \simeq \mathcal{AS}\mathbf{trat}_P$ be presented through Lurie's stratified singular simplicial set (Exit-path) construction, and does Sing_s create stratum-preserving categorical equivalences, in the sense that a stratum-preserving map is a categorical equivalence, if and only if $\mathrm{Sing}_s(f)$ is a Joyal equivalence? If yes, do analogous results hold for the homotopy theories of stratified spaces with varying stratification posets, thereby presenting a global version of the topological stratified homotopy hypothesis?

Q(2) Without additional structure, ∞-categorical localizations of 1-categories are generally difficult objects to study. What can we say about the homotopy theories of stratified spaces from a stratified topological perspective? For example, can we express $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ as a subcategory of stratified spaces $\mathbf{C}$ localized at stratified homotopy equivalences, analogously to the situation of topological spaces and CW complexes? Such a category $\mathbf{C}$ should contain as many stratified spaces of classical interest as possible, to allow us to investigate the stratified homotopy-theoretic properties of geometrically interesting examples through the language of ∞-categories.

Q(3) In the same line of questioning as in the previous question: The category **Strat** admits a naive notion of mapping space, with points given by stratified maps and paths given by stratified homotopies. For what stratified spaces, $\mathcal{X}$ and $\mathcal{Y}$, can we expect the mapping space $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}(\mathcal{X},\mathcal{Y})$ to have the homotopy type of this naive mapping space, or similarly, which classical examples of stratified spaces are contained in $\mathbf{C}$?

Q(4) What are the precise relationships between the several stratified homotopy theories introduced above and the more classical approach due to Quinn? In particular, how large are the differences when restricting to classical, geometrical examples of stratified spaces?

Q(5) How can ∞-categorical construction, such as colimits in a quasi-category (i.e., homotopy colimits), be interpreted in terms of the 1-category **Strat**? For instance, when is a pushout diagram of stratified spaces homotopy pushout in one of the homotopy theories of stratified spaces above?

Q(6) The sets of stratified maps can even be equipped with the structure of a stratified space, with each stratum corresponding to a map of the underlying posets (see [Nan19]). In [Hug99b], such stratified (decomposed) mapping spaces and their relations with stratified notions of fibrations were studied. The central result states a stratified exit path version of the classical path-space fibration. Can we replicate this result in the stratified homotopy theories described above, using methods of modern abstract homotopy theory?

Q(7) What is the relationship of the homotopy theories described above with the homotopy theory of conically smooth stratified spaces discussed in [AFR19]?

Classically, such questions concerning the relationship of a homotopy theory and a 1-category from which it is obtained by localization are answered through the language of model categories. Thus, we may at least partially rephrase the questions above:

Question 7.1.0.3. Do the classes of weak equivalences defining the homotopy theories $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$, $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ (and their fixed, poset counterparts) extend to model structures on **Strat** ($\mathbf{Strat}_P$)? Specifically, do model structures exist in which stratified spaces of classical interest – for example, Whitney stratified spaces or, more recently, conically smooth stratified spaces – are bifibrant? Supposing an affirmative answer, what are the properties of this model category, such as admitting a simplicial structure, cartesian closedness, cofibrant generation? Can one prove an answer to the stratified homotopy hypothesis in terms of Quillen equivalences through the adjunction of stratified realization and stratified singular simplicial sets?

In [Dou21c; Hen] model structures on **Strat** presenting $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ were defined. However, these model structures fail the criterion of having convenient bifibrant objects: Not even the stratified cone on a closed manifold is a cofibrant object. In [Nan19], the author defined a model structure for the subcategory of such stratified spaces, $\mathcal{X}$, for which $\mathrm{Sing}_s(\mathcal{X})$ is a quasi-category and used it to prove a Whitehead theorem for stratified spaces. The existence of a (semi-)model structure as above was left open as a conjecture ([Nan19, p. 8.4.1]). While this makes, for example, (appropriately stratified) piecewise linear pseudo-manifolds bifibrant, a full model category (in particular fibrant replacement) is needed to present the stratified homotopy hypothesis. Finally, in [DW22], we showed that such model structures cannot exist (see Proposition 7.4.0.1 for a detailed proof adapted to the setting of this paper).

7.1.1 Content of this article

One main result of this paper is to prove that we can answer Question 7.1.0.3 affirmatively if we generalize to (left) semi-model categories (see, for example, [Bar10; BW24]). We note that, in practice, these usually turn out to be just as powerful as model categories (see, for example, [BW24, Rem. 4.5]. In particular, the existence of left semi-model structures (just semi-model structures henceforth) allows us to obtain answers to Question 7.1.0.2. Building on the work of [Hai23], [Dou21c], and [Dou21b], we have laid the foundation to prove the existence of these semi-model structures in the two preceding papers: [Waa24b] and [Waa24a]. The remaining task is to combine these results as follows:

1. In [Dou21b] and [Hai23] Douteau and Haine each defined simplicial counterparts to their stratified homotopy theories for the case of a fixed poset P. In [Waa24a], we extended these models to model structures for stratified simplicial sets with varying stratification poset, the category of which we denote s**Strat**,. Doing so, we obtain simplicial model categories $\underline{\mathbf{sStrat}}^{\mathfrak{d},\mathfrak{p}}$, $\underline{\mathbf{sStrat}}^{\mathfrak{d}}$, $\underline{\mathbf{sStrat}}^{\mathfrak{c},\mathfrak{p}}$ and $\underline{\mathbf{sStrat}}^{\mathfrak{c}}$. The latter two of these do, respectively, present the homotopy theories of abstract stratified homotopy types (with varying stratification poset) and of (small) layered ∞-categories, which are the ∞-categories in which every endomorphism is invertible. We recall these results in Section 7.3.2.

2. We want to construct model structures for $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}, \mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}, \mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ and their fixed poset analogues by transferring the model structures on the simplicial side along the stratified singular simplicial set and realization adjunction. To this end, we need to prove that the functor $\mathrm{Sing}_s\colon \mathbf{Strat} \to \mathrm{s}\mathbf{Strat}$ creates weak equivalences for the respective theories. This is proven in Section 7.3.3. In fact, more than that, we show that the functor induces homotopy equivalences of categories with weak equivalences (Theorem 7.3.3.1).
3. Then, in Section 7.4, we transfer the model structures from the combinatorial framework to the topological one, through a transfer lemma for semi-model categories (Lemma 7.4.1.6). Applying this lemma requires a deeper understanding of the interaction of the homotopy link functor with topological stratified constructions, such as pushouts along inclusions of cell complexes. This is achieved through Theorem HB, which establishes the existence of certain regular neighborhoods for multistrata interactions in stratified cell complexes. In many ways, these are the topological (as opposed to smooth) analogues of the Unzip construction of [AFT17]: They provide the necessary glue to connect the geometry/topology of stratified spaces with their homotopy theory.

We then combine these insights to obtain the following result, which – using standard results on (semi-)model categories – ultimately addresses Questions Q(2), Q(5) and Q(6) (see, particularly, Section 7.4.4). For the sake of conciseness, we only cover the case of varying posets in this introduction.

Theorem A (Theorems 7.3.3.1, 7.4.2.7 and 7.4.2.10 and Corollaries 7.4.2.3 and 7.4.3.3)**.** *The category of stratified topological spaces* **Strat** *admits the structure of three distinct simplicial, cofibrantly generated, and cartesian closed left semi-model categories:* $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$, $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$*, and* $\underline{\mathbf{Strat}}^{\mathfrak{c}}$*. Weak equivalences are given, respectively, by the poset-preserving diagrammatic, poset-preserving categorical, and categorical equivalences. They are right-transferred along the adjunction*

$$|-|_s\colon \mathrm{s}\mathbf{Strat} \rightleftharpoons \mathbf{Strat}\colon \mathrm{Sing}_s$$

from their respective counterpart on $\underline{\mathrm{s}\mathbf{Strat}}$*. Moreover, with respect to these transferred structures, the adjunction defines simplicial Quillen equivalences (between the respective topological and simplicial counterparts) that create weak equivalences in both directions.*

$\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ is obtained from $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ in terms of a left Bousfield localization at stratified inner horn inclusions (providing a first step towards an answer to Question Q(4)). In turn, $\underline{\mathbf{Strat}}^{\mathfrak{c}}$ is obtained from $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ in terms of a right Bousfield localization (Theorem 7.3.3.1). As a corollary of this result and Theorem 5.3.3.6, we obtain the following version of the stratified homotopy hypothesis, providing answers to Question Q(1):

Theorem B (Theorem 7.4.4.4)**.** *Mapping a simplicial set to its stratified realization (as in [Nan19]) and conversely mapping a stratified space $\mathcal{X}$ to the underlying simplicial set of* $\mathrm{Sing}_s\mathcal{X}$ *induces a Quillen equivalence of (semi-)model categories*

$$\mathbf{sSet}^{\mathfrak{L}} \overset{\simeq}{\rightleftharpoons} \mathbf{Strat}^{\mathfrak{c}}$$

that creates weak equivalences in both directions.
Here, $\mathbf{sSet}^{\mathfrak{L}}$ *is the left Bousfield localization of the Joyal model structure on* **sSet** *that presents layered* ∞*-categories.*

To obtain a similarly convenient situation to the setup of the classical homotopy hypothesis, an answer to Question Q(3) remains to be obtained. We do so in Section 7.5. It is an immediate consequence of the way that the model structures are constructed via transfer from stratified simplicial sets that any stratified space which admits a piecewise linear structure (or, more generally, a cell structure) that is compatible with the stratification is cofibrant in the semi-model categories presenting $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$, namely $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ and $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$. Hence, for example, by [Gor78], all Whitney stratified spaces are cofibrant. Even more, we

show Proposition 7.5.2.10, from which it follows that any stratified space (over a finite poset) whose strata are manifolds that additionally admit a stratified notion of mapping cylinder neighborhood is cofibrant. In particular, assuming the correctness of a result of [AFR19], this implies that conically smooth stratified spaces are cofibrant. The semi-model category $\underline{\mathbf{Strat}}^{\mathfrak{c}}$ turns out to be a right Bousfield localization of $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$, whose cofibrant objects are precisely the cofibrant objects in $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ that fulfill a weak version of the classical frontier condition and have connected strata (Proposition 7.5.4.5). Conceptually, these are the stratified spaces (cofibrant in $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$) in which the stratification poset structure arises entirely in terms of the topological relations of the strata.
Faced with several different model structures and homotopy theories for stratified spaces, the obvious question about the precise relationship between these theories, in particular in application to geometric examples, arises. We have already illustrated above that the passage from $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ to $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ essentially amounts to requiring that the poset structure is intrinsic to the topology of the space. The difference between $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ (that is, answering Question Q(4)) is more subtle. As the cofibrant objects in the categorical and the diagrammatic semi-model categories agree, the difference between the resulting homotopy theories must lie in the conditions for fibrancy. It is a result of [Lur17] that conically smooth stratified spaces are fibrant in $\underline{\mathbf{Strat}}^{\mathfrak{c}}$, and hence also in $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ and $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$. In [Nan19], it was shown that the same holds for Quinn's homotopically stratified spaces ([Qui88]). This already covers most classically relevant examples of stratified spaces. Thus, it appears that at least in a geometric scenario there is not much difference between the two homotopy theories $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ at all. In fact, we make this result rigorous in terms of the following answer to Questions Q(3) and Q(4).

Theorem C (Proposition 7.5.1.4 and Theorem 7.5.1.6)**.** *Let $\mathcal{X} \in \mathbf{Strat}$ be a metrizable stratified space. Then the following conditions are equivalent:*

(i) $\mathcal{X}$ is fibrant in $\underline{\mathbf{Strat}}^{\mathfrak{c}}$;

(ii) $\mathcal{X}$ is fibrant in $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$;

(iii) $\mathcal{X}$ is fibrant in $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$;

(iv) For any pair of strata $[p < q]$, the starting point evaluation map $\mathcal{H}o\mathrm{Link}_{p<q}(\mathcal{X}) \to \mathcal{X}_p$ is a Serre fibration.

In particular, when restricted to metrizable stratified spaces, the homotopy theories defined by $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ and $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ (in terms of simplicial categories of bifibrant objects) agree.

Similarly to the case of the classical homotopy hypothesis, it can be useful to restrict to a class of particularly convenient stratified spaces that mimic some of the properties of CW complexes. This is handled by Corollary 7.5.4.8, which in particular states that the bifibrant stratified spaces in $\underline{\mathbf{Strat}}^{\mathfrak{c}}$ are precisely the retracts of the so-called CFF stratified spaces, i.e., the stratified spaces $\mathcal{X} \in \mathbf{Strat}$ fulfilling:

1. $\mathrm{Sing}_s\mathcal{X}$ is a quasi-category.
2. $\mathcal{X}$ admits cell structure that is (in a sense which we will specify) compatible with the stratification.
3. $\mathcal{X}$ has non-empty connected strata, and the structure of the stratification poset arises from the classical frontier condition.

It follows that we may rephrase Theorem B as the following statement:

Theorem B'. *Denote by* **CFF** *the full subcategory of* **Strat** *given by CFF stratified spaces and let H_s be the class of stratified homotopy equivalences in* **CFF**. *Lurie's exit-path construction ([Lur17]) induces an equivalence of ∞-categories*

$$\mathrm{Exit}\colon \mathbf{CFF}[H_s^{-1}] \xrightarrow{\simeq} \mathcal{L}\mathbf{ay}_\infty$$

where $\mathcal{L}\mathbf{ay}_\infty$ denotes the ∞-category of small layered ∞-categories.

At the end of Section 7.5 in Section 7.5.5, we study the stratified homotopy link fibrations of [Hug99b] from a model categorical perspective, thus providing further connections between our model categorical approach to stratified homotopy theory and more classical approaches.

Finally, let us comment on the relationship of the homotopy theories investigated in this article with the homotopy theory of conically smooth stratified spaces investigated in [AFR19] (Question Q(7)). Supposing the existence of stratified mapping cylinder neighborhoods (tubular neighborhoods), as asserted in [AFT17, Prop. 8.2.3], it follows from Proposition 7.5.2.10, Proposition 7.5.4.5 and Theorem C that conically smooth stratified spaces are bifibrant in $\underline{\mathbf{Strat}}^{\mathfrak{c}}$. Note that it follows from this result and Theorem B that the following two statements are equivalent:

1. The exit path construction induces a fully faithful embedding from the ∞-category of conically smooth stratified spaces into ∞-categories (as asserted in [AFR19]).
2. For any pair of conically smooth stratified spaces $\mathcal{X}, \mathcal{Y}$, the natural map

$$\underline{\mathbf{Strat}}_{\mathcal{C}^\infty}(\mathcal{X}, \mathcal{Y}) \to \underline{\mathbf{Strat}}(\mathcal{X}, \mathcal{Y})$$

   from the mapping space of conically smooth maps (defined in [AFR19]) into the mapping space of continuous stratified maps is a homotopy equivalence of Kan complexes.

In this sense, the smooth version of the stratified homotopy hypothesis as it is stated above is equivalent to a stratified smooth approximation theorem, as conjectured in [AFT17, Conjecture 1.5.1].

7.2 Preliminaries

In this section, we introduce and recall some of the necessary language and notation, especially for 1-categorical aspects of stratified spaces and higher homotopical frameworks.

7.2.1 Models for $(\infty, 1)$-categories

This paper is concerned with investigating several homotopy theories of stratified spaces. By a *homotopy theory* we mean an $(\infty, 1)$-category, not restricting to a specific model for $(\infty, 1)$-categories. For the sake of readability, we will always drop the 1 and use the term ∞-category and homotopy theory synonymously. For our purposes, it is going to be extremely useful to have several different models for ∞-categories available. Specifically, we are going to use the following models.

Notation 7.2.1.1. 1. The relative categories of [BK12b], given by pairs $(\mathbf{C}, W)$ of a 1-category $\mathbf{C}$ with a wide subcategory $W \subset \mathbf{C}$. Names for relative categories will always begin with an italic letter, that is, they will be of the form $\boldsymbol{N}$**ame**.

2. Categories enriched over simplicial sets, also called simplicial categories (see [DK80c; Ber07a]): Names for simplicial categories will always be underlined, that is, written in the form $\underline{\mathbf{Name}}$.
3. Quasi-categories in the sense of [Lur09]: Names for quasi-categories will always be bold and begin with a calligraphic letter, i.e. they will be of the form $\mathcal{N}$**ame**.

If $\mathbf{C}$ is an ∞-category (in particular a 1-category), the mapping spaces between objects $X, Y \in \mathbf{C}$ will be denoted by $\mathbf{C}(X, Y)$. In the case of simplicial categories, these have an explicit model in the obvious way. For relative categories, they are given by the hammock localization of [DK80a; BK12a]. For quasi-categories, we use any of the equivalent models of mapping spaces of [Lur09].

Notation 7.2.1.2. To study ∞-categories through some underlying 1-category which they are a localization of, we are going to make use of the language of (semi)model categories (see [Hir03] for model categories and [Bar10; BW24] for a good overview of semi-model categories). We will usually denote ordinary categories in bold letters in the form **Name**). The model structure will always be marked by adding some additional ornamentation - in the form $\underline{\mathbf{Name}}^{\mathfrak{s}}$ or $\mathbf{Name}^{\mathfrak{s}}$ - to the name of the underlying simplicial or 1-category.

Notation 7.2.1.3. Functor categories, between two 1-categorical, simplicial categories or quasi-categories $\mathbf{C}, \mathbf{D}$ will always be denoted in the form $\mathbf{Fun}(\mathbf{C}, \mathbf{D})$. This is to be understood in the sense that the type of categories inserted into $\mathbf{Fun}(-,-)$ specifies whether the resulting functor category is itself a 1-category, quasi-category, or simplicial category. In case that the source is a 1-category, and the target a quasi or simplicial category, we treat the 1-category, respectively, as a quasi-category (via its nerve) or as a simplicial category with discrete mapping spaces. At times, we will also use exponential notation $\mathbf{D}^{\mathbf{C}}$ to refer to functor categories.

Notation 7.2.1.4. Often we are going to pass between different models of a homotopy theory. These passages will always follow the following ruleset for nomenclature:

1. Starting with a relative category $\boldsymbol{N}\mathbf{ame}$ or simplicial category $\underline{\mathbf{Name}}$, the underlying 1-category will be denoted **Name**. Similarly, if $\underline{\mathbf{Name}}^{\mathfrak{s}}$ is a simplicial model category, then we denote its underlying 1-model category by $\mathbf{Name}^{\mathfrak{s}}$.

2. Starting with a model category $\mathbf{Name}^{\mathfrak{s}}$, we denote by $\boldsymbol{N}\mathbf{ame}^{\mathfrak{s}}$ the relative category obtained by its underlying 1-category, together with the wide subcategory of weak equivalences.

3. Starting with a relative category $\boldsymbol{N}\mathbf{ame} = (\mathbf{Name}, W)$, we denote by $\mathcal{N}\mathbf{ame}$ the quasi-category $\mathbf{Name}[W^{-1}]$ obtained by taking the nerve of the underlying 1-category of $\boldsymbol{N}\mathbf{ame}$, and then localizing at W.

4. In particular, following this language, if we start with a model category $\mathbf{Name}^{\mathfrak{s}}$, then the relative category $\boldsymbol{N}\mathbf{ame}^{\mathfrak{s}}$ and the quasi-category $\mathcal{N}\mathbf{ame}^{\mathfrak{s}}$ model the same ∞-category.

5. If we start with a simplicial model category $\underline{\mathbf{Name}}^{\mathfrak{s}}$, then we denote by $\underline{\mathbf{Name}}^{\mathfrak{s},o}$ the full simplicial subcategory of bifibrant objects. Following our notation, under the Quillen equivalence between simplicial categories and quasi-categories of [Ber07b], $\underline{\mathbf{Name}}^{\mathfrak{s},o}$ models the same ∞-category as $\mathcal{N}\mathbf{ame}^{\mathfrak{s}}$ (this follows from [DK80b, Prop. 4.8] together with [Hin16, Prop. 1.2.1]).

For the sake of completeness, one could of course also introduce notation to pass from relative to simplicial categories, etc., but we will not make use of this change of model in this article, and thus do not take this extra step.

7.2.2 Poset-stratified spaces

Next, let us introduce the basic objects of study: topological spaces that are stratified over a partially ordered set (poset). We begin with notation for the world of posets.

Notation 7.2.2.1. We are going to use the following terminology and notation for partially ordered sets, drawn partially from [Dou21a] and [Hai23]:

- We denote by **Pos** the category of partially ordered sets, with morphisms given by order-preserving maps.
- We denote by Δ the full subcategory of **Pos** given by the finite, nonempty, linearly ordered posets of the form $[n] \coloneqq \{0, \cdots, n\}$, for $n \in \mathbb{N}$.
- Given $P \in \mathbf{Pos}$, we denote by Δ_P the slice category $\Delta_{/P}$. That is, objects are given by arrows $[n] \to P$ in **Pos**, $n \in \mathbb{N}$, and morphisms are given by commutative triangles.

- We denote by $\mathrm{sd}(P)$ the *subdivision of* P, given by the full subcategory of Δ_P of such arrows $[n] \to P$, which are injective.
- The objects of Δ_P are called *flags of* P. We represent them by strings $[p_0 \leq \cdots \leq p_n]$, of $p_i \in P$.
- Objects of $\mathrm{sd}(P)$ are called regular *flags of* P. We represent them by strings $[p_0 < \cdots < p_n]$, of $p_i \in P$.

Notation 7.2.2.2. **Top** is going to denote either of the following three categories of topological spaces.

1. The category of *all topological spaces*, which we will also refer to as *general* topological spaces.
2. The category of compactly generated topological spaces, i.e. such spaces which have the final topology with respect to compact Hausdorff spaces (see, for example, [Rez17]).
3. The category of Δ-generated topological spaces, i.e. such spaces which have the final topology with respect to realizations of simplices, or equivalently just with respect to the unit interval (compare [Dug03; Gau21]).

We denote by $\underline{\mathbf{sSet}}$ the simplicial category of simplicial sets, i.e. the category of set-valued presheaves on Δ^{op}, equipped with the canonical simplicial structure induced by the product (see [Lur09] for all of the standard notation used for simplicial sets). We denote by $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ the realization functor of simplicial sets and by $\mathrm{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$ its right adjoint. **Top** naturally carries the structure of a simplicial category, tensored and powered over **sSet**, induced by left Kan extension of the construction

$$T \otimes \Delta^n := T \times |\Delta^n|.$$

We denote the resulting simplicial category by $\underline{\mathbf{Top}}$. Furthermore, we will always consider $\underline{\mathbf{Top}}$ to be equipped with the Quillen model structure [Qui67], which makes $|-| \dashv \mathrm{Sing}$ a simplicial Quillen equivalence, between $\underline{\mathbf{Top}}$ and $\underline{\mathbf{sSet}}$ (with the latter equipped with the Kan-Quillen model structure), which creates all weak equivalences in both directions.

Remark 7.2.2.3. Note that one commonly only defines the simplicial structure for compactly or Δ-generated spaces. However, this is mostly due to the fact that for general topological spaces T and infinite simplicial sets K, the tensor $T \otimes K$ does not agree with the inner product $T \times |K|$. Instead, it is given by a colimit of products of T with the simplices of K. Similarly, the power T^K is not given by an internal mapping space (which does not even necessarily exist for arbitrary K) but by the limit of the mapping spaces with source given by simplices of K, which are equipped with the compact-open topology. We want to emphasize that for the resulting homotopy theory the choice in underlying set-theoretic assumptions on topological spaces is immaterial.

For the remainder of this subsection, we fix some category of topological spaces **Top** as in Notation 7.2.2.2.

Notation 7.2.2.4. Having fixed a category of topological spaces **Top**, we then use the following notation for stratified topological spaces (all of these constructions already appear in [Dou21c] among other places). The notation and language is mostly analogous with the language for the simplicial framework.

- We think of the 1-category **Pos** as fully faithfully embedded in **Top**, via the Alexandrov topology functor $\mathcal{A}\colon \mathbf{Pos} \to \mathbf{Top}$, equipping a poset P with the topology where the closed sets are given by the downward closed sets. By abuse of notation, we usually just write P, for the Alexandrov space corresponding to the poset P (compare [DW22, Def. 2.2], which is Definition 3.2.1.2 here).

- For $P \in \mathbf{Pos}$, we denote by $\mathbf{Strat}_P$ the slice category $\mathbf{Top}_{/P}$. We treat $\mathbf{Strat}_P$ as a simplicial category, denoted $\underline{\mathbf{Strat}}_P$, with the structure inherited from $\underline{\mathbf{Top}}$ (see [DW22, Recol. 2.13], which is Recollection 3.2.2.4 in this text, for a detailed definition in the Δ-generated case).
- Objects of $\underline{\mathbf{Strat}}_P$ are called *P-stratified spaces*. They are given by a tuple $(\mathcal{X}, s: X \to P)$. In the literature, a P-stratified space $(\mathcal{X}, s: X \to P)$ is often simply referred to by its underlying space X, omitting the so-called *stratification* $s: X \to P$. We are not going to adopt this notation here and will generally use calligraphic letters for stratified spaces and stick to the notational convention $\mathcal{X} = (X, s_{\mathcal{X}})$ to refer to the underlying space and stratification.
- Morphisms in $\mathbf{Strat}_P$ are called *stratum-preserving maps.*
- Given a map of posets $f: Q \to P$ and $\mathcal{X} \in \mathbf{Strat}_P$, we denote by $f^*\mathcal{X} \in \mathbf{Strat}_Q$ the stratified space $X \times_P Q \to Q$. We are mostly concerned with the case where f is given by the inclusion of a singleton $\{p\}$, of a subset $\{q \sim p \mid q \in P\}$, for $p \in P$ and $\sim$ some relation on the partially ordered set P (such as $\leq$), or more generally, a subposet $Q \subset P$. We then write $\mathcal{X}_p$ (or, respectively, $\mathcal{X}_{\sim p}$, $\mathcal{X}_Q$) instead of $f^*\mathcal{X}$. The spaces $\mathcal{X}_p$, for $p \in P$ are called the *strata of $\mathcal{X}$.*
- For $f: Q \to P$ in $\mathbf{Pos}$, we denote by $f_!$ the left adjoint to the simplicial functor $f^*: \underline{\mathbf{Strat}}_Q \to \underline{\mathbf{Strat}}_P$. $f_!$ is given on objects by $(s_{\mathcal{X}}: X \to Q) \mapsto (f \circ s_{\mathcal{X}}: X \to Q \to P)$.
- Let $\mathbf{Top}^{[1]}$ be the category of arrows of topological spaces. We denote by $\mathbf{Strat}$ the category of all (poset-)stratified spaces, given by the full sub-category of $\mathbf{Top}^{[1]}$ of such arrows $X \to P$, where $X \in \mathbf{Top}$ and $P \in \mathbf{Pos}$ is a poset (equipped with the Alexandrov topology). In particular, every object of $\mathbf{Strat}$ is given by a P-stratified space, for some $P \in \mathbf{Pos}$, and a morphism $(X \to P) \to (Y \to Q)$ is given by a pair of morphisms $f: X \to Y$ and $g: P \to Q$, where f is a continuous map and g can be seen as a map of posets, making the obvious square commute (see also [DW22, Def. 2.11], which is Definition 3.2.2.2 in this text). Morphisms are called *stratified* maps.
- Given $\mathcal{X} \in \mathbf{Strat}$, we are going to use the notations $\mathcal{X} = (X, P_{\mathcal{X}}, s_{\mathcal{X}})$ to refer, respectively, to the underlying space, the poset, and the stratification and proceed analogously for morphisms.
- $\mathbf{Strat}$ is equipped with the structure of a simplicial category (tensored and powered over $\underline{\mathbf{sSet}}$), $\underline{\mathbf{Strat}}$, with the simplicial structure induced by $\mathcal{X} \otimes \Delta^n = (X \times |\Delta^n| \to X \to P_{\mathcal{X}})$. Note that since $\mathcal{X} \otimes -$ preserves colimits, this means that $\mathcal{X} \otimes \varnothing$ is not stratified over P, but instead over the empty poset. Simplicial homotopies, that is, homotopies with respect to the cylinder $- \otimes \Delta^1$ in $\underline{\mathbf{Strat}}$, are called *stratified homotopies.* Simplicial homotopy equivalences are called *stratified homotopy equivalences.*
- The forgetful functor $\underline{\mathbf{Strat}} \to \underline{\mathbf{Top}}$, $\mathcal{X} \mapsto X$, has a right adjoint. It is given by mapping $T \in \underline{\mathbf{Top}}$ to the trivially stratified space $(T \to [0])$. By abuse of notation, we will often write T to refer to the trivially stratified space associated to a space T.

Remark 7.2.2.5. Both $\mathbf{Strat}$ and $\mathbf{Strat}_P$, for $P \in \mathbf{Pos}$, are bicomplete categories (see, for example, [Dou21c]). Limits and colimits in $\mathbf{Strat}_P$ are simply given by the limits and colimits in a slice category. Colimits in $\mathbf{Strat}$ are computed by taking the colimit both on the space and on the poset level. Limits in $\mathbf{Strat}$ are computed by taking the limit on the space and poset level, and then pulling back the map $\varprojlim_i X_i \to \varprojlim \mathcal{A}(P_i)$ along the natural comparison map $\mathcal{A}(\varprojlim_i P_i) \to \varprojlim_i \mathcal{A}(P_i)$, which is an isomorphism for finite diagrams.

Construction 7.2.2.6. If our choice of $\mathbf{Top}$ is Cartesian closed (that is, if we are in the compactly generated or Δ-generated case) then $\mathbf{Strat}$ is also a Cartesian closed category (see also [Nan19], for the slightly different setting of surjectively stratified spaces). We use the notation X^Y to refer to exponential in a Cartesian closed category. To be able to distinguish exponential objects in posets from those in topological spaces, we use the notation $\mathcal{A}(P)$ to denote the Alexandrov space associated to a poset P. Given a stratified space $\mathcal{X} \in \mathbf{Strat}$, the right adjoint $-^{\mathcal{X}}$ to the functor $- \times \mathcal{X} \colon \mathbf{Strat} \to \mathbf{Strat}$ is constructed as follows. Given $\mathcal{Z}$ in $\mathbf{Strat}$, consider a pullback square in $\mathbf{Top}$

$$\begin{tikzcd}
\mathcal{A}({P_{\mathcal{Z}}}^{P_{\mathcal{X}}}) \times_{\mathcal{A}(P_{\mathcal{Z}})^X} Z^X \arrow[rr] \arrow[d] & & Z^X \arrow[d] \\
\mathcal{A}({P_{\mathcal{Z}}}^{P_{\mathcal{X}}}) \arrow[r] & \mathcal{A}(P_{\mathcal{Z}})^{\mathcal{A}(P_{\mathcal{X}})} \arrow[r] & \mathcal{A}(P_{\mathcal{Z}})^X ,
\end{tikzcd} \tag{7.1}$$

where $P_{\mathcal{Z}}^{P_{\mathcal{X}}}$ is the poset obtained by equipping $\mathbf{Pos}(P_{\mathcal{X}}, P_{\mathcal{Z}})$ with the poset structure given by

$$f \leq g :\iff f(p) \leq g(p), \forall p \in P_{\mathcal{X}},$$

which defines the exponential object $P_{\mathcal{Z}}^{P_{\mathcal{X}}}$ in $\mathbf{Pos}$. The lower left horizontal map between the resulting space equipped with the Alexandrov topology into the mapping space $\mathcal{A}(P_{\mathcal{Z}})^{\mathcal{A}(P_{\mathcal{X}})}$ is always continuous, but generally only a homeomorphism if $P_{\mathcal{X}}$ and $P_{\mathcal{Z}}$ are finite ([May16, Cor. 2.2.11]). The stratified mapping space $\mathcal{Z}^{\mathcal{X}}$ is defined by the left vertical of this pullback diagram [3]. This construction (and its functoriality in stratified maps) induces the right adjoint $-^{\mathcal{X}} \colon \mathbf{Strat} \to \mathbf{Strat}$ to $- \times \mathcal{X}$. In particular, we may treat $\mathbf{Strat}$ as a symmetric monoidal category (with monoidal structure induced by the product) enriched over itself.

7.2.3 Stratified simplicial sets and stratified realization

The approach to constructing and investigating homotopy theories of stratified spaces that we take in this article is to transfer homotopy theoretic structure from the combinatorial to the topological world. Let us quickly recall some notation and terminology concerning stratified simplicial sets, as introduced and investigated in [Dou21a; Hai23]. We have surveyed and expanded the results on homotopy theories of stratified simplicial sets in [Waa24a] (which is Chapter 5 in this text), to which we refer for more details.

Notation 7.2.3.1. We use the following terminology and notation for (stratified) simplicial sets, drawn partially from [Dou21a] and [Hai23]:

- When we treat $\underline{\mathbf{sSet}}$ as a model category, this will generally be with respect to the Kan-Quillen model structure (see [Qui67]), unless otherwise noted. When we use Joyal's model structure for quasi-categories ([JT08]) instead, we will denote this model category by $\mathbf{sSet}^{\mathfrak{J}}$.
- We think of $\mathbf{Pos}$ as being fully faithfully embedded in $\underline{\mathbf{sSet}}$, via the nerve functor (compare [Hai23]). By abuse of notation, we just write P, for the simplicial set given by the nerve of $P \in \mathbf{Pos}$.
- For $P \in \mathbf{Pos}$, we denote by $\mathrm{s}\mathbf{Strat}_P$ the slice category $\mathbf{sSet}_{/P}$, which is equivalently given by the category of set-valued presheaves on Δ_P. We treat $\mathrm{s}\mathbf{Strat}_P$ as a simplicial category, denoted $\underline{\mathrm{s}\mathbf{Strat}}_P$, with the structure inherited from $\underline{\mathbf{sSet}}$ (see [DW22, Recol. 2.21.], which is Definition 3.2.3.1 in this text). Objects of $\underline{\mathrm{s}\mathbf{Strat}}_P$ are called *P-stratified simplicial sets.*
- Most of the remaining language and notation we are going to use for stratified simplicial sets can be copied mutatis mutandis from the topological setting in Section 7.2.2. See Section 5.1.1, for a detailed list.

[3]Note that the underlying set of $\mathcal{A}({P_{\mathcal{Z}}}^{P_{\mathcal{X}}}) \times_{\mathcal{A}(P_{\mathcal{Z}})^X} Z^X$ is in natural bijection with $\mathbf{Strat}(\mathcal{X}, \mathcal{Z})$, which shows that we can interpret this construction as a choice of topology on $\mathbf{Strat}(\mathcal{X}, \mathcal{Z})$, which will generally be finer than the initial topology inherited from Z^X.

- The forgetful functor s**Strat** → **sSet**, $\mathcal{X} \mapsto X$, which will be denoted $\mathcal{F}$, has a right adjoint and a left adjoint. The left adjoint is given by left Kan extending the functor on simplices: $\Delta^n \mapsto \{\Delta^n \xrightarrow{1_{\Delta^n}} \Delta^n = [n]\}$. We denote it by $\mathcal{L}\colon \mathbf{sSet} \to \mathrm{s}\mathbf{Strat}$. The right adjoint is given by mapping $K \in \mathbf{sSet}$ to the trivially stratified simplicial set $\{K \to [0]\}$. By abuse of notation, we will often write K to refer to the trivially stratified simplicial set associated to a simplicial set K.

Notation 7.2.3.2. We are going to need some additional notation for flags and stratified simplices.

- For a flag $\mathcal{J} = [p_0 \leq \cdots \leq p_n] \in \Delta_P$, we write $\Delta^{\mathcal{J}}$ for the image of $\mathcal{J}$ in s$\mathbf{Strat}_P$ under the Yoneda embedding $\Delta_P \hookrightarrow \mathrm{s}\mathbf{Strat}_P$. Equivalently, $\Delta^{\mathcal{J}}$ is given by the unique simplicial map $\Delta^n \to P$ mapping $i \mapsto p_i$. $\Delta^{\mathcal{J}}$ is called the *stratified simplex associated to* $\mathcal{J}$.
- Given a stratified simplex $\Delta^{\mathcal{J}}$, for $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, we write $\partial\Delta^{\mathcal{J}}$ for its *stratified boundary*, given by the composition $\partial\Delta^n \to \Delta^n \to P$.
- Furthermore, for $0 \leq k \leq n$, we write $\Lambda^{\mathcal{J}}_k \subset \Delta^{\mathcal{J}}$, for the stratified subsimplicial set given by the composition $\Lambda^n_k \to \Delta^n \to P$ (we use the horn notation as in [Lur09]). The stratum-preserving map $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{I}}$ is called the *stratified horn inclusion associated to* $\mathcal{J}$ and k. The inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{I}}$ is called *admissible*, if $p_k = p_{k+1}$ or $p_k = p_{k-1}$. The inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{I}}$ is called *inner* if $0 < k < n$.
- It will also be convenient to have a concise notation for the images of simplices, horns, and boundaries under $\mathcal{L}\colon \mathbf{sSet} \to \mathrm{s}\mathbf{Strat}$. These are denoted by replacing the exponent $n \in \mathbb{N}$, by the poset $[n]$. That is, we write $\Delta^{[n]} := \mathcal{L}(\Delta^n)$, $\partial\Delta^{[n]} := \mathcal{L}(\partial\Delta^n) =$, $\Lambda^{[n]}_k := \mathcal{L}(\Lambda^n_k)$, for $0 \leq k \leq n$.

Recollection 7.2.3.3. [Dou21a] For fixed $P \in \mathbf{Pos}$, the two simplicial categories $\underline{\mathbf{Strat}}_P$ and $\underline{\mathrm{s}\mathbf{Strat}}_P$ are connected through a realization and functor of singular simplices type of adjunction, denoted

$$|-|_s\colon \underline{\mathrm{s}\mathbf{Strat}}_P \rightleftharpoons \underline{\mathbf{Strat}}_P\colon \mathrm{Sing}_s.$$

The left adjoint is constructed by mapping a stratified simplex $\Delta^{\mathcal{J}} \to P$, with $\mathcal{J} = [p_0 \leq \dots, \leq p_n]$, to the stratified space

$$\begin{aligned} |\Delta^n| &\to P \\ x &\mapsto \sup\{p_i \in \mathcal{J} \mid x_i > 0\}, \end{aligned}$$

where we consider $|\Delta^n|$ as embedded in $\mathbb{R}^{n+1} \cong \mathbb{R}^{\mathcal{J}}$. If we consider s$\mathbf{Strat}_P$ as the category of set-valued presheaves on Δ_P, then by the logic of a nerve and realization functor, $\mathrm{Sing}_s\mathcal{X}$ is hence given the stratified simplicial set

$$\mathrm{Sing}_s\mathcal{X}(\mathcal{J}) = \mathbf{Strat}_P(|\Delta^{\mathcal{J}}|_s, \mathcal{X})$$

with the obvious structure morphisms. The adjunction $|-|_s \dashv \mathrm{Sing}_s$ is simplicial.

Recollection 7.2.3.4 ([DW22, Recol. 2.23] which is Recollection 3.2.3.5 here)**.** The two functors $|-|_s$ and Sing_s are compatible with functoriality in the poset, in the sense that for any morphism $f\colon P \to Q$ there are natural isomorphisms

$$\begin{aligned} |-|_s f_! &\cong f_!|-|_s \\ \mathrm{Sing}_s f^* &\cong f^*\mathrm{Sing}_s. \end{aligned}$$

It follows that the adjunctions

$$|-|_s\colon \underline{\mathrm{s}\mathbf{Strat}}_P \rightleftharpoons \underline{\mathbf{Strat}}_P\colon \mathrm{Sing}_s,$$

extend to a global adjunction

$$|-|_s\colon \mathrm{s}\underline{\mathbf{Strat}} \rightleftarrows \underline{\mathbf{Strat}}\colon \mathrm{Sing}_s,$$

denoted the same, by a slight abuse of notation. Specifically, the maps of simplicial mapping spaces are given by applying the fixed poset versions of $|-|_s \dashv \mathrm{Sing}_s$ component-wise, under the identifications

$$\underline{\mathbf{Strat}}(\mathcal{S},\mathcal{T}) \cong \bigsqcup_{f\in\mathbf{Pos}(P_{\mathcal{S}},P_{\mathcal{T}})} \underline{\mathbf{Strat}}_{P_{\mathcal{S}}}(\mathcal{S},f^*\mathcal{T}) \cong \bigsqcup_{f\in\mathbf{Pos}(P_{\mathcal{S}},P_{\mathcal{T}})} \underline{\mathbf{Strat}}_{P_{\mathcal{T}}}(f_!\mathcal{S},\mathcal{T});$$

$$\mathrm{s}\underline{\mathbf{Strat}}(\mathcal{X},\mathcal{Y}) \cong \bigsqcup_{f\in\mathbf{Pos}(P_{\mathcal{X}},P_{\mathcal{Y}})} \mathrm{s}\underline{\mathbf{Strat}}_{P_{\mathcal{X}}}(\mathcal{X},f^*\mathcal{Y}) \cong \bigsqcup_{f\in\mathbf{Pos}(P_{\mathcal{X}},P_{\mathcal{Y}})} \mathrm{s}\underline{\mathbf{Strat}}_{P_{\mathcal{Y}}}(f_!\mathcal{X},\mathcal{Y}).$$

7.3 Homotopy theories of stratified objects

Equipping **Strat** with a simplicial structure and powers (even an enrichment over Kan complexes) allows us to study stratified spaces from a homotopy-theoretic perspective. It follows from [DK87, §2.5, 2.6] that the ∞-category defined by $\underline{\mathbf{Strat}}$ is equivalently given by localizing **Strat** at stratified homotopy equivalences H_s[4]. It is, however, a general paradigm in abstract homotopy theory that it is often more fruitful to consider a class of weak equivalences larger than the homotopy equivalences with respect to the simplicial structure (for example, weak homotopy equivalences of topological spaces, instead of homotopy equivalences). We may encode this information in terms of a relative category given by equipping **Strat** with the additional data of a class of weak equivalences W. The ∞-categorical localization $\mathbf{Strat}[W^{-1}]$ then defines a homotopy theory of stratified spaces. In the presence of sufficient extra structure, this process of localizing a larger class of maps is often equivalent to restricting the simplicial category to a subclass of particularly nice (i.e., bifibrant) objects (formally, this is [GH05, Prop. 1.1.10] or, more classically, [DK80b, §7]). For example, if one is interested in studying topological manifolds, by Whitehead's theorem (and the fact that topological manifolds have the homotopy type of CW complexes), it is often perfectly sufficient to do so in the setting of weak homotopy equivalences. Following this perspective from classical homotopy theory, the question of a good notion of weak equivalence of stratified spaces and hence after a convenient homotopy theory of (poset) stratified spaces arose.

7.3.1 Homotopy theories of stratified topological spaces

The question after a convenient homotopy theory of stratified spaces was already investigated in [Qui88] by Quinn, who took the approach of restricting the class of objects, rather than increasing the class of weak equivalences. Quinn followed the insight that in geometric scenarios a stratified space $\mathcal{W}$, with finitely many strata and minimal stratum p, can be decomposed into a diagram

$$\mathcal{W}' \hookleftarrow \mathcal{L} \to \mathcal{W}_p$$

with $\mathcal{X} \to \mathcal{W}_p$ a fiber bundle with fiber stratified over $P_{>p}$, and $\mathcal{W}' \hookleftarrow \mathcal{L}$ a stratum-preserving boundary inclusion over $P_{>p}$. If we inductively repeat this process with $\mathcal{W}_{>p}$ and $\mathcal{L}$, we ultimately end up with a diagram of spaces indexed over the regular flags P, that is, over $\mathrm{sd}(P)^{\mathrm{op}}$. In a less geometric scenario, we may not have access to the geometric link $\mathcal{L}$ and its further decomposition into smaller pieces. However, we may still consider a homotopy-theoretic analogue.

Recollection 7.3.1.1. Let $\mathcal{X} \in \mathbf{Strat}_P$, and $\mathcal{I} \in \mathrm{sd}(P)$. The $\mathcal{I}$-th homotopy link of $\mathcal{X}$, $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \in \mathbf{Top}$, is the topological space obtained by equipping $\mathbf{Strat}_P(|\Delta^{\mathcal{I}}|_s,\mathcal{X})$ with

[4]We make now use of this statement, so we omit a proof, but this formally follows from the fact that the powering of $\underline{\mathbf{Strat}}$ under **sSet** can be used to define an equivalence of relative categories between $(\mathbf{Strat}, H_s)$ and the flattening of $\underline{\mathbf{Strat}}$.

the subspace topology inherited from the compact-open topology (or the respective Kelley-fication thereof, to end up in compactly generated or Δ-generated spaces). If $\mathcal{I} = \{p\}$, then $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) = \mathcal{X}_p$ and if $\mathcal{I} = [p < q]$, then $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$ is the space of paths which start in $\mathcal{X}_p$ and immediately exit into the q-stratum. This construction is functorial, both in $\mathcal{I}$ and $\mathcal{X}$, inducing a right-adjoint, simplicial functor

$$\mathcal{H}o\mathrm{Link}\colon \underline{\mathbf{Strat}}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}},\ \underline{\mathbf{Top}}).$$

It will often be more convenient to present $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X}$ as a simplicial set. By abuse of notation, we will also denote by $\mathcal{H}o\mathrm{Link}$ the composition

$$\mathcal{H}o\mathrm{Link}\colon \underline{\mathbf{Strat}}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}},\ \underline{\mathbf{Top}}) \xrightarrow{\mathrm{Sing}_*} \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \underline{\mathbf{sSet}}).$$

If $\mathcal{W}$ is a Whitney stratified space with two strata, then the diagram $\mathcal{H}o\mathrm{Link}(\mathcal{W})$ is weakly equivalent to the geometric decomposition diagram illustrated above.

Quinn studied a class of (metrizable) stratified spaces, for which the natural evaluation maps $\mathcal{H}o\mathrm{Link}_{p<q}\mathcal{X} \to \mathcal{X}_p$ are Hurewicz fibrations and which additionally fulfill a cofibrancy condition for inclusions of strata (see [Qui88] for details), so-called homotopically stratified spaces (also called homotopically stratified sets). In this type of framework he showed, for example, a stratified analogue of the s-cobordism theorem. In [Mil13], Miller proved a Whitehead-style theorem for homotopically stratified spaces: A stratum-preserving map between the latter is a stratified homotopy equivalence if and only if it induces homotopy equivalences on all pairwise homotopy links and strata. Inspired by this, both Douteau and Henriques (see [Dou21c; Hen]) independently suggested the following class of weak equivalences for stratified spaces.

Recollection 7.3.1.2. [Dou21c; DW22] We call a stratum-preserving map $\mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ a *diagrammatic equivalence*, if and only if it induces weak homotopy equivalences on all generalized homotopy links $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}$, $\mathcal{I} \in \mathrm{sd}(P)$. We denote by $\boldsymbol{Strat}^{\mathfrak{d}}_P$, the relative category obtained by equipping $\mathbf{Strat}_P$ with the class of diagrammatic equivalences. It follows from [Dou21c, Thm. 3] that the homotopy link functor

$$\mathcal{H}o\mathrm{Link}\colon \underline{\mathbf{Strat}}_P \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}},\ \underline{\mathbf{Top}})$$

induces an equivalence of quasi-categories

$$\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \simeq \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{Top})[\{\text{pointwise weak equivalences}\}^{-1}] \simeq \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces}).$$

This result can be extremely useful insofar, as it allows one to study stratified spaces in terms of presheaves on a fairly simple category, which is a homotopy theoretical setting that is well understood. At first glance, if one takes the perspective of Miller's Whitehead theorem for homotopically stratified sets, it may seem surprising that generalized homotopy links of flags $\mathcal{I}$ containing more than two elements are part of the data detecting weak equivalences. Roughly speaking, higher homotopy links cannot be ignored, as even though they are not necessary to detect stratified homotopy equivalences between two homotopically stratified spaces, they nevertheless provide obstructions for such maps to exist at all. Compare this to the situation of equivalences of categories: Whether a functor is an equivalence of categories may be detected in terms of objects and hom-sets, but whether a functor exists at all requires us to consider higher-dimensional data. Namely, one needs to take into account the composition laws in the categories involved.

There is, however, another way of homotopically approximating the framework of homotopically stratified spaces. It follows from [Lur17, A.5] or [Nan19, Thm. 8.1.2.6.] that the homotopy link diagrams arising from classical examples of stratified spaces usually have the following property:

Recollection 7.3.1.3. [Hai23; Waa24a] A diagram $D \in \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ is called a *décollage*, if for every regular flag $\mathcal{I} = [p_0 < \cdots < p_n] \in \mathrm{sd}(P)$, the canonical morphism

$$D(\mathcal{I}) \to D(p_0 < p_1) \times_{D(p_1)} \cdots \times_{D(p_{n-1})} D(p_{n-1} < p_n).$$

is a weak homotopy equivalence (that is, an isomorphism in the ∞-category of spaces). If D is presented by a commutative diagram in topological spaces or simplicial sets, this condition is equivalent to the natural map

$$D(\mathcal{I}) \to D(p_0 < p_1) \times^{H}_{D(p_1)} \cdots \times^{H}_{D(p_{n-1})} D(p_{n-1} < p_n).$$

into the iterative homotopy pullback being a weak homotopy equivalence. In particular, if the homotopy link diagram $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ of a stratified space $\mathcal{X}$ is a décollage, then, roughly speaking, exit paths in $\mathcal{X}$ admit compositions which are unique up to higher coherence. The homotopy theory of such décollages can be constructed in terms of a left Bousfield localization of $\underline{\mathbf{Diag}}_P \coloneqq \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \underline{\mathbf{sSet}})$, equipped with the injective model structure, by localizing at such weak equivalences, which are local with respect to décollages (see Section 5.2.2). In other words, $f\colon E \to E'$ is a weak equivalence in the model structure that presents décollages, if and only if for every injectively fibrant diagram $D \in \underline{\mathbf{Diag}}_P$ that has the décollage property the induced map of mapping spaces

$$\underline{\mathbf{Diag}}_P(E', D) \to \underline{\mathbf{Diag}}_P(E, D)$$

is a weak homotopy equivalence. We denote this left-Bousfield localization of the injective model structure on $\underline{\mathbf{Diag}}_P$ by $\underline{\mathbf{Diag}}_P^{\mathrm{dé}}$. We say that a stratum-preserving map $f\colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ is a *categorical equivalence*, if and only if the induced morphism $\mathcal{H}o\mathrm{Link}(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}(\mathcal{Y})$ is a weak equivalence in $\underline{\mathbf{Diag}}_P^{\mathrm{dé}}$, and denote the corresponding relative category by $\mathbf{Strat}_P^{\mathfrak{c}}$. Categorical weak equivalences are precisely the weak equivalences of stratified spaces suggested by Haine in [Hai23]. It follows by construction and [Dou21c, Thm. 3] that the induced functor of quasi-categories

$$\mathcal{H}o\mathrm{Link}\colon \mathcal{S}\mathbf{trat}_P^{\mathfrak{c}} \to \mathcal{D}\mathbf{iag}_P^{\mathrm{dé}}$$

is an equivalence. That is, if we localize categorical equivalences in $\mathbf{Strat}_P$, we obtain the homotopy theory of décollages. [BGH18; Hai23, Thm. 1.1.7] states that $\mathcal{D}\mathbf{iag}_P^{\mathrm{dé}}$ is in turn equivalent to the homotopy theory of quasi-categories with a conservative functor in P, so-called *abstract stratified homotopy types*. This already shows that $\mathcal{S}\mathbf{trat}_P^{\mathfrak{c}}$ fulfills a version of a stratified homotopy hypothesis ([Hai23]).

However, in [Hai23] it was not yet known whether the equivalence between abstract stratified homotopy types and $\mathcal{S}\mathbf{trat}_P^{\mathfrak{c}}$ could be constructed on the nose through the stratified singular simplicial set construction. In fact, it was not known whether any categorical equivalence even induced a categorical equivalence (Joyal-equivalence) on stratified singular simplicial sets. We are going to prove that this is indeed the case (see Theorem 7.3.3.1). Before we do so, let us generalize from the setting of stratum-preserving maps to flexible posets. There are two apparent ways to generalize to the case of a flexible poset. The first essentially amounts to first requiring an isomorphism on the poset level, and then a weak equivalence under the resulting of homotopy theories over fixed posets:

Recollection 7.3.1.4. Let $f\colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}$. We say f is a *poset-preserving diagrammatic (categorical) equivalence* if the underlying map of posets $P(f)\colon P_{\mathcal{X}} \to P_{\mathcal{Y}}$ is an isomorphism and the induced stratum-preserving map $P(f)_!\mathcal{X} \to \mathcal{Y}$ is a diagrammatic (categorical) equivalence. We denote the resulting relative categories on $\mathbf{Strat}$ by $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$. This is how weak equivalence in $\mathbf{Strat}$ are defined, respectively, in [Dou21a; Hai23].

This definition of homotopy theories on $\mathbf{Strat}$ has the advantage that by making use of the Grothendieck bifibration $\mathbf{Strat} \to \mathbf{Pos}$, most questions about the global homotopy theories may be reduced to questions of the fibers. From a conceptual point of view, however, it has the side effect that the resulting homotopy theory contains a lot of highly pathological stratified spaces. Namely, both $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}}$ contain a fully faithful copy of **Pos**, given by empty stratified spaces. If we are looking to obtain a theory which embeds fully faithfully into the homotopy theory of (small) quasi-categories $\mathcal{C}\mathbf{at}_\infty$, i.e., fulfills a homotopy hypothesis closer to the classical one, then the stratification poset should generally not be a homotopy invariant, but merely an additional piece of data used to specify the allowable paths in a stratified space. This can be achieved by following the other possible approach to generalize the definitions in Recollections 7.3.1.2 and 7.3.1.3. Namely, one can transfer weak equivalences along a functor defined on stratified maps:

Construction 7.3.1.5. The extended homotopy link $\hat{\mathcal{H}o}\mathrm{Link}(\mathcal{X})$ of a stratified space $\mathcal{X} \in$ **Strat** is the bisimplicial set given by

$$n \mapsto \underline{\mathbf{Strat}}(|\Delta^{[n]}|_s, \mathcal{X}),$$

with the obvious functoriality in n. Note that

$$\begin{aligned}\underline{\mathbf{Strat}}(|\Delta^{[n]}|_s, \mathcal{X}) &= \bigsqcup_{\mathcal{I} \in N(P)_n} \underline{\mathbf{Strat}}_{P_{\mathcal{X}}}(|\Delta^{\mathcal{I}}|_s, \mathcal{X}) \\ &= \bigsqcup_{\mathcal{I} \in \mathrm{sd} P_{\mathcal{X}}} \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \sqcup \bigsqcup_{\mathcal{J} \in N(P_{\mathcal{X}})_n, \text{degenerate}} \underline{\mathbf{Strat}}_{P_{\mathcal{X}}}(|\Delta^{\mathcal{J}}|_s, \mathcal{X}).\end{aligned}$$

In particular, $\hat{\mathcal{H}o}\mathrm{Link}(\mathcal{X})$ contains the data of all homotopy links of $\mathcal{X}$. Furthermore, whenever $\mathcal{J}$ is a flag that degenerates from a regular flag $\mathcal{I}$, then the degeneracy map induces weak equivalences $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \to \underline{\mathbf{Strat}}_{P_{\mathcal{X}}}(|\Delta^{\mathcal{J}}|_s, \mathcal{X})$. In this sense, if we remove homotopically redundant data, then $\hat{\mathcal{H}o}\mathrm{Link}(\mathcal{X})$ stores exactly the data of all homotopy links of $\mathcal{X}$. At the same time, allowing for degenerate simplices in homotopy links makes $\hat{\mathcal{H}o}\mathrm{Link}$ functorial in stratified maps, not only stratum-preserving ones.

Definition 7.3.1.6. A stratified map $f\colon \mathcal{X} \to \mathcal{Y}$ is called a *diagrammatic equivalence*, if it induces weak homotopy equivalences of simplicial sets on all extended homotopy links $\hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{X}) \to \hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{Y})$, for $n \in \mathbb{N}$. We denote by $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}$ the relative category given by **Strat** together with diagrammatic equivalences.

Finally, let us consider the categorical analogue of $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}$.

Recollection 7.3.1.7. In [Nan19], Nand-Lal defined a stratified map $f\colon \mathcal{X} \to \mathcal{Y}$ to be a *categorical equivalence*, if and only if the underlying simplicial map of

$$\mathrm{Sing}_s(f)\colon \mathrm{Sing}_s(\mathcal{X}) \to \mathrm{Sing}_s(\mathcal{Y})$$

is a categorical equivalence of simplicial sets (also called Joyal equivalences). We denote by $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ the relative category given by **Strat** together with categorical equivalences.

As we have already illustrated in the introduction, defining a homotopy theory of stratified spaces in terms of an ∞-categorical localization, especially in terms of maps which are themselves obtained through a localization, comes with an immediate series of questions. To just name a few: Are the stratum-preserving categorical equivalences detected by Sing_s? Do we have any intrinsic description of the mapping spaces in the ∞-categorical localizations? More precisely, how do they relate to the classical mapping spaces of stratified maps given by the simplicial structure on $\underline{\mathbf{Strat}}$? In the same vein of questioning, if we restrict ourselves to classical examples of stratified space, do we obtain the same homotopy theory as is investigated in [Qui88]? And if this is the case, how come [Qui88] has no need for higher homotopy links in his definition of a good class of stratified spaces? The classical way of making localizations of 1-categories more tractable is, of course, the theory of model categories. In [Dou21c], the relative categories $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$, were extended to model structures. However, these model structures have only limited expressive power when it comes to investigating classical examples of stratified spaces, since these are almost never bifibrant (for example, no stratified cone on a non-empty manifold is ever bifibrant). As we have already illustrated in the introduction, the goal of this work is to construct (semi-)model structures for the homotopy theories defined in this section that make as many of such classical examples bifibrant.

7.3.2 Combinatorial models for stratified homotopy theory

One general strategy of constructing model categories is to first construct a model structure in a framework where this can easily be done, i.e., a category of presheaves or a similarly convenient setting, and then to transfer the theory along some adjunction. In this section, we will discuss simplicial analogues to the homotopy theories defined in the last section. We have investigated these theories in detail in [Waa24a], connecting the theories of [Hai23; Dou21a] and constructing combinatorial models for $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}$. Let us begin with the case of $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$.

Recollection 7.3.2.1 ([Dou21a; DW22])**.** The *Douteau-Henriques model structure* on $\underline{\mathrm{s}\mathbf{Strat}}_P$, defined first in [Dou21a], is the Cisinski model structure (see [Cis19, Thm. 2.4.19]) induced by the simplicial cylinder $X \mapsto X \otimes \Delta^1$, with the empty set of anodyne extensions. This defines a combinatorial, cofibrant, simplicial model structure on $\underline{\mathrm{s}\mathbf{Strat}}_P$, which may be characterized as the minimal model structure (in the sense of the smallest possible class of weak equivalences) in which the cofibrations are the monomorphisms and all stratified simplicial homotopy equivalences are weak equivalences. We denote the resulting simplicial model category $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$. $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ is cofibrantly generated by the classes of stratified boundary inclusions and admissible horn inclusions. It follows purely abstractly that weak equivalences between stratified simplicial sets $\mathcal{X}, \mathcal{Y}$ that have the horn filling property with respect to admissible horn inclusions are precisely such stratum-preserving simplicial maps $\mathcal{X} \to \mathcal{Y}$, for which the induced simplicial map of the simplicial homotopy links

$$\mathrm{HoLink}_{\mathcal{I}}(\mathcal{X}) \coloneqq \underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathcal{X}) \to \underline{\mathrm{s}\mathbf{Strat}}_P(\Delta^{\mathcal{I}}, \mathcal{Y}) \eqqcolon \mathrm{HoLink}_{\mathcal{I}}(\mathcal{Y})$$

is a weak homotopy equivalence, for all $\mathcal{I} \in \mathrm{sd}(P)$. More surprisingly, in [DW22] (specifically Corollary 3.6.0.2 in this text) it was shown that this detection criterion holds for all $\mathcal{X}, \mathcal{Y}$ and that no fibrancy assumptions are necessary. Furthermore, mapping $\mathcal{X} \in \mathrm{s}\mathbf{Strat}$ to the simplicial presheaf on $\mathrm{sd}(P)$ given by $\mathcal{I} \mapsto \underline{\mathbf{Strat}}(\Delta^{\mathcal{I}}, \mathcal{X})$ induces a Quillen equivalence

$$\mathbf{Diag}_P \overset{\simeq}{\rightleftharpoons} \mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P : \mathrm{HoLink},$$

where the left-hand side denotes the category of simplicial presheaves, $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ equipped with the injective (or projective) model structure. Even more, this Quillen equivalence creates weak equivalences in both directions (see Eq. (3.2)). In particular, $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ presents the ∞-category of space-valued diagrams indexed over $\mathrm{sd}(P)^{\mathrm{op}}$.

Recollection 7.3.2.2. [Hai23] The *Joyal-Kan model structure* on $\underline{\mathrm{s}\mathbf{Strat}}_P$ is obtained by left Bousfield localizing $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ at the class of stratified inner horn inclusions. Fibrant objects are precisely the stratified simplicial sets $\mathcal{X}$, for which the underlying simplicial set X is a quasi-category and $s_{\mathcal{X}} \colon X \to P_{\mathcal{X}}$ is a conservative functor. It follows that $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ presents the ∞-category of conservative functors from a quasi-category into P, also called *abstract stratified homotopy types* over P. We can explicitly present the equivalence between décollages and abstract stratified homotopy types of [Hai23] in terms of a Quillen equivalence. In Section 5.2.2, we have shown that the adjunction

$$\underline{\mathbf{Diag}}^{\mathrm{d\acute{e}}}_P \rightleftharpoons \underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P : \mathrm{HoLink}$$

induces a Quillen equivalence between $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ and the model structure for décollages, which creates weak equivalences in both directions. This lifts the equivalence between abstract stratified homotopy types and décollages already proven in [BGH18] to the level of model categories.

Recollection 7.3.2.3. [Waa24a] Both the Douteau-Henriques as well as the Joyal-Kan model structures admit a global analogue on $\underline{\mathrm{s}\mathbf{Strat}}$ (named accordingly) obtained by gluing the model structures together along the Grothendieck bifibration $\mathrm{s}\mathbf{Strat} \to P$, using [CM20, Thm

4.4]. We denote by $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ the simplicial model categories with underlying category $\underline{\mathrm{s}\mathbf{Strat}}$, defined by applying [CM20, Thm 4.4] to the forgetful functor

$$P\colon \underline{\mathrm{s}\mathbf{Strat}} \to \mathbf{Pos},$$

with the fiberwise model structures given by $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{d}}$ and $\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{c}}$, for $P \in \mathbf{Pos}$, respectively. In more detail, a stratified simplicial map $f\colon \mathcal{X} \to \mathcal{Y}$ is a cofibration, if and only if the induced map $P(f)_!\mathcal{X} \to \mathcal{Y}$ is a monomorphism, and weak equivalences are precisely such maps for which $P(f)$ is an isomorphism, and $P(f)_!\mathcal{X} \to \mathcal{Y}$ is a weak equivalence in $\underline{\mathrm{s}\mathbf{Strat}}_{P_{\mathcal{Y}}}^{\mathfrak{d}}$ ($\underline{\mathrm{s}\mathbf{Strat}}_{P_{\mathcal{Y}}}^{\mathfrak{c}}$). $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ are cofibrant combinatorial model categories. A set of generating cofibrations for both model categories is given by the set of stratified boundary inclusions $\{\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]} \mid n \in \mathbb{N}\}$, together with the two morphisms

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ \varnothing \arrow[r] & {[0]}, \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ {[0] \sqcup [0]} \arrow[r, hook] & {[1]}. \end{tikzcd} \tag{7.2}$$

Furthermore, in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ acyclic cofibrations are generated by the stratified admissible horn inclusions

$$\begin{tikzcd} \Lambda^n_k \arrow[rr] \arrow[dr] & & \Delta^n \arrow[dl] \\ & {[m]} & \end{tikzcd} \quad , \tag{7.3}$$

for $n, m \in \mathbb{N}$.

Generally, the poset of a stratified simplicial set may contain a lot of redundant elements and relations that do not reflect in the simplicial set itself. For example, $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ does not present a subcategory of $\mathcal{C}\mathbf{at}_\infty$, but instead an ∞-category of categories together with a conservative functor into a poset, and contains a fully faithful copy of the category of posets, given by mapping a poset P to the functor $\varnothing \to P$. If we are looking to obtain a version of a stratified homotopy hypothesis, referring only to layered ∞-categories without a choice of additional data, a further (right) localization is necessary. This uses the fact that any layered ∞-category naturally comes with a functor to the poset of its isomorphism classes, with relations induced by morphisms between the latter.

Recollection 7.3.2.4 (Section 5.3.2)**.** Given a stratified simplicial set $\mathcal{X} \in \mathbf{Strat}$, its refined poset $P_{\mathcal{X}^{\mathfrak{r}}}$ is the poset generated with elements the vertices of $\mathcal{X}$, and a relation $x \leq y$, if and only if there is a path of 1-simplices

$$x = x_0 \leftrightarrow x_1 \leftrightarrow x_2 \leftrightarrow \cdots \leftrightarrow x_n = y$$

where only simplices that are contained within a stratum of $\mathcal{X}$ are allowed to point in direction of x. In other words, $P_{\mathcal{X}^{\mathfrak{r}}}$ is the poset whose elements are the path component of strata of $\mathcal{X}$ and whose relations are generated by exit paths in $\mathcal{X}$. The stratification map of $\mathcal{X}$, $s_{\mathcal{X}}\colon X \to P_{\mathcal{X}}$, factors through $P_{\mathcal{X}^{\mathfrak{r}}}$, inducing a new stratification of X. The resulting stratified simplicial set is called *the refinement of* $\mathcal{X}$ and denoted $\mathcal{X}^{\mathfrak{r}}$, and comes with a natural stratified simplicial map $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$. Stratified simplicial sets for which this map is an isomorphism are called *refined.* We denote by $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}$ ($\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$) the two (simplicial) right Bousfield localizations of respectively $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ and $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ obtained by localizing the refinement maps (the existence of these localizations was proven in Theorem 5.3.2.19). They are respectively called the *diagrammatic and categorical* model structure on $\underline{\mathrm{s}\mathbf{Strat}}$. Both form a combinatorial model category, with cofibrations generated by the class of stratified boundary inclusions $\{\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]} \mid n \in \mathbb{N}\}$, together with the boundary inclusion $\partial\Delta^{[1]} \to \Delta^1$, into the trivially stratified simplex. The fibrant objects of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$, i.e., the refined abstract stratified homotopy types are precisely what [BGH18] call the 0-*connected abstract stratified homotopy types* - such abstract stratified homotopy types which have 0-connected strata and pairwise homotopy links. Forgetting the underlying stratification poset induces an equivalence

$$\mathcal{A}\mathcal{S}\mathbf{trat}^{\mathfrak{r}} \simeq \mathcal{L}\mathbf{ay}_{\infty}$$

between the the homotopy theory of refined abstract stratified homotopy types $\mathcal{A}\mathcal{S}\mathbf{trat}^{\mathfrak{r}}$ and the homotopy theory of layered ∞-categories $\mathcal{L}\mathbf{ay}_{\infty}$ (see [BGH18] and Proposition 5.3.1.8 and Theorem 5.3.3.6).

7.3.3 Equivalences of homotopy theories of stratified objects

Now that we have introduced both topological as well as simplicial homotopy theories of stratified spaces, the obvious question concerning the relation between the two arises. More precisely, what are the homotopy-theoretic properties of the adjunctions $|-|_s \dashv \mathrm{Sing}_s$? We prove that each homotopy theory of stratified topological spaces is, in fact, equivalent to its simplicial counterpart, with the equivalence induced through the adjunctions $|-|_s \dashv \mathrm{Sing}_s$.

Theorem 7.3.3.1 (Partially already found in [DW22; Hai23])**.** *Let $\boldsymbol{R}$ be any of the relative categories of topological stratified spaces of Section 7.3.1 and let* s$\boldsymbol{R}$ *be its simplicial counterpart from Section 7.3.2. Then, the adjunction $|-|_s \dashv \mathrm{Sing}_s$ preserves all weak equivalences in both directions (i.e. induces functors of relative categories), and has unit and counit a weak equivalence (in other words, it is a strict homotopy equivalence of relative categories in the sense of [BK12b]). In particular, the adjunctions induce equivalences of quasi-categories*

$$\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P; \qquad \mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P\,,$$

for each $P \in \mathbf{Pos}$, as well as equivalences of quasi-categories

$$\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}} \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}; \qquad \mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}} \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{c},\mathfrak{p}};$$
$$\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}} \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{d}}; \qquad \mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}} \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{c}}.$$

Before we give a proof, let us remark on the history of these results.

Remark 7.3.3.2. The existence of an equivalence of quasi-categories $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}} \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ was first shown in [Dou21b], in terms of a Quillen equivalence. [Hai23] used the equivalence between abstract stratified homotopy types and décollages, $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P \simeq \mathcal{D}\mathbf{iag}^{\text{dé}}_P$, proven in [BGH18], as well as [Dou21c, Thm. 3], to subsume that $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ was equivalent to a left localization of $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$. If one carefully follows the argument in [Hai23], this localization turns out to be precisely $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$. However, a priori, the resulting equivalence $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P \simeq \mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ is not induced by the adjunction $|-|_s \dashv \mathrm{Sing}_s$, but rather as a composition of equivalences $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P \xrightarrow{\mathcal{H}o\mathrm{Link}} \mathcal{D}\mathbf{iag}^{\text{dé}}_P \xrightarrow{\simeq} \mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$. At this point, it was not yet known that $\mathrm{Sing}_s\colon \mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P$ maps categorical equivalences to weak equivalences in the Joyal-Kan model structure. The strict homotopy equivalence of relative categories version for the cases $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ was first shown in [DW22, Sec. 5] (see Section 3.5 here), albeit it is stated there in slightly different language. As we will see, all other cases can be derived from this one quite readily.

Let us first show that the nomenclature for categorical equivalences in $\mathbf{Strat}_P$ is justified:

Lemma 7.3.3.3. *Let $f\colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ be a stratum-preserving map. Then the following are equivalent:*

1. *f is a categorical equivalence;*
2. *$\mathrm{Sing}_s(f)$ is an equivalence in the Joyal-Kan model structure;*
3. *The underlying simplicial map of $\mathrm{Sing}_s(f)$ is a categorical equivalence.*

Proof. There is a natural isomorphism of functors of 1-categories $\mathrm{HoLink}_{\mathcal{I}} \circ \mathrm{Sing}_s \cong \mathcal{H}o\mathrm{Link}_{\mathcal{I}}$. By definition, f is a categorical equivalence if and only if $\mathcal{H}o\mathrm{Link}(f)$ is an equivalence in the model structure for décollages. Hence, to prove that equivalence of the first two conditions holds, it suffices to see that HoLink creates weak equivalences, which is part of the statement of Theorem 5.2.2.20. The final equivalence is a consequence of [Hai23, Thm. 0.2.2]. □

As an immediate corollary, we obtain:

Corollary 7.3.3.4. *Let $f\colon \mathcal{X} \to \mathcal{Y} \in$ **Strat** be a stratified map. Then the following are equivalent:*

1. *f is a poset-preserving categorical equivalence;*
2. *$\mathrm{Sing}_s(f)$ is an equivalence in the poset-preserving Joyal-Kan model structure;*
3. *f is a categorical equivalence and induces an isomorphism on posets.*

Furthermore, the following equivalence for diagrammatic equivalences holds:

Lemma 7.3.3.5. *A stratified map $f\colon \mathcal{X} \to \mathcal{Y}$ in* **Strat** *is a poset-preserving diagrammatic equivalence if and only if $\mathrm{Sing}_s(f)$ is a weak equivalence in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$.*

Proof. This follows from Proposition 5.3.4.4 together with the natural isomorphism of $\hat{\mathcal{H}o}\mathrm{Link}$ with the composition of its simplicial counterpart with Sing_s. □

Furthermore, we are going to make use of the following lemma:

Lemma 7.3.3.6. *Given two categories with weak equivalences $(\mathbf{C}, W_{\mathbf{C}})$, $(\mathbf{D}, W_{\mathbf{D}})$ suppose that a pair of functors*

$$L\colon \mathbf{C} \rightleftharpoons \mathbf{D}\colon R,$$

together with natural transformations $\varepsilon\colon 1_{\mathbf{C}} \to RL$ and $\eta\colon LR \to 1_{\mathbf{D}}$ define a strict homotopy equivalence of relative categories. Let $W'_{\mathbf{C}}$ be any wide subcategory such that $W_{\mathbf{C}} \subset W'_{\mathbf{C}}$, and such that $(\mathbf{C}, W'_{\mathbf{C}})$ is again a category with weak equivalences. Denote $W'_{\mathbf{D}} = R^{-1}(W'_{\mathbf{C}})$. Then the quadruple $(L, R, \varepsilon, \eta)$ also defines a strict homotopy equivalence between $(\mathbf{C}, W'_{\mathbf{C}})$ and $(\mathbf{D}, W'_{\mathbf{D}})$.

Proof. Clearly, $W_{\mathbf{D}} \subset W'_{\mathbf{D}}$ and $R(W'_{\mathbf{D}}) \subset W'_{\mathbf{C}}$. Really, the only thing one needs to verify is that L remains a functor of relative categories, i.e., maps $W'_{\mathbf{C}}$ into $W'_{\mathbf{D}}$. So let $w\colon X \to Y \in W'_{\mathbf{C}}$. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "w"'] & RL(X) \arrow[d, "LR(w)"] \\ Y \arrow[r, "\varepsilon_Y"'] & RL(Y)\,. \end{tikzcd} \tag{7.4}$$

By the two-out-of-three property for weak equivalences, and the assumption that $\varepsilon_X, \varepsilon_Y \in W_{\mathbf{C}} \subset W'_{\mathbf{C}}$, it follows that w being in $W'_{\mathbf{C}}$ implies that $RL(w) \in W'_{\mathbf{C}}$. As R creates weak equivalences, this is the case if and only if $L(w)$ is in $W'_{\mathbf{D}}$. This finishes the proof. □

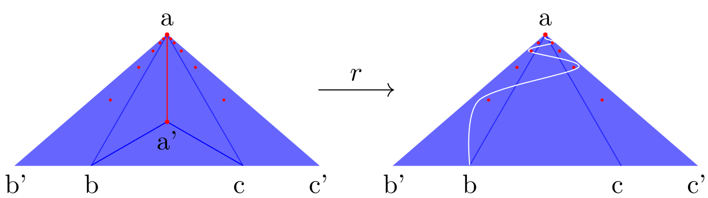
Proof of Theorem 7.3.3.1. Note that all model structures on the simplicial side are obtained in terms of Bousfield localizations from $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$ or $\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ (see Propositions 5.2.1.3 and 5.3.2.26). It follows from Lemma 7.3.3.6 together with Lemmas 7.3.3.3 and 7.3.3.5 and Corollary 7.3.3.4 that it suffices to show the cases of $\boldsymbol{\mathcal{S}}\mathbf{trat}_P^{\mathfrak{d}}$ and $\boldsymbol{\mathcal{S}}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$. These were proven in the proof of [DW22, Thm. 5.1] (which is Theorem 3.5.1.1 in this text). Specifically, see Lemmas 3.5.1.2 and 3.5.1.3 and their proofs. □

7.4 (Semi-)model categories of stratified spaces

The difficulty with the results in Theorem 7.3.3.1 is their interpretability. Without some additional structure on the side of **Strat**, the only real information that we have is that the homotopy theories we defined are equivalent to the combinatorial world we transferred them from. We thus encounter the difficulty of interpreting what these results actually say about classical examples of stratified spaces. That is, we need to determine how the localizations of Theorem 7.3.3.1 relate to stratified maps, stratified homotopies or more generally the stratified simplicial mapping spaces of $\underline{\mathbf{Strat}}$. The usual approach to relating a localization to a simplicial category is the language of simplicial model categories. In [Dou21c], such simplicial model structures were suggested for $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$. The difficulty with these model structures, however, is that classically relevant examples such as (for example) Whitney stratified spaces are not bifibrant in the latter (in fact, no stratified cone on a manifold is cofibrant). Hence, the expressive power of these model structures when it comes to investigating mapping spaces for such spaces is limited. In [Lur17, A.5] or [Nan19, Thm. 8.1.2.6.], it was shown that for essentially all stratified spaces of classical relevance $\mathcal{X}$ the stratified singular simplicial set $\mathrm{Sing}_s(\mathcal{X})$ is fibrant in $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$, and hence in any of the model structures for stratified simplicial sets. Furthermore, at least Whitney stratified spaces or PL-pseudo manifolds admit triangulations compatible with their stratifications ([Gor78]), and are thus stratum-preserving homeomorphic to the realization of a stratified simplicial set. This suggests the approach of transferring the simplicial model structures on stratified simplicial sets to the topological setting. Constructing such a model structure in the case of $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ was one of the stated goals and conjectures of [Nan19], which was ultimately left as a conjecture. In fact, it turns out that the model structures in the combinatorial setting do not transfer to the topological framework, which was first shown in [DW22, A.5] (see Section 3.A in this text). We repeat the result here, in slightly different form and with a more detailed proof, as it illustrates well the difficulties which may arise in the topological but not in the combinatorial world.

Proposition 7.4.0.1 ([DW22, Prop. A.1, Rem. A.4, Rem. A.5])**.** *There does not exist a model structure on* **Strat***, for which all poset-preserving diagrammatic equivalences are weak equivalences, and the inclusion* $|\Lambda^{[3]}_1 \hookrightarrow \Delta^{[3]}|_s$ *is a cofibration. The analogous result for* $\mathbf{Strat}_P$*, for* P *non-discrete, also holds.*

Proof. Consider the stratum-preserving map over $\{p < q\}$ illustrated in the following picture:



The picture shows two stratifications over $\{p < q\}$ of the realization of simplicial complexes (ignoring the white path from a to b for now), embedded in $\mathbb{R}^2$. Denote, from right to left, the corresponding stratified spaces by $\mathcal{X}$ and $\mathcal{Y}$. The p-stratum is marked in red. In $\mathcal{X}$ the p-stratum consists of a vertical line and two sequences, given in barycentric coordinates with respect to a, b, b' and a, c, c' by $(1 - \frac{1}{2^n}, \frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})$. In $\mathcal{Y}$, the vertical line is replaced by a single point. Denote these sequences, respectively, by b_n and c_n. The map r is given by convexly extending the map which keeps all vertices besides a' the same and maps the latter to a. This map is not a categorical equivalence. Indeed, consider the induced map of simplicial sets

$$\mathrm{Sing}_s(\mathcal{X}) \to \mathrm{Sing}_s(\mathcal{Y})$$

and the induced map of hom-sets

$$\mathrm{hoSing}_s(\mathcal{X})(a, b) \xrightarrow{r_*} \mathrm{hoSing}_s(\mathcal{Y})(a, b)$$

in the associated homotopy categories. First, note that $\mathrm{Sing}_s(\mathcal{Y})$ is, in fact, a quasi-category. This can be seen, for example, from the fact that every non-trivially stratified horn $|\Lambda^{\mathcal{J}}|_s \to \mathcal{X}$, $\mathcal{J} = [p \leq p \leq \cdots \leq q]$ is constant on the p-stratum. It follows that it suffices to show that $\mathcal{Y}$ admits extensions with respect to $|\Lambda^{\mathcal{J}}/\Delta^{\mathcal{J}_p}|_s \hookrightarrow |\Delta^{\mathcal{J}}/\Delta^{\mathcal{J}_p}|_s$, which always admits a stratified retraction. Consequently, elements of $\mathrm{Sing}_s(\mathcal{Y})(a,b)$ are given by stratified homotopy classes (rel boundary) of exit paths from a to b. It turns out that r_* is not surjective: Consider a path γ from a to b that (described as starting from b) ascends monotonously in height and passes to the left of b_1, to the right of c_2, to the left of b_3 etc., as illustrated in the right picture. The class of γ does not lie in the image of r_*. Roughly speaking, this is due to the fact that no exit path in $\mathcal{X}$ can oscillate around the points of both sequences (b_n) and (c_n). For a formal proof, see Section 7.C. Next, let $\mathcal{J} = [p \leq p < q \leq q]$ and $\mathcal{J}' = [p < q \leq q]$ consider the following diagram of pushout squares

$$\begin{tikzcd} |\Lambda^{[3]}_1|_s \arrow[r, hook] \arrow[d] & |\Delta^{[3]}|_s \arrow[r, "|s^1|_s"] \arrow[d] & |\Delta^{[2]}|_s \arrow[d] \\ |\Lambda^{\mathcal{J}}_1|_s \arrow[r, hook] \arrow[d, hook] & |\Delta^{\mathcal{J}}|_s \arrow[r] \arrow[d, hook] & |\Delta^{\mathcal{I}}|_s \arrow[d, hook] \\ \mathcal{X} \arrow[r] & \mathcal{Z} \arrow[r, "r'"] & \mathcal{Y}, \end{tikzcd} \tag{7.5}$$

with the upper verticals induces by the identity on spaces and the flags $\mathcal{J}$ and $\mathcal{J}'$, the lower left vertical given by affinely extending the map of vertices $e_0 \mapsto a'$, $e_1 \mapsto a$ and $e_2 \mapsto b$ and $e_3 \mapsto c$, and with s^1 the degeneracy map collapsing the edge $\{1,2\}$ to $\{1\}$. The bottom composition is precisely r, by construction of the latter. Now, the middle left horizontal is the realization of a diagrammatic equivalence and hence a diagrammatic equivalence. It follows that if a model structure as claimed in the statement of the proposition existed, then the left lower horizontal would be an acyclic cofibration. Now, suppose that the lower right vertical r' was also a categorical equivalence. Then it would follow that r is also a categorical equivalence, in contradiction to what we have just shown. In fact, the lower left vertical is even a stratum-preserving homotopy equivalence, and hence a categorical equivalence. To see this, note that r' admits a section, induced by the pushout

$$\begin{tikzcd} |\Delta^{\mathcal{I}}|_s \arrow[r, hook, "|d^0|_s"] \arrow[d] & |\Delta^{\mathcal{J}}|_s \arrow[d] \\ \mathcal{Y} \arrow[r, "s'"] & \mathcal{Z}. \end{tikzcd} \tag{7.6}$$

$|d_0|_s$ is even a stratum-preserving strong deformation retract (that is, admits a deformation retraction which is also stratum-preserving). It follows that s' is also a stratum-preserving deformation retraction, and in particular a stratum-preserving homotopy equivalence. Since r' is a retraction of s', it follows that r' is a stratified homotopy equivalence. For the case of a fixed poset P, construct $\mathcal{X}$ and $\mathcal{Y}$ over any trivial relation $p < q$ in P, and simply repeat the argument omitting the first line in Diagram (7.5). $\square$

7.4.1 A transfer lemma for cofibrantly generated semi-model structures

If one takes a precise look at the counterexample in Proposition 7.4.0.1, one will notice that it involves a pushout of an acyclic cofibration along a stratified map whose target is somewhat pathologically stratified. In particular, we cannot expect the target to be cofibrant. Hence, what one may nevertheless hope to obtain is a left semi-model structure. Semi-model structures provide a weaker version of model structures in which one can only expect pushouts of acyclic cofibrations to remain acyclic if all objects involved are cofibrant. Despite having slightly weaker axioms, for almost all intents and purposes, semi-model categories (cofibrantly generated ones to be more precise) are just as useful as regular model categories. Essentially, every theorem about model categories admits an analogue in the world of semi-model categories, as long as one takes care that sources of morphisms often need to be assumed to be cofibrant (see, for example, [BW24] for an overview of some results). We will not go through the ordeal of reproducing every necessary result from the model category world in the semi-model category world here. Instead, when necessary, we will often reference a proof in the model-categorical setting and explain what needs to be adapted in the setting of semi-model categories. Since we only ever make use of the cofibrantly generated scenario in this paper, and the latter tends to be significantly more well-behaved, we are going to use the following definition. Recall first the notion of weak factorization system (for example, from [Rie14, Ch. 11])

Definition 7.4.1.1. Let $\mathbf{M}$ be a bicomplete category. A cofibrantly generated left semi-model structure on $\mathbf{M}$ is the data of three classes (C, W, F), called respectively cofibrations, weak equivalences, and fibrations, such that:

1. W contains all isomorphisms and is closed under two-out-of-three and retracts;
2. $(C, W \cap F)$ is a (functorial) weak factorization system;
3. F consists exactly of those morphisms which have the right lifting property with respect to all morphisms in $W \cap C$ with cofibrant source;
4. Every morphism with cofibrant source factors (functorially) into a morphism in $W \cap C$, followed by a fibration.

Furthermore, we assume that there exist sets of cofibrations I and of acyclic cofibrations with cofibrant source J, such that F is the class of morphisms that have the right lifting property with respect to J, and $F \cap W$ is precisely the class of morphisms that have the right lifting property with respect to I. A (cofibrantly generated, left) semi-model category is the data of a complete and cocomplete category $\mathbf{C}$, together with a (cofibrantly generated) model structure on $\mathbf{C}$. Since we are only concerned with left semi-model categories in this work, we will just speak of semi-model categories when we mean left semi-model category.

It is a routine exercise to show that since we assumed cofibrant generation, our definition is equivalent to [BW24, Def. 2.1]. Note that this definition is slightly stronger than the one used, for example, in [Fre09, p. 12.1.1]. We use the following definition of simpliciality:

Definition 7.4.1.2. Let $\mathbf{M}$ be a bi-complete simplicial category, admitting powers and tensors, with the underlying 1-category equipped with a semi-model structure. We say that $\mathbf{M}$ is a *simplicial semi-model* category, if for every cofibration $i: A \hookrightarrow B$ in the Kan-Quillen model structure on simplicial sets (that is, for every monomorphism), and for every fibration $f: X \to Y$ in $\mathbf{M}$, the induced morphism

$$X^B \to X^A \times_{Y^A} Y^B$$

is a fibration, and furthermore a weak equivalence if i or f is a weak equivalence.

Remark 7.4.1.3. Note that this is a slightly different definition of simplicial semi-model category than the one in [BW24, Def. 2.3] or [GH05, Def. 1.1.8]. Our definition is a priori stronger than the ones in [BW24; GH05]. The latter only implies that when i is a cofibration and f a fibration, then $Y^B \to X^B \times_{X^A} Y^A$ has the right lifting property with respect to all acyclic cofibrations with cofibrant source. However, since in our cases all relevant semi-model categories are generated by (acyclic) cofibrations with cofibrant source, for the semi-model categories we are concerned with, all of these definitions agree.

Definition 7.4.1.4. Let $\mathbf{D}$ be a bicomplete category. Consider an adjunction

$$L: \mathbf{C} \rightleftharpoons \mathbf{D}: R$$

and assume that $\mathbf{C}$ is a model category. We say that the *right transferred semi-model structure on* $\mathbf{D}$ *along* R *exists*, if the following classes form a semi-model structure on $\mathbf{C}$:

1. Fibrations (weak equivalences) in $\mathbf{D}$ are precisely such $f \in \mathbf{D}$, for which $R(f)$ is a fibration (weak equivalence).

2. Cofibrations in $\mathbf{D}$ are those morphisms that have the left lifting property with respect to all acyclic fibrations (i.e. morphisms which are both a weak equivalence and a fibration).

We then call this structure the *right transferred semi-model structure* on $\mathbf{D}$ (with respect to R).

Given a class of morphisms I in a bicomplete category $\mathbf{C}$, we denote by $\mathrm{cell}(I)$ the class of relative cell complexes with respect to I (see [Hir03, Def. 10.5.8]). Similarly to the scenario of regular model categories (see, for example, [nLa23] for a proof), one obtains:

Lemma 7.4.1.5. *In the situation of Definition 7.4.1.2, suppose that $\mathbf{C}$ is cofibrantly generated by cofibrations I and acyclic cofibrations with cofibrant source J. Suppose furthermore that:*

1. *Every morphism in $\mathbf{D}$ factors into an element of $\mathrm{cell}L(I)$, followed by an acyclic fibration.*

2. *Every morphism in $\mathbf{D}$ with cofibrant source factors into an element of $\mathrm{cell}L(J)$, followed by a fibration.*

3. *Every element of $\mathrm{cell}L(J)$ with cofibrant source is a weak equivalence.*

Then the right transferred model structure on $\mathbf{D}$ exists and is cofibrantly generated by $L(I)$ and $L(J)$.

Let us now consider the following transfer lemma for semi-model structures. We note that this result (omitting the simplicial part) can also be obtained as a special case of [WY18, Thm. 2.2.2] (see also [Fre10, Thm. 3.3]). We provide a proof here, purely in order for the reader not to have to translate back and forth between the slightly different setup in [WY18].

Lemma 7.4.1.6. *Let $\mathbf{D}$ be a complete and cocomplete category. Consider an adjunction*

$$L\colon \mathbf{C} \rightleftharpoons \mathbf{D}\colon R$$

and assume that $\mathbf{C}$ is a cofibrantly generated model category, with cofibrations generated by a set I and acyclic cofibrations generated by a set of morphisms with cofibrant source J, arrows in which have κ-small source, for some cardinal κ. Assume furthermore the following:

(i) R preserves transfinite compositions of morphisms in $\mathrm{cell}L(I)$ and $\mathrm{cell}L(J)$ respectively;

(ii) Weak equivalences in $\mathbf{C}$ are stable under transfinite composition;

(iii) For any a pushout diagram in $\mathbf{D}$

$$\begin{tikzcd} L(A) \arrow[r, "L(j)"] \arrow[d] & L(B) \arrow[d] \\ X \arrow[r, "j'"] & Y \end{tikzcd} \tag{7.7}$$

with $j \in J$ and $X \in \mathrm{cell}(L(I))$ an absolute cell complex, the map j' is a weak equivalence.

Then the right transferred semi-model structure on $\mathbf{D}$ exists, and it is furthermore cofibrantly generated by $L(I)$ and $L(J)$. Even more, if $L \dashv R$ is an adjunction of simplicial categories, with $\mathbf{D}$ admitting a simplicial tensoring and powering, and the model structure on $\mathbf{C}$ is simplicial, then so is the induced semi-model structure on $\mathbf{D}$.

Proof. It follows from Assumption (i) and the assumption on small sources that the classes $L(I)$ and $L(J)$ permit the small object argument. In fact, it follows under the adjunction $L \dashv R$ that the sources of $L(I)$ and $L(J)$ are κ small, with respect to cell$L(I)$ and cell$L(J)$. Consequently, the only remaining requirement of Lemma 7.4.1.5 is that every element $j' \in \mathrm{cell}L(J)$ with cofibrant source is a weak equivalence. It follows by Assumptions (i) and (ii) that we only need to consider the case of a pushout diagram

$$\begin{tikzcd} L(A) \arrow[r, "L(j)"] \arrow[d] & L(B) \arrow[d] \\ X \arrow[r, "j'"] & Y \end{tikzcd} \tag{7.8}$$

with $j \in J$ and X cofibrant. Furthermore, since X is cofibrant, it is the retract of an absolute $L(I)$ cell complex X', through morphisms $X \overset{i}{\hookrightarrow} X'$ and $X' \overset{r}{\to} X$, with $r \circ i = 1_X$. We may extend the square in question to a diagram of pushout squares

$$\begin{tikzcd} L(A) \arrow[r] \arrow[d] & L(B) \arrow[d] \\ X \arrow[r, "j'"] \arrow[d, "i"] \arrow[dd, bend right, "1_X"'] & Y \arrow[d] \arrow[dd, bend left, "1_Y"] \\ X' \arrow[r, "j''"] \arrow[d, "r"] & Y' \arrow[d] \\ X \arrow[r, "j'"] & Y\,. \end{tikzcd} \tag{7.9}$$

In particular j' is a retract of j'', and it suffices to show that the latter is a weak equivalence. By the composability of pushout squares, it follows that we have thus reduced to the case where X is an absolute $L(I)$-cell complex. This case is precisely the content of Assumption (iii). Simpliciality follows immediately by using that

$$R(X^B \to X^A \times_{Y^A} Y^B) \cong \big(R(X)^B \to R(X)^A \times_{R(Y)^A} R(Y)^B\big),$$

and the explicit definition of (acyclic) fibrations in $\mathbf{D}$ in terms of R. □

7.4.2 Semi-model structures on Strat

To transfer the model structures from the combinatorial world to $\underline{\mathbf{Strat}}_P$ and $\underline{\mathbf{Strat}}$, we now need to verify the assumptions of Lemma 7.4.1.6. Most of these will turn out to be fairly standard abstract arguments. The difficult part is showing Assumption (iii). This requires an in-depth study of the behavior of homotopy links of absolute cell complexes, with respect to realizations of stratified boundary inclusions, which was performed in [Waa24b] (Chapter 6 in this text).

Definition 7.4.2.1. Denote $I \coloneqq \{|\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P\} \subset \mathbf{Strat}_P$. Absolute cell complexes with respect to I are called *cellularly stratified spaces.* If $\mathcal{A}$ and $\mathcal{X}$ are cellularly stratified spaces, then we call a stratum-preserving map $f\colon \mathcal{A} \hookrightarrow \mathcal{X}$ which is a relative I-cell complex an *inclusion of cellularly stratified spaces.*

The following result is the decisive argument to apply Lemma 7.4.1.6 to transfer the model structure on combinatorially stratified objects to the stratified framework. It is a corollary of Theorem HB, which is one of the main result of [Waa24b].

Lemma 7.4.2.2. *Consider a pushout diagram in* $\mathbf{Strat}$*,*

$$\begin{tikzcd} |\mathcal{A}|_s \arrow[r, "|i|_s"] \arrow[d] & |\mathcal{B}|_s \arrow[d] \\ \mathcal{X} \arrow[r, "i'"] & \mathcal{Y} \end{tikzcd} \tag{7.10}$$

with $\mathcal{X}$ cellularly stratified. Suppose that i is an acyclic cofibration in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$. *Then, i' is a weak equivalence in* $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$*, i.e., an isomorphism in* $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$*. The analogous result holds for any of the model categories of stratified simplicial sets and their corresponding topological ∞-category in Notation 7.4.2.5.*

Before we give a proof, note that Lemma 7.4.2.2 can be taken as a statement about certain pushout diagrams of stratified spaces being homotopy pushout, i.e. pushout in the associated ∞-category obtained by inverting weak equivalences. This already shows that the existence of semi-model structures is closely related with Question Q(5).

Proof. Let us begin with the case over a fixed poset P, and all morphisms on posets given by the identity. We claim that the diagram in the statement of the proposition is a pushout diagram in the quasi-categories $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$. We already know that $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ is a left-localization of $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ (by Theorem 7.3.3.1 and the fact that $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}_P$ is a left Bousfield localization of $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$, together with [Lur09, Prop. 5.2.4.6]) and in particular that the canonical functor $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \to \mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$ is left adjoint. Therefore, by [Lur09, p. 5.2.3.5], which states that left-adjoint functors preserve colimits, it suffices to show the case of $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$. Recall from Recollection 7.3.2.1 that taking homotopy links induces equivalences between $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ and $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$. Thus, it suffices to show that the diagram becomes pushout after applying homotopy links. Now, by [Lur09, Cor. 5.1.2.3] a diagram in a functor quasi-category is pushout if and only if it is so after evaluating at every point. Thus, it suffices to show that the diagram in $\mathcal{S}\mathbf{paces}$

$$\begin{tikzcd} \mathcal{H}o\mathrm{Link}_{\mathcal{I}}|\mathcal{A}|_s \arrow[r] \arrow[d] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}|\mathcal{B}|_s \arrow[d] \\ \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{X} \arrow[r] & \mathcal{H}o\mathrm{Link}_{\mathcal{I}}\mathcal{Y} \end{tikzcd} \tag{7.11}$$

is pushout. Now, the quasi-category $\mathcal{S}\mathbf{paces}$ is given by the homotopy coherent nerve of $\underline{\mathbf{Top}}^{o}$. Theorem HB states that the diagram in $\underline{\mathbf{Top}}$ corresponding to Diagram (7.11) is a homotopy pushout square. Finally, by [Lur09, Thm. 4.2.4.1.], this is equivalent to Diagram (7.11) being a pushout diagram in $\mathcal{S}\mathbf{paces}$.

We have established that the diagram in the statement of the lemma is pushout in $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ and $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$. Consequently, by Theorem 7.3.3.1, so is its image under Sing_s in $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ ($\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$). By assumption, i defines an isomorphism in $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ ($\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$). It follows, again by Theorem 7.3.3.1, that $\mathrm{Sing}_s(|i|_s)$ also defines an isomorphism in $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ ($\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$). Consequently, since pushouts preserve isomorphisms, it follows that $\mathrm{Sing}_s(i')$ is an isomorphism in the quasi-category $\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$ ($\mathrm{s}\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_P$), and thus a weak equivalence in $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$ ($\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}_P$) (by [Hir03, Thm. 8.3.10.] and the fact that the homotopy category functor from quasi-categories to 1-categories commutes with localization). It follows that i is a weak equivalence, as was to be shown.

Next, let us cover the poset-preserving cases in $\mathrm{s}\mathbf{Strat}$. Note that i induces an isomorphism on posets. By pushing forward with $P_{\mathcal{A}} \to P_{\mathcal{X}}$, and using the fact that this preserves realizations of acyclic cofibrations (by Proposition 7.4.2.8), we may assume that the whole diagram lies in $\mathbf{Strat}_{P_{\mathcal{X}}}$. Hence, this case follows from the previous cases. Finally, for the non-poset preserving cases, note that the model structure $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}$ ($\mathrm{s}\mathbf{Strat}^{\mathfrak{c}}$) on $\mathrm{s}\mathbf{Strat}$ has the same acyclic cofibrations as $\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ ($\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$) as it is obtained via right Bousfield localization. Hence, this case follows from the case $\mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ ($\mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$). ☐

We may now prove the following.

Corollary 7.4.2.3. *Let $\underline{\mathbf{cS}}$ be any of the simplicial model categories of stratified simplicial sets listed in Section 7.3.2. Let $\underline{\mathbf{S}}$ be correspondingly the simplicial category of topological stratified spaces $\underline{\mathbf{Strat}}_P$ or $\underline{\mathbf{Strat}}$. Then, along the adjunction*

$$|-|_s\colon \underline{\mathbf{cS}} \rightleftarrows \underline{\mathbf{S}}\colon \mathrm{Sing}_s,$$

the simplicial model structure on $\underline{\mathbf{cS}}$ right-transfers to a cofibrantly generated simplicial semi-model structure on $\underline{\mathbf{S}}$.

Proof. We verify the assumptions of Lemma 7.4.1.6. Note first that $\underline{\mathbf{cS}}$ is a cofibrantly generated simplicial model category (with acyclic generators with cofibrant source) as we have recalled in Section 7.3.2. Assumption (i) follows from the classical fact that any compactum in a relative cell complex only intersects finitely many cells (see, for example, [Hir03, Prop. 10.7.4]). Assumption (ii) was verified in Propositions 5.2.1.6 and 5.3.1.5 and Lemma 5.3.2.23. Finally, Assumption (iii) follows from Lemma 7.4.2.2. □

Remark 7.4.2.4. We note that the analogous proof of Corollary 7.4.2.3 also applies to the category of surjectively stratified spaces, as investigated in [Nan19], $\underline{\mathbf{Strat}}^s$, and one may replace $\underline{\mathrm{s}\mathbf{Strat}}$ by **sSet** with the Joyal-model structure (omitting the simpliciality statement). Indeed, Lemma 7.4.2.2 also applies to surjectively stratified spaces and realizations of Joyal acyclic cofibrations, using Proposition 7.A.0.2 and that surjectively stratified spaces form a coreflective subcategory of $\underline{\mathbf{Strat}}$. In particular, this also gives an affirmative answer to the semi-model structure conjectured in [Nan19, Subsec. 8.4.1]. As a consequence of Proposition 7.A.0.2, the resulting semi-model category is Quillen equivalent to the one on all stratified spaces $\underline{\mathbf{Strat}}$ transferred from $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{c}}$. We also note that the alternative method suggested for the proof in [Nan19] cannot succeed, as [Nan19, Conjecture 2] is false (realizations of fibrant simplicial sets are generally not fibrant). We will provide a counterexample in future work.

Notation 7.4.2.5. We denote by $\underline{\mathbf{Strat}}^{\mathfrak{d}}_P$ ($\underline{\mathbf{Strat}}^{\mathfrak{c}}_P$), $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ ($\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$) and $\underline{\mathbf{Strat}}^{\mathfrak{d}}$ ($\underline{\mathbf{Strat}}^{\mathfrak{c}}$), respectively, the corresponding simplicial semi-model categories whose existence is guaranteed by Corollary 7.4.2.3. We, respectively, call the corresponding semi-model structures the *diagrammatic (categorical), poset-preserving diagrammatic (poset-preserving categorical) and diagrammatic (categorical)* model structures.

We use the remainder of the section to make Corollary 7.4.2.3 more explicit by restricting it to its special cases. From Recollections 7.3.2.1 and 7.3.2.2 and Corollary 7.4.2.3 we obtain:

Theorem 7.4.2.6. *Let $P \in$ **Pos**. The simplicial category $\underline{\mathbf{Strat}}_P$ admits the structures of cofibrantly generated simplicial semi-model categories, $\underline{\mathbf{Strat}}^{\mathfrak{d}}_P$ and $\underline{\mathbf{Strat}}^{\mathfrak{c}}_P$ - called the diagrammatic and categorical model structure - defined by the following classes:*

1. *Cofibrations are generated by the set of stratified boundary inclusions*
$$\{|\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P\}.$$
2. *Weak equivalences are the stratum-preserving diagrammatic equivalences or respectively the stratum-preserving categorical equivalences of P-stratified spaces.*
3. *Fibrations are the stratum-preserving maps that have the right lifting property with respect to all acyclic cofibrations with cofibrant source. In $\underline{\mathbf{Strat}}^{\mathfrak{d}}_P$ they are equivalently characterized by having the right lifting property with respect to all realizations of admissible stratified horn inclusions $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$, $\mathcal{J} \in \Delta_P$.*

Next, let us consider the global version of $\underline{\mathbf{Strat}}_P$, allowing for varying stratification posets. From Recollection 7.3.2.1 and Corollary 7.4.2.3 we obtain the following two results:

Theorem 7.4.2.7. *The simplicial category $\underline{\mathbf{Strat}}$ admits the structures of cofibrantly generated simplicial semi-model categories, $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ and $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ - called the poset-preserving diagrammatic and poset-preserving categorical model structure - with the following classes:*

1. *Cofibrations are generated by the set of stratified boundary inclusions*
$$\{|\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]}|_s \mid n \in \mathbb{N}\},$$
together with the maps of empty stratified spaces over $[0] \sqcup [0] \hookrightarrow \{0 < 1\}$ and $\varnothing \hookrightarrow [0]$.
2. *Weak equivalences are the poset-preserving diagrammatic equivalences or, respectively, the poset-preserving categorical equivalences of stratified spaces.*

3. *Fibrations are the stratum-preserving maps that have the right lifting property with respect to all acyclic cofibrations with cofibrant source. In* $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ *they are equivalently characterized by having the right lifting property with respect to all realizations of admissible stratified horn inclusions* $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$*,* $\mathcal{J} \in \Delta_{[m]}$*, for* $m \in \mathbb{N}$*.*

Similarly, to the case of stratified simplicial sets, the global diagrammatic and categorical model structures can essentially be understood entirely from the ones over a fixed poset and the functoriality of base changes along the poset. This is due to the fact that they may again be interpreted as being pieced together from the local pieces along a cartesian bifibration:

Proposition 7.4.2.8. *The model structure on* $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ *(* $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ *) is the unique semi-model structure on* $\mathbf{Strat}$ *pieced together (in the sense of [CM20], using notation from there)*[5] *from the diagrammatic (categorical) model structures on the fibers* $\mathbf{Strat}_P$ *of the Grothendieck bifibration*

$$P\colon \mathbf{Strat} \to \mathbf{Pos}.$$

In particular, a stratified map $f\colon \mathcal{X} \to \mathcal{Y}$ *is a fibration if and only if the canonical map* $f^{\triangleleft}\colon \mathcal{X} \to P(f)^*\mathcal{Y}$ *is a fibration, and* f *is a cofibration if and only if* $f_{\triangleright}\colon P(f)_!\mathcal{X} \to \mathcal{Y}$ *is a cofibration.*

Proof. We cover the case of $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$, the case of $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ is completely analogous. Consider the forgetful functor $P\colon \mathbf{Strat} \to \mathbf{Pos}$, mapping a stratified space $\mathcal{X}$ to its stratification poset $P_{\mathcal{X}}$. It is a Grothendieck bifibration ([CM20]), with the left action given by $f \mapsto f_!$ and the right action given by $g \mapsto g^*$. If we equip $\mathbf{Pos}$ with the trivial model structure, with all morphisms bifibrations and weak equivalences given by isomorphisms, then $P\colon \mathbf{Strat} \to \mathbf{Pos}$ tautologically forms a Quillen bifibration in the sense of [CM20] (replacing model categories by cofibrantly generated semi-model categories). We obtain two commutative diagrams (one for each horizontal) with diagonals Quillen bifibrations

$$\begin{tikzcd}
\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}} \arrow[rr, shift left, "\simeq"] \arrow[dr] & & \mathrm{s}\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}} \arrow[ll, shift left] \arrow[dl] \\
& \mathbf{Pos} &
\end{tikzcd} \quad , \tag{7.12}$$

where the right adjoint horizontal preserves the right action, and the left adjoint the left actions. The fibers of the right hand vertical are the model categories $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$ (see the construction of the global model categories in Section 5.3.1). Since the semi-model structure on $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ is transferred along the horizontal adjunction, it follows that its restriction to the fibers of the left vertical $\mathbf{Strat}_P$, are precisely the transfers of the model structure on $\mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P$ to $\mathbf{Strat}_P$, which is $\mathbf{Strat}^{\mathfrak{d}}_P$. Using the obvious cofibrantly generated semi-model categorical version of [CM20, Prop. 3.4, 3.5] (see also [BDW23]), which is proven identically, the characterization of cofibrations and fibrations follows. ◻

Finally, let us state the explicit results for the refined framework. We also are going to need the following construction.

Construction 7.4.2.9. Consider the pushout diagram in **Strat**

$$\begin{tikzcd}
|\partial\Delta^{[1]}|_s \arrow[r, hook] \arrow[d, hook, "t"] & |\Delta^{[1]}|_s \arrow[d, hook] \\
|\Delta^{[1]}|_s \arrow[r, hook] & S^1
\end{tikzcd} , \tag{7.13}$$

where the vertical t is given by the opposite inclusion of the boundary points of the stratified interval. The pushout is a trivially stratified S^1. In particular, the diagonal of this diagram - which is the inclusion of two points lying over separate strata into a trivially stratified S^1 - is a cofibration in the refined model structures. The realization of the stratified boundary inclusion $\partial\Delta^{[1]} \hookrightarrow \Delta^1$, where Δ^1 is trivially stratified, is a retract of the diagonal of this diagram.

[5][CM20] covers the case of model structures. For a generalization to semi-model structures see [BDW23].

The following are a consequence of Theorem 5.3.2.19, which we recalled in Recollection 7.3.2.4, and Corollary 7.4.2.3. Note that, by Construction 7.4.2.9, the topological framework needs one less generator of cofibrations than the combinatorial framework.

Theorem 7.4.2.10. *The simplicial category $\underline{\mathbf{Strat}}$ admits the structures of cofibrantly generated simplicial semi-model categories, $\underline{\mathbf{Strat}}^{\mathfrak{d}}$ and $\underline{\mathbf{Strat}}^{\mathfrak{c}}$ - called the diagrammatic and categorical model structures - with the following classes:*

1. *Cofibrations are generated by the set of stratified boundary inclusions*
$$\{|\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]}|_s \mid n \in \mathbb{N}\}.$$
2. *Weak equivalences are the diagrammatic equivalences or respectively the categorical equivalences of stratified spaces.*
3. *Fibrations are the stratum-preserving maps that have the right lifting property with respect to all acyclic cofibrations with cofibrant source. In $\underline{\mathbf{Strat}}^{\mathfrak{d}}$, they are equivalently characterized by having the right lifting property with respect to all realizations of admissible stratified horn inclusions $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$, $\mathcal{J} \in \Delta_{[m]}$, for all $m \in \mathbb{N}$.*

Finally, we may summarize the connections between the various semi-model categories of stratified objects in the following result, which follows by Propositions 5.2.1.3, 5.3.1.11, 5.3.2.24 and 5.3.2.26 and Theorem 5.3.2.19 and Proposition 7.5.3.7 and Theorems 7.3.3.1 and 7.4.2.10. Here, by a Quillen equivalence of (cofibrantly generated left) semi-model categories, we mean an adjunction fulfilling any (or equivalently all) of the classical characterizations for model categories (see, for example, [nLa25f]). In the definition of a Quillen adjunction of cofibrantly generated left semi-model categories (see [nLa25e]), one only needs to take care that the left adjoint generally only is required to preserve acyclic cofibrations with cofibrant source.

Theorem 7.4.2.11. *The various simplicial semi-model categories discussed in this section fit into a diagram*

$$\begin{tikzcd}
\underline{s\mathbf{Strat}}^{\mathfrak{d}}_P \arrow[rr, shift left, "\simeq"] \arrow[dd, hook] \arrow[dr, Rightarrow] & & \underline{\mathbf{Strat}}^{\mathfrak{d}}_P \arrow[ll, shift left] \arrow[dd, hook] \arrow[dr, Rightarrow] & \\
& \underline{s\mathbf{Strat}}^{\mathfrak{c}}_P \arrow[rr, shift left, "\simeq"] \arrow[dd, hook] & & \underline{\mathbf{Strat}}^{\mathfrak{c}}_P \arrow[ll, shift left] \arrow[dd, hook] \\
\underline{s\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}} \arrow[rr, shift left, "\simeq"] \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & & \underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}} \arrow[ll, shift left] \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & \\
& \underline{s\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}} \arrow[rr, shift left, "\simeq"] \arrow[dd, Rightarrow] & & \underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}} \arrow[ll, shift left] \arrow[dd, Rightarrow] \\
\underline{s\mathbf{Strat}}^{\mathfrak{d}} \arrow[rr, shift left, "\simeq"] \arrow[dr, Rightarrow] & & \underline{\mathbf{Strat}}^{\mathfrak{d}} \arrow[ll, shift left] \arrow[dr, Rightarrow] & \\
& \underline{s\mathbf{Strat}}^{\mathfrak{c}} \arrow[rr, shift left, "\simeq"] & & \underline{\mathbf{Strat}}^{\mathfrak{c}} \arrow[ll, shift left]
\end{tikzcd} \tag{7.14}$$

of simplicial functors, with all horizontals induced by the adjunction $|-|_s \dashv \mathrm{Sing}_s$, and all remaining functors given by inclusions or the identity. This diagram has the following properties.

1. *The arrows pointing in two directions are simplicial Quillen adjunctions, where the upper (left) arrow is the left part of the adjunction.*
2. *The left adjunction part (and, respectively, the right adjunction part) together with the inclusions form a commutative diagram.*

3. *All horizontals are Quillen equivalences, which create weak equivalences in both directions.*

4. *The upper vertical maps are given by including the fibers under the Quillen bifibrations to* **Pos***.*

5. *The diagonals pointing downward are given by left Bousfield localization at inner horn inclusions.*

6. *The lower verticals that point downward are given by right Bousfield localization at the refinement maps* $(-)^{\mathfrak{r}} \to 1$ *(see Section 7.5.3, for the topological case).*

7.4.3 Topological stratified mapping spaces

Give two layered ∞-categories X and Y, the ∞-category of functors Y^X is itself layered. Indeed, it follows from the fact that isomorphisms of functors are detected pointwise that it even suffices for Y to be layered. Together with Theorem 7.3.3.1, this already shows that at least the homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ admits a notion of internal mapping space on the ∞-categorical level. Similarly, in the world of topological stratified spaces (more specifically homotopically stratified spaces) [Hug99b] equipped the space of stratified maps with a natural decomposition (which generally may not be a stratification) and investigated the lifting properties of such mapping spaces.
If we are looking to bring these two notions of internal (i.e. stratified) mapping spaces - one on the ∞-categorical and one on the 1-categorical level together - the language of (semi-)model categories provides the notion of a cartesian closed (semi-)model category (see, for example, [Rez10] for the case of model categories).

Definition 7.4.3.1. Let $\mathbf{M}$ be a semi-model category the underlying 1-category of which is cartesian closed. We say that $\mathbf{M}$ is a *cartesian closed semi-model category*, if for every cofibration $i\colon A \to B$ and every fibration $f\colon X \to Y$, the induced morphism of exponential objects

$$X^B \to Y^B \times_{Y^A} X^A$$

is a fibration and furthermore is a weak equivalence, if additionally f is a weak equivalence, or i is a weak equivalence with cofibrant source.

In Section 5.3.5, we proved that all of the model structures for stratified homotopy theory defined on s**Strat** (see Section 7.3.2) are, in fact, cartesian closed. We may use the following lemma to transfer this result to the topological setting.

Lemma 7.4.3.2. *Let* $\mathbf{M}$ *be a cartesian semi-model category and* $\mathbf{N}$ *a cartesian closed, bicomplete category. Suppose that there is an adjunction*

$$L\colon \mathbf{M} \rightleftarrows \mathbf{N}\colon R,$$

such that L *preserves finite products and the right transferred semi-model structure on* $\mathbf{N}$ *exists. Then, together with this model structure,* $\mathbf{N}$ *is a cartesian closed semi-model category.*

Proof. Note first that if L preserves products, then there are canonical isomorphisms

$$R(X^{L(A)}) \cong R(X)^A$$

for $A \in \mathbf{M}$ and $X \in \mathbf{N}$. Now, let us verify the hypothesis in the definition of a cartesian closed semi-model category. Suppose that we are given a cofibration $i\colon A \to B$ and a fibration $f\colon X \to Y$ in the transferred semi-model structure on $\mathbf{N}$. In the context of Definition 7.4.3.1 in $\mathbf{N}$, let us first reduce to the case where $A \to B$ is of the form $L(A' \to B')$, for $A \to B$ a cofibration (acyclic cofibration with cofibrant source). For the general case, we need to show that $R(X^B \to Y^B \times_{Y^A} X^A)$ is a fibration in $\mathbf{M}$, or equivalently, that every lifting problem

$$\begin{tikzcd} L(A') \arrow[r] \arrow[d] & X^B \arrow[d] \\ L(B') \arrow[r] \arrow[ur, dashed] & Y^B \times_{Y^A} X^A \end{tikzcd} \tag{7.15}$$

with $A' \to B'$ an (acyclic) cofibration (with cofibrant source) has a solution. Every such lifting problem is, in turn, equivalent to a lifting problem of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X^{L(B')} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y^{L(B')} \times_{Y^{L(A')}} X^{L(A')}. \end{tikzcd} \tag{7.16}$$

Hence, it suffices to see that the right-hand map is an (acyclic) fibration, which implies the reduction. Finally, under the natural isomorphism we mentioned at the beginning of the proof, using the fact that right adjoint functors preserve fiber products, the map

$$R(X^{L(B')} \to R(Y^{L(B')} \times_{Y^{L(A')}} X^{L(A')})$$

is equivalently given by the induced morphism

$$R(X)^{B'} \to R(Y)^{B'} \times_{R(Y)^{A'}} R(X)^{A'},$$

which is an (acyclic) fibration in $\mathbf{M}$ by the assumption that the latter is a cartesian semi-model category. □

Corollary 7.4.3.3. *Whenever* **Top** *is a cartesian closed category (i.e., in the compactly generated or Δ-generated case), the semi-model categories* $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{d}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c}}$ *are cartesian closed.*

From this corollary, one may deduce that whenever $\mathcal{X}$ is a fibrant stratified space and $\mathcal{A}$ a cofibrant stratified space, in any of the semi-model structures on **Strat**, then the stratified mapping space $\mathcal{X}^{\mathcal{A}}$ presents the ∞-categorical internal mapping space in the corresponding ∞-category. In Section 7.5.4, we discuss the relationship with the stratified mapping spaces of [Hug99b].

7.4.4 Applying the semi-model structures of stratified spaces

Corollary 7.4.2.3 opens up the study of homotopy theories of stratified spaces to all of the tools available for cofibrantly generated simplicial semi-model categories. Notably, these essentially include all of the tools available for model categories, as long as one is willing to assume cofibrancy in the appropriate places (see, for example, [Bar10; BW24] for a good overview of the literature). In this section, we are going to name a few that stand out, as they provide answers to some questions that have been open for a while (see Questions Q(1), Q(2) and Q(5) in the introduction). Let us begin by noting that the situation is even better than what one can usually expect from a semi-model category. Namely, the simplicial structure gives us simplicial resolutions of fibrant objects (see [DK80b]), without having to cofibrantly replace first.

Corollary 7.4.4.1. *Let $\underline{\mathbf{S}}$ be any of the simplicial semi-model categories for stratified spaces of Notation 7.4.2.5. Then, for any $\mathcal{X} \in \underline{\mathbf{S}}$ which is fibrant, the functor*

$$\Delta^{\mathrm{op}} \to \mathbf{Strat}$$
$$[n] \mapsto \mathcal{X}^{\Delta^n}$$

defines a simplicial resolution of $\mathcal{X}$.

In particular, even non-cofibrant objects that are fibrant admit path-objects, and right and left homotopy classes of morphisms with a cofibrant source and fibrant target agree. We may now compute ∞-categorical mapping spaces of stratified spaces in terms of the simplicial structures.

Corollary 7.4.4.2. *Let $\underline{\mathbf{S}}$ be any of the simplicial semi-model categories for stratified spaces defined in Section 7.4.2. Let W be the class of weak equivalences in $\mathbf{S}$ and denote $\mathbf{S}[W^{-1}]$ the quasi-categorical localization of $\mathbf{S}$ at W. Let $\mathcal{A} \in \mathbf{S}$ be cofibrant and $\mathcal{X}$ be fibrant. Then there is a natural (zigzag) of weak equivalences between mapping spaces*

$$\mathbf{S}[W^{-1}](\mathcal{A}, \mathcal{X}) \simeq \underline{\mathbf{S}}(\mathcal{A}, \mathcal{X}).$$

In particular, the canonical map bijection

$$[\mathcal{A}, \mathcal{X}]_s \to \pi_0(\mathbf{S}[W^{-1}](\mathcal{A}, \mathcal{X}))$$

is a bijection, where the right-hand expression denotes homotopy classes of stratified (stratum-preserving when the poset is fixed) maps with respect to stratified homotopies.

Proof. This follows from [GH05, Prop. 1.1.10] which is the left semi-model category version of the argument in [DK80b, §7], together with [Hin16, p. 1.2]. □

Note that Corollary 7.4.4.2 applies to all piecewise linear pseudomanifolds or, more generally, conically stratified spaces, (equipped with the refined stratification, when necessary) and provides, in particular, a strengthened version of [DW22, Theorem 5.7].
From a global perspective, we obtain the following result, which states that the homotopy theory of stratified spaces obtained by inverting weak equivalences may equivalently be described in terms of the simplicial categories of bifibrant stratified spaces, or equivalently a localization of bifibrant stratified spaces at stratified homotopy equivalences, answering Question Q(2).

Corollary 7.4.4.3. *In the setting of Corollary 7.4.4.2, denote by $\underline{\mathbf{S}}^o$ the full simplicial subcategory of bifibrant objects. Denote by H_s the class of stratified homotopy equivalences between bifibrant stratified spaces. There are canonical equivalences of ∞-categories*

$$\underline{\mathbf{S}}^o \simeq \mathbf{S}^o[H_s^{-1}] \simeq \mathbf{S}[W^{-1}]$$

where equivalences between simplicial and quasi-categories are to be understood in terms of the Quillen equivalence between quasi-categories and simplicial categories of [Ber07a].

Proof. This is the semi-model category version of [DK80b, Prop. 4.8] the proof of which is identical. We note that the first equivalence also follows from the existence of cofibrant replacements and the powering of the stratified categories over **sSet**, using a similar flattening argument as in [DK87, p. 2.5]. Therefore, the full power of a semi-model category is not necessary to obtain this result. The second equivalence, however, needs both fibrant replacement and cofibrant replacement, as well as the Whitehead theorem in a semi-model category. □

Furthermore, we obtain the following two versions of the stratified homotopy hypothesis, providing an answer to Question Q(1).

Theorem 7.4.4.4. *Mapping a simplicial set to K to the stratified realization of its stratified simplicial set $\mathcal{L}(K)$, and conversely mapping a stratified space $\mathcal{X}$ to the underlying simplicial set of $\mathrm{Sing}_s\mathcal{X}$, $\mathcal{F}(\mathrm{Sing}_s\mathcal{X})$, induces a Quillen equivalence of (semi-)model categories*

$$\mathbf{sSet}^{\mathfrak{D}} \overset{\simeq}{\rightleftharpoons} \mathbf{Strat}^{\mathfrak{c}},$$

that creates weak equivalences in both directions.
The left-hand model category $\mathbf{sSet}^{\mathfrak{D}}$ is the left Bousfield localization of $\mathbf{sSet}^{\mathfrak{J}}$ which presents the full reflective subcategory $\mathcal{L}\mathbf{ay}_\infty \subset \mathcal{C}\mathbf{at}_\infty$ of small ∞-categories in which every endomorphism is an isomorphism.

Proof. This is the combination of Corollary 7.4.2.3 and Theorem 7.4.2.10 and Theorem 5.3.3.6. The only thing we need to pay extra attention to is the statement that both functors create weak equivalences. In the case of the right adjoint, note that the forgetful functor

$$\mathrm{s}\mathbf{Strat}^{\mathfrak{c}} \to \mathbf{sSet}^{\mathfrak{O}}$$

does not create weak equivalences. However, it creates weak equivalences between such stratified simplicial sets whose strata are Kan-complexes (Proposition 5.3.2.25). Since Sing_s has image in this category, it follows that the composition of Sing_s with the functor forgetting the stratification creates weak equivalences. Conversely, note that $|-|_s \circ \mathcal{L}$ is left Quillen, with source a cofibrant model category. Since the adjunction above is a Quillen equivalence (as the composition of two Quillen equivalences), the derived unit of adjunction is a weak equivalence. For $X \in \mathbf{sSet}$, It is given by

$$X \to \mathcal{F}(\mathrm{Sing}_s|\mathcal{L}(X)|_s) \to \mathcal{F}(\mathrm{Sing}_s\mathcal{Y})$$

where $|\mathcal{L}(X)|_s \to \mathcal{Y}$ is a fibrant replacement of $|\mathcal{L}(X)|_s$. Note, however, that as we have already shown that $\mathcal{F} \circ \mathrm{Sing}_s$ preserves weak equivalences, it follows by two-out-of three that the ordinary unit of adjunction is also a weak equivalence. It follows, again by two-out-of three, that $f\colon X \to Y$ in $\mathbf{sSet}^{\mathfrak{O}}$ is a weak equivalence, if and only if $\mathcal{F}(\mathrm{Sing}_s|\mathcal{L}(f)|_s)$ is a weak equivalence. Finally, since $\mathcal{F} \circ \mathrm{Sing}_s$ creates weak equivalences, this is equivalent to $|-|_s \circ \mathcal{L}$ creating weak equivalences. □

A quasi-categorical counterpart of this result, using Corollary 7.4.4.3 is:

Corollary 7.4.4.5. *Denote by* $\mathbf{Strat}^o$ *the full subcategory of* $\mathbf{Strat}^{\mathfrak{c}}$ *of bifibrant stratified spaces. Denote by* H_s *the wide subcategory of stratified homotopy equivalences.* Sing_s *(i.e., Lurie's exit-path construction of [Lur17]) induces an equivalence of quasi-categories*

$$\mathrm{Exit}\colon \mathbf{Strat}^o[H_s^{-1}] \xrightarrow{\simeq} \mathcal{L}\mathbf{ay}_\infty,$$

where $\mathcal{L}\mathbf{ay}_\infty$ *denotes the quasi-category of small quasi-categories, in which every endomorphism is an isomorphism.*

Finally, let us comment on Question Q(5), concerning the relationship between homotopy colimits and 1-categorical colimits of stratified spaces. The main application for the methods of regular neighborhoods in stratified cell complexes we developed in [Waa24b] (Chapter 6) was to prove that certain pushout diagrams of stratified spaces are homotopy pushout. Note that, a posteriori, this also follows from the existence of semi-model structures making inclusions of cellularly stratified spaces cofibrations. In fact, we can now use the full machinery for the computation of homotopy colimits in a simplicial (semi-)model category (see, for example, [Hir03]) to compute the latter. As a particular consequence, we obtain the following result.

Corollary 7.4.4.6. *Suppose we are given a pushout diagram of in* $\mathbf{Strat}_P$

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "c"] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{X} \arrow[r, hook] & \mathcal{Y}, \end{tikzcd} \tag{7.17}$$

all of which are cofibrant in $\mathbf{Strat}_P^{\mathfrak{d}}$ *or equivalently* $\mathbf{Strat}_P^{\mathfrak{c}}$ *(that is, retracts of cellularly stratified spaces, by Proposition 7.5.2.1), with the upper vertical given by a cofibration. Then this diagram descends to a pushout diagram in* $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}}$ *and* $\mathcal{S}\mathbf{trat}_P^{\mathfrak{c}}$.
In particular, this holds when all spaces involved are cellularly stratified spaces and c is an inclusion of cellularly stratified spaces. The analogous claim for $\mathbf{Strat}$ *and* $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$, $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ *holds.*

Clearly, Lemma 7.4.2.2 is implied by this result. Lemma 7.4.2.2 was used to prove the existence of semi-model structures. Now, however, we can see that the existence of semi-model structures is essentially equivalent to diagrams of stratified cell complexes as in Diagram (7.17) being homotopy pushout, which illustrates the importance of Question Q(5).

7.5 (Co)fibrant stratified spaces

In the previous section, we have established the existence of semi-model structures of stratified spaces, which were transferred from the combinatorial setting of stratified simplicial sets. These were constructed with the specific goal of connecting the various homotopy theories of stratified spaces with their topology and geometry. If we are looking to deepen our understanding of this connection, the obvious question at hand is how classical examples of stratified spaces interact with these model structures. In particular, which classical examples of stratified spaces are bifibrant (Question Q(3)). For example, [Mil13; Dou21c; Nan19] all showed Whitehead theorems for certain stratified spaces, which characterize the stratified homotopy equivalences between them in terms of a priori weaker conditions. Every simplicial semi-model category comes with its own Whitehead theorem, stating that the homotopy equivalences with respect to the simplicial cylinder between the bifibrant objects are precisely the weak equivalences. The case of s$\mathbf{Strat}^{\mathfrak{c}}$ was already proven through a direct proof in [Nan19]. Summarizing old and proving new criteria for (co)fibrancy and studying how these properties relate to more classical properties of stratified spaces is the main content of this section. Finally, at the end of this section, we provide a short investigation of stratified versions of the homotopy link as investigated in [Hug99a] in our model categorical framework.

7.5.1 Fibrant stratified spaces and Quinn's homotopically stratified sets

Let us begin by studying the class of fibrant objects in the model structures constructed in Section 7.4.2. The question of what classical examples are fibrant in $\mathbf{Strat}^{\mathfrak{c}}$ was already investigated in [Nan19] and [Lur17], with the core results that both Quinn's homotopically stratified spaces and Siebenmann's conically stratified spaces (see later in this section) are such that the associated stratified singular simplicial set is a quasi-category. Hence, it turns out that most examples of stratified spaces of classical geometrical interest fall into the class of fibrant objects in the semi-model structures we defined. The central new contribution from our side in this section is that we prove that, as long as one restricts to metrizable spaces, the class of fibrant objects is in fact independent of the choice of model structure we presented here (see Proposition 7.5.1.4). A consequence of this is Theorem 7.5.1.6, which may be taken as the statement that in any geometric scenario there really is no difference between the diagrammatic homotopy theories for stratified spaces and their categorical counterparts at all.

Before we begin, note that we really only need to study fibrancy in the scenario of a fixed poset, due to the following fact.

Lemma 7.5.1.1. *Let $\mathcal{X} \in \mathbf{Strat}$. Then the following statements are equivalent:*

1. *$\mathcal{X}$ has the horn filling property over $P_{\mathcal{X}}$ with respect to all realizations of admissible stratified horn inclusions. That is, it admits fillers*

$$\begin{tikzcd} {|\Lambda^{\mathcal{J}}_k|_s} \arrow[r] \arrow[d] & \mathcal{X} \\ {|\Delta^{\mathcal{J}}|_s} \arrow[ur, dashed] & \end{tikzcd} \tag{7.18}$$

   in $\mathbf{Strat}_{P_{\mathcal{X}}}$, for all admissible pairs $\mathcal{J} \in \Delta_P$, $0 \leq k \leq n_{\mathcal{J}}$.

2. *$\mathcal{X}$ is fibrant in all of the semi-model categories $\mathbf{Strat}^{\mathfrak{d}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{d}}$.*

3. *$\mathcal{X}$ is fibrant in one of the semi-model categories $\mathbf{Strat}^{\mathfrak{d}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{d}}$.*

Furthermore, the following analogous statements about the categorical semi-model are equivalent:

1. *$\mathcal{X}$ has the horn filling property over $P_{\mathcal{X}}$ with respect to all realizations of admissible stratified horn inclusions, and all inner stratified horn inclusions.*

2. *$\mathcal{X}$ has the horn filling property over $P_{\mathcal{X}}$ with respect to all realizations of inner stratified horn inclusions.*

3. *$\mathcal{X}$ is fibrant in all of the semi-model categories $\mathbf{Strat}^{\mathfrak{c}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c}}$.*

4. *$\mathcal{X}$ is fibrant in one of the semi-model categories $\mathbf{Strat}^{\mathfrak{c}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c}}$.*

Proof. The diagrammatic case is immediate by the construction of the left model structures via left-transfer and Proposition 7.4.2.8 together with the fact that $\mathbf{Strat}^{\mathfrak{d}}$ is obtained from $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ via a right Bousfield localization Theorem 7.4.2.11. For the categorical statement, the proof is almost identical; special attention only needs to be paid to the implication from the second to the first property. Note that for any stratified space $\mathcal{X}$ the strata of the stratified-simplicial set $\mathrm{Sing}_s\mathcal{X}$ are Kan-complexes. Hence, the implication follows from [Hai23, Prop. 2.2.3]. ☐

Definition 7.5.1.2. We call a stratified topological space $\mathcal{X}$ that satisfies any of the equivalent conditions of Lemma 7.5.1.1 concerning the diagrammatic (categorical) semi-model structures *diagrammatically* (*categorically*) fibrant.

Clearly, every categorically fibrant stratified space is diagrammatically fibrant. The converse must necessarily be false. If it were true, then by Corollary 7.4.4.3 and Theorem 7.3.3.1 together with [Hai23, Thm. 1.1.7] and [Dou21c, Thm. 3], it would imply that the inclusion from the ∞-category of décollages into $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ is an equivalence of ∞-categories[6]. However, it is surprisingly hard to provide a geometrical example for a stratified space that is diagrammatically fibrant and not categorically fibrant (depending on your definition of geometric, it is not possible at all). Before we investigate this question further, let us first recall results on which classical examples of stratified space we can expect to be fibrant.

Remark 7.5.1.3. It was first shown in [Lur17, A.5] that all conically stratified spaces - roughly a stratified space that locally has the structure $U \times C(\mathcal{L})$, where $C(\mathcal{L})$ is the stratified cone of a stratified space $\mathcal{L}$ - are categorically fibrant. In particular, this implies that all topological pseudomanifolds (see, for example, [Ban07]), as well as all conically smooth stratified spaces (see [AFT17]) and hence also all Whitney stratified spaces ([NV23] or classically [Tho69]), are categorically and hence also diagrammatically fibrant.

It turns out, however, that fibrancy can already be obtained under significantly less geometric conditions. One of the crucial insights contributing to the foundations of stratified homotopy theory was Quinn's observation that a powerful homotopy theoretical setup for stratified spaces can already be achieved by only posing requirements on pairs of strata ([Qui88]). Quinn defined a notion of homotopically stratified set (with slightly more restrictive conditions on the stratification poset) by requiring $\mathcal{X} \in \mathbf{Strat}$ to be metrizable, and requiring that for any two-element flag $[p < q] \in \Delta_P$ it holds that:

1. $\mathcal{X}_p \hookrightarrow \mathcal{X}_{p<q}$ is tame, i.e. admits a nearly strict neighborhood deformation retraction (see [Qui88]).

2. $\mathcal{H}o\mathrm{Link}_{p<q}\mathcal{X} \to \mathcal{X}_p$ is a Hurewicz fibration.

Together, these assumptions provided an excellent homotopy theoretic framework for classification results concerning manifold stratified spaces (see [Qui88; Wei94]). Note that both of these assumptions are more in line with the Hurewicz approach (see [Str72]), than with the Serre-Quillen approach to homotopy theory ([Qui67]). Since the approach we pursue here is combinatorial (i.e. following Serre and Quillen), we should expect cofibrancy in our semi-model

[6] It is easy to write down an example of a diagram which is not a décollage. For example, take the constant diagram over $(\mathrm{sd}[2])^{\mathrm{op}}$ and replace the entry at $[0 < 1 < 2]$ by the empty set.

categories to be a stronger condition than condition (1) above. Indeed, it is a consequence of Proposition 6.4.2.4 that any locally compact cofibrant stratified space (in any of the model structures of Section 7.4.2) satisfies the tameness condition above. Conversely, in the two strata case it is not hard to see that $\mathcal{H}o\mathrm{Link}_{p<q}\mathcal{X} \to \mathcal{X}_p$ being a Serre fibration is equivalent to diagrammatic and equivalently categorical fibrancy (see the proof of Proposition 7.5.1.4 below). Surprisingly, it follows from [Mil09, Thm. 4.9] that every homotopically stratified set is, in fact, categorically fibrant ([Nan19, Prop. 8.1.2.6]), without any restrictions on the stratification poset. Note that being fibrant is manifestly a statement about the interaction of more than two strata, which makes it so surprising that it can be achieved by only making requirements on two-strata interaction. Even more, this result does not leave much space for diagrammatically fibrant spaces that are not categorically fibrant at all (see Question Q(4)). If we assume pairwise tameness and metrizability, then it would follow that any counterexample would involve $\mathcal{H}o\mathrm{Link}_{p<q}\mathcal{X} \to \mathcal{X}_p$ being a Serre fibration, but not a Hurewicz fibration. It turns out that no cofibrancy assumptions are necessary whatsoever, and in fact in any geometric setting diagrammatic and categorical fibrancy are equivalent conditions, which are furthermore equivalent to the Serre fibration analogue of Quinn's pairwise homotopy link condition. Furthermore, this result does not require any additional tameness assumptions.

Proposition 7.5.1.4. *Let $\mathcal{X} \in$ **Strat** be a metrizable stratified space. Then the following conditions are equivalent:*

(i) $\mathcal{X}$ is categorically fibrant.

(ii) $\mathcal{X}$ is diagrammatically fibrant.

(iii) For any pair $[p < q] \in \Delta_{P_{\mathcal{X}}}$, the starting point evaluation map $\mathcal{H}o\mathrm{Link}_{p<q}(\mathcal{X}) \to \mathcal{X}_p$ is a Serre fibration.

(iv) For any pair $[p < q] \in \Delta_{P_{\mathcal{X}}}$, and for any flag $\mathcal{J} = [p = \cdots = p_k < q = \cdots = q = p_{n_{\mathcal{J}}}]$ with $k \geq 1$, $\mathcal{X}$ has the horn filling property with respect to $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$.

Before we give a proof, we note the implications of this result. By Theorem 7.4.2.11, the categorical setting is always obtained as a left Bousfield localization of the diagrammatic one. In particular, the two settings have the same cofibrations; furthermore, we obtain:

Corollary 7.5.1.5. *Let $\mathcal{X}, \mathcal{Y} \in$ **Strat** be metrizable stratified spaces, $\mathcal{X}$ diagrammatically fibrant and $f\colon \mathcal{Y} \to \mathcal{X}$ a stratified map. Then there are equivalences:*

1. *f is a fibration in $\mathbf{Strat}^{\mathfrak{d}}_P \iff f$ is a fibration in $\mathbf{Strat}^{\mathfrak{c}}_P$;*
2. *f is a fibration in $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}} \iff f$ is a fibration in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$;*
3. *f is a fibration in $\mathbf{Strat}^{\mathfrak{d}} \iff f$ is a fibration in $\mathbf{Strat}^{\mathfrak{c}}$.*

Proof. This is immediate from Proposition 7.5.1.4, together with Theorem 7.4.2.11, and the characterization of fibrations between fibrant objects in a left Bousfield localizations (the semi-model category version of [Hir03, Prop. 3.3.16.], see [WY18, Rem. 4.5].) □

We may hence interpret Proposition 7.5.1.4 as stating that in any geometrical framework the diagrammatic approach is really the same as the categorical approach, obtaining an answer to Question Q(4). More rigorously, we can phrase this insight as follows.

Theorem 7.5.1.6. *Let $\underline{\mathbf{D}}$ be any of the diagrammatic simplicial semi-model categories of Section 7.4.2 and let $\underline{\mathbf{C}}$ be its categorical pendant. Denote by $\underline{\mathbf{D}}^f_m$ and $\underline{\mathbf{C}}^f_m$ the respective restrictions of the semi-model structures to the full simplicial subcategory of fibrant, metrizable stratified spaces. Then equality*

$$\underline{\mathbf{D}}^f_m = \underline{\mathbf{C}}^f_m$$

holds (on the nose). In particular, if we denote by $\underline{\mathbf{D}}^o_m$ and $\underline{\mathbf{C}}^o_m$ the corresponding simplicial categories of bifibrant metrizable stratified spaces, then

$$\underline{\mathbf{D}}^o_m = \underline{\mathbf{C}}^o_m$$

and hence the full sub-∞-categories of bifibrant metrizable stratified spaces agree.

Proof. Equality on the object level follows by Proposition 7.5.1.4. Equality of cofibrations holds even without additional assumptions. Equality of fibrations was shown in Corollary 7.5.1.5. It remains to verify the equality of weak equivalences. We prove the stratum-preserving case; the others are analogous: Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a stratum-preserving map in $\underline{\mathbf{D}}^f_m$. We may factorize f as a cofibration and an acyclic fibration

$$\mathcal{X} \to \mathcal{Z} \to \mathcal{Y},$$

with $\mathcal{Z}$ not necessarily metrizable. Note that since cofibrations in both structures agree, so do acyclic fibrations. By Proposition 7.5.1.4, it follows that $\mathcal{Z}$ is also categorically fibrant (this did not use any metrizability assumptions on $\mathcal{Z}$). f is a weak equivalence if and only if the first of these two maps is a weak equivalence. However, the latter is a map between categorically fibrant spaces. In particular, by the semi-model category version of the Whitehead theorem for Bousfield localizations (see [Hir03, Thm 3.2.13] for the model category version)[7], it is a diagrammatic equivalence if and only if it is a categorical equivalence. □

In other words, for geometric examples, the two homotopy theories agree.
Before we provide a proof of Proposition 7.5.1.4, we need the following elementary set-theoretic topological lemma, the proof of which is provided in Section 7.B. It is the only place where the metrizability of $\mathcal{X}$ comes into play. In the following, D^{n+1} denotes the euclidean unit disk of dimension $n+1$, for $n \in \mathbb{N}$, S^n denotes its boundary, and $\mathring{D}^{n+1} = D^{n+1} \setminus S^n$ its interior.

Lemma 7.5.1.7. *Let $n \geq 0$, X a metrizable topological space and suppose that we are given a solid commutative diagram of (general) topological spaces of the following form:*

$$\begin{tikzcd}
 & D^{n+1} \times \{0\} \cup S^n \times [0,1] \arrow[d, hook] \arrow[dr, "f"] & \\
D^{n+1} \times \{0\} \cup S^n \times [0,1) \arrow[ur, hook] \arrow[dr, hook] & D^{n+1} \times [0,1] \arrow[r, dashed, "\tilde{f}"] & X \\
 & D^{n+1} \times [0,1) \arrow[u, hook] \arrow[ur, "f'"'] &
\end{tikzcd} \quad (7.19)$$

Then there exists $\tilde{f}\colon D^{n+1} \times [0,1] \to X$ making the upper triangle commute, and furthermore such that the inclusion

$$\tilde{f}(\mathring{D}^{n+1} \times [0,1]) \subset f'(\mathring{D}^{n+1} \times [0,1))$$

holds.

As a consequence of Lemma 7.5.1.7 we obtain:

Lemma 7.5.1.8. *Let $\mathcal{X} \in \mathbf{Strat}_P$ be a metrizable stratified space and $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ be a flag in P. Let $0 < k < n$ and denote by $\mathcal{I}$ the regular flag containing all $p_i \in \mathcal{J}$ with $i \geq k$. If $\mathcal{X}_{\mathcal{I}}$ is categorically fibrant, then $\mathcal{X}$ has the horn filling property with respect to $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$.*

[7][BW24, Rem. 4.5] states the semi-model category case for bifibrant objects in the localized model structure. Note, however, that we may always assume bifibrancy, by replacing both source and target cofibrantly first, which is done through acyclic fibrations, which are the same in both model structures. Alternatively, we can just use the full model categorical result and pass to the simplicial setting using the fact that all semi-model structures are transferred.

Proof. Let $\mathcal{J}_k$ be the flag obtained by removing p_k from $\mathcal{J}$, corresponding to the k-th face of $\Delta^{\mathcal{J}}$. We may identify $|\Delta^{\mathcal{J}}|_s$ with an (appropriately stratified) join $|\Delta^{[p_k]}|_s \star |\Delta^{\mathcal{J}_k}|_s$. This identification induces join coordinates $[s,x]$, $s \in [0,1]$, $x \in |\Delta^{\mathcal{J}_k}|_s$ on $|\Delta^{\mathcal{J}}|_s$, with the point $[0,x]$ corresponding to the unique element of $|\Delta^{[p_k]}|_s$[8]. Under this identification, $|\Lambda^{\mathcal{J}}_k|_s$ corresponds to the join $|\Delta^{p_k}|_s \star |\partial\Delta^{\mathcal{J}_k}|_s$. For $0 \leq \alpha \leq \alpha' < 1$, we write

$$S^{\leq \alpha'}_{\geq \alpha} = \{[x,s] \in |\Delta^{[p_k]}|_s \star |\Delta^{\mathcal{J}_k}|_s \mid \alpha \leq s \leq \alpha'\},$$

and use analogous notation replacing $\leq$ by $<$ and $\geq$ by $>$. We may then decompose $|\Delta^{\mathcal{J}}|_s$ into

$$|\Delta^{\mathcal{J}}|_s = S^{\leq \frac{1}{2}}_{\geq 0} \cup S^{\leq 1}_{\geq \frac{1}{2}}.$$

It is immediate by the definition of the stratified simplex $|\Delta^{\mathcal{J}}|_s$, that the inclusion

$$|\Lambda^{\mathcal{J}}_k| \cap S^{\leq \frac{1}{2}}_{\geq 0} \hookrightarrow S^{\leq \frac{1}{2}}_{\geq 0}$$

is stratum-preserving homeomorphic to the inclusion

$$|\Lambda^{\mathcal{J}'}_k|_s \hookrightarrow |\Delta^{\mathcal{J}'}|_s,$$

where $\Delta^{\mathcal{J}'}$ is obtained by replacing every entry p_i in $\mathcal{J}$ with $i < k$ by p_k. In particular, by the assumption on $\mathcal{X}_{\mathcal{I}}$, $\mathcal{X}$ admits a filler with respect to the latter inclusion. Thus, it suffices to show that $\mathcal{X}$ admits fillers with respect to

$$A := S^{\leq 1}_{\geq \frac{1}{2}} \cap |\Lambda^{\mathcal{J}}_k| \cup S^{\leq \frac{1}{2}}_{\geq \frac{1}{2}} \hookrightarrow S^{\leq 1}_{\geq \frac{1}{2}} =: B.$$

Choosing any homeomorphism $|\Delta^n| \cong D^{m+1}$, where $m = n-1$, and using the affine order preserving homeomorphism $[\frac{1}{2},1] \cong [0,1]$ we may identify the latter inclusion with a non-stratified inclusion

$$D^{m+1} \times \{0\} \cup S^m \times [0,1] \hookrightarrow D^{m+1} \times [0,1].$$

Since $p_k \leq p_n$, under this identification $\mathring{D}^{m+1} \times [0,1]$ is entirely contained in the p_n-stratum of $|\Delta^{\mathcal{J}}|_s$. Thus, given a stratum-preserving map $f\colon A \to \mathcal{X}$, an extension of f to B is the same data as an extension of f to $D^{m+1} \times [0,1]$ mapping $\mathring{D}^{m+1} \times [0,1]$ to $\mathcal{X}_{p_n}$. We may now apply Lemma 7.5.1.7, from which it follows that it suffices to obtain an extension of $f_{D^{m+1}\times\{0\}\cup S^m \times [0,1)}$ to $D^{m+1} \times [0,1) \cong S^{<1}_{\frac{1}{2}}$. Now write

$$S^{<1}_{\frac{1}{2}} = \bigcup_{l \geq 1} S^{\leq 1-\frac{1}{2^{l+1}}}_{\geq 1-\frac{1}{2^l}},$$

which defines a locally finite, closed cover of $S^{<1}_{\frac{1}{2}}$. Inductively, we may hence reduce to solving extension problems with respect to

$$S^{\leq 1-\frac{1}{2^l}}_{\geq 1-\frac{1}{2}^l} \cup \left(|\Lambda^{\mathcal{J}}_k|_s \cap S^{\leq 1-\frac{1}{2^{l+1}}}_{\geq 1-\frac{1}{2^l}}\right) \hookrightarrow S^{\leq 1-\frac{1}{2^{l+1}}}_{\geq 1-\frac{1}{2^l}}.$$

Finally, if we denote by $\mathcal{J}'_k$ the k-th face of $\mathcal{J}'$, then the latter inclusion is stratum-preserving homeomorphic to the simplicial box product

$$|\Delta^{\mathcal{J}'_k}|_s \otimes \Delta^0 \cup_{|\partial\Delta^{\mathcal{J}'_k}|_s \otimes \Delta^0} |\partial\Delta^{\mathcal{J}'_k}|_s \otimes \Delta^1 \hookrightarrow |\Delta^{\mathcal{J}'_k}|_s \otimes \Delta^1,$$

for the inclusion of Δ^0 into Δ^1 at 0. Since $\underline{\mathbf{Strat}}^{\mathfrak{c}}_{\mathcal{I}}$ is a simplicial semi-model category, the latter defines an acyclic cofibration in $\underline{\mathbf{Strat}}^{\mathfrak{c}}_{\mathcal{I}}$. Since $\mathcal{X}_{\mathcal{I}}$ was assumed to be fibrant, the existence of extensions with respect to such an inclusion follows. $\square$

[8] Beware that at other points in the text, we have parametrized joins the wother way around.

We may now complete the following proof.

Proof of Proposition 7.5.1.4. Clearly, Property (i) implies Property (ii). To see that Property (ii) implies Property (iii), note that any lifting problem

$$\begin{tikzcd} {|\Lambda^n|} \arrow[r] \arrow[d, hook] & \mathcal{H}o\mathrm{Link}_{p<q}(\mathcal{X}) \arrow[d] \\ {|\Delta^n|} \arrow[r] \arrow[ur, dashed] & \mathcal{X}_p \end{tikzcd} \tag{7.20}$$

is equivalent to a lifting problem (over $P_{\mathcal{X}}$)

$$\begin{tikzcd} {|\Delta^{[p]}|_s \otimes \Delta^n \cup_{|\Delta^{[p]}|_s \otimes \Lambda^n} |\Delta^{[p<q]}|_s \otimes \Lambda^n} \arrow[r] \arrow[d, hook] & \mathcal{X} \\ {|\Delta^{[p<q]}|_s \otimes \Delta^n} \arrow[ur, dashed] & \end{tikzcd} \tag{7.21}$$

Hence, Property (iii) follows from the simpliciality of the semi-model category $\underline{\mathbf{Strat}}^{\mathfrak{d}}_{P_{\mathcal{X}}}$. For the implication Property (iii) $\implies$ Property (iv), note that using the identification $|\Delta^{\mathcal{J}}|_s$ with the (appropriately stratified) join $|\Delta^{\mathcal{J}_p}|_s \star |\Delta^{\mathcal{J}_q}|_s$, we can interpret $|\Delta^{\mathcal{J}}|_s$ as a stratified quotient space of

$$|\Delta^{[p<q]}|_s \otimes (\Delta^k \times \Delta^{n-k-1}),$$

obtained by collapsing $|\Delta^{[p]}|_s \otimes (\Delta^k \times \Delta^{n-k-1})$ to $|\Delta^{[p]}|_s \otimes \Delta^k$ and $|\Delta^{[q]}|_s \otimes (\Delta^k \times \Delta^{n-k-1})$ to $|\Delta^{[q]}|_s \otimes \Delta^{n-k-1}$. Within $|\Delta^{[p<q]}|_s \otimes (\Delta^k \times \Delta^{n-k-1})$, $|\Lambda^{\mathcal{J}}_k|_s$ corresponds to

$$\begin{tikzcd} {|\Delta^{[p<q]}|_s \otimes (\Lambda^k_k \times \Delta^{n-k-1}) \cup |\Delta^{[p]}|_s \otimes (\Delta^k \times \Delta^{n-k-1}) \cup |\Delta^q|_s \otimes (\Delta^k \times \Delta^{n-k-1})} \arrow[d, hook] \\ {|\Delta^{[p<q]}|_s \otimes (\Delta^k \times \Delta^{n-k-1})} \end{tikzcd} \tag{7.22}$$

Hence, it suffices to show that $\mathcal{X}$ has the extension property with respect to this stratum-preserving map over $P_{\mathcal{X}}$. Denote $A = \Lambda^k_k \times \Delta^{n-k-1}$ and $B = \Delta^k \times \Delta^{n-k-1}$. Under the identification $|\Delta^{[p<q]}|_s \cong |\Delta^{[p<q]}|_s \cup_{|\Delta^{[q]}|_s} |\Delta^{[q\leq q]}|_s$ we may decompose (7.22) into pushouts of inclusions

$$|\Delta^{[p]}|_s \otimes B \cup_{|\Delta^{[p]}|_s \otimes A} |\Delta^{[p<q]}|_s \otimes A \hookrightarrow |\Delta^{[p<q]}|_s \otimes B \tag{7.23}$$

and

$$|\partial\Delta^{[q\leq q]}|_s \otimes B \cup_{|\partial\Delta^{[q\leq q]}|_s \otimes A} |\Delta^{[q<q]}|_s \otimes A \hookrightarrow |\Delta^{[q\leq q]}|_s \otimes B. \tag{7.24}$$

The inclusion $A \hookrightarrow B$ is an anodyne map of simplicial sets. It follows that the inclusion (7.24) which lies entirely over one stratum is given by an acyclic Quillen cofibration. Consequently, $\mathcal{X}$ has the right lifting property with respect to the inclusion (7.24). Finally, lifting problems with respect to (7.23) are equivalent to lifting problems of the form

$$\begin{tikzcd} {|A|} \arrow[r] \arrow[d, hook] & \mathcal{H}o\mathrm{Link}_{p<q}(\mathcal{X}) \arrow[d] \\ {|B|} \arrow[r] \arrow[ur, dashed] & \mathcal{X}_p\,, \end{tikzcd} \tag{7.25}$$

since $A \hookrightarrow B$ is an anodyne extension, the left-hand vertical is an acyclic cofibration in the Quillen model structure, showing the existence of a lift.

It remains to show that Property (iv) implies Property (i). It suffices to show that $\mathcal{X}_{\mathcal{I}}$ is categorically fibrant for all regular flags $\mathcal{I} = [q_0 < \cdots < q_n]$ of $P_{\mathcal{X}}$. We only need to cover the case where $n \geq 1$, that is, where $\mathcal{J}$ has at least two different elements (otherwise, we are in the trivially stratified case, that is, the classical case). First, note that Property (i) already implies the case where $\mathcal{I} = [p < q]$. Indeed, let $\mathcal{J} = [p_0 \leq \cdots \leq p_{n_\mathcal{J}}]$ be a flag that degenerates from $\mathcal{I}$. When $p_k = p$ we may without loss of generality assume that k is maximal with the property that $p_k = p$, by permuting the corners of $|\Delta^\mathcal{J}|_s$ by a stratum-preserving homeomorphism. Then we are in the situation of Property (iv). In case when $p_k = q$, the inclusion $|\Lambda^\mathcal{J}_k \hookrightarrow \Delta^\mathcal{J}|_s$ admits a stratum-preserving retraction. (Consider a homeomorphism of $|\Delta^\mathcal{J}|_s$ with D^n, mapping the k-th face to the northern hemisphere. Then the p-stratum is entirely contained in the equator, and projecting vertically down to the southern hemisphere defines a retraction.) We now proceed to show the horn filling property with respect to arbitrary $\mathcal{J}$, by induction over $n_\mathcal{I}$, the length of $\mathcal{I}$. We have already covered the case $n_\mathcal{I} = 1$. Now for the inductive step let $\mathcal{J} = [p_0 \leq \cdots \leq p_m]$ be a flag degenerating from a subflag of $\mathcal{I}$, and $0 < k < m$. For such $\mathcal{J}$, denote by $t_\mathcal{J}$ the length of $\mathcal{J}_{p_0}$ and by $o_\mathcal{J}$ the length of $\mathcal{J}_{q_{n-1}}$. We proceed by double induction on $t_\mathcal{J}$ and $o_\mathcal{J}$, keeping k flexible. If $o_\mathcal{J} = -1$, then $\mathcal{J}$ degenerates from a proper subflag of $\mathcal{I}$, and we are reduced to the inductive assumption in the induction on $n_\mathcal{I}$. If $p_0 \neq p_k$, which is in particular the case when $t_\mathcal{J} = 0$ (since $0 < k < n$), then $\{q_i \mid i \in [n_\mathcal{I}], q_i \geq p_k\}$ has cardinality smaller than $\mathcal{I}$, and by inductive assumption (in $n_\mathcal{I}$) we can apply Lemma 7.5.1.8 proving the existence of a filler with respect to $\mathcal{J}, k$. So, suppose $t_\mathcal{J} \geq 1$, $p_0 = p_k$ and we have already proven the result for all cases $\tilde{\mathcal{J}}, \tilde{k}$ where either $t_{\tilde{\mathcal{J}}} < t_\mathcal{J}$ or $o_{\tilde{\mathcal{J}}} < o_\mathcal{J}$. Let $s \in [n_\mathcal{J}]$ be such that $e_s \in |\Delta^\mathcal{J}|_s$ is the maximal vertex lying in the $q_{n_\mathcal{I}-1}$ stratum and $e_{n_\mathcal{J}} \in |\Delta^\mathcal{J}|_s$ the $n_\mathcal{J}$-th vertex of $|\Delta^\mathcal{J}|_s$. We may assume that $s \geq 2$, as $t_\mathcal{J} \geq 1$ and $n_\mathcal{I} \geq 2$. Furthermore, we may assume that $e_{n_\mathcal{J}}$ lies in the q_n stratum, as otherwise $\mathcal{J}$ degenerates from a proper subflag of $\mathcal{I}$, and we are reduced to the inductive assumption. Now, let

$$e' = \frac{1}{2}e_s + \frac{1}{2}e_{n_\mathcal{J}}$$

be the halfway point between these two vertices, which by assumption also lies in the $q_{n_\mathcal{I}}$ stratum. Consider the two affine stratum-preserving embeddings

$$i_1 \colon |\Delta^\mathcal{J}|_s \hookrightarrow |\Delta^\mathcal{J}|_s$$
$$e_i \mapsto \begin{cases} e_i & i < n_\mathcal{J} \\ e' & i = n_\mathcal{J} \end{cases}$$

and

$$i_2 \colon |\Delta^{\mathcal{J}'}|_s \hookrightarrow |\Delta^\mathcal{J}|_s$$
$$e_i \mapsto \begin{cases} e_i & i \neq s \\ e' & i = s \end{cases}$$

where $\mathcal{J}'$ is obtained from $\mathcal{J}$ by replacing the s-th entry by $q_{n_\mathcal{I}}$. The images of these two embeddings cover $|\Delta^\mathcal{J}|_s$ and intersect in the convex span

$$< \{e', e_i \mid i \in [n_\mathcal{J}], i \neq n_\mathcal{J}, s\} > .$$

Under i_1 this span corresponds to the s-th face of $\Delta^\mathcal{J}$, and under i_2 to the $n_\mathcal{J}$-th face of $\Delta^{\mathcal{J}'}$, which are both given by the flag $\mathcal{J}''$ obtained by removing p_s from $\mathcal{J}$. We obtain an induced stratum-preserving homeomorphism (see Fig. 7.1 for an illustration)

$$|\Delta^\mathcal{J} \cup_{\Delta^{\mathcal{J}''}} \Delta^{\mathcal{J}'}|_s \cong |\Delta^\mathcal{J}|_s \cup_{|\Delta^{\mathcal{J}''}|_s} |\Delta^{\mathcal{J}'}|_s \xrightarrow{(i_1,i_2)} |\Delta^\mathcal{J}|_s.$$

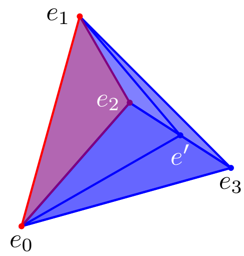


Figure 7.1: Depiction of $\Delta^{\mathcal{J}} \cup_{\Delta^{\mathcal{J}''}} \Delta^{\mathcal{J}'}$ with $\Delta^{\mathcal{J}}$ spanned by e_0, e_1, e_2, e' and $\Delta^{\mathcal{J}'}$ spanned by e_0, e_1, e', e_3.

Under this identification $\Lambda^{\mathcal{J}}_k$ corresponds to the subcomplex A of $\Delta^{\mathcal{J}} \cup_{\Delta^{\mathcal{J}''}} \Delta^{\mathcal{J}'} =: B$ given by removing:

- The simplices corresponding to $\Delta^{\mathcal{J}}$ and $\Delta^{\mathcal{J}'}$;
- The k-th face of $\Delta^{\mathcal{J}}$, $\Delta^{\mathcal{J}_k}$, and of $\Delta^{\mathcal{J}'}$, $\Delta^{\mathcal{J}'_k}$;
- The s-th face of $\Delta^{\mathcal{J}}$, $\Delta^{\mathcal{J}_s}$, which is equivalently the $n_{\mathcal{J}}$-th face of $\Delta^{\mathcal{J}'}$, or the top dimensional simplex of $\Delta^{\mathcal{J}''}$.
- The k-th face of $\Delta^{\mathcal{J}''}$, $\Delta^{\mathcal{J}''_k}$, which is equivalently the $(s-1)$-th face of $\Delta^{\mathcal{J}_k}$.

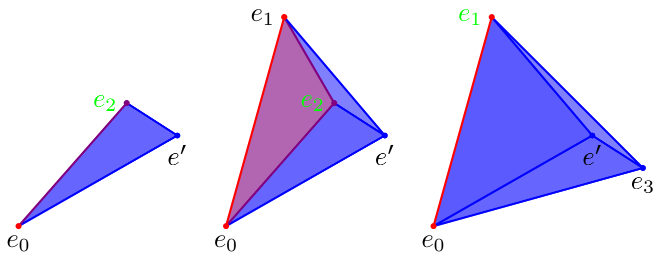


Figure 7.2: Depiction of $\Delta^{\mathcal{J}_k}, \Delta^{\mathcal{J}}$ and $\Delta^{\mathcal{J}'}$ with the vertices respectively opposite to $\Delta^{\mathcal{J}''_k}, \Delta^{\mathcal{J}''}$ and $\Delta^{\mathcal{J}'_k}$ marked in green.

Next, consider the inclusions

1. $j_1 \colon \Lambda^{\mathcal{J}_k}_{s-1} \hookrightarrow \Delta^{\mathcal{J}_k}$;
2. $j_2 \colon \Lambda^{\mathcal{J}}_s \hookrightarrow \Delta^{\mathcal{J}}$;
3. $j_3 \colon \Lambda^{\mathcal{J}'}_k \hookrightarrow \Delta^{\mathcal{J}'}$.

Gluing $\Delta^{\mathcal{J}_k}$ to A along j_1 adds in the missing simplices $\Delta^{\mathcal{J}_k}$ and $\Delta^{\mathcal{J}''_k}$. Then, gluing in $\Delta^{\mathcal{J}}$ along j_2 adds the missing simplices $\Delta^{\mathcal{J}}$ and $\Delta^{\mathcal{J}''}$ (consider Fig. 7.2, for an illustration). Finally, gluing in $\Delta^{\mathcal{J}'}$ along j_3 adds the missing simplices $\Delta^{\mathcal{J}'}$ and $\Delta^{\mathcal{J}'_k}$. We have thus exposed $A \hookrightarrow B$ as a composition of the pushouts of horn inclusions j_1, j_2 and j_3. Since $|-|_s$ preserves pushouts, it suffices to show that $\mathcal{X}$ has the horn filling property with respect to (the realizations of) j_1, j_2 and j_3. Since $s \geq 2$, j_1 is an inner horn inclusion, and furthermore since $p_0 = p_k$ we have $t_{\mathcal{J}_k} < t_{\mathcal{J}}$, which is covered by the inductive assumption. j_2 is an inner horn inclusion, in which the s-th entry is not minimal, which we have already covered above through Lemma 7.5.1.8. Finally, j_3 is an inner horn inclusion with $o_{\mathcal{J}'} < o_{\mathcal{J}}$, and hence is also covered by the inductive assumption. This finishes the induction. □

7.5.2 Cofibrant stratified spaces and cellularly stratified spaces

Next, let us investigate the class of cofibrant objects in the stratum-preserving and poset-preserving semi-model categories, both categorical and diagrammatic.

Proposition 7.5.2.1. *Let $\mathcal{X} \in$ **Strat**. Then the following statements are equivalent:*

1. *$\mathcal{X}$ is a retract in $\mathbf{Strat}_{P_{\mathcal{X}}}$ of a cellularly stratified space;*
2. *$\mathcal{X}$ is cofibrant in all of the semi-model categories $\mathbf{Strat}^{\mathfrak{d}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{c}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$;*
3. *$\mathcal{X}$ is cofibrant in one of the semi-model categories $\mathbf{Strat}^{\mathfrak{d}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{c}}_{P_{\mathcal{X}}}, \mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$.*

Proof. Note that there is no need to differentiate between the diagrammatic and categorical scenarios, as the latter are obtained from the former via left Bousfield localization (Theorem 7.4.2.11), and hence both settings have the same cofibrant objects. That being a retract of a cellularly stratified space is equivalent to being cofibrant over $P_{\mathcal{X}}$ is immediate by the cofibrant generators of Theorem 7.4.2.7. It remains to see that $\mathcal{X}$ being cofibrant in $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ is equivalent to $\mathcal{X}$ being cofibrant in $\mathbf{Strat}^{\mathfrak{d}}_{P_{\mathcal{X}}}$. This follows immediately from Proposition 7.4.2.8. □

Definition 7.5.2.2. We call a stratified topological space $\mathcal{X}$ that satisfies any of the equivalent conditions of Lemma 7.5.1.1 *triangularly cofibrant.*

Remark 7.5.2.3. Clearly every stratified space that admits a triangulation compatible with the stratification (i.e., in particular is the realization of a stratified simplicial set) is triangularly cofibrant. In particular, this holds for piece-linear pseudomanifolds (see, for example, [Ban07]) or locally compact stratified spaces which are definable (with definable stratification) in some o-minimal structure on the reals ([Dri98, Thm. 1.7]). Furthermore, by [Gor78], all Whitney stratified spaces are in this class.

However, one should not restrict oneself to stratified spaces that admit a cell structure. Topological manifolds, for example, are (to the best of our knowledge) not known to admit CW structures in dimension 4 (see [KS69] for all other dimensions). However, every topological manifold is a Euclidean neighborhood retract ([Han51]) and thus a retract of a CW complex. Supposing the existence of certain stratified cylinder neighborhoods, our theory also covers examples of stratified spaces with manifold strata (see Proposition 7.5.2.10). Let us begin by proving that stratified neighborhood retracts of triangularly cofibrant stratified spaces are triangularly cofibrant:

Proposition 7.5.2.4. *Let $\mathcal{X} \in$ **Strat** be a triangularly cofibrant stratified space. Let $A \subset \mathcal{X}$ be a closed subspace such that the following holds: There is a neighborhood $U \subset \mathcal{X}$ of A, such that $\mathcal{A} = \big(A, s_{\mathcal{X}}(U), s_{\mathcal{X}}|_A : A \to s_{\mathcal{X}}(U)\big)$ is a stratum-preserving retract of $\mathcal{U} = \big(U, s_{\mathcal{X}}(U), s_{\mathcal{X}}|_U : U \to s_{\mathcal{X}}(U)\big)$. Then A is also triangularly cofibrant.*

Proof. We will make frequent use of results on stratified cell complexes developed in [Waa24b] (see Chapter 6). Let $\mathcal{Y}$ be a cellularly stratified space and $i: \mathcal{X} \hookrightarrow \mathcal{Y}$, $r: \mathcal{Y} \to \mathcal{X}$ such that $r \circ i = 1_{\mathcal{X}}$. We claim that there is a subcomplex $\mathcal{B}$ of a cell structure on $\mathcal{Y}$ that contains $i(A)$ and is entirely contained in $r^{-1}(U)$. Then we can compose

$$\mathcal{B} \to \mathcal{U} \to \mathcal{A},$$

where $\mathcal{B}$ is considered a stratified space over $P_{\mathcal{U}}$, to obtain a retraction of $\mathcal{A} \hookrightarrow \mathcal{B}$. Fix a stratified cell structure $(\sigma_i : |\Delta^{\mathcal{J}_i}|_s \to \mathcal{Y})_{i \in I}$ on $\mathcal{Y}$ (see Section 6.2.3, for a definition and basic properties). Every cell in $\mathcal{Y}$ defined by a map σ_i is contained in a finite subcomplex of $\mathcal{Y}$ (Lemma 6.2.3.6). We proceed to inductively construct a subdivision of $\mathcal{Y}$ (in the sense of Definition 6.3.2.1), via induction over the minimal number of cells, n_i, required in a subcomplex that contains a cell σ_i. Denote by $\mathcal{Y}^{(n-1)}$ the subcomplex constructed from cells σ_i with $n_i = n$. We construct a subdivision of the cell structure of $\mathcal{Y}$, denoted $\mathcal{Y}'$, through induction on n. For $n = 1$, we simply take the cell structure on $\mathcal{Y}^{(0)}$ inherited from $\mathcal{Y}$. Now, suppose that we have already defined a cell structure $\mathcal{Y}'^{(n)}$ on $\mathcal{Y}^{(n)}$ for $n > 1$, and a subcomplex $\mathcal{B}^n \subset \mathcal{Y}^{(n)}$ of this cell structure, fulfilling the following properties:

1. $r^{-1}(A) \cap \mathcal{Y}^{(n)} \subset \mathcal{B}^n$ and $\mathcal{B}^n$ is a Δ_P-neighborhood of $r^{-1}(A) \cap \mathcal{Y}^{(n)}$ (in the sense of Definition 6.2.4.1);

2. $\mathcal{B}^n$ is contained in the interior of $r^{-1}(U)$.

Next, let us construct $\mathcal{Y}'^{(n+1)}$ and $\mathcal{B}^{n+1}$, i.e., we now add cells σ_i with $n_i = n+2$ to $\mathcal{Y}^{(n)}$. For any cell σ_i with $n_i = n+2$, we can barycentrically subdivide $|\Delta^{\mathcal{I}_i}|_s$ (sufficiently many times) so that for every $\tau \subset |\Delta^{\mathcal{J}_i}|_s$ that is contained in a simplex τ' intersecting $(r \circ \sigma)^{-1}(A)$ we have the following.

1. τ is entirely contained in the interior of $(r \circ \sigma)^{-1}(U)$;

2. $\tau \cap |\partial\Delta^{\mathcal{I}_i}|_s$ is entirely contained in $\sigma^{-1}(\mathcal{B}^n)$.

Indeed, the first property is possible, since $(r \circ \sigma)^{-1}(A)$ is compact and contained in the interior of $(r \circ \sigma)^{-1}(U)$ by (inductive) assumption. It follows that $(r \circ \sigma)^{-1}(A)$ has a positive distance (for any choice of compatible metric on $|\Delta^{\mathcal{J}_i}|_s$), to the complement of the interior of $(r \circ \sigma)^{-1}(U)$, which implies that simplices of sufficiently small diameter cannot intersect $(r \circ \sigma)^{-1}(A) \cup \sigma^{-1}(\mathcal{B}^n)$ and the complement of the interior of $(r \circ \sigma)^{-1}(U)$. That the second property can be assumed is argued similarly. Having chosen such a subdivision for any cell σ_i with $n_i = n+2$, we then obtain an induced cell structure $\mathcal{Y}'^{(n+1)}$ on $\mathcal{Y}^{(n+1)}$ given by the cells of $\mathcal{Y}'^{n}$ and the cells corresponding to (open) simplices in the interior of the subdivisions of $|\Delta^{\mathcal{I}_i}|_s$, for $n_i = n+2$. Finally, we let $\mathcal{B}^{n+1}$ be the subcomplex given by adding to $\mathcal{B}^n$ all such cells of $\mathcal{Y}'^{(n+1)} \setminus \mathcal{Y}'^{(n)}$, σ_i, which correspond to an (open) simplex in the interior of some $|\Delta^{\mathcal{I}_i}|_s$, contained in a closed simplex intersecting $(r \circ \sigma)^{-1}(A)$. Note that this does indeed define a cell complex, since the intersection of the boundary of such a simplex with the boundary of $|\Delta^{\mathcal{I}_i}|_s$ is assumed to be mapped to $\mathcal{B}^n$. $\mathcal{B}^{n+1}$ defined in this fashion fulfills the inductive assumptions by construction and Lemma 6.2.4.4. Then, finally, set $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}^n$, with the induced cell structure. □

Construction 7.5.2.5 (Stratified mapping cylinders)**.** Given $f\colon \mathcal{X} \to \mathcal{Y}$ in **Strat**, the *stratified mapping cylinder* of f, $M_s(f)$, is defined as the pushout

$$\begin{tikzcd} \mathcal{X} \times |\Delta^0|_s \arrow[r, hook, "1 \times i_0"] \arrow[d, "f"] & \mathcal{X} \times |\Delta^{[1]}|_s \arrow[d] \\ \mathcal{Y} \arrow[r, hook] & M_s(f). \end{tikzcd} \tag{7.26}$$

Under the (nonstratified) identification $|\Delta^{[1]}|_s = [0,1]$, this equips the classical mapping cylinder of f with an alternative stratification over the lower right corner of the following pushout diagram of posets.

$$\begin{tikzcd} P_{\mathcal{X}} \arrow[r, hook, "1 \times i_0"] \arrow[d, "f"] & P_{\mathcal{X}} \times [1] \arrow[d] \\ P_{\mathcal{Y}} \arrow[r, hook] & P_{\mathcal{Y}} \sqcup_{\leq f} P_{\mathcal{X}}\,. \end{tikzcd} \tag{7.27}$$

Here, $P_{\mathcal{Y}} \sqcup_{\leq f} P_{\mathcal{X}}$ is explicitly given by equipping the disjoint union $P_{\mathcal{Y}} \sqcup P_{\mathcal{X}}$, with an additional relation $p \leq q$, for $p \in P_{\mathcal{Y}}$ and $q \in P_{\mathcal{X}}$, whenever $p \leq f(q)$. In the special case where $P_{\mathcal{Y}}$ is a poset with one element, this amounts to adjoining a minimal element to $P_{\mathcal{X}}$.

Remark 7.5.2.6. Note that $M_s(f)$ is generally not a mapping cylinder of f with respect to any of the model structures on **Strat**. Indeed, the inclusion $\mathcal{Y} \hookrightarrow M_s(f)$ can only be a weak equivalence (in any of the model structures on **Strat**) if $\mathcal{X}$ is empty. From the perspective of ∞-categories of exit paths, no point $x \in \mathcal{X} \times |\Delta^{\{1\}}|_s \subset M_s(f)$ can lie in the essential image corresponding to the map $\mathcal{Y} \hookrightarrow M_s(f)$, as this would imply a non-stratified path from a point $y \in \mathcal{Y}$ to x.

Corollary 7.5.2.7. *Suppose that $\mathcal{L}, \mathcal{X}, \mathcal{Y} \in$ **Strat** are triangularly cofibrant. Let $f: \mathcal{L} \to \mathcal{Y}, g: \mathcal{L} \to \mathcal{X}$ be stratified maps. Then the stratified double mapping cylinder $M_s(f) \cup_g \mathcal{X}$ is triangularly cofibrant.*

Proof. $M_s(f) \cup_g \mathcal{X}$ fits into a pushout diagram of stratified spaces

$$\begin{tikzcd} \mathcal{L} \times |\partial\Delta^{[1]}|_s = \mathcal{L} \sqcup \mathcal{L} \arrow[r, hook] \arrow[d, "f \sqcup g"] & \mathcal{L} \times |\Delta^{[1]}|_s \arrow[d] \\ \mathcal{Y} \sqcup \mathcal{X} \arrow[r, hook] & M_s(f) \cup_g \mathcal{X}. \end{tikzcd} \tag{7.28}$$

Note that we may, without loss of generality, assume that we are in the Δ-generated scenario, as every triangularly cofibrant stratified space is Δ-generated. By Corollary 7.4.3.3 the upper horizontal is a cofibration, from which it follows that the lower horizontal is a cofibration. $\mathcal{Y} \sqcup \mathcal{X}$ is cofibrant by assumption, which, together with the cofibrancy $\mathcal{Y} \sqcup \mathcal{X} \hookrightarrow$ of $M_s(f) \cup_g \mathcal{X}$, implies the claim. ☐

Definition 7.5.2.8. Let $U \subset \mathcal{X}$ and $A \subset U$ be a closed subset contained in the interior of U. Denote $\mathcal{A} = (s_{\mathcal{X}} |_A: A \to s_{\mathcal{X}}(A))$ and $\mathcal{U} = (s_{\mathcal{X}} |_U: U \to s_{\mathcal{X}}(U))$. We say that U is a *(closed) stratified mapping cylinder neighborhood of* A, if the following holds:
There exists a stratified space $\mathcal{L}$ together with a stratified map $f: \mathcal{L} \to \mathcal{A}$, as well as a stratum-preserving homeomorphism $\phi: M_s(f) \hookrightarrow \mathcal{U}$ under $\mathcal{A}$, such that $\phi^{-1}(\partial U) = \mathcal{L} \times \{1\}$.

Definition 7.5.2.9. Let P be a poset and $p \in P$. We call

$$\sup\{n \in \mathbb{N} \mid \exists p_0, \cdots, p_n: p = p_0 < p_1 < \cdots < p_n\}$$

the *depth* of p in P. We call the supremum over the depths of all $p \in P$ the *depth* of P. Given $k \in \mathbb{N}$ and $\mathcal{X} \in$ **Strat**, we denote by $\mathcal{X}_k$ the stratified subspace given by restricting $\mathcal{X}$ along

$$(P_{\mathcal{X}})_k := \{p \in P \mid p \text{ has depth } k\} \hookrightarrow P.$$

Similarly, we denote by $\mathcal{X}_{\underline{k}}$ the stratified subspace of $\mathcal{X}$ obtained by restricting $\mathcal{X}$ along

$$(P_{\mathcal{X}})_{\underline{k}} := \{p \in P \mid p \text{ has depth } \leq k\} \hookrightarrow P.$$

Proposition 7.5.2.10. *Let $\mathcal{X}$ be a stratified space with $P_{\mathcal{X}}$ finite depth. Suppose that the following holds:*

1. *Each stratum of $\mathcal{X}$ is cofibrant in the Quillen model structure, i.e. a retract of a non-stratified absolute cell complex.*
2. *For each $k \in \mathbb{N}$, there is a family of pairwise disjoint subsets of $\mathcal{X}$, $(U^p)_{p \in (P_{\mathcal{X}})_k}$, indexed over strata of depth k such that, for each p, U^p is a stratified mapping cylinder neighborhood of $\mathcal{X}_p$.*

Then $\mathcal{X}$ is triangularly cofibrant.

Proof. For ease of notation, we denote $P := P_{\mathcal{X}}$. We proceed via induction over the depth of P, denoted d. In the case $d = 0$, $\mathcal{X}$ is simply a disjoint union of trivially stratified spaces, each of which is cellularly stratified by assumption. Now, for the inductive step d to $d+1$: By assumption, there are stratified maps $f^p: \mathcal{L}^p \to \mathcal{X}_p$, for each $p \in P_{d+1}$, together with an injective (on the space level) stratified map

$$\bigsqcup_{p \in (P_{d+1})} M(f^p) \hookrightarrow \mathcal{X},$$

which restricts to an open inclusion on the open cylinders (obtained by removing $\mathcal{X}L^p \times \{1\}$ and denoted by $\mathring{M}(f^p)$). Setting, $\mathcal{L} = \bigsqcup_{p \in P_{d+1}} \mathcal{L}_p$ and $f = \bigsqcup_{p \in P_{d+1}} f^p$, we obtain a stratified open inclusion

$$i: \bigsqcup_{p \in (P_{d+1})} \mathring{M}(f^p) \cong \mathring{M}_s(f) \hookrightarrow \mathcal{X},$$

which defines a neighborhood of $\mathcal{X}_{d+1}$. Note that i is not necessarily an inclusion on the poset level. Now, consider $\mathring{M}_s(f)$ as reparametrized (over $[0,2)$), denoted $\mathring{M}'_s(f)$, such that we may consider $M_s(f)$ (with the usual parametrization) as a closed stratified subspace of $\mathring{M}_s(f)$. With this new notation, we have inclusion $\mathring{M}_s(f) \hookrightarrow M_s(f) \hookrightarrow \mathring{M}'_s(f) \hookrightarrow \mathcal{X}$. Then $\mathring{M}_s(f) \subset \mathring{M}'_s(f)$ is an open neighborhood of $\mathcal{X}_{d+1}$ with boundary $\mathcal{L} \times \{1\}$. We obtain a commutative diagram of stratified maps

$$\begin{tikzcd} \mathcal{L} \times \{1\} \arrow[r, "g"] \arrow[d, hook] & \mathcal{X} \smallsetminus \mathring{M}_s(f) \arrow[d] \\ M_s(f) \arrow[r, hook] & \mathcal{X}, \end{tikzcd} \tag{7.29}$$

where $\mathcal{X} \smallsetminus \mathring{M}_s(f)$ is stratified over $P_{\underline{d}}$. Denote by $\tilde{\mathcal{X}}$ the pushout in **Strat**. Now, on the level of topological spaces, this diagram is clearly pushout. Therefore, the induced map $g: \tilde{\mathcal{X}} \to \mathcal{X}$ is a homeomorphism on the underlying spaces. Consider the commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[r] \arrow[dr] & P(g)_!\tilde{\mathcal{X}} \arrow[d] \\ & \mathcal{X}. \end{tikzcd} \tag{7.30}$$

Since g is a homeomorphism on the underlying spaces, it follows that the right vertical is an isomorphism. The upper vertical is always a cofibration (Proposition 7.4.2.8), hence it follows that $\tilde{\mathcal{X}} \to \mathcal{X}$ is a cofibration. Therefore, it suffices to show that $\tilde{\mathcal{X}}$ is cofibrant. For this, in turn, it suffices to show that the left-hand vertical is a cofibration and $\mathcal{X} \smallsetminus \mathring{M}_s(f)$ is cofibrant. The latter is a retract of $\mathcal{X}_{\underline{d}}$. Hence, cofibrancy follows by inductive assumption. For the left vertical, by Corollary 7.5.2.7, it suffices to see that $\mathcal{L}_p$ and $\mathcal{X}_p$ are cofibrant. The latter is cofibrant by assumption. By construction, $\mathcal{L}^p$ embeds into $\mathcal{X}$ with an open (stratified) neighborhood of the form $\mathcal{L}^p \times (0,1)$. Clearly, $\mathcal{L}^p$ is a retract of the latter, making $\mathcal{L}^p$ a stratified neighborhood retract of $\mathcal{X}_{\underline{d}}$, which is cofibrant by inductive assumption. Hence, the case of $\mathcal{L}^p$ follows by Proposition 7.5.2.4. □

Remark 7.5.2.11. Every topological manifold is a Euclidean neighborhood retract ([Han51]) and since every open subset of Euclidean space can be triangulated, it follows that every topological manifold is cofibrant in the Quillen model structure. Consequently, it follows by Proposition 7.5.2.10 that every stratified space with manifold strata, each of which admits appropriate stratified mapping cylinder neighborhoods, is cofibrant. [AFT17, Prop. 8.2.3] asserts the existence of stratified mapping cylinder neighborhoods for conically smooth stratified spaces, which would make the latter triangularly cofibrant. More generally, for homotopically stratified spaces with manifold strata there are obstructions to the existence of (pairwise) stratified mapping cylinder neighborhoods ([Qui88, Thm. 1.7]).

7.5.3 Refined stratified spaces

Let us take a more detailed look at the cofibrant objects in the semi-model categories $\underline{\textbf{Strat}}^{\mathfrak{d}}$ and $\underline{\textbf{Strat}}^{\mathfrak{c}}$. In particular, our aim is to relate them to more classical properties of stratified spaces.

Definition 7.5.3.1. A stratified space $\mathcal{X} \in$ **Strat** is called *refined* if the following holds:

- $\mathcal{X}$ is *surjectively stratified*, that is, $s_{\mathcal{X}}: X \to P_{\mathcal{X}}$ is surjective.
- For each pair $x, y \in \mathcal{X}$, the relation $s_{\mathcal{X}}(x) \leq s_{\mathcal{X}}(y)$ holds if and only if there is a finite sequence of stratified maps $\gamma_i: |\Delta^{[1]}|_s \to \mathcal{X}$, $i = 1, \dots, n$, with $\gamma_1(0) = x$, $\gamma_n(1) = y$ and $\gamma_i(1) = \gamma_{i+1}(0)$, for $i < n$.

We can think of being refined as the poset $P_{\mathcal{X}}$ being completely reflected in the stratified paths of $P_{\mathcal{X}}$. Next, note the following elementary properties about refinedness.

Proposition 7.5.3.2. *Let $\mathcal{X}, \mathcal{Y} \in$ **Strat** and $\mathcal{A} \in$ s**Strat**. Then the following holds:*

1. *$\mathcal{X}$ is refined, if and only if $\mathrm{Sing}_s(\mathcal{X})$ is refined;*
2. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a poset-preserving categorical equivalence, then $\mathcal{X}$ is refined, if and only if $\mathcal{Y}$ is refined;*
3. *$\mathcal{A}$ is refined, if and only if $|\mathcal{A}|_s$ is refined;*
4. *If $\mathcal{A} \in \mathrm{s}\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ is fibrant, i.e. A a quasi category and $A \to P$ a conservative functor, then $\mathcal{A}$ is refined if and only if $\mathcal{A}$ is 0-connected, in the sense of [BGH18], as an abstract stratified homotopy type.*

It follows that being refined is really purely a property of the homotopy type in $\mathcal{S}\mathbf{trat}^{\mathfrak{d},\mathfrak{p}}$ defined by $\mathcal{X}$. The model categories $\mathbf{Strat}^{\mathfrak{c}}$ and $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}$ may now be interpreted as the respective right Bousfield localizations presenting the full sub-∞-categories of refined stratified spaces. Even more, we may essentially construct the left adjoints of this adjunction on the 1-categorical level.

Definition 7.5.3.3. A stratified space $\mathcal{X} \in$ **Strat** is said to carry the Δ_s topology if one of the following equivalent conditions holds:

1. X has the final topology with respect to the set of stratified maps $|\Delta^{[n]}|_s \to \mathcal{X}$, for $n \in \mathbb{N}$.
2. X has the final topology with respect to the set of stratified maps $|\Delta^{[1]}|_s \to \mathcal{X}$.

Proof. Let us show that these conditions are equivalent. We need to see that every stratified simplex $|\Delta^{[n]}|_s$, for $n \in \mathbb{N}$, carries the final topology with respect to stratified maps with source $|\Delta^{[1]}|_s$. Note that it suffices to see that for any convergent sequence in $|\Delta^{[n]}|_s$, there exists a subsequence $(x_i)_{i\in\mathbb{N}}$, as well as a stratified map $\gamma\colon [0,1] \cong |\Delta^{[1]}|_s \to |\Delta^{[n]}|_s$, with $\gamma(\frac{1}{2^i}) = x_i$, for $i \in \mathbb{N}$. Indeed, the topology on $|\Delta^{[n]}|_s$ is entirely determined by convergent sequences, and the latter conditions mean that these are detected by $|\Delta^{[1]}|_s$. So suppose that we are given a convergent sequence $(\hat{x}_i)_{i\in\mathbb{N}}$ in $|\Delta^{[n]}|_s$. By passing to a subsequence, we may assume that the sequence is contained in a single stratum $j \in [n]$. Then, the limit point lies in some stratum $k \leq j$. Now define $\gamma\colon [0,1] \cong |\Delta^{[1]}|_s \to |\Delta^{[n]}|_s$ by setting $\gamma(\frac{1}{2^i}) \coloneqq x_i$, convexly interpolating between x_i and x_{i+1} and sending 0 to the limit point of x_i. This map is continuous. Furthermore, as the strata of $|\Delta^{[n]}|_s$ are convex, it is also stratified. ☐

Definition 7.5.3.4. A stratified space $\mathcal{X} \in$ **Strat** is called *strongly refined*, if $\mathcal{X}$ is refined and carries the Δ_s-topology.

Construction 7.5.3.5. For a stratified space $\mathcal{X} \in$ **Strat**, its *refined poset*, denoted $P_{\mathcal{X}^{\mathfrak{r}}}$, is the poset generated by the following elements and relations:

- An element for each $x \in \mathcal{X}$.
- A relation $x \leq y$, whenever there is a sequence of stratified paths from x to y, as in Definition 7.5.3.1.

Equivalently, $P_{\mathcal{X}^{\mathfrak{r}}}$ is given by the set of path components of strata of $\mathcal{X}$, together with a generating relation whenever there is an exit path from one component to another. Consider the (generally non-continuous) map $X \to P_{\mathcal{X}^{\mathfrak{r}}}$, given by $x \mapsto [x]$. It provides a factorization of $s_{\mathcal{X}}$ through the map of posets

$$\begin{aligned} P_{\mathcal{X}^{\mathfrak{r}}} &\to P_{\mathcal{X}} \\ [x] &\mapsto s_{\mathcal{X}}(x) \end{aligned}$$

as follows:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr, "s_{\mathcal{X}}"'] & P_{\mathcal{X}^{\mathfrak{r}}} \arrow[d] \\ & P_{\mathcal{X}} \end{tikzcd} \tag{7.31}$$

Although $X \to P_{\mathcal{X}^{\mathfrak{r}}}$ is not necessarily continuous, its precomposition with any stratified map $|\Delta^{[1]}|_s \to \mathcal{X}$ is continuous, by construction. We denote by $\mathcal{X}^{\mathfrak{r}}$ the stratified space given by $X^{\mathfrak{r}} \to P_{\mathcal{X}^{\mathfrak{r}}}$, where $X^{\mathfrak{r}}$ has the same underlying set as X and is equipped with the final topology with respect to stratified maps $|\Delta^{[1]}|_s \to \mathcal{X}$. By construction, $X^{\mathfrak{r}} \to P_{\mathcal{X}^{\mathfrak{r}}}$ is indeed continuous and $\mathcal{X}^{\mathfrak{r}}$ is a strongly refined stratified space. This construction induces a simplicial functor from **Strat** into the full subcategory of strongly refined stratified spaces, called *the refinement functor*, which exposes the latter as a full coreflective subcategory of **Strat**. The counit of adjunction, given by the commutative squares

$$\begin{tikzcd} X^{\mathfrak{r}} \arrow[r] \arrow[d, "s_{\mathcal{X}^{\mathfrak{r}}}"] & X \arrow[d, "s_{\mathcal{X}}"] \\ P_{\mathcal{X}^{\mathfrak{r}}} \arrow[r] & P_{\mathcal{X}} , \end{tikzcd} \tag{7.32}$$

is called the *refinement map*.

As a consequence of the existence of the refinement functor, which makes the inclusion of strongly refined stratified spaces a left adjoint, we obtain:

Corollary 7.5.3.6. *Every colimit of strongly refined stratified spaces in* **Strat** *is again strongly refined.*

The refinement construction of Construction 7.5.3.5 is really the topological analogue of Definition 5.3.2.12.

Proposition 7.5.3.7 (See Section 5.3.2, for notation)**.** *For any $\mathcal{X} \in$* **Strat***, applying* Sing_s *to the refinement map induces a natural transformation (dashed) that makes the diagram.*

$$\begin{tikzcd} (\mathrm{Sing}_s \mathcal{X})^{\mathfrak{r}} \arrow[r, dashed] \arrow[dr] & \mathrm{Sing}_s(\mathcal{X}^{\mathfrak{r}}) \arrow[d] \\ & \mathrm{Sing}_s(\mathcal{X}) \end{tikzcd} \tag{7.33}$$

commute. The dashed transformation is an isomorphism.

Proof. $\mathrm{Sing}_s \mathcal{X}^{\mathfrak{r}}$ is refined by Proposition 7.5.3.2. Hence, the dashed map is induced by the fact that (simplicial) refinement is right adjoint to the inclusion of refined stratified simplicial sets. That this map is an isomorphism on posets follows immediately by the construction of $P_{\mathcal{X}^{\mathfrak{r}}}$ in Construction 7.5.3.5 and Proposition 5.3.2.9. Furthermore, since every stratified simplex $|\Delta^{[n]}|_s$ is refined, note that the map is an isomorphism on simplicial sets, given by

$$(\mathrm{Sing}_s \mathcal{X})^{\mathfrak{r}}[n] = (\mathrm{Sing}_s \mathcal{X}) = \mathbf{Strat}(|\Delta^{[n]}|_s, \mathcal{X}) = \mathbf{Strat}(|\Delta^{[n]}|_s, \mathcal{X}^{\mathfrak{r}}) = \mathrm{Sing}_s(\mathcal{X}^{\mathfrak{r}})[n].$$

□

We may think of Proposition 7.5.3.7 as stating that Sing_s sends the topological refinement map to the simplicial refinement map. As an immediate corollary, using Recollection 7.3.2.4, we obtain the missing part of Theorem 7.3.3.1:

Corollary 7.5.3.8. *The simplicial semi-model categories* $\underline{\mathbf{Strat}}^{\mathfrak{d}}$ *and* $\underline{\mathbf{Strat}}^{\mathfrak{c}}$ *are obtained, respectively, from* $\underline{\mathbf{Strat}}^{\mathfrak{d},\mathfrak{p}}$ *and* $\underline{\mathbf{Strat}}^{\mathfrak{c},\mathfrak{p}}$ *by right Bousfield localizing at the refinement maps* $\mathcal{X}^{\mathfrak{r}} \to \mathcal{X}$.

In particular, we have the following description of cofibrant objects in $\mathbf{Strat}^{\mathfrak{d}}$ and $\mathbf{Strat}^{\mathfrak{c}}$:

Proposition 7.5.3.9. *Let $\mathcal{X} \in$* **Strat***. Then the following statements are equivalent:*

1. *$\mathcal{X}$ is cofibrant in* $\mathbf{Strat}^{\mathfrak{d}}$*;*

2. *$\mathcal{X}$ is cofibrant in* $\mathbf{Strat}^{\mathfrak{c}}$*;*

3. *$\mathcal{X}$ is triangularly cofibrant and refined;*

4. *$\mathcal{X}$ is a retract of a refined, cellularly stratified space.*

Proof. The equivalence between the first two statements is immediate from both semi-model categories having the same generating cofibrations. Since every cellularly stratified space carries the Δ_s-topology (by definition), it follows that a refined cellularly stratified space is strongly refined. Since the latter form a full coreflective subcategory, by Construction 7.5.3.5, any retract of strongly refined spaces is strongly refined. In particular, the fourth statement implies the third. To see that the third implies the second, note that cofibrant objects $\mathcal{X}$ in $\mathbf{Strat}^{\mathfrak{c}}$ are characterized by $\emptyset \hookrightarrow \mathcal{X}$ having the left lifting property, with respect to all stratified maps $f: \mathcal{Y} \to \mathcal{Z}$, which are acyclic fibrations in $\mathbf{Strat}^{\mathfrak{c}}$. By Proposition 7.5.3.7 and Theorem 5.3.2.19, this, in turn, is equivalent to $f^{\mathfrak{r}}$ being an acyclic fibration in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$. Since $\mathcal{X}$ is assumed to be refined, a lifting diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r] \arrow[ur, dashed] & \mathcal{Z} \end{tikzcd} \tag{7.34}$$

admits a solution if and only if the induced diagram

$$\begin{tikzcd} & \mathcal{Y}^{\mathfrak{r}} \arrow[d, "f^{\mathfrak{r}}"] \\ \mathcal{X} \arrow[r] \arrow[ur, dashed] & \mathcal{Z}^{\mathfrak{r}}. \end{tikzcd} \tag{7.35}$$

admits a solution. The latter holds, as $\mathcal{X}$ was assumed cofibrant in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$. Finally, to see that the first characterization implies the fourth, note that every cofibrant object in $\mathbf{Strat}^{\mathfrak{c}}$ is a retract of an absolute cell complex with respect to the stratified boundary inclusions $\{|\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]}|_s \mid n \in \mathbb{N}\}$. It follows by Proposition 7.5.3.2, stratified simplices carrying the Δ_s-topology and Corollary 7.5.3.6, that every such absolute cell complex is a refined cellularly stratified space. ☐

7.5.4 Frontier conditions and refinement

It turns out that being refined is strongly related to the way the poset structure on the strata of a stratified space is classically constructed:

Recollection 7.5.4.1. Classically, stratifications often arise from the so-called *frontier condition* (see, for example, [Mat12]). Namely, one starts with a topological space X and a locally finite decomposition into nonempty locally closed pieces $(X_i)_{i \in I}$. One assumes that for any $i \in I$, the closure $\overline{X_i}$ is given by the disjoint union $(X_j)_{j \in J}$, for some subset $J \subset I$. Then, I naturally carries the structure of a poset, setting $i \leq j$, whenever $X_i \subset \overline{X_j}$, and $X \to I$ defines a stratification of X.

In our framework, such poset structures induced by closure relations can be constructed as follows.

Construction 7.5.4.2. Let $\mathcal{X} \in \mathbf{Strat}$, and denote

$$I = \{S \subset X \mid S \neq \emptyset \exists p \in P_{\mathcal{X}} : S = \mathcal{X}_p.\}$$

We consider $I_{\mathcal{X}}$ as equipped with the structure of a poset, by equipping it with the relation generated by

$$S \leq S' \iff S \cap \overline{S} \neq \emptyset.$$

It follows immediately by construction that mapping S to the unique $p \in P_{\mathcal{X}}$, for which $S = \mathcal{X}_p$, induces a map of posets

$$I_{\mathcal{X}} \to P_{\mathcal{X}}.$$

Definition 7.5.4.3. We say that $\mathcal{X}$ is *weakly frontier stratified*, if the induced map $I_{\mathcal{X}} \to P_{\mathcal{X}}$ is an isomorphism of posets. We say that $\mathcal{X}$ is *frontier stratified*, if in addition to this it fulfills

$$\mathcal{X}_p \cap \overline{\mathcal{X}_q} \neq \varnothing \implies \mathcal{X}_p \subset \overline{\mathcal{X}_q},$$

for all $p, q \in P_{\mathcal{X}}$.

Remark 7.5.4.4. If one describes stratified spaces $\mathcal{X}$ and $\mathcal{Y}$ as the data of spaces X and Y equipped, respectively, with decompositions $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ into non-empty pieces, then classically the morphisms which are considered between such spaces are given by continuous maps $f\colon X \to Y$, such that for each $i \in I$ there exists a $j \in J$ with $f(X_i) \subset Y_j$ [Hug99b]. Let us call such objects decomposition spaces, and such maps decomposed maps, and denote the corresponding category by $\mathbf{D}$. Homotopies in this setting are defined through the cylinder given by equipping $X \times [0,1]$ with the decomposition $(X_i \times [0,1])_{i\in I}$. There is an obvious forgetful functor from the category $\mathcal{D}\colon \mathbf{Strat} \to \mathbf{D}$, given by equipping a stratified space with its decomposition into nonempty strata. Now, if $\mathcal{X}$ is weakly frontier stratified, then it is not hard to see that for any $\mathcal{Y} \in \mathbf{Strat}$ the induced map

$$\mathbf{Strat}(\mathcal{X}, \mathcal{Y}) \to \mathbf{D}(\mathcal{X}, \mathcal{Y})$$

is a bijection. Furthermore, the forgetful functor commutes with cylinders. It follows that as long as one restricts to bifibrant objects (in $\mathbf{Strat}^{\mathfrak{d},\mathfrak{p}}, \mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$) that are weakly frontier stratified then the resulting homotopy theory agrees with the classical homotopy theory of stratified spaces as studied in [Qui88; Hug99b; Mil13].

It turns out that the triangularly cofibrant objects that have path-connected strata and are weakly frontier stratified are precisely the cofibrant objects in the refined setting (see Proposition 7.5.4.5 below).
Note that for strongly frontier stratified spaces the generating relations of Construction 7.5.4.2 already define a partial order. This leads to the following alternative characterization of refinedness, for specific stratified spaces.

Proposition 7.5.4.5. *Let $\mathcal{X} \in$ **Strat**, and consider the following conditions:*

1. *$\mathcal{X}$ is refined.*
2. *$\mathcal{X}$ has path-connected strata and is frontier stratified.*
3. *$\mathcal{X}$ has path-connected strata and is weakly frontier stratified.*

The first and the second property imply the third. Furthermore, if $\mathcal{X}$ is triangularly cofibrant, then the third property implies the first. Finally, if $\mathcal{X}$ is categorically fibrant, then the first property implies the third. In particular, for bifibrant stratified spaces in $\mathbf{Strat}^{\mathfrak{c},\mathfrak{p}}$ *all three properties are equivalent.*

Proof. The first two implications are obvious, using the fact that for any continuous map $f\colon X \to Y$, $x \in \overline{S}$ implies $f(x) \in \overline{f(S)}$. Suppose the third condition holds and $\mathcal{X}$ is triangularly cofibrant. To see that $\mathcal{X}$ is refined, we need to expose for any $x, y \in \mathcal{X}$ with $p := s_{\mathcal{X}}(x) < s_{\mathcal{X}}(y) := q$ a sequence of stratified paths $\gamma_i\colon |\Delta^{[1]}|_s \to \mathcal{X}$ from x to y, as in Definition 7.5.3.1. Since $\mathcal{X}$ is weakly frontier stratified, it suffices to show that for any pair $p, q \in P_{\mathcal{X}}$ and $x \in \mathcal{X}_p, y \in \mathcal{X}_q$ with

$$\mathcal{X}_p \cap \overline{\mathcal{X}_q} \neq \varnothing,$$

there is a stratified path from x to y. Furthermore, since strata are assumed to be path-connected, it suffices to construct a path from any element in $\mathcal{X}_p$ to any element in $\mathcal{X}_q$. Now, let $\mathcal{X} \xrightarrow{i} \mathcal{Y} \xrightarrow{r} \mathcal{X}$ expose $\mathcal{X}$ as a retract of a cellularly stratified space $\mathcal{Y}$ (Proposition 7.5.2.1). Any nonempty closure intersection

$$\mathcal{X}_p \cap \overline{\mathcal{X}_q} \neq \varnothing$$

in $\mathcal{X}$ implies a non-empty closure intersection

$$\mathcal{Y}_{i(p)} \cap \overline{\mathcal{Y}_{i(q)}} \neq \varnothing.$$

Conversely, every sequence of stratified paths in $\mathcal{Y}$, starting and ending in $\mathcal{X}$, descends to a sequence of stratified paths in $\mathcal{X}$ with the same starting and end points. Hence, it suffices to show that a closure relation

$$\mathcal{Y}_{i(p)} \cap \overline{\mathcal{Y}_{i(q)}} \neq \varnothing,$$

implies the existence of a concatenable sequence of stratified paths starting in $\mathcal{Y}_{i(p)}$ and ending in $\mathcal{Y}_{i(q)}$. Choose a cell structure for $\mathcal{Y}$ (see Definition 6.2.3.1), with open cells e_i, $i \in I$. Note that each open cell is entirely contained in only one stratum. We write $s_{\mathcal{Y}}(e_i) \in P_{\mathcal{Y}}$ to denote the latter. Furthermore, note that if e_i intersects the closure of e_j, then $s_{\mathcal{Y}}(e_i) \leq s_{\mathcal{Y}}(e_j)$. Suppose that $x \in \mathcal{Y}_{i(p)} \cap \overline{\mathcal{Y}_{i(q)}}$. In particular, x is contained in the minimal subcomplex of $\mathcal{Y}$ that contains $\mathcal{Y}_{i(q)}$. This implies that there is a sequence of cells $e_0, e_1, e_2, \dots, e_n$, such that $x \in e_0$ and $e_n \subset \mathcal{Y}_q$, and for each $i \in [n-1]$, e_i intersects the closure of e_{i+1}. For $i \in [n-1]$, let $x_i \in e_i \cap \overline{e_{i+1}}$. Since $\overline{e_{i+1}}$ is the quotient of a stratified simplex $|\Delta^{\mathcal{J}}|_s$ over $P_{\mathcal{Y}}$, there is a stratified path $\gamma_i' \colon |\Delta^{[1]}|_s \to \mathcal{Y}$, starting in x_i, immediately entering and staying in e_{i+1}, and ending in x_{i+1}.
The converse implication is immediate. Finally, assume that $\mathcal{X}$ is categorically fibrant and refined (and is thus weakly frontier stratified). Then, whenever $s_{\mathcal{X}}(x) \leq s_{\mathcal{X}}(y)$, any concatenable sequence of stratified paths from x to y induces a stratified path, $\gamma \colon |\Delta^{[1]}|_s \to \mathcal{X}$ from x to y, which implies

$$x = \gamma(0) \subset \overline{\gamma(0,1]} \subset \overline{\mathcal{X}_{s_{\mathcal{X}}(y)}}.$$

In particular, it follows that

$$\mathcal{X}_p \cap \overline{\mathcal{X}_q} \neq \varnothing \implies \mathcal{X}_p \subset \overline{\mathcal{X}_q}.$$

□

The characterization of refinedness in Proposition 7.5.4.5 allows us to represent all homotopy types in $\mathbf{Strat}^{\mathfrak{c}}$ through the following particularly convenient stratified spaces.

Definition 7.5.4.6. A stratified space $\mathcal{X} \in \mathbf{Strat}$ is called *CFF stratified* (C for cellular, first F for frontier, second F for fibrant), if it fulfills the following conditions:

1. $\mathcal{X}$ is cellularly stratified.
2. $\mathcal{X}$ has nonempty, connected strata and is frontier stratified.
3. $\mathcal{X}$ has the horn filling property with respect realizations of inner stratified horn inclusions $|\Lambda^{[n]}_k \hookrightarrow \Delta^{[n]}|_s$, $0 < k < n$.

We denote the full simplicial subcategory of $\underline{\mathbf{Strat}}$ given by CFF stratified spaces by $\underline{\mathbf{CFF}}$.

CFF stratified spaces can be seen as an analogue to CW complexes in the classical scenario. However, note that no assumptions are made that the attaching maps of cells only map to cells of certain dimensions. This is necessary for the small object argument to be able to produce CFF stratified spaces.

Example 7.5.4.7. Every stratified space that admits a PL structure that is compatible with its stratification is cellularly stratified. Furthermore, in increasing order of generality, every Whitney stratified space, Thom-Mather stratified space, pseudomanifold, or conically stratified space (in the sense of [Lur17, A.5]) has the inner horn filling property, by [Lur17, Thm. A.6.5]. Hence, it follows that if we equip such spaces with the refined stratification (induced by the frontier condition, and by taking path components of strata) and additionally assume a piecewise linear structure, then they provide examples of CFF stratified spaces. Note also that for Thom-Mather (and hence for Whitney stratified spaces) a piecewise linear structure (compatible with the stratification) always exists ([Gor78]).

Let us now characterize the bifibrant objects of $\underline{\mathbf{sStrat}}^{\mathfrak{c}}$ in terms of CFF stratified spaces. Using the small object argument on the generating classes in Theorem 7.4.2.10, together with Propositions 7.5.3.9 and 7.5.4.5, we obtain the following corollary.

Corollary 7.5.4.8. *A stratified space $\mathcal{X} \in \underline{\mathbf{Strat}}^{\mathfrak{c}}$ is bifibrant if and only if it is a retract of a CFF stratified space. Furthermore, every bifibrant stratified space is stratified homotopy equivalent to a CFF stratified space. Every stratified space is categorically equivalent to a CFF stratified space.*

In particular, it follows that the homotopy theory defined by $\mathbf{Strat}^{\mathfrak{c}}$ may equivalently be interpreted in terms of CFF stratified spaces:

Corollary 7.5.4.9. *Denote by H_s the class of stratified homotopy equivalences between CFF stratified spaces. The inclusion* $\mathbf{CFF} \to \mathbf{Strat}$ *induces an equivalence of ∞-categories*

$$\mathbf{CFF}[H_s^{-1}] \xrightarrow{\simeq} \mathcal{S}\mathbf{trat}^{\mathfrak{c}}.$$

Combining this result with Corollary 7.4.4.5, we obtain the following version of the stratified homotopy hypothesis:

Corollary 7.5.4.10. *Luries exit-path construction induces an equivalence of ∞-categories*

$$\mathbf{CFF}[H_s^{-1}] \xrightarrow{\simeq} \mathcal{L}\mathbf{ay}_\infty$$

between CFF stratified spaces localized at stratified homotopy equivalences and layered ∞-categories.

7.5.5 Stratified homotopy link fibrations

Much of the literature on stratified spaces takes an inductive approach to the study of stratified spaces. This follows the observation that in many geometric scenarios a stratified space $\mathcal{X}$ over a finite linear poset P with minimal element p can be decomposed into a diagram

$$\mathcal{X}_p \leftarrow \mathcal{E} \to \mathcal{Y}$$

where $\mathcal{E}$ and $\mathcal{Y}$ are stratified over $P_{>p}$ and $\mathcal{E} \to \mathcal{X}_p$ is a kind of *fibration with stratified fiber* or a retraction associated to some stratified notion of a block bundle. See [Wei94] for a good overview of such phenomena and [Tho69; Sto72; Wei94; Hug99b], for some examples of this approach. If one keeps inductively repeating this kind of procedure with $\mathcal{E}$ and $\mathcal{Y}$, one ultimately ends up with a diagram of stratified spaces indexed over the poset of linear flags in P, $\mathrm{sd}(P)$. Whatever geometric construction one uses to decompose one's stratified space, at least from a homotopy-theoretic perspective, one ultimately ends up with the associated diagram of generalized homotopy links $\mathcal{H}o\mathrm{Link}(\mathcal{X}) \in \mathbf{Fun}((\mathrm{sd}(P))^{\mathrm{op}}, \mathcal{S}\mathbf{paces})$ [9]. In order to capture the homotopy-theoretic essence of these types of constructions, Hughes (see [Hug99b]) used a stratified version of the pairwise homotopy links of [Qui88], thus obtaining a functorial, homotopy-theoretic version of such decompositions. In this subsection, we replicate this construction using the cartesian structure on **Strat**, and derive a series of homotopy-theoretic consequences. Most of these are probably known to the expert, at least in the alternative framework of homotopically stratified spaces. However, we think they may also help to connect the classical inductive approach to stratified algebraic topology with our results on stratified homotopy theory.

Notation 7.5.5.1. In the following, we will often cover both the categorical as well as the diagrammatic cases in one statement. When we add the prefix diagrammatic or categorical to names of stratum-preserving maps, such as fibrations, we mean that they are fibrations in the respective model structure on $\mathbf{Strat}_P$. We will then often add the alternative prefix in parenthesis to indicate that both cases hold.

[9] This is more of a meta theorem, which at least holds in all of the cases known to the author. The reader should think of it as a heuristic to motivate the rigorous mathematics performed below.

Construction 7.5.5.2. Let P be some poset, and let $\mathcal{X} \in \mathbf{Strat}_P$. Consider the evaluation map

$$\mathcal{X}^{|\Delta^{[1]}|_s} \to \mathcal{X}^{|\partial\Delta^{[1]}|_s} = \mathcal{X} \times \mathcal{X},$$

where the left component of this map is given by evaluation of a stratified path at 0 and the right component by evaluation at 1. We denote by $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$ the $P_{>p}$ stratified space, obtained via the following diagram of pullback squares in **Strat**.

$$\begin{tikzcd} \mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}^{|\Delta^{[1]}|_s} \arrow[d] \\ \mathcal{X}_p \times \mathcal{X}_{>p} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \times \mathcal{X} \arrow[d] \\ P_{>p} \cong \{p\} \times P_{>p} \arrow[r, hook] & P \times P \end{tikzcd} \tag{7.36}$$

In other words, we may either think of $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$ as a restriction of $|\mathcal{X}|_s{}^{|\Delta^1|_s}$ to $\{p\} \times \mathcal{X}_{>p}$, or as the stratified space of such stratified paths in $\mathcal{X}$ that start in the p-stratum, and immediately exit. We call this stratified space the *p-th stratified homotopy link of* $\mathcal{X}$. We may then compose $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to \mathcal{X}_p \times \mathcal{X}_{>p}$ with the projections to $\mathcal{X}_p$ and $\mathcal{X}_{>p}$, to obtain a diagram in **Strat**

$$\mathcal{X}_p \twoheadleftarrow \mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to \mathcal{X}_{>p}$$

where the two maps are respectively given by evaluating a stratified path in $\mathcal{X}$ at the start and end-point. This construction extends to a functor from $\mathbf{Strat}_P$ into the category of spans in **Strat**

$$B \leftarrow \mathcal{E} \to \mathcal{Y}$$

where T is trivially stratified, and $\mathcal{E} \to \mathcal{Y}$ is a stratum-preserving map over $P_{>p}$.

As a corollary of the cartesianity of the semi-model structures on **Strat**, one obtains:

Lemma 7.5.5.3. *Let $\mathcal{X} \in \mathbf{Strat}_P$ be diagrammatically fibrant (categorically fibrant). Then the starting point evaluation map*

$$\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to \mathcal{X}_p$$

is a diagrammatic (categorical) fibration.

Proof. It follows from the cartesianity of the model structures on **Strat**, that the right vertical in the pullback square

$$\begin{tikzcd} \mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}^{|\Delta^{[1]}|_s} \arrow[d] \\ \mathcal{X}_p \times \mathcal{X}_{>p} \arrow[r, hook] & \mathcal{X} \times \mathcal{X} \end{tikzcd} \tag{7.37}$$

is a diagrammatic (categorical) fibration. Consequently, so is the left vertical. As $\mathcal{X}$ is diagrammatically (categorically) fibrant, the projection map $\mathcal{X}_p \times \mathcal{X}_{>p} \to \mathcal{X}_p$ is also a diagrammatic (categorical) fibration. The evaluation map in question is now the composition of these two diagrammatic (categorical) fibrations. ☐

Let us quickly compare our framework to the stratified path spaces studied in [Hug99b]. This first requires another remark on spaces with decompositions:

Remark 7.5.5.4. Remark 7.5.4.4 also holds in a self-enriched sense, providing an answer to Question Q(6): In [Hug99b] the author studied stratified notions of mapping space, given by equipping the set of decomposition maps $\mathbf{C}D(\mathcal{X}, \mathcal{Y})$ with the subspace topology of the compact-open topology, and the decomposition over $P_{\mathcal{Y}}^{P_{\mathcal{X}}}$ induced by mapping a stratified space to its underlying map of posets. If we work in the setting where **Top** is the category of Δ-generated spaces or compactly generated spaces, and thus replace the subspace topology with its respective Kelleyfication, then this construction defines the internal mapping space in the category of decomposition spaces. There is an obvious map of decomposition spaces

$$\mathcal{D}(\mathcal{Y}^{\mathcal{X}}) \to \mathcal{D}(\mathcal{Y})^{\mathcal{D}(\mathcal{X})},$$

given on the set level by the map $\mathbf{Strat}(\mathcal{X},\mathcal{Y}) \to \mathbf{D}(\mathcal{X},\mathcal{Y})$. Therefore, whenever $\mathcal{X}$ is refined, the comparison map of stratified mapping spaces above is bijective. Furthermore, by construction of $\mathcal{Y}^{\mathcal{X}}$ and [May16, Cor. 2.2.11], it is a decomposition preserving homeomorphism, whenever $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ are finite. In other cases, it is the map that refines the topology on $\mathcal{D}(\mathcal{Y})^{\mathcal{D}(\mathcal{X})}$ such that the induced map

$$\mathcal{D}(\mathcal{Y})^{\mathcal{D}(\mathcal{X})} \to P_{\mathcal{Y}}^{P_{\mathcal{X}}}$$

is continuous.

Remark 7.5.5.5. In [Hug99b], Hughe's main object of study was the space of stratified paths in a stratified space $\mathcal{X}$ with finitely many strata, which start in a closed union of strata $\mathcal{A} \subset \mathcal{X}$, which we denote by $\mathrm{Path}_{nsp}(\mathcal{X},\mathcal{A})$. In the special case when $\mathcal{A} = \mathcal{X}_p$, for some $p \in P$, then we can think of $\mathrm{Path}_{nsp}(\mathcal{X},\mathcal{A})$ as the union of $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$ with the space of paths entirely contained in $\mathcal{X}_p$. One of the main results of [Hug99b] is that when $\mathcal{X}$ is a homotopically stratified space, then the starting point evaluation map $\mathrm{Path}_{nsp}(\mathcal{X},\mathcal{A}) \to \mathcal{A}$ lifts stratified homotopies. Using the cartesian closedness of the semi-model structures on **Strat** we may recover a version of this result in **Strat**: Given a categorically (respectively, diagrammatically) fibrant stratified space $\mathcal{X} \in \mathbf{Strat}$, the starting point evaluation map $\mathcal{X}^{|\Delta^{[1]}|_s} \to \mathcal{X}$ is a stratified fibration. Now, for any subspace $\mathcal{A} \subset \mathcal{X}$, equipped with the induced stratification, we may consider the pullback diagram

$$\begin{tikzcd} \mathcal{X}^{|\Delta^{[1]}|_s} \times_{\mathrm{ev}_0} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{X}^{|\Delta^{[1]}|_s} \arrow[d, "\mathrm{ev}_0"] \\ \mathcal{A} \arrow[r] & \mathcal{X}\,. \end{tikzcd} \tag{7.38}$$

It follows that the right vertical is a categorical (or, respectively, diagrammatic) fibration. In particular, it has the lifting property with respect to stratified homotopies with cofibrant source. Since $|\Delta^{[1]}|_s$ is clearly strongly frontier stratified, the natural comparison map in (7.5.5.4) induces a natural continuous bijection

$$\mathcal{D}(\mathcal{X}^{|\Delta^{[1]}|_s} \times_{\mathrm{ev}_0} \mathcal{A}) \to \mathrm{Path}_{nsp}(\mathcal{X},\mathcal{A}),$$

which refines the topology on $\mathrm{Path}_{nsp}(\mathcal{X},\mathcal{A})$ in order to turn it into a poset-stratified space (and make it compactly or Δ-generated). Hence, one obtains a Serre (as opposed to Hurewicz-style homotopy theory) version of Hughes's result, replacing homotopically stratified spaces with the weaker condition of diagrammatic fibrancy and not requiring that $\mathcal{A}$ is a closed union of strata, but only obtaining the homotopy lifting property with respect to triangularly cofibrant stratified spaces.

Much like generalized homotopy links, stratified homotopy links can be computed in terms of appropriately stratified regular neighborhoods. Let us first observe that stratified homotopy links can be computed locally:

Lemma 7.5.5.6. *Let $\mathcal{X} \in \mathbf{Strat}_P$, $p \in P$ and $\mathcal{N} \subset \mathcal{X}$ be a neighborhood of the p-stratum $\mathcal{X}_p \subset \mathcal{X}$. Then the induced map*

$$\mathcal{H}o\mathrm{Link}^s_p\mathcal{N} \to \mathcal{H}o\mathrm{Link}^s_p\mathcal{X}$$

is a diagrammatic equivalence.

Proof. This is a consequence of Proposition 6.3.1.21 and Lemma 7.5.5.9 below. □

Furthermore, one obtains the following stratified analogue of the two strata case of Proposition 6.4.0.7.

Lemma 7.5.5.7. *Let $\mathcal{N}$ be a stratified space, and suppose we are given a stratified map*

$$R: \mathcal{N}_{\geq p} \times |\Delta^{[1]}|_s \to \mathcal{N}_{\geq p}$$

such that

1. *$R(x,t) = x$, for $x \in \mathcal{N}_p$ or $t = 1$;*
2. *$R(x,0) \in \mathcal{N}_p$, for all $x \in \mathcal{X}$.*

Then the evaluation map

$$\mathrm{ev}_1: \mathcal{H}o\mathrm{Link}^s_p(\mathcal{N}) \to \mathcal{N}_{>p}$$

is a stratum-preserving homotopy equivalence.

Proof. A section $s: \mathcal{N}_{>p} \to \mathcal{H}o\mathrm{Link}^s_p(\mathcal{N})$ of this map is provided by

$$x \mapsto \{t \mapsto R(x,t)\}.$$

A homotopy between $s \circ \mathrm{ev}_1$ and $1_{\mathcal{H}o\mathrm{Link}^s_p(\mathcal{N})}$ is constructed exactly as in the two strata case of Proposition 6.4.0.9. One easily verifies that the homotopy given there is stratum-preserving. □

Combining these two results, one obtains:

Corollary 7.5.5.8. *Let $\mathcal{X} \in \mathbf{Strat}_P$, $p \in P$ and $\mathcal{N} \subset \mathcal{X}$ be a neighborhood of the p-stratum $\mathcal{X}_p \subset \mathcal{X}$. Suppose that it admits a stratified map $R: \mathcal{N}_{\geq p} \times |\Delta^{[1]}|_s \to \mathcal{N}_{\geq p}$ as in Lemma 7.5.5.7. Then there is a zig-zag of diagrammatic equivalences*

$$\mathcal{N}_{>p} \xleftarrow{\mathrm{ev}_1} \mathcal{H}o\mathrm{Link}^s_p\mathcal{N} \xrightarrow{\simeq} \mathcal{H}o\mathrm{Link}^s_p\mathcal{X}.$$

In the introduction, we already alluded to the fact that one can think of generalized homotopy links as arising as the iterated stratified homotopy links of a stratified space. This follows from the following observation:

Lemma 7.5.5.9. *Let $\mathcal{I} = [p_1 < \cdots < p_n] \in \mathrm{sd}(P_{>p})$ be a regular flag in P of length $n-1$, containing only elements larger than $p \in P$ and let $\mathcal{X} \in \mathbf{Strat}_P$. There is a natural weak homotopy equivalence of simplicial sets*

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})) \simeq \mathcal{H}o\mathrm{Link}_{\{p\} \cup \mathcal{I}}(\mathcal{X}).$$

Proof. By definition of $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})$ as a restriction of $\mathcal{X}^{|\Delta^{[1]}|_s}$ to certain strata, we can identify the $\mathcal{I}$-th homotopy link $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}))$ with the $\mathcal{I}'$-th homotopy link of $\mathcal{X}^{|\Delta^{[1]}|_s}$, where $\mathcal{I}'$, is the flag

$$[(p,p_1) < \cdots < (p,p_n)]$$

in $P \times P$. Observe that under the natural isomorphism

$$\underline{\mathbf{Strat}}(|\Delta^{[n-1]}|_s, \mathcal{X}^{|\Delta^{[1]}|_s}) \cong \underline{\mathbf{Strat}}(|\Delta^{[n-1]}|_s \times |\Delta^{[1]}|_s, \mathcal{X}) \cong \underline{\mathbf{Strat}}(|\Delta^{[n-1]} \times \Delta^{[1]}|_s, \mathcal{X})$$

the component of $\underline{\mathbf{Strat}}(|\Delta^{[n-1]}|_s, \mathcal{X}^{|\Delta^{[1]}|_s})$ which is given by $\mathcal{H}o\mathrm{Link}_{\mathcal{I}'}(\mathcal{X}^{|\Delta^{[1]}|_s})$ is identified with the component of $\underline{\mathbf{Strat}}(|\Delta^{[n-1]} \times \Delta^{[1]}|_s, \mathcal{X})$ given by such stratified maps whose underlying map of posets is given by

$$(k,i) \mapsto \begin{cases} p_k & \text{, if } i = 1 \\ p & \text{otherwise.} \end{cases}$$

We may equivalently identify this component with the simplicial mapping space

$$\underline{\mathbf{Strat}}_P(\mathcal{Y}, \mathcal{X})$$

where $\mathcal{Y}$ is the stratified space obtained by equipping $|\Delta^{n-1} \times \Delta^1|$ with the stratification

$$(x,t) \mapsto \begin{cases} s_{|\Delta^{\mathcal{I}}|_s}(x) & \text{, if } t > 0 \\ p, \text{ otherwise.} \end{cases}$$

In this way, we have obtained an identification

$$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})) \cong \underline{\mathbf{Strat}}_P(\mathcal{Y}, \mathcal{X})$$

Now, consider $|\Delta^{\mathcal{I}}|_s$ as embedded into $|\Delta^{\{p\}\cup\mathcal{I}}|_s$ as its $\mathcal{I}$-face. The collapsing map

$$\mathcal{Y} \to |\Delta^{\{p\}\cup\mathcal{I}}|_s$$
$$(x,t) \mapsto ((1-t)e_0 + tx)$$

fits into a pushout square

$$\begin{tikzcd} |\Delta^{n-1}| \times \{p\} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \{p\} \arrow[d] \\ \mathcal{Y} \arrow[r] & |\Delta^{\{p\}\cup\mathcal{I}}|_s \end{tikzcd} \tag{7.39}$$

of cofibrant objects in $\mathbf{Strat}^{\mathfrak{d}}_P$, with verticals given by cofibrations. In particular, this square is a homotopy pushout. As the upper horizontal is a weak diagrammatic equivalence, so is the lower horizontal. As both stratified spaces in the lower horizontal are diagrammatically fibrant and triangularly cofibrant, it follows that the lower horizontal is a stratified homotopy equivalence. Hence, precomposing with this map defines a homotopy equivalence of simplicial sets

$$\underline{\mathbf{Strat}}_P(\mathcal{Y}, \mathcal{X}) \simeq \underline{\mathbf{Strat}}_P(|\Delta^{\{p\}\cup\mathcal{I}}|_s, \mathcal{X}) = \mathcal{H}o\mathrm{Link}_{\{p\}\cup\mathcal{I}}(\mathcal{X}).$$

This completes the proof. □

Remark 7.5.5.10. Applying Lemma 7.5.5.9 inductively, one obtains that the generalized homotopy link $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$ (modeled by a topological mapping space instead of a simplicial set, as in [DW22]) associated to a stratified space $\mathcal{X} \in \mathbf{Strat}_P$ and a flag $\mathcal{I} = [p_0 < \cdots < p_n]$ can be homotopically identified with the iterated stratified homotopy link

$$(\mathcal{H}o\mathrm{Link}^s_{p_n} \circ \cdots \circ \mathcal{H}o\mathrm{Link}^s_{p_0})(\mathcal{X}).$$

As an immediate corollary of Lemma 7.5.5.9, one obtains:

Corollary 7.5.5.11. *Let $w\colon \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ be a stratum-preserving map. w is a diagrammatic equivalence if and only if, for each $p \in P$, the following holds:*

1. *$w_p\colon \mathcal{X} \to \mathcal{Y}$ is a weak homotopy equivalence;*
2. *The induced map of stratified homotopy links $\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X}) \to \mathcal{H}o\mathrm{Link}^s_p(\mathcal{Y})$ is a diagrammatic equivalence.*

The true power of this criterion lies in the fact that it allows one to think of stratum-preserving maps between fibrant objects as having a *tangential component* along the strata, as well as a *normal component*, given by the fibers of stratified homotopy links. To explain this more rigorously, we need the following lemma:

Lemma 7.5.5.12. *Suppose we are given a commutative square in* **Strat**

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\tilde{w}"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & B & \end{tikzcd} \tag{7.40}$$

with B trivially stratified and $\mathcal{X} \to \mathcal{Y}$ a stratum-preserving map over $P_{\mathcal{X}} = P_{\mathcal{Y}}$. Suppose that both diagonals are diagrammatic (categorical) fibrations.

1. *For every $x \in B$, the induced map on fibers $\mathcal{X}_x \to \mathcal{Y}_x$ is a diagrammatic (categorical) equivalence;*

2. *For every path component of B, there exists a representative $x \in B$ such that the induced map on fibers $\mathcal{X}_x \to \mathcal{Y}_x$ is a diagrammatic (categorical) equivalence;*

3. *$\tilde{w}$ is a diagrammatic (categorical) equivalence.*

Proof. Observe that we only need to prove the diagrammatic case. The categorical case then follows by the Whitehead theorem for left Bousfield localizations. That the first condition implies the second is trivial. That the last condition implies the first follows by observing that the two squares

$$\begin{tikzcd} \mathcal{X}_x \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \star \arrow[r, hook, "x"] & B \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Y}_x \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \star \arrow[r, hook, "x"] & B \end{tikzcd} \tag{7.41}$$

are not just pullback, but (by the assumption on fibrancy of the right verticals and since every trivially stratified space is fibrant) even homotopy pullback, in the sense that the underlying square in the associated ∞-category $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}$ is pullback. (There are many ways to see this. One of them is applying the equivalence of ∞-categories induced by Sing_s together with the dual of [Lur09, Thm. 4.2.4.1.].) Thus it follows that the induced map on fibers $\mathcal{X}_x \to \mathcal{Y}_x$ is an isomorphism in $\mathcal{S}\mathbf{trat}^{\mathfrak{d}}$ and hence a diagrammatic equivalence. To see the final remaining implication, observe that since diagrammatic equivalences can be verified on connected components, we may without loss of generality assume that B is connected. Now, let $n \in \mathbb{N}$. Observe that $\hat{\mathcal{H}o}\mathrm{Link}_n(\star) = \Delta^0$. Applying $\hat{\mathcal{H}o}\mathrm{Link}_n$ to the pullback squares above, and using the commutativity of $\hat{\mathcal{H}o}\mathrm{Link}_n$ with pullbacks, we thus obtain a morphism of fiber sequences

$$\begin{tikzcd} \hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{X}_x) \arrow[r, "\simeq"] \arrow[d] & \hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{Y}_x) \arrow[d] \\ \hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{X}) \arrow[r] \arrow[d] & \hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{Y}) \arrow[d] \\ \hat{\mathcal{H}o}\mathrm{Link}_n(B) \arrow[r, equal] & \hat{\mathcal{H}o}\mathrm{Link}_n(B). \end{tikzcd} \tag{7.42}$$

Observe that by simpliciality of the model structure on **Strat**, both lower verticals are Kan-fibrations. Hence, the left and the right vertical sequence are homotopy fiber sequences. Furthermore, as B is trivially stratified, we have that $\hat{\mathcal{H}o}\mathrm{Link}_n(B) \simeq \underline{\mathbf{Top}}(|\Delta^n|, B) \simeq \mathrm{Sing}_s(B)$ is path-connected. Hence, it follows by the classical fiberwise characterization of weak equivalences between Kan-fibrations that $\hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{X}) \to \hat{\mathcal{H}o}\mathrm{Link}_n(\mathcal{Y})$ is a weak homotopy equivalence. ☐

Notation 7.5.5.13. Given a diagrammatically fibrant stratified space $\mathcal{X}$, we call the (homotopy) fiber of $\mathcal{H}o\mathrm{Link}^s_p\mathcal{X} \to \mathcal{X}_p$ at $x \in \mathcal{X}_p$ the *local homotopy link* of $\mathcal{X}$ at x.

As a corollary, combining Lemma 7.5.5.12 and Corollary 7.5.5.11, as well as the stability of weak equivalences between fibrant objects under pullbacks along fibrations, one obtains the following detection criterion for diagrammatic equivalences.

Corollary 7.5.5.14. *Let $w: \mathcal{X} \to \mathcal{Y} \in \mathbf{Strat}_P$ be a stratum-preserving map of diagrammatically fibrant stratified spaces. Then the following are equivalent:*

1. *w is a diagrammatic equivalence;*
2. *For each $p \in P$, the induced map $w_p: \mathcal{X}_p \to \mathcal{Y}_p$ is a weak homotopy equivalence, and for each $x \in \mathcal{X}_p$ (or just for a representative system of path components) the induced map on local homotopy links*
$$\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})_x \to \mathcal{H}o\mathrm{Link}^s_p(\mathcal{Y})_{w(x)}$$
is a diagrammatic equivalence.

The analogous equivalence holds for categorically fibrant spaces and categorical equivalences. Even more, in this case it follows from the décollage condition that these two statements are furthermore equivalent to

3. *For each $p \in P$, the induced map $w_p: \mathcal{X}_p \to \mathcal{Y}_p$ is a weak homotopy equivalence, and for each $x \in \mathcal{X}_p$ (or just for a representative system of path components) the induced map on local homotopy links*
$$\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})_x \to \mathcal{H}o\mathrm{Link}^s_p(\mathcal{Y})_{w(x)}$$
induces weak equivalences on strata.

Using an analogous (but significantly easier) argument to the proof of Proposition 6.3.1.21, one can show that local homotopy links are indeed *local*, in the following sense.

Lemma 7.5.5.15. *Let $\mathcal{X}$ be a diagrammatically fibrant stratified space. Let $p \in P$, $x \in \mathcal{X}_p$ and let $\mathcal{N}$ be a neighborhood of $x \in \mathcal{X}$. Then the inclusion of local homotopy links*
$$\mathcal{H}o\mathrm{Link}^s_p(\mathcal{N})_x \to \mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})_x$$
is a diagrammatic equivalence.

Remark 7.5.5.16. Together with Corollary 7.5.5.14, Lemma 7.5.5.15 allows us to verify stratified homotopy equivalence between bifibrant stratified spaces in terms of a two-step procedure, involving a global computation, which is only concerned with strata, and a purely local computation, concerning the local homotopy links. In the case of categorically fibrant stratified spaces, one even only needs to verify strata-wise weak equivalence on the local level.

In many cases of geometric stratified spaces, such as topological pseudo manifolds, local stratified homotopy links admit explicit geometric models.

Remark 7.5.5.17. Let $\mathcal{X} \in \mathbf{Strat}_P$ be a conically stratified space (see [Lur17]). Recall that this means that, for every $p \in P$ and every $x \in \mathcal{X}_p$, there exists a neighborhood $\mathcal{N}$ of x, that is locally stratum-preserving homeomorphic to a product $U_x \times C\mathcal{L}_x$, where U_x is a trivially stratified space and $C\mathcal{L}_x$ is the stratified cone on a stratified space $\mathcal{L}_x \in \mathbf{Strat}_{P_{>p}}$ obtained by equipping the (teardrop) cone on L_x with the stratification
$$[y,t] \mapsto \begin{cases} p & \text{, if } t = 0 \\ s_{\mathcal{L}_x}(y) & \text{, otherwise.} \end{cases}$$
Suppose, furthermore, that the strata of $\mathcal{X}$ are locally (weakly) contractible, such that we may choose U_x to be (weakly) contractible. Then, combining Lemmas 7.5.5.7 and 7.5.5.15 we obtain a sequence of diagrammatic equivalences over P:
$$\begin{aligned}\mathcal{H}o\mathrm{Link}^s_p(\mathcal{X})_x &\simeq \mathcal{H}o\mathrm{Link}^s_p(\mathcal{N})_x \\ &\cong \mathcal{H}o\mathrm{Link}^s_p(U_x \times C\mathcal{L}_x)_x \\ &\simeq \mathcal{H}o\mathrm{Link}^s_p(U_x \times C\mathcal{L}_x) \\ &\simeq U_x \times (C\mathcal{L}_x)_{>p} \\ &\simeq (C\mathcal{L}_x)_{>p} \\ &\cong (0,1] \times \mathcal{L}_x \\ &\simeq \mathcal{L}_x.\end{aligned}$$

It follows that the local homotopy link can be homotopically identified with what is often referred to as the (local) link of a topological pseudomanifold (see, for example [Ban07]).

7.A Refined stratified spaces and Nand-Lal's homotopy theory

In this section, we relate the homotopy theory $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$ to the work of [Nan19]:

Recollection 7.A.0.1 ([Nan19])**.** To avoid dealing with empty strata, [Nan19] introduced the notion of a *surjectively stratified space*, defined as follows: Denote by $\mathbf{Strat}_s$ the full subcategory of $\mathbf{Strat}$ given by such stratified spaces $\mathcal{X}$, for which the stratification $s_{\mathcal{X}}: X \to P_{\mathcal{X}}$ is surjective. In other words, such stratified spaces whose stratification poset contains no redundant elements (but possibly redundant relations). Such a stratified space is called *surjectively stratified*. In [Nan19], Nand-Lal constructs a homotopy theory for $\mathbf{Strat}_s$ via transfer along the composition

$$\mathbf{Strat}_s \hookrightarrow \mathbf{Strat} \xrightarrow{\mathrm{Sing}_s} \mathrm{s}\mathbf{Strat} \xrightarrow{\mathcal{F}} \mathbf{sSet},$$

using the Joyal model structure on $\mathbf{sSet}$. In other words, a weak equivalence in the resulting homotopy theory is a map of stratified spaces $f: \mathcal{X} \to \mathcal{T}$, for which the underlying simplicial map of the stratified simplicial map $\mathrm{Sing}_s(f)$ is a categorical equivalence. It follows that this class of weak equivalences is precisely the class of categorical equivalences with source and target in $\mathbf{Strat}_s$. We denote by $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_s$ the relative category given by $\mathbf{Strat}_s$, together with such weak equivalences.

Let us now relate $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_s$ to $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}$.

Proposition 7.A.0.2. *The inclusion* $\mathbf{Strat}_s \hookrightarrow \mathbf{Strat}$ *induces a homotopy equivalence of relative categories*

$$\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_s \xrightarrow{\simeq} \mathcal{S}\mathbf{trat}^{\mathfrak{c}},$$

and hence an equivalence of the corresponding ∞*-categories.*

Proof. The functor $(-)^{\mathfrak{r}}: \mathbf{Strat} \to \mathbf{Strat}$ has image in $\mathbf{Strat}_s$. By Proposition 7.5.3.7, there is a natural isomorphism $\mathrm{Sing}_s \circ (-)^{\mathfrak{r}} \cong (-)^{\mathfrak{r}} \circ \mathrm{Sing}_s$, which shows that $(-)^{\mathfrak{r}}$ preserves refined categorical equivalences. Furthermore, since (again by Proposition 7.5.3.7) the refinement map is a categorical equivalence, it follows that $(-)^{\mathfrak{r}}$ defines a homotopy inverse to the inclusion of relative categories $\mathcal{S}\mathbf{trat}^{\mathfrak{c}}_s \hookrightarrow \mathcal{S}\mathbf{trat}^{\mathfrak{c}}$. One homotopy is given by the refinement map and one by the identity. □

7.B An elementary extension lemma

In this section, we give a proof of Lemma 7.5.1.7. Before we give a proof, let us introduce some notation and quickly illustrate where metrizability comes into play. It is not hard to see that there is a continuous map $i: D^{n+1} \times [0,1] \to D^{n+1} \times [0,1) \cup S^n \times [0,1]$, which is the identity on $D^{n+1} \times \{0\} \cup S^n \times [0,1]$ and maps $\mathring{D}^{n+1} \times [0,1]$ into $\mathring{D}^{n+1} \times [0,1)$. Hence, the obvious thing to do would be to simply glue f and f' along $D^{n+1} \times \{0\} \cup S^n \times [0,1)$ to a map $\hat{f}: D^{n+1} \times [0,1) \cup S^n \times [0,1] \to X$ and then set $\tilde{f} = \hat{f} \circ i$. The issue with this approach is that the map $\hat{f}$ obtained in this way will generally not be continuous. Indeed, it is obtained by gluing two maps defined respectively on an open and a closed subset of $D^{n+1} \times [0,1) \cup S^n \times [0,1]$. To illustrate this point a little better, consider the homeomorphism

$$\begin{aligned} D^{n+1} &\to S^n \times [0,1] \big/ S^n \times \{1\} \\ y &\mapsto [\frac{y}{||y||}, 1 - ||y||] \end{aligned}$$

mapping 0 to the point given by $S^n \times \{1\}$. We obtain a change of coordinates $y \triangleq [x,s]$. By setting

$$f'_{x,t}(s) = f'([x,s],t)$$

we can interpret the data of a map $f' \colon D^{n+1} \times [0,1] \to X$ as a continuous family of paths $f'_{x,t} \colon [0,1] \to X$, indexed over $(x,t) \in S^n \times [0,1)$, which fulfill

$$\gamma_{x,t}(1) = \gamma_{x',t}(1),$$

for all $x, x' \in S^n, t \in [0,1]$. If we want $\hat{f} = f \cup f'$ to be continuous, then we precisely need convergence

$$f'_{x_n,t_n}(s_n) \to f([x,0],1)$$

for sequences $([x_n,s_n],t_n) \to ([x,0],1)$. Let $d \colon X \times X \to [0,\infty)$ denote a metric that induces the topology on X and suppose that there is a uniform bound

$$d(f'_{x,t}(s), f'_{x,t}(0)) = d(f'_{x,t}(s), f([x,0],t)) \leq \varphi(s)$$

by some continuous function $\varphi \colon [0,1] \to [0,1]$ with $\varphi(0) = 0$, at least for $(s,t) \in [0,\frac{1}{2}] \times [\frac{1}{2},1]$. It follows from the triangle inequality that

$$d(f'_{x_n,t_n}(s_n), f([x,0],1)) \leq \varphi(s_n) + d(f([x_n,0],t_n), f([x,0],1)),$$

for (s_n,t_n) close to $(0,1)$. In particular, it then follows from the continuity of f that this expression converges to 0. If the speed at which the paths f'_{x_n,t_n} leave $f'_{x_n,t_n}(0)$ becomes arbitrarily large as $(x_n,t_n) \to (x,1)$, then such a bound φ may generally not exist. We may, however, use the metrizability of X to reparametrize the paths $f'_{x,t}$ in a way to enforce such global bounds. We proceed as follows.

Proof of Lemma 7.5.1.7. Using the notation above, i is given by

$$\begin{aligned} i : D^{n+1} \times [0,1] &\to D^{n+1} \times [0,1) \cup S^n \times [0,1] \\ ([x,s],t) &\mapsto ([x,s], s\frac{t}{2} + (1-s)t). \end{aligned}$$

It remains to show that we may without loss of generality assume that f' admits a bounding function $\varphi \colon [0,1] \to [0,1]$ as above. For two topological spaces T, T', denote by $T^{T'}$ the mapping space equipped with the compact open topology. If T' is locally compact Hausdorff, then this construction defines the right adjoint to the functor $- \times T'$ ([DK70]). Furthermore, if T is metrizable, then the topology on $T^{T'}$ is easily seen to be the topology of uniform convergence on every compactum. Finally, denote by $\mathrm{Aut}(\mathbb{R}_{\geq 0})$ the subspace of self-homeomorphisms of $\mathbb{R}_{\geq 0}$. Consider the following three maps.

$$\begin{aligned} M^{[0,1]} &\to \mathrm{Aut}(\mathbb{R}_{\geq 0}) \\ \gamma &\mapsto \sigma_\gamma := \{s \mapsto \sup_{t \leq s,1} \mathrm{d}(\gamma(t), \gamma(0)) + s\}; \end{aligned}$$

$$\begin{aligned} \mathrm{Aut}(\mathbb{R}_{\geq 0}) &\to \mathrm{Aut}(\mathbb{R}_{\geq 0}) \\ \sigma &\mapsto \sigma^{-1}; \end{aligned}$$

$$\begin{aligned} \mathrm{Aut}(\mathbb{R}_{\geq 0}) &\to [0,1]^{[0,1]} \\ \sigma &\mapsto \{s \mapsto \min\{\sigma(s), 1\}\}. \end{aligned}$$

It is not hard to see, using the topology of uniform convergence, that the first and last of these maps are continuous. That inversion is continuous follows from [Are46, Thm. 4]. Now, let

$$\rho : M^{[0,1]} \to [0,1]^{[0,1]}.$$

be the composition of these three maps. Then ρ has the following properties:

$(\mathrm{i})_\rho$ $\rho(\gamma)(0) = 0$ if and only if $s = 0$;

$(\mathrm{ii})_\rho$ $d\big(\gamma(\rho(\gamma)(s)), \gamma(0)\big) \leq s$,

for all $s \in [0,1], \gamma \in X^{[0,1]}$. The first property is immediate by construction of ρ. The second inequality is obtained from

$$\begin{aligned} d\big(\gamma(\rho(\gamma)(s)), \gamma(0)\big) &\leq \sup_{t \leq \rho(\gamma)(s), 1} d\big(\gamma(t), \gamma(0)\big) + \rho(\gamma)(s) = \sigma_\gamma(\rho(\gamma)(s)) \\ &\leq \sigma_\gamma(\sigma_\gamma^{-1}(s)) \\ &= s. \end{aligned}$$

Next, denote by $\phi : [0,1]^2 \to [0,1]$ a function fulfilling

$(\mathrm{i})_\phi$ $\phi(s,t) = 0$, $(s,t) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$;

$(\mathrm{ii})_\phi$ $\phi(s,t) = 1$, $(s,t) \in [0,1] \times \{0\} \cup \{1\} \times [0,1]$,

and, define

$$\begin{aligned} \Phi : D^{n+1} \times [0,1) &\to D^{n+1} \times [0,1) \\ ([x,s],t) &\mapsto ([x, (1-\phi(s,t))\rho(f'_{x,t}, s) + \phi(s,t)s], t). \end{aligned}$$

Note first that Φ is well defined, that is, its value is independent of $x \in S^n$ when $s = 1$, since then also $(1-\phi(s,t))\rho(f'_{x,t}, s) + \phi(s,t)s = 1$. Furthermore, Φ, has the following properties:

$(\mathrm{i})_\Phi$ $\Phi([x,s],t) = ([x,s],t)$, for $([x,s],t) \in D^{n+1} \times \{0\} \cup S^n \times [0,1)$;

$(\mathrm{ii})_\Phi$ $\Phi(\mathring{D}^{n+1} \times [0,1)) \subset \mathring{D}^{n+1} \times [0,1)$;

$(\mathrm{iii})_\Phi$ $\Phi([x,s],t) = \rho(f'_{x,t}, s)$, for $(s,t) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$.

Property $(\mathrm{i})_\Phi$ follows from Property $(\mathrm{i})_\phi$ and Property $(\mathrm{i})_\rho$. Property $(\mathrm{ii})_\Phi$ follows from Property $(\mathrm{i})_\rho$, and finally Property $(\mathrm{iii})_\Phi$ follows from Property $(\mathrm{ii})_\phi$. We may then replace f' by $f'' = f' \circ \Phi$, obtaining

$(\mathrm{i})_{f''}$ $f''([x,s],t) = f'([x,s],t) = f([x,s],t)$, for $([x,s],t) \in D^{n+1} \times \{0\} \cup S^n \times [0,1)$;

$(\mathrm{ii})_{f''}$ $f''(\mathring{D}^{n+1} \times [0,1)) \subset f'(\mathring{D}^{n+1} \times [0,1))$;

$(\mathrm{iii})_{f''}$ $d(f'_{x,t}(s), f'_{x,t}(0)) \leq s$, for $(s,t) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$.

Properties $(\mathrm{i})_{f''}$ and $(\mathrm{ii})_{f''}$ are immediate by Properties $(\mathrm{i})_\Phi$ and $(\mathrm{ii})_\Phi$ respectively, while Property $(\mathrm{iii})_{f''}$ follows from Property $(\mathrm{iii})_\Phi$ and Property $(\mathrm{ii})_\rho$. □

7.C Remaining part of the proof of the nonexistence proposition

In Proposition 7.4.0.1 (using the notation there) we claimed that a path γ from a to b that (described as starting from b) ascends monotonously in height and passes to the left of b_1, to the right of c_2, to the left of b_3 etc., as illustrated, cannot lie in the image of r_* (even up to stratified homotopy). Here, we provide a rigorous proof of this statement.

Proof. This insight can be formalized as follows: For $n \in \mathbb{N}$, denote by $\mathcal{X}_{b_n}$ the stratified spaces obtained from $\mathcal{X}$ by taking only the point b_n as the p-stratum. The identity at the space level 1_X does not induce stratified maps $\mathcal{X}, \mathcal{Y} \to \mathcal{X}_{b_n}$. However, it nevertheless induces (non-)stratified maps of stratified singular simplicial sets. Furthermore, by the same argument as above, $\mathrm{Sing}_s(\mathcal{X}_{b_n})$ are quasi-categories. In addition to this, on $\mathcal{X}_{b_n}$, r is stratified homotopic to the identity relative to a and b. It follows that we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{hoSing}_s(\mathcal{X})(a,b) \arrow[rr, "r_*"] \arrow[dr, "\tau_{c_n}"'] & & \mathrm{hoSing}_s(\mathcal{Y})(a,b) \arrow[dl, "\sigma_{b_n}"] \\ & \mathrm{hoSing}_s(\mathcal{X}_{b_n})(a,b) & \end{tikzcd} \tag{7.43}$$

Now, $\mathrm{Sing}_s(\mathcal{X}_{b_n})(a,b)$ is simply the set of homotopy classes of paths in $X \setminus \{b_n\}$, which is homotopy equivalent to S^1. Under post-composition with the straight line path from b to a, we may thus identify

$$\mathrm{hoSing}_s(\mathcal{X}_{b_n})(a,b) \cong \mathrm{hoSing}_s(\mathcal{X}_{b_n})(a,a) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

We may proceed mutatis mutandis replacing b_n with c_n. Then, under this identification with the integers, the sequences $\sigma_{b_n}([\gamma]), \sigma_{c_n}([\gamma]) \in \mathbb{Z}$ are (modulo signs, originating from choices of orientation) given by:

$$(1,0,1,0,1,\cdots)$$
$$(0,1,0,1,0,\cdots).$$

However, for every morphism $g \in \mathrm{hoSing}_s(\mathcal{X})(a,b)$ at least one of the sequences $\tau_{b_n}(g)$ or $\tau_{c_n}(g)$ has only finitely many non-zero values. Indeed, assume that g is such that both $\tau_{b_n}(g)$ and $\tau_{c_n}(g)$ have infinitely many non-zero values. Every such g can be represented by a path γ' which stays in $\mathcal{X}_p$ for $[0,\frac{1}{2}]$ and then exits into $\mathcal{X}_q$. For $\tau_{b_n}(g)$ ($\tau_{c_n}(g)$) to be non-zero, γ' must intersect the line segment connecting a and b_n (c_n), as otherwise γ' maps into a contractible subspace of $\mathcal{X}_{b_n}$ ($\mathcal{X}_{c_n}$). Since these intersection points necessarily converge to a as $n \to \infty$, it follows that $a \in \gamma[\frac{1}{2},1]$ and therefore $\gamma(\frac{1}{2}) = a$. It follows that $\gamma'(t)$ lies strictly above a' (in direction of the y-axis), for all $t > \frac{1}{2}$ greater than some $t_h \in \frac{1}{2}$. Furthermore, it follows from the stratification of γ and the assumption on infinitely many intersection points with the line segment b_n and a above, that we find $t_b, t_c \in (\frac{1}{2}, t_h)$, such that $\gamma'(t_b)$ lies on a line segment between a and b_n for some $n > 0$, and $\gamma'(t_c)$ lies on a line segment between a and c_m, for some $m > 0$. To see this (in the case of (b_n)), consider a sequence t_i such that each $\gamma'(t_i)$ lies on the line segment between a and b_i. By compactness of $[\frac{1}{2},1]$, we may assume that t_i converges to some $t' \in [\frac{1}{2},1]$. Since the intersection points of the line segments from a to b_i converge to a, it follows that $\gamma'(t') = a$. But $\frac{1}{2}$ is the only value in $[\frac{1}{2},1]$ with $\gamma'(t') \in \mathcal{X}_p$, hence $t' = \frac{1}{2}$, and it follows that t_i converges to $\frac{1}{2}$. Finally, the set of points strictly above a' in $\mathcal{X}_q$ is disconnected, with the line segments between a and b_m and a and c_n lying in different components, for all m,n, in contradiction to the connectedness of $\gamma'(\frac{1}{2},t_h)$. ☐

7.D On the proof of [Hai23, Thm. 0.1.1/3.2.4]

In this section, we take a detailed look at the proof of [Hai23, Thm. 0.1.1/3.2.4], pointing out a gap in the latter which stems from different uses of the terminology *Segal space*.

Remark 7.D.0.1. On the proof of [Hai23, Thm. 0.1.1/3.2.4]: Let $\mathbf{Strat}_P^{\mathrm{ex}}$ denote the full subcategory of $\mathbf{Strat}_P$ given by the fibrant stratified spaces in the categorical model structure and denote by W_{ex} the class of categorical equivalences between the latter. It was already asserted in [Hai23, Thm. 0.1.1/3.2.4] that Sing_s induces an equivalence of quasi-categories

$$\mathbf{Strat}_P^{\mathrm{ex}}[W_{\mathrm{ex}}^{-1}] \xrightarrow{\simeq} \mathcal{A}\mathcal{S}\mathbf{trat}_P.$$

While this result is a consequence of the existence of the categorical semi-model structure on $\mathbf{Strat}_P$ together with Theorem 7.3.3.1, [Hai23] suggested a proof that did not assume the existence of such a structure. We want to point out two gaps within this proof, which seem to require a notion of fibrant replacement in $\mathbf{Strat}_P^{\mathfrak{c}}$ to be closed. Let us first sketch the proof in [Hai23]. Some of the equivalences of ∞-categories in [Hai23] are not made explicit. We will choose explicit models which, to the best of our knowledge, present the intended functors of ∞-categories.

1. Denote by $\mathrm{s}\mathbf{Strat}_P^{\mathrm{ex}}$ the full subcategory of $\mathrm{s}\mathbf{Strat}_P$ given by the (bi)fibrant objects in $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{c}}$ (i.e. quasi-categories with a conservative functor into P) and by H the class of equivalences of quasi-categories over P, between the latter. The quasi-category $\mathrm{s}\mathbf{Strat}_P^{\mathrm{ex}}[H^{-1}]$ is one possible model for the ∞-category of abstract stratified homotopy types over P, $\mathcal{AS}\mathbf{trat}_P$. One may then consider a diagram of 1-categories (commutative up to natural isomorphism)

$$\begin{tikzcd} \mathbf{Strat}_P^{\mathrm{ex}} \arrow[r] \arrow[d, hook] & \mathrm{s}\mathbf{Strat}_P^{\mathrm{ex}} \arrow[d, "\mathrm{HoLink}"] \\ \mathbf{Strat}_P \arrow[r, "\mathcal{H}o\mathrm{Link}"] & \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})\,. \end{tikzcd} \tag{7.44}$$

Now, let $W_{\mathrm{dé}}$ be the class of morphisms in $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ that are weak equivalences in the model structure presenting décollages over P. The class of categorical equivalences in $\mathbf{Strat}_P$, denoted $W_{\mathfrak{c}}$ is precisely the inverse image of $W_{\mathrm{dé}}$ under $\mathcal{H}o\mathrm{Link}$. Furthermore, [Hai23] proves that H is precisely the inverse image of $W_{\mathrm{dé}}$ under the right vertical, and that W_{ex} is precisely the inverse image of H under Sing_s. Hence, there is an induced diagram of quasi-categories

$$\begin{tikzcd} \mathbf{Strat}_P^{\mathrm{ex}}[W_{\mathrm{ex}}^{-1}] \arrow[r, "\mathrm{Sing}_s"] \arrow[d, hook] & \mathrm{s}\mathbf{Strat}_P^{\mathrm{ex}}[H^{-1}] \simeq \mathcal{AS}\mathbf{trat}_P \arrow[d, "\mathrm{HoLink}"] \\ \mathcal{S}\mathbf{trat}_P^{\mathfrak{c}} = \mathbf{Strat}_P[W_{\mathfrak{c}}^{-1}] \arrow[r, "\mathcal{H}o\mathrm{Link}"] & \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})[W_{\mathrm{dé}}^{-1}]\,, \end{tikzcd} \tag{7.45}$$

commutative up to natural isomorphism. It follows from [Dou21c, Thm. 3], that the lower horizontal is an equivalence. Furthermore, it follows either from Theorem 5.2.2.20 or from [BGH18, Thm 2.7.4] that the right vertical is an equivalence. This already proves one of the central results of [Hai23], namely the existence of an equivalence $\mathcal{S}\mathbf{trat}_P^{\mathfrak{c}} \simeq \mathcal{AS}\mathbf{trat}_P$. In addition to this result, [Hai23] aims to show that the upper horizontal in Diagram (7.45) is also an equivalence. This follows from commutativity of Diagram (7.45), if one can show the following two additional claims:

2. The left vertical in Diagram (7.45)

$$\mathbf{Strat}_P^{\mathrm{ex}}[W_{\mathrm{ex}}^{-1}] \to \mathbf{Strat}_P[W_{\mathfrak{c}}^{-1}] = \mathcal{S}\mathbf{trat}_P^{\mathfrak{c}}$$

is fully faithful: This is not commented on further in [Hai23]. Note that there is generally no reason to assume that in a category with weak equivalences $(\mathbf{C}, W)$ with full subcategory $\mathbf{A}$ the natural functor $\mathbf{A}[\mathbf{A} \cap W^{-1}] \to \mathbf{C}[W^{-1}]$ is an equivalence of quasi-categories. Consider, for example, the flattening of a quasi-category $\mathcal{C} \in \underline{\mathbf{sSet}}$, given by the 1-category $(\Delta_{/\mathcal{C}})^{\mathrm{op}}$. The flattening of $\mathcal{C}$ has a full discrete subcategory $\mathcal{C}_0$ given by the objects of $\mathcal{C}$. If we denote by W the class of morphisms in $(\Delta_{/\mathcal{C}})^{\mathrm{op}}$ that map 0 to 0, there is a natural equivalence of quasi-categories, $(\Delta_{/\mathcal{C}})^{\mathrm{op}}[W^{-1}] \simeq \mathcal{C}$, which fixes the objects in $\mathcal{C}_0$ (see, for example, [Lan21, Thm. 3.3.8]). However, $\mathcal{C}_0[W \cap \mathcal{C}_0^{-1}]$ remains a discrete category. A classical condition to even ensure equivalence between $\mathbf{C}[W^{-1}]$ and $\mathbf{A}[\mathbf{A} \cap W^{-1}]$ is the existence of an endofunctor $F \colon \mathbf{C} \to \mathbf{C}$, with image in $\mathbf{A}$, together with a zig-zag of natural transformations $1 \leftrightarrow F$, given by morphisms in W. In the situation of Diagram (7.45) this amounts to constructing a fibrant replacement in $\mathbf{Strat}_P^{\mathfrak{c}}$, which we have done in this work, but whose existence was not yet known at the time of writing of [Hai23].

3. The diagonal in Diagram (7.45)

$$\mathbf{Strat}_P^{\mathrm{ex}}[W_{\mathrm{ex}}^{-1}] \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})[W_{\mathrm{dé}}^{-1}]$$

(and hence also the left vertical) is essentially surjective: Again, this claim is close to requiring fibrant replacements in $\mathbf{Strat}_P^{\mathfrak{c}}$. Recall that $\mathcal{D}\mathbf{iag}_P$ denotes the localization of $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ at pointwise-weak homotopy equivalences of diagrams. It follows from the equivalence $\mathcal{S}\mathbf{trat}_P^{\mathfrak{d}} \simeq \mathcal{D}\mathbf{iag}_P$, that every décollage in $\mathcal{D}\mathbf{iag}_P$ lies in the image of

$$\mathbf{Strat}_P \xrightarrow{\mathcal{H}o\mathrm{Link}} \mathbf{Fun}(\mathrm{sd}P, \mathbf{sSet}) \to \mathcal{D}\mathbf{iag}_P.$$

In other words, if we denote by $\mathbf{Strat}_P^{\mathrm{dé}}$ the full subcategory of stratified spaces $\mathcal{X}$ for which $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ is a décollage, then the restricted functor

$$\mathbf{Strat}_P^{\mathrm{dé}} \to \mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})[W_{\mathrm{dé}}^{-1}]$$

is essentially surjective. [Hai23] then claims that $\mathbf{Strat}_P^{\mathrm{dé}} = \mathbf{Strat}_P^{\mathrm{ex}}$, from which essential surjectivity of the diagonal would follow. With the definition of décollages we used here and which is also used in [Hai23] this is incorrect. The confusion arises from two possible definitions of décollages and complete Segal spaces: One is expressed purely intrinsically to the ∞-categories $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{G}\mathbf{rpd}_\infty)$ and $\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{G}\mathbf{rpd}_\infty)$ and one expressed in terms of injective model structures on the 1-categories $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})$ and $\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{sSet})$. Here, we have defined décollages intrinsically to $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathcal{G}\mathbf{rpd}_\infty)$ and complete Segal spaces in [Hai23] are also defined intrinsically to $\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{G}\mathbf{rpd}_\infty)$. Alternatively, one can define décollages as fibrant objects in the model category $\mathbf{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, \mathbf{sSet})^{\mathrm{dé}}$ (defined in Section 5.2.2, as a left Bousfield localization of the injective model structure) and, as it is classically done, complete Segal spaces as (certain) fibrant objects in the injective model structure on $\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{sSet})$. Let us call the former definition ∞-categorical and the latter 1-categorical. [Hai23] makes use of a result of Joyal and Tierney concerning the relationship of quasi-categories and 1-categorical Segal spaces [JT07, Cor. 3.6], from which it follows that $\mathbf{Strat}_P^{\mathrm{ex}}$ consists exactly of such stratified spaces for which $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ is a 1-categorical décollage. Due to the overlap in terminology, [Hai23] then reasons that $\mathbf{Strat}_P^{\mathrm{dé}} = \mathbf{Strat}_P^{\mathrm{ex}}$. This is false. Indeed, every stratified space with two strata, $\mathcal{X}$ over $P = \{p < q\}$, has the property that $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ is an ∞-categorical décollage (there are no conditions to be verified). But for $\mathcal{H}o\mathrm{Link}(\mathcal{X})$ to be injectively fibrant the map $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \to \mathcal{X}_p$, for $\mathcal{I} = [p < q]$, needs to be a Serre fibration. For example, $|\Lambda_0^{\mathcal{J}}|_s$ with $\mathcal{J} = [p \leq p < q]$ fails to have this property.

Part III

A model categorical approach to generalized simple homotopy theories

Note to the reader: The following part of this thesis presents our results on generalized simple homotopy theory. For a less technical and more straight-to-the-point account, not containing any proofs, we refer to Chapter 2. This part of this thesis is written as one cohesive text, and results in one chapter often strongly rely on the notation, language, and results of the previous chapters.
Our model categorical approach to simple homotopy theory required the development of a calculus of structured cell complexes in an abstract categorical setting. This language is covered in Chapter 8. This chapter is rather technical in nature, and, to a large part, re-frames existing theory on cell complexes in model categories in a setting that is suited to our purposes of simple homotopy theory. For a more motivated reading experience, we certainly recommend first reading Chapter 2. Furthermore, the subsequent chapters Chapters 9 and 10 do not rely on Section 8.3 of Chapter 8. As Section 8.3 is particularly technical, we recommend skipping it first, and then returning to it when reading Chapter 11. The next three chapters Chapters 9 to 12 all build upon each other and are best read in linear order. Finally, Chapter 13 relies on results spread all over Parts II and III, and should probably only be read after reading Part III and at least the summary of Part II in Chapter 2.
We will introduce a rather large amount of new notation in this part of the thesis. In order for this to not pose an obstruction to readability, we have added a notation list with page references that cover most of the recurring notation. It can be found after Chapter 13.

7.5 Category-theoretical notation

Finally, we will constantly and freely make use of the language of category theory (see, for example, [Mac13; Rie17] for an overview) and in the latter half of this part of model categories and higher categories (see [Hir03; Lur17; Bar10]). Let us quickly fix some notation. When we say *category* in the following notation section, we mean a *locally small* 1*-category.* When we speak of *a category in the extended sense*, this may not just refer to 1-categories but also to all sorts of other variations, such as *enriched categories*, *quasi-categories* or *bicategories* (see [JY20], for an introduction to bicategories).

Notation 7.5.0.1. Among the most commonly occurring categories in this part will be the following:

1. $\varnothing$ will denote the empty category; $\star$ will denote the terminal category with one object and one arrow; $[1] = \{0 \to 1\}$, will denote the category with two objects and one non-identity arrow from 0 to 1;
2. **Set** will denote the category of sets;
3. **Top** will denote the category of compactly or Δ-generated spaces (see [Rez17; Dug03])[10];
4. **Ab** will denote the category of abelian groups and **AbMon** will denote the category of abelian, unital monoids;
5. Given a (not necessarily commutative) unital ring R, $\mathbf{Ch}_{\geq 0}(R)$ will denote the category of non-negatively graded chain complexes of (left) R-modules;
6. Δ will denote the category of finite linear posets of the form $[n] \coloneqq \{0, \dots, n\}$, for $n \geq 0$ and **sSet** will denote the category of simplicial sets, i.e., of **Set** valued presheaves on Δ;

We will generally assume the existence of Grothendieck universes (see [nLa25c], for an overview), and not pay much heed to set theoretical issues of size, passing to a larger Grothendieck universe whenever necessary. We ensure the reader that at least the Whitehead

[10] If we use compactly or Δ-generated spaces is mainly irrelevant, the only important part will be that we work in a cartesian closed category of topological spaces. Assuming Δ-generation may at time be useful, in order to obtain local presentability (see [nLa25d]).

groups and Whitehead monoids we construct later on will be small in the sense that they are of set size. We will furthermore make use of the following (large) 2-category (by 2-categories we mean categories enriched over the 1-category of sufficiently small categories and functors).

Notation 7.5.0.2. **Cat** will refer to the (large) 2-category of (sufficiently small) categories, with 1-morphisms given by functors and 2-morphisms given by natural transformations.

We we speak of a *bicategory*, we will mean a category that is weakly enriched over the category of (sufficiently small) categories, in the sense that associativity and unit laws are only guaranteed up to certain canonical isomorphisms (see [nLa25a], for a concrete definition and [JY20], for an introduction). The term $(\infty,1)$-category, or just ∞-category, for short, will refer to quasi-categories as in [Lur09]. Often, it will be useful to distinguish between higher categories, such as quasi-categories or simplicial categories, and 1-categories, which both arise from some underlying 1-category. In this case, we use the following notation.

Notation 7.5.0.3.

1. Categories will usually be denoted by bold letters, i.e., in the form $\mathbf{C}$.

2. Quasi-categories will be denoted by calligraphic letters, i.e., in the form $\mathcal{C}$.

3. Simplicial categories will be denoted by underlining bold letters, i.e., in the form $\underline{\mathbf{C}}$.

When we treat a 1-category as a quasi-category or simplicial category, this will mean we are referring to the associated nerve or, respectively, the associated simplicial category with discrete mapping spaces. In a context where multiple such letters are used, and there is a preferred way of passing between the different settings, the quasi-category $\mathcal{C}$ will always be associated to the 1-category $\mathbf{C}$, $\mathbf{C}$ will be the underlying 1-category of $\underline{\mathbf{C}}$, and so on.

We furthermore use the following notation for constructions on categories in the extended sense.

Notation 7.5.0.4. Many of the following constructions we will only need in the case of 1-categories, and hence not introduce them for the extended case.

1. Given a category $\mathbf{C}$, the associated (possibly large) set of objects will be denoted by $\mathrm{Ob}(\mathbf{C})$.

2. Given a category in the extended sense $\mathbf{C}$ and objects $X, Y \in \mathbf{C}$, the notation $\mathbf{C}(X,Y)$ will refer to the set (simplicial set, space, object in some category, ...) of morphisms $X \to Y$.

3. Dual categories will be denoted in the form $\mathbf{C}^{\mathrm{op}}$.

4. Given two categories in the (same) extended sense, $\mathbf{C}$ and $\mathbf{D}$, the notation $\mathbf{Fun}(\mathbf{C},\mathbf{D})$ or $\mathbf{D}^{\mathbf{C}}$ refers to the (respective) extended category of functors from $\mathbf{C}$ to $\mathbf{D}$.

5. In particular, given a 1-category $\mathbf{C}$, the notation $\mathbf{C}^{[1]}$ will refer to the category of arrows in $\mathbf{C}$, with morphisms from $f\colon X_0 \to X_1$ to $g\colon Y_0 \to Y_1$ given by commutative squares

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f"'] & Y_0 \arrow[d, "g"] \\ X_1 \arrow[r] & Y_1 . \end{tikzcd} \tag{7.46}$$

   The evaluation functors $\mathbf{C}^{[1]} \to \mathbf{C}$ will be denoted by ev_0 and ev_1, respectively.

6. Given a functor of 1-categories $F\colon \mathbf{C} \to \mathbf{D}$ and $X \in \mathbf{D}$, we denote by $F_{X/}$ the comma category, whose objects are pairs $(Y, f\colon X \to F(Y))$ with $Y \in \mathbf{C}$ and $f \in \mathbf{D}$ and whose

morphisms $(Y_0, f_0: X_0 \to F(Y_0)) \to (Y_1, f_1: X_1 \to F(Y_1))$ are arrows $g: Y_0 \to Y_1$ in $\mathbf{C}$, such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\ F(Y_0) \arrow[rr, "F(g)"'] & & F(Y_1) \end{tikzcd} \tag{7.47}$$

commutes. The dual construction, using arrows $F(Y) \to X$ instead is denoted $F_{/X}$. At times, when there is a preferred functor $F: \mathbf{C} \to \mathbf{D}$, we will also write $\mathbf{C}_{X/}$ and $\mathbf{C}_{/X}$ instead. In the particular case where $F = 1_{\mathbf{D}}$, this produces *the under and overcategories* of X, $\mathbf{D}_{X/}$ and $\mathbf{D}_{/X}$, also called *coslice and slice categories.*

7. Given a 1-category or quasi-category $\mathbf{C}$, the notation $\mathbf{C}^{\simeq}$ will refer to the groupoid-core, given by the wide subcategory of $\mathbf{C}$ of isomorphisms.

8. Given a functor of 1-categories $F: \mathbf{I} \to \mathbf{J}$, and a third category $\mathbf{C}$, we denote by

$$F^*: \mathbf{C}^{\mathbf{J}} \to \mathbf{C}^{\mathbf{I}}$$

the precomposition functor, mapping $D: \mathbf{J} \to \mathbf{C}$ to $D \circ F: \mathbf{I} \to \mathbf{C}$ and acting in the obvious way on morphisms. Supposing that it exists (for example if $\mathbf{I}$ is small and $\mathbf{C}$ has small colimits), we denote by

$$F_!: \mathbf{C}^{\mathbf{I}} \to \mathbf{C}^{\mathbf{J}}$$

the left adjoint functor of F^*, given by left Kan extension.

9. Given a 1-category or quasi-category $\mathbf{C}$ and a class of morphism $W \subset \mathbf{C}$, the notation $\mathbf{C}[W^{-1}]$ will denote either the 1-categorical or quasi-categorical localization of (the nerve of) $\mathbf{C}$ at W. We will always explicitly make clear what localization we mean.

10. Homotopy categories (to be understood in the respective sense) of a quasi-category, simplicial category or (semi-)model-category $\mathbf{C}$ will be denoted by ho$\mathbf{C}$. In the case of a quasi-category, this means considering the category obtained by considering 1-morphisms given by simplices subject to the relations generated by 2-simplices (see [Lur09]). In the case of a simplicial category, this means we consider the 1-category obtained by taking path-components of mapping spaces. In the case of a (semi-)model-category, we mean the 1-category obtained by localizing at weak equivalences.
We follow the general convention that, if not stated otherwise, any other operation on categories is to be executed *before* passing to homotopy categories. For example, ho$\mathbf{C}^{\mathbf{I}}$ refers to a homotopy category of a functor category $\mathbf{C}^{\mathbf{I}}$, and not to a functor category with values in the homotopy category ho$\mathbf{C}$.

Chapter 8

On the yoga of general cell complexes

In many areas of mathematics, one of the core techniques of proof and investigation is the idea of a presentation of a complicated object in terms of simpler, elementary objects. In the world of linear algebra, this is most prominent in the concept of a basis, in group theory in the concept of a presentation of a group and in (classical) homotopy theory in the notion of a CW-complex. More recently, and abstractly speaking, in the language of ∞-categories this principle is often captured in the notion of a presentable ∞-category and in the language of model categories, it is usually captured in the notions of combinatorial or cellular model categories and (absolute) cell complexes (see, for example, [Hir03; Lur09]). Before we recall an explicit definition of abstract cell complexes, recall the following two guiding examples:

1. Let **Top** be the category of topological spaces and D^n be the n-dimensional disk. A CW-complex $\mathfrak{X}$, consisting of a space X and a set of characteristic maps $\mathfrak{C}_{\mathfrak{X}} \subset \coprod_{n \in \mathbb{N}} \mathbf{Top}(D^n, X)$, is - roughly speaking - a space that can be built inductively by gluing in cells D^n along the boundaries of the characteristic maps $\partial\sigma\colon \partial D^n \hookrightarrow D^n \xrightarrow{\sigma} X$, for $\sigma \in \mathfrak{C}_{\mathfrak{X}}$. Explicitly, we may then write X as a transfinite composition of the so-called skeletons of $\mathfrak{X}$,
$$\varnothing = X^{-1} \hookrightarrow X^0 \hookrightarrow X^1 \hookrightarrow \cdots \hookrightarrow X^\infty = X$$
where the inclusions $X^n \hookrightarrow X^{n+1}$ are inductively defined through pushout squares
$$\begin{tikzcd} \coprod_{\sigma \in \mathfrak{C}^n_{\mathfrak{X}}} \partial D^n \arrow[r, hook] \arrow[d] & \coprod_{\sigma \in \mathfrak{C}^n_{\mathfrak{X}}} D^n \arrow[d] \\ X^{n-1} \arrow[r, hook] & X^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{8.1}$$
with the left vertical induced from an (assumed) factorization
$$\begin{tikzcd} \partial D^n \arrow[r, hook] \arrow[d, dashed] & D^n \arrow[d, "\sigma"] \\ X^{n-1} \arrow[r] & X \end{tikzcd} \tag{8.2}$$
of the boundaries of the characteristic maps $\sigma \in \mathfrak{C}^n_{\mathfrak{X}}$ of dimension n.

2. Let $\mathbf{Ch}_{\geq 0}(R)$ be the category of positively graded chain complexes of R-modules, over some not-necessarily commutative unital ring R. A free chain-complex $F_\bullet = (F_i, d_i) \in \mathbf{Ch}_{\geq 0}(R)$ can be presented through homological spheres and disks as follows: Observe first that $F_\bullet$ may be written as transfinite composition
$$0 = F^{-1}_\bullet \hookrightarrow F^0_\bullet \hookrightarrow F^1_\bullet \hookrightarrow \cdots \hookrightarrow F^\infty_\bullet = F_\bullet$$

where we denote by $F^n_\bullet$ the subcomplex of $F_\bullet$ obtained by setting groups of degree greater than n to 0. As in the case of CW-complexes, the inclusion $F^{n-1}_\bullet \hookrightarrow F^n_\bullet$ can be interpreted as a gluing of cells procedure:
For $n \geq 1$, denote by $D^n_\bullet$ the chain complex

$$\cdots \to 0 \to R \xrightarrow{1} R \to 0 \cdots \to 0$$

which is non-zero exactly in degree n and $n-1$. Denote by $\partial D^n_\bullet = R[-n+1]$, the subcomplex given by R at $n-1$ and 0 everywhere else. For $n = 0$, denote by $D^n_\bullet$ the chain complex $R[0]$, given by R in degree 0 and set $\partial D^{n-1}_\bullet = 0$.
A choice of basis $(b_{i,n})_{i \in I_n} \subset F_n$, in each degree n, specifies the same data as a map $\bigoplus_{i \in I_n} D^n_\bullet \to F_\bullet$ (given by mapping $1 \in R = D^n_n$ in the i-th component of $\bigoplus_{i \in I_n} D^n_\bullet$ to $b_{i,n}$). Then the induced diagram

$$\begin{tikzcd} \bigoplus_{i \in I_n} \partial D^n_\bullet \arrow[r, hook] \arrow[d, dashed] & \bigoplus_{i \in I_n} D^n_\bullet \arrow[d] \\ F^{n-1}_\bullet \arrow[r, hook] & F^n_\bullet \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{8.3}$$

is a pushout square. In this sense, we can think of a basis of a free chain complex as a choice of cell structure on the latter.

The decisive feature of such cell-structures, exhaustively exploited all over the world of algebra and homotopy theory, is that they allow for the reduction of theorems and constructions concerning complicated objects to statements about simple objects, on which the statement often becomes elementary to handle. In the world of classical homotopy theory, this is often summarized under what people call *induction by cells* (see any introductory textbook on algebraic topology, such as [Hat02]). It turns out that these cellular techniques are quite general and can be applied in many homotopy theoretic contexts (see, for example, [Hir03]). To successfully apply these techniques, one first needs a solid understanding of the fundamental notions, operations, and properties in a context of cell complexes – such as subcomplexes, finiteness, and gluing operations. One goal of this section is to expose these basic principles in a general, 1-categorical context. More specifically, we will do so while keeping track of fixed cell structures on the objects. This will be of importance in our investigations of generalized simple homotopy theory in the subsequent chapters, when the questions of (homotopical) uniqueness of cell structures (up to certain elementary operations) will be the core line of investigation. Most of what we present here is certainly known in some form and has occurred in the literature numerous times in different shapes and forms. The goal here is mainly to present the relevant structures and results in a cohesive context and have a rigorous and compatible source for citations available. Our investigations here will explicitly not yet be homotopy-theoretic or higher categorical. However, they are clearly performed while having the homotopical context of (cofibrantly generated or cellular) model categories in mind.

8.1 Cellularized categories

In this section, we introduce the language of presentations, structured cell complexes and cellularized categories, as well as the basic operations within the latter.

8.1.1 Presentations and structured relative cell complexes

Let us first introduce the definition of a presentation of a relative cell complex. This notion is inspired by [Hir03, p. 10.6.2] however, it is not entirely equivalent (see Remark 8.1.1.8).

Definition 8.1.1.1. Let $\mathbf{C}$ be a cocomplete category and let $\mathbb{B}$ be a set of morphisms in $\mathbf{C}$. Let $c\colon A \to B$ be a morphism in $\mathbf{C}$. By a $\mathbb{B}$-filtration-presentation $\mathfrak{p}$ of c we mean the following data:

- A transfinite composition diagram
$$A = B^0 \to \cdots \to B^\alpha \to B^{\alpha+1} \to \cdots \to B^\lambda = B\,,$$
indexed over some ordinal λ, such that c is the (transfinite) composition $A \to B$.

- For every $\alpha \in \lambda$, a family $(\iota_i{:}\partial D_i \to D_i, \sigma_i{:}D_i \to B^{\alpha+1})_{i\in I^\alpha}$, with $\iota_i \in \mathbb{B}$, such that factorizations
$$\begin{tikzcd} \partial D_i \arrow[r, "\iota_i"] \arrow[d, dashed, "\varphi_i"] & D_i \arrow[d] \\ B^\alpha \arrow[r] & B^{\alpha+1} \end{tikzcd} \tag{8.4}$$
exist. Furthermore, we require that the induced diagram
$$\begin{tikzcd} \coprod_{i\in I_\alpha} \partial D_i \arrow[r, hook, "\sqcup \iota_i"] \arrow[d, "\sqcup \varphi_i"] & \coprod D_i \arrow[d, "\sqcup \sigma_i"] \\ B^\alpha \arrow[r, hook] & B^{\alpha+1} \end{tikzcd} \tag{8.5}$$
is a pushout square.

Remark 8.1.1.2. It follows from the general commutativity properties of colimits and pushouts that a morphism admits a $\mathbb{B}$-filtration-presentation if and only if it lies in the smallest class of morphisms which contains $\mathbb{B}$ and is closed under transfinite composition and pushouts. This class of morphisms is also called the class of relative $\mathbb{B}$-cell complexes (see [Hir03, p. 10.5.8]).

Remark 8.1.1.3. The reason we use the terminology *filtration-presentation*, instead of just *presentation* is because we want to distinguish this notion from a more homotopy-theoretic notion of presentation that appears later in the text.

Observation 8.1.1.4. Observe that an empty filtration-presentation, with ordinal $\lambda = 0$, is simply the data of the identity morphisms $A \to A$. A filtration-presentation with ordinal $\lambda = 1$ and $I^0 = \varnothing$ specifies the data of a morphism $A = B^0 \to B^1 = B$, which fits into a pushout square
$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & B \end{tikzcd} \tag{8.6}$$
where $\varnothing = \coprod_\varnothing$ is the empty coproduct, which specifies the initial object of $\mathbf{C}$. As the upper horizontal is an isomorphism, so is the lower horizontal. It follows that an empty filtration-presentation with $\lambda = 1$ simply specifies the data of an isomorphism $A \cong B$.

Example 8.1.1.5. As we have discussed in the introduction of this section, any CW-complex admits a $\mathbb{B}$-filtration-presentation, where $\mathbb{B}$ is the class of boundary inclusions, and the presentation ordinal λ is ω_0, the first infinite ordinal.

There is a general ambiguity in the language of algebraic topology, whether the term CW-complex should refer to a space, together with a choice of cell structure, or just to a space which *admits* the structure of a CW-complex. Of course, for many theorems this linguistic ambiguity is entirely irrelevant, but in other contexts, such as the definition of cellular chain-complexes, the cell structure is clearly part of the data required to obtain well-defined definitions. For the purpose of studying different possible presentations of an object (especially in the homotopy theoretic sense, when doing simple homotopy theory), the presentation clearly needs to be part of the data in some sense. However, what one is really interested in is not the whole data of a filtration-presentation, but only the data of the *cells or characteristic maps*. In other words, we do not want to distinguish between two filtration-presentations which are obtained by gluing things in a different order, or through different choices of pushout squares. This is captured in the following definitions.

Definition 8.1.1.6. Let $\mathbf{C}$ be a cocomplete category and let $\mathbb{B}$ be a set of morphisms in $\mathbf{C}$.

1. Let $c: A \to B$ be a morphism in $\mathbf{C}$ and let $\mathfrak{p}$ be a $\mathbb{B}$-filtration-presentation of c. Using the notation of Definition 8.1.1.1, the *set of characteristic maps* associated to $\mathfrak{p}$, is the set

$$\mathfrak{C}_{\mathfrak{p}} = \{(\iota_i: \partial D_i \hookrightarrow D_i, D_i \xrightarrow{\sigma_i} B^{\alpha+1} \to B) \mid i \in I^\alpha, \alpha \in \lambda\} \subset \mathbb{B} \times \coprod_{\partial D \hookrightarrow D \in \mathbb{B}} \mathbf{C}(D, B).$$

2. A *structured relative* $\mathbb{B}$*-cell complex* $\mathfrak{c} = (c: A \hookrightarrow B, \mathfrak{C}_{\mathfrak{c}})$ consists of the data of

   - A morphism $c: A \to B$ in $\mathbf{C}$;
   - A set of morphisms $\mathbb{B}_{\mathfrak{c}} \subset \mathbb{B} \times \coprod_{\partial D \hookrightarrow D \in \mathbb{B}} \mathbf{C}(D, B)$, such that $\mathbb{B}_{\mathfrak{c}}$ is the set of characteristic maps with respect to some filtration-presentation of c.

   At times, we will refer to the elements of $\mathfrak{C}_{\mathfrak{c}}$ as the *characteristic maps of* $\mathfrak{c}$ and also as the *the cells of* $\mathfrak{c}$. By a presentation $\mathfrak{p}$ of a cell complex $\mathfrak{c}$ we mean any filtration-presentation of c, such that $\mathfrak{C}_{\mathfrak{p}} = \mathfrak{C}_{\mathfrak{c}}$.

Notation 8.1.1.7. We will follow the convention of writing $\mathfrak{c}: A \to X$, when we want to refer to a cell complex as a structured object, and $c: A \to X$, when we think of the underlying morphism. This general convention of changing to regular font, when referring to an underlying object, will be used all over this thesis.

Remark 8.1.1.8. A remark comparing the definitions of presentation in [Hir03, p. 10.6.2] and Definition 8.1.1.1 is in order. The crucial difference is that the data of a presentation in Hirschhorn emphasizes the *boundary maps* $\partial D_i \to B^\alpha$, while our definition emphasizes the *characteristic maps* $D_i \to B^{\alpha+1}$. In all scenarios which we are interested in, the morphisms $B^\alpha \to B^{\alpha+1}$ are monomorphisms. Hence, the characteristic maps recall the boundary maps. However, the converse is clearly false. This can already be illustrated with the following simple example from algebra, which illustrates well why the definition of a presentation in [Hir03] does not encapsulate the concept of presentation which we had in mind: Let k be a field and let $\mathbf{C}$ be the category of k-vector spaces. We can equip $\mathbf{C}$ with the singleton of boundary inclusions $\mathbb{B} = \{0 \to k\}$. Then a cell structure on a vector space V with respect to $\mathbb{B}$ is simply a choice of unordered basis $B \subset V$. In Hirschhorn's sense, however, a presentation only recalls the maps $0 \to k \to V$, which contain no information. For Hirschhorn's purposes, this notion is entirely sufficient, as [Hir03] only uses presentations as a tool whose existence has useful implications for homotopical arguments, but does not study them as objects of interest on their own. In this sense, the two notions are often equivalent when it comes to existence claims, but may differ when one is looking to classify cell-structures up to some notion of equivalence. The questions of simple homotopy theory that we have developed the theory of abstract cell complexes for are of the latter type. Hence, we will take great care to mind the differences in notions, when citing results from Hirschhorn.

Notation 8.1.1.9. It is unlikely that all throughout this work, we will recall to add the (somewhat superfluous) word *structured* whenever a structured cell complex occurs. We will, however, make sure to use fraktur font for all structured relative cell complexes, in order to remind the reader of the cell structure hidden in the background.

In order for the theory of structured relative cell complexes associated with a set of boundary inclusions $\mathbb{B}$ to be well-behaved, we will make a series of categorical assumptions.

Definition 8.1.1.10. A *celullarized category* consists of the following data:

1. A category $\mathbf{C}$, having all colimits;
2. A set $\mathbb{B}$ of morphisms in $\mathbf{C}$, elements of which will be denoted in the form $\partial D \to D$ and called *boundary inclusions*.

Such that the following holds:

P(i) $\mathbb{B}$ contains no isomorphisms;

P(ii) All relative $\mathbb{B}$-cell complexes a have the property that every pushout square

$$\begin{tikzcd} A \arrow[r, hook, "a"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, hook, "a'"] & B' \end{tikzcd} \tag{8.7}$$

in $\mathbf{C}$ is also a pullback square.

Example 8.1.1.11. The category of sets, **Set**, equipped with the single boundary inclusion $\varnothing \to *$, is easily verified to be a cellularized category. The relative cell complexes are precisely the inclusions of sets. Every inclusion of sets $A \hookrightarrow X$ admits exactly one cell structure, namely the one where the cells are given by the elements of $X \setminus A$.

Example 8.1.1.12. The category of topological spaces (or one of its appropriately generated derivatives, see [Dug03]) **Top**, equipped with the set of boundary inclusions $\mathbb{B} = \{\partial D^n \hookrightarrow D^n \mid n \geq 0\}$ is a cellularized category. The relative cell complexes are essentially CW-complexes where the assumption that gluing maps $\partial D^n \to X$ need to map into X^{n-1} are dropped. Observe that every such relative cell complex is a closed inclusion. Modifying the source by a homeomorphism if necessary, we may assume that for a relative cell complex $A \hookrightarrow X$ we actually have $A \subset X$. Then, in a pushout diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d, "f'"] \\ A' \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & X' \end{tikzcd} \tag{8.8}$$

one may easily compute that $f'^{-1}(A') = A$, which shows Property P(ii).

Example 8.1.1.13. If we are looking to recover CW-complexes, we need to introduce additional data which ensures that dimensions are preserved in the gluing process. Consider the category of filtered topological spaces **Filt**, defined as follows. An object of **Filt** is a space $T \in \mathbf{Top}$, together with a family of subsets $T^0 \subset \cdots \subset T^n \subset \cdots \subset T^\infty = T$. We will usually just write T to refer to the whole filtered space. indexed over the naturals. Morphisms from $f\colon (T_0, (T_0^n)_{n \in \mathbb{N}})$ to $(T_1, (T_1^n)_{n \in \mathbb{N}})$ are given by continuous maps $f\colon T_0 \to T_1$, such that $f(T_0^n) \subset T_1^n$, for all $n \in \mathbb{N}$. This category is cocomplete. The colimit of a diagram $F\colon I \to \mathbf{Filt}$ is given by equipping the topological space $\varinjlim F_i^\infty$ with the filtration

$$\bigcup_{i \in I} \operatorname{im}(F_i^0 \to \varinjlim F_i^0) \subset \cdots \subset \bigcup_{i \in I} \operatorname{im}(F_i^n \to \varinjlim F_i^n) \subset \cdots \subset \varinjlim F_i^\infty = \varinjlim F_i \,.$$

Denote by E^n the filtered space obtained by equipping D^n with the filtration.

$$k \mapsto \begin{cases} \varnothing & k < n-1 \\ \partial D^n & k = n-1 \\ D^n & k \geq n \,. \end{cases}$$

Furthermore, denote by ∂E^n the filtered subspace of E^n given by

$$k \mapsto \begin{cases} \varnothing & k < n-1 \\ \partial D^n & k \geq n-1 . \end{cases}$$

If we set $\mathbb{B} = \{\partial E^n \hookrightarrow E^n \mid n \geq 0\}$, then a structured $\mathbb{B}$-complex $\mathfrak{X}\colon \varnothing \to X$ specifies exactly the same data as a classical CW-complex. Furthermore, the morphisms in **Filt** between two such complexes $\mathfrak{X}$ and $\mathfrak{Y}$ are precisely the cellular maps. Similarly, using constant filtered space $(A, (A)_{n \in \mathbb{N}})$, for $A \in \mathbf{Top}$, one can recover relative CW-complexes. Having this information at hand, one may easily verify Property P(ii).

Example 8.1.1.14. The category of simplicial sets **sSet** equipped with the set of boundary inclusions $\mathbb{B} = \{\partial\Delta^n \to \Delta^n \mid n \geq 0\}$ is a cellularized category (where Property P(ii) is inherited from the analogous property in **Set**, using that limits and colimits in functor categories can be computed pointwise). The relative cell complexes are precisely the inclusions of simplicial sets $A \hookrightarrow X$. Every such inclusion admits exactly one cell structure, namely the one given by the non-degenerate simplices $X_{n.d.} \setminus A_{n.d.}$. This phenomenon of uniqueness of cell structures for certain presheaf categories will be studied in more detail in Section 8.3.

Example 8.1.1.15. The category of positively graded chain complexes of modules over some (not-necessarily commutative) ring R, $\mathbf{Ch}_{\geq 0}(R)$, admits the structure of a cellularized category. Using notation as in the introduction of this chapter, set $\mathbb{B} = \{\partial D^n_\bullet \hookrightarrow D^n_\bullet \mid n \geq 0\}$. An (absolute) structured cell complex $\mathfrak{X}_\bullet \colon 0 \to X_\bullet$ specifies precisely the same data as a choice of basis for a free chain complex $X_\bullet$ in each degree. A structured relative cell complex $\mathfrak{X} \colon A_\bullet \hookrightarrow X_\bullet$ specifies the same data as a family of elements $\{b_i\} \subset X_n$, for each $n \geq 0$, such that $\{[b_i]\}$ is a basis of X_n / A_n.

Notation 8.1.1.16. Given a cellularized category $(\mathbf{C}, \mathbb{B})$, we will generally refer to the whole tuple by $\mathbf{C}$, and sometimes write $\mathbb{B}_{\mathbf{C}}$ for the set of boundary inclusions in order to circumvent any possible ambiguities. Furthermore, given the context of a cellularized category $(\mathbf{C}, \mathbb{B})$, we will often omit $\mathbb{B}$ from the language, and simply speak of structured cell complexes instead of structured $\mathbb{B}$-cell complexes.

Remark 8.1.1.17. The assumption that $\mathbb{B}$ contains no isomorphisms is purely in order to assure that cell-structures contain no redundant data. This ensures that isomorphisms of absolute cell complexes can be identified on the level of the sets of characteristic maps (see Corollaries 8.1.4.1 and 8.1.4.5). The second condition ensures that every relative $\mathbb{B}$-cell complex is an effective monomorphism. Hence this assumption is slightly stronger than the ones made in the context of a cellular model category in [Hir03, Def.12.1.1]. Having relative cell complexes be effective monomorphisms ensures that we have a well-behaved theory of subcomplexes, as discussed in Proposition 8.1.3.1. We use the slightly stronger assumption in order to ensure that filtration-presentations cannot contain the same characteristic map twice, i.e., that it is sensible to work with a set of characteristic maps instead of a family of characteristic maps.

Remark 8.1.1.18. Observe that any category with colimits necessarily has an initial object: The colimit over the empty diagram. Up to equivalence of categories, there is really no harm in assuming that this initial object is unique, which we will always do in the following, and denote the initial object by $\varnothing$. This has the added advantage of all absolute cell complexes, i.e., relative cell complexes with source an initial object, all having the same source.

Lemma 8.1.1.19. *Let and $c \colon A \to X$ be a relative cell complex. Given $(\iota \colon \partial D \to D) \in \mathbb{B}$, suppose that there is a filtration-presentation $\mathfrak{p}$ of c and indices $\alpha, \alpha' < \lambda_{\mathfrak{p}}$, $i \in I_\alpha$, $i' \in I_{\alpha'}$, such that*

$$\begin{tikzcd} D \arrow[r, "\sigma_i"] \arrow[d, "\sigma_{i'}"] & X^{\alpha+1} \arrow[d] \\ X^{\alpha'+1} \arrow[r] & Y \end{tikzcd} \tag{8.9}$$

commutes. Then $(i, \alpha) = (i', \alpha')$, or ι is an isomorphism.

Proof. Without loss of generality, we may assume $\alpha \leq \alpha'$. Furthermore, by refining $\mathfrak{p}$ if necessary, we may even assume $\alpha < \alpha'$. We are thus in the situation of the following solid diagram

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "1"] & \partial D \arrow[r, hook] \arrow[d] & D \arrow[d] & \\ X^{\alpha+1} \arrow[r, hook] & X^{\alpha'} \arrow[r] & X^{\alpha'+1} \arrow[r] & X \end{tikzcd} \tag{8.10}$$

with a pushout square to the left and all paths ending in X commuting. Since $X^{\alpha'+1} \to X$ is a monomorphism, the whole diagram commutes. It follows that we obtain a dashed arrow making the diagram commute. In particular, ∂D is a retract of D. However, since $\partial D \hookrightarrow D$ was assumed to be a monomorphism, it follows that $\partial D \hookrightarrow D$ is an isomorphism. □

In particular, every characteristic map in $\mathfrak{C}_{\mathfrak{c}}$ can only appear once in a filtration-presentation of $\mathfrak{c}$. We are going to make a second assumption, which is purely for notational reasons, but will significantly simplify notation.

Assumptions 8.1.1.20. Given a cellularized category $\mathbf{C}$, we will assume that $\mathbb{B}$ has the additional property that whenever $\iota\colon \partial D \to D$ and $\iota'\colon \partial D' \to D'$ in $\mathbb{B}$ fulfill $D = D'$, then $\iota = \iota'$.

Remark 8.1.1.21. Assumptions 8.1.1.20 is really not a substantial assumption to make. Up to equivalence of categories, we may always assume that there are **Set** many representatives in any given isomorphism class of an object. Then, as $\mathbb{B}$ was assumed to be a set, we can hence always modify the targets of morphisms in $\mathbb{B}$ up to an isomorphism, in order for this property to hold. The crucial notation advantage of Assumptions 8.1.1.20, is that we can uniquely identify a cell $(\iota\colon \partial D_\iota \to D_\iota, \sigma\colon D_\iota \to X)$ of a relative cell complex $A \overset{\mathfrak{c}}{\hookrightarrow} X$, with the map $\sigma\colon D_\iota \to X$. We will often use this to our notational advantage, simply writing $\sigma\colon D_\iota \to X$, instead of $(\iota\colon \partial D_\iota \to D_\iota, \sigma\colon D_\iota \to X)$.

8.1.2 Categories of structured (relative) cell complexes

Let us now define the associated categories of relative cell complexes, not the least to have a notion of isomorphism, and hence a concrete idea of when two structured relative complexes are considered the same from the perspective we take here. For the remainder of this subsection, we fix the context of a cellularized category $\mathbf{C}$.

Definition 8.1.2.1. By a *structure preserving morphism* of two structured relative cell complexes $A_1 \overset{\mathfrak{c}_1}{\hookrightarrow} X_1$ and $A_1 \overset{\mathfrak{c}_1}{\hookrightarrow} X_1$, we mean a pair of morphisms $(f_A\colon A_0 \to A_1, f_X\colon X_0 \to X_1)$ in $\mathbf{C}$ fitting into a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "f_A"] \arrow[d, hook, "c_0"] & A_1 \arrow[d, hook, "c_1"] \\ X_0 \arrow[r, "f_X"] & X_1 , \end{tikzcd} \tag{8.11}$$

in $\mathbf{C}$ such that, for every characteristic map $\sigma \in \mathfrak{C}_{\mathfrak{c}_0}$ the induced map $f_X \circ \sigma$ is in $\mathfrak{C}_{\mathfrak{c}_1}$. We denote by $\mathbf{RCell}(\mathbf{C})$ the category of structured relative $\mathbb{B}$-complexes and structure preserving morphisms, with the obvious notions of identity and composition.

Notation 8.1.2.2. Generally, we will use the notational convention that given a cell complex $\mathfrak{c}_i$, its source will be denoted A_i and its target X_i. Similarly, given a morphism $f\colon \mathfrak{c}_i \to \mathfrak{c}_j$, we denote the underlying pair of morphisms in $\mathbf{C}$ in the form (f_A, f_X).

Clearly, we are not only interested in studying (proper) relative cell complexes, but also absolute cell complexes, obtained by letting the source be the initial object $\varnothing \in \mathbf{C}$.

Definition 8.1.2.3. We denote by $\mathbf{Cell}(\mathbf{C})$ the full subcategory of $\mathbf{RCell}(\mathbf{C})$, given by such structured relative cell complexes $A \overset{\mathfrak{c}}{\hookrightarrow} X$, for which $A \cong \varnothing$ is initial in $\mathbf{C}$. Elements of $\mathbf{Cell}(\mathbf{C})$ are called *absolute structured cell complexes.*

Notation 8.1.2.4. We will denote absolute structured cell complexes in the form $\mathfrak{X} = (X, \mathfrak{C}_{\mathfrak{X}})$.

Next, let us list some elementary properties of structured relative cell-complexes which are easily verified.

Notation 8.1.2.5. It will be convenient to treat the set of characteristic maps $\mathfrak{C}$ as a functor into the overcategory $\mathbf{Set}_{/\mathbb{B}}$, with functoriality induced by postcomposition of characteristic maps:

$$\begin{aligned} \mathfrak{C}\colon \mathbf{RCell}(\mathbf{C}) &\to \mathbf{Set}_{/\mathbb{B}} \\ \mathfrak{c} &\mapsto \mathfrak{C}_{\mathfrak{c}} \\ (f\colon \mathfrak{c}_0 \to \mathfrak{c}_1) &\mapsto \mathfrak{C}(f) \end{aligned}$$

with $\mathfrak{C}(f)$ given by

$$\begin{aligned} \mathfrak{C}(f_A, f_X)\colon \mathfrak{C}_{\mathfrak{c}_0} &\to \mathfrak{C}_{\mathfrak{c}_1} \\ (\iota, D_\iota \to X_0) &\mapsto (\iota, D_\iota \to X_0 \xrightarrow{f_X} X_1)\,. \end{aligned}$$

Observation 8.1.2.6. When working with morphisms of relative cell complexes $f \in \mathbf{RCell}(\mathbf{C})$, it is generally convenient to observe that as every structured relative cell complex $\mathfrak{c}_0\colon A_0 \to X_0$ can be constructed inductively in terms of colimits of its cells, a morphism $(f_A, f_B)\colon \mathfrak{c}_0 \to (A_1 \xrightarrow{\mathfrak{c}_1} B_1)$ of cell complexes is uniquely determined by the data of:

1. The morphism $f_A\colon A_0 \to A_1$;
2. The morphism $\mathfrak{C}(f)$.

In other words, we obtain an inclusion

$$\begin{aligned} \mathbf{RCell}(\mathbf{C})(\mathfrak{c}_0, \mathfrak{c}_1) &\hookrightarrow \mathbf{C}(A_0, A_1) \times \mathbf{Set}_{/\mathbb{B}}(\mathfrak{C}_{\mathfrak{c}_0}, \mathfrak{C}_{\mathfrak{c}_1}) \\ f &\mapsto (f_A, \mathfrak{C}(f)) \end{aligned}$$

and thus a faithful functor

$$\begin{aligned} \mathbf{RCell}(\mathbf{C}) &\to \mathbf{C} \times \mathbf{Set}_{/\mathbb{B}} \\ (A \overset{\mathfrak{c}}{\hookrightarrow} X) &\mapsto (A, \mathfrak{C}_{\mathfrak{c}}) \\ f &\mapsto (f_A, \mathfrak{C}(f)) \end{aligned}$$

It particular, in the case when f_A is the identity, or unique as $A = \varnothing$, this means we can entirely think of a morphism of relative cell complexes as a map of sets, which can be significantly easier to work with.

Next, let us consider the two fundamental constructions to any theory of relative cell complexes: Cobase change and transfinite (vertical) composition:

Construction 8.1.2.7. Given a structured relative cell complex $A_0 \overset{\mathfrak{c}_0}{\hookrightarrow} X_0$, and an arrow $g\colon X_0 \to X_1$, we denote

$$g\mathfrak{C}_{\mathfrak{c}_0} := \{g \circ \sigma \mid \sigma \in \mathfrak{C}_{\mathfrak{c}_0}\} \subset \coprod_{\partial D \to D \in \mathbb{B}} \mathbf{C}(D, X_1).$$

Generally, there is no reason why $g\mathfrak{C}_{\mathfrak{c}_0}$ should define the structure of a cell complex on X_1, or on some relative cell complex $A_1 \hookrightarrow X_1$. However, in the following situations, this is indeed the case.

1. Suppose that the diagram
$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "c_0"] & A_1 \arrow[d, "c_1"] \\ X_0 \arrow[r, "f'"] & X_1\,. \end{tikzcd} \tag{8.12}$$
is a pushout square. Then it follows from the compatibility relations of transfinite compositions and pushouts, that c_1 is a relative cell complex and $f'\mathfrak{C}_{\mathfrak{c}_0}$ defines a cell

structure on c_1, such that (f, f') defines a morphism of structured relative cell complexes. These types of morphisms in $\mathbf{RCell(C)}$ will be called *cobase change morphisms*. We will also call the associate squares with verticals given by structured relative cell complexes *cobase change squares*.

2. Consider a transfinite composition

$$c: A = X^0 \to X^1 \to \cdots \to X^\lambda = X$$

of relative cell complexes, together with, for each $\alpha \in \lambda$, a cell structure $\mathfrak{C}^\alpha$ on $X^\alpha \to X^{\alpha+1}$, defining a structured relative cell complex $\mathfrak{c}^\alpha$. Denote by f^α the canonical morphism $X^\alpha \to X$. Then it follows from the composability of transfinite compositions, that the set $\bigcup_{\alpha\in\lambda} f^{\alpha+1}\mathfrak{C}^\alpha$ defines a cell structure on c. We call the resulting cell complex the *vertical transfinite composition* of the structured relative cell complexes $\mathfrak{c}^\alpha: X^\alpha \hookrightarrow X^{\alpha+1}$. In the case, $\lambda = 2$, i.e., the case of an ordinary composition, we will write $\mathfrak{c}^1 \circ \mathfrak{c}^0$ to denote the induced relative structured cell complex. We also use this notation when $\mathfrak{c}_0 = \mathfrak{X}$ is an absolute structured cell complex, treating it as a relative structured cell complex with source $\varnothing$, and thus obtaining a new absolute structured cell complex $\mathfrak{c}_1 \circ \mathfrak{X}$.

Lemma 8.1.2.8. *Given a cobase change square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{c}"] & A' \arrow[d, "\mathfrak{c}'"] \\ X \arrow[r, "f'"] & X'. \end{tikzcd} \tag{8.13}$$

the associated morphism $(f, f'): \mathfrak{c} \to \mathfrak{c}'$ has the following universal property:
Given any further morphism of cell complexes $(g, g_1): \mathfrak{c} \to (B \overset{\mathfrak{d}}{\hookrightarrow} Y)$, together with a factorization

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & B \\ & A' \arrow[ur, dotted, "h"'] & \end{tikzcd} \tag{8.14}$$

there exists a unique morphism $h_1: X' \dashrightarrow Y$, such that (h, h_1) defines a morphism of structured relative cell complexes $\mathfrak{c}' \to \mathfrak{d}$ making the diagram

$$\begin{tikzcd} \mathfrak{c} \arrow[rr, "(g,g_1)"] \arrow[dr, "(f,f')"'] & & \mathfrak{d} \\ & \mathfrak{c}' \arrow[ur, dotted, "(h,h_1)"'] & \end{tikzcd} \tag{8.15}$$

commute. In particular $\mathfrak{c}'$ is uniquely determined by $\mathfrak{c}$ and f, up to canonical isomorphism of relative structured cell complexes.

Proof. This is immediate from the universal property of the pushout and the definition of the cobase change cell structure. ☐

Notation 8.1.2.9. In the situation of Lemma 8.1.2.8, we will write $f_{¡}\mathfrak{c} := \mathfrak{c}'$ for the induced relative structured cell complex (determined up to canonical isomorphism).

Remark 8.1.2.10. The reason why we do not use a regular shriek to denote cobase changes is that the notation $f_!$ is usually used to suggest that $f_!$ is left adjoint to some base change functor f^* depending on f. However, in the case of structured cell complexes, there is no such right adjoint in sight. Hence, we use ¡, in order to not give any false intuition.

Construction 8.1.2.11. Consider the forgetful functor

$$\begin{aligned} \mathbf{RCell}(\mathbf{C}) &\to \mathbf{C} \\ (A \xrightarrow{\mathfrak{c}} X) &\mapsto A\,. \end{aligned}$$

We denote its fiber at $A \in \mathbf{C}$ by $\mathbf{RCell}(\mathbf{C})_A$. Morphisms in the fiber can be identified with commutative diagrams

$$\begin{tikzcd} & A \arrow[dl, "\mathfrak{c}_0"'] \arrow[dr, "\mathfrak{c}_1"] & \\ X_0 \arrow[rr, "f"'] & & X_1 \end{tikzcd} \tag{8.16}$$

f such that $f\mathfrak{C}_{\mathfrak{c}_0} \subset \mathfrak{C}_{\mathfrak{c}_1}$. It is precisely the content of Lemma 8.1.2.8, together with the assumption that $\mathbf{C}$ has pushouts, that

$$\begin{aligned} \mathbf{RCell}(\mathbf{C}) &\to \mathbf{C} \\ (A \xrightarrow{\mathfrak{c}} X) &\mapsto A\,. \end{aligned}$$

is a cocartesian fibration (see [GR04], for the original source, and [nLa24g] for an excellent overview). The cocartesian arrows are precisely the cobase change morphisms (squares) of Construction 8.1.2.7. It follows from the fundamental theorem of (co)cartesian fibrations (see [nLa24g]) that we obtain an induced pseudo-functor

$$\begin{aligned} \mathbf{RCell}(\mathbf{C})_{-} \colon \mathbf{C} &\to \mathbf{Cat} \\ A &\mapsto \mathbf{RCell}(\mathbf{C})_A \\ (A \xrightarrow{f} A') &\mapsto \begin{cases} f_{!} \colon \mathbf{RCell}(\mathbf{C})_A & \to \mathbf{RCell}(\mathbf{C})_{A'} \\ \mathfrak{c} & \mapsto f_{!}\mathfrak{c} \end{cases}. \end{aligned}$$

In this sense, we can think of $\mathbf{C}$ as acting on the category of structured relative cell complexes $\mathbf{RCell}(\mathbf{C})$ via cobase change.

As we have identified cobase change squares as cocartesian morphisms with respect to the evaluation at 0-functor $\mathbf{RCell}(\mathbf{C}) \to \mathbf{C}$, it follows that they fulfill the horizontal pasting law:

Corollary 8.1.2.12. *Given a commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\mathfrak{c}"'] & A' \arrow[r] \arrow[d, hook, dashed, "\mathfrak{c}'"] & A'' \arrow[d, hook, "\mathfrak{c}''"] \\ X \arrow[r] & X' \arrow[r] & X'' \end{tikzcd} \tag{8.17}$$

defining morphisms of structured relative cell complexes $\mathfrak{c} \to \mathfrak{c}', \mathfrak{c}' \to \mathfrak{c}''$, where the left square is a cobase change, then the right square is a cobase change, if and only if the composition of the two squares is a cobase change.

Observation 8.1.2.13. The two constructions in Construction 8.1.2.7 are compatible with each other, in the sense that if we are given a diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, hook] & X_1 \arrow[d, hook] \\ X_0^1 \arrow[r] \arrow[d, hook] & X_1^1 \arrow[d, hook] \\ \dots \arrow[r] \arrow[d, hook] & \dots \arrow[d, hook] \\ X_0^\lambda \arrow[r] & X_1^\lambda \end{tikzcd} \tag{8.18}$$

with vertical transfinite compositions and all successor squares pushout, with left horizontals relative cell complexes, then the two constructions commute, in the sense that the vertical transfinite composition of the cell structures obtained via cobase change is the cobase change of the transfinite composition on the left.

Remark 8.1.2.14. From a categorical perspective, it would, of course, be preferable to have a universal characterization of vertical transfinite composition as defined in Construction 8.1.2.7. There is a conceptual reason why this is hard to do with what we have defined so far. Observe that vertical composition, restricted to the case $\lambda = 2$, equips the category $\mathbf{RCell(C)}$ with another notion of composition, given by composing relative cell complexes. Formally, the correct language to describe a category with two directions of composition is the language of double categories, i.e., a category *internal to categories.* While it seems extremely plausible that much of what we say here has an elegant and concise interpretation in this language, we did not want to burden this expository material with another level of categorical complexity. Furthermore, there is a way around this issue. Namely, we can reinterpret the vertical composition as a horizontal composition, as soon as we have exposed an appropriate theory of subcomplexes. This will be the content of the next section.

8.1.3 Subcomplexes and their properties

One of the main classes of objects we are going to study, in particular in the context of simple homotopy theory, are inclusions of subcomplexes. These were studied in an abstract categorical context in [Hir03, Def. 10.6.7]. Modulo the difference in focusing on characteristic or boundary maps, and [Hir03] only defining what a subcomplex is with respect to a fixed presentation, our theory of subcomplexes will essentially agree with the one in [Hir03]. Let us now study what morphisms of cell complexes arise from sub-presentations.

Proposition 8.1.3.1. *Given a morphism* $i\colon (A \overset{\tilde{\mathfrak{c}}}{\hookrightarrow} \tilde{X}) \to (A \overset{\mathfrak{c}}{\hookrightarrow} X)$ *in* $\mathbf{RCell(C)}_A$*, the following conditions are equivalent:*

(i) $i\colon \tilde{X} \to X$ *itself defines a relative cell complex, with cell structure given by* $\mathfrak{C}_{\mathfrak{c}} \setminus i\mathfrak{C}_{\tilde{\mathfrak{c}}}$*.*

(ii) i *is a monomorphism in* $\mathbf{C}$*.*

(iii) The map $\mathfrak{C}(i)\colon \mathfrak{C}_{\tilde{\mathfrak{c}}} \to \mathfrak{C}_{\mathfrak{c}}$ *is injective.*

(iv) i *is a monomorphism in* $\mathbf{RCell(C)}$*.*

Proof. We show equivalence of the first three conditions first. Equivalence with the final condition is shown in Lemma 8.1.3.3. That Property (i) implies Property (ii) was discussed in Remark 8.1.1.17. The implication Property (ii) to Property (iii) is trivial. Finally, to see that Property (iii) implies Property (iv), choose filtration-presentations $\tilde{\mathfrak{p}}$ of $\tilde{\mathfrak{c}}$ and $\mathfrak{p}$ of $\mathfrak{c}$ indexed over ordinals $\tilde{\beta}$, β, such that exactly one cell is added in each step. Via these filtration-presentations, we can think of the sets of characteristic maps as being identified with $\tilde{\beta}$ and β, respectively. $\mathfrak{C}(i)$ then induces an injective map

$$f\colon \tilde{\beta} \to \beta.$$

We will write

1. $\tilde{\alpha} \leq \alpha$ if $f(\tilde{\alpha}) \leq \alpha$;

2. And $\alpha \leq \tilde{\alpha}$ if there exists $\tilde{\alpha}' \leq \tilde{\alpha}$ such that $f(\tilde{\alpha}') = \alpha$.

We claim the existence of the following extension of the filtration-presentation of $\tilde{\mathfrak{c}}$ and $\mathfrak{c}$ to a commutative diagram over X

$$\begin{tikzcd}
\tilde{X}^{\tilde{\beta}} \arrow[r] & \tilde{X}^{\tilde{\beta},2} \arrow[r] & \tilde{X}^{\tilde{\beta},1} \arrow[r] & \dots \arrow[r] & X^{\beta} \\
\dots \arrow[r] \arrow[u] & \dots \arrow[r] \arrow[u] & \dots \arrow[r] \arrow[u] & \dots \arrow[r] \arrow[u] & X^{\beta} \arrow[u, "1"'] \\
\tilde{X}^{1} \arrow[r] \arrow[u] & \tilde{X}^{2,1} \arrow[r] \arrow[u] & \tilde{X}^{2,2} \arrow[r] \arrow[u] & \dots \arrow[r] \arrow[u] & X^{\beta} \arrow[u, "1"'] \\
\tilde{X}^{0} \arrow[r] \arrow[u] & \tilde{X}^{1,1} \arrow[r] \arrow[u] & \tilde{X}^{1,2} \arrow[r] \arrow[u] & \dots \arrow[r] \arrow[u] & X^{\beta} \arrow[u, "1"'] \\
A = X^{-1} \arrow[r] \arrow[u] & X^{0} \arrow[r] \arrow[u] & X^{1} \arrow[r] \arrow[u] & \dots \arrow[r] \arrow[u] & X^{\beta} \arrow[u, "1"']
\end{tikzcd} \tag{8.19}$$

with the following properties:

1. Each row and each column is a transfinite composition of identities, or pushouts of one boundary inclusion in $\mathbb{B}$.

2. Given a commutative square of successors

$$\begin{tikzcd}
\tilde{X}^{\tilde{\alpha},\alpha+1} \arrow[r] & \tilde{X}^{\tilde{\alpha}+1,\alpha+1} \\
\tilde{X}^{\tilde{\alpha},\alpha} \arrow[r] \arrow[u] & \tilde{X}^{\tilde{\alpha},\alpha+1} \arrow[u]
\end{tikzcd} \tag{8.20}$$

   there are the following cases:

   (a) If $f(\tilde{\alpha}) \neq \alpha$, then the square is a pushout.

   (b) If $\tilde{\alpha} \leq \alpha$ then the right vertical is given by the identity.

   (c) If $\alpha \leq \tilde{\alpha}$, then the upper horizontal is given by the identity.

3. The left column and the bottom horizontal row are, respectively, given by the filtration-presentations $\tilde{\mathfrak{p}}$ and $\mathfrak{p}$.

In such a diagram the top row defines a filtration-presentation of $\tilde{X} \to X$ with characteristic maps $\mathfrak{C}_{\mathfrak{c}} \setminus i\mathfrak{C}_{\tilde{\mathfrak{c}}}$. Note, furthermore, that every row defines a filtration-presentation of a relative cell complex, making all arrows pointing into $X^{\beta} = X$ monomorphisms. Hence, the diagram is entirely a diagram of subobjects of X. Note that, if $X' \to X$ is a monomorphism, then there can be at most one morphism $A' \to X'$ over X. Hence, we may safely identify subobjects of X which are isomorphic as identical, and need not verify any commutativity conditions or specify morphisms involving such objects at the target. Let us construct such a diagram via transfinite induction over $\beta_0 \leq \tilde{\beta}$. In particular, this allows us to inductively assume that below β_0, the whole diagram is given by subobjects. In case of a limit ordinal β_0, we may use uniqueness of morphisms between subobjects over X to glue the subdiagrams for $\tilde{\alpha} < \beta_0$ together and set $\tilde{X}^{\beta_0,\alpha} = \varinjlim_{\tilde{\alpha}<\beta_0} X^{\tilde{\alpha},\alpha}$. We need to verify that both rows and columns still fulfill the properties above. For the columns this is immediate. For the new row, the commutativity of transfinite compositions shows it is a transfinite composition. It remains to show that every successor morphism is either an identity or a pushout of a cell, as required above. Consider

$$\tilde{X}^{\beta_0,\alpha} \to \tilde{X}^{\beta_0,\alpha+1}.$$

There are two cases to consider. If there exists $\alpha' < \beta_0$ such that $\alpha \leq \alpha'$, then the colimit over $X^{\tilde{\alpha}',\alpha} \to X^{\tilde{\alpha}',\alpha+1}$ is ultimately given by the colimit over identities, and hence given by an identity. Otherwise, it follows that at each $\tilde{\alpha} < \beta_0$, we have that $f(\tilde{\alpha}) \neq \alpha$. Hence, we may write $\tilde{X}^{\beta_0,\alpha} \to \tilde{X}^{\beta_0,\alpha+1}$ as fitting into a colimit diagram

$$\begin{tikzcd} \tilde{X}^{\beta_0,\alpha} \arrow[r] & \tilde{X}^{\beta_0,\alpha+1} \\ \dots \arrow[u] \arrow[r] & \dots \arrow[u] \\ X^{\alpha} \arrow[u] \arrow[r] & X^{\alpha+1} \arrow[u] \end{tikzcd} \tag{8.21}$$

which, at each successor step, is given by a pushout. In particular, it follows by the transfinite composability of pushout squares that $\tilde{X}^{\beta_0,\alpha} \to \tilde{X}^{\beta_0,\alpha+1}$ is the pushout of $\partial D^{\alpha} \to D^{\alpha}$ along $\partial D^{\alpha} \to X^{\alpha} \to \tilde{X}^{\beta_0,\alpha}$, as required.

Next, assume that $\beta_0 = \tilde{\alpha} + 1$ is a successor. We may then proceed to construct $X^{\beta_0,\alpha}$ (together with the respective structure morphisms between the latter), for $\alpha \leq \beta$ via transfinite induction as follows. For α a limit ordinal, set $X^{\beta_0,\alpha}$ to the colimit of its predecessors. In the case of a successor, $\alpha + 1$, there are the following cases to consider:

1. If $\alpha \neq f(\tilde{\alpha})$, then define $X^{\beta_0,\alpha+1}$ via the pushout square

$$\begin{tikzcd} X^{\tilde{\alpha}+1,\alpha} \arrow[r] & X^{\tilde{\alpha}+1,\alpha+1} \\ X^{\tilde{\alpha},\alpha} \arrow[u] \arrow[r] & X^{\tilde{\alpha},\alpha+1} \arrow[u] \end{tikzcd} . \tag{8.22}$$

2. If $f(\tilde{\alpha}) = \alpha$, then we claim $X^{\tilde{\alpha}+1,\alpha+1} = X^{\tilde{\alpha}+1,\alpha}$ and set $X^{\tilde{\alpha}+1,\alpha+1} = X^{\tilde{\alpha}+1,\alpha}$;

Then, supposing that this definition is well defined, one may easily show via inductive assumption that the requirements about the properties of the squares are met. It remains to show that with the family of subobjects $X^{\tilde{\beta}_0,\alpha}$ constructed in this fashion, the target of their transfinite composition is $X^{\beta} = X$. In the limit ordinal case, this is immediate from the commutativity of colimits. In the case of a successor $\beta_0 = \tilde{\alpha} + 1$, note that by construction, for values of α greater than $f(\tilde{\alpha})$ the morphism $X^{\tilde{\alpha},\alpha} \to X^{\tilde{\alpha}+1,\alpha}$ is given by the identity. In particular, so is the induced morphism

$$X^{\tilde{\alpha},\beta} = X^{\beta} \to X^{\tilde{\alpha}+1,\beta}.$$

Let us finally show that if $f(\tilde{\alpha}) = \alpha$ then $X^{\tilde{\alpha},\alpha+1} = X^{\tilde{\alpha}+1,\alpha}$. Note that in this case, by inductive assumption, both $X^{\tilde{\alpha},\alpha} \to X^{\tilde{\alpha},\alpha+1}$ as well as $X^{\tilde{\alpha},\alpha} \to X^{\tilde{\alpha}+1,\alpha}$, are given by a cobase change of $(\partial D \hookrightarrow D) := (\partial D^{\alpha} \to D^{\alpha}) = (\partial D^{\tilde{\alpha}} \to D^{\tilde{\alpha}})$ along a map $\partial D \to X^{\tilde{\alpha},\alpha}$ (over X). We are looking to show that the two maps $\sigma\colon D \to X^{\tilde{\alpha}+1,\alpha} \to X$ and $\sigma^{\alpha}\colon D \to X$ are the same as objects in the slice category over X. Then the identity $X^{\tilde{\alpha},\alpha+1} = X^{\tilde{\alpha}+1,\alpha}$ follows as we have identified isomorphic subobjects, and from the fact that in any pushout square over X of the form

$$\begin{tikzcd} \partial D \arrow[r, hook] \arrow[d] & D \arrow[d] \\ X' \arrow[r, hook] & \hat{X} \end{tikzcd} \tag{8.23}$$

with $X', \hat{X}$ subobjects of X, $\hat{X}$ is already uniquely determined by $\partial D \to D$ and $D \to X$, up to natural isomorphism over X. Now, to see that $\sigma = \sigma^{\alpha}$, consider the following diagram in which all (small) quadrilaterals and triangles are commutative by construction

$$
\begin{tikzcd}
 & \tilde{X} \arrow[rrrd, "i"] & & & \\
 & \tilde{X}^{\tilde{\alpha}+1} \arrow[u] \arrow[r] & X^{\tilde{\alpha}+1,\alpha} \arrow[rr] & & X \\
 & \tilde{X}^{\tilde{\alpha}} \arrow[u] \arrow[r] & X^{\tilde{\alpha},\alpha} \arrow[u] \arrow[r] & X^{\tilde{\alpha},\alpha+1} \arrow[ru] & \\
 & & X^{\alpha} \arrow[u] \arrow[r] & X^{\alpha+1} \arrow[u] \arrow[ruu] & \\
D \arrow[ruuuu, bend left, "\sigma^{\tilde{\alpha}}"] \arrow[ruuu, dashed] \arrow[rrru, dashed] \arrow[rrrruuu, bend right, "\sigma_{\alpha}"'] & & & &
\end{tikzcd}
\tag{8.24}
$$

The morphism σ is given by the composition $D \to \tilde{X}^{\tilde{\alpha}+1} \to X^{\tilde{\alpha}+1,\alpha} \to X$. By commutativity, this is the same as the path $D \xrightarrow{\sigma^{\alpha}} \tilde{X} \to X$. But the latter is, by definition of f in terms of i, the same as $\sigma^{\alpha}: D \to X$. This finishes the proof. $\square$

Definition 8.1.3.2. A morphism $i: (A \overset{\tilde{\mathfrak{c}}}{\hookrightarrow} \tilde{X}) \to (A \overset{\mathfrak{c}}{\hookrightarrow} X)$ in $\mathbf{RCell(C)}_A$ is called *an inclusion of a relative subcomplex*, if it fulfills one of the equivalent conditions in Proposition 8.1.3.1.

Lemma 8.1.3.3. *A morphism $i: \tilde{\mathfrak{c}} \to \mathfrak{c}$ in $\mathbf{RCell(C)}_A$ is a monomorphism in $\mathbf{RCell(C)}$ if and only if it is an inclusion of a subcomplex.*

Proof. Observe that, clearly every morphism in $\mathbf{RCell(C)}_A$, whose underlying morphism is a monomorphism is also a monomorphism in $\mathbf{RCell(C)}_A$. Hence, by the equivalence of the first three conditions of Proposition 8.1.3.1, the *if-direction* follows. It remains to show the *only-if* part. We use the criterion of Proposition 8.1.3.1 which characterizes a subcomplex via injectivity on the level of cells. We proceed via transfinite induction over the minimal presentation ordinal β of $\tilde{\mathfrak{c}}$. In the case $\beta = 0$, $\tilde{\mathfrak{c}}$ is given by the identity and the result is obvious. Now, if β is a limit ordinal, then $\mathfrak{C}(i)$ fits into a diagram

$$
\begin{tikzcd}
S^0 \arrow[r, hook] \arrow[rrrrd] & \dots \arrow[r, hook] & S^{\alpha} \arrow[r, hook] \arrow[rrd] & \dots \arrow[r, hook] & \mathfrak{C}_{\tilde{\mathfrak{c}}} \arrow[d, dashed] \\
 & & & & \mathfrak{C}_{\mathfrak{c}}
\end{tikzcd}
\tag{8.25}
$$

of sets, with the upper vertical a transfinite composition, and all solid diagonals an injection. In follows that the dashed induced morphism is also an injection. Now, suppose that $\beta = \alpha + 1$ is a successor ordinal, and fix a filtration-presentation of $A \to X^{\alpha+1}$, of $\tilde{\mathfrak{c}}$, denoted as in Definition 8.1.1.1. We are now in the following situation

$$
\begin{tikzcd}
 & A \arrow[d, hook, "c_{\alpha}"'] \arrow[rdd, hook] & \\
\bigsqcup_I \partial D_j \arrow[r] \arrow[d, hook] & X^{\alpha} \arrow[d, hook] \arrow[rd, hook] & \\
\bigsqcup_I D_j \arrow[r] & X^{\alpha+1} \arrow[r, "i"] & X
\end{tikzcd}
\tag{8.26}
$$

with the upper vertical admitting a filtration-presentation of length α, and the square pushout. Furthermore, we have assumed that $(1_A, X^{\alpha+1} \to X)$ induces a monomorphism. Since $(1_A, X^{\alpha} \to X^{\alpha+1})$ is the inclusion of a subcomplex, it is also a monomorphism. It follows that $(1_A, X^{\alpha} \to X)$ is a monomorphism. By inductive assumption, it is thus injective on cells and its underlying map in $\mathbf{C}$ is a monomorphism. Now, let $\sigma_0, \sigma_1 \colon D \to X^{\alpha+1}$ be two characteristic maps, which are identified under composition with $X^{\alpha+1} \to X$. There are two cases to consider:

1. The two cells σ_0 and σ_1 which are identified in $\mathfrak{C}_{\mathfrak{c}}$ have characteristic maps factoring through X^{α}. Then they are already identical, by inductive assumption.

2. The cell σ_1 is (without loss of generality) added in the step α to $\alpha + 1$, and the cell σ_0 is added in some step $\alpha' \leq \alpha$. In this case, consider the diagram

$$\begin{tikzcd}
\partial D \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, dashed, "\partial\sigma_1"] & D \arrow[r, dashed, "\sigma_0"] \arrow[d, equal] \arrow[dr, "\sigma_0"'] & X^{\alpha} \arrow[r, hook] \arrow[d, hook] & X \\
& D \arrow[r, "\sigma_1"'] & X^{\alpha+1} \arrow[ur] &
\end{tikzcd} \tag{8.27}$$

in which all paths ending in X commute. Chasing the diagram, and using that by inductive assumption $X^{\alpha} \to X$ is a monomorphism, it follows that $\partial\sigma_1 = \partial\sigma_0$. In particular, we may find an alternative filtration-presentation of $\tilde{\mathfrak{c}}$, indexed again over $\alpha + 1$, in which σ_1 is attached at an earlier step. We may thus assume, without loss of generality, that this case does not occur.

3. Both σ_0 and σ_1 are added in the final step α to $\alpha+1$, i.e., respectively present a component j_0 and j_1 of $\bigsqcup_I D_j \to X^{\alpha+1}$. Observe that, again as in the previous diagram chase, it follows that the (induced morphisms) $\partial\sigma_i \dashrightarrow X^{\alpha}$ agree. Hence, we obtain a well defined automorphism of $X^{\alpha+1}$, induced by the automorphism of $\bigsqcup_I D_j$ exchanging j_0 and j_1. If $\sigma_0 \neq \sigma_1$, this defines a non-trivial (by Observation 8.1.2.6) automorphism of $\tilde{\mathfrak{c}}$. However, under composition with i, this automorphism agrees with the identity (by the universal property of the pushout). As i is assumed to be a monomorphsm, it thus follows that $\sigma_0 = \sigma_1$, as was to be shown.

□

Remark 8.1.3.4. It follows from Proposition 8.1.3.1 that a morphism $\tilde{\mathfrak{c}} \to \mathfrak{c}$ in $\mathbf{RCell(C)}_A$ defines a subcomplex in our sense, if and only if it is a subcomplex in the sense of [Hir03, Def. 10.6.7] with respect to any filtration-presentation of the cell structure $\mathfrak{C}_{\mathfrak{c}}$, or equivalently with respect to one fixed filtration-presentation of the cell structure $\mathfrak{C}_{\mathfrak{c}}$. Indeed, an argument via transfinite induction shows that the conditions in Proposition 8.1.3.1 imply the requirements in [Hir03], with respect to any presentation of $\mathfrak{C}_{\mathfrak{c}}$. In turn, if the condition in Hirschhorn holds for one fixed filtration-presentation, then i is trivially injective on characteristic maps.

Let us make some observations concerning the interaction of cell complexes and the constructions in Construction 8.1.2.7.

Observation 8.1.3.5. Given two relative cell complexes $A \xhookrightarrow{\tilde{\mathfrak{c}}} \tilde{X} \xhookrightarrow{\mathfrak{i}} X$, equip the composition $i \circ \tilde{\mathfrak{c}}$ with the induced cell structure of Construction 8.1.2.7. Then, by Proposition 8.1.3.1, i defines an inclusion of subcomplexes

$$\tilde{\mathfrak{c}} \hookrightarrow \mathfrak{i} \circ \tilde{\mathfrak{c}}.$$

Conversely, by Proposition 8.1.3.1, given any inclusion of cell complexes $i \colon \tilde{\mathfrak{c}} \hookrightarrow \mathfrak{c}$, $(\tilde{X} \hookrightarrow X)$ equipped with $\mathfrak{C}_{\mathfrak{c}} \setminus \mathfrak{C}_{\tilde{\mathfrak{c}}}$ defines a relative cell complex. These two constructions are inverse to each other, inducing a bijection

$$\begin{array}{c} \{(\tilde{\mathfrak{c}}, i) \mid \tilde{\mathfrak{c}} \in \mathbf{RCell(C)}, \tilde{\mathfrak{c}} \xhookrightarrow{i} \mathfrak{c} \text{ is an inclusion of subcomplexes } \} \\ \updownarrow{\scriptstyle 1:1} \\ \{(\tilde{\mathfrak{c}}, \mathfrak{i}) \mid (A \xrightarrow{\tilde{\mathfrak{c}}} \tilde{X}), (\tilde{X} \xrightarrow{\mathfrak{i}} X) \in \mathbf{RCell(C)}, \mathfrak{i} \circ \tilde{\mathfrak{c}} = \mathfrak{c}\} \end{array} \tag{8.28}$$

between decompositions of $\mathfrak{c}$ into relative cell complexes and subcomplexes of $\mathfrak{c}$. In this sense, many statements about vertical composition can alternatively be expressed in terms of inclusions of subcomplexes.

Observation 8.1.3.6. Suppose we are given a transfinite composition diagram

$$c: A = X^0 \to X^1 \to \cdots \to X^\lambda = X$$

together with, for each $\alpha \in \lambda$, the structure of a structured relative cell complex $\mathfrak{i}^\alpha$ on $X^\alpha \to X^{\alpha+1}$. Then the structured relative cell complex $\mathfrak{c}: A \to X$, constructed in Construction 8.1.2.7 can be seen as a transfinite composition of inclusions of subcomplexes. For $\alpha < \lambda$, denote by $\mathfrak{c}^\alpha: A \to X^\alpha$ the relative cell complex obtained by vertical transfinite composition up to α. In fact, for any $\alpha \leq \alpha' < \lambda$, the canonical map $X^\alpha \to X^{\alpha'}$ defines an inclusion of subcomplexes $\mathfrak{c}^\alpha \hookrightarrow \mathfrak{c}^{\alpha'}$. Then the induced diagram

$$\mathfrak{c}^0 \hookrightarrow \mathfrak{c}^1 \hookrightarrow \cdots \hookrightarrow \mathfrak{c}^\lambda = \mathfrak{c}$$

in $\mathbf{RCell(C)}$ is a transfinite composition diagram.

It can be convenient to have language available which mimics the classical scenario of CW-complexes as closely as possible. To this end, we use the following language.

Notation 8.1.3.7. Given a structured relative cell complex $\mathfrak{c}$, and a subcomplex $i: \tilde{\mathfrak{c}} \hookrightarrow \mathfrak{c}$, we will generally think of the cells of $\mathfrak{C}_{\tilde{\mathfrak{c}}}$ as being a subset of $\mathfrak{C}_{\mathfrak{c}}$, via $\mathfrak{C}(i)$. We say that a cell $\sigma \in \mathfrak{C}_{\mathfrak{c}}$ is contained in $\tilde{\mathfrak{c}}$, if $\sigma \in \mathfrak{C}_{\tilde{\mathfrak{c}}}$, under this identification. We will say that the boundary of a cell σ, $\partial\sigma$, is contained in $\tilde{\mathfrak{c}}$, if $\partial\sigma: \partial D \to X$ factors through i.

Another way of thinking of a subcomplex of a structured relative cell complex $\mathfrak{c}$ is in terms of a universal object, associated to a subset of the cells of $\mathfrak{c}$.

Proposition 8.1.3.8. *Given a structured relative cell complex $\mathfrak{c}: A \hookrightarrow X$, together with a subcomplex $\tilde{\mathfrak{c}} \xhookrightarrow{i} \mathfrak{c}$. Suppose we are given another morphism of relative cell complexes $f = (f_A, f_X): (A_0 \xrightarrow{\mathfrak{c}_0} X_0) \to \mathfrak{c}$. Then, f factors through i, if and only if $\mathfrak{C}(f)$ factors through $\mathfrak{C}(i)$,*

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{c}_o} \arrow[r, "\mathfrak{C}(f)"] \arrow[d, dashed] & \mathfrak{c} \\ \mathfrak{C}_{\tilde{\mathfrak{c}}} \arrow[ur, hook, "\mathfrak{C}(i)"'] & \end{tikzcd} \tag{8.29}$$

i.e., if every cell in $\mathfrak{C}_{\mathfrak{c}_o}$ maps to a cell contained in $\mathfrak{C}_{\tilde{\mathfrak{c}}}$. In other words, if we denote $\mathbf{D}_{\mathfrak{C}_{\tilde{\mathfrak{c}}}}$ the full subcategory of overcategory $\mathbf{RCell(C)}_{/\mathfrak{c}}$, consisting only of morphisms $f: \mathfrak{c}_0 \to \mathfrak{c}$, fulfilling

$$\mathrm{im}\mathfrak{C}(f) \subset \mathfrak{C}_{\tilde{\mathfrak{c}}}$$

then i is a terminal object in this category.
Conversely, if we are given a subset $\tilde{\mathfrak{C}} \subset \mathfrak{C}_{\mathfrak{c}}$, then $\tilde{\mathfrak{C}} = \mathfrak{C}_{\tilde{\mathfrak{c}}}$, for some subcomplex $i: \tilde{\mathfrak{c}} \to \mathfrak{c}$ of $\mathfrak{c}$, if and only if the analogously constructed category $\mathbf{D}_{\tilde{\mathfrak{C}}}$ has a terminal object and has at least one object $f: \tilde{\mathfrak{c}} \to \mathfrak{c}$, such that $(im)\mathfrak{C}(f) = \tilde{\mathfrak{C}}$.

Proof. The only if part of the first statement is immediate from the functoriality of $\mathfrak{C}$. For the converse, we proceed with transfinite induction over the minimal ordinal β of filtration-presentations of $\mathfrak{c}_0$. In the case of $\beta = 0$, there is nothing to be shown. Now, if $\beta = \alpha + 1$ is a successor ordinal, then we may consider the subcomplex $\mathfrak{c}^\alpha \xhookrightarrow{j} \mathfrak{c}_0$, given by all cells but the one glued in at the step α. By inductive assumption, we have a factorization

$$\begin{tikzcd}
\mathfrak{c}_0^\alpha \arrow[r, "f \circ j"] \arrow[dr, dashed] & \mathfrak{c} \\
& \tilde{\mathfrak{c}} \arrow[u, "i"']
\end{tikzcd} \tag{8.30}$$

Now, j, interpreted as a structured relative cell complex, is by definition given by gluing in a single cell, along a cobase change square, as pictured below.

$$\begin{tikzcd}
\partial D \arrow[r] \arrow[d, hook] & X^\alpha \arrow[d, hook, "j"] \arrow[ddr, dashed, bend left] \arrow[dddrr, bend left] & & \\
D \arrow[r, "\sigma"] \arrow[drr, dashed, bend right] \arrow[ddrrr, bend right, "\mathfrak{C}(f)(\sigma)"'] & X_0 \arrow[dr, dashed] & & \\
& & \tilde{X} \arrow[dr, hook, "i"] & \\
& & & X
\end{tikzcd} \tag{8.31}$$

The outer diagram commutes by assumption. The two bend factorizations exist, respectively, by inductive assumption and the assumption that $\mathfrak{C}(f)$ factors through $\mathfrak{C}(i)$. Now, using the fact that i is a monomorphism, we obtain commutativity of all paths from ∂D to $\tilde{X}$. In particular, using that cobase change squares define cocartesian arrows (over the evaluation at A functor $\mathbf{RCell(C)} \to \mathbf{C}$), we obtain a dashed arrow $X_0 \to \tilde{X}$, as required. The case of a limit ordinal is handled similarly using Observation 8.1.3.6.
Now, conversely, suppose that a terminal object $i\colon \tilde{\mathfrak{c}}_1 \to \mathfrak{c}$ in $\mathbf{D}_{\mathfrak{C}_{\tilde{\mathfrak{c}}}}$ exists. Then, as it is a terminal object, it must be given by a monomorphism. Furthermore, using trivial relative cell complexes $A' \xrightarrow{1} A'$, we may easily see that $i_A\colon A_0 \to A$ must be given by the identity (up to changing A_0 by an isomorphism, if necessary). Thus, by Proposition 8.1.3.1, i is a subcomplex of $\mathfrak{c}$. By the assumption of it being terminal, and there existing an object mapping surjectively onto $\mathfrak{C}_{\tilde{\mathfrak{c}}}$ we have $\mathfrak{C}_{\tilde{\mathfrak{c}}} = \tilde{\mathfrak{C}}$. ☐

Notation 8.1.3.9. As the category of subobjects of an object is always equivalent to a poset (i.e., admits at most one morphism between any two objects), it is safe to use notation like $\tilde{\mathfrak{c}} \subset \tilde{\mathfrak{c}}_1$, when we want to express that there exists a morphism over between two subcomplexes over a structured relative cell complex $\mathfrak{c}$.

Observation 8.1.3.10. One immediate consequence of Proposition 8.1.3.8 is that two subcomplexes $i_0\colon \mathfrak{c}_0 \hookrightarrow \mathfrak{c}$ and $i_1\colon \mathfrak{c}_1 \hookrightarrow \mathfrak{c}$ of a structured relative cell complex $\mathfrak{c}$ are isomorphic as subobjects of $\mathfrak{c}$, if and only if they determine the same subset of cells in $\mathfrak{c}$. Up to canonical isomorphism, a sub-complex $i\colon \tilde{\mathfrak{c}} \hookrightarrow \mathfrak{c}$ is thus uniquely determined by its set of cells $\mathfrak{C}_{\tilde{\mathfrak{c}}} \subset \mathfrak{C}_{\mathfrak{c}}$, and the category of subobjects of $\mathfrak{c}$ in $\mathbf{RCell(C)}_A$ is equivalent to a sub-poset of the power set of $\mathfrak{C}_{\mathfrak{c}}$, via the functor associating to a subcomplex its set of cells.

8.1.4 Reduction to sets of cells

One central advantage of cellular frameworks is that categorical statements about cell complexes can often be verified purely at the level of sets of cells. As the category of sets is generally extremely well-behaved, with universal constructions being easy to handle, this can often significantly reduce the complexity of proofs. First, the forgetful functor $\mathfrak{C}\colon \mathbf{Cell(C)}_A \to \mathbf{Set}$ is conservative, which is a corollary of Proposition 8.1.3.1:

Corollary 8.1.4.1. *For $f\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ in $\mathbf{RCell}(\mathbf{C})_A$ the following conditions are equivalent*

1. *f is an isomorphism;*
2. *The underlying morphism in $\mathbf{C}$, $f\colon X_0 \to X_1$ is an isomorphism;*
3. *$\mathfrak{C}(f)$ is a bijection.*

Proof. That the first condition implies the two others is an immediate consequence of the functoriality of $\mathfrak{C}$ and the forgetful functor to $\mathbf{C}$. By Proposition 8.1.3.1, the second condition implies the third. Now, conversely, if f induces a bijection on the set of cells, then by Proposition 8.1.3.1, there exists, in particular, a filtration-presentation of $f\colon X_0 \to X_1$ with an empty set of cells. It follows that f is an isomorphism in $\mathbf{C}$. Then, an inverse to f in $\mathbf{C}$ is also structure preserving (as it induces $\mathfrak{C}(f)^{-1}$ on the level of cells) and defines an inverse in $\mathbf{RCell}(\mathbf{C})_A$. ☐

Corollary 8.1.4.2. *For $f\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ in $\mathbf{RCell}(\mathbf{C})$ the following conditions are equivalent.*

1. *f is a cobase change (i.e., a cocartesian arrow with respect to $\mathbf{RCell}(\mathbf{C}) \xrightarrow{\mathrm{ev}_0} \mathbf{C}$);*
2. *The underlying diagram of f in $\mathbf{C}$ is a pushout square;*
3. *The map on cells $\mathfrak{C}(f)$ is a bijection.*

Proof. By invariance of cocartesian arrows under isomorphism over A_0, the f is cocartesian, if and only if the dashed arrow in the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "f_A"] \arrow[d, hook, "\mathfrak{c}_0"'] & A_1 \arrow[d, hook, "(f_A)_! \mathfrak{c}_0"'] \arrow[ddr, bend left, "\mathfrak{c}_1"] \arrow[dl, phantom, "\ulcorner"] & \\
X_0 \arrow[r] \arrow[drr, bend right, "f_X"'] & X_1' \arrow[dr, dashed] & \\
 & & X_1
\end{tikzcd} \tag{8.32}$$

defines an isomorphism in $\mathbf{RCell}(\mathbf{C})_{A_1}$. By Corollary 8.1.4.1, this is in turn equivalent to the remaining two conditions. ☐

As Corollary 8.1.4.2 allows us to verify cobase changes entirely on the level of the underlying commutative squares, we also obtain a vertical pasting law:

Corollary 8.1.4.3. *Suppose we are given a commutative diagram with verticals given by relative cell complexes*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "\tilde{\mathfrak{c}}"] \arrow[dd, bend right=60, "\mathfrak{i} \circ \tilde{\mathfrak{c}}"'] & A' \arrow[d, hook, "\tilde{\mathfrak{c}}'"'] \arrow[dd, bend left=60, "\mathfrak{i}' \circ \tilde{\mathfrak{c}}'"] \\
\tilde{X} \arrow[r] \arrow[d, hook, "\mathfrak{i}"] & \tilde{X}' \arrow[d, "\mathfrak{i}'"'] \\
X \arrow[r] & X',
\end{tikzcd} \tag{8.33}$$

with the upper square cobase change. Then the lower square is a cobase change if and only if the outer square, given by the vertical composition of the two squares is a cobase change.

As another immediate consequence of Corollary 8.1.4.2 and Proposition 8.1.3.1, we obtain

Corollary 8.1.4.4. *For $f\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ in $\mathbf{RCell}(\mathbf{C})$ the following conditions are equivalent:*

1. *The map on cells $\mathfrak{C}(f)$ is an injection;*

2. *The canonical arrow* $(f_A)_{¡}\mathfrak{c}_0 \to \mathfrak{c}_1$ *is an inclusion of cell complexes.*

Furthermore, we may combine Corollary 8.1.4.2 and Corollary 8.1.4.1, together with the fact that a cocartesian lift of an isomorphism along a cocartesian fibration is an isomorphism, to obtain:

Corollary 8.1.4.5. *For* $f\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ *in* $\mathbf{RCell(C)}$ *the following conditions are equivalent*

1. f *is an isomorphism;*

2. *The underlying morphisms in* $\mathbf{C}$, $f_A\colon A_0 \to A_1$ *and* $f\colon X_0 \to X_1$ *are isomorphisms;*

3. $f_A\colon A_0 \to A_1$ *is an isomorphism and* $\mathfrak{C}(f)$ *is a bijection.*

To finish this section, let us say a few words about colimits in the categories of structured relative cell complexes. Let us begin with the following lemma:

Notation 8.1.4.6. Given a category I, we will use the notation $I^{\triangleright}$ for the right cone on I, obtained by adjoining a formal terminal object ∞ to I.

Lemma 8.1.4.7. *Let* $A \in \mathbf{C}$. *Suppose that we are given a diagram* $\mathfrak{c}^{\bullet}\colon I \to \mathbf{RCell(C)}$ *(*$\mathbf{RCell(C)}_A$*), together with a cocone on it, specified by an absolute cell complex* $\mathfrak{c}^{\infty}$*, and morphisms* $\mathfrak{c}^i \to \mathfrak{c}^{\infty}$*, which agglomerate into an extension of* $\mathfrak{c}^{\bullet}$ *to a functor,* $\overline{\mathfrak{c}}^{\bullet}\colon I^{\triangleright} \to \mathbf{RCell(C)}$ *(*$\mathbf{RCell(C)}_A$*). Suppose, furthermore, that this cocone has the following two properties:*

1. *The induced map on cells* $\bigsqcup_{i\in I} \mathfrak{C}_{\mathfrak{c}^i} \to \mathfrak{C}_{\mathfrak{c}^{\infty}}$ *is surjective.*

2. *The underlying cocone* $\overline{c}^{\bullet}$ *of* $\overline{\mathfrak{c}}^{\bullet}$ *in* $\mathbf{C}$ *(*$\mathbf{C}_{A/}$*) is a colimit diagram.*

Then $\overline{\mathfrak{c}}^{\bullet}$ *defines a colimit diagram in* $\mathbf{RCell(C)}$ *(*$\mathbf{RCell(C)}_A$*), that is, the colimit of* $\mathfrak{c}^{\bullet}$ *exists and is given by* $\mathfrak{c}^{\infty} = \varinjlim_I \mathfrak{c}^i$.

Proof. The proof in both cases (the one for fixed A and the flexible one) is completely analogous. We only cover the former. This is essentially immediate from the definition of a morphism of relative structured cell complexes. By the second condition, any further cone $\hat{\mathfrak{c}}^{\bullet}$ admits a unique morphism of cones on $\mathfrak{c}^{\bullet}$, $\overline{\mathfrak{c}}^{\bullet} \to \hat{\mathfrak{c}}^{\bullet}$. As a morphism of structured cell complexes is entirely determined by its underlying morphism in the arrow category $\mathbf{C}^{[1]}$, this already shows the uniqueness part of the universal property of the colimit. To prove existence, it suffices to show that the unique morphism $f\colon (A^{\infty} \xhookrightarrow{c^{\infty}} X^{\infty}) \to (\hat{A}^{\infty} \xhookrightarrow{\hat{d}^{\infty}} \hat{X}^{\infty})$ defines a morphism of relative structured cell complexes. In other words, we need to show that the underlying map $f_X\colon \hat{X}^{\infty} \to X^{\infty}$ fulfills,

$$f_X \circ \sigma \in \mathfrak{C}_{\hat{\mathfrak{c}}^{\infty}},$$

for each $\sigma \in \mathfrak{C}_{\mathfrak{c}^{\infty}}$. By assumption, each such σ is of the form $\iota_i \circ \sigma'$, for some $i \in I$ and $\sigma \in \mathfrak{C}_{\mathfrak{c}^i}$,with $\iota^i\colon X^i \to X^{\infty}$ induced by the structure morphism $\mathfrak{c}^i \to \mathfrak{c}^{\infty}$. As f was assumed to define a morphism of cocones, we obtain

$$f_X \circ \iota^i \circ \sigma' = \hat{\iota}^i \circ \sigma',$$

with $\hat{\iota}^i\colon X^i \to \hat{X}^{\infty}$ induced by the structure morphism $\mathfrak{c}^i \to \hat{\mathfrak{c}}^{\infty}$. In particular, as the latter is a morphism in $\mathbf{RCell(C)}$, it follows that $f_X \circ \iota^i \circ \sigma' = \hat{\iota}^i \circ \sigma' \in \mathfrak{C}_{\hat{\mathfrak{c}}^{\infty}}$, as was to be shown. □

Corollary 8.1.4.8. *The category of structure relative cell complexes under* A, $\mathbf{RCell(C)}_A$*, has pushouts for solid spans of the form*

$$\begin{tikzcd} \tilde{\mathfrak{c}} \arrow[r, "f"] \arrow[d, hook, "i"] & \tilde{\mathfrak{c}}' \arrow[d, hook, dotted, "i'"] \\ \mathfrak{c} \arrow[r, dotted, "f'"] & \mathfrak{c}' \end{tikzcd} \tag{8.34}$$

with $\tilde{\mathfrak{c}} \overset{i}{\hookrightarrow} \mathfrak{c}$ an inclusion of a subcomplex. Explicitly, a pushout is given by the diagram of relative structured cell complexes.

$$\begin{tikzcd} \tilde{\mathfrak{c}} \arrow[r, "f"] \arrow[d, hook, "i"] & \tilde{\mathfrak{c}}' \arrow[d, hook] \\ \mathfrak{c} \arrow[r] & f_{\mathfrak{i}}\mathfrak{i} \circ \tilde{\mathfrak{c}}' \end{tikzcd} \tag{8.35}$$

where $\mathfrak{i}$ is the relative cell complex associated to i under Observation 8.1.3.5.

Proof. It is immediate by the construction of cobase change and vertical composition, that the canonical diagram

$$\begin{tikzcd} \tilde{\mathfrak{c}} \arrow[r, "f"] \arrow[d, hook, "i"] & \tilde{\mathfrak{c}}' \arrow[d, hook] \\ \mathfrak{c} \arrow[r] & f_{\mathfrak{i}}\mathfrak{i} \circ \tilde{\mathfrak{c}}' \end{tikzcd} \tag{8.36}$$

fulfills the requirements of Lemma 8.1.4.7. □

Observe that diagrams as in Corollary 8.1.4.8 are also pullback.

Lemma 8.1.4.9. *Any pushout diagram as in Corollary 8.1.4.8 is pullback in* $\mathbf{RCell(C)}_A$*, and its underlying diagram in* $\mathbf{C}$ *is also pullback.*

Proof. To see the latter claim, observe that the underlying diagram in $\mathbf{C}$ is a pushout diagram with vertical given by a relative $\mathbb{B}$-cell complex. Hence it is a pullback diagram by the axioms of a cellularized category. Now, suppose we are given a solid commutative diagram

$$\begin{tikzcd} \mathfrak{d} \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \tilde{\mathfrak{c}} \arrow[r] \arrow[d] & \tilde{\mathfrak{c}}' \arrow[d] \\ & \mathfrak{c} \arrow[r] & \mathfrak{c}' . \end{tikzcd} \tag{8.37}$$

As the underlying diagram in $\mathbf{C}$ of the lower square is pullback, any dotted arrow making the diagram commute is necessarily unique. To see that such an arrow exists, it suffices to verify that the arrow induced by the pullback property in $\mathbf{C}$ defines a morphism in $\mathbf{RCell(C)}_A$. By Proposition 8.1.3.8, this is the case if and only if the morphism $\mathfrak{d} \to \mathfrak{c}$ factors through $\tilde{\mathfrak{c}}$ on the cell level.

Hence, it suffices to verify that the associated diagram of sets of cells

$$\begin{tikzcd} \mathfrak{C}_{\tilde{\mathfrak{c}}} \arrow[r] \arrow[d, hook] & \mathfrak{C}_{\tilde{\mathfrak{c}}'} \arrow[d, hook] \\ \mathfrak{C}_{\mathfrak{c}} \arrow[r] & \mathfrak{C}_{\mathfrak{c}'} \end{tikzcd} \tag{8.38}$$

is also pullback. This is immediate by the explicit description of the cell structure on the pushout in Corollary 8.1.4.8. □

Corollary 8.1.4.10. *Let $A \in \mathbf{C}$, and suppose we are given a any commutative square of inclusions of subcomplexes*

$$\begin{tikzcd} \mathfrak{c}_0 \arrow[r, hook] \arrow[d, hook] & \mathfrak{c}_2 \arrow[d, hook] \\ \mathfrak{c}_1 \arrow[r, hook] & \mathfrak{c} \end{tikzcd} \tag{8.39}$$

in $\mathbf{RCell(C)}_A$*, such that the associated square of sets of cells is pullback, i.e., such that* $\mathfrak{C}_{\mathfrak{c}_0} = \mathfrak{C}_{\mathfrak{c}_1} \cap \mathfrak{C}_{\mathfrak{c}_2}$*. Then the underlying square in* $\mathbf{C}$*, and the square itself are pullback.*

Proof. Using Corollary 8.1.4.8, we may factor the square in question as

$$\begin{tikzcd}
\mathfrak{c}_0 \arrow[r, hook] \arrow[d, hook] & \mathfrak{c}_1 \arrow[d, hook] \arrow[ddr, hook, bend left] & \\
\mathfrak{c}_2 \arrow[r, hook] \arrow[drr, hook, bend right] & \mathfrak{c}_1 \cup_{\mathfrak{c}_0} \cup \mathfrak{c}_2 \arrow[dr, dashed] & \\
 & & \mathfrak{c}
\end{tikzcd} \tag{8.40}$$

with the inner square pushout. By Lemma 8.1.4.9 and the fact that a square is pullback, if and only if a modification of the lowest corner by a monomorphism is pullback, it suffices to see that the dashed arrow is a monomorphism, i.e., by (Proposition 8.1.3.1), is injective on cells. The claim now follows from the elementary fact that in a commutative diagram of inclusions of sets

$$\begin{tikzcd}
U_0 \arrow[r, hook] \arrow[d, hook] & U_1 \arrow[d, hook] \arrow[ddr, hook, bend left] & \\
U_2 \arrow[r, hook] \arrow[drr, hook, bend right] & U_1 \cup_{U_0} U_2 \arrow[dr, dashed] & \\
 & & U
\end{tikzcd} \tag{8.41}$$

with inner square pushout and outer square pullback, the induced dashed arrow is an injection. ☐

As a further corollary of Lemma 8.1.4.7, we obtain the following:

Corollary 8.1.4.11. *Suppose we are given two inclusions of subcomplexes $\mathfrak{X}_\mathfrak{i} \hookrightarrow \mathfrak{X}$ in* $\mathbf{Cell}(\mathbf{C})$*, with $i = 1, 2$, such that $\mathfrak{C}_{\mathfrak{X}_1} \cup \mathfrak{C}_{\mathfrak{X}_2} = \mathfrak{C}_{\mathfrak{X}}$. Then the pullback diagram*

$$\begin{tikzcd}
\mathfrak{X}_1 \cap \mathfrak{X}_2 \arrow[r, hook] \arrow[d, hook] & \mathfrak{X}_2 \arrow[d, hook] \\
\mathfrak{X}_1 \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" very near end] & \mathfrak{X}
\end{tikzcd} \tag{8.42}$$

is also pushout.

Corollary 8.1.4.12. *The category* $\mathbf{RCell}(\mathbf{C})$ *has pushouts for solid spans of the form*

$$\begin{tikzcd}
\tilde{\mathfrak{c}} \arrow[r, "f"] \arrow[d, "i"] & \tilde{\mathfrak{c}}' \arrow[d, dotted] \\
\mathfrak{c} \arrow[r, dotted, "f'"] & \mathfrak{c}'
\end{tikzcd} \tag{8.43}$$

with $\tilde{\mathfrak{c}} \xrightarrow{i} \mathfrak{c}$ injective on the level of cells.

Proof. Consider first the underlying (solid) span at the sources of the relative cell complexes

$$\begin{tikzcd}
\tilde{A} \arrow[r, "f_A"] \arrow[d, "i_A"'] & \tilde{A}' \arrow[d, dotted, "i'_A"] \\
A \arrow[r, dotted, "f'_A"'] \arrow[ur, phantom, "\ulcorner" very near end] & A' .
\end{tikzcd} \tag{8.44}$$

As $\mathbf{C}$ was assumed to be cocomplete, it admits a pushout as indicated in the dotted part of the diagram above. Now, using the universal property of the cobase change, consider the induced solid diagram

$$\begin{tikzcd}
(f'_A \circ i_A)_{\mathfrak{i}} \tilde{\mathfrak{c}} \arrow[r, "f"] \arrow[d, "i"] & (i'_A)_{\mathfrak{i}} \tilde{\mathfrak{c}}' \arrow[d, dotted] \\
(f'_A)_{\mathfrak{i}} \mathfrak{c}' \arrow[r, dotted] & \mathfrak{c}'
\end{tikzcd} \tag{8.45}$$

in $\mathbf{RCell(C)}_A$. By assumption, the left vertical is injective on cells, and hence defines the inclusion of a subcomplex. Thus, by Corollary 8.1.4.8, we may complete the diagram to a colimit diagram as indicated. Using the universal property of the cobase change, we may extend this diagram to the following commutative diagram

$$\begin{tikzcd}
\tilde{\mathfrak{c}} \arrow[r] \arrow[d] & (f_A)_{\mathrm{i}}\tilde{\mathfrak{c}} \arrow[r] \arrow[d] & \tilde{\mathfrak{c}}' \arrow[d] \\
(i_A)_{\mathrm{i}}\tilde{\mathfrak{c}} \arrow[r] \arrow[d] & (f'_A \circ i_A)_{\mathrm{i}}\tilde{\mathfrak{c}} \arrow[r] \arrow[d] & (i'_A)_{\mathrm{i}}\tilde{\mathfrak{c}}' \arrow[d] \\
\mathfrak{c} \arrow[r] & (f'_A)_{\mathrm{i}}\mathfrak{c}' \arrow[r] & \mathfrak{c}' .
\end{tikzcd} \tag{8.46}$$

We claim that the outer large square defines the required pushout. That the underlying diagram in $\mathbf{C}$ is a pushout follows by the construction of the cobase change and the elementary pasting laws for pushouts. To see that on the level of cells, the right vertical composition and the lower horizontal composition are jointly surjective, note that this holds in the lower corner square, by the explicit construction in Corollary 8.1.4.8 and that the upper right vertical and lower left horizontal are bijective on cells. We may now apply Lemma 8.1.4.7, from which it follows that the large composed square is a pushout. □

Observation 8.1.4.13. For every $A \in \mathbf{C}$, the morphism from the empty relative cell complex $(A \xrightarrow{1} A)$ to a relative cell complex $\mathfrak{c}$ defines the inclusion of a subcomplex. It follows by Corollary 8.1.4.8 that the category of relative cell complexes $\mathbf{RCell(C)}_A$ has finite coproducts. In particular, it has a canonical symmetric monoidal structure, given by the coproduct.

Lemma 8.1.4.14. *For every morphism $f\colon A \to A'$ in $\mathbf{C}$, the induced functor*

$$f_{\mathrm{i}}\colon \mathbf{RCell(C)}_A \to \mathbf{RCell(C)}_{A'}$$

preserves coproducts.

Proof. Consider the following commutative diagram of categories

$$\begin{tikzcd}
\mathbf{RCell(C)}_A \arrow[r, "f_{\mathrm{i}}"] \arrow[d] & \mathbf{RCell(C)}_{A'} \arrow[d] \\
\mathbf{C}_{A/} \arrow[r, "f_!"'] & \mathbf{C}_{A/}
\end{tikzcd} \tag{8.47}$$

with vertical given by forgetful functors. By Corollary 8.1.4.1, the verticals are conservative. The lower horizontal is a left adjoint functor, and hence preserves coproducts. By Corollary 8.1.4.8, the underlying diagram in $\mathbf{C}$ of a coproduct diagram is a coproduct in $\mathbf{C}_{A/}$, which shows that the verticals preserve coproducts. Consequently, the upper horizontal preserves coproducts. □

Corollary 8.1.4.15. *Equipping the categories of relative cell complexes $\mathfrak{c}_A$, for $A \in \mathbf{C}$, with the symmetric monoidal structure induced by the coproduct, induces a canonical lift of the pseudo functor*

$$\begin{aligned}
\mathbf{RCell(C)}_{-}\colon \mathbf{C} &\to \mathbf{Cat} \\
A &\mapsto \mathbf{RCell(C)}_A \\
f &\mapsto f_{\mathrm{i}} .
\end{aligned}$$

to the bicategory of (sufficiently small) symmetric monoidal categories equipped with symmetric monoidal functors (denoted **SymMonCat***)*

$$\begin{tikzcd} & \mathbf{SymMonCat} \arrow[d] \\ \mathbf{C} \arrow[ur, dashed] \arrow[r, "\mathbf{RCell(C)}_{-}"'] & \mathbf{Cat}\,. \end{tikzcd} \tag{8.48}$$

8.1.5 Constructing subcomplexes

By Observation 8.1.3.10, we can safely identify subcomplexes of a structured relative cell complex $\mathfrak{c}$ with certain subsets of $\mathfrak{C}_{\mathfrak{c}}$. The obvious question arises, what requirements such a subset needs to fulfill in order to arise from a subcomplex of $\mathfrak{c}$. Moreover, one may ask what operations on these subsets, such as intersection and union, do define new sub-complexes. To this end, let us introduce the notion of a sub-presentation, which is strongly related to the more rigid way subcomplexes are handled in [Hir03]. Let us first introduce the analogue to Hirschhorn's definition of a subcomplex (while keeping track of characteristic maps). Again, throughout this subsection, we assume the context of a cellularized category $\mathbf{C}$.

Notation 8.1.5.1. For the remainder of this subsection, we will always treat the disjoint union of the indexing sets I^α of a filtration-presentation $\mathfrak{p}$ of a structured cell complex $\mathfrak{c}: A \to X$ as identified with $\mathfrak{C}_{\mathfrak{c}}$, via

$$\begin{aligned} \bigsqcup_{\alpha<\lambda} I^\alpha &\xrightarrow{\cong} \mathfrak{C}_{\mathfrak{c}} \\ j &\mapsto (\sigma_j : D \to X)\,. \end{aligned}$$

Definition 8.1.5.2. Let $c: A \hookrightarrow X$ be a relative cell complex and let $\mathfrak{p}$ (using notation as in Definition 8.1.1.1), be a filtration-presentation of c. A *sub-presentation* of $\mathfrak{p}$ consists of the data of:

- A morphism $i: (\tilde{c}: A \to \tilde{X}) \to (c: A \to X)$ in $\mathbf{C}_{A/}$.
- A filtration-presentation $\tilde{\mathfrak{p}}$, of $\tilde{c}$ (using notation as in Definition 8.1.1.1, but adding "$\tilde{-}$" in the obvious way), such that $\tilde{\lambda} = \lambda$ and $\tilde{I}^\alpha \subset I^\alpha$, for all $\alpha < \lambda$.
- A morphisms of transfinite composition diagrams

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & \tilde{X}^0 \arrow[r] \arrow[d, equal, "i^0"'] & \tilde{X}^1 \arrow[r] \arrow[d, "i^1"'] & \dots \arrow[r] & \tilde{X}^\lambda \arrow[r, equal] \arrow[d, "i^\lambda"'] & \tilde{X} \arrow[d, "i"] \\ A \arrow[r, equal] & X^0 \arrow[r] & X^1 \arrow[r] & \dots \arrow[r] & X^\lambda \arrow[r, equal] & X \end{tikzcd} \tag{8.49}$$

  such that $i^\alpha \circ \tilde{\sigma}^i_\alpha = \sigma^i_\alpha$, for all $\alpha < \lambda$ and $i \in \tilde{I}^\alpha \subset I^\alpha$.

Observation 8.1.5.3. In the situation of Definition 8.1.5.2, it is immediate from the definition of a sub-presentation and Proposition 8.1.3.1 that $(\tilde{c}: A \to \tilde{X}) \to (c: A \to X)$ defines the inclusion of a subcomplex with respect to the cell structures given by $\tilde{\mathfrak{p}}$ and $\mathfrak{p}$. The set of cells of the subcomplex $\tilde{\mathfrak{c}}$ defined by $\tilde{\mathfrak{p}}$ is precisely given by the subset $\bigsqcup_{\alpha<\lambda} \tilde{I}^\alpha$ under the canonical identification $\bigsqcup_{\alpha<\lambda} I^\alpha \cong \mathfrak{C}_{\mathfrak{c}}$.

Notation 8.1.5.4. Let $\mathfrak{c}$ be a structured relative cell complex with a filtration-presentation $\mathfrak{p}$ inducing $\mathfrak{c}$. We say that a subset $\tilde{\mathfrak{C}} \subset \mathfrak{C}_{\mathfrak{c}}$ is the set of cells of a sub-presentation $\tilde{\mathfrak{p}}$ of $\mathfrak{p}$ (with notation as in Definition 8.1.5.2) if $\tilde{\mathfrak{C}} = \bigsqcup_{\alpha<\lambda} \tilde{I}^\alpha$.

Next, let us study the relationship between subcomplexes and sub-presentations. We first need the following proposition.

Proposition 8.1.5.5. *Let $\mathfrak{c}\colon A \to X$ be a structured relative cell complex and $\tilde{\mathfrak{C}} \subset \mathfrak{C}_{\mathfrak{c}}$. The following are equivalent:*

1. *$\tilde{\mathfrak{C}}$ defines a subcomplex of $\mathfrak{c}$.*
2. *There exists a filtration-presentation $\mathfrak{p}$ of $\mathfrak{c}$ together with a sub-presentation $\tilde{\mathfrak{p}}$ of $\mathfrak{p}$, such that $\tilde{\mathfrak{C}}$ is the set of cells of $\tilde{\mathfrak{p}}$.*
3. *For any filtration-presentation $\mathfrak{p}$ of $\mathfrak{c}$, there exists a sub-presentation $\tilde{\mathfrak{p}}$ of $\mathfrak{p}$, such that $\tilde{\mathfrak{C}}$ is the set of cells of $\tilde{\mathfrak{p}}$.*

Proof. Clearly, the third claim implies the first and the second implies the first (see Observation 8.1.5.3). Now, suppose that $\tilde{\mathfrak{C}}$ defines a subcomplex of $\mathfrak{c}$ and fix some filtration-presentation $\mathfrak{p}$. We are going to construct a sub-presentation of $\mathfrak{p}$, which gives rise to $\tilde{\mathfrak{C}}$ via a transfinitely inductive process. Denote by $S \subset \lambda + 1$, the set of ordinals β for which $\tilde{\mathfrak{C}} \cap \bigsqcup_{\alpha<\beta} I^\alpha$ does not arise from a sub-presentation of $\mathfrak{p}$. The complement of S is closed below. Indeed, if $\tilde{\mathfrak{C}} \cap \bigsqcup_{\alpha<\beta} I^\alpha$ arises from a sub-presentation $\tilde{\mathfrak{p}}^\beta$ a sub-presentation for $\tilde{\mathfrak{C}} \cap \bigsqcup_{\alpha<\beta'} I^\alpha$, for $\beta' \leq \beta$, is obtained by restricting $\tilde{\mathfrak{p}}^\beta$. If S is empty, then $\lambda \notin S$, and we are done. Otherwise, let β be a minimal element of S. For $\beta' \leq \lambda$ not in S, denote by $\tilde{\mathfrak{c}}^{\beta'}$ the thus defined subcomplex of $\mathfrak{c}$ and by $\mathfrak{c}^{\beta'}$ the subcomplex of $\mathfrak{c}$ defined by the set of cells $\bigsqcup_{\alpha<\beta'} I^\alpha$. Now, if β is a limit ordinal, then for all $\beta' < \beta$, the set $\tilde{\mathfrak{C}} \cap \bigsqcup_{\alpha<\beta'} I^\alpha$ arises from a sub-presentation, and thus defines a subcomplex of $\mathfrak{c}$, by the implication which we have already shown. By Proposition 8.1.3.8, these subcomplexes of $\mathfrak{c}$ arrange into a commutative diagram over $\mathfrak{c}$,

$$\begin{tikzcd}
\tilde{\mathfrak{c}}^0 \arrow[r] \arrow[d, equal] & \tilde{\mathfrak{c}}^1 \arrow[r] \arrow[d, hook] & \dots \arrow[d] & & & \\
\mathfrak{c}^0 \arrow[r, hook] & \mathfrak{c}^1 \arrow[r, hook] & \dots \arrow[r, hook] & \mathfrak{c}^\beta \arrow[r, hook] & \dots \arrow[r, hook] & \mathfrak{c}^\lambda
\end{tikzcd} \tag{8.50}$$

where the upper row is indexed over β. Using Observation 8.1.3.6, we may pass to the colimit in the upper row, to extend this diagram to a diagram of subobjects

$$\begin{tikzcd}
\tilde{\mathfrak{c}}^0 \arrow[r] \arrow[d, equal] & \tilde{\mathfrak{c}}^1 \arrow[r] \arrow[d, hook] & \dots \arrow[r] & \tilde{\mathfrak{c}}^\beta \arrow[r, equal] \arrow[d, hook] & \dots \arrow[r, equal] & \tilde{\mathfrak{c}}^\beta \arrow[d, hook] \\
\mathfrak{c}^0 \arrow[r, hook] & \mathfrak{c}^1 \arrow[r, hook] & \dots \arrow[r, hook] & \mathfrak{c}^\beta \arrow[r, hook] & \dots \arrow[r, hook] & \mathfrak{c}^\lambda
\end{tikzcd} \tag{8.51}$$

Now, we claim that in each successor step, the associated structured relative complex $\tilde{X}^\alpha \to X^{\alpha+1}$ fits into a cobase change square

$$\begin{tikzcd}
\bigsqcup_{j\in I^\alpha \cap \tilde{\mathfrak{C}}} \partial D_j \arrow[r, hook] \arrow[d, dotted] & \bigsqcup_{j\in I^\alpha \cap \tilde{\mathfrak{C}}} D_j \arrow[d, "(\sigma_j)_{j\in I^\alpha\cap\tilde{\mathfrak{C}}}"] \\
\tilde{X}^\alpha \arrow[r, hook] & \tilde{X}^{\alpha+1}
\end{tikzcd} \tag{8.52}$$

where we denote factorizations of characteristic maps by the same name, by abuse of notation. It follows, that the underlying diagram of Definition 8.1.5.2 gives rise to a sub-presentation of $\mathfrak{C}_{\tilde{\mathfrak{c}}^\beta} = \tilde{\mathfrak{C}} \cap \bigsqcup_{\alpha<\beta} I^\alpha$, in contradiction to the assumption. To see that cobase change diagrams as in Diagram (8.52) exist note that $I^\alpha \cap \tilde{\mathfrak{C}}$ specifies precisely the set of cells missing in $\tilde{\mathfrak{c}^\alpha} \subset \widetilde{\mathfrak{c}^{\alpha+1}}$. It follows by Corollary 8.1.4.2, that it suffices to expose a commutative square as in Diagram (8.52), and we need not show that the latter is a pushout. To expose such a square, it suffices to expose the dotted left hand vertical, making the diagram commute. Consequently, it suffices to see that the underlying commutative square

$$\begin{tikzcd}
\tilde{X}^\alpha \arrow[r, hook] \arrow[d, hook] & \tilde{X}^{\alpha+1} \arrow[d, hook] \\
X^\alpha \arrow[r, hook] & X
\end{tikzcd} \tag{8.53}$$

is pullback. This follows by Corollary 8.1.4.10 Now, finally assume that $\beta = \alpha + 1$ is a successor ordinal. Arguing analogously to how we did in the case of a limit ordinal, it suffices to see that $\mathfrak{C}_{\tilde{\mathfrak{c}}} \cap \bigsqcup_{\alpha' \leq \alpha} I^{\alpha'}$ defines a sub-complex of $\mathfrak{c}$. One may easily reduce this claim to showing that the boundary maps $\partial\sigma_j : \partial D_j \to X^\alpha$ factor through $\tilde{X}^\alpha$. This, in turn, follows by showing that

$$\begin{tikzcd} \tilde{X}^\alpha \arrow[r, hook] \arrow[d, hook] & \tilde{X} \arrow[d, hook] \\ X^\alpha \arrow[r, hook] & X \end{tikzcd} \tag{8.54}$$

is pullback, which again follows by Corollary 8.1.4.10. □

If we carefully trace the proof of Proposition 8.1.5.5, we obtain the following general detection principle for subcomplexes.

Corollary 8.1.5.6. *Let $\mathfrak{c} : A \to X$ be a structured relative cell complex and $\tilde{\mathfrak{C}} \subset \mathfrak{C}_{\mathfrak{c}}$. Fix some filtration-presentation $\mathfrak{p}$ of $\mathfrak{c}$. For $\beta < \lambda$, denote $\tilde{\mathfrak{C}}^\beta := \tilde{\mathfrak{C}} \cap \bigsqcup_{\alpha < \beta} I^\alpha$. Then the following are equivalent:*

1. *$\tilde{\mathfrak{C}}$ defines a subcomplex of $\mathfrak{c}$;*
2. *For all $\beta < \lambda$, $\tilde{\mathfrak{C}}^\beta$ defines a subcomplex $(\mathfrak{c}^\beta : A \to \tilde{X}^\beta) \subset \mathfrak{c}$, and for every cell $\sigma : D \to X$ in $I^\beta \cap \tilde{\mathfrak{C}}$, the associated boundary map $\partial\sigma : D \to X$ factors through $\tilde{X}^\beta$.*

As an immediate corollary, we obtain that the union of subcomplexes always exists.

Corollary 8.1.5.7. *Let $(\mathfrak{c}_i)_{j \in I}$ be a family of subcomplexes of a structured relative cell complex $\mathfrak{c}$. Then the union $\bigcup_{i \in I} \mathfrak{C}_{\mathfrak{c}_i} \subset \mathfrak{C}_{\mathfrak{c}}$ defines a subcomplex in $\mathfrak{c}$.*

Notation 8.1.5.8. In the situation of Corollary 8.1.5.7, we are going to denote the associated subcomplex specified by $\bigcup_{i \in I} \mathfrak{C}_{\mathfrak{c}_i} \subset \mathfrak{C}_{\mathfrak{c}}$ by $\bigcup_{i \in I} \mathfrak{c}_i$.

Furthermore, we obtain from Corollaries 8.1.4.10 and 8.1.5.6 that the intersection of two subcomplexes exists. Such a claim was first proven in [Hir03, Thm. 12.2.4]. The proof is essentially identical to the one in [Hir03], using Corollaries 8.1.4.10 and 8.1.5.6.

Corollary 8.1.5.9. *Let $\mathfrak{c}_1, \mathfrak{c}_2 \subset \mathfrak{c}$ be subcomplexes of a structured relative cell complex $\mathfrak{c}$. Then the set of cells $\mathfrak{C}_{\mathfrak{c}_1} \cap \mathfrak{C}_{\mathfrak{c}_2} \subset \mathfrak{C}$ defines a subcomplex of $\mathfrak{c}$, denoted $\mathfrak{c}_1 \cap \mathfrak{c}_2$. Furthermore, the induced diagram of subcomplexes*

$$\begin{tikzcd} \mathfrak{c}_1 \cap \mathfrak{c}_2 \arrow[r, hook] \arrow[d, hook] & \mathfrak{c}_2 \arrow[d, hook] \\ \mathfrak{c}_2 \arrow[r, hook] & \mathfrak{c}_1 \cup \mathfrak{c}_2 \end{tikzcd} \tag{8.55}$$

is bicartesian in $\mathbf{RCell(C)}_A$ *and* $\mathbf{RCell(C)}$ *and its underlying diagram in* $\mathbf{C}$ *is also bicartesian.*

Corollary 8.1.5.10. *Let $\mathfrak{c} : A \to X$ be a structured relative cell complex and let I be a directed set. Furthermore, let $\mathfrak{c}^\bullet : I^\triangleright \to \mathbf{RCell(C)}_A$ send the terminal object ∞ of $I^\triangleright$ to $\mathfrak{c}$, be furthermore such that each arrow $i \to \infty$ is mapped to an inclusion $\mathfrak{c}^i \to \mathfrak{c}$. Then the colimit of $\mathfrak{c} \mid_I$ in $\mathbf{RCell(C)}_A$ and $\mathbf{RCell(C)}_A$ exists, and is given by extending $\mathfrak{c}^\bullet|_I$ to $I^\triangleright$ by $\infty \mapsto \bigcup_{i \in I} \mathfrak{c}^i \subset \mathfrak{c}$, with all arrows pointing into i, given by the canonical inclusions of subcomplexes $\mathfrak{c}_i \to \bigcup_{i \in I} \mathfrak{c}^i$. Furthermore, the forgetful functor $\mathbf{RCell(C)}_A \to \mathbf{C}$ preserves this colimit.*

Proof. By Lemma 8.1.4.7, we only need to show that the specified cocone on $\mathfrak{c}^\bullet|_I$ is such that its underlying diagram in $\mathbf{C}$ defines the colimit of the underlying diagram in $\mathbf{C}$ of $\mathfrak{c}^\bullet|_I$. In other words, if we denote by $\tilde{c}(A \hookrightarrow X) \hookrightarrow c$ the underlying subobject of c specified by $\bigcup_{i \in I} \mathfrak{c}^i \subset \mathfrak{c}$, then we need to show that the induced arrow $\varinjlim_I c^i \to \tilde{c}$ is an isomorphism. An inverse can be constructed via transfinite induction, by fixing a filtration-presentation of $\tilde{\mathfrak{c}}$, and inductively constructing a coherent family of morphisms $\tilde{X}^\beta \to \varinjlim_I X^i$, as follows. If $\beta = 0$, then $\tilde{X}^\beta = A$ and there is a canonical morphism to choose. If β is a limit ordinal, then we may use that $\tilde{X}^\beta = \varinjlim_{\alpha<\beta} X^\alpha$ to obtain an induced morphism $\tilde{X}^\beta \to X^i$. Finally, if $\beta = \alpha + 1$ is a successor ordinal, then for each $j \in \tilde{I}^\alpha$, it holds by assumption that there exists some $i \in I$, such that $\sigma_j \colon D_j \to X^{\alpha+1} \to X$ factors through $X^i \to X$. Furthermore, as $X^i \to X$ is a monomorphism, the resulting composition $\sigma'_j \colon X \dashrightarrow X^i \to \varinjlim_I X^i$ is independent of the choice of factorization and i. By the universal property of the colimit, we obtain an induced dashed morphism

$$\begin{tikzcd} \bigsqcup_{j\in\tilde{I}^\alpha} \partial D_j \arrow[r, hook] \arrow[d] & \bigsqcup_{j\in\tilde{I}^\alpha} D_j \arrow[d] \arrow[ddr, dotted, bend left, "(\sigma'_j)_{j\in\tilde{I}^\alpha}"] & \\ \tilde{X}^\alpha \arrow[r, hook] \arrow[drr, bend right] & \tilde{X}^{\alpha+1} \arrow[dr, dashed] & \\ & & \varinjlim X^i \,. \end{tikzcd} \tag{8.56}$$

That the thus inductively defined morphism $\tilde{X} \to \varinjlim_I X^i$ defines an inverse to the canonical morphism $\varinjlim_I X^i \to \tilde{X}$, follows from it being constructed in a way such that it commutes with the (induced) morphism $\sigma \colon D \to \tilde{X}$ and $D \xrightarrow{\sigma} X^i \to \varinjlim_I X^i$, for $\sigma \in \mathfrak{C}_{\tilde{\mathfrak{c}}}$, and from the fact that the induced arrows $\bigsqcup_{\sigma\in\mathfrak{C}_{\tilde{\mathfrak{c}}}} D_\sigma \to \tilde{X}, \varinjlim_I X^i$ are both epimorphisms. $\square$

Notation 8.1.5.11. A cocone $\mathfrak{c}^\bullet \colon I^\triangleright \to \mathbf{RCell}(\mathbf{C})_A$ as in Corollary 8.1.5.10 such that $\bigcup_{i\in I} \mathfrak{c}^i = \mathfrak{c}$ will be called a *filtration of* $\mathfrak{c}$ by subcomplexes.

8.1.6 On compactness

Recall the classical fact from algebraic topology that every compact subset of a CW-complex is contained in a finite subcomplex. For our purposes, it will be convenient to have a similar tool available in the context of general structured cell complexes.

Remark 8.1.6.1. There are several incompatible notions of compactness in category theory. One commonly used notion is that an object X in a category $\mathbf{C}$ is called *compact*, if the hom-functor $\mathbf{C}(X, -)$ preserves filtered colimits. This notion is, however, often too strong for the purpose of abstract homotopy theory. For example, compact topological spaces are not *compact* in this sense, and it is thus not helpful to recover the classical fact we have mentioned above. One way to weaken this notion is to only require that the hom-functor preserves filtered colimits of a certain shape. Namely, given a regular cardinal κ, an object X is often called *κ-compact*, if $\mathbf{C}(X, -)$ preserves such filtered colimits, where the indexing poset I is such that every subset $J \subset I$ of size strictly smaller than κ admits an element $p \in I$, such that $q \leq p$, for all $q \in J$. Another way to generalize compactness is that one often only requires the compactness of the object relative to certain other objects and diagrams. One such relative notion of compactness is the one introduced in [Hir03, Prop. 10.8.1.], where the author defines an object B to be κ-compact with respect to a class of boundary inclusions $\mathbb{B}$, if for every structured relative $\mathbb{B}$-complex $\mathfrak{c} \colon A \to X$, and every morphism $f \colon B \to X$, there exists a subcomplex $(A \xhookrightarrow{\tilde{\mathfrak{c}}} \tilde{X}) \hookrightarrow \mathfrak{c}$ with a set of cells smaller or equal to κ, such that f factors through $\tilde{X} \hookrightarrow X$. [Hir03] then defines an object to be compact, if it is compact with respect to *any* cardinal κ. We find this nomenclature somewhat inconvenient, both for our purposes, as well as in comparison to the competing definitions, for two reasons:

1. In the first definition, being κ-compact is a weaker notion than being compact. In the definition of [Hir03], it is a stronger notion.
2. Using the language of [Hir03], there is no way to express that arrows should factor through finite subcomplexes. One would have to amend his definition by requiring the set of cells to be *strictly smaller* than κ.

While the latter limitation is no real hurdle when the specific size of the cardinal one refers to is irrelevant, for our investigations of finite cell complexes in the context of simple homotopy theory it cannot be used. We thus introduce the following notion of compactness.

Definition 8.1.6.2. Let $\mathbf{C}$ be a cellularized category. We say that an object $B \in \mathbf{C}$ is filtration compact, if the following condition holds: Let $\mathfrak{c}\colon A \to X$ be a structured relative cell complex and let $\mathfrak{c}^\bullet\colon I^\triangleright \to \mathbf{RCell}(\mathbf{C})_A$ be a filtration of $\mathfrak{c}$ by subcomplexes. Denote by $X^\bullet\colon I \to \mathbf{C}$ the underlying diagram of $\mathfrak{c}^\bullet\,|_I$ in $\mathbf{C}$. Then the canonical morphism

$$\varinjlim_I \mathbf{C}(B, X^i) \to \mathbf{C}(B, X)$$

is an isomorphism. We say that $\mathbf{C}$ has filtration compact boundaries, if the source generating boundary inclusion $b \in \mathbb{B}$ is filtration compact.

Remark 8.1.6.3. Observe that, as every inclusion of a subcomplex defines a monomorphism in $\mathbf{C}$, the defining condition in Definition 8.1.6.2 is equivalent to saying that every morphism $f\colon B \to X$ factors through $X^i \to X$, for some $i \in I$.

Let us give a second notion of compactness, which is the finite version of the notion of compactness considered in [Hir03].

Notation 8.1.6.4. Let $\mathbf{C}$ be a cellularized category and let $\mathfrak{c}\colon A \to X$ be a structured relative cell complex in $\mathbf{C}$. We say that a morphism $f\colon B \to X$ in $\mathbf{C}$ factors through a subcomplex $(\tilde{\mathfrak{c}}\colon A \to \tilde{X}) \subset \mathfrak{c}$, if f factors through $\tilde{X} \hookrightarrow X$.

Definition 8.1.6.5. Let $\mathbf{C}$ be a cellularized category. We say that an object $B \in \mathbf{C}$ is subcomplex compact, if the following condition holds. For every structured relative cell complex $\mathfrak{c}\colon A \to X$ in $\mathbf{C}$ and for every morphism $B \to X$, there exists a finite subcomplex $\tilde{\mathfrak{c}} \subset \mathfrak{c}$, such that f factors through $\tilde{\mathfrak{c}}$. We say that $\mathbf{C}$ has subcomplex compact boundaries, if the source of every generating boundary inclusion $b \in \mathbb{B}$ is subcomplex compact.

Remark 8.1.6.6. Clearly, Definitions 8.1.6.2 and 8.1.6.5 have analogues where $\aleph_0$ is replaced with any regular cardinal. The generalization of the theory we describe here to this case is essentially purely formal and we will have no need for it here.

Let us now study the relation between these two notions of compactness. For the remainder of this section, fix some cellularized category $\mathbf{C}$. We begin with an elementary observation.

Lemma 8.1.6.7. *Suppose we are given a pushout diagram in* $\mathbf{C}$

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \end{tikzcd} \tag{8.57}$$

with X_1 and X_2 filtration compact (subcomplex compact). Then X is also filtration compact (subcomplex compact).

Proof. This is an immediate consequence of the universal property of the pushout and every inclusion of a subcomplex inducing a monomorphism in $\mathbf{C}$. ☐

Proposition 8.1.6.8. *Let $B \in \mathbf{C}$. Then if B is subcomplex compact, it is also filtration compact.*

Proof. Let $\mathfrak{c}^\bullet\colon I^\triangleright \to \mathbf{RCell}(\mathbf{C})_A$ be a filtration of $\mathfrak{c}$ by subcomplexes. Let $\tilde{\mathfrak{c}}$ be a finite subcomplex of f through which f factors. Then, for every cell σ of $\tilde{\mathfrak{c}}$, there exists an $i \in I$, such that $\sigma \in \mathfrak{C}_{\mathfrak{c}^i}$. Consequently, as I is filtered and $\tilde{\mathfrak{c}}$ finite, there exists an $m \in I$, such that $\sigma \in \mathfrak{C}_{\mathfrak{c}^m}$ for all $\sigma \in \mathfrak{C}_{\tilde{\mathfrak{c}}}$. It follows by Observation 8.1.3.10 that $\tilde{\mathfrak{c}} \subset \mathfrak{c}^m$, and hence f factors through $X^m \to X$, as required. ☐

Proposition 8.1.6.9. *If* $\mathbf{C}$ *has filtration compact boundaries or subcomplex compact boundaries, then the following holds. An object* $B \in \mathbf{C}$ *is filtration compact, if and only if it is subcomplex compact.*

Proof. We have already seen that subcomplex compactness implies filtration compactness in Proposition 8.1.6.8. We proceed to show the remaining implication from the filtration factorization condition to the subcomplex factorization condition, where we induce over the minimal presentation ordinal λ of the target structured relative cell complexes $\mathfrak{c}: A \to X$, and we only allow for such filtration-presentations, which only add a single cell in each step. Clearly, for $\lambda = 0$, there is nothing to be shown. Now, let λ be a a limit ordinal, and $\mathfrak{c}: A \to X$ and $f: B \to X$ in $\mathbf{C}$. Let us show that the filtration condition implies the subcomplex condition. Then fixing some filtration-presentation of X

$$A = X^0 \hookrightarrow X^1 \hookrightarrow \ldots X^\lambda = X\,.$$

denote, for $\beta < \lambda$, by $\mathfrak{c}^\beta: A \to X^\beta$ the induced subcomplex of $\mathfrak{c}$. Then $\varinjlim c^\lambda = c$, and by definition of filtration compactness, it follows that f factors through X^β, for some $\beta < \lambda$. Consequently, we find a finite subcomplex of $\mathfrak{c}^\beta$, $\tilde{\mathfrak{c}}: A \to \tilde{X}$, such that f factors through $\tilde{X}$. But then $\tilde{\mathfrak{c}} \hookrightarrow \mathfrak{c}^\beta \hookrightarrow \mathfrak{c}$ defines a finite subcomplex of $\mathfrak{c}$, through which f factors.
Now, for the case of a successor ordinal $\lambda = \beta + 1$, observe first that, by the inductive assumption applied to the sources of generating boundary inclusions $b \in \mathbb{B}$, under any of the two assumptions it follows by Corollary 8.1.5.6 that every cell of $\mathfrak{c}$ is contained in a finite subcomplex of $\mathfrak{c}$.

Let S be the set of all finite subcomplexes of $\mathfrak{c}$ (choosing one representative for each subobjects). Equip S with the partial order given by inclusion of subcomplexes. By Corollary 8.1.5.7, S is a filtered poset. Now, consider the filtration by subcomplexes of $\mathfrak{c}$, $\mathfrak{c}^\bullet: S^\triangleright \to \mathbf{RCell}(\mathbf{C})_A$, given by $\mathfrak{c}^s = s$, for a finite subcomplex $s \in S$, equipped with the obvious functoriality on objects. Since we have shown that every cell of $\mathfrak{c}$ is contained in a finite subcomplex of $\mathfrak{c}$, it follows that this cocone does indeed define a filtration of subcomplexes. Hence, by definition of filtration compactness, every morphism $f: B \to X$ factors through some $X^s \hookrightarrow X$, for some $s \in S$.

□

Corollary 8.1.6.10. *A cellularized category* $\mathbf{C}$ *has filtration compact boundaries if and only if it has subcomplex compact boundaries.*

Notation 8.1.6.11. If a cellularized category $\mathbf{C}$ has filtration compact or equivalently subcomplex compact boundaries, we will just say that it has *strictly compact boundaries*.

As we have seen in the proof of Proposition 8.1.6.9, in the scenario of filtration (or equivalently subcomplex) compact boundaries, cells are always contained in finite subcomplexes.

Corollary 8.1.6.12. *If a cellularized category* $\mathbf{C}$ *has strictly compact boundaries, then every cell* $\sigma \in \mathfrak{C}_\mathfrak{c}$ *of a structured relative cell complex* $\mathfrak{c}$ *is contained in some finite subcomplex* $\tilde{\mathfrak{c}} \hookrightarrow \mathfrak{c}$.

We have also seen in the proof of Proposition 8.1.6.9:

Corollary 8.1.6.13. *In a cellularized category* $\mathbf{C}$ *with strictly compact boundaries, every structured relative cell complex* $\mathfrak{c}$ *admits a filtration by its finite subcomplexes.*

Next, let us investigate when cell complexes are filtered or subcomplex compact:

Proposition 8.1.6.14. *Let* $\mathbf{C}$ *be a cellularized category. Assuming that, for every* $b: \partial D \to D \in \mathbb{B}$*, it holds that* D *is subcomplex compact, then an absolute structured cell complex* $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C})$ *is subcomplex compact, if and only if it is finite. If every target* D *of* $b \in \mathbb{B}$ *is filtration compact, then every finite absolute cell complex is filtration compact. If* $\mathbf{C}$ *has strictly compact boundaries, then the converse statement for filtration compactness holds.*

Proof. The final claim is immediate from the first claim, by Proposition 8.1.6.9. Now, for the first statement, the only if statement is trivially true. It remains to show the if case for subcomplex and filtration compactness. Inducing over the number of cells, this is an immediate consequence of Lemma 8.1.6.7. □

Another claim which is immediate from the definition of filtration and subcomplex compactness is the following.

Lemma 8.1.6.15. *Let* $\mathbf{C}$ *be a cellularized category. Any retract of a filtration compact (subcomplex compact) object is again filtration compact (subcomplex compact).*

Finally, let us end this subsection with two observations on compactness which will be extremely useful when investigating the homotopy theory of finite complexes in a cofibrantly generated model category.

Observation 8.1.6.16. Let $\mathbf{C}$ be a cellularized category. It is immediate from the definition of filtration compactness that every filtration compact object is $\aleph_0$-small with respect to relative cell complexes. It follows that if $\mathbb{E}_{\mathbf{C}}$ is a set of morphisms in $\mathbf{C}$, each of which admits the structure of a finite relative structured cell complex, and is such that every source of a morphism in $\mathbb{E}_{\mathbf{C}}$ is filtration compact, then $\mathbb{E}_{\mathbf{C}}$ permits the small object argument (see [Hir03, pp. 10, 5]), and the latter can be performed with countable transfinite compositions indexed over the naturals.

Proposition 8.1.6.17. *Let* $\mathbf{C}$ *be a cellularized category. Let* $\mathbb{E}_{\mathbf{C}}$ *be a set of finite structured relative cell complexes in* $\mathbb{E}_{\mathbf{C}}$*, each of which has filtration compact source. Let* $\mathbf{E}$ *be the smallest class of structured relative cell complexes in* $\mathbf{C}$*, such that*

1. $\mathbf{E}$ *contains* $\mathbb{E}_{\mathbf{C}}$*;*
2. $\mathbf{E}$ *is closed under cobase changes;*
3. $\mathbf{E}$ *is closed under vertical transfinite composition.*

Let $\mathfrak{e}: A \to X \in \mathbf{E}$ *let* $B \in \mathbf{C}$ *be filtration compact, and let* $f: B \to X$ *be a morphism in* $\mathbf{C}$*. Then there exists a finite subcomplex* $(\tilde{\mathfrak{e}}: A \to \tilde{X}) \subset \mathfrak{e}$*, such that* f *factors through* $\tilde{X} \hookrightarrow X$ *and such that* $\tilde{\mathfrak{e}} \in \mathbf{E}$*.*

Remark 8.1.6.18. Carefully tracing the proof of Proposition 8.1.6.17 below, one observes that it suffices for the sources of $\mathbb{E}_{\mathbf{C}}$ to be filtration compact with respect to $\mathbb{E}_{\mathbf{C}}$ (after removing isomorphisms).

Proof of Proposition 8.1.6.17. By the composability of transfinite vertical compositions and Observation 8.1.2.13, every morphism in $\mathbf{E}$ is a transfinite composition of cobase changes of morphisms in $\mathbb{E}_{\mathbf{C}}$. Observe that, in any such transfinite composition, we may always reduce the latter such that only the first morphism is a cobase change of an isomorphism. Clearly, precomposing with an isomorphism has no impact on the claimed factorization conditions. Hence, we may assume that $\mathbb{E}_{\mathbf{C}}$ contains no isomorphism. In particular, it follows that if we equip $\mathbf{C}$ with the class of generating boundary inclusions given by the underlying morphisms of $\mathbb{E}_{\mathbf{C}}$, then this defines the structure of a cellularized category with strictly compact boundaries. As B is filtration compact with respect to $\mathbb{B}$ and every filtration by $\mathbb{E}_{\mathbf{C}}$ subcomplexes refines to a filtration of $\mathbb{B}$-subcomplexes, B is also filtration compact with respect to $\mathbb{E}_{\mathbf{C}}$. Now, as $\mathfrak{e} \in \mathbf{E}$, we may write the latter in terms of a transfinite composition of cobase changes of elements of $\mathbb{E}_{\mathbf{C}}$. This defines a coarsening of the $\mathbb{B}$-cell structure on $\mathfrak{e}$ to a $\mathbb{E}_{\mathbf{C}}$-cell structure, which we denote $\mathfrak{e}'$. As B is filtration compact with respect to $\mathbb{E}_{\mathbf{C}}$, it follows by Proposition 8.1.6.9 that B is also subcomplex compact with respect to $\mathbb{E}_{\mathbf{C}}$. Hence, $f: B \to X$ (as in the claim of the proposition), factors through a finite structured $\mathbb{E}_{\mathbf{C}}$-subcomplex $\tilde{\mathfrak{e}}': A \to \tilde{X}$ of $\mathfrak{e}'$. We may now refine the cell structure on $\mathfrak{e}'$ to a $\mathbb{B}$-cell structure again, by using the cell structures on the generators $\mathfrak{e}_0 \in \mathbb{E}_{\mathbf{C}}$. This equips $e': A \to \tilde{X}$ with the structure of a $\mathbb{B}$-subcomplex of $\mathfrak{e}$, denoted $\tilde{\mathfrak{e}}$. As $\tilde{e}'$ was finite and every $\mathfrak{e}_0 \in \mathbb{E}_{\mathbf{C}}$ is assumed to be finite, $\tilde{\mathfrak{e}}$ is finite. □

8.2 Cellularized functors: Bookkeeping

In this section, we develop the theory of functors between cellularized categories. As our notion of structured cell complexes was developed in the full generality of relative structured cell complex, our notion of functor will consequently also be relative. We emphasize that this section summarizes a lot of basic facts about the interaction of colimit preserving functors with cell complexes that are well known and used in the theory of model categories (see, for example, [Hir03]). The purpose is mainly to keep to list these classical facts in a coherent form, while keeping track of cell structures. Hence, we will often only provide sketches of proofs.

8.2.1 Some generalities on Leibniz-style constructions

The guiding example of a relative cellularized functor is the actions of a simplicial category on a simplicial model category, or of a topological category on a (tensored and powered) topological category. As to connect to the classical scenario, however, let us recall the case of CW-complexes. To simplify things a little bit, in particular in order to not have to keep track of dimension conditions, we will describe the case of topological cell complexes, omitting any requirements on the gluing maps.

Example 8.2.1.1. Given a structured topological cell complex $\mathfrak{X}$ (equipped with a choice of cell structure), the functor

$$X \times -\colon \mathbf{Top} \to \mathbf{Top}$$

(of compactly generated topological spaces) lifts to a functor of relative structured cell complexes, which can be seen as follows. To define a cell structure on $X \times B \to X \times Y$, for a relative structured cell complex $\mathfrak{d}\colon B \hookrightarrow Y$, observe first that (as we are working in compactly generated spaces) $X \times c$ can be written as a transfinite composition

$$X \times A \to X \times (Y^0 \cup A) \to X \times (Y^1 \cup A) \to \cdots \to X \times Y^{\lambda} = X \times Y.$$

For each $\alpha < \lambda$, we obtain an induced pushout diagram

$$\begin{tikzcd} \bigsqcup X \times \partial D^{n_{i,\alpha}} \arrow[r, hook] \arrow[d, dashed] & \bigsqcup X \times D^{n_{i,\alpha}} \arrow[d, "X\times\bigsqcup\sigma"] \\ X \times (Y^{\alpha} \cup A) \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & X \times (Y^{\alpha+1} \cup A) \end{tikzcd} \tag{8.58}$$

Using cobase change of cell structures and transfinite composition, it follows that to expose a cell structure on $X \times Y$, it suffices to expose the structure of a relative cell complex on $X \times (\partial D^n \to D^n)$. Making this a bit more formal, we will see that, a lift of the functor on arrows induced by $X \times -$ to a functor

$$\mathbf{RCell(Top)} \to \mathbf{RCell(Top)}$$

which preserves vertical composition, is entirely determined by a choice of relative cell structures on $X \times (\partial D^n \to D^n)$. So far we have not used the cell structure on $\mathfrak{X}$ at all. Taking it into account, we can choose a filtration-presentation of $\mathfrak{X}$ and observe that we may write may write $X \times (\partial D^n \to D^n)$ as a transfinite composition

$$(X \times \partial D^n) \to (X \times \partial D^n) \cup (X^0 \times D^n) \to (X \times \partial D^n) \cup (X^1 \times D^n) \to \cdots \to X^{\beta} \times D^n\,.$$

Again, we obtain pushout squares

$$\begin{tikzcd} \bigsqcup_{i\in I^{\alpha}} (D^{k_{i,\alpha}} \times \partial D^n) \cup (\partial D^{k_{i,\alpha}} \times D^n) \arrow[r, hook] \arrow[d] & \bigsqcup_{i \in I^{\alpha}} D^{k_i,\alpha} \times D^n \arrow[d] \\ (X \times \partial D^n) \cup (X^{k-1} \times D^n) \arrow[r, hook] & (X \times \partial D^n) \cup (X^k \times D^n) \end{tikzcd} \tag{8.59}$$

for $\alpha < \beta$. Hence, we may reduce to exposing a cell structure on

$$\partial(D^{k+n}) \cong (D^k \times \partial D^n) \cup (\partial D^k \times D^n) \hookrightarrow D^k \times D^n \cong D^{k+n}.$$

In this case, having chosen an identification $D^k \times D^n \cong D^{k+n}$, there is of course a canonical one available. However, there is a deeper principle to be studied here: When defining functors of cellularized frameworks, the decisive structures to study are Leibniz formula morphisms of the form

$$X \times B \cup_{A\times B} A \times Y \to X \times Y$$

induced by morphisms $A \to X$ and $B \to Y$.

Notation 8.2.1.2. Suppose we are given a natural transformation

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=L}] & \mathbf{D} \arrow[from=U, to=L, Rightarrow, "\iota"] \end{tikzcd} \tag{8.60}$$

where $\mathbf{D}$ is a category that admits pushouts. Given $f\colon X_0 \to X_1$ in $\mathbf{C}$, consider the induced diagram

$$\begin{tikzcd} F(X_0) \arrow[r, "\iota_{X_0}"] \arrow[d, "F(f)"'] & G(X_0) \arrow[d] \arrow[ddr, bend left, "G(f)"] & \\ F(X_1) \arrow[r] \arrow[drr, bend right, "\iota_{X_1}"'] & F(X_1) \cup_{F(X_0)} G(X_0) \arrow[dr, dashed, "\hat{\iota}(f)"] & \\ & & G(X_1). \end{tikzcd} \tag{8.61}$$

We denote by $\hat{\iota}$ the induced functor

$$\mathbf{C}^{[1]} \to \mathbf{D}^{[1]}$$
$$f \mapsto \hat{\iota}(f)$$

with functoriality induced by the universal property of the pushout. We call $\hat{\iota}$ the *Leibniz construction* associated to $\iota\colon F \to G$.

Let us recall some of the elementary properties of the Leibniz construction, which show up in one form or another in most textbooks on model categories (see for example [Hir03, p. 10.2.17] where the simplicial cases are proven and [RV13], for some context where these properties are excessively used).

Recollection 8.2.1.3. Let $\iota\colon F \Rightarrow G$ a natural transformation of functors $F, G\colon \mathbf{C} \to \mathbf{D}$ be such that F, G both preserve colimits. The Leibniz construction has the following properties:

1. Given a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X' \end{tikzcd} \tag{8.62}$$

   in $\mathbf{C}$, the induced commutative square in $\mathbf{D}$

$$\begin{tikzcd} F(X) \cup_{F(A)} G(A) \arrow[r] \arrow[d] & F(X') \cup_{F(A')} G(A') \arrow[d] \\ G(X) \arrow[r] & G(X') \end{tikzcd} \tag{8.63}$$

   is again pushout.

2. Given a (transfinite) composition

$$X^0 \longrightarrow X^1 \longrightarrow \dots \longrightarrow X^\lambda \tag{8.64}$$

with pairwise arrows denoted $f^{\alpha,\alpha'}: X^\alpha \to X^{\alpha'}$ then $\hat{\iota}(f^{0,\lambda})$ is given by a transfinite composition

$$\begin{tikzcd} F(X^\lambda)\cup_{F(X^0)} G(X^0) \arrow[r] \arrow[rrr, bend right, "\hat{\iota}(f^{0,\lambda})"'] & F(X^\lambda)\cup_{F(X^1)} G(X^1) \arrow[r] \arrow[rr, bend left, "\hat{\iota}(f^{1,\lambda})"] & \dots \arrow[r] & G(X^\lambda) \end{tikzcd} \tag{8.65}$$

where the arrows $F(X^\lambda)\cup_{F(X^\alpha)} G(X^\alpha) \to F(X^\lambda)\cup_{F(X^{\alpha'})} G(X^{\alpha'})$ fit into pushout squares

$$\begin{tikzcd} F(X^{\alpha'})\cup_{F(X^\alpha)} G(X^\alpha) \arrow[r, "\hat{\iota}(f^{\alpha,\alpha'})"] \arrow[d] & G(X^{\alpha'}) \arrow[d] \\ F(X^\lambda)\cup_{F(X^\alpha)} G(X^\alpha) \arrow[r] & F(X^\lambda)\cup_{F(X^{\alpha'})} G(X^{\alpha'})\,. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{8.66}$$

Notation 8.2.1.4. To keep the language analogous to the framework of cellularized categories, we will call a natural transformation $\iota: F \Rightarrow G$ a *relative functor*. We are going to consider the following two notions of composition of relative functors.

1. The regular composition of natural transformations

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\iota"] \arrow[rr, Rightarrow, bend right, "\tau\circ\iota"'] & G \arrow[r, Rightarrow, "\tau"] & H \end{tikzcd} \tag{8.67}$$

will be called vertical composition and denoted by $\circ$.

2. We are furthermore going to consider an additional Leibniz-style composition of functors, constructed in Construction 8.2.1.5, denoted $\hat{\circ}$.

Construction 8.2.1.5. Suppose we are given a diagram of categories which have pushouts

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F_0"] \arrow[r, bend right, "G_0"'] \arrow[r, phantom, "\Downarrow\iota_0"] & \mathbf{D} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\iota_1"] & \mathbf{E}\,. \end{tikzcd} \tag{8.68}$$

We denote by $\iota_1\hat{\circ}\iota_0$ the induced horizontal Leibniz composition,

$$\begin{tikzcd} F_1\circ F_0 \arrow[r, "F_1\iota_0"] \arrow[d, "\iota_1 F_0"'] & F_1\circ G_0 \arrow[d] \arrow[ddr, bend left, "\iota_1 G_0"] & \\ G_1\circ F_0 \arrow[r] \arrow[rrd, bend right, "G_1\iota_0"'] & F_1\circ G_0\cup_{F_1\circ F_0} G_1\circ F_0 \arrow[dr, dashed, "\iota_1\hat{\circ}\iota_0"] \arrow[ul, phantom, "\ulcorner", very near start] & \\ & & G_1\circ G_0 \end{tikzcd} \tag{8.69}$$

induced through the universal property of the pushout of functors. This construction is associative and unital up to canonical isomorphism. We may summarize the whole situation in the following bicategory (see [JY20], for basic language and definition in bicategories), which we denote by **Leib**:

1. Objects of **Leib** are categories $\mathbf{C}$ with finite colimits.

2. The category of morphisms between $\mathbf{C}$ and $\mathbf{D}$ is the 1-category of relative functors $\mathbf{Fun}_{\mathrm{col}}(\mathbf{C},\mathbf{D})^{[1]}$ which preserve colimits, with unit object given by the unique transformation $\varnothing \to 1_{\mathbf{C}}$.

3. Composition
$$\mathbf{Leib}(\mathbf{D},\mathbf{E}) \times \mathbf{Leib}(\mathbf{C},\mathbf{D}) \to \mathbf{Leib}(\mathbf{C},\mathbf{E})$$
is given by the Leibniz compostion $-\hat{\circ}-$.

4. All associators and unitors are induced by the universal property of the pushout.

Observation 8.2.1.6. Given a diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F_0"] \arrow[r, bend right, "G_0"'] \arrow[r, phantom, "\Downarrow \iota_0"] & \mathbf{D} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow \iota_1"] & \mathbf{E}\,. \end{tikzcd} \tag{8.70}$$

as in Construction 8.2.1.5, the universal property of the pushout induces a canonical isomorphism
$$\widehat{(-)}(\iota_1 \hat{\circ} \iota_0) \cong \hat{\iota}_1 \circ \hat{\iota}_0.$$

Checking the relevant identities, we may summarize this as the Leibniz construction inducing a pseudo-functor

$$\begin{aligned} \mathbf{Leib} &\to \mathbf{Cat} \\ \mathbf{C} &\mapsto \mathbf{C}^{[1]} \\ \iota &\mapsto \hat{\iota} \end{aligned}$$

where functoriality on 2-morphisms is again induced by the universal property of the pushout.

Recollection 8.2.1.7. Finally, recall that the symmetric analogue of Recollection 8.2.1.3 holds, with the roles of relative functors and morphisms exchanged (using cobase changes and transfinite compositions of natural transformations).

8.2.2 Cellularizing functors

Having recalled the Leibniz calculus of functors, let us now study cellularized versions of functors as suggested by the guiding examples of the tensor actions on topological or simplicial model categories:

Definition 8.2.2.1. Let $\mathbf{C}$ and $\mathbf{D}$ be cellularized categories. A *relative cellularized functor* $\mathfrak{i}$ from $\mathbf{C}$ to $\mathbf{D}$ consists of the data of:

1. A natural transformation of colimit preserving functors $\iota\colon F \Rightarrow G$.

2. A lift
$$\begin{tikzcd} \mathbf{RCell}(\mathbb{B}_{\mathbf{C}}) \arrow[r, dashed, "\hat{\mathfrak{i}}"] \arrow[d] & \mathbf{RCell}(\mathbb{B}_{\mathbf{D}}) \arrow[d] \\ \mathbf{C}^{[1]} \arrow[r, "\hat{\iota}"] & \mathbf{D}^{[1]}\,, \end{tikzcd} \tag{8.71}$$

such that, for any inclusion of relative cell complexes $j\colon (A \overset{\tilde{\mathfrak{c}}}{\hookrightarrow} \tilde{X}) \hookrightarrow (A \overset{\mathfrak{c}}{\hookrightarrow} X)$, $\hat{\mathfrak{i}}(\mathfrak{c})$ carries the cell structure given by the right vertical composition

$$\begin{tikzcd} F(\tilde{X}) \cup_{F(A)} G(A) \arrow[r] \arrow[d, hook, "\hat{\mathfrak{i}}(\tilde{\mathfrak{c}})"] & F(X) \cup_{F(A)} G(A) \arrow[d, hook] \arrow[dd, bend left, "\hat{\imath}(\mathfrak{c})"] \\ G(\tilde{X}) \arrow[r] & F(X) \cup_{F(\tilde{X})} G(\tilde{X}) \arrow[d, hook, "\hat{\mathfrak{i}}(\mathfrak{j})"] \\ & G(X) \end{tikzcd} \tag{8.72}$$

with the square cobase change.

By an *(absolute) cellularized functor*, we mean a relative cellularized functor $\mathfrak{i}$ of the form $\iota\colon \varnothing \Rightarrow F$.

Notation 8.2.2.2. We follow the notational convention of denoting structured objects by special fonts, such as calligraphic or fraktur-font, and the underlying object in regular font. Hence, given an absolute cellularized functor $\mathfrak{F}$, we refer to the underlying (ordinary) functor by F.

Observation 8.2.2.3. Observe that, by the essential uniqueness of initial objects, we may simply think of an absolute cellularized functor as a functor $F\colon \mathbf{C} \to \mathbf{D}$, together with a lift

$$\begin{tikzcd} \mathbf{RCell}(\mathbb{B}_{\mathbf{C}}) \arrow[r, dashed, "\hat{\mathfrak{F}}"] \arrow[d] & \mathbf{RCell}(\mathbb{B}_{\mathbf{D}}) \arrow[d] \\ \mathbf{C}^{[1]} \arrow[r, "\hat{F}"] & \mathbf{D}^{[1]} \end{tikzcd}, \tag{8.73}$$

Observe, furthermore, that for a relative cell complex $c\colon A \to X$, $\hat{F}(c) = (F(A) \cup_{\varnothing} \varnothing \to F(X))$ is canonically isomorphic to $F(c)$. Hence, the data of a cellularization of F is the same as choosing for each structured relative cell complex $\mathfrak{c}$ a cell structure on $F(c)$, in a way that is compatible with the functoriality of F and vertical composition.

Notation 8.2.2.4. Given a cellularized category $\mathbf{C}$, and an element $b\colon \partial D \to D$ of $\mathbb{B}$, we will denote by $\mathfrak{b}\colon \partial D \to D$, the associated structured relative cell complex, with exactly one cell given by $D \xrightarrow{1_D} D$.

Lemma 8.2.2.5. *Given a relative cellularized functor* $\mathfrak{i}\colon F \Rightarrow G$ *, we may consider the family*

$$(\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})})_{b \in \mathbb{B}_{\mathbf{C}}} .$$

This construction induces a bijection

$$\{\hat{\mathfrak{i}}\colon \mathbf{RCell}(\mathbb{B}_{\mathbf{C}}) \to \mathbf{RCell}(\mathbb{B}_{\mathbf{D}}) \mid \hat{\mathfrak{i}} \text{ cellularizes } \iota\}$$
$$\cong$$
$$\{(\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})})_{b \in \mathbb{B}} \mid \mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})} \text{ defines a cell structure on } \hat{\iota}(b)\} .$$

The inverse is defined as follows. Suppose we are given cell structures $(\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})})_{b \in \mathbb{B}}$*, on* $\hat{\iota}(b)$*, for* $b \in \mathbb{B}$*. For* $\mathfrak{c}\colon A \hookrightarrow X$*, the cell structure on* $\hat{\iota}(c)$ *is explicitly given by a disjoint union*

$$\bigsqcup_{\sigma \in \mathfrak{C}_{\mathfrak{c}}} G(\sigma) \mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b}_{\sigma})} ,$$

with b_σ *the generating boundary inclusion associated to* σ*.*

Proof. We first show that the explicit construction above does indeed define a cell structure. That it then defines a cellularized functor is easily verified from the construction. Let $\mathfrak{c}\colon A \hookrightarrow X$ be a structured relative cell complex, and let

$$A = X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \dots \hookrightarrow X^\lambda \xrightarrow{\cong} X \tag{8.74}$$

together with

$$\begin{tikzcd} \partial D^\alpha \arrow[r, hook] \arrow[d] & D^\alpha \arrow[d] \\ X^\alpha \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & X^{\alpha+1}, \end{tikzcd} \tag{8.75}$$

for $\alpha < \lambda$, be a filtration-presentation of $\mathfrak{c}$, with one cell in each degree. It follows by Recollection 8.2.1.3, that we obtain an induced transfinite composition

$$F(X) \cup_{F(A)} G(A) \hookrightarrow F(X) \cup_{F(X^1)} G(X^1) \hookrightarrow \dots \hookrightarrow F(X) \cup_{F(X^\lambda)} G(X^\lambda) \xrightarrow{\cong} G(X) \tag{8.76}$$

together with composed pushout squares

$$\begin{tikzcd} F(D^\alpha) \cup_{F(\partial D^\alpha)} G(\partial D^\alpha) \arrow[r, "\hat{\iota}(b^\alpha)"] \arrow[d] & G(D^\alpha) \arrow[d] \\ F(X^{\alpha+1}) \cup_{F(X^\alpha)} G(X^\alpha) \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & G(X^{\alpha+1}) \arrow[d] \\ F(X) \cup_{F(X^\alpha)} G(X^\alpha) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & F(X) \cup_{F(X^{\alpha+1})} G(X^{\alpha+1}) \end{tikzcd} \tag{8.77}$$

for $\alpha < \lambda$. Hence, we can write $\hat{\iota}(c)$ as a transfinite composition of cobase changes of the relative cell complexes $\hat{\iota}(b)$, for $b \in \mathbb{B}$. The induced cell structure is precisely the one described in the statement of the proposition. It is not hard to verify, that the thus constructed functor fulfills the vertical composition law stated above. Now, for the uniqueness statement, we show that the set of cells $\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{c})}$ associated to a structured relative cell complex $\mathfrak{c}$ needs to contain $\bigcup_{(b,\sigma)\in\mathfrak{C}_\mathfrak{c}} G(\sigma)\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})}$. Using the first part of this proof, we then obtain an alternative structured relative cell complex $\mathfrak{c}'\colon A \to X$, together with a morphism inclusion of cell complexes $\mathfrak{c}' \hookrightarrow \mathfrak{c}$, given by the identity on objects. By Corollary 8.1.4.5, it follows that $\mathfrak{c}' = \mathfrak{c}$. To see that this containment holds, again choose a filtration-presentation

$$A = X^0 \to \dots \to X^\lambda \cong X,$$

of $\mathfrak{c}\colon A \to X$ as above, and proceed to prove the statement via transfinite induction over λ. For $\lambda = 1$, i.e., the case of a single cell, the claim is immediate since, by the functoriality of $\hat{\mathfrak{i}}$, the induced diagram

$$\begin{tikzcd} F(D) \cup_{F(\partial D)} G(\partial D) \arrow[r] \arrow[d, hook, "\hat{\mathfrak{i}}(\mathfrak{b})"'] & F(X) \cup_{F(A)} G(A) \arrow[d, hook, "\hat{\mathfrak{i}}(\mathfrak{c})"] \\ G(D) \arrow[r, "G(\sigma)"'] & G(X) \end{tikzcd} \tag{8.78}$$

defines a morphism of structured relative cell complexes. In the case where λ is a limit ordinal, for every cell (b, σ) of $\mathfrak{c}$, there is some X^α, $\alpha < \lambda$, through which σ factors. Using Proposition 8.1.3.1 and Observation 8.1.3.5, we may factor $\mathfrak{c}$ into a subcomplex $\mathfrak{c}^\alpha\colon A \to X^\alpha$, together with an inclusion of subcomplexes $i\colon \mathfrak{c}^\alpha \hookrightarrow \mathfrak{c}$, and we can think of σ as a cell of $\mathfrak{c}^\alpha$. We denote this factorization of σ by σ'. By the functoriality of $\hat{\mathfrak{i}}$, the diagram

$$\begin{tikzcd} F(X^\alpha) \cup_{F(A)} G(A) \arrow[r] \arrow[d, "\hat{\mathfrak{i}}(\mathfrak{c}^\alpha)"] & F(X) \cup_{F(A)} G(A) \arrow[d, "\hat{\mathfrak{i}}(\mathfrak{c})"] \\ G(X^\alpha) \arrow[r] & G(X) \end{tikzcd} \tag{8.79}$$

defines a morphism of cell complexes. By inductive assumption $G(\sigma')\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})}$ is contained in the set of cells of the left hand side morphism. It follows, that

$$G(i)G(\sigma')\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})} = G(\sigma)\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})}$$

is contained in the cell of the right hand vertical, as was to be shown. Finally, the case of a successor ordinal follows from the vertical composition rule for cellularized functors, as well as the case of a single cell. □

Corollary 8.2.2.6. *In the special case of a relative functor $\iota: \varnothing \to F$, which we may just think of as a functor F, we obtain from Lemma 8.2.2.5 a unique lift*

$$\begin{tikzcd} \mathbf{RCell}(\mathbf{C}) \arrow[r, dashed, "\hat{\mathfrak{F}}"] \arrow[d] & \mathbf{RCell}(\mathbf{D}) \arrow[d] \\ \mathbf{C}^{[1]} \arrow[r, "F\circ-"] & \mathbf{D}^{[1]} \end{tikzcd} \tag{8.80}$$

which also behaves functorial under vertical composition.

Example 8.2.2.7. The cellularized identity functor $\mathbb{1}_{\mathbf{C}}$ of a cellularized category $\mathbf{C}$, is given by equipping $\iota: \varnothing \Rightarrow 1_{\mathbf{C}}$ with the cell structures given by the canonical isomorphism

$$\hat{\iota}(b) = (\partial D \cup_{\varnothing} \varnothing \to D) \cong (\partial D \xrightarrow{b} D)$$

for $b: \partial D \to D \in \mathbb{B}$.

Example 8.2.2.8. Not every relevant functor of structured cell complexes necessarily arises by cellularizing a functor of the underlying categories. Consider, for example, the reduced cellular chain-complex functor

$$\mathrm{C}: \mathbf{Cell}(\mathbf{Filt}) \to \mathbf{Cell}(\mathbf{Ch}(\mathbb{Z})_{\geq 0})$$

mapping a CW-complex $\mathfrak{X}$ to is cellular chain complex $\mathrm{C}_\bullet(\mathfrak{X})$, with the cell structure on the cellular chain complex $\mathrm{Cell}_\bullet(\mathfrak{X})$ given by the canonical basis induced by the cells of $\mathfrak{X}$. It can be obtained as a lift of the functor

$$\mathbf{Filt} \to \mathbf{Ch}(\mathbb{Z})_{\geq 0}$$

which sends a filtered space

$$\varnothing \xrightarrow{f_0} X^0 \xrightarrow{f^1} X^1 \to \dots$$

to the chain complex whose n-th entry is the reduced homology of the homotopy cofiber of f_n

$$\tilde{H}_n(C(f_n))$$

with boundary maps induced by the Barrat-Puppe maps

$$C(f_n) \to \Sigma(X_{n-1}) \to \Sigma(C(f_{n-1})).$$

This functor is, however, not colimit preserving (homology does not preserve pushouts), and it seems likely that a case can be made that no colimit preserving extension of the cellular chain-complex to **Filt** exists. (At least if we assume preservation of monomorphisms, then one can derive non-existence from the fact that not every pushout of inclusions of spaces induces a homotopy pushout of chain complexes.)

Cellularized functors have the following elementary properties, each of which are direct consequences of Recollection 8.2.1.3 and Lemma 8.2.2.5.

Corollary 8.2.2.9. *Given a cellularized functor* $\mathfrak{i}\colon F \Rightarrow G$*, then the functor*

$$\hat{\mathfrak{i}}\colon \mathbf{RCell}(\mathbf{C}) \to \mathbf{RCell}(\mathbf{D})$$

has the following properties:

1. *It is compatible with (transfinite) vertical composition in the sense that the analogue of Recollection 8.2.1.3 holds (where all pushouts are replace by cobase change squares and all transfinite compositions by transfinite vertical compositions of structured relative cell complexes).*
2. *It preserves cobase change morphisms.*
3. *If* $f\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ *is injective on cells, then so is* $\hat{\mathfrak{i}}(f)$*.*

We can now arrange cellularized categories and relative cellularized functors into a bicategory (see [JY20] for an introduction to the theory and language of bicategories) as follows. First, let us define the category of cellularized functors with fixed source and target:

Definition 8.2.2.10. By a morphism of relative cellularized functors $\mathfrak{i}_0\colon F_0 \to G_0$ and $\mathfrak{i}_1\colon F_1 \to G_1$, we mean a pair of natural transformations $\eta_F\colon F_0 \Rightarrow F_1$ and $\eta_G\colon G_0 \Rightarrow G_1$, such that the diagram

$$\begin{tikzcd} F_0 \arrow[r, Rightarrow, "\eta_F"] \arrow[d, Rightarrow, "\iota_0"'] & F_1 \arrow[d, Rightarrow, "\iota_1"] \\ G_0 \arrow[r, Rightarrow, "\eta_G"'] & G_1 \end{tikzcd} \tag{8.81}$$

commutes, and such that, for every relative cell complex $\mathfrak{c} \in \mathbf{RCell}(\mathbf{C})$, the induced morphism

$$\hat{\mathfrak{i}_0}(\mathfrak{c}) \to \hat{\mathfrak{i}_1}(\mathfrak{c})$$

is a morphism of structured relative cell complexes. Given two cellularized categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\mathbf{CellCat}^{\to}(\mathbf{C}, \mathbf{D})$ the 1-category of cellularized functors and morphisms of cellularized functors (with the obvious choice of composition and identities).

Observation 8.2.2.11. By the same inductive argument used in the proof of Lemma 8.2.2.5 it follows that to verify whether a pair of natural transformations (η_F, η_G) as in Definition 8.2.2.10 defines a morphism of cellularized functors, it suffices to verify that the induced morphism

$$\hat{\mathfrak{i}_0}(\mathfrak{b}) \to \hat{\mathfrak{i}_1}(\mathfrak{b})$$

is a morphism of structured relative cell complexes for all $b \in \mathbb{B}$.

8.2.3 Categories of cellularized categories

It will be useful to think of cellularized categories and relative cellularized functors as themselves forming a kind of category. The following subsection is mainly bookkeeping (even more so than the previous one). While conceptually nice to have, and useful in some arguments in simple homotopy theory, parts of it are unsatisfying in the sense that there is no non-surprising argument in sight, and in fact, we will omit most of the straightforward verifications. We recommend returning to this section when a reference to it occurs later in the text.
To assemble all cellularized categories into one larger category, we may modify the Leibniz category of Construction 8.2.1.5 as follows:

Construction 8.2.3.1. We denote by $\mathbf{CellCat}^{\to}$ the following bicategory, obtained by equipping the constructions in **Leib** with cell structures.

- Objects are given by (sufficiently small) cellularized categories $\mathbf{C}$.

- The category of morphism from $\mathbf{C}$ to $\mathbf{D}$, is given by the category of relative cellularized functors $\mathbf{CellCat}^{\to}(\mathbf{C},\mathbf{D})$. The identity object in $\mathbf{CellCat}^{\to}(\mathbf{C},\mathbf{C})$ is given by $\imath_{\mathbf{C}}\colon\varnothing\to 1_{\mathbf{C}}$.
- The composition functor

$$\mathbf{CellCat}^{\to}(\mathbf{D},\mathbf{E}))\times\mathbf{CellCat}^{\to}(\mathbf{C},\mathbf{D})\to\mathbf{CellCat}^{\to}(\mathbf{C},\mathbf{E})$$

  is given by cellularizing the Leibniz compositions $\iota_1\hat{\circ}\iota_0$ (see Construction 8.2.1.5) via the canonical isomorphisms

$$\hat{\iota_1}(\hat{\iota_0}(b))\cong\big(\widehat{(-)}(\iota_1\hat{\circ}\iota_0)\big)(b)$$

  for $b\in\mathbb{B}_{\mathbf{C}}$.
- Unitors and associators are inherited from **Leib**.

We may then think of

$$\begin{aligned}\mathbf{RCell}(-)\colon\mathbf{CellCat}^{\to}&\to\mathbf{Cat}\\(\mathbf{C},\mathbb{B})&\mapsto\mathbf{RCell}(\mathbf{C})\\\mathfrak{i}&\mapsto\hat{\mathfrak{i}}\end{aligned}$$

as a functor of bicategories.
We denote by **CellCat** the wide subcategory, of (absolute) cellularized functors. Note that **CellCat** can be made into a strict bicategory, by simply omitting the initial objects of absolute cellularized functors, and observing that in this case Leibniz composition comes down to regular composition. Restricting to absolute cellularized functors, we obtain an induced functor of strict bicategories

$$\begin{aligned}\mathbf{Cell}(-)\colon\mathbf{CellCat}&\to\mathbf{Cat}\\\mathbf{C}&\mapsto\mathbf{Cell}(\mathbf{C})\,.\end{aligned}$$

Observation 8.2.3.2. The two functors

$$\mathbf{CellCat}\xrightarrow{\mathbf{Cell}(-)}\mathbf{Cat};$$
$$\mathbf{CellCat}\hookrightarrow\mathbf{CellCat}^{\to}\xrightarrow{\mathbf{RCell}(-)}\mathbf{Cat}$$

are representable. Denote by $\star$ the terminal object in **Set**, given by a set with one element. By abuse of notation, we also denote the category with one object and one morphism by $\star$. The first of these two functors is represented by the cellularized category **Set** with the equivalence given via the evaluation functor

$$\begin{aligned}\mathbf{CellCat}(\mathbf{Set},\mathbf{C})&\to\mathbf{Cell}(\mathbf{C})\\\mathfrak{F}&\mapsto\mathfrak{F}(\star).\end{aligned}$$

That this defines an equivalence of categories follows by left Kan extension along the inclusion of the terminal category as the terminal object of **Set**, $\star\hookrightarrow\mathbf{Set}$, together with Lemma 8.2.2.5 and Observation 8.2.2.11.
To present the second functor, consider the category of arrows in **Set**, $\mathbf{Set}^{[1]}$, and equip it with the single generating boundary inclusion given by the (unique) natural transformation of representables

$$u:\Delta^1(1,-)\to\Delta^1(0,-).$$

Then, again using left Kan extension together with Lemma 8.2.2.5 and Observation 8.2.2.11, one obtains an equivalence of categories

$$\begin{aligned}\mathbf{CellCat}((\mathbf{Set}^{[1]},\{u\}),\mathbf{C})&\xrightarrow{\simeq}\mathbf{RCell}(\mathbf{C})\\\mathfrak{F}&\mapsto\mathfrak{F}(u)\,.\end{aligned}$$

Let us end this subsection with an observation on the invariance of cellularized categories under replacing a generating boundary inclusion by an isomorphic boundary inclusion.

Remark 8.2.3.3. Especially in the case of the category of topological spaces, the choice of generating boundary inclusions and generating expansions is extremely non-canonical. For example, it seems to depend on what precise choice for the disk D^n we have in mind. We could use the subspace of $\mathbb{R}^n$ given by the vectors of Euclidean norm lesser or equal to 1, but we may just as well use the n-dimensional cube $[0,1]^n$ or, which seems preferable from the perspective of simplicial homotopy theory, the topological n-simplex $|\Delta^n|$. These choices are essentially irrelevant, however. To see this, observe the following. Suppose we are given a category $\mathbf{C}$, equipped with two different sets of generating boundary inclusions $\mathbb{B}_0$ and $\mathbb{B}_1$, which both make it a cellularized category. We denote the resulting cellularized categories by $\mathbf{C}_0$ and $\mathbf{C}_1$, respectively. Suppose, furthermore, that we are given the data of

1. A bijection $\Phi: \mathbb{B}_0 \to \mathbb{B}_1$;
2. For every $b \in \mathbb{B}_0$, a choice of isomorphism of arrows $\phi^b: \Phi(b) \cong b$.

For $b \in \mathbb{B}$, let us denote the induced isomorphism on targets associated to ϕ^b by ϕ^b_1. Let F be the identity functor

$$\mathbf{C} \to \mathbf{C},$$

which we denote F, to make clear if we think about the source or the target category. Under Lemma 8.2.2.5, we obtain a cellularization of F, by equipping $F(b: \partial D^b \to D^b)$, for $b \in \mathbb{B}_0$ with the cell structure given by the single cell $\phi^b_1: D^{\Phi(b)} \xrightarrow{\cong} D^b$. The resulting cellularized functor $\mathfrak{F}$ is an isomorphism of cellularized categories. An inverse is constructed, by applying the analogous construction to Φ^{-1} and $\big((\phi^{\Phi^{-1}(b)})^{-1}\big)_{b \in \mathbb{B}_1}$. This shows that the associated cellularized category is effectively independent of the choice of representative of the isomorphism class of generating boundary inclusions.

8.2.4 Operations of cellularized functors

It is a general principle in category theory that a functor category with targets $\mathbf{D}$ generally inherits a large class of constructions and properties from $\mathbf{D}$. Such a claim also holds in the cellularized world. Namely, the category of cellularized functors $\mathbf{CellCat}^{\to}(\mathbf{C}, \mathbf{D})$ between two cellularized categories behaves itself much like a cellularized category. To be more precise, it inherits notions of cobase change, vertical (transfinite) compositions and subcomplexes.

Construction 8.2.4.1. Consider the left evaluation functor

$$\begin{aligned} \mathbf{CellCat}^{\to}(\mathbf{C}, \mathbf{D}) &\to \mathbf{Cat}(\mathbf{C}, \mathbf{D}) \\ (F \overset{\iota}{\Rightarrow} G, \hat{\mathfrak{i}}) &\mapsto F \\ (\eta_F, \eta_G) &\mapsto \eta_F \end{aligned}$$

It is a cocartesian fibration. Indeed, given a relative cellularized functor $(F \overset{\iota}{\Rightarrow} G, \hat{\mathfrak{i}})$ and given a natural transformation $\eta: F \Rightarrow F'$ (with F' colimit preserving), we can consider a pushout square of functors

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\iota"] & F' \arrow[d, Rightarrow, "\iota'"] \\ G \arrow[r, Rightarrow, "\eta'"] & G'\,. \end{tikzcd} \tag{8.82}$$

Then, by Recollection 8.2.1.7, for every relative cell complex $\mathfrak{c}: A \hookrightarrow X$ in $\mathfrak{c}$ the induced square

$$\begin{tikzcd} F(X) \cup_{F(A)} G(A) \arrow[r] \arrow[d, "\iota"] & F'(X) \cup_{F'(A)} G'(A) \arrow[d] \\ G(X) \arrow[r] & G'(A)\,. \end{tikzcd} \tag{8.83}$$

is again pushout. In particular, the right hand vertical inherits the structure of a relative cell complex from the left hand vertical. In this fashion, ι' inherits the structure of a relative cellularized functor, and (η, η') becomes a morphism of relative cellularized functors. The lift (η, η') of η obtained in this fashion is by construction cocartesian. We also call this construction the *cobase change* of cellularized functors, and will denote it in the form $\eta_{\mathfrak{i}}$.

Construction 8.2.4.2. Cellularized relative functors also admit a notion of vertical composition. Given two cellularized relative functors $\mathfrak{i}_0 \colon F \Rightarrow G$ and $\mathfrak{i}_1 \colon G \Rightarrow H$, a cell structure on $\iota_1 \circ \iota_0$ is obtained as follows. Given $b \in \mathbb{B}$, consider the commutative diagram

$$\begin{tikzcd}
F(D) \cup_{F(\partial D)} G(\partial D) \arrow[r] \arrow[d, hook, "\hat{\mathfrak{i}}_0(\mathfrak{b})"'] \arrow[dr, phantom, "\ulcorner" very near end] & F(D) \cup_{F(\partial D)} H(\partial D) \arrow[d, hook] \arrow[ddr, bend left, "\widehat{\iota_1 \circ \iota_0}(b)"] & \\
G(D) \arrow[r] & G(D) \cup_{G(\partial D)} H(\partial D) \arrow[dr, hook, "\hat{\mathfrak{i}}_1(\mathfrak{b})"'] & \\
 & & H(D)\,,
\end{tikzcd} \tag{8.84}$$

for $b \in \mathbb{B}$. The left square is a pushout. In particular, we may equip the right vertical with the relative cell structure induced via the cobase change. Then, $\widehat{\iota_1 \circ \iota_0}(b)$ can be equipped with the cell structure given by vertical composition of the induced relative cell complex with $\hat{\mathfrak{i}}_1(\mathfrak{b})$. By Lemma 8.2.2.5, this induces the structure of a cellularized functor on $\iota_1 \circ \iota_0$. One can verify that the resulting cell structure is such that for any $\mathfrak{c} \in \mathbf{RCell(C)}$, $(\widehat{\mathfrak{i}_1 \circ \mathfrak{i}_0})(\mathfrak{c})$ is given by the right vertical composition

$$\begin{tikzcd}
F(X) \cup_{F(A)} G(A) \arrow[r, "F(X)\cup_{F(A)}(\iota_1)"] \arrow[d, hook, "\hat{\mathfrak{i}}_0(\mathfrak{c})"'] \arrow[dr, phantom, "\ulcorner" very near end] & F(X) \cup_{F(A)} H(A) \arrow[d, hook, "(F(X)\cup_{F(A)}(\iota_1))_{\mathfrak{i}}\hat{\mathfrak{i}}_0(\mathfrak{c})"'] \arrow[ddr, bend left, "(\widehat{\mathfrak{i}_1 \circ \mathfrak{i}_0})(\mathfrak{c})"] & \\
G(X) \arrow[r] & G(X) \cup_{G(A)} H(A) \arrow[dr, "\hat{\mathfrak{i}}_1(\mathfrak{c})"'] & \\
 & & H(X)\,,
\end{tikzcd} \tag{8.85}$$

where the left square is a cobase change. In particular, by Proposition 8.1.3.1, ι_1 induces a morphism of cellularized functors $\mathfrak{i}_0 \to \mathfrak{i}_1 \circ \mathfrak{i}_0$, such that for every $\mathfrak{c} \in \mathbf{RCell(C)}$, the induced morphism of structured relative cell complexes

$$\hat{\mathfrak{i}}_0(\mathfrak{c}) \to \widehat{\mathfrak{i}_1 \circ \mathfrak{i}_0}(\mathfrak{c})$$

induces an injection on the level of cells.

Construction 8.2.4.2 together with Proposition 8.1.3.1, suggests the following definition of a cellularized subfunctor:

Definition 8.2.4.3. A morphism of cellularized functors $(1, \eta) \colon (F \overset{\mathfrak{i}}{\Rightarrow} \tilde{G}) \to (F \overset{\mathfrak{j}}{\Rightarrow} G)$ is called an *inclusion of cellularized relative functors*, if one of the following equivalent conditions holds.

1. The cellularized functor $\mathfrak{j}$ is given by the vertical composition of $\tilde{\mathfrak{i}}$ and a (necessarily unique) cell structure on $\tilde{G} \overset{\eta}{\Rightarrow} G$.
2. For $b \in \mathbb{B}_{\mathbf{C}}$, the induced morphism on cells $\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{b})} \to \mathfrak{C}_{\hat{\mathfrak{j}}(\mathfrak{b})}$ is injective on the level of cells.
3. For $\mathfrak{c} \in \mathbf{RCell(C)}$, the induced morphism on cells $\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{c})} \to \mathfrak{C}_{\hat{\mathfrak{j}}(\mathfrak{c})}$ is injective.

Observation 8.2.4.4. By Proposition 8.1.3.1, an inclusion of (absolute) cellularized functors $\eta\colon \mathfrak{F} \hookrightarrow \mathfrak{G}$ induces inclusions of absolute cell complexes

$$\mathfrak{F}(\mathfrak{X}) \hookrightarrow \mathfrak{G}(\mathfrak{X})$$

for every absolute cell complex $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C})$.

We may now use Observation 8.1.3.6, to define a notion of transfinite composition of cellularized functors.

Construction 8.2.4.5. Let $\mathbf{C}$ and $\mathbf{D}$ be cellularized functors. Suppose we are given a transfinite composition of functors from $\mathbf{C}$ to $\mathbf{D}$

$$F = F^0 \Rightarrow F^1 \Rightarrow \cdots \Rightarrow F^\lambda = G$$

together with, for each $\alpha < \alpha' \in \lambda$, the structure of a relative cellularized functor on

$$\iota_{\alpha,\alpha'}\colon F^\alpha \Rightarrow F^{\alpha'}$$

compatible under vertical composition. By Definition 8.2.4.3, we may equivalently think of such a diagram as a diagram of inclusions of cellularized subfunctors $\mathfrak{i}^\alpha\colon F \to F^\alpha$

$$\mathfrak{i}^0 \hookrightarrow \mathfrak{i}^1 \hookrightarrow \mathfrak{i}^2 \dots .$$

The colimit of this diagram exists, and specifies the structure of a cellularized functor $\mathfrak{i}$ on $F \Rightarrow G$. For $\mathfrak{c} \in \mathbf{RCell}(\mathbf{C})$, the cell structure $\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{c})}$ is explicitly given by

$$\bigcup_{\alpha<\lambda} \iota_{\alpha+1,\lambda} \mathfrak{C}_{\hat{\mathfrak{i}}^{\alpha+1}(\mathfrak{c})} \, .$$

Lemma 8.2.4.6. *Suppose we are given a solid span of absolute cellularized functors*

$$\begin{tikzcd} \mathfrak{F} \arrow[r, "\eta"] \arrow[d, hook, "\iota"] & \mathfrak{F}' \arrow[d, hook, dotted, "\iota'"] \\ \mathfrak{G} \arrow[r, dotted] \arrow[ur, phantom, "\ulcorner"] & \eta_{\mathfrak{i}}\mathfrak{i} \circ \mathfrak{F}' \end{tikzcd} \tag{8.86}$$

in $\mathbf{CellCat}(\mathbf{C}, \mathbf{D})$ *where the vertical morphism is an inclusion of cellularized functors. Then the pushout of this span exists and is given by equipping G' in the pushout diagram of functors*

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\iota"'] & F' \arrow[d, "\iota'"] \\ G \arrow[r, "\eta'"] \arrow[ur, phantom, "\ulcorner"] & G' \end{tikzcd} \tag{8.87}$$

with the cellularization given by $\eta_{\mathfrak{i}}\mathfrak{i} \circ \mathfrak{F}'$, where $\mathfrak{i}$ is the relative cellularized functor associated to the inclusion $i\colon \mathfrak{F} \hookrightarrow \mathfrak{G}$. In other words, the complete diagram of functors Diagram (8.86) *is a pushout.*

Proof. Unravelling the definitions, observe that if evaluate Diagram (8.86) at a structured relative cell complex $\mathfrak{c}\colon A \hookrightarrow X$, the cell structure on $\eta_{\mathfrak{i}}\mathfrak{i} \circ \mathfrak{F}'(\mathfrak{c})$ is given by the set

$$\iota' \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})} \cup \eta' \mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{c})}.$$

Since $\eta \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})} \subset \mathfrak{C}_{\mathfrak{F}'(\mathfrak{c})}$ and $\mathfrak{C}_{\hat{\mathfrak{i}}(\mathfrak{c})} = \mathfrak{C}_{\mathfrak{G}(\mathfrak{c})} \smallsetminus \iota \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})}$, we may furthermore rewrite this union as

$$\begin{aligned} (\iota' \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})} \cup \iota' \eta \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})}) \cup \eta' (\mathfrak{C}_{\mathfrak{G}(\mathfrak{c})} \smallsetminus \iota \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})}) &= (\iota' \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})}) \cup \eta' \iota \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})} \cup \eta' (\mathfrak{C}_{\mathfrak{G}(\mathfrak{c})} \smallsetminus \iota \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})}) \\ &= (\iota' \mathfrak{C}_{\mathfrak{F}(\mathfrak{c})}) \cup \eta' \mathfrak{C}_{\mathfrak{G}(\mathfrak{c})} \, . \end{aligned}$$

In particular, it follows that Diagram (8.86) is indeed a well-defined commutative diagram in $\mathbf{CellCat}(\mathbf{C},\mathbf{D})$. Furthermore, using Lemma 8.1.4.7, it also follows from this that for each $\mathfrak{c} \in \mathbf{RCell}(\mathbf{C})$, the induced diagram

$$\begin{tikzcd} \mathfrak{F}(\mathfrak{c}) \arrow[r, "\eta"] \arrow[d, hook, "\iota"] & \mathfrak{F}'(\mathfrak{c}) \arrow[d, hook, dotted, "\iota'"] \\ \mathfrak{G}(\mathfrak{c}) \arrow[r, dotted] \arrow[ur, phantom, "\ulcorner"] & \eta_{\mathfrak{i}}\mathfrak{i} \circ \mathfrak{F}'(\mathfrak{c}) \end{tikzcd} \tag{8.88}$$

is a pushout. From this, one may immediately derive that $\eta_{\mathfrak{i}}\mathfrak{i} \circ \mathfrak{F}'$ does not only fulfill the universal property of the pushout in $\mathbf{Fun}(\mathbf{C},\mathbf{D})$, but that for any solid commutative diagram in $\mathbf{CellCat}(\mathbf{C},\mathbf{D})$

$$\begin{tikzcd} \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F}' \arrow[ddr, bend left] & \\ \mathfrak{G} \arrow[drr, bend right] & & \\ & & \mathfrak{H} \end{tikzcd} \tag{8.89}$$

the unique dashed natural transformation

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[d] \arrow[ddr, bend left] & \\ G \arrow[r] \arrow[drr, bend right] & G' \arrow[dr, dashed] & \\ & & H \end{tikzcd} \tag{8.90}$$

making the diagram commute, defines a morphism of cellularized functors $\eta_{\mathfrak{i}}\mathfrak{i} \circ \mathfrak{F}' \to \mathfrak{H}$, which (by the faithfulness of the functor forgetting cellularization) is unique with the property that it makes the diagram

$$\begin{tikzcd} \mathfrak{F} \arrow[r] \arrow[d, hook] & \mathfrak{F}' \arrow[d, hook] \arrow[ddr, bend left] & \\ \mathfrak{G} \arrow[r] \arrow[ur, phantom, "\ulcorner"] \arrow[drr, bend right] & \eta_{\mathfrak{i}}\mathfrak{i} \circ \mathfrak{F}' \arrow[dr, dashed] & \\ & & \mathfrak{H} \end{tikzcd} \tag{8.91}$$

commute. □

8.2.5 Cellularized bifunctors

Many examples of cellularized functors, for example ones arising from actions of the category of simplicial sets on a cofibrantly generated model category, arise by fixing one of the variables in a bivariate functor.

Notation 8.2.5.1. Given a bivariate functor $-\otimes -\mathbf{C}\colon \times \mathbf{D} \to \mathbf{E}$ into a category with finite colimits $\mathbf{E}$, we denote by

$$-\hat{\otimes}-\colon \mathbf{C}^{[1]} \times \mathbf{D}^{[1]} \to \mathbf{E}^{[1]}$$

the functor obtained by mapping a pair of arrows $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ to the canonical induced arrow

$$X_1 \otimes Y_0 \cup_{X_0 \otimes Y_0} X_0 \otimes Y_1 \to X_1 \otimes Y_1$$

with functoriality induced by the universal property of the pushout.

Observation 8.2.5.2. Observe that the notation for Leibniz tensors is compatible with the notation of the Leibniz construction in Notation 8.2.1.2, in the sense that the Leibniz construction $\widehat{f \otimes -}$ associated to the relative functor $X_0 \otimes - \xrightarrow{f\otimes -} X_1 \otimes -$, for an arrow $f \colon X_0 \to X_1$ in $\mathbf{C}$, is equivalently given by the functor $f\hat{\otimes} - \colon \mathbf{D}^{[1]} \to \mathbf{E}^{[1]}$ obtained by fixing f.

Definition 8.2.5.3. Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be cellularized categories. By a *cellularized bifunctor* of celularized categories, $\underline{\otimes} \colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$, we mean a bifunctor

$$- \otimes - \colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}\,,$$

preserving colimits in both arguments, together with a fixed lift

$$\begin{tikzcd} \mathbf{RCell}(\mathbf{C}) \times \mathbf{RCell}(\mathbf{D}) \arrow[r, dashed, "\hat{\underline{\otimes}}"] \arrow[d] & \mathbf{RCell}(\mathbf{E}) \arrow[d] \\ \mathbf{C}^{[1]} \times \mathbf{D}^{[1]} \arrow[r, "-\hat{\otimes}-"] & \mathbf{E}^{[1]} \end{tikzcd} \tag{8.92}$$

such that, for each $A_1 \overset{\mathfrak{c}_1}{\hookrightarrow} X_1 \in \mathbf{RCell}(\mathbb{B}_{\mathbf{C}})$ and $A_2 \overset{\mathfrak{c}_2}{\hookrightarrow} X_2 \in \mathbf{RCell}(\mathbb{B}_{\mathbf{D}})$, the induced functors

$$\mathfrak{c}_1 \hat{\underline{\otimes}} -, - \hat{\underline{\otimes}} \mathfrak{c}_2$$

do, respectively, define the structure of a cellularized functor on the relative functors $c_1 \otimes -$ and $- \otimes c_2$. The lift $-\hat{\underline{\otimes}}-$ is called the *cellularization* $- \otimes -$.

Observation 8.2.5.4. It follows by Lemma 8.2.2.5 that a choice of cellularization of bifunctor $- \otimes - \colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is specified uniquely by the data of a choice of cell structure $\mathfrak{C}_{\mathfrak{b}_1 \hat{\underline{\otimes}} \mathfrak{b}_2}$ on

$$b_1 \hat{\otimes} b_2 \colon D_1 \otimes \partial D_2 \cup_{\partial D_1 \otimes \partial D_2} \partial D_1 \otimes D_2 \to D_1 \otimes D_2\,,$$

for each pair of boundary inclusions $\partial D_1 \overset{b_1}{\hookrightarrow} D_1 \in \mathbb{B}_{\mathbf{C}}$ and $\partial D_2 \overset{b_2}{\hookrightarrow} D_2 \in \mathbb{B}_{\mathbf{D}}$.

Example 8.2.5.5. Let **Set** be the cellularization of the category of sets given by the single boundary inclusion $\varnothing \hookrightarrow \star$. Let $\mathbf{C}$ be a cellularized category. Consider the canonical actions of **Set** on $\mathbf{C}$, that is, the bifunctor

$$- \ast - \colon \mathbf{Set} \times \mathbf{C} \to \mathbf{C}$$

given by mapping $(S, X) \mapsto \bigsqcup_{s \in S} X$, with functoriality defined by mapping $(f, g) \colon (S_0, X_0) \to (S_1, X_1)$ to the morphism

$$\bigsqcup_{s \in S_0} X_0 \to \bigsqcup_{t \in S_1} X_1,$$

mapping the component associated to $s \in S_0$ to the $f(s)$ component, via g. Observe that

$$\{\varnothing \to \star\} \hat{\otimes} a \cong a\,.$$

Hence, there is a canonical cell structure on $- \ast -$, given by equipping

$$\{\varnothing \to \star\} \hat{\ast} b \cong b$$

with the tautological cell structure, for $b \in \mathbb{B}$. Observe that this cellularization is such that for any relative cellularized functor $\mathfrak{i} \colon F \to G$, from a cellularized category $\mathbf{C}$ to a cellularized category $\mathbf{D}$, and every $a \in \mathbf{RCell}(\mathbf{Set})$ the canonical isomorphism, induced by the commutativity of F and G with coproducts,

$$(a \ast -) \hat{\circ} \mathfrak{i} \cong \mathfrak{i} \hat{\circ} (a \ast -)$$

(using the Leibniz composition of Construction 8.2.1.5) is an isomorphism of cellularized functors.

Example 8.2.5.6. The product functor

$$- \times -\colon \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$$

is canonically equipped with the structure of a cellularized bifunctor. To verify this, observe that every morphism in **sSet** can admit at most one cell structure, and is a relative cell complex if and only if it is a monomorphism (see also Section 8.3.5). From this, the claim that $- \times -$ has a unique structure of a cellularized bifunctor is easily verified.

The exponential law for functor categories has the following analogue in the world of cellularized functors:

Construction 8.2.5.7. Recall that in the 1-category of small categories, there is a isomorphism of categories

$$\begin{aligned} \Phi\colon \mathbf{Fun}(\mathbf{C} \times \mathbf{D}, \mathbf{E}) &\cong \mathbf{Fun}(\mathbf{C}, \mathbf{E}^{\mathbf{D}}) \\ F &\mapsto \{X \mapsto F(X, -)\} \end{aligned}$$

Under this isomorphism, a bivariate functor which is cocontinuous in both variables corresponds to a cocontinuous functor into the full subcategory $\mathbf{Cocon}(\mathbf{E}^{\mathbf{D}}) \subset \mathbf{E}^{\mathbf{D}}$ of cocontinuous functors. If we are given a cellularization $\underline{\otimes}$ of a bivariate functor $\otimes$ (cocontinuous in both variables), then, by definition, for any fixed relative cell complex $(A \overset{c}{\hookrightarrow} B) \in \mathbf{RCell}(\mathbf{C})$, the induced natural transformation

$$\Phi(\otimes)_c\colon A \otimes - \xrightarrow{c \otimes -} B \otimes -$$

is canonically equipped with the structure of a cellularized functor, and similarly any morphism of cell complexes is mapped to a transformation of cellularized functors.

In this fashion, we obtain a map

$$\underline{\Phi}\colon [CellBiFun](\mathbf{C} \times \mathbf{D}, \mathbf{E}) \to \mathbf{CellCat}^{\to}(\mathbf{D}, \mathbf{E})^{\mathbf{RCell}(\mathbf{C})},$$

where $[CellBiFun](\mathbf{C} \times \mathbf{D}, \mathbf{E})$ denotes the (possibly large) set of cellular bifunctors from $\mathbf{C} \times \mathbf{D}$ to $\mathbf{E}$. Observe, that given a fixed cellularized bifunctor $\underline{\otimes}$, the associated functor $R \coloneqq \underline{\Phi}(\underline{\otimes})$ has the following properties

(i) R preserves vertical composition.

(ii) The diagram

$$\begin{tikzcd} \mathbf{RCell}(\mathbf{C}) \arrow[r, "R"] \arrow[d] & \mathbf{CellCat}^{\to}(\mathbf{D}, \mathbf{E}) \arrow[d] \\ \mathbf{C}^{[1]} \arrow[r, "\Phi(\otimes)^{[1]}"] & \mathbf{Fun}(\mathbf{D}, \mathbf{E})^{[1]} \end{tikzcd} \tag{8.93}$$

commutes.

Proposition 8.2.5.8. *In the framework of Construction 8.2.5.7, the assignment $\underline{\otimes} \mapsto \underline{\Phi}(\underline{\otimes})$ induces a bijection between cellularizations of $-\otimes-$ and functors $F\colon \mathbf{RCell}(\mathbf{C}) \to \mathbf{CellCat}^{\to}(\mathbf{D}, \mathbf{E})$, that fulfill Properties (i) and (ii).*

Proof. The inverse is constructed by composing

$$F\colon \mathbf{RCell}(\mathbf{C}) \to \mathbf{CellCat}^{\to}(\mathbf{D}, \mathbf{E})$$

with the functor

$$\mathbf{RCell}(-)\colon \mathbf{CellCat}^{\to}(\mathbf{D}, \mathbf{E}) \to \mathbf{Fun}(\mathbf{RCell}(\mathbf{D}), \mathbf{RCell}(\mathbf{E})),$$

and then considering the associated functor

$$\mathbf{RCell}(\mathbf{C}) \times \mathbf{RCell}(\mathbf{D}) \to \mathbf{RCell}(\mathbf{E}).$$

By Property (ii), it specifies a lift of

$$-\hat{\otimes}-\colon \mathbf{C}^{[1]} \times \mathbf{D}^{[1]} \to \mathbf{E}^{[1]}.$$

Furthermore, by Property (i) and the assumption that R had image in cellularized functors, it follows that this lift defines a cellularization of $-\otimes-$. One may now verify that this construction defines the required inverse. ☐

Finally, let us observe the following properties of the bijection constructed in Construction 8.2.5.7, the elementary verifications of which we will not perform here.

Lemma 8.2.5.9. *Given a cellularized bifunctor $\underline{\otimes}$, as in Construction 8.2.5.7, the associated functor $\underline{\Phi}(\underline{\otimes})\colon \mathbf{RCell}(\mathbf{C}) \to \mathbf{CellCat}^{\to}(\mathbf{D},\mathbf{E})$ has the following properties:*

1. *It maps absolute cell complexes into absolute cellularized functors.*
2. *It maps cobase changes into cobase changes.*
3. *It maps inclusions of relative cell complexes into inclusions of cellularized functors.*
4. *It maps (transfinite) vertical composition into (transfinite) vertical compositions.*

Next, let us categorify the class of cellularized bifunctors.

Construction 8.2.5.10. Given three cellularized categories $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, we equip the class of cellular bifunctors

$$\mathbf{CellBiFun}(\mathbf{C} \times \mathbf{D}, \mathbf{E})$$

with the structure of a category, by taking as morphisms such natural transformations $\eta\colon \otimes_0 \Rightarrow \otimes_1$, which have the property that, for each pair of relative cell complexes $\mathfrak{c} \in \mathbf{RCell}(\mathbf{C})$ and $\mathfrak{d} \in \mathbf{RCell}(\mathbf{D})$, the induced morphism

$$c\hat{\otimes}_0 d \to c\hat{\otimes}_1 d$$

defines a morphism of structured relative cell complexes. In analogy to Observation 8.2.2.11, this holds for all $\mathfrak{c}, \mathfrak{d}$ as above if and only if it holds for generating boundary inclusions $b_1 \in \mathbb{B}_{\mathbf{C}}$ and $b_2 \in \mathbb{B}_{\mathbf{D}}$.

Observation 8.2.5.11. One would expect that a cellularized bifunctor with no degree of freedom in the second argument should be essentially the same as a cellularized functor. Indeed, similarly to Observation 8.2.3.2, one can use the fact that the category **Set** is generated under coproducts by the singleton to see that the functor

$$\begin{aligned} \mathbf{CellBiFun}(\mathbf{C} \times \mathbf{Set}, \mathbf{D}) &\to \mathbf{CellCat}(\mathbf{C}, \mathbf{D}) \\ \underline{\otimes} &\mapsto -\underline{\otimes} \star \ . \end{aligned}$$

induces an equivalence of categories. In this sense, **Set** behaves like a unital object, if cellularized bifunctors were indeed presented by a monoidal structure. Similarly, using the cellularized category $(\mathbf{Set}^{[1]}, \{u\})$ of Observation 8.2.3.2, one obtains an equivalence of categories

$$\begin{aligned} \mathbf{CellBiFun}(\mathbf{C} \times (\mathbf{Set}^{[1]}, \{u\}), \mathbf{D}) &\xrightarrow{\simeq} \mathbf{CellCat}^{\to}(\mathbf{C}, \mathbf{D}) \\ \underline{\otimes} &\mapsto -\underline{\otimes} u \, . \end{aligned}$$

Observation 8.2.5.12. Finally, observe that, given a cellularized bifunctor $\underline{\otimes}\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ and cellularized functors $\mathfrak{F}\colon \mathbf{C}_0 \to \mathbf{C}$ and $\mathfrak{G}\colon \mathbf{E} \to \mathbf{E}_1$, the composed bifunctor

$$G(- \otimes F(-))$$

inherits a canonical cellularization through the composition

$$\mathbf{RCell}(\mathbf{C}_0) \times \mathbf{RCell}(\mathbf{D}) \xrightarrow{\hat{\mathfrak{F}} \times 1} \mathbf{RCell}(\mathbf{C}) \times \mathbf{RCell}(\mathbf{D}) \xrightarrow{\hat{\otimes}} \mathbf{RCell}(\mathbf{E}) \xrightarrow{\hat{\mathfrak{G}}} \mathbf{RCell}(\mathbf{E}_1)\,. \tag{8.94}$$

Setting $F = 1$, this composition is compatible with the Leibniz composition of cellularized relative functors insofar, as that for any fixed relative cell complex $\mathfrak{d} \in \mathbf{RCell}(\mathbf{D})$, there is a canonical isomorphism (in fact, an identity for appropriately defined choices of pushouts in the Leibniz composition) of relative cellularized functors between $\mathfrak{G}\hat{\circ}(-\underline{\otimes}\mathfrak{d})$ and the relative cellularized functor obtained by fixing $\mathfrak{d}$ in the second variable of $\mathfrak{G}(-\underline{\otimes}-)$.

8.3 Diagrams and colimits of cell complexes

We have seen in Corollary 8.1.4.8 that the category of absolute cell complexes $\mathbf{Cell}(\mathbf{C})$ associated to a cellularized category $\mathbf{C}$ has pushouts of spans of two inclusions of subcomplexes. More generally, using the language of cobase changes (Construction 8.1.2.7), we have also observed that given a more general (solid) span

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d, hook] & \mathfrak{X}' \arrow[d, hook, dotted] \\ \mathfrak{Y} \arrow[r, dotted] & \mathfrak{Y}' \end{tikzcd} \tag{8.95}$$

where only the vertical is an inclusion of subcomplexes and the horizontal is an arbitrary morphism in $\mathbf{C}$, the pushout in $\mathbf{C}$ inherits a canonical cell structure. We may ask this question more generally: For what kind of diagrams of structured cell complexes, using morphisms in $\mathbf{C}$, can we expect the colimit to again come with a canonical cell structure? This question is of particular interest when one is looking to understand the behavior of simple homotopy equivalences under colimits. One way to approach this question is to ask when the colimit functor, $\mathbf{C}^{\mathbf{I}} \to \mathbf{C}$, for some small indexing category $\mathbf{I}$, can be equipped with the structure of a cellularized functor. To even make this question well defined, we need a good theory of what a structured cell complex in $\mathbf{C}^{\mathbf{I}}$ should be. It turns out that such a theory was already developed in-depth in the context of model structures on functor categories: Namely, the theory of Reedy categories and their associated model structure (see [RV13; Hir03]). In this section, we will investigate the functor categories on a Reedy category from the perspective of cellularized categories and cellularized functors. This will strongly rely on the insights and calculus of [RV13; Bar07].

8.3.1 Weighted colimits

Before we focus on the setting of Reedy categories, let us recall some general notation and techniques concerning functor categories and weighted colimits which can, for example, be found in [RV13]. The latter article follows the paradigm "*It's all in the weights*". We will strongly embrace this philosophy in our study of cellularized diagram categories and the functors between them. Ultimately, this calculus will allow us to reduce all the statements we need to well-known results about presheaves. A detailed overview, also surveying claims below can also be found in [nLa24e].

Notation 8.3.1.1. Throughout this section, we will be dealing with functors of the form $X\colon \mathbf{R} \to \mathbf{C}$, where $\mathbf{R}$ is a small category, which we think of as diagrams in $\mathbf{C}$. For such functors, we denote the value of X at r by X^r, and the morphism induced by $f\colon r \to r'$ by X^f or by $X(f)$. In the contravariant case, i.e. the case of a functor $X\colon \mathbf{R}^{\mathrm{op}} \to \mathbf{C}$, we will use subscript notation – X_r, X_f – instead. We will sometimes also use $\bullet$-notation (in the form $X^\bullet$) to indicate that we treat a construction as a functor in the variable marked by $\bullet$. We are aware that this notation can, at times, be ambiguous, in the sense that it may sometimes be unclear whether one should insert a variable for $\bullet$ first, or perform some operation on the functor first. If in doubt, any operation is to be performed before a variable is inserted. Usually, there should be no room for confusion, however.

Notation 8.3.1.2. Given some indexing category $\mathbf{R}$, we will often denote the bivariate hom functor $\mathbf{R}(-,-)$ in the form

$$\mathbf{R}_r^{\overline{r}} \coloneqq \mathbf{R}(r, \overline{r}) \,.$$

Following this notation, we write $\mathbf{R}^{\overline{r}} \colon \mathbf{R}^{\mathrm{op}} \to \mathbf{Set}$ for the contravariant representable associated to $\overline{r}$ and $\mathbf{R}_r$ for the covariant representable.

Notation 8.3.1.3. Given a covariant functor $W \colon \mathbf{R} \to \mathbf{Set}$, we denote by $\mathbf{el}(W)$ its category of elements, whose objects are given by pairs (r, x), with $r \in \mathbf{R}$ and $x \in W^r$, and whose morphisms $(r, x) \to (r', x')$ are given by arrows $f \colon r \to r'$, such that $W^f(x) = x$. In the context of a fixed category $\mathbf{R}$, we will also denote by $\mathbf{el}(U)$ the category of elements of a contravariant functor $U \colon \mathbf{R} \to \mathbf{Set}$, i.e.,a covariant functor $U' \colon \mathbf{R}^{\mathrm{op}} \to \mathbf{Set}$, given by $\mathbf{el}(U) \coloneqq \mathbf{el}(U')^{\mathrm{op}}$. In particular, the forgetful functor $\mathbf{el}(U) \to \mathbf{R}$ is always covariant, no matter if U is co- or contravariant. Observe that for contravariant U, $\mathbf{el}(U)$ is equivalently the comma category $\mathbf{R}_{/U}$, under the Yoneda embedding $\mathbf{R} \hookrightarrow \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$.

Recollection 8.3.1.4. Let $\mathbf{R}$ be some small category, and $\mathbf{C}$ an arbitrary category.

1. Given a diagram $X \in \mathbf{C}^{\mathbf{R}}$ and a so-called *diagram of weights* $W \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ one may ask the question, whether the functor

$$\begin{aligned} \mathbf{C} &\to \mathbf{Set} \\ C &\mapsto \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}(W, \mathbf{C}(X^\bullet, C)) \end{aligned}$$

is representable. I.e., does there exist an object $U \in \mathbf{C}$, such that there is a canonical natural isomorphism

$$\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}(W, \mathbf{C}(X^\bullet, C)) \cong \mathbf{C}(U, C)?$$

For example, if W is the constant weight $r \mapsto \star$, then this is the defining universal property of the colimit of the diagram X in $\mathbf{C}$. If such an object exists we denote it in the form $W \circledast X$, and call it the *colimit of X weighted by W.* If $\mathbf{C}$ has all weighted colimits, indexed over $\mathbf{R}$, then applying the Yoneda lemma to the universal property makes $- \circledast -$ a bivariate functor

$$- \circledast - \colon \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \times \mathbf{C}^{\mathbf{R}} \to \mathbf{C} \,.$$

2. If (and only if) $\mathbf{C}$ has all colimits, then $W \circledast X$ always exists and is explicitly given by the coend formula

$$\int^{r \in \mathbf{R}} W_r * X^r = \mathrm{coequ}(\coprod_{f \colon r \to \overline{r} \in \mathbf{R}} (W_{\overline{r}} * X^r) \rightrightarrows \coprod_{r \in \mathbf{R}} (W_r * X^r))$$

with the two arrows induced respectively by the families of arrows

$$\{W_{\overline{r}} * X^{r_0} \xrightarrow{W_f * X^{r_0}} W_{r_0} * X^{r_0} \hookrightarrow \coprod_{r \in \mathbf{R}} (\coprod_{W_r} X^r))\}_{f \colon r_0 \to \overline{r} \in \mathbf{R}, w \in W_{\overline{r}}}$$

and

$$\{W_{\overline{r}} * X^{r_0} \xrightarrow{W_{\overline{r}} * X^f} W_{\overline{r}} * X^{\overline{r}} \hookrightarrow \coprod_{r \in \mathbf{R}} (\coprod_{W_r} X^r))\}_{f \colon r_0 \to \overline{r} \in \mathbf{R}, w \in W_{\overline{r}}} \,.$$

In the special case where $\mathbf{R} = \star$, and hence $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \cong \mathbf{Set}$ and $\mathbf{C}^{\mathbf{R}^{\mathrm{op}}} \cong \mathbf{C}$, we may thus identify $\circledast$ with the weighting tensor $*$ of Example 8.2.5.5.

3. There is another useful construction of the colimit of $X \in \mathbf{C}^{\mathbf{R}}$ weighted by $W \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, supposing that $\mathbf{C}$ is cocomplete. Namely, consider the covariant category of elements $\mathbf{el}(W)$ of W. Then $W \circledast X$ is equivalently given by the colimit of the composition

$$\mathbf{el}(W) \to \mathbf{R} \xrightarrow{X} \mathbf{C}\,.$$

4. Using this language, the Yoneda lemma gives us natural isomorphisms

$$\mathbf{C}(\mathbf{R}^r \circledast X, C) \cong \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}(\mathbf{R}^r, \mathbf{C}(X^\bullet, C)) \cong \mathbf{C}(X^r, C)$$

and thus a canonical isomorphism

$$\mathbf{R}^r \circledast X \cong X^r.$$

In this sense, taking weighted colimits by $\mathbf{R}^r$ is simply evaluation at r.

We are going to make use of the following expanded version of the weighted colimit, which, we will refer to as the *composition tensoring.*

Construction 8.3.1.5. Given a cocomplete category $\mathbf{C}$, and three small categories $\mathbf{R}$ and $\mathbf{T}$ and $\mathbf{S}$, consider the bi-functor

$$\begin{aligned} - \circledcirc -\colon \mathbf{Set}^{\mathbf{S}^{\mathrm{op}}\times\mathbf{T}} \times \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}} &\to \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}} \\ (W, X) &\mapsto \{(r,t) \mapsto W^t_\bullet \circledast X^\bullet_r\} \end{aligned}$$

with the obvious functoriality and action on morphisms induced by the functoriality of $\circledast$. It follows from the elementary properties of weighted colimits (specifically the Fubini-theorem found in [RV13], for example), that this construction is associative in the sense that, given another small category $\mathbf{U}$, there is a canonical natural isomorphism making the diagram

$$\begin{tikzcd} \mathbf{Set}^{\mathbf{T}^{\mathrm{op}}\times\mathbf{U}} \times \mathbf{Set}^{\mathbf{S}^{\mathrm{op}}\times\mathbf{T}} \times \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}} \arrow[r, "(-\circledcirc-)\circledcirc-"] \arrow[d, "-\circledcirc(-\circledcirc-)"'] \arrow[dr, phantom, "\cong"] & \mathbf{Set}^{\mathbf{S}^{\mathrm{op}}\times\mathbf{U}} \times \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}} \arrow[d, "-\circledcirc-"] \\ \mathbf{Set}^{\mathbf{T}^{\mathrm{op}}\times\mathbf{U}} \times \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}} \arrow[r, "-\circledcirc-"'] & \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{U}} \end{tikzcd} \tag{8.96}$$

commute.

Example 8.3.1.6. Special examples of the composition tensoring in Construction 8.3.1.5 include the following more elementary constructions, where we identified $\mathbf{D}^\star \cong \mathbf{D}$ and $\mathbf{D}\times\star \cong \mathbf{D}$, for the terminal category $\star$.

(i) If we set $\mathbf{R} = \mathbf{S} = \mathbf{T} = \star$, then $\circledcirc = *$.

(ii) If we set $\mathbf{R} = \mathbf{T} = \star$, then $\circledcirc = \circledast$.

(iii) If we set $\mathbf{S} = \star$, and replace $\mathbf{R}$ with $\mathbf{R}^{\mathrm{op}}$ then $\circledcirc$ is given by the *outer weighting*

$$\begin{aligned} \mathbf{Set}^{\mathbf{T}} \times \mathbf{C}^{\mathbf{R}} &\to \mathbf{C}^{\mathbf{R}\times\mathbf{T}} \\ (W, X) &\mapsto \{(t,r) \mapsto W^t * X^r\}\,. \end{aligned}$$

(iv) Set $\mathbf{R} = \star$. Suppose we are given a functor $F\colon \mathbf{S} \to \mathbf{T}$. Consider the two functors $\mathbf{T}^{F(\bullet)}_\bullet \in \mathbf{Set}^{\mathbf{T}^{\mathrm{op}}\times\mathbf{S}}$ and $\mathbf{T}^\bullet_{F(\bullet)} \in \mathbf{Set}^{\mathbf{S}^{\mathrm{op}}\times\mathbf{T}}$. Then there are canonical isomorphisms

$$\begin{tikzcd} \mathbf{C}^{\mathbf{T}} \arrow[r, bend left, "F^*"{name=U}] \arrow[r, bend right, "\mathbf{T}^{F(\bullet)}_\bullet\circledcirc-"'{name=D}] \arrow[from=U, to=D, phantom, "\sim"] & \mathbf{C}^{\mathbf{S}} \end{tikzcd} \tag{8.97}$$

and

$$\begin{tikzcd} \mathbf{C}^{\mathbf{S}} \arrow[r, bend left, "F_!"] \arrow[r, bend right, "\mathbf{T}^{\bullet}_{F(\bullet)} \circledcirc -"'] \arrow[r, phantom, "\cong"] & \mathbf{C}^{\mathbf{T}} \end{tikzcd} . \tag{8.98}$$

where $F_!$ is the left adjoint to the precomposition functor F^* given by left Kan extension. To see that these canonical isomorphisms hold, observe first that for $s \in \mathbf{S}$, the category of elements $\mathbf{el}(\mathbf{T}^{F(s)}_{\bullet})$ is equivalently given by the slice category $\mathbf{T}_{/F(s)}$ and has a terminal object given by $1_{F(s)}$. Thus

$$\varinjlim(\mathbf{el}(\mathbf{T}^{F(s)}_{\bullet}) \to \mathbf{T} \xrightarrow{X} \mathbf{C}) = X^{F(r)} = (F^* X)^r$$

which provides the first canonical isomorphism. For the second isomorphism, observe that, for $t \in \mathbf{T}$, the category of elements $\mathbf{el}(\mathbf{T}^{t}_{F(\bullet)})$ is equivalently given by the comma category $F_{/t}$, and for $X \in \mathbf{C}^{\mathbf{S}}$, the colimit of $F_{/t} \to \mathbf{S} \xrightarrow{X} \mathbf{C}$ is precisely the explicit description of the left Kan extension $(F_! X)^t$ found for example in [nLa24e], exposing the second canonical isomorphism.

Notation 8.3.1.7. Following Example (iv), we will at times denote all of the constructions in Example (iv) in the form $- \circledcirc -$. This will usually require that some of the categories in Construction 8.3.1.5 are set to the terminal category. It will always be clear from context, which categories those are.

8.3.2 Reedy categories

The canonical way of making diagrams compatible with homotopy theoretic constructions such as colimits is to replace cofibrantly (or fibrantly) in the projective or injective model structure (see [Hir03]). For many examples, however, one can get away with significantly less. For example, for a pushout square of cofibrant objects

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y' \end{tikzcd} \tag{8.99}$$

in some model category $\mathbf{M}$ to be a homotopy pushout diagram, it suffices for one of the two maps in the span to be a cofibration. This additional degree of asymmetry, of only certain parts of a diagram needing to be well behaved, can be encoded by equipping the indexing category with additional structure. The following notion, due to Reedy (unpublished, but well summarized in [RV13; Hir03; Bar07]), has turned out to be particularly well suited in order to perform inductive arguments:

Definition 8.3.2.1. A *Reedy category* consists of the data of:

1. A small category $\mathbf{R}$;
2. A map $\deg\colon \mathrm{Ob}(\mathbf{R}) \to \mathbb{N}$, called the *degree function*;
3. Two wide subcategories $\mathbf{R}^+, \mathbf{R}^- \subset \mathbf{R}$. Morphisms in $\mathbf{R}^+$ are sometimes called *face maps*, and morphisms in $\mathbf{R}^-$ are sometimes called *degeneracy maps*[1];

such that the following conditions hold:

1. For every non-identity morphism $f\colon r \to r'$ in $\mathbf{R}^+$, it holds that $\deg(r) < \deg(r')$.

[1] At times, the prefix **co** is added, in order to emphasize that the actual correct face and degeneracy maps should be given by the functoriality of some presheaf on $\mathbf{R}$. We will not follow this language.

2. For every non-identity morphism $f: r \to r'$ in $\mathbf{R}^-$, it holds that $\deg(r') < \deg(r)$.
3. Every morphism $f \in \mathbf{R}$ admits a unique factorization $f = f^+ \circ f^-$ with $f^+ \in \mathbf{R}^+$ and $f^- \in \mathbf{R}^-$.

Example 8.3.2.2. Let us give some guiding examples to keep in mind when thinking about Reedy categories:

1. The terminal category $\star$ is a Reedy category, with the degree function taking the unique object to 0, and $\star^+ = \star^- = \star$.
2. The category Δ, equipped with the obvious degree function $[n] \mapsto n$, and with Δ^- the subcategory of order preserving surjections and Δ^+ the category of order preserving injections is a Reedy category.
3. The poset $\mathbb{N}$, with $\deg(n) = n$, and $\mathbb{N}^+ = \mathbb{N}$ (and consequently $\mathbb{N}^-$ the discrete category) is a Reedy category.
4. The category
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \tag{8.100}$$
with (disjoint) choice of assignments + and − to the non-identity arrows defines a Reedy category (for appropriate choices of degree function).

More generally, it can be useful to know how to generate new Reedy categories from old ones:

Example 8.3.2.3. If $\mathbf{R}$ is a Reedy category, then the following categories inherit the structure of a Reedy category from $\mathbf{R}$:

1. The opposite category $\mathbf{R}^{\mathrm{op}}$, using the same degree function, and swapping the roles of $\mathbf{R}^+$ and $\mathbf{R}^-$.
2. For $n \in \mathbb{N}$, the full subcategory $\mathbf{R}_{\leq n} \subset \mathbf{R}$, of all objects of degree lesser or equal to n is equipped with the structure of a Reedy category, by restricting the degree function to $\mathbf{R}_{\leq n}$, and taking $\mathbf{R}^+_{\leq n} = \mathbf{R}_{\leq n} \cap \mathbf{R}^+$ and $\mathbf{R}^-_{\leq n} = \mathbf{R}_{\leq n} \cap \mathbf{R}^-$.
3. More generally, any full subcategory $\mathbf{S} \subset \mathbf{R}$, which has the property that, for every morphism $f \in \mathbf{S}$, the factorization morphisms f^+ and f^- are again in $\mathbf{S}$, inherits the structure of a Reedy category from $\mathbf{R}$, by restricting the degree function to $\mathbf{S}$ and defining $\mathbf{S}^+ = \mathbf{S} \cap \mathbf{R}^+$ and $\mathbf{S}^- = \mathbf{S} \cap \mathbf{R}^-$.
4. Given a functor $F: \mathbf{R} \to \mathbf{C}$, into any category $\mathbf{C}$, then for any object $C \in \mathbf{C}$, the comma category $F_{C/}$, whose objects are arrows of the form $C \to F(r)$, inherits the structure of a Reedy category, with the degree of $(r, C \to F(r)) \in F_{C/}$ given by $\deg(r)$, and with a morphism $f \in F_{C/}$ in $(F_{C/})^+$ $((F_{C/})^-)$ if and only if the underlying morphism in $\mathbf{R}$ is in $\mathbf{R}^+$ $(\mathbf{R}^-)$. The analogous construction for the comma category $F_{/C}$ also defines a Reedy category. In particular, we may use this construction to equip the category of elements $\mathbf{el}(U)$ of a co- or contravariant functor $U: \mathbf{R} \to \mathbf{Set}$ with the structure of a Reedy category.
5. Given two Reedy categories $\mathbf{R}$ and $\mathbf{S}$, their product $\mathbf{R} \times \mathbf{S}$ inherits the structure of a Reedy category, by setting $(\mathbf{R} \times \mathbf{T})^+ = \mathbf{R}^+ \times \mathbf{T}^+$, $(\mathbf{R} \times \mathbf{T})^- = \mathbf{R}^- \times \mathbf{T}^-$, and $\deg(r, s) = \deg(r) + \deg(s)$.

Next, let us recall some of the basic constructions involving Reedy categories, which can be found in detail in [RV13]. For the sake of dealing with cell complexes, we will essentially only need half of the constructions usually relevant for Reedy categories (the parts interacting with cofibrations).

Recollection 8.3.2.4. Let $\mathbf{R}$ be a reedy category and $\mathbf{C}$ be a category which has all colimits. For $n \in \mathbb{N}$, denote by $i_n : \mathbf{R}_{\le n} \hookrightarrow \mathbf{R}$ the obvious inclusion functor.

1. We write $i_n^* : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{R}_{\le n}}$ for the restriction functor, and denote by $(i_n)_! : \mathbf{C}^{\mathbf{R}_{\le n}} \to \mathbf{C}^{\mathbf{R}}$ its left adjoint (whose existence is guaranteed by Kan extension). The functor $(i_n)_!$ defines a fully faithful embedding
$$\mathbf{C}^{\mathbf{R}_{\le n}} \hookrightarrow \mathbf{C}^{\mathbf{R}},$$
or, in other words, the unit of adjunction $1 \to i_n^* \circ (i_n)_!$is an isomorphism. We are mainly interested in the functor sk_n, when it is precomposed with i_n^*. We denote the composition $(i_n)_! \circ i_n^*$ by
$$\mathrm{sk}_n : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{R}}.$$

2. Now, let $r \in \mathbf{R}$ be of degree n. Recall that the latching object of $X \in \mathbf{C}^{\mathbf{R}}$ at r is defined as $L^r X := (\mathrm{sk}_{n-1} X)^r$. The counit
$$L^r X = (\mathrm{sk}_{n-1} X)^r = ((i_{n-1})_! i_{n-1}^* X)^r \to X^r .$$
is then referred to as the *latching map*. These latching objects (and their duals, the matching objects) are at the core of the inductive arguments used in the yoga of Reedy model structures (see [RV13]).

3. For $n \ge 0$, denote the inclusion $\mathbf{R}_{\le n} \hookrightarrow \mathbf{R}$ by i_n. By Example (iv), we may present the functor $\mathrm{sk}_n : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{R}}$ in the form
$$\mathrm{sk}_n X \cong (\mathbf{R}^\bullet_{i_n(\bullet)} \circledcirc \mathbf{R}^{i_n(\bullet)}_\bullet) \circledcirc X .$$
The diagram
$$(\mathbf{R}^\bullet_{i_n(\bullet)} \circledcirc \mathbf{R}^{i_n(\bullet)}_\bullet) \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{R}}$$
is thus of particular importance. It can explicitly be computed as
$$(\mathbf{R}^\bullet_{i_n(\bullet)} \circledcirc \mathbf{R}^{i_n(\bullet)}_\bullet)^r_{\overline{r}} \cong (\mathrm{sk}_n \mathbf{R}^\bullet_{\overline{r}})^r \cong \{ f : \overline{r} \to r \mid f \text{ factors through } \mathbf{R}_{\le n} \},$$
which canonically identifies it with the subfunctor of $\mathbf{R}^\bullet_\bullet \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{R}}$ given by such arrows that factor through $\mathbf{R}_{\le n}$. We will also denote this subfunctor in the form $\mathrm{sk}_n \mathbf{R}^\bullet_\bullet$.

4. Consequently, we may identify the latching object $L^r X$, for $r \in \mathbf{R}$ with $\deg(r) = n$ as:
$$(\mathbf{R}^r \circledcirc \mathrm{sk}_{n-1} \mathbf{R}^\bullet_\bullet) \circledcirc X$$
and we denote
$$\partial \mathbf{R}^r := (\mathbf{R}^r \circledcirc \mathrm{sk}_{n-1} \mathbf{R}^\bullet_\bullet).$$
By the explicit description of $\mathrm{sk}_n \mathbf{R}^r_{\overline{r}}$ above, we obtain that the morphism $\partial \mathbf{R}^r \to \mathbf{R}^r$ obtained by evaluating $\mathrm{sk}_n \mathbf{R}^\bullet_\bullet \to \mathbf{R}^\bullet_\bullet$ at r is given by the inclusion of the subfunctor of $\mathbf{R}^r \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ specified by
$$(\partial \mathbf{R}^r)_{\overline{r}} = \{ f : \overline{r} \to r \mid f^+ \neq 1_r \} .$$

5. Dually, (replacing $\mathbf{R}$ by $\mathbf{R}^{\mathrm{op}}$) we denote by $\partial \mathbf{R}_{\overline{r}} \hookrightarrow \mathbf{R}_{\overline{r}} \in \mathbf{Set}^{\mathbf{R}}$ the subfunctor of $\mathbf{R}_{\overline{r}}$ given by
$$(\partial \mathbf{R}_{\overline{r}})^r = \{ f : \overline{r} \to r \mid f^- \neq 1_{\overline{r}} \} .$$

Observation 8.3.2.5. Observe that the claim that $\{ f : \overline{r} \to r \mid f^+ \neq 1_r \}$ is functorial in $\overline{r}$ does in particular require that any precomposition of a morphism f with $f^+ \neq 1$, $f \circ g$, again has the property that $(f \circ g)^+ \neq 1$. Indeed, the unique factorization of $f \circ g$ is given by $(f^+ \circ (f^- \circ g)^+, (f^- \circ g)^-)$, and as f^+ increases the degree, so does $f^+ \circ (f^- \circ g)^+$.

Remark 8.3.2.6. One should be careful to note that, for $\overline{r}, r \in \mathbf{R}$, the equality $(\partial \mathbf{R}^r)_{\overline{r}} = (\partial \mathbf{R}_{\overline{r}})^r$ is generally false.

Let us furthermore investigate the relations between the boundary inclusions $\partial\mathbf{R}^r \to \mathbf{R}^r$ and $\partial\mathbf{R}_r \to \mathbf{R}_r$ under the operation $\circledcirc$.

Notation 8.3.2.7. In the following, we will frequently be concerned with the bivariate Yoneda functor associated to $\mathbf{Q} = \mathbf{R}^{\mathrm{op}} \times \mathbf{S}$. Formally speaking, the latter is an object of $\mathbf{Set}^{(\mathbf{R}^{\mathrm{op}}\times\mathbf{S})^{\mathrm{op}}\times(\mathbf{R}^{\mathrm{op}}\times\mathbf{S})}$. As a quadrivariate functor, we will denote the value of this functor at $(\overline{x}, x) = ((r,\overline{s}),(\overline{r},s))$ in the form

$$\mathbf{Q}^x_{\overline{x}} = \mathbf{R}^r_{\overline{r}} \times \mathbf{S}^s_{\overline{s}}\,.$$

Remark 8.3.2.8. There is a notational hurdle that seems hard to overcome in a clean way here. Namely, given $x = (\overline{r}, s), \overline{x} = (r, \overline{s}) \in \mathbf{R}^{\mathrm{op}} \times \mathbf{S} \in \mathbf{Q}$, one has $\mathbf{Q}^x_{\overline{x}} = \mathbf{R}^r_{\overline{r}} \times \mathbf{S}^s_{\overline{s}}$, leading to a wholly unpleasant cross-exchange of variables. We will thus also use the notation $\mathbf{Q}^{r,s}_{\overline{r},\overline{s}}$ to refer to $\mathbf{Q}^x_{\overline{x}}$. Consequently, we will also use the notation $\partial\mathbf{Q}^{\bullet,s}_{\overline{r},\bullet}$ in order to refer to $\partial\mathbf{Q}^x$ and similarly $\partial\mathbf{Q}^{r,\bullet}_{\bullet,\overline{s}}$ for $\partial\mathbf{Q}_{\overline{x}}$.

Example 8.3.2.9. Let $r, \overline{r} \in \mathbf{R}$. In the following, we use the canonical isomorphism $\mathbf{R}^r \circledcirc \mathbf{R}_{\overline{r}} \cong \mathbf{R}^r_{\overline{r}}$ to identify the two objects. Then the following explicit descriptions hold:

1. The induced morphism
$$\partial\mathbf{R}^r \circledcirc \partial\mathbf{R}_{\overline{r}} \to \mathbf{R}^r \circledcirc \mathbf{R}_{\overline{r}}$$
is (canonically isomorphic to) the suboject of $\mathbf{R}^r_{\overline{r}}$ given by the inclusion
$$\{f\colon \overline{r} \to r \mid f^- \neq 1_{\overline{r}}, f^+ \neq 1_r\} = (\partial\mathbf{R}^r)_{\overline{r}} \cap (\partial\mathbf{R}_{\overline{r}})^r \hookrightarrow \mathbf{R}^r_{\overline{r}}\,.$$

2. The Leibniz tensor
$$(\partial\mathbf{R}^r \hookrightarrow \mathbf{R}^r)\hat{\circledcirc}(\partial\mathbf{R}_{\overline{r}} \hookrightarrow \mathbf{R}_{\overline{r}})$$
is (canonically isomorphic to) the subobject of $\mathbf{R}^r_{\overline{r}}$ given by the inclusion
$$\{f\colon \overline{r} \to r \mid f \neq 1_r\} = (\partial\mathbf{R}^r)_{\overline{r}} \cup (\partial\mathbf{R}_{\overline{r}})^r \hookrightarrow \mathbf{R}^r_{\overline{r}}\,.$$

3. Let $\mathbf{S}$ be another Reedy category and let $\mathbf{Q} = \mathbf{R}^{\mathrm{op}} \times \mathbf{S}$. Consider the exterior action
$$\begin{aligned}\circledcirc\colon \mathbf{Set}^{\mathbf{S}} \times \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} &\to \mathbf{Set}^{\mathbf{Q}}\\ (U,V) &\mapsto \{(\overline{r}, s) \mapsto U^s * V_{\overline{r}} = V_{\overline{r}} \times U^s\}\,.\end{aligned}$$
as described in Example 8.3.1.6 (iv). Let $\overline{x} = (r, \overline{s}) \in \mathbf{R}^{\mathrm{op}} \times \mathbf{S}$. Then the induced morphism
$$(\partial\mathbf{S}_{\overline{s}} \hookrightarrow \mathbf{S}_{\overline{s}})\hat{\circledcirc}(\partial\mathbf{R}^r \hookrightarrow \mathbf{R}^r)\colon \mathbf{S}_{\overline{s}} \circledcirc \partial\mathbf{R}^r \cup_{\partial\mathbf{S}_{\overline{s}}\circledcirc\partial\mathbf{R}^r} \partial\mathbf{S}_{\overline{s}} \circledcirc \mathbf{R}^r \to \mathbf{S}_{\overline{s}} \circledcirc \mathbf{R}^r$$
is canonically isomorphic to the inclusion
$$\partial\mathbf{Q}_{\overline{x}} \hookrightarrow \mathbf{Q}_{\overline{x}}\,.$$

Remark 8.3.2.10. Mnemonically, we can think of the third identity in Example 8.3.2.9 as the Leibniz formula, stating that the boundary of $A \times B$ is given by the union $A \times \partial B \cup \partial A \times B$ with intersection $\partial A \times \partial B$.

Let us explain how the identities above can be obtained:

Proof of the identities in Example 8.3.2.9. For the first identity, observe that $\partial\mathbf{R}^r \circledcirc \partial\mathbf{R}_{\overline{r}} = \partial\mathbf{R}^r \circledast \partial\mathbf{R}_{\overline{r}}$ through the coequalizer formula for weighted colimits, may explicitly be written as the quotient of the set $\bigsqcup_{r' \in \mathbf{R}} (\partial\mathbf{R}^r)_{r'} \times (\partial\mathbf{R}_{\overline{r}})^{r'}$ by the relation generated by

$$(f \circ h, g) \sim (f, h \circ g)\,.$$

The canonical map into $\mathbf{R}^r_{\overline{r}}$ is given by composition. Now, for any such element $(f,g) \in (\partial \mathbf{R}_{\overline{r}})^{r'}$, consider the commutative diagram

$$
\begin{tikzcd}
\overline{r} \arrow[rr, "g"] \arrow[dr, "g^-"'] & & r' \arrow[rr, "f"] \arrow[dr, "f^-"'] & & r \\
& \bullet \arrow[ur, "g^+"'] \arrow[dr, "(f^- \circ g^+)^-"'] & & \bullet \arrow[ur, "f^+"'] & \\
& & \bullet \arrow[ur, "(f^- \circ g^+)^+"'] & &
\end{tikzcd}
\tag{8.101}
$$

It provides a sequence of relations

$$(f,g) \sim (f^+, f^- \circ g^+ \circ g^-) \sim (f^+, (f^- \circ g^+)^+ \circ (f^- \circ g^+)^- \circ g^-) \sim (f^+ \circ (f^- \circ g^+)^+, (f^- \circ g^+)^- \circ g^-)\,.$$

By uniqueness of factorizations, we have

$$(f^+ \circ (f^- \circ g^+)^+, (f^- \circ g^+)^- \circ g^-) = ((f \circ g)^+, (f \circ g)^-).$$

It follows from this that every equivalence class $[(f,g)]$ admits a canonical form, which only depends on $f \circ g$. Consequently, the induced map

$$\partial \mathbf{R}^r \circledast \partial \mathbf{R}_{\overline{r}} \to \mathbf{R}^r_{\overline{r}}$$

is injective. By the functoriality of $\partial \mathbf{R}^r$ and $\partial \mathbf{R}_{\overline{r}}$, its image is contained in the intersection $(\partial \mathbf{R}^r)_{\overline{r}} \cap (\partial \mathbf{R}_{\overline{r}})^r$. Furthermore, every element in this intersection f can be written as $f^+ \circ f^-$, with $f^+ \neq 1$ and $f^- \neq 1$. Hence, the image of the injection

$$\partial \mathbf{R}^r \circledast \partial \mathbf{R}_{\overline{r}} \hookrightarrow \mathbf{R}^r_{\overline{r}}$$

is precisely $(\partial \mathbf{R}^r)_{\overline{r}} \cap (\partial \mathbf{R}_{\overline{r}})^r$. For the second identity, we may use that $- \circledast \mathbf{R}_{\overline{r}}$ and $\mathbf{R}^r \circledast -$ correspond respectively to contra and covariant evaluation at $\overline{r}$ and r, and use the previous identity to compute the Leibniz tensor in question as

$$(\partial \mathbf{R}^r)_{\overline{r}} \cup_{\partial \mathbf{R}^r \cap \partial \mathbf{R}_{\overline{r}}} (\partial \mathbf{R}_{\overline{r}})^r \to \mathbf{R}^r_{\overline{r}}.$$

In other words, the Leibniz tensor is given by the inclusion of the subset

$$(\partial \mathbf{R}^r)_{\overline{r}} \cup (\partial \mathbf{R}_{\overline{r}})^r \hookrightarrow \mathbf{R}^r_{\overline{r}}\,.$$

Now, observe that as morphisms in $\mathbf{R}^+$ increase degree and morphisms in $\mathbf{R}^-$ decrease degree, a morphism $f\colon \overline{r} \to r$ is an identity, if and only if f^+ and f^- are identities. Hence, it follows that

$$(\partial \mathbf{R}^r)_{\overline{r}} \cup (\partial \mathbf{R}_{\overline{r}})^r = \{f\colon \overline{r} \to r \mid f \neq 1_r\}\,.$$

Finally, let us prove the remaining identity. To this end, observe first that as the incarnation of $\circledcirc$ is given by pointwise products, we may identify the Leibniz tensor in question at a point $x = (\overline{r}, s)$ as the canonical morphism

$$\mathbf{R}^r_{\overline{r}} \times (\partial \mathbf{S}_{\overline{s}})^s \cup_{(\partial \mathbf{R}^r)_{\overline{r}} \times (\partial \mathbf{S}_{\overline{s}})^s} (\partial \mathbf{R}^r)_{\overline{r}} \times \mathbf{S}^s_{\overline{s}} \to \mathbf{R}^r_{\overline{r}} \times \mathbf{S}^s_{\overline{s}} = \mathbf{Q}^x_{\overline{x}}\,.$$

The claim is now immediate from the definition of the product Reedy category. □

8.3.3 Cellularizing the weighted colimit calculus

Next, let us investigate diagrams of structured cell complexes indexed over a Reedy category. As we explained in the introduction of this section, we aim to answer the question of what a *good* diagram of cell complexes should be. The approach we are going to take is to embrace the following guiding principle which will turn out to be quite fruitful, in particular when applied to our investigations of the simple homotopy theory of diagrams:

A diagram of complexes should be the same thing as a complex in diagrams.

To make this more rigorous, we need to cellularize the categories $\mathbf{C}^{\mathbf{R}}$, for $\mathbf{C}$ a cellularized category and $\mathbf{R}$ a Reedy category.

Remark 8.3.3.1. As the dual of a Reedy category is again a Reedy category in a canonical way, every statement we make concerning covariant functor categories $\mathbf{C}^{\mathbf{R}}$, where $\mathbf{R}$ is Reedy category, will have an analogue for contravariant functor categories. We will not go through the trouble of spelling every result out in both variances. It is, however, also not going to be possible to stick to a single variance, as weighted colimits will generally involve functor colimits of both variances. In this subsection, we will generally state things for the covariant case, and (keeping calm and minding the variances) the contravariant cases can be derived by replacing $\mathbf{R}$ by $\mathbf{R}^{\mathrm{op}}$, and replacing $\mathbf{R}^r$ by $\mathbf{R}_r$, etc.

The cellularizations of functor categories over a Reedy category $\mathbf{R}$ are all derived from cellularizing categories of set-valued functors $\mathbf{Set}^{\mathbf{R}}$.

Notation 8.3.3.2. Given a fixed Reedy category $\mathbf{R}$, we will denote the inclusion $\partial\mathbf{R}^r \hookrightarrow \mathbf{R}^r$ by ι^r, and the inclusion $\partial\mathbf{R}_r \hookrightarrow \mathbf{R}_r$ by ι_r.

Construction 8.3.3.3. Given a Reedy category $\mathbf{R}$, the category of set valued functors $\mathbf{Set}^{\mathbf{R}}$ admits a canonical cellularization, with the generating boundary inclusions given by

$$\{\partial\mathbf{R}_r \hookrightarrow \mathbf{R}_r \mid r \in \mathbf{R}\}\,.$$

Observe that, for every pair $r, \overline{r} \in \mathbf{R}$, the induced map

$$(\partial\mathbf{R}_{\overline{r}})^r \hookrightarrow \mathbf{R}_{\overline{r}}^r$$

is the inclusion of a subset, i.e., a structured cell complex in $\mathbf{Set}$. It follows from the stability of monomorphisms in $\mathbf{Set}$ under cobase change and transfinite composition that every structured relative cell complex $\mathfrak{c}\colon A \hookrightarrow X$ in $\mathbf{Set}^{\mathbf{R}}$ with respect to these boundary inclusions is a pointwise monomorphism, and hence a monomorphism. Consequently, using that limits and colimits can be detected pointwise, it follows that the second defining property of a cellularized category (bicartesianity of pushouts of relative cell complexes, see Definition 8.1.1.10) is fulfilled. Clearly,

$$\{\partial\mathbf{R}_r \hookrightarrow \mathbf{R}_r \mid r \in \mathbf{R}\}$$

does not contain any isomorphisms, as $(\partial\mathbf{R}_r)^r$ does not contain the identity. Hence, we have verified that with respect to the above set of boundary inclusions $\mathbf{Set}^{\mathbf{R}}$ is indeed a cellularized category.

Notation 8.3.3.4. We will see later on in Corollary 8.3.4.11 that cell structures on relative cell complexes $c \in \mathbf{Set}^{\mathbf{R}}$ are unique, if they exist. Therefore, in this case we will not distinguish between a structured relative cell complex and a relative cell complex, and not make use of the notation $\mathfrak{c}$ to indicate the presence of a fixed structure.

Example 8.3.3.5. If $\mathbf{R} = \Delta^{\mathrm{op}}$, then the set of boundary inclusions defined in Construction 8.3.3.3 is simply the set of boundary inclusions of simplices

$$\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}\,.$$

Next, let us extend this cellularization to functor categories $\mathbf{C}^{\mathbf{R}}$, where $\mathbf{C}$ is a cellularized category.

Proposition 8.3.3.6. *Let $\mathbf{C}$ be a cellularized category and $\mathbf{R}$ a Reedy category. Consider the bifunctor*

$$\begin{aligned} - \circledcirc -\colon \mathbf{Set}^{\mathbf{R}} \times \mathbf{C} &\to \mathbf{C}^{\mathbf{R}} \\ (W, X) &\mapsto \{r \mapsto W^r * X\} \end{aligned}$$

constructed in Construction 8.3.1.5 (by replacing $\mathbf{S}$ *with* $\star$*,* $\mathbf{R}$ *with* $\star$ *and* $\mathbf{T}$ *with* $\mathbf{R}$ *there). The set of boundary inclusions*

$$\mathbb{B}_{\mathbf{C}^{\mathbf{R}}} \coloneqq \{\iota_r \hat{\circledcirc} b \colon \mathbf{R}_r \circledcirc D \cup_{\partial \mathbf{R}_r \circledcirc \partial D} \partial \mathbf{R}_r \circledcirc D \to \mathbf{R}_r \circledcirc D \mid r \in \mathbf{R}, b \colon \partial D \to D \in \mathbb{B}_{\mathbf{C}}\}$$

defines the structure of a cellularized category on $\mathbf{C}^{\mathbf{R}}$*.*

Proof. Observe first that by the associativity and compatibility with colimits of $\circledcirc$, for $\overline{r}, r \in \mathbf{R}$ and $c \colon A \to X \in \mathbf{C}$, there are canonical isomorphisms

$$(\iota_{\overline{r}} \hat{\circledcirc} c)^r \cong \mathbf{R}^r \circledast (\iota_{\overline{r}} \hat{\circledcirc} c) \cong (\mathbf{R}^r \circledast \iota_{\overline{r}}) \hat{\circledcirc} c = ((\iota_{\overline{r}})^r \colon (\partial \mathbf{R}_{\overline{r}})^r \hookrightarrow \mathbf{R}^r_{\overline{r}}) \hat{*} c\,.$$

Taking $\overline{r} = r$, it follows from the proper containment $(\partial \mathbf{R}^r)_r \hookrightarrow \mathbf{R}^r_r$ that, for any $b \in \mathbb{B}_{\mathbf{C}}$, the associated structured cell complex $(\iota_r \hat{\circledcirc} b)_r$ is a relative cell complex with at least one cell. Consequently, by Corollary 8.1.4.1, it follows that $\iota_r \hat{\circledcirc} b$ is not an isomorphism, which verifies the first defining property for a cellularized category. Furthermore, it follows from Recollection 8.2.1.3 and the canonical cellularization for $- * -$ (see Example 8.2.5.5) that, for any $\overline{r} \in \mathbf{R}$, the image of a relative structured cell complex with respect to $\mathbb{B}_{\mathbf{C}^{\mathbf{R}}}$ under the evaluation functor at $\overline{r}$ is a relative structured cell complex in $\mathbf{C}$. In particular, for any pushout diagram

$$\begin{tikzcd} A \arrow[r, hook, "c"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & X \arrow[d] \\ A' \arrow[r, hook] & X' \end{tikzcd} \tag{8.102}$$

with c a relative $\mathbb{B}_{\mathbf{C}^{\mathbf{R}}}$-cell complex, the induced pushout diagram

$$\begin{tikzcd} A^r \arrow[r, hook, "c^r"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & X^r \arrow[d] \\ A'^r \arrow[r, hook] & X'^r \end{tikzcd} \tag{8.103}$$

has upper horizontal a relative $\mathbb{B}_{\mathbf{C}}$ cell complex. It follows that it is also a pullback diagram. As r was arbitrary and limits in functor categoies are detected pointwise, it follows that Diagram (8.102) is a pullback diagram, which verifies the second defining property of a cellularized category. □

Observation 8.3.3.7. If $\mathbf{C} = \mathbf{Set}$, then the boundary inclusions in defined in Proposition 8.3.3.6 are (up to canonical isomorphism) given by

$$\iota_r \hat{\circledcirc} (\varnothing \to *) \cong \iota_r\,.$$

Observation 8.3.3.8. Given two Reedy categories $\mathbf{R}, \mathbf{T}$, the canonical isomorphism of categories

$$(\mathbf{C}^{\mathbf{R}})^{\mathbf{T}} \cong \mathbf{C}^{\mathbf{R} \times \mathbf{T}}$$

lifts to an isomorphism of cellularized categories, via the identifications (see Construction 8.3.1.5 and Example 8.3.2.9), taking care to exchange variances as needed.

$$\iota_t \hat{\circledcirc} (\iota_r \hat{\circledcirc} b) \hat{=} (\iota_t \hat{\circledcirc} \iota_r) \hat{\circledcirc} b \cong \iota_{t,r} \hat{\circledcirc} b\,,$$

for $r \in \mathbf{R}, t \in \mathbf{T}$ and $b \in \mathbb{B}_{\mathbf{C}}$.

Notation 8.3.3.9. From here on out, given a Reedy category $\mathbf{R}$ and a cellularized category $\mathbf{C}$, when we refer to $\mathbf{C}^{\mathbf{R}}$ as a cellularized category, it will always be with respect to the cell structure constructed in Proposition 8.3.3.6.

Proposition 8.3.3.10. *Given a cellularized category* $\mathbf{C}$*, the bifunctors*

$$- \circledcirc - \colon \mathbf{Set}^{\mathbf{T} \times \mathbf{S}^{\mathrm{op}}} \times \mathbf{C}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{S}} \to \mathbf{C}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{T}}$$

constructed in Construction 8.3.1.5, where $\mathbf{R}$ *and* $\mathbf{T}$ *and* $\mathbf{S}$ *are Reedy categories, admit a canonical cellularization, which is uniquely determined by the following two properties.*

1. *For the case* $\mathbf{T} \cong \mathbf{S} \cong *$, *the cellularization is the one given by*

$$- \circledast - = - * -\colon \mathbf{Set} \times \mathbf{C}^{\mathbf{R}^{\mathrm{op}}} \to \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$$

   constructed in Example 8.2.5.5.

2. *For every quadruple of Reedy categories* $\mathbf{R}, \mathbf{S}, \mathbf{T}, \mathbf{U}$ *and every triple of structured relative cell complexes* $a \in \mathbf{RCell}(\mathbf{Set}^{\mathbf{T}^{\mathrm{op}} \times \mathbf{U}})$, $b \in \mathbf{RCell}(\mathbf{Set}^{\mathbf{S}^{\mathrm{op}} \times \mathbf{T}})$, $\mathfrak{c} \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{S}}$ *the canonical isomorphism*

$$a \hat{\circledcirc} (b \hat{\circledcirc} c) \cong a \hat{\circledcirc} (b \hat{\circledcirc} c)$$

   induced by the associator isomorphisms of Construction 8.3.1.5 defines an isomorphism of structured relative cell complexes.

Proof. Following the notation of Notation 8.3.2.7, we denote the generating boundary inclusions associated to $\mathbf{Set}^{\mathbf{S}^{\mathrm{op}} \times \mathbf{T}}$ in the form $\iota^{s,\bullet}_{\bullet,\bar{t}}$, for $\bar{t} \in \mathbf{T}, s \in \mathbf{S}$. To construct such cellularizations, we may use Observation 8.2.5.4, which states that is suffices to determine cell structures on

$$\iota^{s,\bullet}_{\bullet,\bar{t}} \hat{\circledcirc} (\iota^{r,\bullet}_{\bullet,\bar{s}} \hat{\circledcirc} b) \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{T}}$$

for $\bar{t} \in \mathbf{T}$, $s, \bar{s} \in \mathbf{S}$, $r \in \mathbf{R}$. By Example 8.3.2.9, we have $\iota^{s,\bullet}_{\bullet,\bar{t}} \cong \iota_{\bar{t}} \circledcirc \iota^{s}$[2] and thus have canonical isomorphisms

$$\begin{aligned}
\iota^{s,\bullet}_{\bullet,\bar{t}} \hat{\circledcirc} (\iota^{r,\bullet}_{\bullet,\bar{s}} \hat{\circledcirc} b) &\cong (\iota_{\bar{t}} \hat{\circledcirc} \iota^{s}) \hat{\circledcirc} ((\iota_{\bar{s}} \hat{\circledcirc} \iota^{r}) \hat{\circledcirc} b \\
&\cong \iota_{\bar{t}} \hat{\circledcirc} (\iota^{s} \hat{\circledcirc} \iota_{\bar{s}}) \hat{\circledcirc} (\iota^{r} \hat{\circledcirc} b) \\
&= \iota_{\bar{t}} \hat{\circledcirc} (\iota^{s} \hat{\circledast} \iota_{\bar{s}}) \hat{*} (\iota^{r} \hat{\circledcirc} b) \\
&\cong (\iota^{s} \hat{\circledast} \iota_{\bar{s}}) \hat{*} ((\iota_{\bar{t}} \circledcirc \iota^{r}) \circledcirc b) \\
&\cong (\iota^{s} \hat{\circledast} \iota_{\bar{s}}) \hat{*} ((\iota^{r,\bullet}_{\bullet,\bar{t}}) \circledcirc b)
\end{aligned}$$

where the second to last lines use the linearity of cellularized functors with respect to $*$ (see Example 8.2.5.5). As $(\iota^{s} \hat{\circledast} \iota_{\bar{s}}) \in \mathbf{Set}$ and $(\iota^{r,\bullet}_{\bullet,\bar{t}}) \circledcirc b \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{T}}$ is a generating boundary inclusion, the assumed compatibility of the cellularizations of $\circledcirc$ with $*$ and the assumption that all of the associator morphisms above define morphisms of structured relative cell complexes leaves a unique possible cellularization for $\iota^{s,\bullet}_{\bullet,\bar{t}} \hat{\circledcirc} (\iota^{r,\bullet}_{\bullet,\bar{s}} \hat{\circledcirc} b)$. Conversely, it is not hard to verify that the cellularization determined in this fashion has the required associativity property. ☐

Notation 8.3.3.11. We are not going to introduce additional notation to denote the cellularized versions of the bifunctors $\circledcirc$. When the latter are treated as cellularized will usually be clear from their arguments.

The crucial use-case for Proposition 8.3.3.10 is that it provides a general machinery to construct cellularized functors $\mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{T}}$ between functor categories from structured cell complexes in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{T}}$.

Observation 8.3.3.12. Given a fixed cellularized category $\mathbf{C}$, and two Reedy categories $\mathbf{R}$ and $\mathbf{T}$, we can apply Construction 8.2.5.7, to obtain a functor

$$\begin{aligned}
\mathbf{RCell}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{T}}) &\to \mathbf{CellCat}^{\to}(\mathbf{C}^{\mathbf{R}}, \mathbf{C}^{\mathbf{T}}) \\
a &\mapsto a \hat{\circledcirc} - \ .
\end{aligned}$$

[2]The exchange in variables stems from the age old misfortune of compositions being spelled from right to left. Since $\circledcirc$ is primarily used to turn functors into cell complexes, it inherits this misfortune, and all the notational errors that come with it. We apologize to the reader, for the mistakes of that type which we have certainly commited.

By Lemma 8.2.5.9, this assignment is essentially compatible with all of the relevant constructions on cell complexes on $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}$. In particular, from any structured cell complex $W \in \mathbf{Cell}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}})$, we obtain an associated (absolute) cellularized functor

$$W \circledcirc -: \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{T}}$$

As we have seen in Example 8.3.1.6, a large class of functors $\mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{T}}$ arise from the $\circledcirc$ construction. It follows that to cellularize these functors, it suffices to construct cell structures on the associated objects in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}$. In particular, if we are looking to obtain a cellularized version of the colimit functor $\mathbf{C}^{\mathbf{R}} \to \mathbf{C}$, this means we need to obtain a cell structure on the constant weight $\star \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$.

Let us finish this subsection by investigating the interaction of the cellularized bifunctors $\circledcirc$ with functors of cellularized categories $\mathbf{C} \to \mathbf{D}$. The following provides a significant expansion of the linearity assertion of Example 8.2.5.5. Roughly speaking, it can be summarized as cellularized functors being linear with respect to $\circledcirc$.

Construction 8.3.3.13. In the situation of Proposition 8.3.3.10, consider a relative cellularized functor $\mathfrak{i}: F \to G$ between cellularized categories $\mathbf{C}$ and $\mathbf{D}$. Applying the functor of strict 2-categories $(-)^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}}$ and $(-)^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}$ to the underlying natural transformation ι of $\mathfrak{i}$, we obtain natural transformations

$$\begin{tikzcd} \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}} \arrow[r, bend left, "F^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}}"] \arrow[r, bend right, "G^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}}"'] \arrow[r, phantom, "\Downarrow \iota^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}}"] & \mathbf{D}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}} \end{tikzcd} \tag{8.104}$$

and

$$\begin{tikzcd} \mathbf{C}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}} \arrow[r, bend left, "F^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}"] \arrow[r, bend right, "G^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}"'] \arrow[r, phantom, "\Downarrow \iota^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}"] & \mathbf{D}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}} \end{tikzcd}. \tag{8.105}$$

An elementary computation using the commutativity of F and G with weighted colimits (using that $- \circledcirc -$ is constructed entirely in terms of weighted colimits) shows that there is a canonical natural isomorphism

$$\iota^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}} \hat{\circ} (f \circledcirc -) \cong (f \circledcirc -) \hat{\circ} \iota^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}}$$

for $f: U \to V \in \mathbf{Set}^{\mathbf{S}^{\mathrm{op}}\times\mathbf{T}}$ (using the Leibniz composition of Construction 8.2.1.5).
In particular, if we set $\mathbf{R}, \mathbf{S} = \star$ and use Construction 8.2.1.5, we obtain a canonical isomorphism

$$\widehat{\iota^{\mathbf{T}}}(\iota_t \hat{\circledcirc} b) \cong \iota_t \hat{\circledcirc} \hat{\iota}(b)$$

for $b \in \mathbb{B}$ and $t \in \mathbf{T}$. By Lemma 8.2.2.5, these canonical isomorphisms uniquely define the structure of a relative cellularized functor on $\iota^{\mathbf{T}}$, which we will denote by $\mathfrak{i}^{\mathbf{T}}$. As a family of cellularized functors (with varying $\mathbf{T}$) these cellularizations are uniquely characterized by the property that, given any pair of Reedy categories $\mathbf{R}$ and $\mathbf{T}$, the induced canonical isomorphism

$$\iota^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}} \hat{\circ} (f \circledcirc -) \cong (f \circledcirc -) \hat{\circ} \iota^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}},$$

for $a \in \mathbf{RCell}(\mathbf{Set}^{\mathbf{S}^{\mathrm{op}}\times\mathbf{T}})$, induces an isomorphism of relative cellularized functors

$$\mathfrak{i}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}} \hat{\circ} (a \circledcirc -) \cong (a \circledcirc -) \hat{\circ} \mathfrak{i}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{S}}.$$

The uniqueness claim is immediate by Lemma 8.2.2.5 and Construction 8.2.1.5, and the resulting isomorphisms

$$\widehat{\mathfrak{i}^{\mathbf{T}}}(\iota_t \hat{\circledcirc} \mathfrak{b}) \cong \iota_t \hat{\circledcirc} \hat{\mathfrak{i}}(\mathfrak{b})$$

for $b \in \mathbb{B}$ and $t \in \mathbf{T}$. That this linearity property holds in general, follows from Observation 8.2.2.11 and a lengthy elementary computation, much in line with the one performed in the proof of Proposition 8.3.3.10.

8.3.4 Structured cell complexes in functor categories

For the remainder of this subsection, we fix some Reedy category $\mathbf{R}$ and some cellularized category $\mathbf{C}$. So far, we have only really discussed the cellularized functor categories $\mathbf{C}^{\mathbf{R}}$, in families, considering the interaction between several such categories in terms of the composition tensoring $\circledcirc$. Now, let us focus on a single such category, and get a better understanding of what it means to be a structured cell complex in $\mathbf{C}^{\mathbf{R}}$. Unsurprisingly, this is going to be strongly related to the way cofibrant objects are detected in Reedy model structures (see, for example, [RV13]). Before we begin, let us alert the reader of a notation change in covariance though:

Remark 8.3.4.1. Throughout this section, we will make a choice which may, at first glance, be surprising. Namely, when investigating cell complexes in functor categories, we are going to change variance, and generally phrase statements as pertaining to contravariant functors, i.e.,elements of $\mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$. From a theoretical point of view, this is no limitation, as we already explained in Remark 8.3.3.1. Nevertheless, this choice of focusing on the contravariant perspective may seem unnecessarily complicated at first glance. We found it preferable, however, mainly for the following three reasons:

1. The main example of structured cell complexes in a functor category which one should have in mind are simplicial sets, i.e.,objects in the covariant functor category (category of presheaves) $\mathbf{Set}^{\Delta^{\mathrm{op}}}$.

2. Similarly, in Section 13.1, when we discuss the simple homotopy theory of stratified spaces, we will be mainly concerned with the category $\mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$, of simplicial presheaves indexed over the subdivision of a poset.

3. Towards the end of this section, the Yoneda embeddings $\mathbf{R} \hookrightarrow \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ will play a major role, and it generally seems preferable for these embeddings to be covariant.

Nevertheless, both variances will frequently occur in this section, as we will make frequent use of the composition tensoring defined in Construction 8.3.1.5.

Notation 8.3.4.2. As a consequence of Example 8.3.1.6(iv), it follows that

$$\mathbf{R}^r \circledcirc - : \mathbf{C} \to \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$$

defines the left adjoint to the evaluation at r, which is equivalently given by $\mathbf{R}_r \circledcirc -$.[3] In particular, there is a canonical natural isomorphism

$$\mathbf{C}^{\mathbf{R}^{\mathrm{op}}}(\mathbf{R}^r \circledcirc D, X) \cong \mathbf{C}(D, X_r)$$

for $D \in \mathbf{C}$ and $X \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$, and $r \in \mathbf{R}$. Under this canonical isomorphism, we will not distinguish notationally between a morphism $\sigma : \mathbf{R}^r \circledcirc D \to X$ and its pendant $D \to X_r$.

Observation 8.3.4.3. Under Notation 8.3.4.2, we may equivalently think of a cell structure $\mathfrak{C}_{\mathfrak{c}}$ on a relative cell complex $c : A \to X \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$ as a family of sets of morphisms $(\mathfrak{C}_{r,b})_{r \in \mathbf{R}, b \in \mathbb{B}_{\mathbf{C}}}$, with $\mathfrak{C}_{r,b} \subset \mathbf{C}(D, X_r)$, for $b : \partial D \to D$ in $\mathbb{B}_{\mathbf{C}}$.

Notation 8.3.4.4. Given a structured relative cell complex $\mathfrak{c} : A \hookrightarrow X$ in $\mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$, we denote by

$$\mathfrak{C}_{\mathfrak{c},r} := \bigcup_{b \in \mathbb{B}} \mathfrak{C}_{r,b}$$

its set of cells of the form $\mathbf{R}^r \circledcirc D \to X$ (or equivalently $D \to X_r$). Elements of this set are called *cells of type r*.

[3] Mind the fact that as opposed to the preceding section, all variances have changed, and hence $\mathbf{R}^r \circledcirc -$ turns objects in $\mathbf{C}$ into functors, as opposed to the other way around.

Before we state the main result concerning cell structures on diagrams, let us prove a useful lemma.

Notation 8.3.4.5. In the following, whenever we consider a characteristic map $\sigma\colon \mathbf{R}^r \circledcirc D \to X$ of a structured cell complex $X \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$, the associated generating boundary inclusion $\partial D \to D$ in $\mathbf{C}$ will be denoted by b. Its associated 1-cell structured relative cell complex will be denoted by $\mathfrak{b}$.

Notation 8.3.4.6. We will constantly encounter the source of the Leibniz construction applied to the relative latching functor $(L_r \to (-)_r) = (\partial\mathbf{R}_r \to \mathbf{R}_r) \circledast -$. It will be useful to have a notation to refer to the latter. Given $r \in \mathbf{R}$ and $c\colon A \to X \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$, we will use the notation

$$s\hat{L}_r(c) \coloneqq L_r(X) \cup_{L_r(A)} A_r .$$

We use analogous notation in the covariant case.

Lemma 8.3.4.7. *Let $\mathfrak{c}\colon A \hookrightarrow X$ be a structured relative cell complex in $\mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$ and let $\overline{r} \in \mathbf{R}$. Then, under the identification $\mathbf{R}_{\overline{r}} \circledast X \cong X_{\overline{r}}$, the set of cells of $\iota_{\overline{r}} \hat{\circledast} \mathfrak{c}\colon s\hat{L}_{\overline{r}}(c) \to X_{\overline{r}}$ is given by*

$$\mathfrak{C}_{\iota_{\overline{r}} \hat{\circledast} \mathfrak{c}} = \mathfrak{C}_{\mathfrak{c},\overline{r}} .$$

Proof. By Lemma 8.2.2.5 the set of cells of $\mathfrak{C}_{\iota_{\overline{r}} \hat{\circledast} \mathfrak{c}}$ is given by

$$\bigsqcup_{\sigma:\mathbf{R}^r \circledcirc D \to X \in \mathfrak{C}_{\mathfrak{c}}} (\mathbf{R}_{\overline{r}} \circledcirc \sigma) \mathfrak{C}_{\iota_{\overline{r}} \hat{\circledcirc} (\iota^r \hat{\circledcirc} b)}$$

where we denoted $b\colon \partial D \to D \in \mathbb{B}$. Under the canonical associator isomorphisms of cellularized bifunctors of Proposition 8.3.3.10, we obtain the equalities

$$\begin{aligned}(\mathbf{R}_{\overline{r}} \circledcirc \sigma\colon (\mathbf{R}_{\overline{r}} \circledcirc (\mathbf{R}^r \circledcirc \sigma) \to \mathbf{R}_{\overline{r}} \circledcirc X)\mathfrak{C}_{\iota_{\overline{r}} \hat{\circledcirc} (\iota^r \hat{\circledcirc} \mathfrak{b})} &= ((\mathbf{R}_{\overline{r}} \circledcirc \mathbf{R}^r) \circledcirc c \to \mathbf{R}_{\overline{r}} \circledcirc X)\mathfrak{C}_{(\iota_{\overline{r}} \hat{\circledcirc} \iota^r) \hat{\circledcirc} \mathfrak{b}} \\ &= (\mathbf{R}^r_{\overline{r}} * D \to X_r)\mathfrak{C}_{(\iota_{\overline{r}} \hat{\circledast} \iota^r) * \mathfrak{b}}\end{aligned}$$

Now, by Example 8.3.2.9, $(\iota_{\overline{r}} \hat{\circledast} \iota^r)$ is a relative cell complex with empty set of cells, whenever $\overline{r} \neq r$, and given by a single cell (the identity on r), otherwise. In other words, it fits into a cobase change square

$$\begin{tikzcd}
\varnothing \arrow[r, hook] \arrow[d] & \mathbf{R}_{\overline{r}} \circledast \partial\mathbf{R}^r \cup_{\partial\mathbf{R}_{\overline{r}} \circledast \partial\mathbf{R}^r} \partial\mathbf{R}_{\overline{r}} \circledast \mathbf{R}^r \arrow[d] \\
\{1_r\} \arrow[r, hook] & \mathbf{R}_{\overline{r}} \circledast \mathbf{R}^r \arrow[d, equal] \\
 & \mathbf{R}^r_{\overline{r}}
\end{tikzcd} \tag{8.106}$$

By the compatibility of cellularized functors with cobase changes, it thus follows that

$$(\mathbf{R}^r_{\overline{r}} * D \to X_r)\mathfrak{C}_{(\iota_{\overline{r}} \hat{\circledast} \iota^r) * \mathfrak{b}} = (\{1_r\} * D \xrightarrow{\sigma} X_r)\mathfrak{C}_{(\varnothing \to \{1_r\}) \hat{*} \mathfrak{b}} = \sigma \mathfrak{C}_{\mathfrak{b}} = \{\sigma\colon D \to X^r\} .$$

Hence,

$$\mathfrak{C}_{\iota_{\overline{r}} \hat{\circledast} \mathfrak{c}} = \bigsqcup_{\sigma:\mathbf{R}^r \circledcirc D \to X \in \mathfrak{C}_{\mathfrak{c}}} (\mathbf{R}_{\overline{r}} \circledcirc \sigma) \mathfrak{C}_{\iota_{\overline{r}} \hat{\circledcirc} (\iota^r \hat{\circledcirc} b)} = \{\sigma\colon D \to X_r \mid r = \overline{r}, \sigma \in \mathfrak{C}_{\mathfrak{c}}\} .$$

□

Theorem 8.3.4.8. *Given a morphism $c\colon A \to X \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$ the following map is a bijection*

$$\begin{aligned}&\{\mathfrak{C}_{\mathfrak{c}} \mid \mathfrak{C}_{\mathfrak{c}} \text{ defines a relative cell structure on } c\} \\ &\to \Pi_{r \in \mathbf{R}} \{\mathfrak{C}_{\mathfrak{d}_r} \mid \mathfrak{C}_{\mathfrak{d}_r} \text{ defines a relative cell structure on } \iota_r \hat{\circledcirc} c : s\hat{L}_r(c) \to X_r\}, \\ &\mathfrak{C}_{\mathfrak{c}} \mapsto (\mathfrak{C}_{\iota_r \hat{\circledcirc} \mathfrak{c}})_{r \in \mathbf{R}} .\end{aligned}$$

The inverse is given by mapping a family of cell structures $(\mathfrak{C}_{\mathfrak{d}_r})_{r \in \mathbf{R}}$ to the cell structure

$$\{\mathbf{R}^r \circledcirc D \xrightarrow{\sigma} X \mid r \in \mathbf{R}, \sigma \in \mathfrak{C}_{\mathfrak{d}_r}\}$$

In other words, to cellularize a morphism $c: A \to X \in \mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$ is the same as to cellularize its relative latching maps.

Proof. Observe that, if we can show that the specified inverse is a well defined map, then by Lemma 8.3.4.7 the two constructions are clearly inverse to each other. To see this, we only need to show that, for a family of cell structures $(\mathfrak{C}_{\mathfrak{d}_r})_{\mathbf{R}}$ with $\mathfrak{d}_r: s\hat{L}_r(c) \to X_r$, the set

$$\{\mathbf{R}_r \circledcirc D \xrightarrow{\sigma} X \mid r \in \mathbf{R}, \sigma \in \mathfrak{C}_{\mathbf{d}_r}\}$$

defines a cell structure on c. In [RV13], it is shown that c can be written as a transfinite composition

$$A \to A \cup_{\mathrm{sk}_0 A} \mathrm{sk}_0 X \to \cdots \to A \cup_{\mathrm{sk}_n A} \mathrm{sk}_n X \to \cdots \to X\,.$$

Furthermore, the morphism $A \cup_{\mathrm{sk}_{n-1} A} \mathrm{sk}_{n-1} \to A \cup_{\mathrm{sk}_n A} \mathrm{sk}_n X$ fits into a pushout diagram

$$\begin{tikzcd} \bigsqcup \partial\mathbf{R}^r \circledcirc X_r \cup_{\partial\mathbf{R}^r \circledcirc s\hat{L}_r(c)} \mathbf{R}^r \circledcirc s\hat{L}_r(c) \arrow[r] \arrow[d, "\iota^r \hat{\circledcirc}(\iota_r \hat{\circledast} c)"] & A \cup_{\mathrm{sk}_{n-1} A} \mathrm{sk}_{n-1} X \arrow[d] \\ \bigsqcup_{\deg(r)=n} \mathbf{R}^r \circledcirc X_r \arrow[r] & A \cup_{\mathrm{sk}_n A} \mathrm{sk}_n X\,. \end{tikzcd} \tag{8.107}$$

Then the claimed cell structure on c, is exactly the vertical transfinite composition of these cell structures on $A \cup_{\mathrm{sk}_n A} \mathrm{sk}_n X \to A \cup_{\mathrm{sk}_{n+1} A} \mathrm{sk}_{n+1} X$. □

Next, let us study the consequences of this result for the special case where $\mathbf{C} = \mathbf{Set}$.

Notation 8.3.4.9. Under the Yoneda lemma, there is a canonical natural isomorphism

$$\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}(\mathbf{R}^r, A) \cong A_r$$

for $A \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ and $r \in \mathbf{R}$, given by evaluating a natural transformation at $1_r \in \mathbf{R}^r_r$. We will generally think of this isomorphism as an identity, and make no distinction between elements of A_r and morphisms $\mathbf{R}^r \to A$. Furthermore, we will always treat $\mathbf{R}$ as a subcategory of $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ via the covariant Yoneda embedding. In particular, it makes perfect sense to write $\sigma \circ f$, for $f: \overline{r} \to r$ and $\sigma \in A_r$, to refer to the element $A_f(\sigma)$.

Notation 8.3.4.10. Given a presheaf $A \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, $r \in \mathbf{R}$, and an element $\sigma \in A_r$, we say that σ is degenerate, if σ is of the form

$$\sigma = \tau \circ f$$

where $f: r \to r'$ in $\mathbf{R}^-$, $f \neq 1$ is a non-trivial degeneracy map and $\tau \in A_{r'}$. We will denote the subset of A_r given by non-degenerate elements by $A_{r,n.d.}$.

In turns out that in the cellularized category $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, being a cell complex is a property, rather than an additional structure. Unlike in the world of simplicial sets, however, not every presheaf $A \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ needs to admit a cell structure.

Corollary 8.3.4.11. *Let $A \xrightarrow{c} Y \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$. Then c admits at most one cell structure. Furthermore, the following are equivalent:*

1. *c admits a cell structure;*

2. *For every $r \in \mathbf{R}$, the relative latching map*

$$\iota_r \hat{\circledast} c: s\hat{L}_r(c) = \partial\mathbf{R}_r \circledast Y \cup_{\partial R_r \circledast A} A_r \to Y_r$$

is injective.

Suppose that one of these two equivalent conditions holds. Then the unique cell structure on c is given by the set of non-degenerate elements in Y that are not in the image of A.

$$\mathfrak{C}_c := \bigsqcup_{r \in \mathbf{R}} \{\sigma \colon \mathbf{R}^r \xrightarrow{\sigma} Y \mid \sigma \in (Y_r \smallsetminus c(A_r)) \cap Y_{r,n.d.})\} .$$

Proof. The first two claims are immediate from Theorem 8.3.4.8, using the fact that in **Set**, every morphism admits at most one cell structure, and the latter exists if and only if the morphisms is an injection. For the remaining claim, again using the bijection in Theorem 8.3.4.8, it suffices to see that, for $r \in \mathbf{R}$, the set

$$S = \{\sigma \colon \mathbf{R}^r \xrightarrow{\sigma} Y \mid \sigma \in Y_{r,n.d.} \smallsetminus c(A_r) \cap Y_{r,n.d.})\}$$

defines a cell structure on $\iota_r \hat{\circledast} c \colon A_r \cup_{\partial \mathbf{R}_r \circledast A} \partial \mathbf{R}_r \circledast A \to Y_r$. In other words, as this is now a statement concerning relative cell complexes in **Set**, we need to show that S is precisely the complement of the image of $\iota_r \hat{\circledast} c$. By definition, the latter complement is given by the intersection

$$Y_r \smallsetminus (c(A_r) \cup \mathrm{im}(\iota_r \circledast Y)) = (Y_r \smallsetminus c(A_r)) \cap (Y_r \smallsetminus \mathrm{im}(\iota_r \circledast Y)).$$

Now, observe that

$$(\iota_r \circledast Y) \colon \partial \mathbf{R}_r \circledast Y \to Y_r$$

(using the explicit description of weighted colimits in terms of coequalizers) is given by

$$[(f, \tau)] \mapsto (\tau \circ f)$$

for $\tau \in Y_{r'}$, $f \in (\partial \mathbf{R}_r)^{r'}$. As $(\partial \mathbf{R}_r)^{r'}$ consists precisely of such morphisms $f \colon r \to r'$, for which $f^- \neq 1$, it follows that the image of $\iota_r \circledast Y$ is the set of degenerate elements of Y_r, and hence that $Y_r \smallsetminus \mathrm{im}(\iota_r \circledast Y)$ is the set of non-degenerate elements. Consequently, we obtain

$$Y_r \smallsetminus (c(A_r) \cup \mathrm{im}(\iota_r \circledast Y)) = (Y_r \smallsetminus c(A_r)) \cap (Y_r \smallsetminus \mathrm{im}(\iota_r \circledast Y)) = (Y_r \smallsetminus c(A_r)) \cap Y_{r,n.d.},$$

as was to be shown. □

In particular, we obtain a simple criterion to identify morphisms of structured relative cell complexes in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$.

Corollary 8.3.4.12. *Given two relative cell complexes $c \colon A \hookrightarrow X$ and $d \colon B \hookrightarrow Y$ in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, then a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "c"] & B \arrow[d, hook, "d"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{8.108}$$

defines a morphism of structured relative cell complexes if and only if f maps the non-degenerate elements of X that are not in A to non-degenerate elements of Y that are not in B.

Remark 8.3.4.13. In particular, it follows from Corollary 8.3.4.12, that every isomorphism between relative cell complexes in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ defines an isomorphism of the associated (unique) structured relative cell complexes.

8.3.5 Elegant Reedy categories

The obvious question arises, for what kind of Reedy categories we can expect every presheaf to admit a (necessarily unique) cell structure, or more generally, when we can expect the monomorphisms to be exactly the relative cell complexes. The latter equivalence holds exactly when working with so-called *elegant Reedy categories*. These were introduced in [BR12]. We recommend [nLa24d] for a concise overview.

Recollection 8.3.5.1. Recall that a reedy category $\mathbf{R}$ is called *elegant*, if one of the following equivalent conditions holds:

1. For every pair of arrows $f_1: r_0 \to r_1, f_2: r_0 \to r_2 \in \mathbf{R}^-$, there exists a diagram
$$\begin{tikzcd} r_0 \arrow[r, "f_1"] \arrow[d, "f_2"] & r_1 \arrow[d] \\ r_2 \arrow[r] & r \end{tikzcd} \tag{8.109}$$
in $\mathbf{R}^-$, which is absolute pushout in $\mathbf{R}$. Recall that a square is called *absolute pushout*, if its image under the covariant Yoneda embedding $\mathbf{R} \to \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ is pushout.

2. For every presheaf X on $\mathbf{R}$, an every element $x \in X_r$, with $r \in \mathbf{R}$, there exists unique arrow $s: r \to r'$ in $\mathbf{R}^-$ and a unique element $x' \in X_{r'}$, such that
$$X(r)x' = x$$
and x' is non-degenerate.

3. For every monomorphism of presheaves $c: X \hookrightarrow Y \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, and every $r \in \mathbf{R}$ the induced morphism
$$X_r \cup_{\partial_r X} \partial_r Y \to Y$$
is a monomorphism.

Example 8.3.5.2. The most prominent example for an elegant Reedy category is probably the Reedy category Δ of finite, non-empty, linear posets of the form $[n] = \{0, \dots, n\}$, with face maps given by inclusions and degeneracy maps given by surjections.

Example 8.3.5.3. Any Reedy category without degeneracies is elegant. In particular, if we equip a poset P with the Reedy structure where all morphisms are face maps, then P forms an elegant Reedy category.

Example 8.3.5.4. Given any presheaf on an elegant Reedy $\mathbf{R}$, $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, its category of elements $\mathbf{el}(X) \cong \mathbf{R}_{/X}$, equipped with the induced Reedy structure (Example 8.3.2.3) is again an elegant Reedy category.

In the context of elegant Reedy categories, the notion of a presheaf which is a cell complex becomes essentially superfluous, as indicated by the following corollary of Corollary 8.3.4.11:

Corollary 8.3.5.5. *Let $\mathbf{R}$ be an elegant Reedy category. Then every presheaf $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ admits a unique cell structure. Furthermore, a morphism of presheaves $X \to Y$ admits the (unique) structure of a relative cell complex, if and only if it is a monomorphism.*

In Section 13.1, we are particularly interested in studying presheaf categories of the form $\mathbf{Set}^{(\mathbf{S}\times\mathbf{R})^{\mathrm{op}}} \cong (\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}})^{\mathbf{S}^{\mathrm{op}}}$, where $\mathbf{R}$ and $\mathbf{S}$ are elegant Reedy categories. For these contexts, it will be useful to have the following result at hand.

Lemma 8.3.5.6. *Let $\mathbf{R}$ and $\mathbf{S}$ be elegant Reedy categories. Then $\mathbf{R} \times \mathbf{S}$, equipped with the inherited Reedy structure, is again an elegant Reedy category.*

Proof. Let $c: X \hookrightarrow Y \in \mathbf{Set}^{(\mathbf{R}\times\mathbf{S})^{\mathrm{op}}}$ be a monomorphism of presheafs. We show that, for any pair $r \in \mathbf{R}, s \in \mathbf{S}$ the latching map
$$X_{r,s} \cup_{\partial_{r,s} X} \partial_{r,s} Y \to Y_{r,s}$$
is a monomorphism. Using Example 8.3.2.9, we may equivalently express this map as
$$\iota_r \hat{\circledast} (\iota_s \hat{\circledcirc} c)$$

in other words, it suffices to see that the relative latching maps of $\iota_s \hat{\circledcirc} c$ are monomorphisms. It follows by [nLa24d, Thm. 2.6], which is a generalization of [BR12, Proposition 3.15], that this is in turn equivalent to $\iota_s \hat{\circledcirc} c$ being a monomorphism. In other words, it suffices to show that

$$\mathbf{R}_r \hat{\circledast} (\iota_s \hat{\circledcirc} c)$$

is a monomorphism. By commutativity of colimits (changing the order of $\mathbf{R}$ and $\mathbf{S}$), the latter map is isomorphic to

$$\iota_s \hat{\circledast} (c_{r,\bullet})$$

which is the relative latching map of $c_{r,\bullet}$. As $\mathbf{S}$ is elegant, the latter latching map is a monomorphism if $c_{r,\bullet}$ is a monomorphism. Since monomorphisms in functor categories are detected pointwise, this is immediate from c being a monomorphism. $\square$

As a consequence of this lemma, together with Observation 8.3.3.8, we obtain:

Corollary 8.3.5.7. *Let $\mathbf{R}$ and $\mathbf{S}$ be elegant Reedy categories. Then the statement of Corollary 8.3.5.5 also applies to the cellularized category of diagrams $(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}})^{\mathbf{S}^{\mathrm{op}}}$.*

Finally, it will be useful to observe that degeneracy maps in an elegant Reedy category admit sections by face maps.

Lemma 8.3.5.8. *Let $\mathbf{R}$ be an elegant Reedy category and let $f: r \to r' \in \mathbf{R}^-$ be a degeneracy map. Then there exists a face map $s: r' \to r \in \mathbf{R}^+$ such that $1_{r'} = f \circ s$.*

Proof. In [nLa24d, Lem. 2.1] it is shown that every degeneracy map in an elegant Reedy category admits a section. Such a section is, by assumption, a split monomorphism. Hence, it remains to show that every split monomorphism is a face map. Indeed, let $s: r' \to r$ be a split monomorphism. Consider the canonical factorization $s = s^+ \circ s^-$ of s into a degeneracy map followed by a face map. It suffices to show that s^- is the identity. Observe that since s is a split monomorphism, so is s^-. As s^- is also a degeneracy map, it follows that s^- is also a split epimorphism. Consequently, s^- is an isomorphism. As every isomorphism in a Reedy category is already the identity, it follows that $s^- = 1$, as was to be shown. $\square$

8.3.6 Functors between cellularized categories of diagrams

Now that we have a better understanding of the precise nature of structured cell complexes in $\mathbf{C}^{\mathbf{R}}$, we obtain the following alternative characterization of the cellularized $\mathbf{C}^{\mathbf{R}}$. Recall the definition of the bicategories of cellularized categories from Construction 8.2.1.5 as well as the definition of the category of cellular bifunctors from Construction 8.2.5.10. We can think of $\mathbf{C}^{\mathbf{R}}$ as representing the functor which associates to a category $\mathbf{D}$ the category of cellular bifunctors $\mathbf{CellBiFun}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \times \mathbf{C}, \mathbf{D})$.

Notation 8.3.6.1. Given a cellularized functor $\mathfrak{F}: \mathbf{C} \to \mathbf{D}$, we denote by

$$- \circledast \mathfrak{F}(-): \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \times \mathbf{C} \to \mathbf{D}$$

the cellularized bifunctor obtained by precomposing the cellularized bifunctor $- \circledast -$ with $\mathfrak{F}$ in the first argument (see Observation 8.2.5.12).

Proposition 8.3.6.2. *Let $\mathbf{C}$ and $\mathbf{D}$ be cellularized categories and $\mathbf{R}$ a Reedy category. The functor*

$$\begin{aligned} \mathbf{CellCat}(\mathbf{C}, \mathbf{D}^{\mathbf{R}}) &\to \mathbf{CellBiFun}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \times \mathbf{C}, \mathbf{D}) \\ \mathfrak{F} &\mapsto - \circledast \mathfrak{F}(-)\,. \end{aligned}$$

acting on morphisms in the obvious fashion, is an equivalence of categories.

Proof. To a cellularized bifunctor $\otimes_{\mathfrak{F}}$, we associate the functor

$$F \colon \mathbf{C} \to \mathbf{D}^{\mathbf{R}}$$
$$X \mapsto \mathbf{R}^{\bullet} \otimes_F X$$

acting on morphisms in the obvious fashion. Let us explain how to cellularize this cocontinuous functor. By Lemma 8.2.2.5 and Theorem 8.3.4.8, to specify a cell structure on F, it suffices to expose cell structures after application of $\iota^r \hat{\circledast} F(b)$, for each $r \in \mathbf{R}$ and $b \in \mathbb{B}_{\mathbf{C}}$. By definition of $F(-)$, we have $F(-)^r = \mathbf{R}^r \otimes_F -$. As $- \otimes_F -$ preserves colimits in the first variable, it follows that there is a canonical isomorphism

$$U \circledast F(X) \cong U \otimes_F X$$

for $X \in \mathbf{C}$ and $U \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$. Consequently, for each $r \in \mathbf{R}$, there is a canonical isomorphism

$$\iota^r \hat{\circledast} F(b) \cong \iota^r \hat{\otimes}_F b \,,$$

for $b \in \mathbb{B}_{\mathbf{C}}$. By Theorem 8.3.4.8, we may thus use the cell structure of $\iota^r \hat{\otimes}_{\mathfrak{F}} \mathfrak{b}$ to cellularize F. Given $\mathfrak{G} \in \mathbf{CellCat}(\mathbf{C}, \mathbf{D}^{\mathbf{R}})$, let us denote the associated cellularized bifunctor by $\otimes_{\mathfrak{G}} \colon \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \times \mathbf{C} \to \mathbf{D}$. Starting with a bivariate functor $\otimes_F$, cocontinuous in both variables we then have canonical natural isomorphisms

$$U \circledast F(X) \cong U \circledast (\mathbf{R}^{\bullet} \otimes_F X) \cong (U \circledast \mathbf{R}^{\bullet}) \otimes_F X \cong U \otimes_F X$$

and conversely starting with $\mathfrak{G} \in \mathbf{CellCat}(\mathbf{C}, \mathbf{D}^{\mathbf{R}})$

$$\mathbf{R}^r \otimes_G X = \mathbf{R}^r \circledast G(X) \cong G(X)^r$$

showing that the two functors $\otimes_{\mathfrak{F}} \mapsto \mathfrak{F}$ and $\mathfrak{G} \mapsto \otimes_{\mathfrak{G}}$ are inverse to each other, up to natural isomorphism, if we ignore cell structures for now. That these isomorphisms are also isomorphism on the cellular level can then be verified directly form the construction via Theorem 8.3.4.8. □

The above proposition gives us one way to characterize the Reedy cellularizations on $\mathbf{C}^{\mathbf{R}}$. Another insightful perspective is given by the following cellularized version of left Kan extension.

Corollary 8.3.6.3. *The functor*

$$\mathbf{RCell}(\mathbf{C}^{\mathbf{R}}) \to \mathbf{CellCat}^{\to}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$$
$$(X \overset{\mathfrak{c}}{\hookrightarrow} Y) \mapsto (- \circledast X \xrightarrow{- \otimes \mathfrak{c}} \circledast Y) \,,$$

acting on morphisms in the obvious way, is an equivalence of categories, which preserves

1. *(transfinite) vertical composition,*
2. *cobase changes,*
3. *inclusions of subcomplexes,*

and restricts to an equivalence

$$\mathbf{Cell}(\mathbf{C}^{\mathbf{R}}) \simeq \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C}) \,.$$

Proof. The preservation properties below are an incarnation of Lemma 8.2.5.9. The two claimed equivalences follow from Proposition 8.3.6.2 by setting, respectively, $\mathbf{C} = (\mathbf{Set}^{[1]}, \{u\})$ and $\mathbf{C} = \mathbf{Set}$ there (with $(\mathbf{Set}^{[1]}, \{u\})$ as in Observation 8.2.3.2) and using Observation 8.2.3.2, together with Observation 8.2.5.11. □

Notation 8.3.6.4. Given a cellularized functor $\mathfrak{F} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$, we denote the associated cellularized functor absolute cell complex in $\mathbf{Cell}(\mathbf{C}^{\mathbf{R}})$ (determined uniquely up to natural isomorphism) by $\mathfrak{F}|_{\mathbf{R}}$. This notation makes sense insofar as the underlying functor of $\mathfrak{F}|_{\mathbf{R}}$ is indeed the restriction of F along the Yoneda embedding $\mathbf{R} \hookrightarrow \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$.

Observation 8.3.6.5. Note that, by definition of the equivalence

$$\mathbf{Cell}(\mathbf{C}^{\mathbf{R}}) \simeq \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$$

one obtains a formula computing the value of a relative cellularized functor $\mathfrak{F} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$ at a relative cell complex, in terms of its associated absolute cell complex $\mathfrak{F}\,|_{\mathbf{R}} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{R}})$. Namely, given $c: U \hookrightarrow V \in \mathbf{RCell}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}})$, we may compute $\mathfrak{F}(c)$ as

$$\mathfrak{F}(c) \cong c \circledast \mathfrak{F}|_{\mathbf{R}}\,.$$

A particularly interesting case of Corollary 8.3.6.3 is the following corollary:

Corollary 8.3.6.6. *Given two Reedy categories* $\mathbf{R}$ *and* $\mathbf{T}$*, the functor*

$$\begin{aligned} \mathbf{RCell}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}) \cong \mathbf{RCell}({\mathbf{Set}^{\mathbf{T}}})^{\mathbf{R}^{\mathrm{op}}}) &\to \mathbf{CellCat}^{\rightarrow}(\mathbf{Set}^{\mathbf{R}}, \mathbf{Set}^{\mathbf{T}}) \\ (X \xhookrightarrow{c} Y) &\mapsto (- \otimes X \xrightarrow{-\otimes c} \otimes Y) \end{aligned}$$

defined in the obvious way on morphisms defines an equivalence of categories, which restricts to a natural equivalence

$$\mathbf{Cell}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}) \simeq \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}}, \mathbf{Set}^{\mathbf{T}})\,.$$

Remark 8.3.6.7. Observe that it follows by Corollary 8.3.4.11 that being a cellularized functor $\mathbf{Set}^{\mathbf{R}} \to \mathbf{Set}^{\mathbf{T}}$ is really a property, and not an additional structure. This mirrors the situation that being a cell complex in $\mathbf{Cell}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{R}})$ is a property.

In particular, specifying a cellularized functor categories $\mathbf{Set}^{\mathbf{R}} \to \mathbf{Set}^{\mathbf{T}}$ is really the same data as specifying a cell complex in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{T}}$. Let us give some specific examples of such functors and their corresponding cell complexes:

Example 8.3.6.8. The identity functor $1_{\mathbf{Set}^{\mathbf{R}}}$ corresponds to $\mathbf{R}^{\bullet}_{\bullet} \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{R}}$. Under Corollary 8.3.4.11, its set of cells is given by the set of elements

$$\{1_r \in \mathbf{R}^r_r \mid r \in \mathbf{R}\}.$$

Example 8.3.6.9. The evaluation at r functor $(-)^r: \mathbf{Set}^{\mathbf{R}} \to \mathbf{Set}$ corresponds to the presheaf $\mathbf{R}^r \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$. Its set of non-degenerate elements is given by the set

$$\coprod_{\overline{r}\in\mathbf{R}} \{f : \overline{r} \to r \mid f \in \mathbf{R}^+\}.$$

Example 8.3.6.10. The latching functor $L^r(-): \mathbf{Set}^{\mathbf{R}} \to \mathbf{Set}$ corresponds to the presheaf $\partial\mathbf{R}^r \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$. Its set of non-degenerate elements is given by the set

$$\coprod_{\overline{r}\in\mathbf{R}} \{f : \overline{r} \to r \mid f \in \mathbf{R}^+, f \neq 1_r\}.$$

We may thus think of the canonical relative cell complex $\iota_r : \partial\mathbf{R}^r \to \mathbf{R}^r$ as the inclusion of a subcomplex.

Example 8.3.6.11. As we have seen in Recollection 8.3.2.4, the skeleton functor $\mathrm{sk}_n : \mathbf{Set}^{\mathbf{R}} \to \mathbf{Set}^{\mathbf{R}}$, for $n \in \mathbb{N}$, corresponds to $\mathrm{sk}_n\mathbf{R}^{\bullet}_{\bullet} \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{R}}$ given by

$$(\mathrm{sk}_n\mathbf{R}^{\bullet}_{\bullet})^r_{\overline{r}} = \{f : \overline{r} \to r \mid f \text{ factors through } \mathbf{R}_{\leq n}\}.$$

Under Corollary 8.3.4.11, its set of cells is given by the set of elements

$$\{1_r \in \mathbf{R}^r_r \mid r \in \mathbf{R}, \deg(r) \le n\}.$$

In particular, we obtain a sequence of inclusions of subcomplexes

$$\varnothing = \mathrm{sk}_{-1}\mathbf{R}^\bullet_\bullet \hookrightarrow \mathrm{sk}_0\mathbf{R}^\bullet_\bullet \hookrightarrow \mathrm{sk}_1\mathbf{R}^\bullet_\bullet \hookrightarrow \cdots \hookrightarrow \mathrm{sk}_n\mathbf{R}^\bullet_\bullet \hookrightarrow \dots$$

the transfinite composition of which is given by $\mathbf{R}^\bullet_\bullet$.

Example 8.3.6.12. Denote $\mathbf{Q} = \mathbf{R}^{\mathrm{op}} \times \mathbf{R}$, using notation as in Notation 8.3.2.7. Observe that there is a cobase change of relative cell complexes in $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}\times\mathbf{R}}$

$$\begin{tikzcd}
\coprod_{r\in\mathbf{R},\deg(r)=n} \partial\mathbf{Q}^{r,\bullet}_{\bullet,r} \arrow[r] \arrow[d, hook] & \mathrm{sk}_{n-1}\mathbf{R}^\bullet_\bullet \arrow[d, hook] \\
\coprod_{r\in\mathbf{R},\deg(r)=n} \mathbf{Q}^{r,\bullet}_{\bullet,r} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{sk}_n\mathbf{R}^\bullet_\bullet
\end{tikzcd} \tag{8.110}$$

with the lower vertical given by $(f,g) \mapsto f \circ g$. Indeed, observe that the lower horizontal induces a bijection on cells mapping the set of non-degenerate elements $\{(1_r, 1_r) \mid r \in \mathbf{R}, \deg(r) = n\}$ on the left-hand side to precisely the set of non-degenerate elements $\{1_r \mid r \in \mathbf{R}, \deg(r) = n\}$, which specifies the relative cell structure on $\mathrm{sk}_{n-1}\mathbf{R}^\bullet_\bullet \hookrightarrow \mathrm{sk}_n\mathbf{R}^\bullet_\bullet$. Hence, the claim follows by Corollary 8.1.4.2.

We obtain the following corollary, which was observed in somewhat different languag in [RV13].

Corollary 8.3.6.13. *Let* $\mathbf{C}$ *be any cellularized category. For any* $n \ge -1$*, the* n*-skeleton functor* $\mathrm{sk}_n \colon \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{R}}$ *carries the structure of a cellularized functor, such that the canonical morphism* $\mathrm{sk}_n \to 1_{\mathbf{C}}$ *is an inclusion of cellularized functors. These cellularized functors assemble into a transfinite composition diagram*

$$\varnothing \to \mathrm{sk}_{-1} \hookrightarrow \mathrm{sk}_0 \hookrightarrow \mathrm{sk}_1 \hookrightarrow \cdots \hookrightarrow \mathrm{sk}_n \hookrightarrow \cdots \hookrightarrow 1_{\mathbf{C}}.$$

of inclusions of cellularized subfunctors. At each $n \ge 0$*, the associated cellularized relative functor* $\mathrm{sk}_n \to \mathrm{sk}_{n+1}$ *(see Definition 8.2.4.3) fits into a cobase change square of cellularized functors*

$$\begin{tikzcd}
\coprod_{\deg(r)=n}(\mathbf{R}_r \circledcirc \partial\mathbf{R}^r \cup_{\partial\mathbf{R}_r\circledcirc\partial\mathbf{R}^r} \partial\mathbf{R}_r \circledcirc \mathbf{R}^r) \circledcirc - \arrow[r] \arrow[d, hook, "(\iota_r \hat{\circledcirc} \iota^r)\circledcirc"] & \mathrm{sk}_{n-1} \arrow[d, hook] \\
\coprod_{\deg(r)=n}(\mathbf{R}_r \circledcirc \mathbf{R}^r) \circledcirc - \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{sk}_n\,.
\end{tikzcd} \tag{8.111}$$

Proof. This follows by applying Lemma 8.2.5.9 to Example 8.3.6.11 and Example 8.3.6.12 and using Example 8.3.2.9. ☐

Remark 8.3.6.14. Evaluating Corollary 8.3.6.13 at a relative cell complex $\mathfrak{c} \colon A \hookrightarrow X$ in $\mathbf{C}^{\mathbf{R}}$, provides the decomposition of c into relative cell complexes

$$A \to A \cup_{\mathrm{sk}_0 A} \mathrm{sk}_0 X \to \cdots \to A \cup_{\mathrm{sk}_n A} \mathrm{sk}_n X \to \cdots \to X\,.$$

with cobase change diagrams

$$\begin{tikzcd}
\coprod_{\deg(r)=n} \partial\mathbf{R}_r \circledcirc X^r \cup_{\partial\mathbf{R}_r\circledcirc s\hat{L}^r(c)} \mathbf{R}_r \circledcirc s\hat{L}^r(c) \arrow[r] \arrow[d, "\iota_r\hat{\circledcirc}(\iota^r\hat{\circledast}\mathfrak{c})"] & A \cup_{\mathrm{sk}_{n-1}A} \mathrm{sk}_{n-1}X \arrow[d] \\
\coprod_{\deg(r)=n} \mathbf{R}_r \circledcirc X^r \arrow[r] & A \cup_{\mathrm{sk}_n A} \mathrm{sk}_n X
\end{tikzcd} \tag{8.112}$$

from [RV13], which we already used in the proof of Theorem 8.3.4.8. In particular, in the case of absolute cell complex $\mathfrak{X}$, we obtain a canonical exhaustion by subcomplexes

$$\varnothing \hookrightarrow \mathrm{sk}_0\mathfrak{X} \hookrightarrow \mathrm{sk}_1\mathfrak{X} \hookrightarrow \cdots \hookrightarrow \mathfrak{X}$$

such that the associated relative cell complexes fit into cobase change squares

$$\begin{tikzcd} \coprod_{\deg(r)=n} \partial\mathbf{R}_r \circledcirc X^r \cup_{\partial\mathbf{R}_r \circledcirc L^r(X)} \mathbf{R}_r \circledcirc L^r(X) \arrow[r] \arrow[d, hook, "\iota_r \hat{\circledcirc}(\iota^r \circledast \mathfrak{X})"] & \mathrm{sk}_{n-1}X \arrow[d, hook] \\ \coprod_{\deg(r)=n} \mathbf{R}_r \circledcirc X^r \arrow[r] & \mathrm{sk}_n X\,. \end{tikzcd} \tag{8.113}$$

8.3.7 Change of Reedy category and cellularized colimits

The obvious question arises, when a functor of Reedy categories $f\colon \mathbf{R} \to \mathbf{T}$ has the property that its induced functors f^*, $f_!$, define cellularized functors. The guiding example which we should have in mind here is the case of the constant functor $f\colon \mathbf{R} \to \star$. Replacing the word cellularized with left Quillen-functors, for a second, this question was discussed in great detail in [Bar07]. It turns out that the answer and methodology to tackle this question is much the same. Hence, we will make frequent use of [Bar07] in this subsection. We will furthermore constantly employ the calculus of final functors (i.e.,such functors which have the property that restriction along them does not change the colimits). A good overview can be found in [nLa24f].

Recollection 8.3.7.1. [Bar07] Recall that a morphism of Reedy categories $F\colon \mathbf{R} \to \mathbf{T}$ is a functor of the underlying categories, such that $F(\mathbf{R}^+) \subset \mathbf{T}^+$ and such that $F(\mathbf{R}^-) \subset \mathbf{T}^-$. We will denote the respective restrictions of F by F^+ and F^-.

Notation 8.3.7.2. Given a Reedy category $\mathbf{R}$ and $r \in \mathbf{R}$, we will denote by $\partial\mathbf{R}_{r/}$ ($\partial\mathbf{R}_{/r}$) the full subcategory of the slice (coslice) category given by only such arrows $f\colon r \to r'$ which fulfill $f^- \neq 1$ ($f\colon r' \to r$ which fulfill $f^+ \neq 1$).

Observation 8.3.7.3. Observe that $\partial\mathbf{R}_{r/}$ is the category of elements of $\partial\mathbf{R}_r \in \mathbf{Set}^{\mathbf{R}}$ and that $\partial\mathbf{R}_{/r}$ is the category of elements of $\partial\mathbf{R}^r \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ (where we use the convention that $\mathbf{el}(W) \to \mathbf{R}$ is always a covariant functor, independently of the variance of the functor $W\colon \mathbf{R} \to \mathbf{Set}$).

Observation 8.3.7.4. Given a Reedy category $\mathbf{R}$, observe that $\mathbf{R}^+$ is again a Reedy category, with the degree function inherited from $\mathbf{R}$ and $(\mathbf{R}^+)^+ = \mathbf{R}^+$ and $(\mathbf{R}^+)^-$ the discrete category given by the objects of $\mathbf{R}$. Then, for $r \in \mathbf{R}$ the inclusion of categories

$$I\colon \partial(\mathbf{R}^+)_{/r} \to \partial\mathbf{R}_{/r}$$

is a final functor, i.e., restricting along it does not change colimits. Indeed, the comma category $I_{f/}$ of I at $f\colon r' \to r$ is the category of factorizations (f_1, f_0), $f = f_1 \circ f_0$, with $f_1 \in \mathbf{R}^+$, which has the terminal object (f^+, f^-). Dually, it follows that

$$\partial(\mathbf{R}^-)_{r/} \to \partial\mathbf{R}_{r/}$$

is an initial functor.

Recollection 8.3.7.5 ([Bar07; Hir03])**.** Recall that a Reedy category $\mathbf{R}$ is called *left fibrant* if one of the following equivalent conditions holds.

1. For every model category $\mathbf{M}$, the colimit functor $\varinjlim\colon \mathbf{M}^{\mathbf{R}} \to \mathbf{M}$ is a left Quillen functor.
2. For every $r \in \mathbf{R}$, the category $\partial\mathbf{R}_{r/}$ is empty or connected.
3. For every $r \in \mathbf{R}$, the category $\partial(\mathbf{R}^-)_{r/}$ is empty or connected.

Dually, $\mathbf{R}$ is called *right fibrant* if one of the following equivalent conditions holds.

1. For every model category $\mathbf{M}$, the constant diagram functor $c^*: \mathbf{M} \to \mathbf{M}^{\mathbf{R}}$ is a left Quillen functor.
2. For every $r \in \mathbf{R}$, the category $\partial \mathbf{R}_{/r}$ is empty or connected.
3. For every $r \in \mathbf{R}$, the category $\partial(\mathbf{R}^+)_{/r}$ is empty or connected.

Fibrant Reedy categories are a special case of fibrations of Reedy categories:

Recollection 8.3.7.6. [Bar07] Recall that a Reedy morphism of Reedy categories $F: \mathbf{R} \to \mathbf{T}$ is called a *left fibration* if one of the following equivalent conditions holds.

1. For every model category $\mathbf{M}$, the left Kan extension functor $F_!: \mathbf{M}^{\mathbf{R}} \to \mathbf{M}^{\mathbf{T}}$ is a left Quillen functor.
2. For every $t \in \mathbf{T}$, the Reedy category $F_{/t}$ is left fibrant.
3. For every $t \in \mathbf{T}$, the Reedy category $F^-_{/t}$ is left fibrant.

Dually, $F: \mathbf{R} \to \mathbf{T}$ is called a *right fibration* if one of the following equivalent conditions holds.

1. For every model category $\mathbf{M}$, the precomposition functor $F^*: \mathbf{M}^{\mathbf{T}} \to \mathbf{M}^{\mathbf{R}}$ is a left Quillen functor.
2. For every $t \in \mathbf{T}$, the Reedy category $F_{t/}$ is right fibrant.
3. For every $t \in \mathbf{T}$, the Reedy category $F^+_{t/}$ is right fibrant.

It follows that F is left fibrant if and only if F^{op} is right fibrant.

Let us now connect these results with the theory of cellularized functors.

Proposition 8.3.7.7. *Given a morphism of Reedy categories $F: \mathbf{R} \to \mathbf{T}$, the following conditions are equivalent:*

1. *F is a left fibration.*
2. *$\mathbf{T}^{\bullet}_{F(\bullet)} \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{T}}$ is an absolute cell complex.*
3. *$F_!: \mathbf{Set}^{\mathbf{R}} \to \mathbf{Set}^{\mathbf{T}}$ defines a cellularized funtor (i.e., by Corollary 8.3.4.11, preserves relative cell complexes).*

Dually, the following conditions are equivalent:

1. *F is a right fibration.*
2. *$\mathbf{T}_{\bullet}^{F(\bullet)} \in \mathbf{Set}^{\mathbf{T}^{\mathrm{op}} \times \mathbf{R}}$ is an absolute cell complex.*
3. *$F^*: \mathbf{Set}^{\mathbf{T}} \to \mathbf{Set}^{\mathbf{R}}$ defines a cellularized functor (i.e., by Corollary 8.3.4.11, preserves relative cell complexes).*

Proof. In both series of equivalences, the latter two conditions are equivalent by Corollary 8.3.6.6 together with the identities $F_! \cong \mathbf{T}^{\bullet}_{F(\bullet)} \circledcirc -$ and $F^* \cong \mathbf{T}_{\bullet}^{F(\bullet)} \circledcirc -$ we explained in Example 8.3.1.6 (iv). For the final remaining two equivalences, observe that (by dualizing) it suffices to show the case of left fibrancy. Observe that, by Corollary 8.3.4.11 being a relative cell complex c in $\mathbf{Set}^{\mathbf{R}}$ or $\mathbf{Set}^{\mathbf{T}}$ is equivalent to the latching maps being injections. This, in turn is equivalent to c being a cofibration in the Reedy model structure on $\mathbf{Set}^{\mathbf{R}}$ (or $\mathbf{Set}^{\mathbf{T}}$), where $\mathbf{Set}$ is equipped with the model structure in which cofibrations are given by injective maps and weak equivalences are given by isomorphisms. Since $F_!$ defines a cellularized functor, if and only if it preserves relative cell complexes, this is in turn equivalent to $F_!$ preserving Reedy cofibrations. In particular, it follows that F being left fibrant implies that $F_!$ is a cellularized functor. Finally, let us show that $\mathbf{T}^{\bullet}_{F(\bullet)}$ being a cell complex implies that F is left fibrant. To this end, denote $\mathbf{Q} = \mathbf{R}^{\mathrm{op}} \times \mathbf{T}$, let $(r,t) \in \mathbf{Q}$, and consider the latching map

$$L^t_r(\mathbf{T}^{\bullet}_{F(\bullet)}) = \partial\mathbf{Q}^{\bullet;t}_{r;\bullet} \circledast \mathbf{T}^{\bullet}_{F(\bullet)} \to \mathbf{Q}^{\bullet;t}_{r;\bullet} \circledast \mathbf{T}^{\bullet}_{F(\bullet)} = \mathbf{T}(F(r),t).$$

By Corollary 8.3.4.11, $\mathbf{T}^{\bullet}_{F(\bullet)}$ is a cell complex, if and only if this map is an injection, for every pair (r,t). Using the description of $\circledast$ in terms of the category of elements (Recollection 8.3.1.4), we may compute $\partial\mathbf{Q}^{\bullet;t}_{r;\bullet} \circledast \mathbf{T}^{\bullet}_{F(\bullet)}$ as the colimit of

$$\mathbf{el}(\partial\mathbf{Q}^{\bullet;t}_{r;\bullet}) \to \mathbf{Q} \xrightarrow{\mathbf{T}(F(-),-)} \mathbf{Set}\,.$$

Explictitly, $\mathbf{el}(\partial\mathbf{Q}^{\bullet;t}_{r;\bullet})$ is the twisted arrow category, with objects given by pairs $(f: r \to r', g: t' \to t)$, such that either $f^- \neq 1$ or $g^+ \neq 1$ and and morphisms from (f_0, g_0) to (f_1, g_1) given by pairs of arrows $(r'_1 \to r'_0, t'_0 \to t'_1)$ such that the diagrams

$$\begin{tikzcd} & r \arrow[dl, "f_1"'] \arrow[dr, "f_0"] & \\ r'_1 \arrow[rr] & & r'_0 \end{tikzcd} \qquad \begin{tikzcd} t'_1 \arrow[dr, "g_1"'] & & t'_0 \arrow[ll] \arrow[dl, "g_0"] \\ & t & \end{tikzcd} \tag{8.114}$$

commute. Using Observation 8.3.7.4, we may furthermore restrict to the full subcategory of $\mathbf{el}(\partial\mathbf{Q}^{\bullet;t}_{r;\bullet})$ given by such pairs (f,g) where $f \in \mathbf{R}^-$ and $g \in \mathbf{R}^+$. Let us denote the latter by $\mathbf{TW}$. We may, in turn, compute the colimit of

$$D: \mathbf{TW} \to \mathbf{el}(\partial\mathbf{Q}^{\bullet;t}_{r;\bullet}) \to \mathbf{Q} \xrightarrow{\mathbf{T}(F(-),-)} \mathbf{Set}\,.$$

as the set of path components of the associated category of elements of $\mathbf{el}(D)$, which is explicitly given by the following category: Objects are triples $(f: r \to r', g: t' \to t, s: F(r') \to t')$, with $f \in \mathbf{R}^-$, $g \in \mathbf{R}^+$ and $f \neq 1$ or $g \neq 1$, and morphisms $(f_0, g_0, s_0) \to (f_1, g_1, s_1)$ are given by pairs of arrows $(r'_1 \to r'_0, t'_0 \to t'_1) \in \mathbf{R}^- \times \mathbf{R}^+$ such that the diagrams

$$\begin{tikzcd} & r \arrow[dl, "f_1"'] \arrow[dr, "f_0"] & \\ r'_1 \arrow[rr] & & r'_0 \end{tikzcd} \qquad \begin{tikzcd} t'_1 \arrow[dr, "g_1"'] & & t'_0 \arrow[ll] \arrow[dl, "g_0"] \\ & t & \end{tikzcd} \qquad \begin{tikzcd} F(r'_1) \arrow[r] \arrow[d, "s_1"'] & F(r'_0) \arrow[d, "s_0"] \\ t'_1 & t_0 \arrow[l] \end{tikzcd} \tag{8.115}$$

commute. From this perspective, the latching map

$$L^t_r(\mathbf{T}^{\bullet}_{F(\bullet)}) = \partial\mathbf{Q}^{\bullet;t}_{r;\bullet} \circledast \mathbf{T}^{\bullet}_{F(\bullet)} \to \mathbf{Q}^{\bullet;t}_{r;\bullet} \circledast \mathbf{T}^{\bullet}_{F(\bullet)} = \mathbf{T}(F(r),t)$$

is given by

$$[(f,g,s)] \mapsto (g \circ s \circ F(f))\,.$$

The fiber of the latching map at $\tilde{s} \in \mathbf{T}(F(r),t)$ may thus be computed as the set of path components of the full subcategory $\mathbf{D}$ of $\mathbf{TW}$, given by such triples (f,g,s), for which $g \circ s \circ F(f) = \tilde{s}$. Now, suppose that $\tilde{s} \in \mathbf{R}^-$. Then, for any such triple, (f,g,s) it follows by uniqueness of factorizations, and the assumption that $g \in \mathbf{R}^+$, that $g = 1$. Consequently, we have $\tilde{s} = s \circ F(s)$. Repeating the argument, we obtain $s \in \mathbf{R}^-$. Hence, in this case we may identify $\mathbf{D}$ with the category whose objects are pairs $(f: r \to r', s: F(r') \to t) \in \mathbf{R}^- \times \mathbf{R}^-$, with $f \neq 1$ and $s \circ F(s) = \tilde{s}$ and whose morphisms from (f_0, s_0) to (f_1, s_1) are given by arrows $r'_1 \to r'_0$, such that the diagrams

$$\begin{tikzcd} & r \arrow[dl, "f_1"'] \arrow[dr, "f_0"] & \\ r'_1 \arrow[rr] & & r'_0 \end{tikzcd} \qquad \begin{tikzcd} F(r'_1) \arrow[rr] \arrow[dr, "s_1"'] & & F(r'_0) \arrow[dl, "s_0"] \\ & t & \end{tikzcd} \tag{8.116}$$

commute. However, this is just the opposite of the category $\partial(F^-_{/t})_{\tilde{s}/}$. By assumption, $\mathbf{D}$ is empty or connected. Hence, it follows that F is a left fibration. ☐

As an immediate corollary, using the cellularized bifunctor $- \circledcirc -$ of Proposition 8.3.3.10 and Example 8.3.1.6 (iv), we obtain:

Corollary 8.3.7.8. *If $F\colon \mathbf{R} \to \mathbf{T}$ is a left fibration, and $\mathbf{C}$ a cellularized category, then the canonical natural isomorphism*

$$\begin{tikzcd} \mathbf{C}^{\mathbf{R}} \arrow[rr, bend left, "F_!"] \arrow[rr, bend right, "\mathbf{T}^{\bullet}_{F(\bullet)} \circledcirc -"'] \arrow[rr, phantom, "\Vert\wr"] & & \mathbf{C}^{\mathbf{T}} \end{tikzcd} \tag{8.117}$$

equips $F_!$ with the structure of a cellularized functor. Dually, if F is a right fibration, then the canonical natural isomorphism

$$\begin{tikzcd} \mathbf{C}^{\mathbf{T}} \arrow[rr, bend left, "F^*"] \arrow[rr, bend right, "\mathbf{T}^{F(\bullet)}_{\bullet} \circledcirc -"'] \arrow[rr, phantom, "\Vert\wr"] & & \mathbf{C}^{\mathbf{R}} \end{tikzcd} \tag{8.118}$$

equips F^* with the structure of a cellularized functor.

It will be useful to have an explicit description of the cell structures that arise from restrictions and left Kan extensions.

Proposition 8.3.7.9. *In the situation of Corollary 8.3.7.8, let F be a left fibration. For $r \in \mathbf{R}$ and $t \in \mathbf{T}$, denote by $C^t_{F(r)} \subset \mathbf{T}^t_{F(r)}$ the set of morphisms $f\colon F(r) \to t$ in $\mathbf{T}^-$ and which are furthermore not of the following form*

- *$f = f' \circ F(g)$, for some $g\colon r \to r' \in \mathbf{R}$, with $g^- \neq 1$ and $f'\colon F(r') \to t \in \mathbf{T}$.*

Then, for any relative cell complex $\mathfrak{c}\colon A \to X$ in $\mathbf{C}^{\mathbf{R}}$, the set of cells of $F_!(X)$ of type t is given by

$$\{D \xrightarrow{\sigma} X^r \xrightarrow{\rho_f} \varinjlim_{F(r') \to t} X^{r'} = (F_! X)^t \mid r \in \mathbf{R}, f \in C^t_{F(r)}, \sigma\colon D \to X^r \in \mathfrak{C}_{\mathfrak{c},r}\}\,.$$

where $\rho_f\colon X^r \to \varinjlim_{F(r') \to t} X^{r'}$ is the canonical morphism associated to the element $f \in F_{/t}$.

Similarly, if F is a right fibration, denote by $C^{F(r)}_t \subset \mathbf{T}^{F(r)}_t$ the set of morphisms $f\colon t \to F(r)$ in $\mathbf{T}^+$ and which are furthermore not of the form

- *$f = F(g) \circ f'$, for some $g\colon r' \to r \in \mathbf{R}$, with $g^+ \neq 1$ and $f'\colon t \to F(r') \in \mathbf{T}$.*

Then, for any relative cell complex $\mathfrak{c}\colon A \to X$ in $\mathbf{C}^{\mathbf{R}}$, the set of cells of $F^*(X)$ of type r is given by

$$\{D \xrightarrow{\sigma} X^t \xrightarrow{X^f} X^{F(r)} \mid f\colon t \to F(r) \in C^{F(r)}_t, \sigma\colon D \to X^t \in \mathfrak{C}_{\mathfrak{c},t}\}\,.$$

Remark 8.3.7.10. Before we give a proof, observe that, by the uniqueness of degeneracy face factorizations in a Reedy category, the sets $C^t_{F(r)}$ and $C^{F(r)}_t$ of Proposition 8.3.7.9 are either empty or singletons.

Proof. We only compute the case of $F_!$, the case of F^* is similar. By Lemma 8.2.2.5, the cell structure on $\iota^t \hat{\circledast} F_! \mathfrak{c}$ is given by

$$\bigcup_{\sigma\colon (\iota_r \circledcirc b, \mathbf{R}_r \circledcirc D \to X) \in \mathfrak{C}_{\mathfrak{X}}} (F_! \sigma)^t \mathfrak{C}_{\iota^t \hat{\circledast} F_!(\iota_r \hat{\circledcirc} \mathfrak{b})}\,.$$

Using the associativity of $- \circledcirc -$ and the canonical isomorphism $F_! \cong \mathbf{T}^{\bullet}_{F(\bullet)} \circledcirc -$, we obtain

$$\iota^t \hat{\circledast} F_!(\iota_r \hat{\circledcirc} \mathfrak{b}) \cong (\iota^t \hat{\circledcirc} \mathbf{T}^{\bullet}_{F(\bullet)} \hat{\circledcirc} \iota_r) \hat{\star} \mathfrak{b}.$$

Under the identification $\mathbf{T}^t \circledcirc \mathbf{T}^{\bullet}_{F(\bullet)} \circledcirc \mathbf{R}_r \cong \mathbf{T}^t_{F(r)}$, the subobject of $\mathbf{T}^t_{F(r)}$ given by

$$\begin{aligned}(\iota^t \hat{\circledcirc} \mathbf{T}^{\bullet}_{F(\bullet)} \hat{\circledcirc} \iota_r) &= \mathbf{T}^t \circledcirc \mathbf{T}^{\bullet}_{F(\bullet)} \circledcirc \partial\mathbf{R}_r \cup_{\partial\mathbf{T}^t \circledcirc \mathbf{T}^{\bullet}_{F(\bullet)} \partial\mathbf{R}_r} \partial\mathbf{T}^t \circledcirc \mathbf{T}^{\bullet}_{F(\bullet)} \circledcirc \mathbf{R}_r \\ &\cong \mathbf{T}^t_{F(\bullet)} \circledast \partial\mathbf{R}_r \cup_{\dots} \partial\mathbf{T}^t \circledast \mathbf{T}_{F(r)}\end{aligned}$$

is given by the inclusion of the subset

$$\{f \circ F(g) \colon F(r) \to t \mid r' \in \mathbf{R}, g \colon r \to r', f \colon F(r') \to r, g^- \neq 1\} \cup \{f \colon F(r) \to t \mid f \colon t' \to t, f^+ \neq 1\}\,.$$

The complement of this set is precisely $C^t_{F(r)}$. It follows, that under the canonical isomorphisms above the set of cells of $(\iota^t \hat{\circledcirc} \mathbf{T}^{\bullet}_{F(\bullet)} \hat{\circledcirc} \iota_r) \hat{*} \mathfrak{b}$, for $b \colon \partial D \to D$, is given by

$$\{D \overset{i_f}{\hookrightarrow} \coprod_{\mathbf{T}^t_{F(r)}} D = \mathbf{T}^t_{F(r)} * D \mid f \in C^t_{F(r)}\}$$

Furthermore, under these canonical isomorphisms, $(F_!\sigma)^t \colon (F_!(\mathbf{R}_r \circledcirc D))^t \to (F_!X)^t = \varinjlim_{F(r') \to t} X^{r'}$ is given by the morphism

$$\mathbf{T}^t_{F(r)} * D = \coprod_{\mathbf{T}^t_{F(r)}} D \to \varinjlim_{F(r') \to t} X^{r'}$$

specified on the $f \in \mathbf{T}^t_{F(r)}$ component by

$$D \to X_{F(r)} \xrightarrow{\rho_f} \varinjlim_{F(r') \to t} X^{r'}$$

where ρ_f is the canonical morphism $X_r \to \varinjlim_{F_{/t}} X^{r'}$ associated to the object $f \colon F(r) \to t \in F_{/t}$. Using this, and chasing the appropriate diagrams of canonical isomorphism, we may compute the set of cells of $\iota^t \hat{\circledast} F_! \mathfrak{c}$ as

$$\{D \xrightarrow{\sigma} X^r \xrightarrow{\rho_f} \varinjlim_{F(r') \to t} X^{r'} = (F_!X)^t \mid r \in \mathbf{R}, f \in C^t_{F(r)}, \sigma \colon D \to X^r \in \mathfrak{C}_{\mathfrak{c},r}\}.$$

□

Example 8.3.7.11. For the special case of Corollary 8.3.7.8 where F is given by the constant functor $\mathbf{R} \to \star$, (and hence $\mathbf{R}$ is a left fibrant Reedy category) we obtain a canonical cellularization of the colimit functor

$$\varinjlim \colon \mathbf{C}^{\mathbf{R}} \to \mathbf{C}\,.$$

By Proposition 8.3.7.9, it follows that the set of cells of $\varinjlim \mathfrak{X}$, for a structured cell complex $\mathfrak{X}$ in $\mathbf{C}^{\mathbf{R}}$, is then explicitly given by

$$\{D \xrightarrow{\sigma} X^r \to \varinjlim X \mid r \in \mathbf{R}, \sigma \in \mathfrak{C}_{\mathfrak{X},r} : \nexists f \colon r \to r' \text{s.t.} f^- \neq 1\}.$$

Example 8.3.7.12. Let $r \in \mathbf{R}$. Denote by $i_r \colon \star \to \mathbf{R}$ the inclusion of the single object at r. Then, under the identification $\mathbf{C}^{\star} = \mathbf{C}$ and $i_r^* \mathfrak{X} = \mathfrak{X}^r = \mathbf{R}^r \circledast \mathfrak{X}$, the set of cells of the structured cell complex $\mathfrak{X}^r$ is given by

$$\{D \xrightarrow{\sigma} X^{\overline{r}} \xrightarrow{X^f} X^r \mid \overline{r} \in \mathbf{R}, \sigma \in \mathfrak{C}_{\mathfrak{X},\overline{r}}, f \colon \overline{r} \to r \in \mathbf{R}^+\}\,.$$

Applying Construction 8.3.3.13, we obtain:

Corollary 8.3.7.13. *Given a relative cellularized functor* $\mathfrak{i}\colon F \to G$ *from and a left fibrant Reedy category* $\mathbf{R}$*, there is a canonical isomorphism of cellularized functors*

$$\mathfrak{i} \circ \varinjlim \cong \varinjlim \circ \mathfrak{i}^{\mathbf{R}},$$

using the cellularization of $\iota^{\mathbf{R}}$ *from Construction 8.3.3.13.*

Notation 8.3.7.14. Suppose we are given a subcategory of a Reedy category $\mathbf{S} \subset \mathbf{R}$ that fulfills the requirements of Example 8.3.2.3 and thus inherits the structure of a Reedy category from $\mathbf{R}$. We say that $\mathbf{S}$ is *−-closed in* $\mathbf{R}^{+}$, if for every morphism $f\colon s \to r$, in $\mathbf{R}^{-}$, with $s \in \mathbf{S}$, it follows that $r \in \mathbf{S}$. Dually, we say that $\mathbf{S}$ is +-closed, if $\mathbf{S}^{\mathrm{op}}$ is −-closed in $\mathbf{R}$. In case both conditions hold, we will call such a subcategory ±-closed.

Example 8.3.7.15. Given a Reedy category $\mathbf{R}$ and $n \geq -1$, the inclusion $\mathbf{R}_{\leq n} \hookrightarrow \mathbf{R}$ is always ±-closed.

Lemma 8.3.7.16. *The inclusion of a −-closed Reedy subcategory* $\mathbf{S} \hookrightarrow \mathbf{R}$ *into a Reedy category* $\mathbf{R}$ *is always a left fibration. Dually, the inclusion of a +-closed subcategory is a right fibration.*

Proof. Let $\mathbf{S}$ be −-closed. We denote the inclusion functor by I. Let $r \in \mathbf{R}$. There are two cases to consider. If $r \notin S$, then the category $I^{-}_{/r}$ is necessarily empty, and hence left fibrant. If $r \in S$, $I^{-}_{/r}$ is a Reedy category with a terminal object. It is immediate from the definition of left fibrancy that such a Reedy category is left fibrant. □

8.3.8 Functorial computation via non-degenerate elements

It is a classical observation that the value of a colimit preserving functor F defined on simplicial sets (or more generally a presheaf on an Eilenberg-Zilber category, see [nLa24c]) at a simplicial set X can be computed in terms of a certain colimit involving the non-degenerate simplices of X. This is a particularly useful result insofar as whenever X is finite, this allows one to reduce the computation of F to the computation of a finite colimit. An analogous result holds if we replace Δ with a more general Reedy category and **Set** with an arbitrary cellularized category. Before we can show this, let us introduce some language and elementary results: Recall that, given a Reedy category $\mathbf{R}$ and a presheaf $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, the category of elements of X, $\mathbf{el}(X)$, inherits the structure of a Reedy category from $\mathbf{R}$ (see Example 8.3.2.3).

Proposition 8.3.8.1. *Let* $\mathbf{R}$ *be a Reedy category and let* $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$. *The following are equivalent:*

1. $\mathbf{el}(X)$ *is left fibrant.*
2. X *is a cell complex.*

Proof. X is a cell complex, if and only if $\iota_r \circledast X\colon L_r(X) \to X_r$ is an inclusion, for each $r \in \mathbf{R}$. $L_r(X)$ is equivalently the set of path components of the category of elements $(f\colon r \to r', \sigma \in X_{r'})$, $f^{-} \neq 1$. From this perspective, the fiber of $\iota_r \circledast X$ at σ is the set of path components of the full subcategory given by such pairs (f, σ'), for which $\sigma' \circ f = \sigma$. Hence, $\iota_r \circledast X$ is injective, if and only if this category is empty or connected. However, the category in question is, up to isomorphism, the category $\partial\mathbf{el}(X)_{\sigma/}$, which shows equivalence of the two properties. □

Notation 8.3.8.2. Given a Reedy category $\mathbf{R}$, a presheaf $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, and $r, r \in \mathbf{R}$ we say that an element $\tau \in X_r$ is a *face* of an element $\sigma \in X_{r'}$, if there exists a morphism $f \in \mathbf{R}^{+}$, such that $\sigma \circ f = \tau$.

Notation 8.3.8.3. Given a Reedy category $\mathbf{R}$ and $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, we denote by $\mathbf{el}_{n.d.}(X) \subset \mathbf{el}(X)$, the full subcategory of $\mathbf{el}(X)$ given by all non-degenerate elements of X and all of their faces. As this category is, by construction, closed under the unique factorization in a Reedy category, by Example 8.3.2.3, it inherits the structure of a Reedy category from $\mathbf{el}(X)$.

Proposition 8.3.8.4. *If $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ is a cell complex, then the inclusion functor*

$$I: \mathbf{el}_{n.d.}(X) \hookrightarrow \mathbf{el}(X)$$

is a final functor.

Proof. The comma category $I_{\sigma/}$ at $(\sigma: \mathbf{R}^r \to X) \in \mathbf{el}(X)$ is the category of factorizations of σ into a pair (f, σ') where σ' is a face of a non-degenerate element. We proceed to prove that $I_{\sigma/}$ is connected via induction over the degree of r. Note that if σ is itself a face of a non-degenerate element, then clearly, this category has an initial object and is thus contractible. If $\deg(r) = 0$, then σ is non-degenerate, which provides the start of the induction. Now, for the inductive step, we may again assume that σ is not the face of a degenerate object. Then for all pairs $(f, \sigma') \in I_{\sigma/}$, it holds that $f^- \neq 1$. Consequently, there is a well-defined inclusion

$$I_{\sigma/} \hookrightarrow \partial(\mathbf{el}(X))_{\sigma/}.$$

By Proposition 8.3.8.1 and Proposition 8.3.7.7, the category $\partial(\mathbf{el}(X))_{\sigma/}$ is connected or empty. Note that, since X is a cell complex, it follows from Corollary 8.3.4.11 that every element of X degenerates from some non-degenerate element. Hence, $I_{\sigma/}$ is non-empty. Consequently, it suffices to show that

$$J: I_{\sigma/} \to \partial(\mathbf{el}(X))_{\sigma/}.$$

induces a bijection on path components. The comma category $J_{(f',\sigma')/}$ at (f', σ') of J is the category of commutative diagrams

$$\begin{tikzcd}
\mathbf{R}^r \arrow[dr, "f'"] \arrow[drr, bend left, "f''"] \arrow[dd, "\sigma"'] & & \\
& \mathbf{R}^{r'} \arrow[r] \arrow[dl, "\sigma'"] & \mathbf{R}^{r''} \arrow[dll, bend left, "\sigma''"] \\
X & &
\end{tikzcd} \tag{8.119}$$

where σ'' is the face of a non-degenerate element. By pulling back along f', the category $J_{(f',\sigma')/}$ is canonically isomorphic to $I_{\sigma'/}$. By assumption, $f'^- \neq 1$. We may thus repeat the same argument with $(f'^-, \sigma' \circ f^+)$, and obtain that

$$J_{(f',\sigma')/} \cong I_{\sigma' \circ f^+/} \cong J_{(f'^-, \sigma' \circ f^+)/}.$$

Note, however, that as $f^- \neq 1$, the source of $\sigma' \circ f^+$ has strictly smaller degree than r. Hence, it is connected by inductive assumption. □

Proposition 8.3.8.5. *Given a presheaf $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, the forgetful functor*

$$F_X: \mathbf{el}(X) \to \mathbf{R}$$

is a right fibration.

Proof. To see that F_X is right fibrant, i.e.,that F_X^* is a cellularized functor, observe that the comma category $(F_X^+)_{r/}$ at r is the category of pairs $(f: r \to r', \sigma: \mathbf{R}^{r'} \to X)$ with $f \in \mathbf{R}^+$, where arrows $(f: r \to r'_0, \sigma_0: \mathbf{R}^{r'_0} \to X) \to (f: r \to r'_1, \sigma_1: \mathbf{R}^{r'_1} \to X)$ are given by arrows $r_0 \to r_1$ making the obvious diagrams commute. Now, fixing such a pair $(f: r \to r', \sigma: \mathbf{R}^{r'} \to X)$, the category

$$\partial\big((F_X^+)_{r/}\big)_{/(f,\sigma)}$$

has as objects given by triples (f', f'', σ'), fitting into commutative diagrams

$$\begin{tikzcd} \mathbf{R}^r \arrow[d, "f'"'] \arrow[dr, "f"] & \\ \mathbf{R}^{r''} \arrow[r, "f''"] \arrow[d, "\sigma'"'] & \mathbf{R}^{r'} \arrow[dl, "\sigma"] \\ X & \end{tikzcd} \tag{8.120}$$

with $f'' \neq 1 \in \mathbf{R}^+$. Morphisms between two such triples (f'_0, f''_0, σ'_0) and (f'_1, f''_1, σ'_1) (with the subscripts inherited by all further notation) are given by arrows $r''_0 \to r''_1$, making the obvious diagram commute. Now, if $f \neq 1$, then this category has a terminal object, given by $(1, f, \sigma)$. If $f = 1$, then the category is empty, by the uniqueness of factorizations. ☐

Proposition 8.3.8.6. *Let $\mathbf{R}$ be a Reedy category and let $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$. The inclusion functor*

$$\mathbf{el}_{n.d.}(X) \to \mathbf{el}(X)$$

is a right fibration. Suppose now, additionally, that $\mathbf{R}$ is such that every morphism in $\mathbf{R}^-$ admits a section by a morphism in $\mathbf{R}^+$. Then $\mathbf{el}_{n.d.}(X) \to \mathbf{el}(X)$ is also a left fibration.

Proof. By definition $\mathbf{el}_{n.d.}(X) \to \mathbf{el}(X)$ is +-closed in $\mathbf{el}(X)$. It follows by Lemma 8.3.7.16, that $\mathbf{el}_{n.d.}(X) \to \mathbf{el}(X)$ is a right fibration. To see that it is also a left fibration, we need to show −-closedness. So suppose we are given a commutative diagram

$$\begin{tikzcd} \mathbf{R}^r \arrow[r, "f", "-"'] \arrow[dr, "\sigma"'] & \mathbf{R}^{r'} \arrow[d, "\sigma'"] \\ & X \end{tikzcd} \tag{8.121}$$

with $\sigma \in \mathbf{el}_{n.d.}(X)$. Then a section of f given by $s \in \mathbf{R}^+$ also defines a section of f as a morphism in $\mathbf{el}(X)$

$$\begin{tikzcd} \mathbf{R}^{r'} \arrow[r, "g", "+"'] \arrow[rr, bend left, "1"] \arrow[dr, "\sigma'"'] & \mathbf{R}^r \arrow[r, "f", "-"'] \arrow[d, "\sigma"] & \mathbf{R}^{r'} \arrow[dl, "\sigma'"] \\ & X & \end{tikzcd} \tag{8.122}$$

.

Consequently, $\sigma' \in \mathbf{el}_{n.d.}(X)$, as was to be shown. ☐

Definition 8.3.8.7. We will say that a Reedy category as in Proposition 8.3.8.6 *admits positive sections.*

Example 8.3.8.8. The Reedy category Δ admits positive sections. Indeed, any order preserving surjection $[n] \to [m]$, for $n, m \in \mathbb{N}$ admits an order preserving section. More generally, every elegant Reedy category (see Section 8.3.5) admits positive sections.

We may now state the main result concerning the computation of cellularized functors of presheaf categories in terms of non-degenerate elements.

Theorem 8.3.8.9. *Let $\mathbf{R}$ be a Reedy category which admits positive sections. Let $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ and suppose that X is a cell complex. Denote by $I_{n.d.}: \mathbf{el}_{n.d.}(X) \to \mathbf{el}(X)$ the obvious inclusion functor, and by $F_X: \mathbf{el}(X) \to \mathbf{R}$ the forgetful functor. Under Corollary 8.3.7.8, all of the functors*

- $F_X^*: \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{el}(X)}$;

- $I^*_{n.d.}\colon \mathbf{C}^{\mathbf{el}(X)} \to \mathbf{C}^{\mathbf{el}_{n.d.}(X)}$;
- $\varinjlim\colon \mathbf{C}^{\mathbf{el}(X)} \to \mathbf{C}$;
- $\varinjlim\colon \mathbf{C}^{\mathbf{el}_{n.d.}(X)} \to \mathbf{C}$,

are canonically equipped with the structure of a cellularized functor.
Given a cellularized functor $\mathfrak{F}\colon \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \to \mathbf{C}$*, there is an isomorphism of structured cell complexes (natural in* $\mathfrak{F}$*)*

$$\mathfrak{F}(X) \cong \varinjlim\left((I_{n.d.} \circ F_X)^*(\mathfrak{F}_{|\mathbf{R}})\right).$$

In other words, the following diagram of categories (using the equivalence of categories from Corollary 8.3.6.3) commutes up to natural isomorphism

$$\begin{tikzcd}
\mathbf{Cell}(\mathbf{C}^{\mathbf{R}}) \arrow[r, "\simeq"] \arrow[d, "F_X^*"] & \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C}) \arrow[d, "\mathrm{ev}_X"] \\
\mathbf{Cell}(\mathbf{C}^{\mathbf{el}(X)}) \arrow[r, "\varinjlim"] \arrow[d, "I^*_{n.d.}"] & \mathbf{Cell}(\mathbf{C})\,. \\
\mathbf{Cell}(\mathbf{C}^{\mathbf{el}_{n.d.}(X)}) \arrow[ur, bend right, "\varinjlim"'] &
\end{tikzcd} \tag{8.123}$$

Proof. Observe first that by Propositions 8.3.7.7 and 8.3.8.6 the functors F_X and $I_{n.d.}$ are right fibrations, and hence induce well defined cellularized functors under precomposition, given by Corollary 8.3.7.8. Furthermore, by Proposition 8.3.8.1, $\mathbf{el}(X)$ is left fibrant, equipping $\varinjlim \mathbf{C}^{\mathbf{el}(X)} \to \mathbf{R}$ with the structure of a cellularized functor (by Corollary 8.3.7.8). Finally, again by Proposition 8.3.8.6 and the assumption on the existence of positive sections, $\mathbf{el}_{n.d.}(X) \to \mathbf{el}(X)$ is a left fibration. In particular, as $\mathbf{el}(X)$ is left fibrant, it follows by Proposition 8.3.7.7 and the composability of Quillen functors that $\mathbf{el}_{n.d.}(X)$ is left fibrant. Consequently, $\varinjlim \mathbf{C}^{\mathbf{el}_{n.d.}(X)} \to \mathbf{R}$ is also equipped with the structure of a cellularized functor under Corollary 8.3.6.3. Now, using Corollary 8.3.6.3 and Observation 8.3.6.5, the paths in the diagram in question are presented under Corollary 8.3.7.8 by

$$X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \tag{8.124}$$

$$\star \circledcirc \mathbf{R}_{\bullet}^{F_X(\bullet)} \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \tag{8.125}$$

$$\star \circledcirc \mathbf{el}(X)_{\bullet}^{I_{n.d.}(\bullet)} \circledcirc \mathbf{R}_{\bullet}^{F_X(\bullet)} \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}\,, \tag{8.126}$$

where $\star$ denotes the respective constant terminal presheaves. Consequently, it suffices to expose an isomorphism between these presheaves. However, under Corollary 8.3.6.6, this is equivalent to showing that the diagram of cellularized functors

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{R}} \arrow[d, "F_X^*"'] \arrow[dr, bend left, "X\circledast-"] & \\
\mathbf{Set}^{\mathbf{el}(X)} \arrow[r, "\varinjlim"] \arrow[d, "I^*_{n.d.}"'] & \mathbf{Set} \\
\mathbf{Set}^{\mathbf{el}_{n.d.}(X)} \arrow[ur, bend right, "\varinjlim"'] &
\end{tikzcd} \tag{8.127}$$

commutes up to isomorphism of cellularized functors. By Corollary 8.3.6.6 and Remark 8.3.4.13, it suffices to expose a natural isomorphism of regular functors. Such an isomorphism is immediate by Proposition 8.3.8.4 and the definition of the weighted colimit in terms of the category of elements. □

Chapter 9

Eckmann and Siebenmann's generalized simple homotopy theory axioms revisited

Note to the reader:

In [Sie70; Eck06] both Eckmann and Siebenmann suggested an axiomatic setup for generalized simple homotopy theory. While this setup seems somewhat too general to derive much beyond the existence of a Whitehead group, its relation to the homotopy category and the basic composition formulas, it will serve as a useful starting point for our model categorical investigations of simple homotopy theory. There is a technical difficulty with the given set of axioms though. Namely, they require the existence of certain pushouts in order to define addition and functoriality of Whitehead groups. The contexts which they then apply their theory to, namely the category whose arrows are inclusions of CW-complexes, does not admit pushouts(see also Section 2.3). However, what is really used in the proofs in [Sie70; Eck06], is not the universal property of the pushout, but only certain well-known consequences of the latter, such as symmetry and the pasting laws for pushout squares. In this chapter, we recover the results of Eckmann and Siebenmann under more general axiomatic assumptions, which apply to the examples we (and they) had in mind. Thus, most of what we present here is not new from a conceptual point of view. Rather, this chapter can be seen as a category theoretical hotfix. Only Section 9.2, which covers the study of functors between simple homotopy details results not already in the literature. Essentially all results before this should, in spirit, be attributed to Eckmann or Siebenmann. Many of the proofs below can be found in [Sie70; Eck06] in different language.

9.1 General Whitehead frameworks

In this section, we replicate the axiomatic approach of Eckmann and Siebenmann under different assumptions, making use of the calculus of cocartesian fibrations, rather than pushouts.

9.1.1 Categories with cobase changes

Let us first give an axiomatic category theoretical setting, in which one has a class of squares that are *almost* like pushout squares. This makes use of Grothendieck's notion of a cocartesian fibration. See [nLa24g] for an overview. Recall that, given a category $\mathbf{C}$, we denote by $\mathbf{C}^{[1]}$ the category of arrows in $\mathbf{C}$, or in other words the category of functors from the category $[1] = \{0 \to 1\}$ to $\mathbf{C}$. We denote the functors $\mathbf{C}^{[1]} \to \mathbf{C}$ given by evaluating at $0, 1 \in [1]$ by ev_0 and ev_1, respectively.

Definition 9.1.1.1. A category with symmetric cobase changes consists of the following data.

1. A category $\mathbf{C}$;
2. A subcategory $\mathbf{Q} \subset \mathbf{C}^{[1]}$, such that the following conditions hold:
   - The composition $F\colon \mathbf{Q} \to \mathbf{C}^{[1]} \xrightarrow{\mathrm{ev}_0} \mathbf{C}$ is a cocartesian fibration.
   - The fiber of F at $X \in \mathbf{C}$ is the whole slice category $\mathbf{C}_{X/}$.
   - If $Q\colon f \to g \in \mathbf{Q}$ is a cocartesian arrow, with respect to the above fibration, presented by a commutative square
   $$\begin{tikzcd} X_0 \arrow[r, "h_0"] \arrow[d, "f"'] \arrow[dr, phantom, "Q"] & Y_0 \arrow[d, "g"] \\ X_1 \arrow[r, "h_1"'] & Y_1 \end{tikzcd} \tag{9.1}$$
   then the mirrored square
   $$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "h_0"'] \arrow[dr, phantom, "Q'"] & X_1 \arrow[d, "h_1"] \\ Y_0 \arrow[r, "g"'] & Y_1 \end{tikzcd} \tag{9.2}$$
   defines a cocartesian morphism in $\mathbf{Q}$ from h_0 to h_1.

Notation 9.1.1.2. For the sake of brevity, we will often omit the *symmetric* and simply speak of a category with cobase changes.

Remark 9.1.1.3. Since $F\colon \mathbf{Q} \to \mathbf{C}$ is a cocartesian fibration, it has an associated pseudo-functor

$$\mathbf{C} \to \mathbf{Cat}$$

associating to $X \in \mathbf{C}$ the slice category $\mathbf{C}_{X/}$, and to a morphism $f\colon X \to X'$, a functor

$$f_{¡}\colon \mathbf{C}_{X/} \to \mathbf{C}_{X'/}$$

associating to a morphism $a\colon X \to Y$ the target of a cocartesian lift $\tilde{f}$ of f at a. One should be careful, however, to note that $f_{¡}$ is not left adjoint to the precomposition functor

$$f^*\colon \mathbf{C}_{X'/} \to \mathbf{C}_{X/}.$$

In fact, if it were, then that would imply that $\mathbf{C}$ has pushouts and that $\mathbf{Q}$ can be taken to be the whole category $\mathbf{C}^{[1]}$. In this case, there is no need to assume the extra structure of the category $\mathbf{Q}$. In order to not produce any confusion, we use ¡ to draw attention to the fact that we do not generally expect $f_{¡}$ to be the left adjoint of f^*. (In fact, for all examples we have in mind, this is certainly not the case).

There is a somewhat more elementary description of a category with symmetric cobase changes, which we describe in the following.

Notation 9.1.1.4. Given a category with cobase changes $(\mathbf{C}, \mathbf{Q})$, denote by $\mathcal{Q}$ the class of all commutative squares

$$\begin{tikzcd} X_0 \arrow[r, "h_0"] \arrow[d, "f"'] \arrow[dr, phantom, "Q"] & Y_0 \arrow[d, "g"] \\ X_1 \arrow[r, "h_1"'] & Y_1 \end{tikzcd} \tag{9.3}$$

which define a cocartesian morphism in $\mathbf{Q}$. Such squares will be called *cobase change squares*. We will, at times, also just say the square is a cobase change, to refer to it being a cobase change square.

Lemma 9.1.1.5. *The class of cobase change squares $\mathcal{Q}$, associated to a category with cobase change squares $(\mathbf{C}, \mathbf{Q})$ has the following properties:*

Q(i) Closure under isomorphism: Suppose we are given a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "Q"] & X_0' \arrow[d] \arrow[ddr, bend left] & \\ X_1 \arrow[r] \arrow[drr, bend right] & X_1' \arrow[dr, dashed, "\cong"] & \\ & & \hat{X}_1' \end{tikzcd} \tag{9.4}$$

with Q cobase change and $X_1' \to \hat{X}_1'$ an isomorphism. Then the outer commutative square is also cobase change.

Q(ii) Existence: For every pair of arrows $X_0 \xrightarrow{f} X_0'$ and $X_0 \xrightarrow{a} Y_2$, there exists a cobase change square

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "a_1"] & X_0' \arrow[d, dotted] \\ X_1 \arrow[r, dotted] & X_1' . \end{tikzcd} \tag{9.5}$$

Q(iii) Uniqueness of arrows: Squares in $\mathcal{Q}$ fulfill the uniqueness part of the universal property of the pushout, i.e., given a solid commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0' \arrow[d] \arrow[ddr, bend left] & \\ X_1 \arrow[r] \arrow[drr, bend right] & X_1' \arrow[dr, dashed] & \\ & & Y_1 \end{tikzcd} \tag{9.6}$$

with the inner square a cobase change, there exists at most one dashed arrow as above making the diagram commute.

Q(iv) Uniqueness: Given a solid commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_0' \arrow[d] \arrow[ddr, bend left] & \\ X_1 \arrow[r] \arrow[drr, bend right] & X_1' \arrow[dr, dashed] & \\ & & \hat{X}_1' \end{tikzcd} \tag{9.7}$$

with both squares cobase change, the unique dashed arrow making the diagram commute exists.

Q(v) Identities: For every morphism $f\colon X_0 \to X_1$, the canonical square with horizontal identities

$$\begin{tikzcd} X_0 \arrow[r, "1"] \arrow[d, "f"] & X_0 \arrow[d, "f"] \\ X_1 \arrow[r, "1"] & X_1 , \end{tikzcd} \tag{9.8}$$

is a cobase change.

Q(vi) Pasting Law: Given a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "Q_1"] & X_0' \arrow[r] \arrow[d] \arrow[dr, phantom, "Q_2"] & X_0'' \arrow[d] \\ X_1 \arrow[r] & X_1' \arrow[r] & X_1'' \end{tikzcd} \tag{9.9}$$

denote by $Q_2 \circ Q_1$ the outer rectangle, given by the horizontal composition of Q_1 and Q_2. Suppose Q_1 is a cobase change. Then Q_2 is a cobase change, if and only if $Q_2 \circ Q_1$ is a cobase change.

Q(vii) Symmetry: If

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y \end{tikzcd} \tag{9.10}$$

is a cobase change, then so is

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y_2 \arrow[d] \\ Y_1 \arrow[r] & Y\,. \end{tikzcd} \tag{9.11}$$

It turns out that this list of axioms uniquely specifies a category with cobase changes. In this sense, Definition 9.1.1.1 can be seen as a concise way of encoding these axioms.

Proposition 9.1.1.6. *Given a category* $\mathbf{C}$*, the construction in Notation 9.1.1.4 induces a bijection*

$$\begin{array}{c} \{\mathbf{Q} \subset \mathbf{C}^{[1]} \mid (\mathbf{C}, \mathbf{Q}) \text{ defines a category with cobase changes } \} \\ \Big\| \cong \\ \{\text{Classes of squares } \mathcal{Q}, \text{ fulfilling Properties Q(i) to Q(vii)}\} \end{array} \tag{9.12}$$

Proof. The inverse is constructed as follows. Given a category $\mathbf{C}$ and a family of squares $\mathcal{Q}$ fulfilling Properties Q(i) to Q(vii), then we can construct a category $\mathbf{Q} \subset \mathbf{C}^{[1]}$ such that $(\mathbf{C}, \mathbf{Q})$ defines a category with cobase change squares as follows: let us say a commutative square in $\mathbf{C}$

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, hook, "a"] & X_0' \arrow[d, hook, "b"] \\ X_1 \arrow[r] & Y_1 \end{tikzcd} \tag{9.13}$$

is *good* if it admits a factorization

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "Q"] & X_0' \arrow[d] \arrow[dr] & \\ X_1 \arrow[r] \arrow[rr, bend right] & X_1' \arrow[r, dashed] & Y_1 \end{tikzcd} \tag{9.14}$$

where $Q \in \mathcal{Q}$. Denote by $\mathbf{Q}$ the wide subcategory of $\mathbf{C}^{[1]}$, whose morphisms are given by good squares. Let us suppose, for now, that we have already shown that $\mathbf{Q}$ does indeed define a category. It then follows by Property Q(v), that the fibers of $F \colon \mathbf{Q} \to \mathbf{C}^{[1]} \xrightarrow{\mathrm{ev}_1} \mathbf{C}$ are the full slice categories $\mathbf{C}_{X/}$. That F defines a cocartesian fibration follows by Properties Q(ii) to Q(iv) and the definition of good squares. By Property Q(i), the cocartesian arrows associated to F are precisely given by the cobase change squares $\mathcal{Q}$. The symmetry axioms holds by Property Q(vii). Hence, we have constructed the required inverse. It remains to show that we have indeed defined a well defined category, i.e., that the (horizontal) composition of two good squares is again good. Observe that the smallest category containing all good squares is the subcategory of $\mathbf{C}^{[1]}$ generated by morphisms of the form

$$\begin{tikzcd} X_0 \arrow[r, equal] \arrow[d] & X_0 \arrow[d] \\ \tilde{X}_1 \arrow[r] & X_1 \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "Q"] & X_0' \arrow[d] \\ X_1 \arrow[r] & X_1' \end{tikzcd} \tag{9.15}$$

with Q cobase change. By Property Q(vi) squares of the second type are closed under composition. Clearly, the same holds for squares of the first type. Hence, to see that $\mathbf{Q}$ is a category, it suffices to see that every horizontal composition of the form

$$\begin{tikzcd} X_0 \arrow[r, equal] \arrow[d] & X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "Q"] & X_0' \arrow[d] \\ \tilde{X}_1 \arrow[r] & X_1 \arrow[r] & X_1' \end{tikzcd} \tag{9.16}$$

with Q cobase change, is again a good square. To see this, consider the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "Q_0"] & X_0' \arrow[r, equal] \arrow[d, dotted] & X_0' \arrow[dd] \\ \tilde{X}_1 \arrow[r, dotted] \arrow[d] \arrow[dr, phantom, "Q_1"] & \tilde{X}_1' \arrow[d, dotted] & \\ X_1 \arrow[r, dotted] \arrow[rr, bend right] & \hat{X}_1' \arrow[r, dashed] & {X_1}' \end{tikzcd} \tag{9.17}$$

with Q_0 and Q_1 induced by Property Q(ii). By Property Q(vii) and Property Q(vi), the vertical composition of Q_0 and Q_1 is again cobase change. Hence, by Property Q(iv), a dashed arrow as indicated exists. The induced composition of squares

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "Q_0"] & X_0' \arrow[rr, equal] \arrow[d, dotted] & & X_0' \arrow[d] \\ \tilde{X}_1 \arrow[r, dotted] \arrow[rrr, bend right] & \tilde{X}_1' \arrow[r, dotted] & \hat{X}_1' \arrow[r, dashed] & X_1' \end{tikzcd} \tag{9.18}$$

exposes the horizontal composition in Diagram (9.16) as a good square. □

Example 9.1.1.7. If $\mathbf{C}$ is a category with pushouts, then the class of pushout squares in $\mathbf{C}$ defines a class of cobase change squares as in Proposition 9.1.1.6. More generally, let $F\colon \mathbf{C} \hookrightarrow \mathbf{D}$ be a faithful functor. Assume, in addition to this, that F has the following properties:

1. Given any solid span in $\mathbf{C}$ as indicated to the left in

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_1 \arrow[d, dashed] \\ X_2 \arrow[r, dashed] & Y \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(X_1) \arrow[d] \\ F(X_2) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & F(Y) \end{tikzcd} \tag{9.19}$$

there exists a completion to a square, as indicated by the dashed arrows, such that the image of this square under F is a pushout square.

2. Given two such completions of a span $\{Y_1 \leftarrow X \rightarrow Y_2\}$ in $\mathbf{C}$ to squares (that map to pushout squares under F)

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y_2 \arrow[d] \arrow[ddr, bend left] & \\ Y_1 \arrow[r] \arrow[drr, bend right] & Y \arrow[dr, dashed] & \\ & & Y', \end{tikzcd} \tag{9.20}$$

the canonical arrow $F(Y) \dashrightarrow F(Y')$ induced by the universal property of the pushout lifts to an arrow $Y \dashrightarrow Y'$.

Then the class of squares Q in $\mathbf{C}$, for which $F \circ Q$ is a pushout square, defines the structure of a category with cobase changes on $\mathbf{Q}$.

Example 9.1.1.8. The following pairs of categories and subcategories fulfill the requirements of Example 9.1.1.7:

1. The category of CW-complexes (equipped with choices of characteristic maps, with morphisms given by cellular maps) and the wide subcategory of inclusions of subcomplexes.

2. The category of simplicial sets, and the wide subcategory of inclusions of sub-simplicial sets.

We will see a significantly larger class of examples and general machinery to generate the latter in Chapter 10.

For many intents and purposes, cobase change squares behave much like pushout squares, just that the existence property only holds with respect to other cobase change squares. One easily derives the following:

Observation 9.1.1.9. Cobase change squares have the following properties:

1. It follows from Properties Q(iii) and Q(iv) that the completion of a span $Y_1 \xleftarrow{a_1} X \xrightarrow{a_2} Y_2$ to a commutative square guaranteed by the existence axiom is uniquely determined up to canonical isomorphism. We denote the thus determined diagonal morphism

$$\begin{tikzcd} X \arrow[r, "a_1"] \arrow[d] \arrow[dr, dotted] & Y_1 \arrow[d, dotted] \\ Y_2 \arrow[r, dotted] & Y \end{tikzcd} \tag{9.21}$$

$X \rightarrow Y$ (canonically determined up to an isomorphism in the slice category $\mathbf{C}_{X/}$) by $a_1 \oplus_Q a_2$. We will often write $Y_1 \cup_X^Q Y_2$ for the target of this morphism (determined up to canonical isomorphism), to suggest the reminiscence to the behavior of a pushout.

2. The list of properties in Lemma 9.1.1.5 mirroring much of the behavior of pushout squares guarantee that cobase change squares fulfill essentially all of the relevant identities known for pushouts. That is, one has canonical isomorphisms.

$$Y_1 \cup_X^Q X \cong Y_1 \tag{9.22}$$
$$Y_1 \cup_X^Q Y_2 \cong Y_2 \cup_X^Q Y_1 \tag{9.23}$$
$$(Y_1 \cup_X^Q Y_2) \cup_{Y_2}^Q Y_3 \cong Y_1 \cup_X^Q Y_3 \tag{9.24}$$
$$(Y_1 \cup_X^Q Y_2) \cup_X^Q Y_3 \cong Y_1 \cup_X^Q (Y_2 \cup_X^Q Y_3), \tag{9.25}$$

commuting with all relevant structure morphisms.

3. Supposing one has made a choice of cobase change square for each span. Then Properties Q(iii) and Q(iv) guarantee that the map

$$\begin{aligned} \oplus_{\mathcal{Q}} : \mathbf{C}_{\hat{X}/} \times \mathbf{C}_{\hat{X}/} &\to \mathbf{C}_{\hat{X}/} \\ (a_1, a_2) &\mapsto a_1 \oplus_{\mathcal{Q}} a_2 \end{aligned}$$

canonically extends to a functor. On morphisms, $\oplus$ acts as follows: Given a solid commutative diagram

$$\begin{tikzcd}
X \arrow[r, "a_2"] \arrow[rr, bend left, "b_2"] \arrow[d, "a_1"'] \arrow[dr] \arrow[dd, bend right=60, "b_1"'] & Y_2 \arrow[r, "f_2"] \arrow[d] & Z_2 \arrow[d, dotted] \arrow[ddr, bend left] & \\
Y_1 \arrow[r] \arrow[d, "f_1"] & Y_1 \cup_X^{\mathcal{Q}} Y_2 \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, dotted] & \bullet \arrow[d, dotted] & \\
Z_1 \arrow[r, dotted] \arrow[drrr, bend right] & \bullet \arrow[r, dotted] & \bullet \arrow[dr, dashed] & \\
& & & Z_1 \cup_X^{\mathcal{Q}} Z_2
\end{tikzcd} \tag{9.26}$$

we may complete it by dotted cobase change squares as indicated, using Property Q(ii). Then, by Property Q(iv), a dashed arrow making the diagram commute exists. The diagonal composition $Y_1 \cup_X^{\mathcal{Q}} Y_2 \to Z_1 \cup_X^{\mathcal{Q}} Z_2$ defines a morphism $a_1 \oplus_{\mathcal{Q}} a_2 \to b_1 \oplus_{\mathcal{Q}} b_2$, induced by the morphisms of arrows under X, f_1 and f_2. Functoriality of this construction follows by Property Q(iii). This functor is independent of the choice of squares, up to canonical isomorphism. $\oplus_{\mathcal{Q}}$ can be extended on $\mathbf{C}_{X/}$ to the structure of a symmetric monoidal category (where the structure is again uniquely defined up to canonical isomorphism). Units are given by $X \xrightarrow{1_X} X$. Identitors, associators and symmetrizers are guaranteed by Eq. (9.22). The relevant coherence axioms are verified by Property Q(iv).

Construction 9.1.1.10. Consider the pseudo-functor

$$\begin{aligned} \mathbf{C} &\to \mathbf{Cat} \\ X &\mapsto \mathbf{C}_{X/} \\ f &\mapsto f_{\mathsf{i}} \end{aligned}$$

associated to the cocartesian fibration $F : \mathbf{Q} \to \mathbf{C}$. For each $X \xrightarrow{f} X' \in \mathbf{C}$, the induced functor

$$f_{\mathsf{i}} : \mathbf{C}_{X/} \to \mathbf{C}_{X'/}$$

is canonically equipped with the structure of a monoidal functor. By Property Q(v), f_{i} can be chosen to preserve unit objects. The structure isomorphism $f_{\mathsf{i}}(a) \oplus_{\mathcal{Q}} f_{\mathsf{i}}(a) \xrightarrow{\cong} f_{\mathsf{i}}(a \oplus_{\mathcal{Q}} b)$ is obtained as follows: Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "f"'] & X \arrow[d, dashed, "f"] & X \arrow[l, "1"'] \arrow[d, "f"] \\
X' \arrow[r, "1"'] & X' & X' \arrow[l, "1"]
\end{tikzcd} \tag{9.27}$$

in $\mathbf{C}$. Fixing cocartesian lifts $\tilde{f}_a, \tilde{f}_{a \oplus_{\mathcal{Q}} b}, \tilde{f}_b$ of f at $a, b, a \oplus_{\mathcal{Q}} b$ respectively, we obtain a solid commutative diagram

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "\tilde{f}"] & a \oplus_{\mathcal{Q}} b \arrow[d, "\tilde{f}"] & b \arrow[l] \arrow[d, "\tilde{f}"'] \\
f_{\mathsf{i}} a \arrow[r, dashed] & f_{\mathsf{i}}(a \oplus_{\mathcal{Q}} b) & f_{\mathsf{i}} b \arrow[l, dashed]
\end{tikzcd} \tag{9.28}$$

which, by assumed cocartesianity, completes to a lift of Diagram (9.27) as indicated with dashed arrows. We may translate this lift into a diagram in $\mathbf{C}$, as a commutative cube

$$
\begin{tikzcd}
X \arrow[rr, "b"] \arrow[dr, "a"] \arrow[drrr, "a \oplus_{\mathcal{Q}} b"] \arrow[dd, "f"'] & & Y \arrow[dr] \arrow[dd] & \\
& Z \arrow[rr] \arrow[dd] & & Y \cup_X^{\mathcal{Q}} Z \arrow[dd] \\
X' \arrow[rr, "f_{¡} b"] \arrow[dr, "f_{¡} a"'] \arrow[drrr, "f_{¡}(a \oplus_{\mathcal{Q}} b)"] & & Y_2' \arrow[dr] & \\
& Y_1' \arrow[rr] & & \hat{Z}
\end{tikzcd}
\tag{9.29}
$$

By assumption, the left face, the back face and the vertical diagonal square are cobase change. By the pasting law Property Q(vi), it follows that the right face and front face are cobase change. By assumption, the top face is a cobase change. Using symmetry of cobase change squares and the pasting law, it follows that the composition of the top face and the front face is a cobase change. Again using the pasting law and that the back face is a cobase change, it follows that the bottom face is a cobase change. In particular, by uniqueness of cobase change squares up to isomorphism, we obtain a canonical isomorphism $f_{¡}(a \oplus_{\mathcal{Q}} b) \cong f_{¡}(a) \oplus_{\mathcal{Q}} f_{¡}(a)$ as required. By Property Q(iii), the latter defines a natural isomorphism. It is not hard to see, using the symmetry axiom and Property Q(iii), that $f_{¡}$ even defines a symmetric monoidal functor. One can verify, again by Property Q(iii), that the associated monoical functors $f_{¡}$ define a lift of the induced pseudo functor into **Cat**, to a functor into the 2-category of symmetric monoidal categories **SymMonCat** (equipped with symmetric monoidal functors and symmetric monoidal natural transformations, see [nLa25g])

$$
\begin{aligned}
(-)_{¡} \colon \mathbf{C} &\to \mathbf{SymMonCat} \\
X &\mapsto \mathbf{C}_{X/} \\
f &\mapsto f_{¡} \,.
\end{aligned}
$$

9.1.2 Adding in expansions: Pre-Whitehead frameworks

The framework of cobase change squares equips a category $\mathbf{C}$ with the necessary amount of algebraic structure to define (covariant) functoriality and the addition in Whitehead groups. Now, let us add the remaining ingredient, namely the class of elementary expansions, which determines what it means for a morphism to be a simple equivalence.

Definition 9.1.2.1. A *pre-Whitehead framework* $\mathbf{W}$ consists of

- a category with cobase changes $(\mathbf{C}, \mathbf{Q})$;
- a wide subcategory $E \subset \mathbf{C}$, called the *expansions*, such that:
    1. E contains all isomorphisms.
    2. Given a cobase change square

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "e"] & Y \arrow[d, "e'"] \\
X' \arrow[r] & Y' \,.
\end{tikzcd}
\tag{9.30}
$$

    if e is an expansion, then so is e'.

For the remainder of this subsection, we fix a pre-Whitehead framework $\mathbf{W} = (\mathbf{C}, \mathbf{Q}, E)$.

Construction 9.1.2.2. Given $X \in \mathbf{C}$, denote by $\mathbf{Wh_W}(X)$ the pullback category

$$\mathbf{Wh_W}(X) := \mathbf{C}_{X/} \times_{\mathbf{C}} E.$$

In other words, this is the wide subcategory of $\mathbf{C}_{X/}$, obtained by allowing as morphisms only such commutative triangles

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y_0 \arrow[rr, "e"] & & Y_1 \end{tikzcd} \tag{9.31}$$

that fulfill $e \in E$. Now, suppose that we are given two morphisms $e_1 \colon a_1 \to a_1'$ and $e_2 \colon a_2 \to a_2'$ in $\mathbf{Wh_W}(X)$. Consider a diagram

$$\begin{tikzcd} X \arrow[r, "a_1"] \arrow[d, "a_2"] & Y_1 \arrow[r, "e_1"] \arrow[d] & Z_1 \arrow[d, dotted] \arrow[ddr] & \\ Y_2 \arrow[r] \arrow[d, "e_2"] & Y_1 \cup_X^{\mathcal{Q}} Y_2 \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, dotted] & \bullet \arrow[d, dotted] & \\ Z_2 \arrow[r, dotted] \arrow[drrr] & \bullet \arrow[r, dotted] & \bullet \arrow[dr, dashed, "\cong"] & \\ & & & Z_1 \cup_X^{\mathcal{Q}} Z_2 \end{tikzcd} \tag{9.32}$$

with all squares cobase change. The functoriality of $- \oplus_{\mathcal{Q}} -$ on e_1, e_2 is then given by the diagonal composition $Y_1 \cup_X^{\mathcal{Q}} Y_2 \to Z_1 \cup_X^{\mathcal{Q}} Z_2$. By stability of E under cobase change, all arrows in the lower right cobase change square are in E. The dashed arrow is an isomorphism, and thus in E. It follows that $Y_1 \cup_X^{\mathcal{Q}} Y_2 \to Z_1 \cup_X^{\mathcal{Q}} Z_2$ is in E. Together with the assumption that all isomorphisms are in E, it follows that the symmetric monoidal structure on $\mathbf{C}_{X/}$ restricts to a symmetric monoidal structure on $\mathbf{Wh_W}(X)$. As $\mathbf{Wh_W}(X)$ is a symmetric monoidal category, the set of path components $\pi_0 \mathbf{Wh_W}(X)$ canonically carries a (well defined) monoidal structure, given by

$$([a],[b]) \mapsto [a \oplus_{\mathcal{Q}} b],$$

with unit element given by $[1_X]$. We denote

$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X) := \pi_0(\mathbf{Wh_W}(X))$$

the resulting monoid, and call it the *Whitehead monoid of X* associated to the pre-Whitehead framework $\mathbf{W}$.

Notation 9.1.2.3. Given a morphism $a \colon X \to Y$, we denote by $\langle a \rangle \in \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$ the path component of a in $\mathbf{Wh_W}(X)$.

Observation 9.1.2.4. A priori, by the definition of path components, two arrows in $a_1 \colon X \to Y_1$ and $a_2 \colon X \to Y_2$ define the same element of $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$, if and only if a_1 can be transformed into a_2 through a zig-zag of expansions. However, any such zigzag can be reduced to a cospan $Y_1 \xrightarrow{e_1} Y \xleftarrow{e_2} Y_2$, by changing directions through cobase change squares. As a consequence, we obtain:

1. Given $a_1 \colon X \to Y_1$ and $a_2 \colon X \to Y_2$, the identity $\langle a_1 \rangle = \langle a_2 \rangle$ in $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$ holds, if and only if there are $e_1 \colon Y_1 \to Z$ and $e_2 \colon Y_2 \to Z$, such that $e_1 \circ a_1 = e_2 \circ a_2$.

2. A morphism $a \colon X \to Y$ maps to 0 in $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$, if and only if there exists a morphism $e \colon Y \to Z$ in E, such that $e \circ a \in E$.

Definition 9.1.2.5. A morphism $a \colon X \to Y$ that fulfills any of the equivalent properties in the second point of Observation 9.1.2.4 is called a *simple equivalence*.

Given a pre-Whitehead framework $\mathbf{W}$, the assignment $X \mapsto \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$ is functorial in two different ways:

Construction 9.1.2.6. The category $\mathbf{C}$ acts on the categories $\mathbf{Wh}_{\mathbf{W}}(X)$ contravariantly by precomposition: To a morphism $f\colon X \to X'$, we may associate the precomposition functor

$$\begin{aligned} f^*\colon \mathbf{Wh}_{\mathbf{W}}(X') &\to \mathbf{Wh}_{\mathbf{W}}(X) \\ (g\colon X' \to Y) &\mapsto g \circ f\,. \end{aligned}$$

In total, these functors agglomerate into a functor

$$\begin{aligned} \mathbf{C}^{\mathrm{op}} &\to \mathbf{Cat} \\ X &\mapsto \mathbf{Wh}_{\mathbf{W}}(X') \\ f &\mapsto f^*. \end{aligned}$$

If we compose with π_0, we obtain a functor valued in sets

$$\begin{aligned} \mathbf{C}^{\mathrm{op}} &\to \mathbf{Set} \\ X &\mapsto \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X') \\ f &\mapsto \{\langle a\rangle \mapsto \langle a \circ f\rangle. \end{aligned}$$

Given a morphism $f\colon X \to X'$ in $\mathbf{C}$, we will also denote the induced map of sets

$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X') \to \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$$

by f^*. Observe, that this is explicitly only a map of sets, not of monoids, as there is no reason to assume that f^* can be given the structure of a symmetric monoidal functor.

Construction 9.1.2.7. The more important notion of functoriality associated to Whitehead monoids will be the covariant one, constructed as follows. Recall the pseudo-functor

$$(-)_{\mathsf{i}}\colon \mathbf{C} \to \mathbf{SymMonCat}$$

of Construction 9.1.1.10. Given $f\colon X \to X'$ in $\mathbf{C}$, and a morphism $a_0 \xrightarrow{e} a_1$ in $\mathbf{Wh}_{\mathbf{W}}(X) \subset \mathbf{C}_{X/}$, the image of e under f_{i} is given by the lower right vertical in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a_0"'] \arrow[dd, bend right=60, "a_1"'] & X' \arrow[d, "a_1'"] \arrow[dd, bend left=60, "a_1'"] \\ Y_0 \arrow[r] \arrow[d, "e"] & Y_0' \arrow[d, dashed, "e'"] \\ Y_1 \arrow[r] & Y_1' \end{tikzcd} \tag{9.33}$$

with upper square and outer rectangle cobase change. By the pasting law, the lower square is again cobase change. Consequently, e' is again an expansion. It follows that the functor f_{i} restricts to a functor $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X) \to \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X')$. By abuse of notation, we will also denote this restriction by f_{i}. As every isomorphism is in E, this restriction also inherits the structure of a symmetric monoidal functor. To summarize, we obtain a covariant functor

$$\begin{aligned} \mathbf{C} &\to \mathbf{SymMonCat} \\ X &\mapsto \mathbf{Wh}_{\mathbf{W}}(X) \\ f &\mapsto f_{\mathsf{i}}. \end{aligned}$$

Composing with π_0, we obtain a covariant functoriality of Whitehead monoids, as a functor valued in the category abelian monoids $\mathbf{AbMon}$,

$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}\colon \mathbf{C} \xrightarrow{\mathbf{Wh}_{\mathbf{W}}} \mathbf{SymMonCat} \xrightarrow{\pi_0 \mathrm{N}} \mathbf{AbMon}$$

associating to a morphism $f\colon X \to X'$ a morphism of monoids

$$\begin{aligned} f_*\colon \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X) &\to \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X') \\ \langle a \rangle &\mapsto \langle f_{\mathrm{i}} a \rangle\,. \end{aligned}$$

Notation 9.1.2.8. When we treat $\widetilde{\mathrm{Wh}}_{\mathbf{W}}$ as a functor $\mathbf{C} \to \mathbf{AbMon}$, we will generally mean with respect to the covariant functoriality.

Remark 9.1.2.9. Note that, for the monoid structure on $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$ the specific choices of cobase change squares in $\mathcal{Q}$ defining the monoidal structure on $\mathbf{C}_{X/}$ is inessential, as it is uniquely defined up to canonical isomorphism. In other words, the equality

$$\langle a_1 \rangle + \langle a_2 \rangle = \langle a \rangle,$$

for $a_1\colon X \to Y_1$, $a_2\colon X \to Y_2$ and $a\colon X \to Y$, holds in $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$, if (up to modifying a by a zig-zag of expansions) there is a cobase change square

$$\begin{tikzcd} X \arrow[r, "a_1"] \arrow[d, "a_2"'] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y \end{tikzcd} \tag{9.34}$$

with diagonal given by a or, again in other words, if a and $a_1 \oplus_{\mathcal{Q}} a_2$ are in the same path component of $\mathbf{Wh}_{\mathbf{W}}(X)$. Similarly, functoriality is induced by the equality

$$\langle a' \rangle = f_* \langle a \rangle,$$

holding, if and only if (up to changing a' by expansions) there is a cobase change square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & X' \arrow[d] \\ Y \arrow[r, "a'"] & Y'\,. \end{tikzcd} \tag{9.35}$$

We will make frequent use of the following elementary lemma concerning the interaction of the two possible functorialities. These are simple consequences of the elementary properties of cobase change squares in Lemma 9.1.1.5.

Lemma 9.1.2.10. *For $a\colon X \to Y$ in $\mathbf{C}$, the equality*

$$a_* a^*(-) = (-) + a_* \langle a \rangle$$

holds.

Proof. Given $b\colon Y \to Z$ in $\mathbf{C}$, consider the composition of cobase change squares

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "a"'] & Y \arrow[r, "b"] \arrow[d, "a_{\mathrm{i}} a"'] \arrow[dr, "b \oplus_{\mathcal{Q}} a_{\mathrm{i}} a"] & Z \arrow[d] \\ Y \arrow[r, "a_{\mathrm{i}} a"'] & Y' \arrow[r] & Z'\,. \end{tikzcd} \tag{9.36}$$

As the composition of two cobase change squares is a cobase change, the lower horizontal composition defines $a_* a^* \langle b \rangle$. By commutativity of the diagram, this composition is equivalently given by the diagonal in the right square. The latter does, by definition, present the class $\langle b \rangle + a_* \langle a \rangle$. ☐

Corollary 9.1.2.11. *For $e\colon X \to X'$ in E, the equalities*

$$\begin{aligned} e_* e^* &= 1_{\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X')} \\ e^* e_* &= 1_{\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)} \end{aligned}$$

hold.

Proof. The first equality follows by Lemma 9.1.2.10, using that $\langle e\rangle = 0$. For the second identity, it now suffices to see that e_* is injective. For $i = 1, 2$, consider $a_i{:}\,X \to Y_i$ in $\mathbf{C}$ fitting into a cobase change square

$$\begin{tikzcd} X \arrow[r, "a_i"] \arrow[d, "e"] & Y_i \arrow[d, "e'"] \\ X' \arrow[r, "a_i'"] & Y_i' \,. \end{tikzcd} \tag{9.37}$$

Then $e_*\langle a_i\rangle = \langle a_i'\rangle$ is given by the lower horizontal. Now, suppose that $e_*\langle a_1\rangle = e_*\langle a_2\rangle$. In particular, by Observation 9.1.2.4, it follows that there exist $e_1{:}\,Y_1' \to Z$ and $e_2{:}\,Y_2' \to Z$ in E such that $e_1 \circ a_1' = e_2 \circ a_2'$. By commutativity of Diagram (9.37), we have

$$(e_1 \circ e') \circ a_1 = e_1 \circ a_1' = e_2 \circ a_2' = (e_2 \circ e') \circ a_2$$

and may surmise that $\langle a_1\rangle = \langle a_2\rangle$. ☐

An immediate consequence of this lemma is that extensions are sent to isomorphisms under $\widetilde{\mathrm{Wh}}_{\mathbf{W}}$. In the following, the notation $\mathbf{C}[E^{-1}]$ will refer to the 1-categorical localization of a category $\mathbf{C}$ at a class of arrows, or a subcategory E.

Corollary 9.1.2.12. *For $e{:}\,X \to X' \in E$, the induced morphism $e_*{:}\,\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X) \to \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X')$ is an isomorphism. In particular, $\widetilde{\mathrm{Wh}}_{\mathbf{W}}$ descends to a functor*

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\widetilde{\mathrm{Wh}}_{\mathbf{W}}"] \arrow[d] & \mathbf{AbMon} \\ \mathbf{C}[E^{-1}] \,. \arrow[ur, dashed] & \end{tikzcd} \tag{9.38}$$

Notation 9.1.2.13. By abuse of notation, we will again denote the induced functor $\mathbf{C}[E^{-1}] \to \mathbf{AbMon}$ by $\widetilde{\mathrm{Wh}}_{\mathbf{W}}$, and again denote the induced functoriality on morphisms in the form α_*.

As a consequence of Corollary 9.1.2.12, Corollary 9.1.2.11 and the covariant functoriality of Whitehead monoids, one can derive the following elementary properties of simple equivalences.

Lemma 9.1.2.14. *Simple equivalences in $\mathbf{C}$ have the following properties:*

1. *Simple equivalences are stable under cobase change;*
2. *Simple equivalences fulfill the two-out-of-three property;*
3. *If $s{:}\,Y \to Z$ is a simple equivalence and $a{:}\,X \to Y$ an arbitrary morphism in $\mathbf{C}$, then*
$$\langle s \circ a\rangle = \langle a\rangle.$$
4. *For simple equivalences s the induced map of Whiteheads monoids s_* is invertible, with inverse given by s^*.*

9.1.3 Relating the Whitehead constructions to a homotopy category

Already having homotopical frameworks in mind, we can now begin to study the relation of $\widetilde{\mathrm{Wh}}_{\mathbf{W}}$ with the associated *homotopy category* $\mathbf{C}[E^{-1}]$.

Notation 9.1.3.1. When referring to the image of a morphism $a \in \mathbf{C}$ in a localization $\mathbf{C}[E^{-1}]$, we will generally just again write a. How exactly equalities are to be understood will always be clear from context.

Let us make an elementary observation about the structure of morphisms in $\mathbf{C}[E^{-1}]$.

Lemma 9.1.3.2. *For every $\alpha: X \to Y$ in $\mathbf{C}[E^{-1}]$, there exist $Y' \in \mathbf{C}$ and morphisms $a: X \to Y'$ in $\mathbf{C}$ and $e: Y \to Y'$, such that*

$$\alpha = e^{-1} \circ a$$

in $\mathbf{C}[E^{-1}]$. If α is in the image of $E[E^{-1}]$, then a may be taken such that $a \in E$.

Proof. We prove the first claim; the second claim is shown analogously. By the classical construction of the localization of $\mathbf{C}[E^{-1}]$ α can be expressed as a composition

$$\alpha = b_n^{\pm 1} \circ \cdots \circ b_0^{\pm 1}$$

where the exponent of b_i is negative only if $b_i \in E$. We proceed via induction over the minimal length n, necessary to produce such a zig-zag. The case $n = 0$ is obvious. Now, for the inductive step from n to $n+1$, by inductive assumption we may rewrite $b_n^{\pm} \circ \cdots \circ b_0^{\pm}$ as $e'^{-1} \circ a'$, for appropriate $a' \in \mathbf{C}$ and $e \in E$. Now, there are two cases to consider. If the exponent of $b_{n+1}^{\pm 1}$ is negative, then $b_{n+1} \in E$ and we may write

$$\alpha = (e' \circ b_{n+1})^{-1} \circ a',$$

with $e \circ b_{n+1} \in E$. In case the exponent of $b_{n+1}^{\pm 1}$ is positive, consider a cobase change square

$$\begin{tikzcd} & Y_{n+1} \arrow[dl, "b_{n+1}"'] \arrow[dr, "e'"] & \\ X \arrow[dr, "e"'] & & Y_n \arrow[dl, "b'_{n+1}"] \\ & Y' & \end{tikzcd} \quad . \tag{9.39}$$

By stability under cobase changes, e is again an expansion. By commutativity of the diagram, it follows that

$$b_{n+1} \circ e'^{-1} = e^{-1} \circ b'_{n+1} \text{ in } \mathbf{C}[E^{-1}].$$

We may thus write α as

$$\alpha = b_{n+1} \circ e'^{-1} \circ a' = e^{-1} \circ (b'_{n+1} \circ a'),$$

as was to be shown. □

As we illustrated in the overview section (Chapter 2), our perspective on simple homotopy theory is that it is concerned with presentations of homotopy types, and whether certain identities between them can be verified in terms of elementary operations. We can think of the objects in the pre-Whitehead frameworks we have defined so far as fixed choices of presentations. We will think of an (alternative) presentation of the homotopy type defined by $X \in \mathbf{C}$, as a choice of isomorphism $X \xrightarrow{\alpha} Y$ in the homotopy category $\mathbf{C}[E^{-1}]$. The role of the *elementary operations* is taken by the morphisms in E. In other words, we are interested in studying the following set:

Construction 9.1.3.3. Let $\overline{E} \subset \mathbf{C}[E^{-1}]$ be the wide subcategory generated by the morphisms in E and their inverses. We denote

$$\widetilde{\mathrm{Pres}}_{\mathbf{W}}(X) := \pi_0(\mathbf{C}[E^{-1}]_{X/} \times_{\mathbf{C}[E^{-1}]} \overline{E})$$

Equivalently, this is the quotient set of the set of morphisms in $\mathbf{C}[E^{-1}]$ with source X, under the relation of post-composing with morphisms in E. As we are mainly interested in studying isomorphisms in $\mathbf{C}[E^{-1}]$, we denote by

$$\mathrm{Pres}_{\mathbf{W}}(X) \subset \widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)$$

the subset given by equivalence classes of arrows $\alpha: X \to Y$ that are isomorphisms. Equivalently, this is the quotient set of the set of isomorphisms in $\mathbf{C}[E^{-1}]$ with source X, under the relation of post-composing with morphisms in E.

The functor
$$\mathbf{Wh}_{\mathbf{W}}(X) = \mathbf{C}_{X/} \times_{\mathbf{C}} E \to \mathbf{C}[E^{-1}]_{X/} \times_{\mathbf{C}[E^{-1}]} \overline{E}$$
induced by $\mathbf{C} \to \mathbf{C}[E^{-1}]$ induces a map on path components
$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X) \to \widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)\,.$$
In order for the Whitehead monoid to answer questions about presentations in a homotopy-theoretic context, we want this map to be a bijection. Conditions for this to hold were first provided by [Eck06; Sie70]:

Theorem 9.1.3.4 ([Eck06; Sie70])**.** *Suppose that* $\mathbf{W}$ *has the following property: For any two morphisms* $a_1, a_2 : X \to Y$ *in* $\mathbf{C}$*, if it holds that*
$$a_1 = a_2,\ \text{in } \mathbf{C}[E^{-1}],$$
then there exists a morphism $e_1, e_2 : Y \to Z$ *in* E*, such that*
$$e_1 \circ a_1 = e_2 \circ a_2 \text{ in } \mathbf{C}\,.$$
Then the following map is a bijection:
$$\begin{aligned}\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X) &\to \widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)\\ \langle a : X \to Y\rangle &\mapsto [a : X \to Y]\end{aligned}$$
Furthermore, this map restricts to a bijection
$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)^* \xrightarrow{1:1} \mathrm{Pres}_{\mathbf{W}}(X),$$
where $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)^* \subset \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$ *is the subgroup of invertible elements in* $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$*.*

Proof. Let us first show that the map
$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X) \to \widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)$$
is surjective. By Lemma 9.1.3.2, every morphism $\alpha : X \to Y$, for $Y \in \mathbf{C}$, is of the form
$$e^{-1} \circ a$$
for (appropriately composable) morphisms $a \in \mathbf{C}$ and $e \in E$. Hence, it follows that
$$e \circ \alpha = a, \text{ in } \mathbf{C}[E^{-1}]$$
and hence $[\alpha] = [a]$ in $\widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)$. To see injectivity, suppose we are given two morphisms $a_1 : X \to Y_1$ and $a_2 : X \to Y_2$ such that $[a_1] = [a_2]$ in $\widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)$, i.e., we find morphisms $e_n, \dots, e_1 \in E$, such that
$$e_n'^{\pm 1} \circ \dots \circ e_1'^{\pm 1} a_1 = a_2 \text{ in } \mathbf{C}[E^{-1}].$$
By Lemma 9.1.3.2, we may write $e_n'^{\pm 1} \circ \dots \circ e_1' = e_2^{-1} e_1$, for appropriate $e_1, e_2 \in E$. Hence, we obtain
$$e_1 \circ a_1 = e_2 \circ a_2 \text{ in } \mathbf{C}[E^{-1}]\,.$$
By the second assumption, we find $\tilde{e}_1, \tilde{e}_2 \in E$, such that
$$(\tilde{e}_1 \circ e_1) \circ a_1 = (\tilde{e}_2 \circ e_2) \circ a_2 \text{ in } \mathbf{C}\,,$$
and in particular, that $\langle a_1\rangle = \langle a_2\rangle$, as was to be shown. In remains to show that we obtain a restricted bijection
$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)^* \xrightarrow{1:1} \mathrm{Pres}_{\mathbf{W}}(X).$$

To see that the map is well defined, suppose that $a: X \to Y$ has an additive inverse, i.e., that there exists $b: X \to Y$, such that

$$\langle a \rangle + \langle b \rangle = 0.$$

By Observation 9.1.2.4, this is equivalent to the existence of an (appropriately composable) $e \in E$, such that

$$e \circ (a \oplus_Q b) \in E.$$

In particular, since every morphism in E maps to an isomorphism in $\mathbf{C}[E^{-1}]$, $a \oplus_Q b$ is an isomorphism in $\mathbf{C}[E^{-1}]$. Consider the cobase change square

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"] & Y \arrow[d, "b'"] \\ X' \arrow[r, "a'"] & Y', \end{tikzcd} \tag{9.40}$$

with diagonal given by $a \oplus_Q b$. It follows that a has a left inverse in $\mathbf{C}[E^{-1}]$ and b' has a right inverse in $\mathbf{C}[E^{-1}]$. By construction of the functoriality of the Whitehead monoid, we have

$$\langle b' \rangle = a_* \langle b \rangle.$$

Consequently, it follows that $\langle b' \rangle$ also has an additive inverse. We may thus repeat the argument, and obtain that b' has a left inverse in $\mathbf{C}[E^{-1}]$, making b' an isomorphism. Since a is the left inverse of an isomorphism (in $\mathbf{C}[E^{-1}]$) it follows that a is an isomorphism in $\mathbf{C}[E^{-1}]$. Now, finally, to prove surjectivity, let $a: X \to Y$ in $\mathbf{C}$ be such that a is an isomorphism in $\mathbf{C}[E^{-1}]$. By Lemma 9.1.3.2 and the assumption on identities in $\mathbf{C}[E^{-1}]$, this implies that we find a morphism $b \in \mathbf{C}$, such that

$$b \circ a \in E.$$

In particular, we may apply Lemma 9.1.2.10, and obtain

$$0 = a_* \langle b \circ a \rangle = a_* \langle a \rangle + \langle b \rangle.$$

It follows that the image of $\langle a \rangle$ under a_* has an additive inverse. However, since a is an isomorphism in $\mathbf{C}[E^{-1}]$, it follows by Corollary 9.1.2.12, that a_* is an isomorphism of monoids. In particular, it follows that $\langle a \rangle$ also has an additive inverse, finishing the proof. $\square$

It follows from Theorem 9.1.3.4, that in order to study presentations of X, we may as well study the group of invertible elements in $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$.

Definition 9.1.3.5. A pre-Whitehead framework $(\mathbf{C}, \mathbf{Q}, E)$ is called a *Whitehead framework* if it fulfills the additional condition of Theorem 9.1.3.4.

Hence, we make the following definition:

Definition 9.1.3.6. The *Whitehead group functor* is defined by composing the covariant Whitehead monoid functor with the group core functor, mapping into the category of abelian groups **AbGrp**:

$$\mathrm{Wh}_{\mathbf{W}}: \mathbf{C} \xrightarrow{\mathbf{Wh_W}} \mathbf{SymMonCat} \xrightarrow{\pi_0 \mathrm{N}} \mathbf{AbMon} \xrightarrow{-^*} \mathbf{AbGrp}.$$

That is, given $X \in \mathbf{C}$, we denote by $\mathrm{Wh}_{\mathbf{W}}(X)$ the group of invertible elements in $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$.

Notation 9.1.3.7. Given Theorem 9.1.3.4, it makes sense to speak of the class in $\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$ associated with a morphism $\alpha: X \to Y$ in $\mathbf{C}[E^{-1}]$. We denote this class by $\langle \alpha \rangle$, and call it the *Whitehead torsion of* α.

Definition 9.1.3.8. Let $\mathbf{W} = (\mathbf{C}, \mathbf{Q}, E)$ be a Whitehead framework. A morphism $\alpha: X \to Y$ in $\mathbf{C}[E^{-1}]$ is called a simple equivalence, if one of the following conditions holds (which are equivalent by Theorem 9.1.3.4):

1. $\langle\alpha\rangle = 0 \in \widetilde{\mathrm{Wh}}_{\mathbf{W}}$.
2. There exist $e_1\colon X \to Z$ and $e_2\colon Y \to Z$ in E such that $\alpha = e_2^{-1}e_1$.

Observe that this definition of simple equivalence is compatible with the definition of a simple equivalence $a\colon X \to Y$ in $\mathbf{C}$. In other words, $a \in \mathbf{C}$ is a simple equivalence if and only if the associated morphism in $\mathbf{C}[E^{-1}]$ is a simple equivalence.

Another consequence of Theorem 9.1.3.4, is that the contravariant functoriality of Whitehead monoids also descends to homotopy categories.

Corollary 9.1.3.9. *Given a Whitehead framework* $\mathbf{W} = (\mathbf{C}, \mathbf{Q}, E)$*, the contravariant functoriality of Whitehead monoids*

$$\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$$

defined in Construction 9.1.2.6, descends to a functor

$$\mathbf{C}[E^{-1}] \to \mathbf{Set}.$$

Proof. It is immediate from the definition of $\widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)$, that the latter defines a contravariant functor of $\mathbf{C}$ under precomposition. This makes the bijection verified in Theorem 9.1.3.4 a natural isomorphism. For $\widetilde{\mathrm{Pres}}_{\mathbf{W}}(X)$, the claim that extensions are mapped into isomorphisms is immediate from the latter being defined entirely on the level of $\mathbf{C}[E^{-1}]$. □

Notation 9.1.3.10. Given a morphism $\alpha\colon X \to Y \in \mathbf{C}[E^{-1}]$, we will also denote the induced functoriality on the level of sets

$$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(Y) \to \widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$$

by α^*.

We then obtain the following extension of the composition formula to the homotopy category.

Lemma 9.1.3.11. *Let* $\mathbf{W} = (\mathbf{C}, \mathbf{Q}, E)$ *be a Whitehead framework, and* $\alpha\colon X \to Y$ *be a morphism in* $\mathbf{C}[E^{-1}]$*. The equality*

$$\alpha_*\alpha^*(-) = (-) + \alpha_*\langle\alpha\rangle$$

holds. In particular, if α *is a simple equivalence, then*

$$\alpha^* = (\alpha_*)^{-1}\,.$$

Proof. By Lemma 9.1.3.2, we may write $\alpha = e^{-1}a$, for morphism $a \in \mathbf{C}$ and $e \in E$. Hence, using Lemma 9.1.2.10 and Corollary 9.1.2.11, we compute

$$\begin{aligned}
\alpha_*\alpha^* &= (e^{-1}a)_*(e^{-1}a)^* \\
&= e_*^{-1}a_*a^*e_* \\
&= e_*^{-1}(1_{\widetilde{\mathrm{Wh}}_{\mathbf{W}}(Y)} + a_*\langle a\rangle)e_* \\
&= e_*^{-1}e_* + e_*^{-1}(a_*\langle a\rangle) \\
&= 1_{\widetilde{\mathrm{Wh}}_{\mathbf{W}}(Y)} + \alpha_*\langle a\rangle) \\
&= 1_{\widetilde{\mathrm{Wh}}_{\mathbf{W}}(Y)} + \alpha_*\langle\alpha\rangle\,.
\end{aligned}$$

□

Corollary 9.1.3.12. *Let* $\gamma : X \to Y$ *be an isomorphism in* $\mathbf{C}[E^{-1}]$*. Then*

$$\gamma^*(\langle\alpha\rangle + \langle\beta\rangle) = \gamma^*\langle\alpha\rangle + \gamma^*\langle\beta\rangle - \langle\gamma\rangle$$

Proof. Since γ_* is an isomorphism, it suffices to show that the equation holds after applying γ_*. Using Lemma 9.1.3.11, we derive

$$\begin{aligned}\gamma_*\gamma^*(\langle\alpha\rangle + \langle\beta\rangle) &= \langle\alpha\rangle + \langle\beta\rangle + \gamma_*\langle\gamma\rangle \\ &= \langle\alpha\rangle + \gamma_*\langle\gamma\rangle + \langle\beta\rangle + \gamma_*\langle\gamma\rangle - \gamma_*\langle\gamma\rangle \\ &= \gamma_*\gamma^*\langle\alpha\rangle + \gamma_*\gamma^*\langle\beta\rangle - \gamma_*\langle\gamma\rangle \\ &= \gamma_*(\gamma^*\langle\alpha\rangle + \gamma^*\langle\beta\rangle - \langle\gamma\rangle)\,.\end{aligned}$$

as was to be shown. □

In other words, we may think of the Whitehead torsion $\langle\gamma\rangle$ as the defect to additivity of γ^*. Finally, one may now easily verify the following elementary properties of this extended definition of simple equivalence.

Observation 9.1.3.13. Let $\mathbf{W} = (\mathbf{C}, \mathbf{Q}, E)$ be a Whitehead framework. Simple equivalences in $\mathbf{C}[E^{-1}]$ have the following properties:

1. Every identity morphism is a simple equivalence;
2. Simple equivalences are isomorphism in $\mathbf{C}[E^{-1}]$;
3. Simple equivalences fulfill the two-out-of-three property;
4. The inverse of a simple equivalence is a simple equivalence;
5. If $\gamma\colon Y \to Z$ is a simple equivalence and $\alpha\colon X \to Y$ an arbitrary morphism in $\mathbf{C}$, then
$$\langle\gamma \circ \alpha\rangle = \langle\alpha\rangle.$$
6. For simple equivalences γ the induced map of Whiteheads monoids γ_* is invertible, with inverse given by γ^*.

9.2 A category of (pre-)Whitehead frameworks

For our investigations of stratified simple homotopy theory, it will be crucial to be able to compare different Whitehead frameworks. In this section, we study notions of functors between Whitehead frameworks and their properties.

Notation 9.2.0.1. For the remainder of this subsection, we denote pre-Whitehead frameworks in the form $\mathbf{W}_i := (\mathbf{C}_i, \mathbf{Q}_i, E_i)$. In other words, $\mathbf{C}_i$ will always be the associated category to some Whitehead framework $\mathbf{W}_i$, etc. The associated class of cobase change squares will be denoted by $\mathcal{Q}_i$.

Definition 9.2.0.2. By a functor of pre-Whitehead frameworks $F\colon \mathbf{W}_0 \to \mathbf{W}_1$ we mean a functor $F\colon \mathbf{C}_0 \to \mathbf{C}_1$, such that the following holds

1. For every $e \in E_0$, the associated morphism $F(e)$ is a simple equivalence.
2. For every cobase change square $Q \in \mathcal{Q}_0$, the associated square $F \circ Q$ is a cobase change in $\mathbf{W}_1$, i.e., $F \circ Q \in \mathcal{Q}_1$.

We denote by **WES** (standing for Whitehead-Eckmann-Siebenmann) the category of pre-Whitehead frameworks, with morphisms given by functors of pre-Whitehead frameworks.

Observation 9.2.0.3. Observe that, as every simple equivalence in $\mathbf{C}_i$ is an isomorphism in $\mathbf{C}_i[E_i^{-1}]$, every Whitehead functor $F\colon \mathbf{W}_0 \to \mathbf{W}_1$ descends to a functor

$$F\colon \mathbf{C}_0[E_0^{-1}] \to \mathbf{C}_1[E_1^{-1}],$$

denoted the same by abuse of notation. If $\mathbf{W}_0$ and $\mathbf{W}_1$ are Whitehead frameworks, then as every simple equivalence in $\mathbf{C}_i[E_i^{-1}]$ can be written as a zig-zag of expansions, it follows that $F\colon \mathbf{C}_0[E_0^{-1}] \to \mathbf{C}_1[E_1^{-1}]$ preserves simple equivalences.

To study the functoriality of the construction of Whitehead monoids, we next need an appropriate target category.

Construction 9.2.0.4. Consider the bicategory of (small) categories $\mathbf{Cat}$, and let $\mathbf{Cat}_{/\mathbf{AbMon}}$ be the slice bicategory of functors into abelian monoids (see, for example, [JY20, Def. 7.1.1], an explicit description will be given below). We denote by $\mathbf{Fun}(-, \mathbf{AbMon})$ the strict $(2,1)$-category obtained by restricting the 2-cells in the slice bicategory $\mathbf{Cat}_{/\mathbf{AbMon}}$ to isomorphisms. In practice, this means an object is a functor $M\colon \mathbf{D} \to \mathbf{AbMon}$ where $\mathbf{D}$ is a (sufficiently small) category. A 1-morphism between two such functors $M_0\colon \mathbf{D}_0 \to \mathbf{AbMon}$ and $M_1\colon \mathbf{D}_1 \to \mathbf{AbMon}$ is given by a functor $G\colon \mathbf{D}_0 \to \mathbf{D}_1$, and a natural transformation $\eta\colon M_0 \Rightarrow M_1 \circ G$

$$\begin{tikzcd}
\mathbf{D}_0 \arrow[rr, "G"] \arrow[dr, "M_0"'] & \arrow[d, phantom, "\overset{\eta}{\Longrightarrow}"] & \mathbf{D}_1 \arrow[dl, "M_1"] \\
& \mathbf{AbMon}\,. &
\end{tikzcd} \tag{9.41}$$

A 2-morphism between two 1-morphisms $(G_0, \eta_0) \Rightarrow (G_1, \eta_1)$ is given by a natural isomorphism $G_0 \overset{\phi}{\Longrightarrow} G_1$, such that the *ice-cream cone equality* of pastings

$$\begin{tikzcd}
\mathbf{D}_0 \arrow[rr, bend left, "G_1"] \arrow[rr, bend right, "G_0"'] \arrow[rr, phantom, "\phi\Uparrow"] \arrow[dr, "M_0"'] & \arrow[d, phantom, "\overset{\eta_0}{\Longrightarrow}"] & \mathbf{D}_1 \arrow[dl, "M_1"] \\
& \mathbf{AbMon} &
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{D}_0 \arrow[rr, bend left] \arrow[dr, "M_0"'] & \arrow[d, phantom, "\overset{\eta_1}{\Longrightarrow}"] & \mathbf{D}_1 \arrow[dl, "M_1"] \\
& \mathbf{AbMon} &
\end{tikzcd} \tag{9.42}$$

or in other words $\eta_1 = M_1\phi \circ \eta_0$, holds. We use analogous language and notation, replacing $\mathbf{AbMon}$ by the category of abelian groups $\mathbf{AbGrp}$.

Remark 9.2.0.5. Given a strict 2-category (strict bicategory) $\mathbf{D}$ and an object $A \in \mathbf{D}$, consider the slice 2-category $\mathbf{D}_{/\mathbf{A}}$. Suppose that we are given a homotopy equivalence in $\mathbf{D}_{/\mathbf{A}}$, specified by the data of two morphisms

$$\begin{tikzcd}
D_0 \arrow[rr, "F"] \arrow[dr, "f_0"'] & \arrow[d, phantom, "\overset{\sigma}{\Longrightarrow}"] & D_1 \arrow[dl, "f_1"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
D_1 \arrow[rr, "G"] \arrow[dr, "f_1"'] & \arrow[d, phantom, "\overset{\tau}{\Longrightarrow}"] & D_0 \arrow[dl, "f_0"] \\
& A &
\end{tikzcd} \tag{9.43}$$

as well as 2-isomorphisms

$$\varepsilon\colon G \circ F \cong 1_{D_0}\,,$$
$$\eta\colon F \circ G \cong 1_{D_1}$$

fulfilling the respective ice-cream cone identities with respect to σ and τ. Then it is an easily verifiable fact about 2-categories that σ (and by symmetry also τ) is an isomorphism. A left inverse to σ is given by the pasting

$$\begin{tikzcd}
D_0 \arrow[d, "F"'] \arrow[dr, bend left, "1"] \arrow[dr, phantom, "\overset{\varepsilon}{\Nearrow}"] & \\
D_1 \arrow[r, "G"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\overset{\tau}{\Searrow}"] & D_0 \arrow[dl, bend left, "f_0"] \\
A &
\end{tikzcd}\,, \tag{9.44}$$

which explicitly is

$$(f_0\varepsilon) \circ (\tau F).$$

Construction 9.2.0.6. Let $F\colon \mathbf{W}_0 \to \mathbf{W}_1$ be a functor of pre-Whitehead frameworks. Since $F(E_0)$ is a subset of the class of simple equivalences, and every simple equivalence defines an isomorphism in $\mathbf{C}_1[E_1^{-1}]$, F descends to a functor of the localization $F\colon \mathbf{C}_0[E_0^{-1}] \to \mathbf{C}_1[E_1^{-1}]$, denoted the same by abuse of notation. Now, for $\hat{X} \in \mathbf{C}$, consider the map

$$\begin{aligned}\widetilde{\mathrm{Wh}}_F\colon \widetilde{\mathrm{Wh}}_{\mathbf{W}_0}(X) &\to \widetilde{\mathrm{Wh}}_{\mathbf{W}_1}(F(X))\\ \langle a\rangle &\mapsto \langle F(a)\rangle\,.\end{aligned}$$

As F maps expansions into simple equivalences, this map is well defined (see Lemma 9.1.2.14). Furthermore, as F preserves cobase change squares and identities, it is a monoid morphism. Even more, again by the assumption that F preserves cobase change squares, $\widetilde{\mathrm{Wh}}_F$ defines a natural transformation

$$\widetilde{\mathrm{Wh}}_{\mathbf{W}_0} \xRightarrow{\widetilde{\mathrm{Wh}}_F} \widetilde{\mathrm{Wh}}_{\mathbf{W}_1} \circ F\,.$$

On a global level, this construction induces a functor of 1-categories

$$\begin{aligned}\widetilde{\mathrm{Wh}}_{-}\colon \mathbf{WES} &\to \mathbf{Fun}(-,\mathbf{AbMon})\\ \mathbf{W} &\mapsto (\widetilde{\mathrm{Wh}}_{\mathbf{W}}\colon \mathbf{C}[E^{-1}] \to \mathbf{AbMon})\\ F &\mapsto \widetilde{\mathrm{Wh}}_F\,.\end{aligned}$$

Restricting to invertible elements we obtain a functor

$$\begin{aligned}\mathrm{Wh}_{-}\colon \mathbf{WES} &\to \mathbf{Fun}(-,\mathbf{Ab})\\ \mathbf{W} &\mapsto (\mathrm{Wh}_{\mathbf{W}}\colon \mathbf{C}[E^{-1}] \to \mathbf{Ab})\\ F &\mapsto \mathrm{Wh}_F\,.\end{aligned}$$

Proposition 9.2.0.7. *Let $F\colon \mathbf{W}_0 \to \mathbf{W}_1$ be a functor of Whitehead frameworks. The following statements are equivalent:*

- *E(i)* $\widetilde{\mathrm{Wh}}_F$ *is a homotopy equivalence in the* $(2,1)$*-category* $\mathbf{Fun}(-,\mathbf{AbMon})$*.*
- *E(ii)* *F descends to an equivalence of categories $\mathbf{C}_0[E_0^{-1}] \xrightarrow{\simeq} \mathbf{C}_1[E_1^{-1}]$ and induces isomorphisms on Whitehead monoids $\widetilde{\mathrm{Wh}}_{\mathbf{W}_0}(X) \xrightarrow{\cong} \widetilde{\mathrm{Wh}}_{\mathbf{W}_1}(F(X))$, for all $X \in \mathbf{C}_0$.*
- *E(iii)* Wh_F *is a homotopy equivalence in the* $(2,1)$*-category* $\mathbf{Fun}(-,\mathbf{Ab})$*;*
- *E(iv)* *F descends to an equivalence of categories $\mathbf{C}_0[E_0^{-1}] \xrightarrow{\simeq} \mathbf{C}_1[E_1^{-1}]$ and induces isomorphisms on Whitehead groups $\mathrm{Wh}_{\mathbf{W}_0}(X) \xrightarrow{\cong} \mathrm{Wh}_{\mathbf{W}_1}(F(X))$, for all $X \in \mathbf{C}_0$.*
- *E(v)* *The induced functor $F\colon \mathbf{C}_0[E_0^{-1}] \to \mathbf{C}_1[E_1^{-1}]$ is an equivalence of categories, and furthermore the following holds. Every object $Y \in \mathbf{C}_1[E_1^{-1}]$ is simply equivalent to an object of the form $F(X)$, for $X \in \mathbf{C}_1[E_1^{-1}]$, and a morphism $\alpha\colon X \to X'$ in $\mathbf{C}_0[E_0^{-1}]$ is a simple equivalence if and only if $F(\alpha)$ is a simple equivalence.*
- *E(vi)* *The induced functor $F\colon \mathbf{C}_0[E_0^{-1}] \to \mathbf{C}_1[E_1^{-1}]$ is an equivalence of categories, and its restriction to the wide subcategories of simple equivalences is an equivalence of categories.*
- *E(vii)* *There exists a functor $G\colon \mathbf{C}_1[E_1^{-1}] \xrightarrow{\simeq} \mathbf{C}_0[E_0^{-1}]$ together with natural isomorphisms*

$$\begin{aligned}\varepsilon\colon G \circ F &\cong 1_{\mathbf{C}_0[E_0^{-1}]}\\ \eta\colon F \circ G &\cong 1_{\mathbf{C}_1[E_1^{-1}]}\end{aligned}$$

such that, for every $X \in \mathbf{C}_0$ and $Y \in \mathbf{C}_1$, ε_X and η_Y are simple equivalences.

Proof. We first show the following sequences of implications

$$\begin{tikzcd}
 & & \mathrm{E(iii)} \arrow[dr, Rightarrow] & & \\
\mathrm{E(vii)} \arrow[r, Rightarrow] & \mathrm{E(i)} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \mathrm{E(iv)} \arrow[r, Rightarrow] & \mathrm{E(vii)} \\
 & & \mathrm{E(ii)} \arrow[ur, Rightarrow] & &
\end{tikzcd} \tag{9.45}$$

For the implication, E(vii) $\Longrightarrow$ E(i), by Remark 9.2.0.5, it suffices to show that ε and η define 2-morphisms in $\mathbf{Fun}(-, \mathbf{AbMon})$, i.e., that the ice-cream cone identities of Remark 9.2.0.5 hold. In more elementary terms, this just means that, for each $X \in \mathbf{C}$ and $a: X \to Y$, we need to verify the identity

$$\varepsilon_* \langle G \circ F(a) \rangle = \langle a \rangle .$$

or equivalently

$$\langle G \circ F(a) \rangle = (\varepsilon_*)^{-1} \langle a \rangle ,$$

(and the respective identity for η). By symmetry, we only verify the case of ε. By Lemma 9.1.3.11, we have

$$(\varepsilon_*)^{-1} \langle a \rangle = \langle a \circ \varepsilon \rangle$$

and by the naturality of ε,

$$\langle a \circ \varepsilon \rangle = \langle \varepsilon \circ (G \circ F(a)) \rangle$$

and finally

$$\langle \varepsilon \circ (G \circ F(a)) \rangle = \langle G \circ F(a) \rangle$$

by assumption on ε being simple. This finishes the proof of the implication E(vii) $\Rightarrow$ E(i). For the implications Property E(i) $\Rightarrow$ Property E(iii) and Property E(ii) $\Rightarrow$ Property E(iv), simply apply the functor associating to a monoid its invertible elements. The impliciations Property E(i) $\Rightarrow$ Property E(ii) and Property E(iii) $\Rightarrow$ Property E(iv), hold by Remark 9.2.0.5. It remains to show the implication Property E(iv) $\Rightarrow$ Property E(vii). Let first construct G and ε. Let $Y \in \mathbf{C}_1$. We have assumed F to be an equivalence of categories. In particular, we find some $X \in \mathbf{C}_1$, such that there is an isomorphism $F(X) \xrightarrow{\alpha} Y$ in $\mathbf{C}[E^{-1}]$. α specifies an element of $\mathrm{Wh}_{\mathbf{W}_1}(FX)$. We have assumed that

$$\mathrm{Wh}_F : \mathrm{Wh}_{\mathbf{W}_0}(X) \to \mathrm{Wh}_{\mathbf{W}_1}(FX)$$

is an isomorphism. Under Theorem 9.1.3.4, there is another way of rewriting this map. Namely, if we identify $\mathrm{Wh}_{\mathbf{W}_0}(X)$ and $\mathrm{Wh}_{\mathbf{W}_1}(FX)$ with equivalence classes of isomorphisms in $\mathbf{C}_0[E_0^{-1}]$ and $\mathbf{C}_1[E_1^{-1}]$, then, for such an isomorphism α with source X,

$$\mathrm{Wh}_F \langle \alpha \rangle = \langle F\alpha \rangle .$$

In particular, we find an imorphism $X \cong X'$ mapping to $\langle \alpha \rangle$. It follows, that there exists a simple equivalence $F(X') \xrightarrow{\eta'} Y$ in $\mathbf{C}_1[E_1^{-1}]$. Now set $G(Y) = X'$, and $\eta_Y = \eta'$, with the functoriality of G uniquely determined by η being natural (and F fully faithful). By definition, $\eta: F \circ G \to 1$ is given by simple equivalences. Now, consider

$$\eta F : F \circ G \circ F \to F .$$

Since F is a fully faithful functor, at each $X \in \mathbf{C}_0$, ηF_X admits a unique inverse image

$$\varepsilon_X : G \circ F(X) \to X .$$

It follows from the fully faithfulness of F, that these ε agglomerate into a natural isomorphism

$$\varepsilon\colon G \circ F \to 1_{\mathbf{C}_0[E_0^{-1}]}.$$

Finally, it follows from the injectivity of Wh_F and Definition 9.1.3.8, that $F(\alpha)$ is a simple equivalence, if and only if α is a simple equivalence. As η was assumed to be given by simple equivalences, and $F\varepsilon_X = \eta_{F(X)}$, it follows that ε is given by simple equivalences. It remains to show that any of the already covered characterizations is equivalent to Property E(v) and Property E(vi). That the latter two are equivalent, is straightforward. Finally, let us show that, Proposition 9.2.0.7 is equivalent to Property E(iv). Under the assumption that F induces an equivalence of categories after localization, injectivity on Whitehead groups corresponds to the condition that F reflects simple equivalences. Finally, we need to show that under these assumptions, surjectivity on Whitehead groups is equivalent to F being surjective up to simple equivalence. Let F have the latter property. Then, by Theorem 9.1.3.4, F induces a surjection on Whitehead groups if and only if, for $X \in \mathbf{C}$ and every isomorphism $F(X) \to Y$ in $\mathbf{C}_1[E_1^{-1}]$, there exists $X' \in \mathbf{C}$, and a simple equivalence $Y \simeq F(X')$, such that $F(X) \to Y \simeq F(X')$ is in the image of F. By assumption, F is essentially surjective. Hence, the latter condition is equivalent to F being surjective up to simple equivalence. ☐

Definition 9.2.0.8. A functor of pre-Whitehead frameworks $F\colon \mathbf{W}_0 \to \mathbf{W}_1$ is called a τ-*equivalence of Whitehead frameworks* if it fulfills Property E(ii).

Remark 9.2.0.9. We speak of a τ-equivalence, instead of an equivalence, as these equivalences are truncated in the sense that they only consider 1-categorical data happening on the level of the associated homotopy categories. There is a case to be made that one should also consider higher categorical perspectives to simple homotopy theory, at some later point in time, and that the name equivalence of Whitehead frameworks should be reserved for this kind of framework.

Corollary 9.2.0.10. *Let $F\colon \mathbf{W}_0 \to \mathbf{W}_1$, be a functor of pre-Whitehead frameworks. Suppose that $\mathbf{C}_0$ has an initial object, $\varnothing$, and suppose that $F(\varnothing)$ is again initial in $\mathbf{C}_1$. Then F is a τ-equivalence if and only if F descends to a fully faithful functor $\mathbf{C}_0[E_0^{-1}] \to \mathbf{C}_1[E_1^{-1}]$ and induces isomorphisms on Whitehead monoids $\widetilde{\mathrm{Wh}}_{\mathbf{W}_0}(X) \xrightarrow{\cong} \widetilde{\mathrm{Wh}}_{\mathbf{W}_1}(FX)$, for all $X \in \mathbf{C}_0$.*

Proof. By Proposition 9.2.0.7, we only need to show that the induced functor $\mathbf{C}_0[E_0^{-1}] \xrightarrow{\simeq} \mathbf{C}_1[E_1^{-1}]$ is essentially surjective. To this end, consider the map of Whitehead monoids

$$\widetilde{\mathrm{Wh}}_{\mathbf{W}_0}(\varnothing) \xrightarrow{\cong} \widetilde{\mathrm{Wh}}_{\mathbf{W}_1}(F(\varnothing))$$

associated to the initial objects. Observe that $\widetilde{\mathrm{Wh}}_{\mathbf{W}_0}(\varnothing)$ is simply the set of equivalence classes of objects in $\mathbf{W}_0$ under simple equivalence. As F preserves initial objects, the analogous statement for $\widetilde{\mathrm{Wh}}_{\mathbf{W}_1}(F(\varnothing))$ holds. Consequently, it follows that F induces a bijection on simple equivalence classes of objects. As every simple equivalence induces an isomorphism in $\mathbf{C}_1[E_1^{-1}]$, it follows that the induced functor $\mathbf{C}_0[E_0^{-1}] \xrightarrow{\simeq} \mathbf{C}_1[E_1^{-1}]$ is essentially surjective. ☐

Chapter 10

A model categorical approach to generalized simple homotopy theory

In this chapter, we finally combine the ingredients developed over the previous chapters to perform generalized simple homotopy theory in the way we envisioned it in Chapter 2: We apply the axiomatic approach of Eckmann and Siebenmann to frameworks of generalized cell complexes arising from certain cellular (semi-)model categories in the sense of [Hir03] (the semi-model category analogue of the latter). If in doubt how to generalize a definition from the model categorical to the semi-model categorical setting, any references to semi-model categories will use the language of Chapter 7. In particular, by a *semi-model category* we will always mean a *cofibrantly generated left semi-model category.*

10.1 From cellularized categories to pre-Whitehead frameworks

Let us first explain how to associate to a cellularized category (see Definition 8.1.1.10) with a notion of expansion a pre-Whitehead framework. This will be a very general procedure that does not require the language of model categories. Only when we are aiming to produce actual Whitehead frameworks, and thus pass to a context where the interactions of Whitehead monoids and homotopy theory become important, will we need the latter. The first step is to associate to a cellularized category a category with cobase changes.

10.1.1 Cellularized categories give rise to categories with cobase changes

The definition of a category with cobase changes is so general that essentially any cellularized category, together with some restriction to the size of structured cell complexes, gives rise to such a framework. Here, we will restrict ourselves to the setting of finite cell complexes, i.e., such cell complexes whose set of cells has cardinality smaller than the countable cardinal $\aleph_0$. We note, however, that there is really nothing special about $\aleph_0$ here, and much of what we will say below has an analogue for larger cardinals.

Notation 10.1.1.1. Let $\mathbf{C}$ be a cellularized category. We denote by $\mathbf{C\vec{e}ll}(\mathbf{C})$ the subcategory of $\mathbf{Cell}(\mathbf{C})$, whose objects are finite absolute structured cell complexes and whose morphisms are inclusions of subcomplexes. We denote by $\mathbf{C\vec{e}ll}_c(\mathbf{C}) \subset \mathbf{C\vec{e}ll}(\mathbf{C})$ the full subcategory given by finite absolute structured cell complexes.

Construction 10.1.1.2. Let $\mathbf{C}$ be a cellularized category. Let $\mathcal{Q}$ be the class of all squares in $\mathbf{C\vec{e}ll}_c(\mathbf{C})$ that are pushout in $\mathbf{Cell}(\mathbf{C})$. Observe that, by Corollary 8.1.4.8, every span in $\mathbf{C\vec{e}ll}_c(\mathbf{C})$, $\mathfrak{Y}_1 \hookleftarrow \mathfrak{X} \hookrightarrow \mathfrak{Y}_2$, completes to a pushout square

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Y}_1 \arrow[d, hook] \\
\mathfrak{Y}_2 \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & \mathfrak{Y}
\end{tikzcd} \tag{10.1}$$

in $\mathbf{Cell}(\mathbf{C})$, with all arrows inclusions of cell complexes. Consequently, the requirements of Example 9.1.1.7 are fulfilled, and under Proposition 9.1.1.6, $\mathcal{Q}$ defines the structure of a category with cobase changes on $\mathbf{C\vec{e}ll}_c(\mathbf{C})$.

Definition 10.1.1.3. Let us say a morphism $d: \mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1$ in $\mathbf{C\vec{e}ll}_c(\mathbf{C})$ is *elementary*, if its relative set of cells is either empty (i.e., if it defines an isomorphism in $\mathbf{C\vec{e}ll}_c(\mathbf{C})$) or given by a single cell.

Clearly, the category $\mathbf{C\vec{e}ll}_c(\mathbf{C})$ is generated by elementary morphisms. Let us now study the relations which need to be added to present $\mathbf{C\vec{e}ll}_c(\mathbf{C})$ by generators and relations.

Lemma 10.1.1.4. *Given a pushout diagram*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hook, "a_1"] \arrow[d, hook, "b_2"'] & \mathfrak{Y}_1 \arrow[d, hook] \\
\mathfrak{Y}_2 \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & \mathfrak{Y}
\end{tikzcd} \tag{10.2}$$

in $\mathbf{Cell}(\mathbf{C})$*, with* a_1 *and* a_2 *elementary, then* b_1 *and* b_1 *are also elementary*

Proof. This follows by Corollary 8.1.4.8. □

Lemma 10.1.1.5. *The category* $\mathbf{C\vec{e}ll}_c(D)$ *is given by the free category generated by the elementary inclusions (taking the identities as identities), subject to the relations*

$$b_1 \circ a_1 \sim b_2 \circ a_2$$

whenever there is a pushout diagram of elementary inclusions

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hook, "a_1"] \arrow[d, hook, "a_2"'] & \mathfrak{Y}_1 \arrow[d, hook, "b_1"] \\
\mathfrak{Y}_2 \arrow[r, hook, "b_2"] \arrow[ur, phantom, "\ulcorner", very near end] & \mathfrak{Y}
\end{tikzcd} \tag{10.3}$$

in $\mathbf{Cell}(\mathbf{C})$.

Proof. Let $\mathbf{T}$ be the category described above. There is an obvious functor

$$\Phi: \mathbf{T} \to \mathbf{C\vec{e}ll}_c(\mathbf{C})$$

given by the identity on objects, and by mapping the generator a to the morphism a. This functor is an isomorphism of categories. It certainly defines an bijection on objects. Furthermore, by Proposition 8.1.3.1, it is full. Now, let us see that is is faithful. To this end, observe first that if $a: \mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}_1$ is elementary and has an empty relative set of cells, i.e., is an isomorphism, then for any morphism $b: \mathfrak{Y}_1 \to \mathfrak{Y}_1'$ the induced square

$$\begin{tikzcd}
\mathfrak{Y}_0 \arrow[r, "a"] \arrow[d, "b \circ a"'] & \mathfrak{Y}_1 \arrow[d, "b"] \\
\mathfrak{Y}_1' \arrow[r, equal] & \mathfrak{Y}_1'
\end{tikzcd} \tag{10.4}$$

is a pushout. In particular, the equality $b \circ a = (b \circ a)$ holds. We may use this rule to remove isomorphisms from a word in the generators of $\mathbf{T}$. Removing isomorphisms, every morphism $a \in \mathbf{T}$ can be represented by composition

$$a = a_{n+1} \circ \cdots \circ a_0,$$

where a_1 to a_n are elementary and a_0 is given by an elementary morphism with empty set of cells, i.e., an isomorphism of cell complexes. Note that n is uniquely determined by $\Phi(a)$, as it specifies the number of cells of $\Phi(a)$. Hence, it suffices to show that whenever two equivalence classes of words

$$a = \mathfrak{X} \xrightarrow[\cong]{a_0} \mathfrak{Y}^1 \hookrightarrow \mathfrak{Y}^n \xhookrightarrow{a_n} \mathfrak{Y}, \tag{10.5}$$

$$a' = \mathfrak{X} \xrightarrow[\cong]{a'_0} \mathfrak{Y}'^1 \hookrightarrow \mathfrak{Y}'^n \xhookrightarrow{a'_n} \mathfrak{Y}, \tag{10.6}$$

presented in the form above, map to the same morphism in $\vec{\mathbf{Cell}}_c(\mathbf{C})$, then $a = a'$. We show this by induction over n. The case $n = 0$ is obvious, as in this case a'_0 and a_0 and a'_0 are given by a single isomorphism, which is necessarily uniquely determined by $\Phi(a_0) = \Phi(b_0)$. Now, for the inductive step, consider the following two cases: If $\mathfrak{Y}_{n+1} \xhookrightarrow{a_{n+1}} \mathfrak{Y}$ and $\mathfrak{Y}'_{n+1} \xhookrightarrow{a'_{n+1}} \mathfrak{Y}$ are given by gluing in the same cell of $\mathfrak{Y}$, then by Observation 8.1.3.10, there is an isomorphism of structured complexes $\mathfrak{Y}_{n+1} \xrightarrow{b} \mathfrak{Y}'_{n+1}$ making the diagram

$$\begin{tikzcd} \mathfrak{Y}_{n+1} \arrow[rr, "b"] \arrow[dr, hook, "a_{n+1}"'] & & \mathfrak{Y}'_{n+1} \arrow[dl, hook', "a'_{n+1}"] \\ & \mathfrak{Y} & \end{tikzcd} \tag{10.7}$$

commute. Consequently, we obtain identities

$$a_{n+1} \circ a_n \circ \cdots \circ a_0 = a'_{n+1} \circ b \circ a_n \cdots \circ a_0 = a'_{n+1} \circ (b \circ a_n) \circ \cdots \circ a_0.$$

Using the fact that $\Phi(a'_{n+1})$ is a monomorphism, it follows that

$$\Phi((b \circ a_n) \circ \cdots \circ a_0) = \Phi(a'_n \circ \cdots \circ a'_0).$$

Hence, the claim follows by inductive assumption. If this is not the case consider the following diagram of pullback squares in $\mathbf{Cell}(\mathbf{C})$.

$$\begin{tikzcd} \mathfrak{X} \cap \mathfrak{Y}'^{n+1} \arrow[r] \arrow[d, "f"] & \dots \arrow[r] \arrow[d] & \mathfrak{Y}^{n} \cap \mathfrak{Y}'^{n+1} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Y}^{n+1} \cap \mathfrak{Y}'^{n+1} \arrow[r, "b_n"] \arrow[d, "c_n"] & \mathfrak{Y}'^{n+1} \arrow[d, hook, "a'_{n+1}"] \\ \mathfrak{X} \arrow[r, hook] & \dots \arrow[r, hook] & \mathfrak{Y}^{n} \arrow[r, hook] & \mathfrak{Y}^{n+1} \arrow[r, hook, "a_{n+1}"] & \mathfrak{Y} \end{tikzcd} \tag{10.8}$$

Observe that the left vertical f is a bijection on cells and hence (by Corollary 8.1.4.1) is an isomorphism. Furthermore, observe that since we assumed that $\mathfrak{Y}^{n+1}$ and $\mathfrak{Y}'^{n+1}$ define different sets of cells, it follows that the upper right horizontal is not an isomorphism. Consequently, counting cells, at least one of the upper horizontals is an isomorphism. We may thus rewrite the composition from the lower left corner, taking the inverse of f and then taking the upper horizontal composition, up to $\mathfrak{Y}'^{n+1} \cap \mathfrak{Y}^{n+1}$ in the form

$$b_{n-1} \circ \cdots \circ b_0$$

with all b_i elementary and b_0 an isomorphism. Then, by commutativity of the diagram,

$$\Phi(c_n \circ b_{n-1} \cdots \circ b_0) = \Phi(a_n \cdots \circ a_0)\,.$$

By inductive assumption, it follows that

$$c_n \circ b_{n-1} \cdots \circ b_0 = a_n \cdots \circ a_0$$

in $\mathbf{T}$. Furthermore, again by the assumption that $\mathfrak{Y}^{n+1}$ and $\mathfrak{Y}'^{n+1}$ define different sets of cells, it follows by Corollary 8.1.4.2 and Lemma 8.1.4.7 that the right square is a pushout. In particular, we have the identity

$$a'_{n+1} \circ b_n = a_{n+1} \circ c_n$$

in $\mathbf{T}$. Together, these two identities show that

$$a_{n+1} \circ a_n \cdots \circ a_0 = a_{n+1} \circ c_n \circ b_{n-1} \cdots \circ b_0 = a'_{n+1} \circ b_n \circ b_{n-1} \cdots \circ b_0 \, .$$

in $\mathbf{T}$. In particular, it suffices to show that

$$a'_{n+1} \circ b_n \circ b_{n-1} \cdots \circ b_0 = a'_{n+1} \circ a'_n \circ \cdots \circ a'_0$$

in $\mathbf{T}$. Now we are again in the situation, where the final elementary morphisms add the same cell, which we have already covered. □

10.1.2 Pre-Whitehead frameworks from elementary expansions

Next, let us turn the category with cobase changes associated to a cellularized category into a Pre-Whitehead framework, by adding a notion of expansion.

Notation 10.1.2.1. Let $\mathbf{C}$ be a cellularized category. When we treat $\ddot{\mathbf{Cell}}_c(\mathbf{C})$ as a category with cobase changes, we always mean with respect to the class of cobase change squares constructed in Construction 10.1.1.2.

Next, let us equip the category with cobase changes $\ddot{\mathbf{Cell}}_c(\mathbf{C})$ with a notion of expansion.

Definition 10.1.2.2. A cellularized category with expansions consists of

1. a cellularized category $\mathbf{C}$;

2. a set $\mathbb{E}_{\mathbf{C}} \subset \mathbf{RCell}(\mathbf{C})$ of finite structured relative cell complexes.

Elements of $\mathbb{E}_{\mathbf{C}}$ are called *generating elementary expansions*.

Example 10.1.2.3. Clearly, any set of relative cell complexes will define a category with expansions. Here are some more natural examples to keep in mind:

- The cellularized category of simplicial sets $\mathbf{sSet}$ (with the standard set of boundary inclusions) is usually equipped with the set of horn inclusions

  $$\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1, 0 \leq k \leq n\},$$

  at least if one is interested in doing classical simple homotopy theory.

- The cellularized category of topological space $\mathbf{Top}$ (with the boundary inclusions given by the boundary inclusions of disks) can be equipped with a class of expansions as follows. By Remark 8.2.3.3, we may replace the boundary inclusions of disks by boundary inclusions $|\partial\Delta^n \hookrightarrow \Delta^n|$, without really changing the theory. In particular, using Lemma 8.2.2.5, this makes $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ a cellularized functor in a canonical way. We may then take

  $$\mathbb{E}_{\mathbf{C}} = \{|\Lambda^n_k \hookrightarrow \Delta^n| \mid n \geq 1, 0 \leq k \leq n\} \, .$$

- The category of positively graded chain complexes $\mathbf{Ch}_{\geq 0}(R)$ over some not-necessarily commutative Ring R, cellularized as in Example 8.1.1.15, can be equipped with the following set of expansions. Denote by $I_\bullet$ the graded chain complex given by

$$\dots 0 \to R \xrightarrow{(-1,1)} R \oplus R \to 0$$

with $R \oplus R$ in degree 0. Denote by e_0 the inclusions of $R[0]$ into $I_\bullet$ via inclusion in the left component and analogously by e_1 the inclusion in the right component. e_0 obtains the structure of a relative cell complex, by taking the cells defined by the remaining basis elements $1 \in R$ in degree 1 and $(0,1) \in R \oplus R$ in degree 0 and one may proceed analogously to construct a cell structure on e_1. Then, we can consider the set of expansions

$$\{\mathfrak{e}_i[-n] \colon R[-n] \hookrightarrow I_\bullet[-n] \mid i = 0, 1, n \in \mathbb{N}\}.$$

Notation 10.1.2.4. Just as in the case of cellularized categories, we will omit the expansions from the notation for a cellularized category with expansions, and just write $\mathbf{C}$, to refer to the latter. The associated set of generating expansions is always denoted by $\mathbb{E}_{\mathbf{C}}$.

Definition 10.1.2.5. Let $\mathbf{C}$ be a cellularized category with expansions.

1. A relative cell complex $\mathfrak{e}' \in \mathfrak{c}_{A'}$ that fits into a cobase change square

$$\begin{tikzcd} \Lambda \arrow[r, "f"] \arrow[d, hook, "\mathfrak{e}"'] & A \arrow[d, hook, "\mathfrak{e}'"] \\ D \arrow[r] & X \end{tikzcd} \tag{10.9}$$

where $\mathfrak{e}$ is either an empty cell complex, or in $\mathbb{E}_{\mathbf{C}}$, is called an *elementary expansion.*

2. A (possibly empty) transfinite vertical composition of elementary expansions is called an *expansion.*

3. The inclusion of a subcomplex $i \hookrightarrow (A \overset{\tilde{\mathfrak{c}}}{\hookrightarrow} \tilde{X}) \hookrightarrow (A \overset{\mathfrak{c}}{\hookrightarrow} X)$ in $\mathbf{RCell(C)}_A$, for $A \in \mathbf{C}$, is called an expansion, if the associated relative cell complex $\tilde{X} \overset{\mathfrak{i}}{\hookrightarrow} X$ is an expansion.

As an immediate consequence of Observation 8.1.2.13, we obtain the following lemma.

Lemma 10.1.2.6. *Let $\mathbf{C}$ be a cellularized category with expansions. The class of expansions contains all empty cell complexes (i.e., isomorphisms) and is closed under cobase change and transfinite vertical composition.*

Construction 10.1.2.7. Let $\mathbf{C}$ be a cellularized category. Denote by $E_{\mathbf{C}}$ the wide subcategory of $\mathbf{C\vec{e}ll}_c(\mathbf{C})$, given by all such inclusions of subcomplexes $i \colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ that are expansions. Observe that, by closure under vertical composition (Lemma 10.1.2.6), this does indeed define a wide subcategory. Since we allowed for empty cell complexes in the definition of expansions, $E_{\mathbf{C}}$ contains all isomorphisms. Finally, for $e \in E_{\mathbf{C}}$, and any cobase change square

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[d, hook] & \mathfrak{X}' \arrow[d, hook] \\ \mathfrak{Y} \arrow[r, hook] & \mathfrak{Y}' \end{tikzcd} \tag{10.10}$$

in $\mathbf{C\vec{e}ll}_c(\mathbf{C})$, then, by Corollary 8.1.4.8, the associated square

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook, "\mathfrak{e}"] & X' \arrow[d, hook, "\mathfrak{e}'"] \\ Y \arrow[r, hook] & Y' \end{tikzcd} \tag{10.11}$$

is a cobase change. Consequently, e' is again an expansion. To summarize, we have shown that $(\mathbf{C\vec{e}ll}_c(\mathbf{C}), E_{\mathbf{C}})$ is a pre-Whitehead framework. We will denote this pre-Whitehead framework by $\mathbf{W}(\mathbf{C})$.

Finally, it can be useful to observe that the covariant Whitehead monoid functor

$$\widetilde{\mathrm{Wh}}_{\mathbf{W(C)}} \colon \mathbf{C\ddot{e}ll}_c(\mathbf{C}) \to \mathbf{AbMon}$$

associated to a cellularized category with expansions can be extended to $\mathbf{C}$.

Construction 10.1.2.8. Let $\mathbf{C}$ be a cellularized category with expansions. For $A \in \mathbf{C}$, denote by $\mathbf{Wh_C}(A)$ the subcategory of $\mathbf{RCell(C)}_A$, whose objects are finite structured relative cell complexes and whose morphisms are such inclusions of subcomplexes $i \colon \tilde{\mathfrak{c}} \hookrightarrow \mathfrak{c}$, for which the associated relative cell complex $\mathfrak{i} \colon X \hookrightarrow Y$ is an expansion. By Lemma 10.1.2.6, this is indeed a well-defined sub-category, which furthermore is closed under co-products and contains all isomorphisms between its objects. As finite structured relative cell complexes and expansions are preserved under cobase change, this construction defines a subpseudo-functor

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "\mathbf{Wh_C}(-)"] \arrow[r, bend right, "\mathbf{RCell(C)}_-"'] \arrow[r, phantom, "\downarrow"] & \mathbf{SymMonCat} \end{tikzcd} \tag{10.12}$$

Now, given a fixed absolute cell complex $\mathfrak{A}$, we may consider the forgetful functor

$$\mathbf{C\ddot{e}ll}_c(\mathbf{C})_{\mathfrak{A}/} \to \mathbf{RCell(C)}_A$$

which associates to an inclusion of subcomplexes $\mathfrak{A} \overset{i}{\hookrightarrow} \mathfrak{X}$, the induced relative cell complex $\mathfrak{i} \colon A \hookrightarrow X$ (under Observation 8.1.3.5) and acts as the identity on morphisms. This functor is injective on objects, and has as image the finite structured relative cell complexes with inclusions of subcomplexes between them. If we restrict the target category to finite structured relative cell complexes and inclusions of subcomplexes, denoted $\mathbf{RCell(C)}_A^{\hookrightarrow,c}$, then it follows from Observation 8.1.3.5 that the induced functor is fully faithful and bijective on objects. In other words, it induces an isomorphism of categories

$$\mathbf{C\ddot{e}ll}_c(\mathbf{C})_{\mathfrak{A}/} \xrightarrow{\cong} \mathbf{RCell(C)}_A^{\hookrightarrow,c}.$$

If we now restrict to only allowing for expansions as morphisms on both sides, we obtain an isomorphism of categories

$$\mathbf{Wh_{W(C)}}(\mathfrak{A}) \xrightarrow{\cong} \mathbf{Wh_C}(A).$$

This construction is natural in A, inducing a natural isomorphism

$$\begin{tikzcd} \mathbf{C\ddot{e}ll}_c(\mathbf{C}) \arrow[r, "\mathbf{Wh_{W(C)}}"] \arrow[d] \arrow[dr, phantom, "\Leftarrow \cong"] & \mathbf{SymMonCat} \\ \mathbf{C}. \arrow[ur, bend right, "\mathbf{Wh_C}"'] & \end{tikzcd} \tag{10.13}$$

If we now pass to path components of categories, we obtain an extension (up to natural isomorphism) of the Whitehead monoid

$$\begin{tikzcd} \mathbf{C\ddot{e}ll}_c(\mathbf{C}) \arrow[r, "\widetilde{\mathrm{Wh}}_{\mathbf{W(C)}}"] \arrow[d] & \mathbf{AbMon}. \\ \mathbf{C} \arrow[ur, dashed, bend right, "\widetilde{\mathrm{Wh}}_{\mathbf{C}}"'] & \end{tikzcd} \tag{10.14}$$

Notation 10.1.2.9. Just as in the case of pre-Whitehead frameworks, we will denote the equivalence class in $\widetilde{\mathrm{Wh}}_{\mathbf{C}}(A)$ associated to a relative cell complex $\mathfrak{c} \colon A \to X$ by $\langle \mathfrak{c} \rangle$, and the covariant functoriality in the form $(-)_*$.

Remark 10.1.2.10. Construction 10.1.2.8 is important insofar as it illustrates that for every pre-Whitehead framework arising from a cellularized framework, the cellular structure of $\mathfrak{X} \in \mathbf{C\ddot{e}ll}_c(\mathbf{C})$ is essentially irrelevant to the monoid

$$\widetilde{\mathrm{Wh}}_{\mathbf{W(C)}}(\mathfrak{X})$$

and consequently also not relevant to the underlying Whitehead group of invertible elements. Nevertheless, the cell structure of $\mathfrak{X}$ is relevant for the sake of simple homotopy theory, in order to associate a well-defined Whitehead torsion to a morphism $\mathfrak{X} \to \mathfrak{Y}$. This is a phenomenon that does not arise in the world of classical simple homotopy theory, as every homeomorphism of finite CW-complexes is a simple equivalence([Cha74]), but is a relevant phenomenon in the stratified world (Section 13.3.2).

Notation 10.1.2.11. Keeping Remark 10.1.2.10 in mind, we will nevertheless make notational use of Construction 10.1.2.8 insofar as we will simply write $\mathrm{Wh}_{\mathbf{C}}(\mathfrak{X})$ to refer to the Whitehead group $\mathrm{Wh}_{\mathbf{W(C)}}(\mathfrak{X})$. The cell structure on X is tracked via the notation here. We use analogous notation for Whitehead monoids.

Completely analogously to the proof of Lemma 9.1.3.11, one may show the following identity.

Lemma 10.1.2.12. *Let* $\mathbf{C}$ *be a cellularized category with expansions. Given two finite relative cell complexes* $\mathfrak{c}\colon A \to X$ *and* $\mathfrak{d}\colon X \to Y$*, the identity*

$$c_*\langle \mathfrak{d} \circ \mathfrak{c} \rangle = \langle \mathfrak{d} \rangle + c_*\langle \mathfrak{c} \rangle$$

holds. If $\mathfrak{e}\colon A \to X$ *is a finite expansion, then* e_* *is invertible, with inverse given by*

$$(e_*)^{-1}\langle \mathfrak{d} \rangle = \langle \mathfrak{d} \circ \mathfrak{e} \rangle \, .$$

We will also make use of the following extension of the definition of simple equivalences.

Definition 10.1.2.13. Let $\mathbf{C}$ be a cellularized category with expansions. We say that a finite structured relative cell complex $\mathfrak{s}\colon A \hookrightarrow X$ is a *structured simple equivalence*, if it defines the 0-element in $\widetilde{\mathrm{Wh}}_{\mathbf{C}}(A)$.

Using analogous arguments as in the case of a pre-Whitehead framework one may then deduce the following properties of structured simple equivalences.

Lemma 10.1.2.14. *Let* $\mathbf{C}$ *be a cellularized category with expansions. The following statements hold:*

1. *A structured relative cell complex* $\mathfrak{c}\colon A \hookrightarrow X$ *is a structured simple equivalence, if and only if there exists a finite expansion* $\mathfrak{e}\colon X \hookrightarrow Y$ *such that* $\mathfrak{e} \circ \mathfrak{c}$ *is a finite expansion.*
2. *Structured simple equivalences are stable under cobase change.*
3. *Structured simple equivalences fulfill the two-out-of-three property for vertical composition.*
4. *Structured simple equivalences are closed under the two-out-of-three property.*

Furthermore, the definition is compatible with Definition 9.1.3.8, in the sense that if $\mathbf{W(C)}$ *is a Whitehead framework, and* $\mathfrak{A}$ *a finite relative cell complex, then a finite relative cell complex* $\mathfrak{s}\colon A \to X$ *is a structured simple equivalence, if and only if the induced inclusion of absolute cell complexes* $\mathfrak{A} \hookrightarrow \mathfrak{s} \circ \mathfrak{A}$ *is a simple equivalence in* $\mathbf{W(C)}$*.*

To keep language a bit more concise, we will use the following language:

Notation 10.1.2.15. A transfinite vertical composition of (structured) simple equivalences will be called a *transfinite* (structured) simple equivalence.

Observation 10.1.2.16. Using the basic interactions between transfinite compositions and cobase changes, it follows that transfinite (structured) simple equivalences are stable under cobase change and vertical transfinite composition. Furthermore, as every transfinite simple equivalence $s\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$, between finite absolute cell complexes $\mathfrak{X}$ and $\mathfrak{Y}$ is necessarily given by a finite composition of simple equivalences, s is already a simple equivalence.

Lemma 10.1.2.17. *Let $A \xhookrightarrow{\mathfrak{s}} X$ be a transfinite structured simple equivalence in a cellularized category with expansion* $\mathbf{C}$. *Then there exists an expansion $\mathfrak{e}'\colon X \hookrightarrow X'$, such that $\mathfrak{e}' \circ \mathfrak{s}$ is an expansion.*

Proof. Fix a filtration-presentation of $\mathfrak{s}$ as a transfinite composition of finite structured simple equivalences

$$A = X^0 \xhookrightarrow{\mathfrak{s}^0} X^1 \hookrightarrow \cdots \hookrightarrow X^\lambda = X.$$

Under these assumptions, one may now produce a diagram with arrows given by structured relative cell complexes (commutative under composition of structured relative cell complexes) indexed over $\{(\alpha, \beta) \mid \alpha, \beta \in \lambda, \alpha \geq \beta\}$

$$\begin{tikzcd}
X^0 \arrow[r, hook] & X^1 \arrow[r, hook] \arrow[d, hook] & X^2 \arrow[r, hook] \arrow[d, hook] & \dots \arrow[r, hook] \arrow[d, hook] & X^\lambda \arrow[d, hook] \\
& X^{1,1} \arrow[r, hook] & X^{2,1} \arrow[r, hook] \arrow[d, hook] & \dots \arrow[r, hook] \arrow[d, hook] & X^{\lambda,1} \arrow[d, hook] \\
& & X^{2,2} \arrow[r, hook] & \dots \arrow[r, hook] & \dots \arrow[d, hook] \\
& & & & X^{\lambda,\lambda}
\end{tikzcd} \tag{10.15}$$

with the following properties:

1. Every horizontal and every vertical restriction of the diagram to a line is a vertical transfinite composition diagram of structured cell complexes.
2. Every square

$$\begin{tikzcd}
X^{\alpha,\beta} \arrow[r, hook] \arrow[d, hook] & X^{\alpha+1,\beta} \arrow[d, hook] \\
X^{\alpha+1,\beta} \arrow[r, hook] & X^{\alpha+1,\beta+1}
\end{tikzcd} \tag{10.16}$$

   forms a cobase change square, and so does its reflection at the diagonal.
3. For every $\alpha < \lambda$, the vertical relative cell complex $X^{\alpha,\alpha+1} \hookrightarrow X^{\alpha+1,\alpha+1}$ is a finite expansion.
4. For every $\alpha < \lambda$, the diagonal composition $X^{\alpha,\alpha} \hookrightarrow X^{\alpha+1,\alpha+1}$ is a finite expansion.

The diagram can be constructed via transfinite induction in the vertical direction, using the inductive assumption that all horizontal successor complexes $X^{\alpha,\alpha} \hookrightarrow X^{\alpha+1,\alpha}$ are finite structured simple equivalences. Then the diagonal of this diagram

$$X^0 \hookrightarrow X^{1,1} \hookrightarrow X^{2,2} \hookrightarrow \cdots \hookrightarrow X^{\lambda,\lambda}$$

is a transfinite composition diagram of finite expansions and equips $X^0 \to X^{\lambda,\lambda}$ with the structure of an expansion, which we denote by $\mathfrak{e}'$. Similarly, the vertical at λ

$$X^\lambda \hookrightarrow X^{\lambda,1} \hookrightarrow X^{\lambda,2} \hookrightarrow \cdots \hookrightarrow X^{\lambda,\lambda}$$

equips $X^\lambda \to X^{\lambda,\lambda}$ with the structure of a structured relative cell complex. Observe that it follows by the compatibility of pushouts and transfinite compositions and Corollary 8.1.4.2

that every successor complex $X^{\lambda,\alpha} \hookrightarrow X^{\lambda,\alpha+1}$ is a finite expansion. Consequently, the induced structured relative cell complex $X^{\lambda} \hookrightarrow X^{\lambda,\lambda}$ is an expansion, which we denote by $\mathfrak{e}$. Finally, observe that, by commutativity of the diagram, it holds that $\mathfrak{e} = \mathfrak{e}' \circ \mathfrak{s}$, which shows the claim. □

Remark 10.1.2.18. One should be careful to note that the converse of Lemma 10.1.2.17 is generally false. Namely, there can be an inclusion of finite cell complexes $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$, such that a composition of the latter with a (transfinite) expansion is an expansion, but i is not a simple equivalence. Counterexamples can be produced by performing Eilenberg Swindles on non-trivial PL-h-cobordisms, but as we have not shown that classical simple homotopy theory fits into the frameworks we discuss here yet, we will not give a detailed example here.

10.1.3 Functors of cellularized categories with expansions

To finish this section, let us study the functoriality of the construction which assigns to a cellularized category with expansions its associated pre-Whitehead framework. This makes use of the calculus of cellularized functors described in Section 8.2. Notation and language can be found there.

Definition 10.1.3.1. Let $\mathbf{C}$ and $\mathbf{D}$ be cellularized categories. We say that a cellularized relative functor $\mathfrak{i}\colon F \to G$, from $\mathbf{C}$ to $\mathbf{D}$, is *finite*, if $\hat{\mathfrak{i}}\colon \mathbf{RCell}(\mathbf{C}) \to \mathbf{RCell}(\mathbf{D})$ preserves finite relative structured cell complexes.

Remark 10.1.3.2. By Corollary 8.2.2.9, a cellularized relative functor $\mathfrak{i}$ is finite, if and only if $\mathfrak{i}$ maps generating boundary inclusions into finite relative structured cell complexes

Definition 10.1.3.3. Let $\mathbf{C}$ and $\mathbf{D}$ be cellularized categories with expansions. We say that a cellularized relative functor $\mathfrak{i}\colon F \to G$, from $\mathbf{C}$ to $\mathbf{D}$, is called a *functor of cellularized categories with expansions*, if the following conditions hold:

1. $\mathfrak{i}$ is finite;
2. $\hat{\mathfrak{i}}\colon \mathbf{RCell}(\mathbf{C}) \to \mathbf{RCell}(\mathbf{D})$ maps transfinite structured simple equivalences into transfinite structured simple equivalences.

Notation 10.1.3.4. The expression "*functor of cellularized categories with expansions*" is significantly too long to keep spelling it out on a regular basis. We will call such a functor a relative W-functor, with W standing for Whitehead.

Example 10.1.3.5. Using the cellularizations from Example 10.1.2.3, the cellularized functor $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ is tautologically a W-functor.

Example 10.1.3.6. Consider the cellularized action of $\mathbf{Set}$ on a category with expansions (and a non-empty set of boundary inclusions) $\mathbf{C}$ (Example 8.2.5.5). Given an inclusion of sets $i\colon S \hookrightarrow S'$, the associated functor is a relative W-functor if and only if the image of i has finite complement. Indeed, given any relative cell complex $\mathfrak{c}$, the relative structured cell complex $i \hat{*} \mathfrak{c}$ is a cobase change of $\bigsqcup_{S' \setminus i(S)} \mathfrak{c}$. Hence, expansions are always preserved, and finite cell complexes are preserved if and only if $S' \setminus i(S)$ is finite.

By Corollary 8.2.2.9 and Construction 8.2.3.1 and stability under cobase change and finite vertical composition of all notions of relative cell complex involved in the definition of Definition 10.1.3.3, one obtains the following elementary properties of W-functors.

Lemma 10.1.3.7. *Let $\mathfrak{i}\colon \mathbf{C} \to \mathbf{D}$ be a cellularized relative functor between two cellularized categories with expansions. $\mathfrak{i}$ is a W-functor if and only if $\hat{\mathfrak{i}}$ maps generating boundary inclusions $b \in \mathbb{B}$ into finite structured relative cell complexes and generating elementary expansions $\mathfrak{e} \in \mathbb{E}_{\mathbf{C}}$ into structured simple equivalences.*

Lemma 10.1.3.8. *The vertical composition of two W-functors (whenever defined) is again a cellularized W-functor.*

Lemma 10.1.3.9. *The Leibniz composition of two W-functors (whenever defined, see Construction 8.2.3.1) is again a cellularized W-functor.*

Construction 10.1.3.10. If $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ is a relative W-functor of the form $\varnothing \Rightarrow F$, i.e., an absolute W-functor (or just a W-functor), then we obtain an induced functor

$$\mathbf{Cell}(\mathfrak{F})\colon \mathbf{Cell}(\mathbf{C}) \to \mathbf{Cell}(\mathbf{D}).$$

By Corollary 8.2.2.9, this functor preserves inclusions of subcomplexes. Furthermore, by assumption, it maps finite absolute cell complexes into finite absolute cell complexes and expansions into simple equivalences. As $\hat{\mathfrak{F}}$ preserves cobase changes (see Corollary 8.2.2.9) and it follows by Corollary 8.1.4.8 that $\mathbf{Cell}(\mathfrak{F})$ preserves pushouts of inclusions of subcomplexes. Consequently, we obtain an induced functor

$$\mathbf{W}(\mathfrak{F})\colon \mathbf{C\ddot{e}ll}_c(\mathbf{C}) \to \mathbf{C\ddot{e}ll}_c(\mathbf{D})$$

which preserves cobase change squares and maps expansions into simple equivalences. In other words, we obtain an induced functor of pre-Whitehead categories. Denote by **CellCatExp** the category whose objects are cellularized categories with expansions and whose morphisms are absolute W-functors. Composition is given by Leibniz composition, which in the case of absolute functors is indeed associative (see Construction 8.2.3.1). Clearly, the identity functor is always a W-functor. By Lemma 10.1.3.9, **CellCatExp** does indeed define a category. Using the functoriality of **Abs**, it follows that we obtain a functor

$$\begin{aligned} \mathbf{CellCatExp} &\to \mathbf{WES} \\ \mathbf{C} &\mapsto \mathbf{W}(\mathbf{C}) \\ \mathfrak{F} &\mapsto \mathbf{W}(\mathfrak{F})\,, \end{aligned}$$

associating to a cellular category with expansions its associated pre-Whitehead framework.

10.2 Simple cylinders and Whitehead model categories

So far, we have not described how the pre-Whitehead framework associated to a cellularized category with expansions relates to homotopy theoretic considerations. Let us now give conditions under which the pre-Whitehead framework becomes a Whitehead framework and the Whitehead groups can be thought of as groups of presentations of a homotopy type, as described in Chapter 2.

10.2.1 Simple cylinders

We are first going to need an appropriate notion of cylinder that mimics the interaction of the topological cylinder with simple homotopy equivalences (see [Coh73; KP86]). To this end, let us introduce some additional language for cellularized functors.

Definition 10.2.1.1. We say that a relative cellularized functor $(\mathfrak{i}\colon F \Rightarrow G)\colon \mathbf{C} \to \mathbf{D}$, with target a cellularized category with expansions, is *simple*, if for every structured relative cell complex $\mathfrak{c}\colon A \to X$ in $\mathbf{RCell}(\mathbf{C})$, the induced relative cell complex

$$\hat{\mathfrak{i}}(\mathfrak{c})\colon F(X) \cup_{F(A)} G(A) \to G(X)$$

is a (possibly transfinite) structured simple equivalence.

Remark 10.2.1.2. It follows, by Corollary 8.2.2.9 and the stability of transfinite structured simple equivalences under cobase change and transfinite composition, that to check whether a cellularized functor $\mathbf{C} \to \mathbf{D}$ is simple, it suffices to verify that the defining property holds for generating boundary inclusion $b \in \mathbb{B}$. Similarly, it is not hard to see that simple cellularized functors are stable under cobase change and vertical transfinite composition.

Recall that the *fold map*, associated to an object $X \in \mathbf{C}$, in a category with coproducts $\mathbf{C}$, is the canonical morphism $X \sqcup X \to X$, induced by the identities on X. The fold map induces a natural transformation of functors $- \sqcup - \Rightarrow 1_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{C}$.

Observation 10.2.1.3. We can think of $- \sqcup - \colon \mathbf{C} \to \mathbf{C}$ as the cellularized functor $\{0,1\} \otimes -$ of Example 8.2.5.5. In particular, the inclusions $\{i\} \hookrightarrow \{0,1\}$ induce inclusions of cellularized functors $\iota_i \colon 1_{\mathbf{C}} \hookrightarrow \{0,1\} \otimes -$, for $i = 0,1$. It is immediate from the definition of Example 8.2.5.5 that ι_i defines a relative W-functor.

Definition 10.2.1.4. Let $\mathbf{C}$ be a cellularized category with expansions. A simple cylinder consists of the data of

1. a factorization of the fold transformation

$$\begin{tikzcd} \{0,1\} \otimes - \arrow[rr, Rightarrow, "\text{fold}"] \arrow[dr, Rightarrow, dashed, "\iota_\partial"'] & & 1_{\mathbf{C}} \\ & \mathrm{Cyl}; \arrow[ur, Rightarrow, dashed, "\pi"'] & \end{tikzcd} \tag{10.17}$$

2. the structure of a cellularized functor on ι_∂, denoted $\mathfrak{i}_\partial$, such that $\mathfrak{i}_\partial$ is a W-functor, that furthermore has the following property: For $i = 0,1$, the vertical composition of W-functors

$$\mathfrak{i}_i \colon \{i\} \otimes - \hookrightarrow \{0,1\} \otimes - \xhookrightarrow{\mathfrak{i}_\partial} \mathrm{Cyl}\,.$$

is simple.

Notation 10.2.1.5. We will use the suggestive notation of denoting simple cylinders in the form $\mathfrak{i}_\partial \colon \{0,1\} \otimes - \hookrightarrow [0,1] \otimes -$. This does not necessarily mean that the functor $[0,1] \otimes -$ comes from some action of spaces or simplicial sets on $\mathbf{C}$. However, in all relevant examples we consider $[0,1] \otimes -$ does indeed arise from a cellularized bifunctor $\mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$, induced by the structure of a simplicial (semi-)model category on $\mathbf{C}$.

For the remainder of this subsection, fix a cellularized category with expansions $\mathbf{C}$ and a simple cylinder $(\mathfrak{i}_\partial \colon \{0,1\} \otimes - \Rightarrow [0,1] \otimes -, \pi \colon [0,1] \otimes \Rightarrow 1_{\mathbf{C}})$ on $\mathbf{C}$. As an immediate consequence of Corollary 8.2.2.9, and the stability of structured transfinite simple equivalences under transfinite compositions and cobase change, one obtains the following lemma.

Lemma 10.2.1.6. *To verify that a W-functor $\mathfrak{i}_\partial$ as in Definition 10.2.1.4 has the second defining property of a simple cylinder, it suffices to verify that, for every boundary inclusion $b \colon \partial D \to D \in \mathbb{B}$, the induced morphism*

$$\hat{\mathfrak{i}}_i(\mathfrak{b}) \colon \{i\} \otimes D \cup_{\{i\} \otimes \partial D} [0,1] \otimes \partial D \hookrightarrow [0,1] \otimes D$$

is a structured simple equivalence.

Observation 10.2.1.7. We have seen in Observation 10.2.1.3 that the inclusions $\{i\} \otimes - \hookrightarrow \{0,\} \otimes -$ are canonically W-functors. As $\{i\} \otimes - \cong 1_{\mathbf{C}}$, the inclusion $\varnothing \hookrightarrow \{i\} \otimes -$ is the cellularized relative identity functor, and hence also carries the structure of a W-functor. Now, let $\mathfrak{i}_\partial \colon \{0,1\} \otimes - \to [0,1] \otimes -$ be part of a simple cylinder. Then, by Lemma 10.1.3.8, the vertical compositions

$$\begin{tikzcd} & \{0\} \otimes - \arrow[dr] & & \\ \varnothing \arrow[ur] \arrow[dr] & & \{0,1\} \otimes - \arrow[r, "\mathfrak{i}_\partial"] & {[0,1] \otimes -} \\ & \{1\} \otimes - \arrow[ur] & & \end{tikzcd} \tag{10.18}$$

equip $\varnothing \to [0,1] \otimes -$ with the structure of an absolute W-functor, which fits into a diagram of inclusions of cellularized functors (see Definition 8.2.4.3)

$$\begin{tikzcd}
1_{\mathbf{C}} \arrow[r, equal, "\sim"] & \{0\} \otimes - \arrow[dr, hook, "i_0"] & & \\
& & \{0,1\} \otimes - \arrow[r, hook] & {[0,1]} \otimes - \\
1_{\mathbf{C}} \arrow[r, equal, "\sim"] & \{1\} \otimes -, \arrow[ur, hook] & &
\end{tikzcd} \tag{10.19}$$

each of which induces a relative W-functor (under Definition 8.2.4.3). In particular, it follows from this observation that given any absolute cell complex $\mathfrak{X}$, we obtain an induced diagram of inclusions of subcomplexes

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, equal, "\sim"] & \{0\} \otimes \mathfrak{X} \arrow[dr, hook, "i_0"] & & \\
& & \{0,1\} \otimes \mathfrak{X} \arrow[r, hook] & {[0,1]} \otimes \mathfrak{X} \\
\mathfrak{X} \arrow[r, equal, "\sim"] & \{1\} \otimes \mathfrak{X}. \arrow[ur, hook] & &
\end{tikzcd} \tag{10.20}$$

Given that we now have cellularized versions of the cylinder available, a cellularized version of the mapping cylinder is, of course, not far off.

Construction 10.2.1.8. Let $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C})$ be an absolute cell complex. Given a morphism $f: X \to Y$ in $\mathbf{C}$, we can consider the cobase change square

$$\begin{tikzcd}
\{0\} \otimes X \arrow[r] \arrow[d, "\hat{\mathfrak{i}}_0(\mathfrak{X})"] & Y \arrow[d, "\mathfrak{i}_Y"] \\
{[0,1]} \otimes X \arrow[r] & Y \cup_{\{0\} \otimes X} [0,1] \otimes X,
\end{tikzcd} \tag{10.21}$$

with $\hat{\mathfrak{i}}_0$ as described in Observation 10.2.1.7. If $\mathfrak{X}$ is finite, then by assumption and stability of simple equivalences under vertical composition and cobase change (Lemma 10.1.2.14) and compatibility of cellularized functors with the latter operations (Corollary 8.2.2.9), the left-hand vertical in Diagram (10.21) is a structured simple equivalence. If $\mathfrak{X}$ is infinite, then it is at least a transfinite vertical composition of structured simple equivalences, which we will keep in mind for later. Consequently, the right vertical is a structured simple equivalence (transfinite structured simple equivalence). If we now assume the existence of a cell structure $\mathfrak{Y}$ on Y, then we may compose $\mathfrak{i}_Y$ with this cell structure to obtain an inclusion of cell complexes

$$\mathfrak{Y} \hookrightarrow \mathfrak{Y} \cup_{\{0\} \otimes \mathfrak{X}} [0,1] \otimes \mathfrak{X}.$$

We denote the right-hand absolute cell complex by $\mathfrak{M}_f$, the associated inclusion by $\mathfrak{i}_{\mathfrak{Y}}$, and the underlying object in $\mathbf{C}$ by M_f. This notation needs to be taken with the caveat that $\mathfrak{M}_f$ is not entirely a functor of f, but of f together with choice of cell structures on X and Y. If $\mathfrak{X}$ and $\mathfrak{Y}$ are finite, then by construction $\mathfrak{i}_{\mathfrak{Y}}: \mathfrak{Y} \hookrightarrow \mathfrak{M}_f$ is a simple equivalence.

Notation 10.2.1.9. As we have already seen in Construction 10.2.1.8 we will from now on often be in the situation where we refer to a morphism $f: X \to Y$ in $\mathbf{C}$, while having two specific cell structures $\mathfrak{X}$ on X and $\mathfrak{Y}$ on Y. The correct category to handle these kinds of arrows is the fiber product $\mathbf{C} \times_{\mathrm{Ob}(\mathbf{C})} \mathrm{Ob}(\mathbf{Cell}(\mathbf{C}))$, whose objects are structured cell complexes, and whose morphisms are morphisms of the underlying object in $\mathbf{C}$ (where all object categories are equipped with the indiscrete structure, making every object terminal). It will be useful to have a shorthand notation for this category available, and we will denote it by $\mathbf{C\bar{e}ll}(\mathbf{C})$. We can think of $\mathbf{Cell}(\mathbf{C})$ as embedded into $\mathbf{C\bar{e}ll}(\mathbf{C})$ as a wide subcategory. We say that a morphism in $\mathbf{C\bar{e}ll}(\mathbf{C})$, $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism of structured cell complexes, if it is in $\mathbf{Cell}(\mathbf{C})$. We furthermore say that f is an

- inclusions of subcomplexes;
- expansion;

if f is in $\mathbf{Cell}(\mathbf{C})$ and has the respective property. We denote by $\mathbf{C\bar{e}ll}_c(\mathbf{C})$ the full subcategory of $\mathbf{C\bar{e}ll}(\mathbf{C})$, whose objects are given by finite structured cell complexes.

Construction 10.2.1.10. In the situation of Construction 10.2.1.8, the morphism $i_Y\colon Y \to Y \cup_{\{0\}\otimes X} [0,1] \otimes X$ admits a retract in $\mathbf{C}$, induced by the universal property of the pushout in the diagram

$$\begin{tikzcd} X \otimes 0 \arrow[r] \arrow[d] & X \otimes [0,1] \arrow[r, "\pi_X"] \arrow[d] & X \arrow[d] \\ Y \arrow[r] \arrow[rr, bend right, "f \sqcup 1_Y"'] & Y \cup_{\{0\}\otimes X} [0,1] \otimes X \arrow[r, dashed, "r_Y"] & Y\,. \end{tikzcd} \tag{10.22}$$

Together with the composition $i_X\colon X \cong \{1\} \otimes X \hookrightarrow [0,1] \otimes X \to M_f$, r_Y defines a functorial factorization of f

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i_X"'] & & Y \\ & M_f \arrow[ur, "r_Y"'] & \end{tikzcd} \tag{10.23}$$

which lifts to a functorial factorization in $\mathbf{C\bar{e}ll}(\mathbf{C})$.

Construction 10.2.1.11. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathbf{C\bar{e}ll}(\mathbf{C})$. Consider the composition of morphisms

$$\{1\} \otimes X \hookrightarrow [0,1] \otimes X \to M_f\,.$$

While the right hand morphism does generally not define a morphism of structured cell complexes $[\mathfrak{0},\mathfrak{1}] \otimes \mathfrak{X} \to \mathfrak{M}_f$, its composition with the left hand morphism nevertheless defines an inclusion of structured cell complexes $\mathfrak{X} \hookrightarrow \mathfrak{M}_f$. To see this, observe first that as $\{\mathfrak{0},\mathfrak{1}\} \otimes \mathfrak{X} \hookrightarrow [\mathfrak{0},\mathfrak{1}] \otimes \mathfrak{X}$ is an inclusion of a subcomplex, the sets of cells of $\{\mathfrak{0}\} \otimes \mathfrak{X}$ and $\{\mathfrak{1}\} \otimes \mathfrak{X}$ in $[\mathfrak{0},\mathfrak{1}] \otimes \mathfrak{X}$ are disjoint. Consequently, the set of cells of the structured relative cell complex associated to $\{\mathfrak{0}\} \otimes \mathfrak{X} \hookrightarrow [\mathfrak{0},\mathfrak{1}] \otimes \mathfrak{X}$ contains all cells of $\{\mathfrak{1}\} \otimes \mathfrak{X}$. In other words, the square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, "\{1\}\otimes\mathfrak{X}"'] & \{0\} \otimes X \arrow[d, "\hat{\mathfrak{i}}_0(\mathfrak{X})"] \\ \{1\} \otimes X \arrow[r] & [0,1] \otimes X \end{tikzcd} \tag{10.24}$$

defines a morphism of relative cell complexes $\{1\} \otimes \mathfrak{X} \to \hat{\mathfrak{i}}_0(\mathfrak{X})$. In particular, the horizontal composition of squares

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, "\{1\}\otimes\mathfrak{X}"'] & \{0\} \otimes X \arrow[r] \arrow[d, "\hat{\mathfrak{i}}_0(\mathfrak{X})"] & \{0\} \otimes Y \arrow[d, "\mathfrak{i}_Y"] \\ \{1\} \otimes X \arrow[r] & [0,1] \otimes X \arrow[r] & Y \cup_{\{0\}\otimes X} [0,1] \otimes X \end{tikzcd} \tag{10.25}$$

defines a morphism of structured cell complexes $\{1\} \otimes \mathfrak{X} \to \mathfrak{i}_Y$, which as both parts of the composition are injective on cells, is also injective on cells. As, by definition, $\mathfrak{C}_{\mathfrak{M}_f}$ contains all cells of $\mathfrak{i}_Y$, it follows in particular that the morphism $\{1\} \otimes X \to M_f$ defines a morphism of cell complexes $\{1\} \otimes \mathfrak{X} \to \mathfrak{M}_f$, which is injective on cells. By Proposition 8.1.3.1, it thus defines the inclusion of a subcomplex. Importantly, using Construction 10.2.1.10, it follows that up to inverting $\mathfrak{i}_Y$ we may represent f by a span of inclusions of subcomplexes

$$\mathfrak{X} \xhookrightarrow{\mathfrak{i}_X} \mathfrak{M}_f \xhookleftarrow{\mathfrak{i}_Y} \mathfrak{Y} \,. \tag{10.26}$$

Similarly, we may represent homotopies in terms of $[0,1] \otimes -$ as inclusions of subcomplexes.

Construction 10.2.1.12. Given a morphism $H\colon [0,1] \otimes \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{C\overline{ell}(C)}$, we denote by $\overline{\mathfrak{M}}_H$, the object in $\mathbf{C\overline{ell}(C)}$ obtained as follows: First, consider the following cobase change

$$\begin{tikzcd} {[0,1] \otimes X} \arrow[r, "\pi"] \arrow[d, hook, "\mathfrak{i}_{[0,1]\otimes \mathfrak{X}}"'] & X \arrow[d, hook] \\ M_H \arrow[r] & \overline{M}_H \end{tikzcd} \tag{10.27}$$

of structured relative cell complexes. Composing the right-hand vertical cell structure with the one on X induces the structure of an absolute cell complex $\overline{\mathfrak{M}}_H$ on $\overline{M}_H$, together with an inclusion of subcomplexes

$$\mathfrak{X} \hookrightarrow \overline{\mathfrak{M}}_H \,.$$

It is then not hard to verify that the composition

$$Y \xhookrightarrow{i_Y} M_H \to \overline{M}_H$$

induces a morphism of cell complexes $\mathfrak{Y} \to \overline{\mathfrak{M}}_H$, which is injective on the level of cells, i.e., we obtain a subcomplex inclusion

$$\mathfrak{Y} \hookrightarrow \overline{\mathfrak{M}}_H \,.$$

Remark 10.2.1.13. In a topological scenario using the canonical simple cylinder $[0,1] \times -$, the space $\overline{\mathfrak{M}}_H$ of Construction 10.2.1.11 is the cell complex obtained by collapsing the subspace $[0,1] \times X$ in M_H to X.

Notation 10.2.1.14. As in the case of a mapping cylinder, we will often encounter the following situation. We are given structured cell complexes $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{X}'$ together with a pushout square in $\mathbf{C}$

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, hook, "a"] \arrow[dr, phantom, "\ulcorner" very near end] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd} \tag{10.28}$$

such that the left vertical defines an inclusion of a subcomplex $\mathfrak{X} \hookrightarrow \mathfrak{Y}$. We may then equip Y' with the cell structure $f_{!}\mathfrak{a} \circ \mathfrak{X}'$. We will write

$$\mathfrak{X}' \cup_{\mathfrak{X}} \mathfrak{Y}$$

for the resulting structured cell complex on Y. We then obtain a pushout diagram in $\mathbf{C\overline{ell}(C)}$ (and $\mathbf{C}$),

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d, hook, "a"] \arrow[dr, phantom, "\ulcorner" very near end] & \mathfrak{X}' \arrow[d, hook] \\ \mathfrak{Y} \arrow[r] & \mathfrak{X}' \cup_{\mathfrak{X}} \mathfrak{Y} \end{tikzcd} \tag{10.29}$$

such that the verticals are both inclusions of subcomplexes, and the relative cell structure on the right is obtained via cobase change from the relative cell structure on the left. Such a diagram will be called *cobase change* in $\mathbf{C\overline{e}ll(C)}$. Note that this notation is slightly misleading insofar as it refers to additional data compared to just being the pushout in $\mathbf{C\overline{e}ll(C)}$ (as pushouts in this category do not determine the cell structure). This notation is compatible with Corollary 8.1.4.8 in the sense that whenever the pushout in $\mathbf{Cell(C)}$ exists, it agrees with this construction.

Next, let us study the interaction of the mapping cylinder construction with the pre-Whitehead framework associated to $\mathbf{C}$. These are reminiscent of the classical interactions of mapping cylinders with simple equivalences, found, for example, in [Coh73]. Note that, in a setting without cell structures, similar results were proven in [KP86].

Proposition 10.2.1.15. *Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{A}, \mathfrak{B} \in \mathbf{Cell(C)}$ be absolute cell complexes. Let $a: \mathfrak{A} \hookrightarrow \mathfrak{B}$ be an inclusion of a subcomplex, and let $s: \mathfrak{B} \hookrightarrow \mathfrak{X}$ be a subcomplex inclusion that is also a transfinite simple equivalence. Finally, let $f: \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathbf{C\overline{e}ll}(C)$ and let $H : [0,1] \otimes \mathfrak{X} \to \mathfrak{Y}$ be a homotopy from f to a morphism g. Then the following holds:*

S(i) *The canonical morphism $X \cup_{\{1\}\otimes B} M_{f\circ s} \to M_f$ induces the inclusion of a subcomplex*

$$\mathfrak{X} \cup_{1\otimes\mathfrak{B}} \mathfrak{M}_{f\circ s} \hookrightarrow \mathfrak{M}_f ,$$

which is a transfinite simple equivalence.

S(ii) *The canonical morphism $M_{f\circ a} \to M_f$ induces the inclusion of a subcomplex $\mathfrak{M}_{f\circ a} \hookrightarrow \mathfrak{M}_f$, which is a transfinite simple equivalence.*

S(iii) *The canonical morphisms*

$$M_f, M_g \to \overline{M}_H$$

induces inclusions of subcomplexes

$$\mathfrak{M}_f, \mathfrak{M}_g \hookrightarrow \overline{\mathfrak{M}}_H,$$

which are transfinite simple equivalences.

In particular, if $\mathfrak{X}$ and $\mathfrak{Y}$ are finite, it follows that all transfinite simple equivalences above define simple equivalences.

Proof. Explicitly, the set of cells of $\mathfrak{X} \cup_{\mathfrak{B}} \mathfrak{M}_{f\circ s}$ can canonically be identified with the set

$$\mathfrak{C}_{\{1\}\otimes\mathfrak{X}} \sqcup (\mathfrak{C}_{[0,1]\otimes\mathfrak{B}} \setminus \mathfrak{C}_{\{0,1\}\otimes\mathfrak{B}}) \sqcup \mathfrak{C}_{\{0\}\otimes\mathfrak{Y}}$$

and $\mathfrak{M}_{f\circ s}$ can be identified with

$$\mathfrak{C}_{\{1\}\otimes\mathfrak{X}} \sqcup (\mathfrak{C}_{[0,1]\otimes\mathfrak{X}} \setminus \mathfrak{C}_{\{0,1\}\otimes\mathfrak{X}}) \sqcup \mathfrak{C}_{\{0\}\otimes\mathfrak{Y}} .$$

By Corollary 8.2.2.9, the induced morphism $[0,1] \otimes \mathfrak{B} \to [0,1] \otimes \mathfrak{X}$ is injective on cells, and maps $\mathfrak{C}_{\{0,1\}\otimes\mathfrak{B}}$ into $\mathfrak{C}_{\{0,1\}\otimes\mathfrak{X}}$. In particular, it induces a well defined injective map on cells $(\mathfrak{C}_{[0,1]\otimes\mathfrak{B}} \setminus \mathfrak{C}_{\{0,1\}\otimes\mathfrak{B}}) \to (\mathfrak{C}_{[0,1]\otimes\mathfrak{X}} \setminus \mathfrak{C}_{\{0,1\}\otimes\mathfrak{X}})$. From this perspective, the morphism $X \cup_{\{1\}\otimes B} M_{f\circ s} \to M_f$ acts on cells as the injection

$$1_{\mathfrak{C}_{\{1\}\otimes\mathfrak{X}}} \sqcup \mathfrak{C}([0,1] \otimes s)|_{\dots} \sqcup 1_{\mathfrak{C}_{\{0\}\otimes\mathfrak{Y}}} .$$

In particular, it induces a well defined inclusion of cell complexes

$$\mathfrak{X} \cup_{1\otimes\mathfrak{B}} \mathfrak{M}_{f\circ s} \hookrightarrow \mathfrak{M}_f .$$

Using the identifications of sets of cells above, the cells of the induced relative cell complex

$$X \cup_{1\otimes B} M_{f\circ s} \hookrightarrow M_f$$

can be canonically identified with

$$\mathfrak{C}_{[0,1]\otimes\mathfrak{X}} \setminus (\mathfrak{C}_{\{0,1\}\otimes\mathfrak{X}} \cup \mathfrak{C}_{[0,1]\otimes\mathfrak{B}}) .$$

Observe that, by definition, this also defines the set of cells of the relative cell complex $\hat{\mathfrak{i}}(\mathfrak{s})$. Hence, the commutative square

$$\begin{tikzcd} \{0,1\}\otimes X \cup_{\{0,1\}\otimes B} \otimes[0,1]\otimes B \arrow[r] \arrow[d, hook, "\hat{\mathfrak{i}}(\mathfrak{s})"] & X \cup_{1\otimes B} M_{f\circ s} \arrow[d, hook] \\ {[0,1]}\otimes X \arrow[r] & M_f \end{tikzcd} \tag{10.30}$$

defines a morphism of structured relative cell complexes, which is bijective on cells. By Corollary 8.1.4.2, it follows that the latter square defines a cobase change. As $[0,1]\otimes -$ is a W-functor, it follows that the left hand vertical is a (transfinite) structured simple equivalence. Consequently, by stability under cobase change, the same holds for the right hand vertical, as was to be shown.
That the remaining morphisms define inclusions of subcomplexes can be shown in a similar manner. We will only verify that these inclusions define simple equivalences. For the second morphism, denote by $\mathfrak{c}\colon M_{f\circ a} \to M_f$ the relative cell complex, induced by the inclusion of cell complexes $\mathfrak{M}_{f\circ a} \hookrightarrow \mathfrak{M}_f$. Then the commutative diagram

$$\begin{tikzcd} \{0\}\otimes X \cup_{\{0\}\otimes A} [0,1]\otimes A \arrow[r] \arrow[d, hook, "\hat{\mathfrak{i}}"'] & \{0\}\otimes Y \cup_{\{0\}\otimes A} [0,1]\otimes A \arrow[d, hook, "\mathfrak{c}"] \\ {[0,1]}\otimes X \arrow[r] & \{0\}\otimes Y \cup_{\{0\}\otimes X} [0,1]\otimes X \end{tikzcd} \tag{10.31}$$

defines a morphism of relative cell complexes, which one may easily verify to be a bijection on cells. Consequently, by Corollary 8.1.4.2, the latter square is a cobase change. By the definition of a simple cylinder, the left hand morphism is a structured transfinite simple equivalence. Consequently, $\mathfrak{c}$ is a transfinite simple equivalence, as was to be shown. Finally, for the remaining pair of inclusions of cell complexes $\mathfrak{M}_f, \mathfrak{M}_g \hookrightarrow \overline{\mathfrak{M}}_H$, we only prove the case of g, which suffices by symmetry of the situation. Denote the associated relative cell complex $M_g \hookrightarrow \overline{M}_H$ by $\mathfrak{c}'$. Consider the following commutative diagram of squares which can all be verified to define morphisms of structured cell complexes.

$$\begin{tikzcd} \{1\}\otimes X \arrow[r, hook] \arrow[d, hook, "\mathfrak{i}_X"'] & {[0,1]}\otimes X \arrow[r, hook] \arrow[d, hook, "\mathfrak{i}_{[0,1]\otimes X}"] & X \arrow[d, hook, "\mathfrak{i}_X"] \\ M_g \arrow[r] & {[0,1]}\otimes X \cup_{\{1\}\otimes X} M_g \arrow[r] \arrow[d, hook, "\mathfrak{c}"] & M_g \arrow[d, hook, "\mathfrak{c}'"] \\ & M_H \arrow[r] & \overline{M}_H . \end{tikzcd} \tag{10.32}$$

Here $\mathfrak{c}$ is the inclusions of subcomplexes associated to the pair of morphisms $\{1\}\otimes\mathfrak{X} \hookrightarrow [0,1]\otimes\mathfrak{X}$, $H\colon X\otimes[0,1] \to Y$, given by Property S(i). Observe that the outer vertical rectangle (which is the defining diagram for $\overline{\mathfrak{M}}_H$) is a cobase change by definition. The left upper square is a cobase change by definition, and the upper horizontal rectangle is a square with horizontals identities (isomorphisms), and hence also cobase change. Using the pasting law for cobase changes (Corollaries 8.1.2.12 and 8.1.4.3) it follows that all squares in the diagram are cobase change. By Property S(i), $\mathfrak{c}$ is a transfinite structured simple equivalence. Consequently, the same holds for $\mathfrak{c}'$, as was to be shown. □

Next, let us use the simple cylinder to connect the Whitehead framework to the associated homotopy category. For the following proofs, it will be useful for the cylinder to be symmetric, in the sense that one can mirror it at $t = \frac{1}{2}$, in a way that preserves cell structures. It turns out that by gluing together two (not necessarily symmetric) cylinders, this can always be achieved.

Proposition 10.2.1.16. *Suppose that* $\mathbf{C}$ *is a cellularized category with expansions which admits a simple cylinder functor. Then* $\mathbf{C}$ *admits a (possibly different) simple cylinder* $(\mathfrak{i}_\partial\colon \{0,1\} \otimes - \Rightarrow [0,1] \otimes -, \pi)$*, such that the following holds: There exists an inclusion of a cellularized subfunctor*

$$i_{\frac{1}{2}}\colon 1_{\mathbf{C}} \hookrightarrow [0,1] \otimes -$$

and an isomorphism of cellularized functors

$$t\colon [0,1] \otimes - \xrightarrow{\cong} [0,1] \otimes -$$

fulfilling the following properties:

1. *The relative cellularized functor* $\mathfrak{i}_{\frac{1}{2}}$ *associated to* $i_{\frac{1}{2}}$ *is simple.*
2. *The identities*

$$t \circ i_{\frac{1}{2}} = i_{\frac{1}{2}}$$
$$t \circ i_0 = i_1$$

hold.

Proof. We will suggestively denote the cylinder whose existence we assume in the form

$$i'_\partial\colon \{0, \frac{1}{2}\} \otimes - \hookrightarrow [0, \frac{1}{2}] \otimes -,$$

where we shifted notation from 0 to 0 and 1 to $\frac{1}{2}$. To make the notation even more suggestive, we will also denote the very same cylinder in the form

$$\{\frac{1}{2}, 1\} \otimes - \hookrightarrow [\frac{1}{2}, 1] \otimes - .$$

where this time, however, we replaced notation from 0 to 1 and from 1 to $\frac{1}{2}$. Now, consider the diagram of absolute cellularized functors

$$\begin{tikzcd}
\{\frac{1}{2}\} \otimes - \arrow[r, hook] \arrow[d, hook] & \{0, \frac{1}{2}\} \otimes - \arrow[r, hook] \arrow[d, hook] & {[0, \frac{1}{2}] \otimes -} \arrow[d, hook] \\
\{\frac{1}{2}, 1\} \otimes - \arrow[r, hook] \arrow[d, hook] & (\{0, \frac{1}{2}\} \cup_{\{\frac{1}{2}\}} \{\frac{1}{2}, 1\}) \otimes - \arrow[r, hook] \arrow[d, hook] & ([0, \frac{1}{2}] \otimes -) \sqcup \{1\} \otimes - \arrow[d, hook] \\
{[\frac{1}{2}, 1] \otimes -} \arrow[r, hook] & ([\frac{1}{2}, 1] \otimes -) \sqcup \{0\} \otimes - \arrow[r, hook] & {[0, 1] \otimes -}
\end{tikzcd} \tag{10.33}$$

in which all squares are given by pushouts (which exist by Lemma 8.2.4.6). The inclusion ι_∂ is given by the composition

$$\{0, 1\} \otimes - \hookrightarrow (\{0, \frac{1}{2}\} \cup_{\{\frac{1}{2}\}} \{\frac{1}{2}, 1\}) \otimes - \hookrightarrow [0, 1] \otimes - .$$

The required natural transformation $[0,1] \otimes - \to 1_{\mathbf{C}}$ is given by the gluing of the associated transformations $[0, \frac{1}{2}] \otimes -, [\frac{1}{2}, 1] \otimes - \to 0$, and together with ι_∂, is easily seen to define a factorization of the fold map. Observe that, by assumption, all upper horizontal and left vertical arrows in Diagram (10.33) define relative W-functors. By Lemma 8.2.4.6, all remaining inclusions are given by cobase changes of these relative W-functors. Hence, every inclusion of cellularized functors in the diagram is a W-functor. In particular, by stability of W-functors under composition, the inclusion

$$\{0,1\}\otimes - \hookrightarrow (\{0,\frac{1}{2}\}\cup_{\{\frac{1}{2}\}}\{\frac{1}{2},1\})\otimes - \hookrightarrow [0,1]\otimes - \,.$$

is a W-functor. Next, consider the commutative diagram

$$\begin{tikzcd}
 & & \{0\}\otimes - \arrow[d, hook] \\
 & \{\frac{1}{2}\}\otimes - \arrow[r, hook] \arrow[d, hook] & {[0,\frac{1}{2}]}\otimes - \arrow[d, hook] \\
\{1\}\otimes - \arrow[r, hook] & {[\frac{1}{2},1]}\otimes - \arrow[r, hook] & {[0,1]}\otimes -
\end{tikzcd} \tag{10.34}$$

with the inner square being pushout. All of the relative cellularized functors (induced by the inclusions of subfunctors) in this diagram, besides the two arrows with target $[0,1]\otimes -$, were assumed to be simple. The remaining two cellularized relative functors, however, are by Lemma 8.2.4.6, given by cobase changes of simple cellularized relative functors. In particular, they are also simple. It follows that all arrows in the latter diagram, and all of their compositions, define simple cellularized relative functors. It remains to expose the automorphism $t\colon[0,1]\to[0,1]$. It is induced by applying the universal property of the pushout to the diagram

$$\begin{tikzcd}
\{\frac{1}{2}\}\otimes - \arrow[r, hook] \arrow[d, hook] & {[0,\frac{1}{2}]}\otimes - \arrow[r, equal] \arrow[d, hook] & {[0,1]'}\otimes - \arrow[d, equal] \\
{[\frac{1}{2},1]}\otimes - \arrow[r, hook] \arrow[d, equal] & {[0,1]}\otimes - \arrow[dr, dashed, "t"] & {[\frac{1}{2},1]}\otimes - \arrow[d, hook] \\
{[0,1]'}\otimes - \arrow[r, equal] & {[0,\frac{1}{2}]}\otimes - \arrow[r, hook] & {[0,1]}\otimes -
\end{tikzcd} \tag{10.35}$$

and is readily verified to have the required properties.

□

We will also need the following lemma:

Lemma 10.2.1.17. *Let $E\subset \mathbf{C\vec{e}ll}(\mathbf{C})$ be the class of expansions. Let $a,b\colon\mathfrak{X}\hookrightarrow\mathfrak{Y}$ be such that $\mathfrak{X}\sqcup\mathfrak{B}\xrightarrow{(a,b)}\mathfrak{Y}$ extends to an inclusion $H\colon[0,1]\otimes\mathfrak{X}\to\mathfrak{Y}$. Then $a=b$ in $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}]$. The analogous statement for $\mathbf{C\vec{e}ll}_c(\mathbf{C})$ holds.*

Proof. Observe first that by the two-out-of-three property for isomorphism, every transfinite simple equivalence is an isomorphism in $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}]$. Next, observe that the two inclusions of subcomplexes

$$\iota_i\colon\{i\}\otimes\mathfrak{X}\to[0,1]\otimes\mathfrak{X}$$

are identified in $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}]$. Indeed, by assumption, we have

$$t\circ i_{\frac{1}{2}} = 1_{\hat{X}\otimes[0,1]}\circ i_{\frac{1}{2}}$$

Furthermore, as $i_{\frac{1}{2}}\colon\{\frac{1}{2}\}\otimes\mathfrak{X}\hookrightarrow[0,1]\otimes\mathfrak{X}$ is a simple equivalence, it defines an isomorphism and in particular an epimorphism in $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}]$. It follows that

$$t = 1_{[0,1]\otimes\mathfrak{X}}$$

in $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}]$. Consequently, we obtain the equalities

$$i_0 = t\circ i_1 = 1_{[0,1]\otimes\mathfrak{X}}\circ i_1 = i_1$$

in $\mathbf{C\vec{e}ll(C)}[E^{-1}]$, as was to be shown. It follows that any two inclusions of cell complexes $a, b\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ (with Y not necessarily fibrant), which extend to the inclusion of subcomplexes $H\colon [\mathfrak{o}, \mathfrak{1}] \otimes \mathfrak{X} \to \mathfrak{Y}$, are identified in $\mathbf{C\vec{e}ll(C)}[E^{-1}]$ via

$$a = H \circ i_0 = H \circ i_1 = b\,.$$

The proof for $\mathbf{C\vec{e}ll}_c\mathbf{(C)}$ is identical. □

In the next section, we will additionally assume that the choices of generating boundary inclusions and expansions can be used as generating data for a (semi-)model category. It turns out that these assumptions guarantee that one obtains a Whitehead framework that interacts well with the associated homotopy category of the model category.

10.2.2 Whitehead model categories

We now state the main result of this section, which shows that in a large class of homotopy theoretical frameworks the associated homotopy category can be described entirely in terms of structured absolute cell complexes and expansions. First, however, recall from Section 8.1.6 the notion of filtration compactness in a cellularized category and from [Hir03] the notion of a class of arrows *permitting the small object argument.*

Theorem 10.2.2.1. *Let* $\mathbf{C}$ *be a cellularized category with expansions. Now, suppose the following additional assumptions are fulfilled:*

- *T(i)* $\mathbf{C}$ *admits the structure of a cofibrantly generated (left) semi-model category (see, for example, Definition 7.4.1.1), with class of weak equivalences* W*.*
- *T(ii)* $\mathbb{B}_{\mathbf{C}}$ *provides a class of generating cofibrations for the semi-model structure.*
- *T(iii) The underlying set of morphisms of* $\mathbb{E}_{\mathbf{C}}$ *permits the small object argument.*
- *T(iv) Every underlying morphism in* $\mathbb{E}_{\mathbf{C}}$ *defines an acyclic cofibration with cofibrant source.*
- *T(v) An object* $X \in \mathbf{C}$ *for which the terminal morphism* $X \to \star$ *has the right lifting property with respect to every morphism in* $\mathbb{E}_{\mathbf{C}}$ *is fibrant.*
- *T(vi)* $\mathbf{C}$ *admits a simple cylinder.*

Denote by $E \subset \mathbf{C\vec{e}ll(C)}$ *the wide subcategory given by such inclusions of subcomplexes which define expansions. Then the forgetful functor*

$$\mathbf{C\vec{e}ll(C)} \to \mathbf{C}$$

descends to an equivalence of categories

$$\begin{tikzcd} \mathbf{C\vec{e}ll(C)} \arrow[r] \arrow[d] & \mathbf{C} \arrow[d] \\ \mathbf{C\vec{e}ll(C)}[E^{-1}] \arrow[r, dashed, "\simeq"'] & \mathbf{C}[W^{-1}]\,. \end{tikzcd} \tag{10.36}$$

Suppose, additionally, that every target of an arrow in $\mathbb{B}_{\mathbf{C}}$ *is filtration compact, and every source of an arrow in* $\mathbb{E}_{\mathbf{C}}$ *is filtration compact. Then the induced functor*

$$\begin{tikzcd} \mathbf{C\vec{e}ll}_c\mathbf{(C)} \arrow[r] \arrow[d] & \mathbf{C} \arrow[d] \\ \mathbf{C\vec{e}ll}_c\mathbf{(C)}[E^{-1}_{\mathbf{W(C)}}] \arrow[r, dashed, "f.f."] & \mathbf{C}[W^{-1}] \end{tikzcd} \tag{10.37}$$

is fully faithful.

Before we give the proof, let us keep track of the following useful facts about fibrant replacement under the conditions of Theorem 10.2.2.1.

Lemma 10.2.2.2. *Under the complete assumptions of Theorem 10.2.2.1 the following two statements hold:*

1. *Let $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C})$. Then there exists an expansion $e: \mathfrak{X} \hookrightarrow \mathfrak{Y}$, such that the underlying object of $\mathfrak{Y}$, Y, is fibrant in $\mathbf{C}$.*

2. *Let $\mathfrak{Z}$ be a finite cell complex and let $e: \mathfrak{X} \hookrightarrow \mathfrak{Y}$ be a (possibly infinite) expansion. Then, for every morphism $f: \mathfrak{Z} \to \mathfrak{Y}$ in $\mathbf{C\bar{e}ll}(\mathbf{C})$, there exists a factorization $\mathfrak{X} \xhookrightarrow{\tilde{e}} \tilde{\mathfrak{Y}} \xhookrightarrow{e'} \mathfrak{Y}$ of e, such that f factors through e', and such that $\tilde{e}$ is a finite expansion.*

Proof. The first claim follows by applying the small object argument (see Observation 8.1.6.16 and [Hir03, Sec. 10.5]) to $\mathbb{E}_{\mathbf{C}}$, and using that $\mathbb{E}_{\mathbf{C}}$ detects fibrant objects. The second claim follows from Propositions 8.1.6.14 and 8.1.6.17. ☐

Let us now provide a proof of Theorem 10.2.2.1.

Proof. We will freely make us of standard arguments involving model categories (see, for example, [Hir03]). As all objects in sight are cofibrant, their analogues for semi-model categories hold. Let us first observe that, by Property T(ii), the underlying object of every absolute cell complex $\mathfrak{X}$ is cofibrant in $\mathbf{C}$. In particular, by Property T(iv), expansions between two such objects are given by transfinite compositions of cobase changes of acyclic cofibrations with cofibrant source, and thus define acyclic cofibrations. It follows that every morphism in E maps to a weak equivalence, i.e., that a factorization as indicated in the statement of the theorem does exist. Observe, furthermore, that by the small object argument and Property T(ii), every object in $\mathbf{C}$ is weakly equivalent to an absolute cell complex. Hence, it follows that the induced functor

$$\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}] \to \mathbf{C}[W^{-1}]$$

is essentially surjective. Next, observe that every transfinite simple equivalence in $\mathbf{C\vec{e}ll}(\mathbf{C})$ descends to an isomorphism in $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}]$, as it admits a composition with an isomorphism (an expansion) which is an isomorphism (see Lemma 10.1.2.17). We may thus just assume that we have inverted all transfinite simple equivalences, instead of only expansions. In particular, every transfinite simple equivalence in $\mathbf{C\vec{e}ll}(\mathbf{C})$ descends to a weak equivalence in $\mathbf{C}$. To see fully-faithfullness, observe that by the small object argument applied to Property T(iii) and and Property T(v), every object in $\mathbf{C\vec{e}ll}(\mathbf{C})$ is, up to an expansion, given by a fibrant object in $\mathbf{C}$. Hence, to show fully-faithfulness, we may restrict to such absolute cell complexes $\mathfrak{X}$ whose underlying object is fibrant. In particular, for such objects $\mathfrak{X}, \mathfrak{Y}$, the set of morphisms $\mathbf{C}[W^{-1}](X, Y)$ is given by homotopy classes of morphisms $f: X \to Y$, with respect to any choice of cylinder $X \sqcup X \hookrightarrow \mathrm{Cyl}(X) \xrightarrow{\simeq} X$, with the right hand morphism given by a weak equivalence and the left hand morphism given by a cofibration. We may use the simple cylinder whose existence is guaranteed by Property T(vi). Indeed, by the assumption that $\{0,1\} \otimes - \to [0,1] \otimes -$ is a cellularized relative functor and Property T(ii), it follows that for any absolute cell complex $\mathfrak{X}$, the induced map $\{0,1\} \otimes X = X \sqcup X \to [0,1] \otimes X$ is a cofibration. Furthermore, the morphism $\pi: [0,1] \otimes X \to X$, admits a section $\{\mathfrak{o}\} \otimes \mathfrak{X} \to [\mathfrak{o}, \mathfrak{1}] \otimes \mathfrak{X}$, which is given by a simple equivalence, and hence induces a weak equivalence in $\mathbf{C}$. By the two-out-of-three property for weak equivalences, it follows that $\pi: [0,1] \otimes X \to X$ is a weak equivalence. Now, to see fullness of the functor, observe that by Construction 10.2.1.11, every morphism $f: X \to Y$ of the underlying objects may be written in $\mathbf{C}[W^{-1}]$ as a composition of an inclusion of structured absolute subcomplexes, followed by a transfinite simple equivalence. To verify that $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}] \to \mathbf{C}[W^{-1}]$ defines an equivalence of categories, it remains to verify faithfullness. Let $\alpha, \beta: \mathfrak{X} \to \mathfrak{Y}$ be two morphisms in $\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}]$ which map to the same element in

$\mathbf{C}[W^{-1}](X, Y)$. Observe that, using cobase changes in $\mathbf{C\vec{e}ll}(C)$, α and β can be represented in the form

$$\alpha = e_a^{-1} a,$$
$$\beta = e_b^{-1} b\,.$$

Consider the commutative diagram

$$\begin{tikzcd}
 & & \mathfrak{Y}_a \arrow[dr, "e'_a"] & \\
\mathfrak{X} \arrow[urr, bend left, "\alpha"] \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[drr, bend right, "\beta"'] & \mathfrak{Y} \arrow[ur, "e_a"] \arrow[rr, "e_0"] \arrow[dr, "e_b"'] & & \mathfrak{Y}' \\
 & & \mathfrak{Y}_b \arrow[ur, "e'_b"'] &
\end{tikzcd} \tag{10.38}$$

with the square induced by the pushout in $\mathbf{Cell(C)}$, which exists by Corollary 8.1.4.8. By stability of expansions under cobase change, every morphism denoted with an e in this diagram is an expansion. Chasing the diagram, to see that $\alpha = \beta$, it suffices to see that

$$e_0 \circ a = e_0 \circ b$$

in $\mathbf{C\vec{e}ll(C)}[E^{-1}]$. Composing with an expansion $\mathfrak{Y}' \hookrightarrow \hat{\mathfrak{Y}}$, we may again assume that the target of $e_0 \circ a, e_0 \circ b$ is fibrant. By assumption, the underlying diagram in $\mathbf{C}[W^{-1}]$ commutes. In particular, we have thus reduced to showing that if $a, b \colon \mathfrak{X} \to \mathfrak{Y}$ are inclusions of subcomplexes which induce the same morphism in $\mathbf{C}[W^{-1}]$, then they induce the same morphism in $\mathbf{C\vec{e}ll(C)}[E^{-1}]$. By assumption that the target is fibrant, a and b are identified if they are homotopic in $\mathbf{C}$, with respect to any choice of cylinder. Now, observe that by Proposition 10.2.1.16 we may without loss of generality assume that the simple cylinder is as in Diagram (10.33). To this end, let $H \colon [0,1] \otimes X \to Y$ be a homotopy between a and b in $\mathbf{C}$, and consider the following diagram in $\mathbf{C\vec{e}ll(C)}$

$$\begin{tikzcd}
 & & \mathfrak{M}_a \arrow[dr, hook, bend left] & \\
\{1\} \otimes \mathfrak{X} \arrow[urr, hook, bend left] \arrow[rr, hook, bend left, "a"] \arrow[rr, hook, bend right, "b"'] \arrow[drr, hook, bend right] & & \mathfrak{Y} \arrow[u, hook] \arrow[d, hook] \arrow[r, hook] & \overline{\mathfrak{M}}_H \\
 & & \mathfrak{M}_b \arrow[ur, hook, bend right] &
\end{tikzcd} \tag{10.39}$$

with morphisms as constructed in Proposition 10.2.1.15. One should be careful to note that this is not a commutative diagram in $\mathbf{C\vec{e}ll(C)}$. Only the right part of the diagram, as well as the two outer composed paths commute. Assume, for now, that we have shown that the left two triangles commute in $\mathbf{C\vec{e}ll(C)}[E^{-1}]$. Observe, furthermore, that by Proposition 10.2.1.15, the right part of the diagram is given by isomorphisms in $\mathbf{C\vec{e}ll(C)}[E^{-1}]$. Then, by chasing the diagram and using the given commutativity, it follows that $a = b$ in $\mathbf{C\vec{e}ll(C)}[E^{-1}]$. Finally, to see that the remaining two triangles commute, consider again the defining pushout square for the mapping cylinder of a (and analogously for b):

$$\begin{tikzcd}
\{0\} \otimes X \arrow[r, hook, "a"] \arrow[d, hook, "i_0"'] & Y \arrow[d, hook] \\
{[0,1]} \otimes X \arrow[r, hook] & M_a \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{10.40}$$

As a defines a morphism of cell complexes, this commutative diagram lifts to a diagram of morphisms of absolute cell complexes

$$\begin{tikzcd} \{\mathfrak{0}\} \otimes \mathfrak{X} \arrow[r, hook, "a"] \arrow[d, hook, "\mathrm{i}_0"'] & \mathfrak{Y} \arrow[d, hook] \\ [\mathfrak{0}, \mathfrak{1}] \otimes \mathfrak{X} \arrow[r, hook] & \mathfrak{M}_a . \end{tikzcd} \tag{10.41}$$

Since a is a relative cell complex, so is its parallel morphism $[0,1] \otimes X \to M_a$, making it a monomorphism, in particular. Consequently, the bottom morphism defines the inclusion of a subcomplex. It follows that

$$[\mathfrak{0}, \mathfrak{1}] \otimes \mathfrak{X} \hookrightarrow \mathfrak{M}_a$$

defines a homotopy that $\mathfrak{X} \xrightarrow{a} \mathfrak{Y} \hookrightarrow \mathfrak{M}_a$ and $\mathfrak{X} = \{\mathfrak{1}\} \otimes \mathfrak{X} \hookrightarrow \mathfrak{M}_a$ as required in Lemma 10.2.1.17, and it follows that the left two triangles in Diagram (10.39) commute. This finishes the proof that

$$\mathbf{C\vec{e}ll}(\mathbf{C})[E^{-1}] \to \mathbf{C}[W^{-1}]$$

defines an equivalence of categories. It remains to show the fully faithfulness claim about

$$\mathbf{C\vec{e}ll}_c(\mathbf{C})[E^{-1}_{\mathbf{W}(\mathbf{C})}] \to \mathbf{C}[W^{-1}].$$

The proof of this claim is essentially identical with the one of the previous fully-faithfulness statement. The only difference is that we may not assume that the target $\mathfrak{Y}$ has underlying fibrant Y. However, by Lemma 10.2.2.2, we may first replace $\mathfrak{Y}$ fibrantly by an expansion $\mathfrak{Y} \hookrightarrow \hat{\mathfrak{Y}}$. Then, again by Lemma 10.2.2.2 it follows that every morphism $\mathfrak{X} \to \hat{\mathfrak{Y}}$ in $\mathbf{C\vec{e}ll}(\mathbf{C})$ and every homotopy $[\mathfrak{0}, \mathfrak{1}] \otimes \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{C\vec{e}ll}(\mathbf{C})$ factors through some $\tilde{\mathfrak{Y}} \subset \hat{\mathfrak{Y}}$ containing $\mathfrak{Y}$, such that $\mathfrak{Y} \hookrightarrow \tilde{\mathfrak{Y}}$ is a finite expansion. As every finite expansion defines an isomorphism in $\mathbf{C\vec{e}ll}_c(\mathbf{C})[E^{-1}]$, it follows that in the proof of fully faithfulness we may, whenever necessary, replace $\mathfrak{Y}$ by an isomorphic $\tilde{\mathfrak{Y}}$, in order to present a morphism in $\mathbf{C}[W^{-1}]$ or a homotopy in $\mathbf{C}$ with respect to $[0,1] \otimes -$. Clearly, this does not change any claims about existence or identity of morphisms. One may then essentially repeat the previous fully faithfulness proof, increasing the size of $\mathfrak{Y}$ by a finite expansion, whenever necessary. ☐

It turns out that the assumptions of Theorem 10.2.2.1 provide a convenient framework to perform simple homotopy theory in. Let us condense them (in slightly strengthened form) in a definition.

Definition 10.2.2.3. A cellularized category with expansions $\mathbf{C}$ is called a *Whitehead semi-model category* if the following axioms are fulfilled.

1. $\mathbf{C}$ admits the structure of a cofibrantly generated semi-model category that is compatible with the generating boundary inclusions $\mathbb{B}_{\mathbf{C}}$ and generating elementary expansions $\mathbb{E}_{\mathbf{C}}$ in the following sense.

2. $\mathbb{B}_{\mathbf{C}}$ provides a set of generating cofibrations and the source and target of every morphisms in $\mathbb{B}_{\mathbf{C}}$ is filtration compact.

3. For every generating elementary expansion $\mathfrak{e}\colon A \hookrightarrow X \in \mathbb{E}_{\mathbf{C}}$, it holds that the underlying morphism $e\colon A \hookrightarrow X$ in $\mathbf{C}$ is an acyclic cofibration whose source, A, is cofibrant and filtration compact.

4. An object $X \in \mathbf{C}$ is fibrant, if and only if the terminal morphism $X \to \star$ has the right lifting property with respect to the set of underlying arrows in $\mathbf{C}$ of elements of $\mathbb{E}_{\mathbf{C}}$.

5. $\mathbf{C}$ admits a simple cylinder.

Notation 10.2.2.4. We will usually just omit the "*semi*" from the name "*Whitehead semi-model category*". In the cellularized context, most objects considered are cell complexes anyway, and the theory of semi-model categories and model categories behaves essentially identically.

Remark 10.2.2.5. Definition 10.2.2.3 is a slight strengthening of the assumptions of Theorem 10.2.2.1. Indeed, observe that for a Whitehead model category, $\mathbb{E}_{\mathbf{C}}$ even permits the small object argument with respect to $\aleph_0$ (by Observation 8.1.6.16). The assumptions of Definition 10.2.2.3 are slightly stronger only insofar as we have assumed that the sources of $\mathbb{B}_{\mathbf{C}}$ are also filtration compact and cofibrant. These assumptions will not be needed in this section, but they will play a crucial role in the core arguments of Theorem 12.1.0.4.

Remark 10.2.2.6. Observe that a Whitehead model category uniquely determines the semi-model structure which it expands to. Indeed, the equivalence of homotopy categories proven in Theorem 10.2.2.1 uniquely determines the class of weak equivalences. The cofibrations are determined by Property T(ii). As the fibrations in a (cofibrantly generated) semi-model category are determined as the morphisms that have the right lifting property with respect to acyclic cofibrations with cofibrant source, it follows that the triple $(\mathbf{C}, \mathbb{B}_{\mathbf{C}}, \mathbb{E}_{\mathbf{C}})$ extends to at most one semi-model structure.

Example 10.2.2.7. The category of simplicial sets, with boundary inclusions and expansions as in Example 10.1.2.3 defines a Whitehead model category. Indeed, the associated combinatorial model category is simply the Kan-Quillen model structure, which is well known to fulfill the required compactness assumptions on generators (indeed, every finite simplicial set is a compact object in the sense that its associated covariant hom-functor preserves all filtered colimits). A simple cylinder is given by $\partial\Delta^1 \times - \to \Delta^1 \times -$. This is canonically a cellularized functor as it is given by the restriction of the cellularized bifunctor given by the product of simplicial sets (see Example 8.2.5.6). The requirements of a simple cylinder were, for example, verified in [Mos19], where it is even shown that the product of a monomorphism with an expansion is again an expansion.

Example 10.2.2.8. The category of topological spaces, with boundary inclusions and expansion as in Example 10.1.2.3 defines a Whitehead model category. The associated combinatorial model category is given by the Quillen model structure on topological spaces. The required compactness assumption is a well-known classical result, which follows, for example, by [Hir03, p. 10.7.4.] and Proposition 8.1.6.8. A simple cylinder is given by cellularizing $\{0,1\} \times - \hookrightarrow [0,1] \times -$, under Lemma 8.2.2.5, as follows. First, fix an (oriented) identification $[0,1] \cong |\Delta^1|$. Then, this choice induces an isomorphism

$$[0,1] \times |\Delta^n| \cong |\Delta^1| \times |\Delta^n| \cong |\Delta^1 \times \Delta^n|$$

which maps $\{0,1\} \times |\Delta^n| \cup [0,1] \times |\partial\Delta^n|$ homeomorphically to $|\partial\Delta^1 \times \Delta^n \cup \Delta^1 \times \partial\Delta^n|$. Hence, one may use Lemma 8.2.2.5 and the cellularization of $|-|$ to obtain a cell structure on $\{0,1\} \times - \hookrightarrow [0,1] \times -$. It is then immediate from Remark 10.2.1.2 and the respective properties for simplicial sets that the thus-defined cellularized cylinder is simple.

Example 10.2.2.9. The category of positively graded chain complexes over some not-necessarily commutative ring $\mathbf{R}$, equipped with the expansions in Example 10.1.2.3 defines a Whitehead model category. The associated model structure is the projective model structure (see [Qui67; Hir03]) on positively graded chain complexes. The compactness assumptions are a classical elementary fact in commutative algebra. As a simple cylinder, one can use the tensor product with the interval complex $I_\bullet \otimes -$. To see this, observe first that the tensor product of chain complexes obtains the structure of a cellularized bifunctor, by equipping $(\partial D^n_\bullet \hookrightarrow D^n_\bullet) \hat{\otimes} (\partial D^m_\bullet \hookrightarrow D^m_\bullet)$ with a single cell given by the basis element $1 \otimes 1 \in R \otimes R \cong R$ in degree $n+m$. It is then not hard to see that tensoring with $I_\bullet$ (with the cell structure given by the cellularization of $\otimes$) defines a simple cylinder.

Corollary 10.2.2.10. *If* $\mathbf{C}$ *is a Whitehead model category, then its associated pre-Whitehead framework* $\mathbf{W}(\mathbf{C})$ *is a Whitehead framework.*

Proof. Let $a, b \colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ be two inclusions of finite cell complexes which agree in $\vec{\mathbf{Cell}}_c(\mathbf{C})[E^{-1}_{\mathbf{W}(\mathbf{C})}]$. We need to show that there exist (appropriately composable) finite expansion e_a, e_b, such that

$e_a a = e_b b$. In other words, we need to show that a and b define the same class in $\widetilde{\mathrm{Wh}}_{\mathbf{W(C)}}$. By faithfulness of the functor

$$\mathbf{C\overset{\rightleftarrows}{e}ll}_c(\mathbf{C})[E^{-1}_{\mathbf{W(C)}}] \to \mathbf{C}[W^{-1}],$$

it follows that $a, b\colon X \to Y$ define the same morphism in $\mathrm{ho}\mathbf{C} = \mathbf{C}[W^{-1}]$. Using Lemma 10.2.2.2, and arguing as in the proof of Theorem 10.2.2.1, it follows that there exists a finite expansion $Y \overset{\mathfrak{e}}{\hookrightarrow} \hat{Y}$ as well as a homotopy $H\colon [0,1] \otimes X \to \hat{Y}$ between ea and eb. We may hence assume without loss of generality that a and b are homotopic through a homotopy $H\colon [0,1] \otimes X \to Y$. By definition of $\widetilde{\mathrm{Wh}}_{\mathbf{W(C)}}$, it suffices to connect a and b through a zig-zag of expansions under $\mathfrak{X}$. More than that, it suffices to expose a zig-zag of simple equivalences in $\mathbf{C\overset{\rightleftarrows}{e}ll}_c(\mathbf{C})_{\mathfrak{X}/}$. By Proposition 10.2.1.15, such a zig-zag is given by the diagram

$$\begin{tikzcd}
 & & \{1\} \otimes \mathfrak{X} \arrow[dll, hook', "a"'] \arrow[ddll, hook'] \arrow[ddl, hook'] \arrow[dd] \arrow[ddr, hook] \arrow[ddrr, hook] \arrow[drr, hook, "b"] & & \\
\{1\} \otimes \mathfrak{Y} \arrow[d, hook, "\simeq"'] & & & & \{1\} \otimes \mathfrak{Y} \arrow[d, hook, "\simeq"] \\
[0,1] \otimes \mathfrak{Y} & \mathfrak{M}_a \arrow[l, hook', "\simeq"] \arrow[r, hook, "\simeq"'] & \overline{\mathfrak{M}}_H & \mathfrak{M}_b \arrow[l, hook', "\simeq"] \arrow[r, hook, "\simeq"'] & [0,1] \otimes \mathfrak{Y}\,.
\end{tikzcd} \tag{10.42}$$

□

10.2.3 Simple equivalences and presentations

Let us now expose some of the consequences of the definition of a Whitehead model category. We remark that most proofs in this section are abstractifications of proofs in [Coh73] from the setting of simple homotopy theory of spaces to general Whitehead model categories. For the remainder of this subsection, fix a Whitehead model category $\mathbf{C}$. We denote the 1-categorical localization of $\mathbf{C}$ at its associated class of weak equivalences by $\mathrm{ho}\mathbf{C}$. The first consequence of Theorem 10.2.2.1 and Corollary 10.2.2.10 is that it allows us to associate a Whitehead torsion to any morphism $\alpha\colon X \to Y$ in $\mathrm{ho}\mathbf{C}$, provided that we have fixed the structure of finite cell complexes $\mathfrak{X}$ and $\mathfrak{Y}$ on X and Y, respectively. To make these types of statements a bit less wordy, let us introduce the following notation.

Notation 10.2.3.1. From an aesthetic perspective, the expressions $\mathbf{C\overset{\rightleftarrows}{e}ll(C)}[E^{-1}]$ and $\mathbf{C\overset{\rightleftarrows}{e}ll}_c(\mathbf{C})[E^{-1}_{\mathbf{W(C)}}]$ can get rather overwhelming. Observe that by Theorem 10.2.2.1, we may respectively identify these categories with $\mathrm{ho}\mathbf{C} \times_{\mathrm{Ob}\mathbf{C}} \times \mathrm{Ob}(\mathbf{Cell(C)})$ (where we equip the categories of objects with the indiscrete structure, making every object terminal) and the full subcategory of this category only given by finite cell complexes. In other words, these are the categories having objects in $\mathbf{Cell(C)}$, but morphisms given by morphisms of the underlying objects in $\mathrm{ho}\mathbf{C}$. We denote the former category by $\mathfrak{ho}\mathbf{C}$, and the latter category, restricting to finite objects, by $\mathfrak{ho}_c\mathbf{C}$, where the fraktur font in $\mathfrak{ho}$ indicates that objects in these categories come with cell structures.

Observation 10.2.3.2. The various localization and forgetful functors agglomerate into a commutative diagram of categories

$$\begin{tikzcd}
\mathbf{C\overset{\rightleftarrows}{e}ll(C)} \arrow[r] \arrow[d] & \mathbf{C\bar{e}ll(C)} \arrow[r] & \mathfrak{ho}\mathbf{C} \arrow[d, hook, "f.f."] \\
\mathbf{C} \arrow[rr] & & \mathrm{ho}\mathbf{C}\,.
\end{tikzcd} \tag{10.43}$$

If we refer to an arrow in some of the upper left categories as an arrow somewhere lower down this commutative diagram, we will mean with respect to these functors.

Notation 10.2.3.3. In the following, we will often refer to the Whitehead framework $\mathbf{W(C)}$ associated to a Whitehead model category $\mathbf{C}$ just by $\mathbf{C}$. In particular, we will write $\mathrm{Wh}_{\mathbf{C}}$ to refer to the Whitehead group functor, and use analogous notation for Whitehead monoids.

Notation 10.2.3.4. Let $\alpha\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathfrak{ho}_c\mathbf{C}$. Under Notation 9.1.3.7, we may associate to α its Whitehead torsion, denoted

$$\langle \mathfrak{X} \xrightarrow{\alpha} \mathfrak{Y} \rangle \in \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X}).$$

Recall that α *is a simple equivalence*, if $\langle \mathfrak{X} \xrightarrow{\alpha} \mathfrak{Y} \rangle = 0$. Given a morphism $\alpha\colon X \to Y$, we will sometimes speak of *the Whitehead torsion of α with respect to the cell structures $\mathfrak{X}$ and $\mathfrak{Y}$*, to refer to the Whitehead torsion of the associated morphism in $\mathfrak{ho}_c\mathbf{C}$.

Observe that this definition of simple equivalence is compatible with previous definitions, in the sense that a morphism $\alpha\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{ho}_c\mathbf{C}$ is a simple equivalence in the sense of Definition 9.1.3.8, if and only if the associated morphism in $\mathfrak{ho}_c\mathbf{C}$ defines a simple equivalence.

Remark 10.2.3.5. There are a series of properties of simple equivalences in $\mathfrak{ho}_c\mathbf{C}$, such as being isomorphisms, stable under inversion and fulfilling the two-out-of-three property, which are immediately inherited from Observation 9.1.3.13 under the fully faithful inclusion of Theorem 9.1.3.4. We will not explicitly list them here and use them freely.

Example 10.2.3.6. Let $\mathfrak{X}, \mathfrak{X}'$ and $\mathfrak{Y}$ be finite structured cell complexes. Let $a\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ be an inclusion of a subcomplex and $f\colon \mathfrak{X} \to \mathfrak{X}'$ be a morphism in $\mathbf{C\overline{e}ll(C)}$. Then the following two morphisms in $\mathbf{C}$ are simple equivalences:

1. The retraction $r\colon \mathfrak{M}_f \to \mathfrak{X}'$ constructed in Construction 10.2.1.11;
2. The canonical morphism $\mathfrak{M}_f \cup_{\{1\}\otimes\mathfrak{X}} \mathfrak{Y} \to \mathfrak{X}' \cup_{\mathfrak{X}} \mathfrak{Y}$, induced by $\mathfrak{M}_f \xrightarrow{r} \mathfrak{X}'$.

The first morphism is a retraction of a simple equivalence (by Construction 10.2.1.11), and hence a simple equivalence by the two-out-of-three property. For the second morphism ...

Proof. ..., consider the following diagram of morphisms in $\mathbf{C}$

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \{0\}\otimes Y \arrow[r, hook] \arrow[d] & {[0,1]\otimes Y} \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r, hook] & X' \cup_X Y \arrow[r, hook, "\phi"] \arrow[rr, bend right, "1"'] & X' \cup_{\{0\}\otimes X} [0,1]\otimes Y \arrow[r, dashed, "\pi'"] & X' \cup_X Y\,.
\end{tikzcd} \tag{10.44}$$

Using cobase changes of the inclusions of subcomplexes in horizontal direction, we obtain an inclusion of subcomplexes

$$\mathfrak{X}' \cup_{\mathfrak{X}} \mathfrak{Y} \hookrightarrow \mathfrak{X}' \cup_{\{0\}\otimes\mathfrak{X}} [0,1] \otimes \mathfrak{Y}$$

the underlying structured relative cell complex of which is a cobase change of a structured simple equivalence. In particular, this inclusion is a simple equivalence. It follows by the two-out-of-three property that its retraction

$$\pi'\colon \mathfrak{X}' \cup_{\{0\}\otimes\mathfrak{X}} [0,1] \otimes \mathfrak{Y} \to \mathfrak{X}' \cup_{\mathfrak{X}} \mathfrak{Y}$$

in $\mathbf{C}$ is a simple equivalence. The latter fits into a commutative diagram in $\mathbf{C\overline{e}ll(C)}$,

$$\begin{tikzcd}
\mathfrak{Y} \cup_{\{1\}\otimes\mathfrak{X}} \mathfrak{M}_f \arrow[d, hook] \arrow[dr, "\pi'"] & \\
{[0,1]\otimes\mathfrak{Y} \cup_{\{0\}\otimes\mathfrak{X}} \mathfrak{X}'} \arrow[r] & \mathfrak{Y} \cup_{\mathfrak{X}} \mathfrak{X}'
\end{tikzcd} \tag{10.45}$$

where the left vertical defines the inclusion of a subcomplex. Consequently, by the two-out-of-three law for simple equivalences, it suffices to show that the left hand vertical is a simple equivalence. The latter morphism fits into a commutative square in $\mathbf{C}$

$$\begin{tikzcd}
{[0,1]\otimes X \cup_{\{1\}\otimes X} \{1\}\otimes Y} \arrow[r] \arrow[d, hook] & Y \cup_{\{1\}\otimes X} M_f \arrow[d, hook] \\
{[0,1]\otimes Y} \arrow[r] & {[0,1]\otimes Y \cup_{\{0\}\otimes X} X'} .
\end{tikzcd} \tag{10.46}$$

One can verify, similarly to the proof of Proposition 10.2.1.15 that the horizontals induce a bijection on cells between the relative structured cell complex

$$\hat{\mathfrak{i}}_1(\mathfrak{a})\colon [0,1]\otimes X \cup_{\{1\}\otimes X} \{1\}\otimes Y \hookrightarrow [0,1]\otimes Y$$

and the relative cell structure associated to the inclusion

$$\mathfrak{Y} \cup_{\{1\}\otimes\mathfrak{X}} \mathfrak{M}_f \hookrightarrow [\mathfrak{0},\mathfrak{1}]\otimes\mathfrak{Y} \cup_{\{\mathfrak{0}\}\otimes\mathfrak{X}} \mathfrak{X}' .$$

It follows, by Corollary 8.1.4.2, that the latter is obtained as a cobase change of the former (alternatively, this is also readily derived from the pasting laws for cobase change squares). By assumption, the former is a structured simple equivalence. As structured simple equivalences are stable under cobase change, it follows that the right vertical defines a simple equivalence, as was to be shown. □

We may now finally connect the Whitehead group to the set of presentations of a homotopy type $Y \in \mathrm{ho}\mathbf{C}$.

Notation 10.2.3.7. Given an object $X \in \mathrm{ho}\mathbf{C}$, a pair $(\mathfrak{Y}, \omega\colon X \xrightarrow{\simeq} Y)$, with $\mathfrak{Y}$ a (finite) structured cell complex and $\omega \in \mathrm{ho}\mathbf{C}(X,Y)$ an isomorphism, will be called a *(finite) presentation of X*. In the following, all presentations will be assumed to be finite.

Construction 10.2.3.8. Let $\mathbf{C}$ be a Whitehead model category and let $X \in \mathrm{ho}\mathbf{C}$. We denote by $\mathrm{Pres}_{\mathbf{C}}(X)$ the quotient set of finite presentations

$$\mathrm{Pres}_{\mathbf{C}}(X) \coloneqq \{(\mathfrak{Y}, \omega\colon X \to Y) \mid \mathfrak{Y} \in \mathfrak{ho}_c\mathbf{C}, \omega \in \mathrm{ho}\mathbf{C}(X,Y) \text{ is an isomorphism. }\}/\sim_s ,$$

where the equivalence relation $\sim_s$ is given by $(\mathfrak{Y}_1,\omega_1) \sim_s (\mathfrak{Y}_2,\omega_2)$, if and only if $\mathfrak{Y}_1 \xrightarrow{\omega_2\omega_1^{-1}} \mathfrak{Y}_2$ is a simple equivalence. This construction is contravariantly functorial in isomorphisms in $\mathrm{ho}\mathbf{C}$, with functoriality given by precomposition, inducing a functor

$$\mathrm{Pres}_{\mathbf{C}}\colon (\mathrm{ho}\mathbf{C})^{\simeq,\mathrm{op}} \to \mathbf{Set} ,$$

defined on the maximal subgroupoid $(\mathrm{ho}\mathbf{C})^{\simeq} \subset \mathrm{ho}\mathbf{C}$, given by taking as morphisms the isomorphisms in $\mathrm{ho}\mathbf{C}$. We may extend $\mathrm{Pres}_{\mathbf{C}}(X)$ to a larger set, functorial in arbitrary morphisms, as follows:
Define

$$\widetilde{\mathrm{Pres}}_{\mathbf{C}}(X) \coloneqq \{(\mathfrak{Y}, \omega\colon X \to Y) \mid \mathfrak{Y} \in \mathfrak{ho}_c\mathbf{C}, \alpha \in \mathrm{ho}\mathbf{C}(X,Y)\}/\sim_{\tilde{s}} ,$$

where, $\sim_{\tilde{s}}$ is the extension of $\sim_s$, given by $(\mathfrak{Y}_1,\omega_1) \sim_s (\mathfrak{Y}_2,\omega_2)$, if and only if there exists a simple equivalence $\gamma\colon \mathfrak{Y}_1 \to \mathfrak{Y}_2$, such that $\gamma \circ \omega_1 = \omega_2$. This construction is contravariantly functorial in morphisms in $\mathrm{ho}\mathbf{C}$, with functoriality given by precomposition, inducing a functor

$$\widetilde{\mathrm{Pres}}_{\mathbf{C}}\colon (\mathrm{ho}\mathbf{C})^{\mathrm{op}} \to \mathbf{Set} .$$

Theorem 10.2.3.9. *Let $\mathbf{C}$ be a Whitehead model category and let $X \in \mathrm{ho}\mathbf{C}$. Let $\omega_0\colon X \xrightarrow{\simeq} \mathfrak{X}_0$ be a finite presentation of X. ω_0 induces a bijection*

$$\widetilde{\mathrm{Pres}}_{\mathbf{C}}(X) \xrightarrow{1:1} \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X}_0)$$

$$(\mathfrak{Y},\alpha) \mapsto \langle \mathfrak{X}_0 \xrightarrow{\alpha\circ\omega_0^{-1}} \mathfrak{Y}\rangle .$$

which restricts to a bijection

$$\begin{aligned}\mathrm{Pres}_{\mathbf{C}}(X) &\xrightarrow{1:1} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}_0)\\ (\mathfrak{Y},\omega) &\mapsto \langle \mathfrak{X}_0 \xrightarrow{\omega\circ\omega_0^{-1}} \mathfrak{Y}\rangle\,.\end{aligned}$$

Proof. This is simply a combination of Theorem 9.1.3.4 with Theorem 10.2.2.1 and the two-out-of-three property for simple equivalences. □

Observation 10.2.3.10. If we fix two different presentations of X, $X \xrightarrow{\omega_i} \mathfrak{X}_i$, for $i = 1,2$, then the diagram

$$\begin{tikzcd} & \mathrm{Pres}_{\mathbf{C}}(X) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}_2) \arrow[rr, "(\omega_2\omega_1^{-1})^*"] & & \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}_1) \end{tikzcd} \tag{10.47}$$

commutes. By Corollary 9.1.3.12, the failure of the lower morphism to be a group homomorphism is precisely measured by the Whitehead torsion $\langle\omega_2\omega_1^{-1}\rangle \in \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}_1)$.

Finally, let us make a few observations on computing the addition and functoriality of Whitehead monoids.

Lemma 10.2.3.11. *Let $\mathfrak{X},\mathfrak{Y}$ be finite structured cell complexes and $f\colon\mathfrak{X}\to\mathfrak{Y}$ a morphism in* $\mathbf{C}$. *Then the equality $\langle f\rangle = \langle \mathfrak{X} \xhookrightarrow{i_{\mathfrak{X}}} \mathfrak{M}_f\rangle$ holds.*

Proof. By Construction 10.2.1.11, f fits into a commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[dr, hook] & & \mathfrak{Y} \\ & \mathfrak{M}_f \arrow[ur, "r_Y"'] & \end{tikzcd} \tag{10.48}$$

where r_Y is a simple equivalence by Example 10.2.3.6. □

Proposition 10.2.3.12. *Let $\mathfrak{X},\mathfrak{X}'$ and $\mathfrak{Y}$ be finite structured cell complexes. Let $a\colon\mathfrak{X}\hookrightarrow\mathfrak{Y}$ be an inclusion of subcomplexes and let $f\colon\mathfrak{X}\to\mathfrak{X}'$ be a morphism in* $\mathbf{C}$. *Consider a cobase change diagram*

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d, hook, "a"] & \mathfrak{X}' \arrow[d, hook, "a'"] \\ \mathfrak{Y} \arrow[r, "f'"] \arrow[ur, phantom, "\ulcorner"] & \mathfrak{X}'\cup_{\mathfrak{X}}\mathfrak{Y} \end{tikzcd} \tag{10.49}$$

in $\mathbf{C\overline{e}ll}(\mathbf{C})$ *(i.e., we take the lower right corner equipped with the induced cell structure $f_{\text{!`}}\mathfrak{a}\circ\mathfrak{X}'$). Then the equalities*

$$\begin{aligned} f_*\langle a\rangle &= \langle a'\rangle;\\ a_*\langle f\rangle &= \langle f'\rangle;\\ \langle a\rangle + \langle f\rangle &= \langle a'\circ f\rangle = \langle f'\circ a\rangle \end{aligned}$$

hold.

Proof. The first equality follows by Construction 10.1.2.8. For the second and the third equality, consider the factorization

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[d, hook] & \mathfrak{M}_f \arrow[r] \arrow[d, hook] & \mathfrak{X}' \arrow[d, hook] \\ \mathfrak{Y} \arrow[r, hook] & \mathfrak{M}_f\cup_{\{1\}\otimes\mathfrak{X}}\mathfrak{Y} \arrow[r] & \mathfrak{X}'\cup_{\mathfrak{X}}\mathfrak{Y} \end{tikzcd} \tag{10.50}$$

of the pushout square Diagram (10.21), with all cell structures induced via vertical composition and cobase change. The left square is a cobase change square in $\vec{\mathbf{Cell}}_c(\mathbf{C})$. By the pasting law for cobase change squares, the right square is again cobase change (in the cellularized category $\mathbf{C}$, where the verticals are equipped with the relative cell structure given by Observation 8.1.3.5). By definition of addition and functoriality of $\widetilde{\mathrm{Wh}}_{\mathbf{C}}$, together with Lemma 10.2.3.11, the middle vertical defines $a_*\langle f\rangle$ and the diagonal of the left square defines $\langle a\rangle + \langle f\rangle$. Consequently, to see that the claimed equalities hold, it suffices to see that the lower right horizontal is a simple equivalence. This was shown in Example 10.2.3.6. ☐

In particular, we obtain the following stability law for cobase changing simple equivalences of the form $f\colon \mathfrak{X} \to \mathfrak{X}'$ in $\mathbf{C}$.

Corollary 10.2.3.13. *In the situation of Proposition 10.2.3.12, if f is a simple equivalence, then so is f', and if a is a simple equivalence, then so is a'.*

10.3 Equivalences of Whitehead model categories

Let us finish this chapter with a detailed look at W-functors in the setting of Whitehead model categories, in particular investigating different resulting notions of equivalence.

10.3.1 W-functors and Quillen functors

Let us first investigate the relationship between W-functors and Quillen functors. For the convenience of the reader, let us spell out some characterizations of Quillen adjunctions for the less common case of semi-model categories.

Recollection 10.3.1.1 (See, for example [BW24])**.** Recall that an adjunction of (cofibrantly generated, left) semi-model categories $L\colon \mathbf{M} \rightleftharpoons \mathbf{N}\colon R$ is called a Quillen adjunction if one of the following equivalent conditions holds:

1. The right adjoint R preserves fibrations and acyclic fibrations.
2. The left adjoint L preserves cofibrations and acyclic cofibrations between cofibrant objects.
3. The left adjoint L preserves cofibrations and weak equivalences between cofibrant objects.

That the first two conditions are equivalent is shown, for example, in [BW24, Lem 4.4]. That the last condition implies the second is immediate. That the second condition implies the last follows by Ken Brown's lemma (see the dual of [nLa25b, Prop.3.1]), applied to the category of cofibrant objects. We will use the basic properties of Quillen adjunctions, such as the construction of the derived functors, freely (see, for example, [Hir03] for the model-category case). When there is a difference between the theory of semi-model categories and model categories we will explicitly remark this.

Remark 10.3.1.2. The first obvious difference between W-functors between Whitehead model categories and left Quillen functors is that the former are only assumed to preserve colimits, while the latter are assumed to be left adjoint. Observe, however, that the main examples of cellularized categories which we consider are locally presentable, or at least can be modified to have this property without changing the categories of structured cell complexes (replacing compactly generated spaces by Δ-generated spaces, for example). In this case, it follows by a version of the adjoint functor theorem (see [nLa24a]) that preserving colimits is equivalent to being left adjoint.

Observation 10.3.1.3. Let $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ be a cellularized functor between Whitehead model categories. Assume F is a left-Quillen functor, and hence preserves weak equivalences between cofibrant objects. Consider the induced functor

$$\begin{aligned} \mathbf{C\ddot{e}ll}(\mathbf{C}) &\to \mathbf{C\ddot{e}ll}(\mathbf{D}) \\ \mathfrak{X} &\mapsto \mathfrak{F}(\mathfrak{X}) \\ a &\mapsto F(a)\,. \end{aligned}$$

It follows by Theorem 10.2.2.1 that localizing $\mathbf{C\ddot{e}ll}(\mathbf{C})$ (or $\mathbf{C\ddot{e}ll}(\mathbf{D})$) at expansions is the same as localizing at such inclusions of subcomplexes which are weak equivalences. As every underlying object of a complex in $\mathbf{C\ddot{e}ll}(\mathbf{C})$ is cofibrant, we obtain an induced functor $\mathfrak{ho}\mathbf{C} \to \mathfrak{ho}\mathbf{D}$, making the diagram

$$\begin{tikzcd} \mathbf{C\ddot{e}ll}(\mathbf{C}) \arrow[r] \arrow[d] & \mathbf{C\ddot{e}ll}(\mathbf{D}) \arrow[d] \\ \mathfrak{ho}\mathbf{C} \arrow[r, dashed] & \mathfrak{ho}\mathbf{D} \end{tikzcd} \tag{10.51}$$

commute. Under the equivalence of categories $\mathfrak{ho}\mathbf{C} \simeq \mathrm{ho}\mathbf{C}$ and $\mathfrak{ho}\mathbf{D} \simeq \mathrm{ho}\mathbf{D}$, this functor computes the left derived functor of F.

As a consequence, we obtain the following equivalent characterization of W-functors.

Proposition 10.3.1.4. *Let $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ a cellularized functor between Whitehead model categories which preserves finite structured relative cell complexes. Then the following are equivalent:*

1. *The underlying functor $F\colon \mathbf{C} \to \mathbf{D}$ preserves cofibrations and weak equivalences between cofibrant objects, and the induced functor*

$$\mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}_c\mathbf{D}$$

preserves simple equivalences.

2. *$\mathfrak{F}$ is a W-functor.*

In particular, it follows that any left adjoint W-functor induces a left Quillen functor of semi-model categories.

Proof. In the following proof, we will often treat structured cell complexes as objects of $\mathbf{C}$ or $\mathbf{D}$. This is to be understood as a reference to their underlying objects. If $\mathfrak{F}$ is a W-functor, then it necessarily preserves relative cell complexes (with respect to the cofibrant generators induced by the structures of cellularized categories on $\mathbf{C}$ and $\mathbf{D}$). As every cofibration is a retract of a relative cell complex, it follows that F preserves cofibrations. To see that F preserves weak equivalences between cofibrant objects, we may first reduce to the case where both the source and the target are cell complexes, as follows. Observe that the property for a weak equivalence, w, to be preserved under F is stable under retracts. It thus suffices to expose w as a retract of a weak equivalence between cell complexes. Now let $w\colon A \to B$ be a weak equivalence between cofibrant objects in $\mathbf{C}$. As A is cofibrant, it follows by the small object argument that A is a retract of a cell complex $\mathfrak{A}$

$$A \hookrightarrow \mathfrak{A} \to A$$

such that $A \hookrightarrow \mathfrak{A}$ is a weak equivalence. Now, consider the diagram of pushout squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "w"'] & \mathfrak{A} \arrow[r] \arrow[d, "w'"'] & A \arrow[d, "w"] \\ B \arrow[r] & B' \arrow[r] & B\,. \end{tikzcd} \tag{10.52}$$

Observe that $A \hookrightarrow \mathfrak{A}$ is a retract of the identity cofibration on $\mathfrak{A}$. In particular, it is a cofibration. Hence, $A \hookrightarrow \mathfrak{A}$ is an acyclic cofibration of cofibrant objects, and thus so is its parallel morphism $B \to B'$. By the two-out-of-three property from weak equivalences, it follows that w' is a weak homotopy equivalence. Hence w is a retract of a weak equivalence with source a cell complex. We have thus reduced to the case where the source of w is a cell complex. Now, to reduce to the case where the target is also a cell complex, we may expose B as a retract of a cell complex

$$B \xrightarrow{i} \mathfrak{B}' \xrightarrow{r} B$$

with $B \xrightarrow{i} \mathfrak{B}'$ a weak equivalence. Then w is a retract of the weak equivalence $w \circ i$, which shows the reduction claim. Now, let $w\colon \mathfrak{A} \to \mathfrak{B}$ be a weak equivalence between cell complexes in $\mathbf{C}$. Applying the small object argument with respect to expansions, we obtain a commutative square

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "w"] \arrow[d, hook] & \mathfrak{B} \arrow[d, hook] \\ \hat{\mathfrak{A}} \arrow[r, "\hat{w}"'] & \hat{\mathfrak{B}} \end{tikzcd} \tag{10.53}$$

with verticals expansions and horizontals weak equivalences, where $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{B}}$ are bifibrant cell complexes. Observe that as $\mathfrak{F}$ is a W-functor, it maps expansions between cofibrant objects into acyclic cofibrations. Hence, by the two-out-of-three property for weak equivalences, it suffices to see that F preserves weak equivalences between bifibrant cell complexes. By the Whitehead Theorem, any such weak equivalence is a homotopy equivalence with respect to any choice of cylinder. In particular, we may use a simple cylinder $\{\mathfrak{o}, \mathfrak{1}\} \to [\mathfrak{o}, \mathfrak{1}] \otimes - \to 1$, whose existence is guaranteed by $\mathbf{C}$ being a Whitehead model category. Given a structured complex $\mathfrak{A}$ in $\mathbf{C}$, the associated cylinder retractions

$$[\mathfrak{o}, \mathfrak{1}] \otimes \mathfrak{A} \to \mathfrak{A}$$

admits a section by a structured simple equivalence. As F is a W-functor, it preserves structured simple equivalences, and hence maps this section into a structured simple equivalence between absolute cell complexes. As every structured simple equivalence between cofibrant objects is an acyclic cofibration (as a transfinite composition of acyclic cofibrations with cofibrant source), it follows that F maps the cylinder retraction $[\mathfrak{o}, \mathfrak{1}] \otimes \mathfrak{A} \to \mathfrak{A}$ into a weak equivalence. As F preserves cofibrations, it thus follows that

$$\mathfrak{F}(\mathfrak{A}) \sqcup \mathfrak{F}(\mathfrak{A}) \cong \mathfrak{F}(\mathfrak{A} \sqcup \mathfrak{A}) \to \mathfrak{F}([\mathfrak{o}, \mathfrak{1}] \otimes \mathfrak{A}) \to \mathfrak{F}(\mathfrak{A})$$

defines a cylinder for $\mathfrak{A}$. In particular, $\mathfrak{F}$ preserves the relation of homotopy of maps between cell complexes. Consequently, $\mathfrak{F}$ preserves homotopy equivalences between cell complexes, and hence weak equivalences between bifibrant cell complexes. The induced functor $\mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}_c\mathbf{D}$ associated to $\mathfrak{F}$ is the functor of homotopy categories associated to the functor of Whitehead frameworks $\mathfrak{F}$ (see Construction 10.1.3.10). Hence, it preserves simple equivalences.

Now, conversely, suppose that F preserves cofibrations and weak equivalences between cofibrant objects, and that

$$\mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}_c\mathbf{D}$$

preserves simple equivalences. We need to show that the image of every elementary expansion $\mathfrak{e}\colon A \hookrightarrow X$ in $\mathbb{E}_{\mathbf{C}}$ under F is a simple equivalence. As A is cofibrant, we may write A as a retract

$$A \xhookrightarrow{i} \mathfrak{A}' \xrightarrow{r} A$$

of a structured cell complex $\mathfrak{A}'$. Furthermore, as A was assumed to be filtration compact, it follows by Proposition 8.1.6.9 that the inclusion $A \hookrightarrow \mathfrak{A}'$ factors through a finite subcomplex of $\mathfrak{A}'$. Hence, we may without loss of generality assume that $\mathfrak{A}'$ is a finite cell complex. Now, consider the commutative diagram of cobase change squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathfrak{e}"'] \arrow[rr, bend left, "1"] & A' \arrow[r] \arrow[d, dashed, "i_{¡}\mathfrak{e}"] & A \arrow[d, "\mathfrak{e}"] \\ X \arrow[r] \arrow[rr, bend right, "1"'] & X' \arrow[r] & X\,. \end{tikzcd} \tag{10.54}$$

If we apply $\mathfrak{F}$ to this diagram, we obtain a diagram of cobase change squares.

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, "\mathfrak{F}(\mathfrak{e})"'] \arrow[rr, bend left, "1"] & F(A') \arrow[r] \arrow[d, dashed, "F(i)_{¡}\mathfrak{F}(\mathfrak{e})"] & F(A) \arrow[d, "\mathfrak{F}(\mathfrak{e})"] \\ F(X) \arrow[r] \arrow[rr, bend right, "1"'] & F(X') \arrow[r] & F(X) \end{tikzcd} \tag{10.55}$$

Using that a morphism is a structured simple equivalence, if and only if defines the 0-element in the extended Whitehead monoids of Construction 10.1.2.8 , it follows that $\mathfrak{F}(\mathfrak{e})$ is a structured simple equivalence, if and only if the middle vertical in the last diagram defines a simple equivalence $\mathfrak{F}(\mathfrak{A}') \to F(i)_{¡}\mathfrak{F}(\mathfrak{e}) \circ \mathfrak{F}(\mathfrak{A}')$. As $\mathfrak{F}$ preserves cobase changes, we have $F(i)_{¡}\mathfrak{F}(\mathfrak{e}) = \mathfrak{F}(i_{¡}\mathfrak{e})$. $i_{¡}\mathfrak{e}$ is a cobase change of a structured simple equivalence, and hence a structured simple equivalence. Hence, again using Construction 10.1.2.8, it follows that $i_{¡}e$ defines a simple equivalence between $\mathfrak{A}' \to i_{¡}\mathfrak{e} \circ \mathfrak{A}'$. In particular, by assumption on F, the image of this morphism under $\mathfrak{F}$, which is $\mathfrak{F}(\mathfrak{A}') \to F(i)_{¡}\mathfrak{F}(\mathfrak{e}) \circ \mathfrak{F}(\mathfrak{A}')$ is a simple equivalence. This finishes the proof. ☐

10.3.2 Equivalences of Whitehead model categories

Next, let us study several possible notions of equivalences between Whitehead model categories. Before we do so, recall the fact that a Quillen functor $\mathbf{C} \to \mathbf{D}$ of (semi-)model categories is a Quillen equivalence (defined just as in the case of ordinary model categories, see [Hir03]) if and only if the induced (respectively left or right derived) functor of ∞-categories is an equivalence of ∞-categories. For the purpose of simple homotopy theory, it can be useful to also have ∞-categorical versions of equivalences available. In the following subsection, localizations are to be understood in the ∞-categorical sense, if not explicitly stated otherwise.

Definition 10.3.2.1. Let $\mathbf{C}$ be a Whitehead model category, with associated ∞-category $\mathcal{C} = \mathbf{C}[W^{-1}]$, which we present as an (∞-categorical) localization of quasi-categories here. Let $\mathrm{Ob}(\mathbf{Cell}(\mathbf{C}))$ be the indiscrete category of absolute structured cell complexes in $\mathbf{C}$ (with objects given by absolute cell complexes, and exactly one morphism between any two objects). We denote by $\mathcal{C}\mathbf{ell}(\mathbf{C})$ the quasi-category obtained via the following pullback of simplicial sets (where we treat categories as quasi-categories, via the nerve construction).

$$\begin{tikzcd} \mathcal{C}\mathbf{ell}(\mathbf{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ \mathrm{Ob}(\mathbf{Cell}(\mathbf{C})) \arrow[r] & \mathrm{Ob}(\mathbf{C}). \end{tikzcd} \tag{10.56}$$

In other words, $\mathcal{C}\mathbf{ell}(\mathbf{C})$ denotes the quasi-category whose objects are given by the absolute structured cell complexes in $\mathbf{C}$, mapping spaces and given by the mapping spaces of the underlying objects in $\mathcal{C}$.

Observation 10.3.2.2. Observe that the forgetful functor

$$\mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}$$

is always an equivalence of ∞-categories. Indeed, by definition, it is fully faithful. The small object argument applied to the generating boundary inclusions in $\mathbf{C}$ shows that it is also essentially surjective.

Definition 10.3.2.3. Given a Whitehead model category $\mathbf{C}$, with associated ∞-category of cell complexes $\mathcal{C}\mathbf{ell}(\mathbf{C})$, we say that a morphism between finite cell complexes $f\colon \mathfrak{X} \to \mathfrak{Y} \in \mathcal{C}\mathbf{ell}(\mathbf{C})$ is a simple equivalence if the associated morphism in $\mathrm{ho}\mathcal{C}\mathbf{ell}(\mathbf{C}) = \mathfrak{ho}\mathbf{C}$ is a simple equivalence. We denote by $\mathcal{S}\mathbf{im}(\mathbf{C})$ the wide subcategory of $\mathcal{C}\mathbf{ell}(\mathbf{C})$, given by finite cell complexes and simple equivalences.

Observation 10.3.2.4. For any cellularized category $\mathcal{C}\mathbf{ell}(\mathbf{C})$, the composition of functors

$$\mathbf{C\overline{ell}}(\mathbf{C}) \to \mathbf{C} \to \mathcal{C}$$

induces a functor

$$\mathbf{C\overline{ell}}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{C}).$$

This functor maps weak equivalences into isomorphisms, and hence induces a functor of quasi-categories

$$\mathbf{C\overline{ell}}(\mathbf{C})[W^{-1}] \to \mathcal{C}\mathbf{ell}(\mathbf{C}).$$

This functor is an equivalence of ∞-categories, which is bijective on objects. Indeed, to see that it is fully faithful, observe that there is a commutative diagram of functors

$$\begin{tikzcd}
\mathbf{C\overline{ell}}(\mathbf{C})[W^{-1}] \arrow[r] \arrow[dr] & \mathcal{C}\mathbf{ell}(\mathbf{C}) \arrow[d] \\
& \mathcal{C}\,.
\end{tikzcd} \tag{10.57}$$

The diagonal is an equivalence of ∞-categories, with inverse induced by replacing an object in $\mathbf{C}$ by an absolute cell complex through the small object argument. The right vertical is fully faithful by construction. Hence, it follows that all functors in the diagram are fully faithful. In this sense, we can also treat $\mathcal{C}\mathbf{ell}(\mathbf{C})$ as the localization of the 1-category $\mathbf{C\overline{ell}}(\mathbf{C})$ at weak equivalences.

Observation 10.3.2.5. It follows by Proposition 10.3.1.4 that every W-functor of Whitehead model categories $\mathfrak{F}\colon \mathbf{C} \to \mathbf{D}$ descends to a functor of ∞-categories.

$$\begin{tikzcd}
\mathbf{C\overline{ell}}(\mathbf{C}) \arrow[r, "\mathfrak{F}"] \arrow[d] & \mathbf{C\overline{ell}}(\mathbf{D}) \arrow[d] \\
\mathcal{C}\mathbf{ell}(\mathbf{C}) \arrow[r, dashed] & \mathcal{C}\mathbf{ell}(\mathbf{D})
\end{tikzcd} \tag{10.58}$$

which preserves simple equivalences, and hence restricts furthermore to a functor

$$\mathcal{S}\mathbf{im}(\mathbf{C}) \to \mathcal{S}\mathbf{im}(\mathbf{D})\,.$$

Notation 10.3.2.6. In the context of Observation 10.3.2.5, we will usually also denote the induced functor (unique up to homotopy equivalence relative to $\mathbf{C\overline{ell}}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{C})$) by $\mathfrak{F}\colon \mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{D})\,.$

There are now three relevant notions of equivalences to study in the context of these functors.

Definition 10.3.2.7. Given two W-functors $\mathfrak{F}, \mathfrak{G}\colon \mathbf{C} \to \mathbf{D}$ between Whitehead model categories $\mathbf{C}$ and $\mathbf{D}$, we say that an isomorphism of the associated functors of ∞-categories $\mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{D})$

$$\mathfrak{F} \simeq \mathfrak{G}$$

is an (∞-categorical) simple equivalence if the induced isomorphism $\mathfrak{F}(\mathfrak{X}) \simeq \mathfrak{G}(\mathfrak{X})$ in $\mathfrak{ho}_c\mathbf{D}$ is a simple equivalence, for each finite structured cell complex $\mathfrak{X}$ in $\mathbf{C}$.

Remark 10.3.2.8. Observe that any ∞-categorical simple equivalence of W-functors $\eta: F \Rightarrow G$ as in Definition 10.3.2.7 descends to a natural isomorphism

$$\begin{tikzcd} \mathfrak{ho}_c\mathbf{C} \arrow[r, bend left, "\mathfrak{F}"] \arrow[r, bend right, "\mathfrak{G}"'] \arrow[r, phantom, "\Downarrow\cong"] & \mathfrak{ho}_c\mathbf{D} \end{tikzcd} \tag{10.59}$$

given by pointwise simple equivalences. It follows from Lemma 9.1.3.11 that η induces a natural isomorphism of Whitehead groups $\mathrm{Wh}_{\mathbf{C}} \circ \mathfrak{F} \cong \mathrm{Wh}_{\mathbf{C}} \circ \mathfrak{G}$ (and monoids), making the diagram

$$\begin{tikzcd} & \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}) \arrow[dl, "\mathrm{Wh}_{\mathfrak{F}}"'] \arrow[dr, "\mathrm{Wh}_{\mathfrak{G}}"] & \\ \mathrm{Wh}_{\mathbf{D}}(\mathfrak{F}(\mathfrak{X})) \arrow[rr, "\eta_*"'] & & \mathrm{Wh}_{\mathbf{D}}(\mathfrak{G}(\mathfrak{X})) \end{tikzcd} \tag{10.60}$$

commute. Hence, for most of the purposes of simple homotopy theory, the two functors $\mathfrak{F}$ and $\mathfrak{G}$ can be identified.

Definition 10.3.2.9. We say that a W-functor of Whitehead model categories $\mathfrak{F}: \mathbf{C} \to \mathbf{D}$ *is an ∞-categorical homotopy equivalence (of Whitehead model categories)*, if there exists another W-functor $\mathfrak{G}: \mathbf{D} \to \mathbf{G}$ together with ∞-categorical simple equivalences of cellularized functors

$$\mathfrak{F} \circ \mathfrak{G} \simeq 1_{\mathcal{C}} \text{ and } \mathfrak{G} \circ \mathfrak{F} \simeq 1_{\mathcal{D}} .$$

Definition 10.3.2.10. We say that a W-functor of Whitehead model categories $\mathfrak{F}: \mathbf{C} \to \mathbf{D}$ *is a weak equivalence (of Whitehead model categories)*, if the induced functor

$$\mathfrak{F}: \mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{D})$$

is an equivalence of ∞-categories, and the induced functor

$$\mathfrak{F}: \mathcal{S}\mathbf{im}(\mathbf{C}) \to \mathcal{S}\mathbf{im}(\mathbf{D})$$

is also an equivalence of ∞-categories.

Let us observe some of the obvious relations of the several different possible notions of equivalence.

Observation 10.3.2.11. Clearly, whenever $\mathfrak{F}: \mathbf{C} \to \mathbf{D}$ is an ∞-categorical homotopy equivalence of Whitehead model categories, then it is also a weak equivalence.

Lemma 10.3.2.12. *Let $\mathfrak{F}: \mathbf{C} \to \mathbf{D}$ be a W-functor of Whitehead model categories. Then the first of the following three conditions implies the remaining two.*

1. *$\mathfrak{F}$ is a weak equivalence of Whitehead model categories.*
2. *The following two conditions hold.*
   - *Every finite structured cell complex $\mathfrak{X} \in \mathfrak{ho}_c\mathbf{D}$ is simply equivalent to a finite cell complex of the form $\mathfrak{F}(\mathfrak{Y})$, for a finite structured cell complex $\mathfrak{Y}$ in $\mathbf{C}$.*
   - *An morphism $\omega: \mathfrak{X} \to \mathfrak{Y} \in \mathfrak{ho}_c\mathbf{C}$ is a simple equivalence, if and only if $\mathfrak{F}(\omega)$ is a simple equivalence.*
3. *The induced functor of Whitehead frameworks $\mathbf{W}(\mathbf{C}) \to \mathbf{W}(\mathbf{D})$ is a τ-equivalence (see Definition 9.2.0.8).*

Suppose, furthermore, that $\mathfrak{F}: \mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{D})$ is an equivalence of ∞-categories; then the converse implications also hold.

Proof. Observe that for any Whitehead model category $\mathbf{C}$ the full subcategory of $\mathrm{ho}\mathcal{C}\mathbf{ell}(\mathbf{C})$ given by the finite cell complexes is canonically identified with $\mathfrak{ho}\mathbf{C}$. By the assumption on the induced functor $\mathfrak{F}$, it follows that $\mathfrak{F}$ induces a fully faithful functor

$$\mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}_c\mathbf{D}\,.$$

Furthermore, by the assumption that $\mathfrak{F}$ restricts to an equivalence of categories on the associated categories of finite cell complexes and simple equivalences, it follows that this functor is also essentially surjective, and thus an equivalence of categories. Proposition 9.2.0.7 states that

$$\mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}_c\mathbf{D}$$

being an equivalence of categories together with the second condition in the statement of the lemma is equivalent to $\mathfrak{F}$ inducing a τ-equivalence. Hence, it remains to prove that the first condition implies the second. In fact, we will show that under the assumption that $\mathfrak{F}\colon \mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{D})$ is an equivalence of categories, the two conditions are equivalent. Indeed,

$$\mathcal{S}\mathbf{im}(\mathbf{C}) \to \mathcal{S}\mathbf{im}(\mathbf{D})$$

being essentially surjective is exactly the claim that every finite cell complex in $\mathfrak{D}$ lies in the image of $\mathfrak{F}$, up to simple equivalence. Next, observe that by the definition of a subcategory in the ∞-categorical sense, the mapping spaces in $\mathcal{S}\mathbf{im}(\mathbf{C})$ and $\mathcal{S}\mathbf{im}(\mathbf{D})$ are given by path components of the mapping spaces in $\mathcal{C}\mathbf{ell}(\mathbf{C})$ and $\mathcal{C}\mathbf{ell}(\mathbf{D})$, respectively. As $\mathfrak{F}\colon \mathcal{C}\mathbf{ell}(\mathbf{C}) \to \mathcal{C}\mathbf{ell}(\mathbf{D})$ is fully faithful, it follows that the functor $\mathcal{S}\mathbf{im}(\mathbf{C}) \to \mathcal{S}\mathbf{im}(\mathbf{D})$ is fully faithful, if and only if it induces a surjection on path components, i.e., is full. In other words, it is fully faithful, if and only if the following holds: A morphism in $\omega\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{ho}_c\mathbf{C}$ is a simple equivalence, if and only if $\mathfrak{F}(\omega)$ is a simple equivalence. $\square$

Chapter 11

Simple homotopy theory of diagrams

It is a classical question in homotopy theory what precise shape colimit diagrams need to have, in order for them to preserve (weak) homotopy equivalences. For example, a classical statement is what is sometimes called the cube lemma (see [KP86], for example): Suppose we are given a commutative diagram of spaces

$$\begin{tikzcd}
\bullet \arrow[rr, "f"] \arrow[dr, "w_0"] \arrow[dd, hook, "a"'] & & \bullet \arrow[dr, "w_2"] \arrow[dd, "a'"] & \\
& \bullet \arrow[rr] \arrow[dd, hook] & & \bullet \arrow[dd] \\
\bullet \arrow[rr, "f'"] \arrow[dr, "w_1"] & & \bullet \arrow[dr, "w"] & \\
& \bullet \arrow[rr] & & \bullet
\end{tikzcd} \tag{11.1}$$

with the front and back square pushout, all hooked arrows closed Hurewicz cofibrations, and w_0, w_1, w_2 homotopy equivalences. Then w is also a homotopy equivalence. Similarly, if the hooked arrows are Serre-cofibrations and w_0, w_1, w_2 are weak homotopy equivalences, then so is w. From a modern perspective, these claims may be interpreted as the back and front square not just being pushout squares, but *homotopy pushout squares* or, even more modernly put, pushout squares in the associated ∞-categories (of general spaces and, respectively, of spaces with the homotopy type of a CW-complex; see, for example, [Lur09], specifically Theorem 4.2.4.1). Now, suppose that all objects in the above cube are equipped with cell structures in **Top**, and with respect to these cell structures, w_0, w_1, w_2 are simple equivalences. One may then again repeat the question and ask whether w is a simple equivalence. In other words, what shape do the front and the back face need to have in order for simple equivalences to be preserved under pushout. More generally, we may ask the question whether we can compute the Whitehead torsion of w in terms of the torsions of w_0, w_1, w_2. In [Coh73, Prop.22], Cohen gives such a criterion for the classical simple homotopy theory of CW-complexes. Namely, if both the front and back face are given by pushout squares such that all arrows are given by inclusions of subcomplexes, then

$$\langle w \rangle = f'_* \langle w_1 \rangle + a'_* \langle w_2 \rangle - (f' \circ a)_* \langle w_0 \rangle .$$

In particular, if w_0, w_1 and w_2 are simple equivalences then the expression above is 0, and w is also a simple equivalence. In [KP86, Theorem (4.34)], the authors gave such a formula (under somewhat stronger conditions) for more general simple homotopy theories. Their frameworks do not cover the combinatorial examples which we are interested in, however. To obtain general answers to this question, for arbitrary Whitehead model categories, we will now study the simple homotopy theory of diagrams. In particular, we will provide similar results for finite colimit diagrams of significantly more general shape, reprove the sum-formula for general Whitehead model categories under weaker assumptions than [Coh73], and compute the general Whitehead groups associated to categories of diagrams indexed over a Reedy category. This will ultimately allow us to compute the stratified Whitehead groups defined in [Waa21].

11.1 Simple homotopy theory of Reedy diagrams

We first need to consider an appropriate framework to specify which shapes of diagrams of cell complexes we allow for. As in Section 8.3, we follow the paradigm: *Good diagrams of cell complexes, are cell complexes of diagrams.* In Section 8.3 we developed the theory of cell complexes in functor categories in much detail. We refer to Section 8.3, for language and notation.

11.1.1 Reedy Whitehead model categories

For the theory of diagrams in a Whitehead model category to be well-behaved, we will need the following slightly stronger assumption on the expansions. When we refer to expansions as morphism in $\mathbf{C}$, this will mean we refer to the underlying morphism of the structured relative cell complex.

Definition 11.1.1.1. Let $\mathbf{C}$ be a Whitehead model category. We say that $\mathbf{C}$ is properly generated if a morphism $f: X \to Y$ in $\mathbf{C}$ is a fibration if and only if it has the lifting property with respect to all morphisms in $\mathbb{E}_{\mathbf{C}}$, or equivalently, if $\mathbb{E}_{\mathbf{C}}$ is a generating set for acyclic cofibrations.

Example 11.1.1.2. The Whitehead model categories **sSet** and **Top** are properly generated.

We will also need the following lemma on the preservation of compactness properties:

Lemma 11.1.1.3. *Suppose that we are given two cellularized categories* $\mathbf{C}, \mathbf{D}$ *as well as a functor* $L: \mathbf{C} \to \mathbf{D}$*, which admits a right adjoint* $L \dashv R$*, such that* R *can be cellularized. Then* L *preserves filtration compact objects.*

Proof. Let $\mathfrak{c}: A \to X$ be a structured relative cell complex in $\mathbf{D}$ and let $B \in \mathbf{C}$ be filtration compact. Now, let $\mathfrak{c}^\bullet: I^\triangleright \to \mathbf{RCell}(\mathbf{C})_A$ be a filtration of $\mathfrak{c}$ by subcomplexes, with underlying cocone $X^\bullet: I^\triangleright \to \mathbf{D}$. Consider the following composition of canonical isomorphisms

$$\begin{aligned}
\varinjlim_I \mathbf{C}(L(B), X^i) &\cong \varinjlim_I \mathbf{C}(B, R(X^i)) \\
&\cong \mathbf{C}(B, \varinjlim_I R(X^i)) \\
&\cong \mathbf{C}(B, R(\varinjlim_I X^i)) \\
&\cong \mathbf{C}(B, R(X)) \\
&\cong \mathbf{C}(L(B), X)\,.
\end{aligned}$$

The first follows by the adjunction $L \dashv R$, the second follows from B being filtration compact and $R(X^\bullet)$ being the underlying diagram in $\mathbf{C}$ of the diagram filtration of subcomplexes $\mathfrak{R} \circ \mathfrak{c}^\bullet$, where $\mathfrak{R}$ is some cellularization of R. The third follows from R being cellularizable and hence preserving colimits. The final isomorphism is again given by adjunction. The composition of these canonical isomorphisms is the canonical comparison morphism $\varinjlim_I \mathbf{C}(L(B), X^i) \to \mathbf{C}(L(B), X)$ induced via the universal property of the colimit. $\square$

We now recommend that the reader recall the general notation for the cellularized composition products $- \circledcirc -$ which we studied in Section 8.3.

Lemma 11.1.1.4. *Let* $\mathbf{R}$ *be a Reedy category* $\mathbf{C}$ *be a cellularized category. Then, for every finite cell complex* $U \in \mathbf{Set}^{\mathbf{R}}$ *and every filtration compact object* $B \in \mathbf{C}$*, the induced diagram* $U \circledcirc B \in \mathbf{C}^{\mathbf{R}}$ *is filtration compact.*

Proof. Via induction over the number of cells of U and Lemma 8.1.6.7, it suffices to see that $\mathbf{R}_r \circledcirc B$ is filtration compact, for each $r \in \mathbf{R}$. This follows by Lemma 11.1.1.3 and Example 8.3.7.12. □

Definition 11.1.1.5. A Reedy category $\mathbf{R}$ will be said to *have locally finitely many degeneracies*, if for every $r \in \mathbf{R}$, there are only finitely many $r' \in \mathbf{R}$, such that there exists a morphism $r \to r'$ in $\mathbf{R}^-$. Dually, a Reedy category $\mathbf{R}$ *have locally finitely many faces* if for every $r \in \mathbf{R}$, there are only finitely many $\overline{r} \in \mathbf{R}$, such that there exists a morphism $\overline{r} \to r$ in $\mathbf{R}^+$.

Observation 11.1.1.6. It follows immediately from Example 8.3.6.9 that a Reedy category $\mathbf{R}$ has locally finitely many degeneracies if and only if, for each $r \in \mathbf{R}$, the cell complex $\mathbf{R}_r \in \mathbf{Set}^{\mathbf{R}}$ has only finitely many cells.

Notation 11.1.1.7. For the remainder of this section, we drop the indices from the sets of generating boundary inclusions and expansions of a Whitehead model category $\mathbf{C}$ and just write $\mathbb{B}$ and $\mathbb{E}$.

Recall that, given r an element of a Reedy category $\mathbf{R}$, we denote by $\iota_r \colon \partial\mathbf{R}_r \hookrightarrow \mathbf{R}_r$ the canonical inclusion (see Notation 8.3.3.2).

Proposition 11.1.1.8. *Let* $\mathbf{C}$ *be a properly generated Whitehead model category, and let* $\mathbf{R}$ *be a Reedy category that has locally finitely many degeneracies. Equip the cellularized category* $\mathbf{C}^{\mathbf{R}}$ *with the set of expansions*

$$\{\iota_r \hat{\circledcirc} \mathfrak{e} \mid \mathfrak{e} \in \mathbb{E}, r \in \mathbf{R}\}\ .$$

Then, equipped with this class of expansions, $\mathbf{C}^{\mathbf{R}}$ *is a properly generated Whitehead model category.*

Proof. The relevant cofibrantly generated (semi)-model structure is the Reedy (semi) model structure as discussed in [Hir03, Thm. 15.6.27] (the case of semi-model categories is proven entirely analogously to the case of model categories). Indeed, as we have assumed that $\mathbb{B}$ and $\mathbb{E}$ provide sets of generating cofibrations and acyclic cofibrations, it follows by the construction of the Reedy (semi)model structure, that

$$\mathbb{B}^{\mathbf{R}} = \{\iota_r \hat{\circledcirc} b \mid b \in \mathbb{B}, r \in \mathbf{R}\}$$

and

$$\mathbb{E}^{\mathbf{R}} = \{\iota_r \hat{\circledcirc} \mathfrak{e} \mid \mathfrak{e} \in \mathbb{E}, r \in \mathbf{R}\}$$

define generating sets of cofibrations and acyclic cofibrations for the Reedy (semi)-model structure. For the filtration compactness assumption on the objects $\partial\mathbf{R}_r \circledcirc A$, observe first that the targets of $\mathbb{B}^{\mathbf{R}}$ are filtration compact by Lemma 11.1.1.4. For the sources of $\mathbb{B}^{\mathbf{R}}$ and $\mathbb{E}^{\mathbf{R}}$, we apply the same argument, and hence only show the case of the former. Let $b \colon \partial D \to D \in \mathbb{B}$ and $r \in \mathbf{R}$. Observe that the source of $\iota_r \hat{\circledcirc} b \mid b$ is a pushout of a span with targets $\partial\mathbf{R}_r \circledcirc D$ and $\mathbf{R}_r \circledcirc \partial D$. By Lemma 8.1.6.7, it suffices to see that the latter two are filtration compact. As $\partial\mathbf{R}_r$ is a subcomplex of $\mathbf{R}_r$, and the latter is assumed to be finite, it follows that $\partial\mathbf{R}_r$ is also finite. Hence the filtration compactness of both $\partial\mathbf{R}_r \circledcirc D$ and $\mathbf{R}_r \circledcirc \partial D$ follows by Lemma 11.1.1.4. Finally, we need to expose a simple cylinder on $\mathbf{C}^{\mathbf{R}}$. To this end, fix a simple cylinder

$$\{0,1\} \otimes - \xhookrightarrow{\mathfrak{i}} [0,1] \otimes -$$

together with

$$\pi \colon [0,1] \otimes - \to 1_{\mathbf{C}}$$

on $\mathbf{C}$. We claim that the associated relative cellularized functor

$$\mathfrak{i}^{\mathbf{R}}\colon (\{0,1\}\otimes -)^{\mathbf{R}} \hookrightarrow ([0,1]\otimes -)^{\mathbf{R}}$$

as constructed in Construction 8.3.3.13, together with $\pi^{\mathbf{R}}$ defines a simple cylinder on $\mathbf{C}^{\mathbf{R}}$. AS $(\{0,1\}\otimes -)^{\mathbf{R}} = (-)\sqcup(-)$ and $1_{\mathbf{C}}^{\mathbf{R}} = 1_{\mathbf{C}^{\mathbf{R}}}$, it is immediate that this provides a factorization of the fold map. To see that the resulting cylinder is simple, use Remark 10.2.1.2 and Construction 8.3.3.13 together with the explicit description of the elementary expansions in $\mathbb{E}^{\mathbf{R}}$. ☐

Notation 11.1.1.9. Given a properly generated Whitehead model category, if we refer to $\mathbf{C}^{\mathbf{R}}$ as a Whitehead model category, it will always be with respect to the structure of a cellularized category with expansions described in Proposition 11.1.1.8.

For the remainder of this section, fix a properly generated Whitehead model category $\mathbf{C}$ and a Reedy category with locally finitely many degeneracies. Let us make a first observation about $\mathbf{C}^{\mathbf{R}}$, which follows by Example 10.1.3.6 and Remark 8.3.6.14:

Corollary 11.1.1.10. *A structured relative cell complex $\mathfrak{c}\colon X \hookrightarrow Y \in \mathbf{C}^{\mathbf{R}}$ is an expansion if and only if, for every $r \in \mathbf{R}$, the associated structured relative cell complex $\iota^r \hat{\circledast} \mathfrak{c}$ is an expansion.*

We will want to think of cell complexes in $\mathbf{C}^{\mathbf{R}}$ as specific types of commutative diagrams, indexed over $\mathbf{R}$, in $\mathbf{C\overline{e}ll(C)}$. This makes rigorous the proclaimed paradigm:
"*Good diagrams in cell complexes are cell complexes in diagrams.*"

Construction 11.1.1.11. Let $r \in \mathbf{R}$. In Example 8.3.6.9 we have seen that the evaluation at r functor $(-)^r\colon \mathbf{C}^{\mathbf{R}} \to \mathbf{C}$ is equipped with the structure of a cellularized functor, given by the canonical isomorphism $(-)^r \cong \mathbf{R}^r \circledast -$. If $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{R}})$ is a structured cell complex, then we may explicitly compute the set of cells of $\mathfrak{X}^r$ as given by the set of compositions

$$\{D \xrightarrow{\sigma} X^{r'} \xrightarrow{X^f} X^r \mid \sigma\colon \mathbf{R}_{r'} \circledcirc D \to X \in \mathfrak{C}_{\mathfrak{X}}, f\colon r' \to r \in \mathbf{R}^+\}$$

using the notational conventions of Notation 8.3.4.2. Considering all evaluation functors at the same time, we obtain a commutative diagram D given by

$$r \mapsto \mathfrak{X}^r$$
$$(r \xrightarrow{f} r') \mapsto (\mathfrak{X}^r \xrightarrow{X^f} \mathfrak{X}^{r'})$$

in $\mathbf{C\overline{e}ll(C)}$. This diagram has the following additional properties:

1. Recall that $\partial\mathbf{R}^+_{/r} \subset \mathbf{R}^+_{/r}$ denotes the full subcategory of the overcategory $\mathbf{R}^+_{/r}$, given by such arrows $f\colon r' \to r \in \mathbf{R}^+$ that are not the identity. For every $r \in \mathbf{R}$, the colimit

$$\varinjlim_{(r' \to r) \in \partial\mathbf{R}^+_{/r}} X^{r'}$$

obtains a cell structure via the set of cells

$$\bigcup_{f\colon r' \to r \in \mathbf{R}^+, f \neq 1_r} X^f \mathfrak{C}_{\mathfrak{X}^{r'}}.$$

(This is simply the cell structure on $\partial\mathbf{R}^r \circledast \mathfrak{X}$).

2. For every $r \in \mathbf{R}$, the canonical morphism

$$\varinjlim_{(r' \to r) \in \partial\mathbf{R}^+_{/r}} X^{r'} \to X^r$$

defines an inclusion of structured cell complexes, with respect to these structures. (This is the cellularized latching map at r, obtained through the cellularized functor $\iota^r \circledast -$).

It follows, by Theorem 8.3.4.8, that the diagrams in $\mathbf{C\overline{e}ll}(\mathbf{C})$, indexed over $\mathbf{R}$ that fulfill these two properties are precisely the diagrams which we can expect to arise from a structured cell complex in $\mathbf{C}^{\mathbf{R}}$.

In the following, we will not specify the degree functions of Reedy categories; it is not relevant to the associated shapes of diagrams.

Example 11.1.1.12. Let $\mathbf{R}$ be the Reedy category of the following shape

$$\begin{tikzcd} \bullet \arrow[r, "-"] \arrow[d, "+"'] & \bullet \arrow[d, "+"] \\ \bullet \arrow[r, "-"'] & \bullet. \end{tikzcd} \tag{11.2}$$

Then the diagrams in $\mathbf{C\overline{e}ll}(\mathbf{C})$ arising from Construction 11.1.1.11 are the diagrams that are simply commutative squares

$$\begin{tikzcd} \mathfrak{X}_0 \arrow[r] \arrow[d, hook] & \mathfrak{X}_2 \arrow[d, hook] \\ \mathfrak{X}_1 \arrow[r] & \mathfrak{X} \end{tikzcd} \tag{11.3}$$

with both vertical arrows inclusions of subcomplexes.

Example 11.1.1.13. Let $\mathbf{R}$ be the Reedy category of the following shape

$$\begin{tikzcd} \bullet \arrow[r, "+"] \arrow[d, "+"'] & \bullet \arrow[d, "+"] \\ \bullet \arrow[r, "+"'] & \bullet \end{tikzcd} \tag{11.4}$$

Then the diagrams in $\mathbf{C\overline{e}ll}(\mathbf{C})$ arising from Construction 11.1.1.11 are the diagrams

$$\begin{tikzcd} \mathfrak{X}_0 \arrow[r, hook] \arrow[d, hook] & \mathfrak{X}_2 \arrow[d, hook] \\ \mathfrak{X}_1 \arrow[r, hook] & \mathfrak{X} \end{tikzcd} \tag{11.5}$$

which are such that all morphisms are inclusions of subcomplexes, and the induced morphism in $\mathbf{Cell}(\mathbf{C})$

$$\mathfrak{X}_1 \cup_{\mathfrak{X}_0} \mathfrak{X}_2 \to \mathfrak{X}$$

is also an inclusion of subcomplexes.

Example 11.1.1.14. Let $\mathbf{R}$ be the Reedy category of the following shape

$$0 \xrightarrow{+} 1 \xrightarrow{+} 2 \xrightarrow{+} 3 \xrightarrow{+} \dots \tag{11.6}$$

with objects indexed over the naturals. The diagrams in $\mathbf{C\overline{e}ll}(\mathbf{C})$ arising from Construction 11.1.1.11 are the diagrams

$$\mathfrak{X}^0 \hookrightarrow \mathfrak{X}^1 \longrightarrow \mathfrak{X}^2 \hookrightarrow \mathfrak{X}^3 \hookrightarrow \dots \tag{11.7}$$

with all morphisms given by inclusions of subcomplexes.

We are now faced with two possible definitions of what a simple equivalence of diagrams should be. If we think of a diagram $\mathfrak{X}$ as a finite structured cell complex in the Whitehead model category $\mathbf{C}^{\mathbf{R}}$, then there is a canonical notion of simple equivalence associated to this simple homotopy theory (Notation 9.1.3.7). However, if we think of this diagram as a diagram valued in $\mathbf{C\overline{e}ll}_c(\mathbf{C})$, then the obvious notion of simple equivalence is pointwise simple equivalence. It will turn out that these two notions are in fact the same, at least under appropriate finiteness assumptions on $\mathbf{R}$.

11.1.2 Computation of the diagrammatic Whitehead group

The final claim of the last subsection will be a consequence of the computation of Whitehead groups in functor categories, which we now perform. To this end, let us first investigate which functors between Reedy categories induce W-functors.

Proposition 11.1.2.1. *Let* $\mathbf{R}$ *and* $\mathbf{T}$ *be Reedy categories with locally finitely many degeneracies. Let* $U \overset{c}{\hookrightarrow} W$ *be a relative cell complex in* $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}} \times \mathbf{T}}$ *(with its necessarily unique structure by Corollary 8.3.4.11). Such that for every* $r \in \mathbf{R}$ *and* $t \in \mathbf{T}$ *images of the relative latching maps*

$$\iota_{r,\bullet}^{\bullet;t} \hat{\circledast} c \colon L_r^t(W) \cup_{L_r^t(U)} U_r^t \to W_r^t$$

has finite complement, which is furthermore non-empty for only finitely many t*. Then*

$$c \circledcirc - \colon U \circledcirc - \to W \circledcirc -$$

defines a relative W-functor from $\mathbf{C}^{\mathbf{R}}$ *to* $\mathbf{C}^{\mathbf{T}}$*. In particular, this is the case whenever* c *is a finite cell complex.*

Proof. By Proposition 8.3.3.10, the functor above is canonically a cellularized functor. We verify the requirements of Lemma 10.2.1.6. Let us first see that $c \hat{\circledcirc} -$ maps elementary expansions into (possibly transfinite) simple equivalences. It thus suffices to show that it maps elementary expansions into simple equivalences. By Corollary 11.1.1.10, it furthermore suffices to see that

$$\iota^t \hat{\circledcirc} (c \hat{\circledcirc} (\iota_r \hat{\circledcirc} \mathfrak{e})) \cong (\iota^t \hat{\circledcirc} c \hat{\circledcirc} \iota_r) \hat{\circledast} \mathfrak{e}$$

is an expansion, for every $r \in \mathbf{R}, t \in \mathbf{T}$ and $\mathfrak{e} \in \mathbb{E}$. As $(\iota^t \circledcirc c \circledcirc \iota_r) \in \mathbf{Set}^{\star} = \mathbf{Set}$, this claim follows by Example 10.1.3.6. Now, to see that $c \hat{\circledcirc} -$ preserves finite cell complexes, let $\mathfrak{b}$ be a generating boundary inclusion of $\mathbf{C}$ and $r \in \mathbf{R}$. We then compute

$$\iota^t \hat{\circledcirc} (c \hat{\circledcirc} (\iota_r \hat{\circledcirc} \mathfrak{b})) \cong (\iota^t \hat{\circledcirc} c \hat{\circledcirc} \iota_r) \hat{\circledast} \mathfrak{b}\,.$$

By Example 10.1.3.6 and Theorem 8.3.4.8, it suffices to see that $(\iota^t \circledcirc c \circledcirc \iota_r) \in \mathbf{Set}$ only has finitely many cells (i.e., has finite complement), and has no cells for only finitely many t. Finally, observe that (by exchanging the order of variables), we have

$$(\iota^t \hat{\circledcirc} c \hat{\circledcirc} \iota_r) \cong (\iota^t \hat{\circledcirc} \iota_r) \hat{\circledast} c.$$

By Example 8.3.2.9, these maps are equivalently given by the relative latching maps

$$(\iota_{r,\bullet}^{\bullet;t}) \hat{\circledast} c,$$

finishing the proof. □

As a corollary of this result and Corollary 8.3.6.3 and Remark 8.3.7.10 as well as the computation of latching maps in the last proof, we obtain W-functor versions of Kan extensions and restrictions.

Corollary 11.1.2.2. *Let* $\mathbf{R}$ *and* $\mathbf{T}$ *be Reedy categories with locally finitely many degeneracies. Let* $F \colon \mathbf{R} \to \mathbf{T}$ *be a functor of Reedy categories, such that the following holds*

1. *F is a left fibration;*
2. *For every* $r \in \mathbf{R}$*, the associated (cardinality* ≤ 1*) set* $C_{F(r)}^t \subset \mathbf{T}_{F(r)}^t$ *(see Proposition 8.3.7.9) is only non-empty for finitely many* $t \in \mathbf{T}$*.*

Then the cellularized functor $F_! \colon \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{T}}$ *is a W-functor.*
Dually, suppose that

1. *F is a right fibration;*

2. *For every $t \in \mathbf{T}$, the associated (cardinality ≤ 1) set $C_t^{F(r)} \subset \mathbf{T}_t^{F(r)}$ (see Proposition 8.3.7.9) is only non-empty for finitely many $r \in \mathbf{R}$.*

Then the cellularized functor $F^*: \mathbf{C}^{\mathbf{T}} \to \mathbf{C}^{\mathbf{R}}$ is a W-functor.

Remark 11.1.2.3. Observe that the first condition of Corollary 11.1.2.2 is trivially met, if $\mathbf{T}$ is a finite Reedy category. However, by the explicit computation in Example 8.3.6.9 together with Proposition 8.3.7.7 they also clearly hold for the inclusion functors $\{r\} \hookrightarrow \mathbf{R}$, for $r \in \mathbf{R}$, if $\mathbf{R}$ has locally finitely many degeneracies. While an easier proof would have certainly been available, this shows that both functors of the adjunction $\mathbf{R}_r \circledcirc - \dashv (-)^r$ are W-functors.

Example 11.1.2.4. By Example 8.3.7.15 and Lemma 8.3.7.16, the inclusion functor $i: \mathbf{R}_{\leq n} \hookrightarrow \mathbf{R}$ is always a bifibration. The set $C^r_{i(\overline{r})}$ computes to

$$\{f: \overline{r} \to r \mid f = 1\}$$

for $\overline{r} \in \mathbf{R}_{\leq n}$ and $r \in \mathbf{R}$ and dually to $C^{i(r)}_{\overline{r}}$

$$\{f: \overline{r} \to r \mid f = 1\}$$

for $r \in \mathbf{R}_{\leq n}$ and $\overline{r} \in \mathbf{R}$. Hence, this set is non-empty if and only if $r = \overline{r}$. Consequently, the restriction and extension functors always define W-functors. We may use these explicit descriptions of the inclusions in Corollary 11.1.2.2 to give (non-surprising) descriptions of the sets of cells of $i^*\mathfrak{c}$, for a relative cell complex $\mathfrak{c}: A \hookrightarrow X$ in $\mathbf{C}^{\mathbf{R}}$. Namely, under Theorem 8.3.4.8, using Notation 8.3.4.2, it is simply given by

$$\{\sigma: D \to X^r \mid r \in \mathbf{R}, \deg(r) \leq n, \sigma \in \mathfrak{C}_{\mathfrak{c},r}\}.$$

Similarly, the set of cells of $i_!\mathfrak{d}$, for $\mathfrak{d}: B \hookrightarrow Y \in \mathbf{C}^{\mathbf{R}_{\leq n}}$ is given by

$$\{D \xrightarrow{\sigma} Y^r \cong (i_! Y)^r \mid r \in \mathbf{R}_{\leq n}, \sigma \in \mathfrak{C}_{\mathfrak{d},r}\} \cong \mathfrak{C}_{\mathfrak{d}}\,.$$

Remark 11.1.2.5. Example 11.1.2.4 generalizes to any subcategory $\mathbf{R}' \subset \mathbf{R}$ which is $\pm$-closed, by exactly the same arguments.

We can now state the main result on the structure of Whitehead groups of diagram categories.

Theorem 11.1.2.6. *Let $\mathbf{C}$ be a properly generated Whitehead model category and let $\mathbf{R}$ be a finite Reedy category. The cellularized evaluation functor $(-)^r: \mathbf{C}^{\mathbf{R}} \to \mathbf{C}$, for $r \in \mathbf{R}$, induces a natural isomorphism*

$$\mathrm{Wh}_{\mathbf{C}^{\mathbf{R}}}(\mathfrak{X}) \xrightarrow{(\mathrm{Wh}_{(-)^r})_{r \in \mathbf{R}}} \prod_{r \in \mathbf{R}} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r)$$

for finite structured cell complexes $\mathfrak{X}$ in $\mathbf{C}^{\mathbf{R}}$.

Remark 11.1.2.7. The reader may also be interested in the analogous statement of Theorem 11.1.2.6, where Whitehead monoids are used instead. This is clearly false. If we set $\mathfrak{X} = \varnothing$, then $\widetilde{\mathrm{Wh}}_{\mathbf{C}^{\mathbf{R}}}(\varnothing)$ is the set of equivalence classes of all cellularized diagrams (with finitely many cells), modulo simple equivalence. $\prod_{r \in \mathbf{R}} \mathrm{Wh}_{\mathbf{C}}(\varnothing)$ is then the product of simple homotopy types in $\mathbf{C}$. Clearly, for two diagrams to be even weakly equivalent, it does not suffice that their values at every $r \in \mathbf{R}$ are simply equivalent. Take, for example, the diagrams on $0 \xrightarrow{-} 1$ given by $S^1 \xrightarrow{1} S^1$ and $S^1 \xrightarrow{f} S^1$, where f is a map of degree 2.

Proof of Theorem 11.1.2.6. We proceed via induction over the maximal degree n of an element in $\mathbf{R}$. In the case where $n = 0$, $\mathbf{R}$ is a discrete category, and the claim is immediate from the construction of the Whitehead framework on $\mathbf{C}^{\mathbf{R}}$. Now, for the inductive step, denote by i^* the cellularized restriction functor associated to $\mathbf{R}_{\leq n} \overset{i}{\hookrightarrow} \mathbf{R}$. Consider the factorization

$$\begin{tikzcd}
\mathrm{Wh}_{\mathbf{C}^{\mathbf{R}}}(\mathfrak{X}) \arrow[rr, "(\mathrm{Wh}_{(-)^r})_{r\in\mathbf{R}}"] \arrow[dr, swap, "(\mathrm{Wh}_{i^*}, \prod_{r\in\mathbf{R},\deg(r)=n+1}\mathrm{Wh}_{(-)^r})"] & & \prod_{r\in\mathbf{R}} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r) \\
& \mathrm{Wh}_{\mathbf{C}^{\mathbf{R}_{\leq n}}}(i^*\mathfrak{X}) \times \prod_{r\in\mathbf{R},\deg(r)=n+1} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r) \arrow[ur, swap, "(\prod_{r\in\mathbf{R}_{\leq n}}\mathrm{Wh}_{(-)^r},1)"] &
\end{tikzcd} \quad . \tag{11.8}$$

Note that the right diagonal morphism is an isomorphism by the inductive assumption applied to $\mathbf{C}^{\mathbf{R}_{\leq n}}$. It thus suffices to show that the left diagonal is an isomorphism. Let us begin, by showing injectivity. So, suppose we are given an inclusion of subcomplex, which is also a weak equivalence, $\mathfrak{X} \overset{a}{\hookrightarrow} \mathfrak{Y}$, such that $i^*(a)\colon i^*\mathfrak{X} \to i^*\mathfrak{Y}$ and $a^r\colon \mathfrak{X}^r \hookrightarrow \mathfrak{Y}^r$, for $\deg(r) = n+1$, have trivial Whitehead torsion. Recall that $i_! i^* = \mathrm{sk}_n$. Consider the natural inclusion of cellularized functors

$$j\colon \mathrm{sk}_n \hookrightarrow \mathrm{sk}_{n+1} = 1_{\mathbf{C}^{\mathbf{R}}} \,,$$

of Example 8.3.6.11. Under Corollary 8.1.4.8, we obtain an induced diagram

$$\begin{tikzcd}
\mathrm{sk}_n\mathfrak{X} \arrow[r, hook] \arrow[d, hook] & \mathfrak{X} \arrow[d, hook, "\mathrm{sk}_n a = i_! i^* a"] \arrow[dr, hook, bend left] & \\
\mathrm{sk}_n\mathfrak{Y} \arrow[r, hook] & \mathrm{sk}_n\mathfrak{Y} \cup_{\mathrm{sk}_n\mathfrak{X}} \mathfrak{X} \arrow[r, hook, "\hat{j}(a)"] & \mathfrak{Y} \,.
\end{tikzcd} \tag{11.9}$$

in $\mathbf{C\ddot{e}ll}_c(\mathbf{C})$, with the left square cobase change. As i^*a was assumed to be a simple equivalence, and $i_!$ is a W-functor (Example 11.1.2.4), it follows that the left hand vertical is a simple equivalence. By stability of simple equivalences under cobase change, it follows that the right hand vertical is a simple equivalence. It hence suffices to show that $\hat{j}(a)$ is a simple equivalence. By Remark 8.3.6.14, it is given by a cobase change of the coproduct of the relative cell complex

$$\coprod_{\deg(r)=n+1} (\partial\mathbf{R}_r \to \mathbf{R}_r) \bar{\circledast}_{\mathbf{R}} (\partial\mathbf{R}^r \to \mathbf{R}^r) \hat{\circledast} a,$$

with $\deg(r) = n+1$. By Example 10.1.3.6, (and stability of simple equivalences under cobase change) it thus suffices to show that $(\partial\mathbf{R}^r \to \mathbf{R}^r)\hat{\circledast} a$ is a simple equivalence. The latter fits into a commutative diagram in $\mathbf{C\ddot{e}ll}_c(\mathbf{C})$

$$\begin{tikzcd}
\partial\mathbf{R}^r \circledast \mathfrak{X} \arrow[r, hook] \arrow[d, hook, "\partial\mathbf{R}^r \circledast a"] & \mathfrak{X}^r \arrow[d] \arrow[dr, hook] & \\
\partial\mathbf{R}^r \circledast \mathfrak{Y} \arrow[r, hook] & \partial\mathbf{R}^r \circledast \mathfrak{Y} \cup_{\partial\mathbf{R}^r \circledast \mathfrak{X}} \mathfrak{X}^r \arrow[r, hook] & \mathfrak{Y}^r
\end{tikzcd} \tag{11.10}$$

with the left square cobase change. Arguing as we have just done for the analogous diagram involving sk_n, it thus suffices to see that the left hand vertical is a simple equivalence. Observe that, by definition of $\partial\mathbf{R}^r$, the identity $\partial\mathbf{R}^r \cong (\mathrm{sk}_n\mathbf{R})^r \cong \mathbf{R}^r \circledcirc (\mathrm{sk}_n\mathbf{R})$ holds. Furthermore, as $\mathrm{sk}_n = i_! i^*$, we way use the associativity law for $\circledcirc$ (Proposition 8.3.3.10) to obtain

$$\begin{aligned}
\partial\mathbf{R}^r &\cong \mathbf{R}^r \circledcirc (\mathrm{sk}_n\mathbf{R}) \\
&\cong \mathbf{R}^r \circledcirc (\mathbf{R}^{\bullet}_{i(\bullet)} \circledcirc \mathbf{R}^{i(\bullet)}_{\bullet}) \\
&\cong (\mathbf{R}^r \circledcirc \mathbf{R}^{\bullet}_{i(\bullet)}) \circledcirc \mathbf{R}^{i(\bullet)}_{\bullet} \,.
\end{aligned}$$

Using Proposition 8.3.3.10, we use associativity up to canonical isomorphism of $\circledcirc$ to omit some brackets from here on. We then obtain

$$\begin{aligned}
\partial\mathbf{R}^r \circledast a &\cong \mathbf{R}^r \circledcirc \mathbf{R}^{\bullet}_{i(\bullet)} \circledcirc \mathbf{R}^{i(\bullet)}_{\bullet} \circledcirc a \\
&\cong \mathbf{R}^r \circledcirc \mathbf{R}^{\bullet}_{i(\bullet)} \circledcirc (i^*a) \\
&\cong ((-)^r \circ i_!)(i^*a) \,.
\end{aligned}$$

By assumption, i^*a is a simple equivalence. Furthermore, by Remark 11.1.2.3 and Example 11.1.2.4, $i_!$ and $(-)^r$ are W-functors, and hence preserve simple equivalences. Hence, it follows that $\partial\mathbf{R}^r \circledast a$ is a simple equivalence, as was to be shown. This finishes the proof of injectivity.

Finally, let us show surjectivity. We first show that, for every element of the form $\langle a' : i^*\mathfrak{X} \hookrightarrow \mathfrak{Y}' \rangle \in \mathrm{Wh}_{\mathbf{R}_{\leq n}}(i^*\mathfrak{X})$, with $a' : i^*\mathfrak{X} \hookrightarrow \mathfrak{Y}'$ a weak equivalence (i.e., an acyclic cofibration) in $\mathbf{RCell}(\mathbb{B}^{\mathbf{R}_{\leq n}})$, there exists an acyclic cofibration $a : \mathfrak{X} \to \mathfrak{Y}$, such that $\mathrm{Wh}_{i^*}\langle a \rangle = \langle a' \rangle$. Indeed, consider the pushout square in $\mathbf{Cell}(\mathbf{C}^{\mathbf{R}})$

$$\begin{tikzcd} i_!i^*\mathfrak{X} \arrow[r, hook] \arrow[d, hook] & \mathfrak{X} \arrow[d, hook, "a"] \\ i_!\mathfrak{Y}' \arrow[r, hook] & \mathfrak{Y}\,. \end{tikzcd} \tag{11.11}$$

Observe that as $i_!$ is a left Quillen functor and $i^*\mathfrak{X} \hookrightarrow \mathfrak{Y}'$ is an acyclic cofibration (with cofibrant source), it follows that the left vertical in Diagram (11.11) is an acyclic cofibration. Hence, by stability of the latter under cobase change it follows that $a : \mathfrak{X} \hookrightarrow \mathfrak{Y}$ is again an acyclic cofibration, i.e., defines an element of the Whitehead group. As cellularized functors preserve pushout squares in $\mathbf{Cell}(\mathbf{C})$, we obtain a cobase change diagram

$$\begin{tikzcd} i^*i_!i^*\mathfrak{X} \arrow[r, "\cong"] \arrow[d, hook] & i^*\mathfrak{X} \arrow[d, hook] \arrow[dr, hook] & \\ i^*i_!\mathfrak{Y}' \arrow[r] \arrow[rr, bend right, "\cong"] & i^*\mathfrak{Y} \arrow[r, dashed] & \mathfrak{Y}'\,. \end{tikzcd} \tag{11.12}$$

Recall, furthermore, that the embedding $i : \mathbf{R}_{\leq n} \hookrightarrow \mathbf{R}$ has the property that $i_!$ is fully faithful, i.e., that unit $1 \to i^*i_!$ is an isomorphism (see, for example, [RV13]). It follows that the upper horizontal and lower bend horizontals are isomorphisms. As the square is a pushout, it follows that the lower horizontal is an isomorphism too. Consequently, by the two-out-of-three property, the dashed morphism is an isomorphism showing that

$$\mathrm{Wh}_{i^*}(\langle \mathfrak{X} \to \mathfrak{Y} \rangle) = \langle i^*\mathfrak{X} \to \mathfrak{Y}' \rangle\,.$$

We have now shown that the composition of

$$\mathrm{Wh}_{\mathbf{C}^{\mathbf{R}}}(\mathfrak{X}) \to \mathrm{Wh}_{\mathbf{C}^{\mathbf{R}_{\leq n}}}(i^*\mathfrak{X}) \times \prod_{r \in \mathbf{R}, \deg(r) = n+1} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r)$$

with projection to the first component is surjective. Hence, showing that elements of the form

$$(0, \dots, 0, \langle \mathfrak{X}^r \overset{a'}{\hookrightarrow} \mathfrak{Z} \rangle, 0, \dots)$$

with $\deg(r) = n+1$ lie in the image of the map finishes proof of surjectivity. Let us show that this holds.

Consider the counit of adjunction $\mathbf{R}_r \circledcirc \mathfrak{X}^r \to \mathfrak{X}$, together with the canonical morphism $\mathbf{R}_r \circledcirc \mathfrak{X}^r \to \mathbf{R}_r \circledcirc \mathfrak{X}^r \cup_{\partial\mathbf{R}_r \circledcirc \mathfrak{X}^r} \partial\mathbf{R}_r \circledcirc \mathfrak{Z}$. The latter is a cobase change of $\partial\mathbf{R}_r \circledcirc \mathfrak{X}^r \to \partial\mathbf{R}_r \circledcirc \mathfrak{Z}$, which is an acyclic cofibration (as by Proposition 11.1.2.1 $\partial\mathbf{R}_r \circledcirc -$ is a W-functor). Let $\mathfrak{e} : \mathfrak{X} \hookrightarrow \hat{\mathfrak{X}}$ be a fibrant replacement of $\mathfrak{X}$ through a (transfinite) expansion (using the small object argument, and the assumption that elementary expansions generate acyclic cofibrations). It follows that the following lifting diagram admits a solution

$$\begin{tikzcd} \mathbf{R}_r \circledcirc \mathfrak{X}^r \arrow[rr] \arrow[d] & & \mathfrak{X} \arrow[r, hook] & \hat{\mathfrak{X}} \\ \mathbf{R}_r \circledcirc \mathfrak{X}^r \cup_{\partial\mathbf{R}_r \circledcirc \mathfrak{X}^r} \partial\mathbf{R}_r \circledcirc \mathfrak{Z} \arrow[urrr, dashed, bend right, "g"'] & & & \end{tikzcd} \tag{11.13}$$

Furthermore, by the compactness assumptions in Theorem 10.2.2.1, it follows that this lift factors through a finite subexpansion of $\mathfrak{X} \hookrightarrow \hat{\mathfrak{X}}$. We may thus assume that $\hat{\mathfrak{X}}$ is given by a finite expansion of $\mathfrak{X}$. Next, observe that as $\mathfrak{e}$, $\mathfrak{e}^r$ and $i^*\mathfrak{e}$ are all simple equivalences, they all induce isomorphisms on Whitehead groups. Performing a quick diagram chase shows that we may without loss of generality replace $\mathfrak{X}$ by $\hat{\mathfrak{X}}$ or, in other words, assume that g factors through $\mathfrak{X}$. Now let $a\colon \mathfrak{X} \to \mathfrak{Y}$ be the structured relative cell complex obtained via the following cobase change:

$$\begin{tikzcd} \mathbf{R}_r \circledcirc \mathfrak{X}^r \cup_{\partial \mathbf{R}_r \circledcirc \mathfrak{X}^r} \partial \mathbf{R}_r \circledcirc \mathfrak{Z} \arrow[r, "g"] \arrow[d, hook] & \mathfrak{X} \arrow[d, hook, "a"] \\ \mathbf{R}_r \circledcirc \mathfrak{Z} \arrow[r] & \mathfrak{Y}\,. \end{tikzcd} \tag{11.14}$$

As the left vertical is an acyclic cofibration, so is a, which shows that it defines a well defined element in the Whitehead group of $\mathfrak{X}$. By construction, the structured relative cell complex associated to a only has cells of the form $\mathbf{R}_r \circledcirc D \to Y$. It thus follows, by the explicit computation of the sets of cells of evaluation and restriction in Construction 11.1.1.11 and Example 11.1.2.4 that the Whitehead torsion $\langle a \rangle$ maps to an element of the form

$$(0, \dots, \langle a^r \rangle, 0, \dots)$$

in $\mathrm{Wh}_{\mathbf{C}^{\mathbf{R}_{\leq n}}}(i^*\mathfrak{X}) \times \prod_{r \in \mathbf{R}, \deg(r) = n+1} \mathrm{Wh}_{\mathbf{C}}(X^r)$. It remains to show that $\langle a^r \rangle = \langle a'\colon \mathfrak{X}^r \to \mathfrak{Z} \rangle$. Applying $(-)^r$ to Diagram (11.14), we obtain the right square in the following diagram of cobase changes of relative cell complexes

$$\begin{tikzcd} \mathfrak{X}^r \arrow[r, hook, "i_{1_r}"] \arrow[d, hook, "a'"] \arrow[rr, bend left, "1_{\mathfrak{X}^r}"] & (\coprod_{\mathbf{R}_r^r, f \neq 1} \mathfrak{Z}) \sqcup \mathfrak{X}^r \arrow[r, "g^r"] \arrow[d, hook, "1 \sqcup a'"] & \mathfrak{X}^r \arrow[d, hook, "a^r"] \\ \mathfrak{Z} \arrow[r, "i_{1_r}"] & \coprod_{\mathbf{R}_r^r} \mathfrak{Z} \arrow[r] & \mathfrak{Y}^r\,. \end{tikzcd} \tag{11.15}$$

Hence, a^r is a cobase change of a' along the identity, and we have

$$\mathrm{Wh}_{(-)^r}(\langle a \rangle) = \langle a^r \rangle = 1_*\langle a' \rangle = \langle a' \rangle\,,$$

as was to be shown. □

As a consequence of this theorem, we obtain the following equivalent characterizations of simple equivalences in a diagram category.

Corollary 11.1.2.8. *Let $\mathbf{C}$ be a properly generated Whitehead model category and let $\mathbf{R}$ be a Reedy category that locally has finitely many degeneracies and faces. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be finite cell complexes in $\mathbf{C}^{\mathbf{R}}$. Finally, let $\omega\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathfrak{ho}_c\mathbf{C}^{\mathbf{R}}$. Then ω is a simple equivalence if and only if, for every $r \in \mathbf{R}$, the associated morphism*

$$\omega^r\colon \mathfrak{X}^r \to \mathfrak{Y}^r$$

in $\mathfrak{ho}_c\mathbf{C}$ is a simple equivalence.

Proof. The finite case is immediate from Theorem 11.1.2.6. For the locally finite case, observe first that the only if part is immediate from $(-)^r$ being a W-functor. For the converse, a straightforward inductive argument shows that every object $r \in \mathbf{R}$ of the Reedy category $\mathbf{R}$ is contained in a finite full subcategory $\mathbf{S}$ ±-closed. Furthermore, the union of any two such sub-categories is again a finite ±-closed subcategory of $\mathbf{R}$. Consequently, any finite subset of objects of $\mathbf{R}$ is contained in a full finite ±-closed subcategory. By Remark 11.1.2.5, it holds that for a ±-closed subcategory $\mathbf{S} \subset \mathbf{R}$ the associated inclusion functor $i: \mathbf{S} \hookrightarrow \mathbf{R}$ induces adjoint W-functors $i_! \dashv i^*$. The counit of adjunction is the inclusion of a cellularized subfunctor

$$i_! \circ i^* \hookrightarrow 1$$

with the set of relative cells of $i_! \circ i^* \mathfrak{X} \hookrightarrow \mathfrak{X}$ and $i_! \circ i^* \mathfrak{Y} \hookrightarrow \mathfrak{Y}$ given by the respective unions over the sets of cells of type r, for $r \notin \mathbf{S}$. Since $\mathfrak{X}$ and $\mathfrak{Y}$ are finite cell complexes, it follows that there exists a finite ±-closed subcategory $\mathbf{S}$, such that $i_! \circ i^* \mathfrak{X} \hookrightarrow \mathfrak{X}$ and $i_! \circ i^* \mathfrak{Y} \hookrightarrow \mathfrak{Y}$ are bijective on cells, and hence isomorphisms of cell complexes by Corollary 8.1.4.1. In particular, we obtain a commutative diagram in $\mathfrak{ho}_c \mathbf{C}^{\mathbf{R}}$,

$$\begin{tikzcd} i_! i^* \mathfrak{X} \arrow[r, "i_! i^* \omega"] \arrow[d, hook, "\cong"] & i_! i^* \mathfrak{Y} \arrow[d, hook, "\cong"'] \\ \mathfrak{X} \arrow[r, "\omega"'] & \mathfrak{Y} \end{tikzcd} \tag{11.16}$$

with verticals given by isomorphisms of cell complexes. As $i_!$ is a W-functor and hence preserves simple equivalences, it follows that ω is a simple equivalence if $i^* \omega$ is a simple equivalence. By assumption, and using the identity $(i^* \omega)^r = \omega^r$, for $r \in \mathbf{S}$, it follows that $(i^* \omega)^r$ is a pointwise simple equivalence. We have thus reduced the assertion to the finite case, which we have already shown. □

11.2 Simple homotopy colimits and their applications

Now, having a good understanding of the simple equivalences in a category of diagrams, we may easily leverage our investigations of cellularized left Kan extensions in Section 8.3.7 to study the interaction of appropriately cellularized colimits with Whitehead torsion.

11.2.1 Simple homotopy colimits and the gluing formula for Whitehead torsions

Let us begin by showing that pointwise simple equivalences between appropriately cellularized diagrams descend to simple equivalences on the appropriately cellularized colimits. This is a special case of Corollary 11.1.2.2, together with Corollary 11.1.2.8.

Corollary 11.2.1.1. *Let $\mathbf{C}$ be a properly generated Whitehead model category. Let $\mathbf{R}$ be a left fibrant Reedy category that has locally finitely many degeneracies. Then the cellularized colimit functor*

$$\varinjlim \mathbf{C}^{\mathbf{R}} \to \mathbf{C}$$

of Example 8.3.7.11 is a W-functor.
Now, let $\mathbf{R}$ have locally finitely many faces and degeneracies and let

$$\alpha: \mathfrak{X} \to \mathfrak{Y}$$

in $\mathfrak{ho}_c \mathbf{C}^{\mathbf{R}}$, such that

$$\alpha^r: \mathfrak{X}^r \to \mathfrak{Y}^r$$

is a simple equivalence in $\mathfrak{ho}_c \mathbf{C}$, for each $r \in \mathbf{R}$. Then the induced morphism

$$\varinjlim \mathfrak{X}^r \to \varinjlim \mathfrak{Y}^r$$

in $\mathfrak{ho}_c \mathbf{C}$ is a simple equivalence.

Let us now derive the cube lemma and sum formula for properly generated Whitehead model categories.

Notation 11.2.1.2. In the following, we denote by $\mathbf{Q}$ the category $\{0 \to 1\} \times \{0 \to 1\}$ equipped with the structure of a Reedy category given by

$$\begin{tikzcd} (0,0) \arrow[r, "-"] \arrow[d, "+"'] & (1,0) \arrow[d, "+"] \\ (0,1) \arrow[r, "-"'] & (1,1) \end{tikzcd} \tag{11.17}$$

(see Example 11.1.1.12). The explicit choice of degree function will not be relevant to our discussion. Denote by $\mathbf{S} \subset \mathbf{Q}$ the full Reedy subcategory

$$\begin{tikzcd} (0,0) \arrow[r, "-"] \arrow[d, "+"'] & (1,0) \\ (0,1) & \end{tikzcd} . \tag{11.18}$$

We denote by $i\colon \mathbf{S} \hookrightarrow \mathbf{Q}$ the inclusion functor.

Let us first compute the Whitehead torsion associated to cellularized pushouts, given by the cellularized functor $\varinjlim\colon \mathbf{C}^{\mathbf{S}} \to \mathbf{C}$. To this end, let us explicitly describe the inverse of the decomposition isomorphism in Theorem 11.1.2.6 for the case $\mathbf{R} = \mathbf{S}$.

Construction 11.2.1.3. Let $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{S}})$ be a finite structured cell complex, with associated diagram

$$\begin{tikzcd} \mathfrak{X}^{(0,0)} \arrow[r, "X^{(0,0)\to(1,0)}"] \arrow[d, hook, "X^{(0,0)\to(0,1)}"'] & \mathfrak{X}^{(1,0)} \\ \mathfrak{X}^{(0,1)} & \end{tikzcd} \tag{11.19}$$

in $\mathbf{C\bar{e}ll}(\mathbf{C})$. We now give an explicit description of the inverse to the isomorphism

$$\mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(\mathfrak{X}) \xrightarrow{(\mathrm{Wh}_{(-)^r})_{r\in\mathbf{S}}} \bigoplus_{r\in\mathbf{R}} \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r)$$

of Theorem 11.1.2.6. Given $r \in \mathbf{S}$ denote by $i_r\colon * \hookrightarrow \mathbf{S}$ the inclusion at r. Now, given the inclusion of a subcomplex $(i_r)^*\mathfrak{X} = \mathfrak{X}^r \hookrightarrow \mathfrak{Y}$, consider the cobase change square

$$\begin{tikzcd} (i_r)_! i_r^* \mathfrak{X} \arrow[r] \arrow[d, hook, "(i_r)_! a"] & \mathfrak{X} \arrow[d, hook, "((i_r)_! a)'"] \\ i_! \mathfrak{Y} \arrow[r] & i_!\mathfrak{Y} \cup_{(i_r)_! i_r^* \mathfrak{X}} \mathfrak{X} \end{tikzcd} \tag{11.20}$$

Let us consider, what happens if we evaluate this diagram at $r' \in \mathbf{S}$. As $i_{r'}^*$ is a cellularized functor, the associated diagram will again be cobase change. Under the canonical identifications $i_{r'}^* \cong \mathbf{R}^{r'} \circledast -$ and $i_r^! \cong \mathbf{R}_r \circledcirc -$, we obtain canonical isomorphisms of cellularized functor

$$i_{r'}^* \circ i_r^! = \mathbf{R}^{r'} \circledcirc \mathbf{R}_r \circledcirc - = \mathbf{R}_r^{r'} * - .$$

Observe that $\mathbf{R}_r^{r'}$ is a singleton, if $r \le r'$ (in the product order on $\mathbf{S} \subset \{0,1\}^2$) and the empty set otherwise. Hence, we obtain identities of cellularized functors

$$i_{r'}^* \circ i_r^! \cong 1_{\mathbf{C}}$$

for $r \le r'$ and

$$i_{r'}^* \circ i_r^! = \varnothing$$

otherwise. Under these identifications, the evaluation of Diagram (11.20) at r' (i.e., its image under $i_{r'}^*$) is given by a cobase change square

$$\begin{tikzcd} \mathfrak{X}^r \arrow[r, "X^{r\to r'}"] \arrow[d, hook, "a"] & \mathfrak{X}^{r'} \arrow[d, hook] \\ \mathfrak{Y} \arrow[r] & \mathfrak{Y}' \end{tikzcd} \tag{11.21}$$

for $r \le r'$ and by

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \mathfrak{X}^{r'} \arrow[d, "1_{\mathfrak{X}^{r'}}"] \\ \varnothing \arrow[r] & \mathfrak{X}^{r'} \end{tikzcd} \tag{11.22}$$

otherwise. Now, suppose that a is also a weak equivalence. Denote by ψ_r the map

$$\begin{aligned} \psi_r\colon \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r) &\to \mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(\mathfrak{X}) \\ \langle a\rangle &\mapsto \langle ((i_r)_! a)'\rangle\,. \end{aligned}$$

That this map is well defined, is easily seen from $(i_r)_!$ being a W-functor. We may then apply the description of the functoriality of the Whitehead groups in Proposition 10.2.3.12 and obtain:

$$\mathrm{Wh}_{(-)^{r'}}\psi_r\langle a\rangle = \mathrm{Wh}_{(-)^{r'}}\langle ((i_r)_! a)'\rangle = (X^{r\to r'})_*\langle a\rangle \tag{11.23}$$

for $r \le r'$ and

$$\mathrm{Wh}_{(-)^{r'}}\psi_r\langle a\rangle = \mathrm{Wh}_{(-)^{r'}}\langle (i_r)_! a)'\rangle = \langle (((i_r)_! a)')^{r'}\rangle = \langle 1_{\mathfrak{X}^r}\rangle = 0 \tag{11.24}$$

otherwise. Now, using this construction we define

$$\Psi\colon \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(1,0)}) \oplus \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(0,1)}) \oplus \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(0,0)}) \to \mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(\mathfrak{X})$$

as the sum of

$$\Psi_r = \psi_r\colon \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^r) \to \mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(\mathfrak{X})$$

for $r = (0,1), (1,0)$ and

$$\Psi_{(0,0)} = \psi_{(0,0)} - \psi_{(0,1)} \circ (X^{(0,0)\to(1,0)})_* - \psi_{(1,0)} \circ (X^{(0,0)\to(0,1)})_*\colon \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(0,0)}) \to \mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(\mathfrak{X})\,.$$

A short elementary computation involving Eqs. (11.23) and (11.24) shows that Ψ is the inverse to $(\mathrm{Wh}_{(-)^r})_{r\in\mathbf{S}}$.

Proposition 11.2.1.4. *Let $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{S}})$ be a finite structured cell complex, with associated diagram*

$$\begin{tikzcd} \mathfrak{X}^{(0,0)} \arrow[r, "f"] \arrow[d, hook, "a"'] & \mathfrak{X}^{(1,0)} \\ \mathfrak{X}^{(0,1)} & \end{tikzcd} \tag{11.25}$$

in $\overline{\mathbf{Cell}}(\mathbf{C})$. Denote by $d\colon \mathfrak{X}^{(0,0)} \to \varinjlim \mathfrak{X}$, $a'\colon \mathfrak{X}^{(0,1)} \to \varinjlim \mathfrak{X}$ and $f'\colon \mathfrak{X}^{(0,1)} \to \varinjlim \mathfrak{X}$ the associated structure morphisms of the colimit.
Then the group homomorphism

$$\mathrm{Wh}_{\varinjlim}\colon \mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(\mathfrak{X}) \to \mathrm{Wh}_{\mathbf{C}}(\varinjlim \mathfrak{X})$$

is given by

$$\mathrm{Wh}_{\varinjlim} = a'_*\mathrm{Wh}_{(-)^{(1,0)}} + f'_*\mathrm{Wh}_{(-)^{(0,1)}} - d_* \circ \mathrm{Wh}_{(-)^{(0,0)}}\,.$$

Proof. We denote $\mathfrak{X}^{(1,1)} := \varinjlim \mathfrak{X}$. The claim is equivalent to the commutativity of the diagram

$$\begin{tikzcd} \mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(\mathfrak{X}) \arrow[rr, "\mathrm{Wh}_{\varinjlim}"] \arrow[d, "\cong"'] & & \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(1,1)}) \\ \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(1,0)}) \oplus \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(0,1)}) \oplus \mathrm{Wh}_{\mathbf{C}}(\varinjlim \mathfrak{X}) \arrow[urr, "a'_*+f'_*-d_*"'] & & \end{tikzcd} \tag{11.26}$$

with the left vertical the isomorphism of Theorem 11.1.2.6 given by pointwise evaluation. To see that this commutativity holds, we verify commutativity after inverting the left hand vertical. We use the notation of Construction 11.2.1.3, with $a = X^{(0,0)\to(0,1)}$ and $f = X^{(0,0)\to(1,0)}$ and explicitly compute the compositions

$$\mathrm{Wh}_{\varinjlim} \circ \Psi_r\,.$$

Recall that $\psi_r\langle b\colon \mathfrak{X}^r \hookrightarrow \mathfrak{Y}\rangle$ is given by the Whitehead torsion of the right hand vertical in the cobase change square

$$\begin{tikzcd} (i_r)_! i_r^* \mathfrak{X} \arrow[r] \arrow[d, hook, "(i_r)_! b"] & \mathfrak{X} \arrow[d, hook, "((i_r)_! b)'"] \\ i_! \mathfrak{Y} \arrow[r] & i_! \mathfrak{Y} \cup_{(i_r)_! i_r^* \mathfrak{X}} \mathfrak{X}\,. \end{tikzcd} \tag{11.27}$$

Now, observe that by functoriality of (cellularized) left Kan extension, we have $\varinjlim (i_r)_! = c_!(i_r)_! \cong (c \circ i_r)_! = 1_! = 1$, for $c\colon \mathbf{S} \to \star$ the constant functor. As cellularized functors preserve cobase changes, the image of Diagram (11.27) under $\varinjlim$ is given by a cobase change square

$$\begin{tikzcd} \mathfrak{X}^r \arrow[r] \arrow[d, hook, "b"'] & \varinjlim \mathfrak{X} \arrow[d, hook] \\ \mathfrak{Y} \arrow[r] & \mathfrak{Y}' \end{tikzcd} \tag{11.28}$$

with the upper horizontal given respectively by a', f' and d. In particular, it follows that

$$\mathrm{Wh}_{\varinjlim} \psi_r\langle b\rangle = g_*\langle b\rangle$$

with $g = d, a', f'$ respectively. We thus compute

$$\begin{aligned} \mathrm{Wh}_{\varinjlim} \circ \Psi_{(0,0)} &= \mathrm{Wh}_{\varinjlim} \circ \psi_{(0,0)} - \mathrm{Wh}_{\varinjlim} \circ \psi_{(0,1)} \circ f_* - \mathrm{Wh}_{\varinjlim} \circ \psi_{(1,0)} \circ a_* \\ &= d_* - a'_* \circ f_* - f'_* \circ a_* = d_* - (a' \circ f)_* - (f' \circ a)_* \\ &= d_* - d_* - d_* \\ &= -d_* \end{aligned}$$

as well as

$$\begin{aligned} \mathrm{Wh}_{\varinjlim} \circ \Psi_{(0,1)} = \mathrm{Wh}_{\varinjlim} \circ \psi_{(0,1)} = a'_*; \\ \mathrm{Wh}_{\varinjlim} \circ \Psi_{(1,0)} = \mathrm{Wh}_{\varinjlim} \circ \psi_{(1,0)} = f'_*\,. \end{aligned}$$

Hence, it follows that

$$\mathrm{Wh}_{\varinjlim} \circ \Psi = \mathrm{Wh}_{\varinjlim} \circ \left(\Psi_{(0,1)} + \Psi_{(1,0)} + \Psi_{(0,0)}\right) = a'_* + f'_* - d_*$$

as was to be shown. □

Construction 11.2.1.5. It is easily verified that the inclusion $i\colon \mathbf{S} \hookrightarrow \mathbf{Q}$ is a left and a right fibration. Let us take an explicit look at the cell structures arising from the associated cellularized functor, from the perspective of Construction 11.1.1.11. We have seen in Example 11.1.1.12, that a structured cell complex $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{Q}})$ corresponds to a commutative square

$$\begin{tikzcd} \mathfrak{X}^{(0,0)} \arrow[r] \arrow[d, hook] & \mathfrak{X}^{(1,0)} \arrow[d, hook] \\ \mathfrak{X}^{(0,1)} \arrow[r] & \mathfrak{X}^{(1,1)} \end{tikzcd} \tag{11.29}$$

in $\mathbf{C\bar{e}ll}(\mathbf{C})$, with both verticals given by inclusions of a subcomplex. Similarly, a structured cell complex $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{S}})$ corresponds to a span

$$\begin{tikzcd} \mathfrak{X}^{(0,0)} \arrow[r] \arrow[d, hook] & \mathfrak{X}^{(1,0)} \\ \mathfrak{X}^{(0,1)} & \end{tikzcd} \tag{11.30}$$

in $\mathbf{C\bar{e}ll}(\mathbf{C})$, with the vertical leg given by an inclusion of a subcomplex. Using Proposition 8.3.7.9, it is a simple computation that the diagram in $\mathbf{C\bar{e}ll}(\mathbf{C})$ associated to the restriction $i^*\mathfrak{X}$, for $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{Q}})$ is simply the restriction of the diagram in $\mathbf{C\bar{e}ll}(\mathbf{C})$ associated to $\mathfrak{X}$, i.e., the cellularized functor i^* corresponds to removing $\mathfrak{X}^{(1,1)}$ from Diagram (11.29). A more interesting computation arises in the case of the cellularized left Kan extension functor $i_!$. The underlying functor of $i_!$ is given by left Kan extension. It follows that the underlying diagram in $\mathbf{C}$ associated to $i_!\mathfrak{X}$, for $\mathfrak{X}$ as in Diagram (11.25) is a pushout square

$$\begin{tikzcd} X^{(0,0)} \arrow[r, "f"] \arrow[d, hook, "a"] & X^{(1,0)} \arrow[d, hook, "a'"] \\ X^{(0,1)} \arrow[r, "f'"] \arrow[ur, phantom, "\ulcorner"] & X^{(1,1)} \end{tikzcd} . \tag{11.31}$$

Let us now use Proposition 8.3.7.9 to compute the cell structure on this square. First, we compute the sets $C^t_{i(r)} \subset \mathbf{Q}^t_r$, for $r \in \mathbf{S}$ and $t \in \mathbf{Q}$.

1. If $t < r$, in the product partial order, then $\mathbf{Q}^t_r$ and hence also $C^t_{i(r)} \subset \mathbf{Q}^t_r$ is empty.

2. If $t \in \mathbf{Q}$, then $C^t_{i(r)}$ is the set of morphisms $f\colon r \to t$ with $f \in \mathbf{Q}^-$ (i.e., $f = f^-$) with $f^- = 1$, i.e can only contain identities. It follows that $C^t_{i(r)}$ is empty, if $r \neq t$ and given by 1_r if $r = t$.

3. Finally, if $t = (1,1)$, then the only object $r \in \mathbf{S}$ which admits a morphism $r \to t$ in $\mathbf{Q}^-$ is $(0,1)$. It follows that $C^t_{i(r)}$ is empty, for $r \neq (0,1)$, and given by the unique morphisms $(0,1) \to (1,1)$ if $r = (0,1)$.

Hence, we may compute the set of cells of $i_!\mathfrak{X}$ as follows:

1. Cells of type r, with $r \in \mathbf{S}$ are

$$\{D \xrightarrow{\sigma} X^r \xrightarrow{X^1} X^r \mid \sigma \in \mathfrak{C}_{\mathfrak{X},r}\} = \mathfrak{C}_{\mathfrak{X},r} \,.$$

2. Cells of type $(1,1)$ are

$$\{D \xrightarrow{\sigma} X^{(0,1)} \xrightarrow{f'} X^{(1,1)} \mid \sigma \in \mathfrak{C}_{\mathfrak{X},(0,1)}\} = f'\mathfrak{C}_{\mathfrak{X},(0,1)} \,.$$

Hence, using Construction 11.1.1.11, we compute

$$\mathfrak{C}_{(i_!\mathfrak{X})^r} = \mathfrak{C}_{\mathfrak{X}^r}$$

for $r \neq (1,1)$ and

$$\mathfrak{C}_{(i_!\mathfrak{X})^r} = a'\mathfrak{C}_{\mathfrak{X}^{(1,0)}} \sqcup f'\mathfrak{C}_{\mathfrak{X},(0,1)}\,.$$

Observe that the cell structure on $\mathfrak{C}_{\mathfrak{X}^{(1,1)}}$ is precisely the cell structure on a cobase change square in $\mathbf{C\overline{e}ll(C)}$ as described in Notation 10.2.1.14. We may summarize this insight as in the following observation, which we spell out separately, in order to make it more easily accessible via cross reference.

Observation 11.2.1.6. In the situation of Construction 11.2.1.5: A commutative diagram

$$\begin{tikzcd} \mathfrak{X}^{(0,0)} \arrow[r] \arrow[d, hook] & \mathfrak{X}^{(1,0)} \arrow[d, hook] \\ \mathfrak{X}^{(0,1)} \arrow[r] & \mathfrak{X}^{(1,1)} \end{tikzcd} \tag{11.32}$$

in $\mathbf{C\overline{e}ll(C)}$ with both verticals inclusions of subcomplexes is a cobase change square, if and only if the associated structured cell complex $\mathfrak{X} \in \mathbf{Cell(C^Q)}$ lies in the essential image of $i_!\colon \mathbf{Cell(C^S)} \to \mathbf{Cell(C^Q)}$, or, again in other words, if the counit of adjunction

$$i_! i^* X \to X$$

induces an isomorphism of cell complexes

$$i_! i^* \mathfrak{X} \cong \mathfrak{X}\,.$$

Observation 11.2.1.6 will be very useful insofar as it allows us to translate back and forth between the more classical context of cobase change squares and their associated cell complexes and our language of cellularized Kan extensions. We can now prove a generalization of Cohen's sum formula [Coh73, Prop.22], that extends the latter in several different directions:

1. Our result holds for general properly generated Whitehead model categories, not just for the one of spaces.
2. Our result has weaker assumptions on the shape of the pushout diagrams involved, only requiring one of the legs of a span to be the inclusion of a subcomplex.
3. Our result allows for morphisms $\mathfrak{ho}_c\mathbf{C^Q}$, not just such isomorphisms which come from a cube in $\mathbf{C\overline{e}ll}_c\mathbf{(C)}$.

Theorem 11.2.1.7. *Let $\mathbf{C}$ be a properly generated Whitehead model category. Let $\mathfrak{X}, \mathfrak{Y}$ be two finite structured cell complexes in $\mathbf{C^Q}$, such that the associated diagrams*

$$\begin{tikzcd} \mathfrak{X}^{(0,0)} \arrow[r, "f"] \arrow[d, hook, "a"] \arrow[dr, "d"] & \mathfrak{X}^{(1,0)} \arrow[d, hook, "a'"] \\ \mathfrak{X}^{(0,1)} \arrow[r, "f'"] & \mathfrak{X}^{(1,1)} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{Y}^{(0,0)} \arrow[r] \arrow[d, hook] & \mathfrak{Y}^{(1,0)} \arrow[d, hook] \\ \mathfrak{Y}^{(0,1)} \arrow[r] & \mathfrak{Y}^{(1,1)} \end{tikzcd} \tag{11.33}$$

in $\mathbf{C\overline{e}ll(C)}$ are cobase change. Suppose that we are given a morphism

$$\omega\colon \mathfrak{X} \to \mathfrak{Y}$$

in $\mathfrak{ho}_c\mathbf{C^Q}$. Suppose, furthermore, that $\omega^{(0,0)}, \omega^{(1,0)}, \omega^{(0,1)}$ are isomorphisms in $\mathfrak{ho}_c\mathbf{C}$ (i.e., come from zig-zags of weak equivalences in ho$\mathbf{C}$*). Then $\omega^{(1,1)}\colon \mathfrak{X}^{(1,1)} \to \mathfrak{Y}^{(1,1)}$ is also an isomorphism in $\mathfrak{ho}_c\mathbf{C}$, and the identity of Whitehead torsions*

$$\langle w^{(1,1)}\rangle = f'_*\langle w^{(0,1)}\rangle + a'_*\langle w^{(1,0)}\rangle - d_*\langle w^{(0,0)}\rangle$$

holds.

Let us first state two immediate corollaries, following from this result and Corollary 11.1.2.8:

Corollary 11.2.1.8. *Under the assumptions of Theorem 11.2.1.7, the following claims are equivalent:*

1. *$\omega^{(0,0)}, \omega^{(1,0)}$ and $\omega^{(0,1)}$ are simple equivalences.*

2. *ω is a simple equivalence.*

3. *$\omega^{(0,0)}, \omega^{(1,0)}, \omega^{(0,1)}$ and $\omega^{(1,1)}$ are simple equivalences.*

We also obtain the following formulation in terms of cubes, which is more akin to the way [Coh73] phrased the sum formula.

Corollary 11.2.1.9. *Let $\mathbf{C}$ be a properly generated Whitehead model category. Suppose we are given a commutative cube in $\overline{\mathbf{Cell}}_c(\mathbf{C})$*

$$\begin{tikzcd}
\mathfrak{X}_0 \arrow[rr, "w_0"] \arrow[dd, hook, "a"'] \arrow[dr, "f"] & & \mathfrak{Y}_0 \arrow[dd, hook, "b"] \arrow[dr] & \\
& \mathfrak{X}'_0 \arrow[rr, "w'_0"] \arrow[dd, "a'"] & & \mathfrak{Y}'_0 \arrow[dd] \\
\mathfrak{X}_1 \arrow[rr, "w_1"] \arrow[dr, "f'"'] & & \mathfrak{Y}_1 \arrow[dr] & \\
& \mathfrak{X}'_1 \arrow[rr, "w'_1"] & & \mathfrak{Y}'_1
\end{tikzcd} \tag{11.34}$$

with left and right face cobase change, a, b inclusion of subcomplexes and w_0, w_1, w_2 weak equivalences in $\mathbf{C}$. Then the equality of Whitehead torsions

$$\langle w'_1 \rangle = f'_* \langle w_1 \rangle + a'_* \langle w'_0 \rangle - (f'a)_* \langle w_0 \rangle$$

holds.

Let us now provide a proof of Theorem 11.2.1.7:

Proof of Theorem 11.2.1.7. The claim that $\omega^{(1,1)}$ is an isomorphism is classical. We provide it here for the convenience of the reader. By Theorem 10.2.2.1, we may identify $\mathfrak{ho}\mathbf{C}^{\mathbf{Q}} = \mathrm{ho}\mathbf{C}^{\mathbf{Q}}$ and $\mathfrak{ho}\mathbf{C}^{\mathbf{S}} = \mathrm{ho}\mathbf{C}^{\mathbf{S}}$. As all structured cell complexes define cofibrant objects, $i_!$ is a left Quillen functor and i^* is both a right and a left Quillen functor (see [Bar07]), it follows that on $\mathfrak{ho}\mathbf{C}^{\mathbf{Q}}$ we have $Li_! = i_!$ and $Ri^* = i^* = Li^*$ (using standard notation for left and right derived functors). Observe, furthermore, that the unit of adjunction $1 \to i^* i_!$ (which agrees with the derived unit here) is an isomorphism. By Observation 11.2.1.6, we may without loss of generality assume that $\mathfrak{X} = i_! i^* \mathfrak{X}$ and $\mathfrak{Y} = i_! i^* \mathfrak{Y}$. Doing so, we obtain $i_! i^* \omega = \omega$. By assumption, $i^* \omega$ evaluates to an isomorphism at each $r \in \mathbf{S}$. Thus, by the definition of weak equivalences in a functor category, it follows that $i^* \omega$ is an isomorphism. Consequently $i_! i^* \omega = \omega$ is an isomorphism, and hence induces an isomorphism after evaluating at $(1,1)$ (using that evaluation can be cellularized, and is thus left Quillen). Let us now compute the Whitehead torsion of $\omega^{(1,1)}$. We have already seen that up to natural isomorphisms we have $\omega = i_! i^* \omega$. Observe now, that we have a commutative diagram of W-functors

$$\begin{tikzcd}
\mathbf{C}^{\mathbf{S}} \arrow[r, "i_!"] \arrow[dr, bend right, "\varinjlim"'] & \mathbf{C}^{\mathbf{Q}} \arrow[d, "(-)^{(1,1)}"] \\
& \mathbf{C}\,.
\end{tikzcd} \tag{11.35}$$

We obtain an induced commutative diagram of Whitehead groups

$$\begin{tikzcd} \mathrm{Wh}_{\mathbf{C}^{\mathbf{S}}}(i^*\mathfrak{X}) \arrow[r, "\mathrm{Wh}_{i_!}"] \arrow[dr, bend right, "\mathrm{Wh}_{\varinjlim}"'] & \mathrm{Wh}_{\mathbf{C}^{\mathbf{Q}}}(\mathfrak{X}) \arrow[d, "\mathrm{Wh}_{(-)^{(1,1)}}"] \\ & \mathrm{Wh}_{\mathbf{C}}(\mathfrak{X}^{(1,1)}) \end{tikzcd} \tag{11.36}$$

In particular, we have

$$\begin{aligned} \langle\omega^{(1,1)}\rangle &= \mathrm{Wh}_{(-)^{(1,1)}}\langle\omega\rangle \\ &= \mathrm{Wh}_{(-)^{(1,1)}}\langle i_! i^*\omega\rangle \\ &= \mathrm{Wh}_{(-)^{(1,1)}} \circ \mathrm{Wh}_{i_!}\langle i^*\omega\rangle \\ &= \mathrm{Wh}_{\varinjlim}\langle i^*\omega\rangle \\ &= \mathrm{Wh}_{\varinjlim} \circ \mathrm{Wh}_{i^*}\langle\omega\rangle\,. \end{aligned}$$

By Proposition 11.2.1.4, we may use the identities $(-)^r i^* = (-)^r$, for $r \in \mathbf{S}$, to compute

$$\begin{aligned} \langle\omega^{(1,1)}\rangle &= \mathrm{Wh}_{\varinjlim} \circ \mathrm{Wh}_{i^*}\langle\omega\rangle \\ &= a'_*\mathrm{Wh}_{(-)^{(1,0)}} \circ \mathrm{Wh}_{i^*}\langle\omega\rangle + f'_*\mathrm{Wh}_{(-)^{(0,1)}} \circ \mathrm{Wh}_{i^*}\langle\omega\rangle - d_* \circ \mathrm{Wh}_{(-)^{(0,0)}} \circ \mathrm{Wh}_{i^*}\langle\omega\rangle \\ &= a'_*\mathrm{Wh}_{(-)^{(1,0)}\circ i^*}\langle\omega\rangle + f'_*\mathrm{Wh}_{(-)^{(0,1)}\circ i^*}\langle\omega\rangle - d_* \circ \mathrm{Wh}_{(-)^{(0,0)}\circ i^*}\langle\omega\rangle \\ &= a'_*\langle\omega^{(1,0)}\rangle + f'_*\langle\omega^{(0,1)}\rangle - d_*\langle\omega^{(0,0)}\rangle \end{aligned}$$

as was to be shown.

□

11.2.2 Subdivisions of structured cell complexes

In many topological scenarios, it can be useful to observe that the simple homotopy type is often invariant under subdivision. Let us investigate this type of scenario in the full generality of Whitehead model categories.

Construction 11.2.2.1. Suppose we are given a cellularized category $\mathbf{C}$ and a structured relative cell complex $\mathfrak{c}: A \hookrightarrow X$ in $\mathbf{C}$. For every cell $(\partial D \to D, \sigma: D \to X)$, we may fix an alternative cell structure on the generating boundary inclusion $\partial D \to D$, denoted $\mathfrak{i}_\sigma$. Then we can consider the set of cells

$$\mathfrak{C}_{\mathfrak{c}'} := \bigcup_{\sigma \in \mathfrak{C}_{\mathfrak{c}}} \sigma\mathfrak{C}_{\mathfrak{i}_\sigma} \subset \coprod_{(b:\partial D \to D \in \mathbb{B})} \mathbf{C}(D, X).$$

This set defines a new cell structure on $c: A \hookrightarrow X$. To see this, fix any filtration-presentation $\mathfrak{p}$ of $\mathfrak{c}$, with one cell in each transfinite inductive step. Denote the associated transfinite composition diagram in the form

$$A = X^0 \to \cdots \to X^1 \to \cdots \to X^\lambda = X.$$

We may then use the associated pushout squares

$$\begin{tikzcd} \partial D^\alpha \arrow[r, hook] \arrow[d, dashed] & D^\alpha \arrow[d, dashed, "\sigma^\alpha"'] \\ X^\alpha \arrow[r, hook] & X^{\alpha+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{11.37}$$

to equip $X^\alpha \to X^{\alpha+1}$ with the structure of a relative cell complex, obtained as the cobase change of the relative structured cell complex $\mathfrak{i}_{\sigma_\alpha}: D^\alpha \to D^\alpha$. Then $\mathfrak{C}_{\mathfrak{c}'}$ is precisely the transfinite composition of all of these cell structures.

Example 11.2.2.2. Let $X \in \mathbf{sSet}$ be a simplicial complex (i.e., a simplicial set whose non-degenerate simplices are uniquely determined by their set of vertices). Then any subdivision of $|X|$ in the classical sense, by some ordered simplicial complex X', defined by a simplexwise affine homeomorphism $|X'| \cong |X|$ defines a subdivision of $|X|$ (by using the cell structure on $|X|$ transported from $|X'|$ along the fixed homeomorphism).

Definition 11.2.2.3. Let $\mathbf{C}$ be a cellularized category and $\mathfrak{c}: A \hookrightarrow X$ be a structured relative cell complex. We call another cell structure $\mathfrak{c}': A \hookrightarrow X$ on the same morphism $c: A \hookrightarrow X$ a *subdivision of* $\mathfrak{c}$, if it arises from Construction 11.2.2.1.

Proposition 11.2.2.4. *Let* $\mathbf{C}$ *be a properly generated Whitehead model category. Suppose that the following holds.*

1. *For every generating boundary inclusion* $\partial D \to D \in \mathbb{B}$*, the source* ∂D *admits the structure of a finite cell complex.*
2. *Let* $\partial D \to D \in \mathbb{B}$ *be a generating boundary inclusion. For some (and hence any) finite cell structure on* ∂D*,* $\partial\mathfrak{D}$*, the following holds: Denote by* $\mathfrak{D}$ *the structured cell complex on* D*, induced by adding the identity cell* $D \to D$ *to* $\partial\mathfrak{D}$*. Then*

$$\mathrm{Wh}_{\mathbf{C}}(\mathfrak{D}) = 0 \,.$$

Then, for any finite structured cell complex $\mathfrak{X}$ *and any finite subdivision* $\mathfrak{X}'$ *of* $\mathfrak{X}$*, the identity map* $1_X: X \to X$*, defines a simple equivalence* $\mathfrak{X}' \to \mathfrak{X}$*.*

Proof. We proceed via induction over the number of cells of $\mathfrak{X}$. The case of an empty cell complex is obvious. For the inductive step, we may write $\mathfrak{X}$ in terms of a cobase change of structured cell complexes

$$\begin{tikzcd} \partial\mathfrak{D} \arrow[r, dashed, "\partial\sigma"'] \arrow[d, hook] & \mathfrak{A} \arrow[d, hook] \\ \mathfrak{D} \arrow[r, "\sigma"] \arrow[ur, phantom, "\ulcorner" very near end] & \mathfrak{X} \,. \end{tikzcd} \tag{11.38}$$

identifying $\mathfrak{X}$ with the structured cell complex $\mathfrak{X} = \mathfrak{A} \cup_{\partial\mathfrak{D}} \mathfrak{D}$. The subdivision of $\mathfrak{X}$, $\mathfrak{X}'$, induces the structure of a subdivision on $\mathfrak{D}$, denoted $\mathfrak{D}'$, which does not subdivide $\partial\mathfrak{D}$. Furthermore, it induces a subdivision $\mathfrak{A}'$ of $\mathfrak{A}$. We obtain a commutative cube

$$\begin{tikzcd} \partial\mathfrak{D} \arrow[rr] \arrow[dd, hook] \arrow[dr] & & \mathfrak{A}' \arrow[dd, hook] \arrow[dr] & \\ & \partial\mathfrak{D} \arrow[rr, crossing over] & & \mathfrak{A} \arrow[dd, hook] \\ \mathfrak{D}' \arrow[rr] \arrow[dr] & & \mathfrak{X}' \arrow[dr] & \\ & \mathfrak{D} \arrow[rr] \arrow[from=uu, hook, crossing over] & & \mathfrak{X} \end{tikzcd} \tag{11.39}$$

with front and back face cobase change squares of structured cell complexes, and all diagonal maps given by the identity on the underlying objects in $\mathbf{C}$. By inductive assumption, it follows that $\mathfrak{A}' \to \mathfrak{A}$ is a simple equivalence. By the assumption on triviality of Whitehead groups of $\mathfrak{D}$, it follows that $\mathfrak{D}' \to \mathfrak{D}$ is a simple equivalence. Thus, it follows by Corollary 11.2.1.9, that $\mathfrak{X}' \to \mathfrak{X}$ is a simple equivalence. □

Example 11.2.2.5. Suppose for a second that we have already shown that the Whitehead groups associated to the properly generated Whitehead model category **Top** agree with the classical Whitehead group (this is the content of Theorem 12.2.0.4). Then we recover from Proposition 11.2.2.4 the classical result that the simple homotopy type is invariant under subdivisions of simplicial complexes.

11.2.3 Simple acyclic models: Constructing simple equivalences of cellularized functors

We will finish our investigation of the simple homotopy theory of diagrams and functors with a simple homotopy theoretic version of the principle of acyclic models. Roughly speaking, we will give a purely abstract criterion for two cellularized functors $\mathfrak{F}$ and $\mathfrak{G}$, defined on a category of presheaves, to be weakly equivalent in a way such that the induced equivalence $\mathfrak{F}(\mathfrak{X}) \simeq \mathfrak{G}(\mathfrak{X})$ is a simple equivalence for every finite cell complex $\mathfrak{X}$. The criterion only requires the computation of the homotopy types and Whitehead groups of the values on $\mathfrak{F}$ and $\mathfrak{G}$ on the representables of the presheaf category. This turns out to be a rather useful result, as it often allows one to prove simple homotopy equivalence of two cell complexes, purely through homotopy theoretic arguments involving elementary objects.

Notation 11.2.3.1. In the following, we will often look at homotopy categories associated to some category of diagrams, denoted in the form $\mathrm{ho}\mathbf{C}^{\mathbf{R}}$. We use the notational convention that $(-)^{\mathbf{R}}$ always has preference over $\mathrm{ho}(-)$ in the order of application, i.e., we always mean the homotopy category of the diagram category, and not diagrams in the homotopy category.

Let us begin with a remark on finiteness.

Lemma 11.2.3.2. *Let $\mathbf{R}$ be a Reedy category which has locally finitely many faces. Let $\mathbf{C}$ be a cellularized category. Let $\mathfrak{F} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$. Then $\mathfrak{F}$ is a finite cellularized functor if and only if, for every $r \in \mathbf{R}$, the structured cell complex $\mathfrak{F}(\mathbf{R}^r)$ is finite. If $\mathbf{R}$ is finite, then $\mathfrak{F}$ is finite if and only if the associated structured cell complex $\mathfrak{F}|_{\mathbf{R}} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{R}})$ (under Corollary 8.3.6.3) is finite.*

Proof. For the first claim, observe that, by the assumption on finiteness of faces $\mathbf{R}^r$ is a finite cell complex. Hence, the only if part of the claim is trivial. For the if part, we need to see that, for each $r \in \mathbf{R}$, $\mathfrak{F}(\iota^r)$ is a finite structured relative cell complex. Observe, however, that $\mathfrak{F}(\iota^r)$ corresponds to the inclusion of subcomplexes $\mathfrak{F}(\partial\mathbf{R}^r) \hookrightarrow \mathfrak{F}(\mathbf{R}^r)$ (under Observation 8.1.3.5). In particular, $\mathfrak{F}(\iota^r)$ is finite if $\mathfrak{F}(\mathbf{R}^r)$ is a finite structured cell complex. For the remaining claim, it suffices to see that a structured cell complex $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C}^{\mathbf{R}})$, for a finite Reedy category $\mathbf{R}$ is finite, if and only if $\mathfrak{X}^r$ is finite, for each $r \in \mathbf{R}$. This is immediate from the explicit description of the cell structure of $\mathfrak{X}^r$ in Construction 11.1.1.11. □

Recall the definition of a Reedy category which admits positive sections from Definition 8.3.8.7. Recall, furthermore, that given a model category $\mathbf{C}$, we denote the associated quasi-category obtained by localizing weak equivalences by $\mathcal{C}$.

Theorem 11.2.3.3. *Let $\mathbf{R}$ be a Reedy category that admits positive sections and has locally finitely many degeneracies and faces. Let $\mathbf{C}$ be a properly generated Whitehead model category, such that the underlying model category $\mathbf{C}$ is simplicial and locally presentable. Let $\mathfrak{F}, \mathfrak{G} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$ be two finite cellularized functors and let $\eta\colon F \Rightarrow G \in \mathcal{C}^{(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}})}$ be a natural transformation of the associated functors of ∞-categories. Then the following are equivalent:*

1. *η_X is a simple equivalence, at each finite $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$.*
2. *$\eta_{\mathbf{R}^r}$ is a simple equivalence for each $r \in \mathbf{R}$.*

Proof. Let us prove the non-trivial implication. Let us first note that the functors of $(\infty,1)$-categories

$$\mathbf{C\bar{e}ll}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}) \hookrightarrow \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \xrightarrow{F,G} \mathbf{C} \to \mathcal{C}$$

are given by ∞-categorical left Kan extension along $\mathbf{R} \hookrightarrow \mathbf{C\bar{e}ll}_c(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}})$. We cover the case of F. Let $X \in \mathbf{C\bar{e}ll}_c(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}})$. By Theorem 8.3.8.9, we have a canonical identification of cell complexes

$$\mathfrak{F}(X) = \varinjlim_{\mathbf{el}(X)} \mathfrak{F}(\mathbf{R}^r).$$

A priori, this colimit is only 1-categorical. The diagram which we are taking the colimit over is

$$D\colon \mathbf{el}(X) \to \mathbf{R} \xrightarrow{F|_{\mathbf{R}}} \mathbf{C}\,.$$

It follows by Corollary 8.3.6.3, that $F|_{\mathbf{R}}$ defines a cofibrant object in the Reedy (semi)model structure on $\mathbf{C}^{\mathbf{R}^{\mathrm{op}}}$. It follows by Proposition 8.3.8.5 that D defines a cofibrant object in the Reedy (semi)model structure on $\mathbf{C}^{\mathbf{el}(X)}$. By Proposition 8.3.8.1 and the (analogously proven) semimodel category version of [Bar07, Theorem 2.7.], the colimit functor $\mathbf{C}^{\mathbf{el}(X)} \to \mathbf{C}$ is a left-Quillen functor, and hence provides a construction for the homotopy colimit. Thus, it follows by [Lur09, Thm. 4.2.4.1] (and the semi-model category version of [DK80b], the proof of which is identical), that

$$\varinjlim_{\mathbf{el}(X)} F(\mathbf{R}^r)$$

is actually a colimit in the ∞-category $\mathcal{C}$, exposing the functor $\mathbf{C\overline{e}ll}_c(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}) \to \mathcal{C}$ induced by F as a Kan extension in the ∞-categorical sense. It follows that, at each finite cell complex $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, η_X is induced by taking the colimit of the natural transformation (of functors of $(\infty,1)$-categories)

$$\eta_{\mathrm{ev}}\colon F \circ f \Rightarrow G \circ f\,,$$

where f denotes the composition $\mathbf{el}(X) \to \mathbf{R} \hookrightarrow \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$.
By Proposition 8.3.8.4, the inclusion $I\colon \mathbf{el}_{n.d.}(X) \hookrightarrow \mathbf{el}(X)$ is a final functor of 1-categories, i.e., $I_{\sigma/}$ is connected, for $\sigma \in \mathbf{el}(X)$. By Proposition 8.3.8.6, $I_{\sigma/}$ is also a right fibrant Reedy category. In particular, the constant diagram of spaces on $I_{\sigma/}$ is cofibrant. At the same time, by Proposition 8.3.8.6, $I_{\sigma/}$ is also left fibrant. Thus, again by [Bar07, Theorem 2.7.], the colimit functor is a left Quillen functor, and the homotopy colimit of the constant diagram in spaces agrees with the usual colimit, i.e., the point. As the homotopy colimit of a constant diagram in $\mathcal{S}\mathbf{paces}$ is equivalently the homotopy type of the nerve of the underlying diagram (this follows, for example, by the cobar construction of the homotopy colimit found in [Hir03, Ch.18]) it follows that that $I_{\sigma/}$ is contractible. Thus, I is also a final functor of ∞-categories. Consequently, we may compute η_X as the ∞-categorical colimit of

$$\eta' \coloneqq \eta_{f \circ I}.$$

Now, denote by $\mathfrak{F}|_{\mathbf{R}}, \mathfrak{G}|_{\mathbf{R}}$ the structured cell complexes in $\mathbf{C}^{\mathbf{R}}$ associated to $\mathfrak{F}$ and $\mathfrak{G}$ under Corollary 8.3.6.3. Observe that by Theorem 8.3.8.9 that $\mathfrak{F}' \coloneqq I^*\mathfrak{F}|_{\mathbf{R}}, \mathfrak{G} \coloneqq I^*\mathfrak{G}|_{\mathbf{R}}$ obtain canonical structures of cell complexes, such that $\varinjlim I^*\mathfrak{F} \cong \mathfrak{F}(\mathfrak{X})$ (and analogously for $\mathfrak{G}$). Note that $\mathbf{el}_{n.d.}(X)$ is a finite Reedy category, by the assumption on finiteness of degeneracies and faces. It follows by Lemma 11.2.3.2, that $\mathfrak{F}'$ and $\mathfrak{G}'$ are finite structured cell complexes. As they define cell complexes, their underlying diagrams $F' = F \circ f \circ I$ and $G' = G \circ f \circ I$ are both cofibrant in $\mathbf{C}^{\mathbf{el}_{n.d.}(X)}$. It follows by (the semi-model category version of) [Lur09, Prop. 4.2.4.4] and the semi-model category version of [DK80b], that we may identify $\mathcal{C}^{\mathbf{el}_{n.d.}(X)}$ with the ∞-categorical localization of $\mathbf{C}^{\mathbf{el}_{n.d.}(X)}$ at pointwise weak equivalences. Using the Reedy semi-model structure on $\mathbf{C}^{\mathbf{el}_{n.d.}(X)}$, and Theorem 10.2.2.1, it follows that η' defines a morphism $\mathfrak{F}' \to \mathfrak{G}'$ in $\mathfrak{ho}_c\mathbf{C}^{\mathbf{el}_{n.d.}(X)}$, which we also denote η', by abuse of notation, such that evaluation of η' at $(\mathbf{R}^r \to X) \in \mathbf{el}_{n.d.}(X)$ is given by $\eta_{\mathbf{R}^r}$. Now, finally passing to the colimit, by Corollary 11.2.1.1 and the left fibrancy of $\mathbf{el}_{n.d.}(X)$ (see Theorem 8.3.8.9), we obtain that

$$\varinjlim \eta'\colon \mathfrak{F}(X) \cong \varinjlim I^*\mathfrak{F} \to \varinjlim I^*\mathfrak{G} \cong \mathfrak{G}(X)$$

is a simple equivalence. As we have computed above, that $\eta_X = \varinjlim \eta'$, also in the ∞-categorical sense, it follows that $\eta_X\colon \mathfrak{F}(X) \to \mathfrak{G}(X)$ is a simple equivalence. □

In the first part of the proof of Theorem 11.2.3.3, we have shown the following statement (which is certainly known in the setting where $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ is equipped with some model structure).

Lemma 11.2.3.4. *Let* $\mathbf{C}$ *be a cofibrantly generated simplicial semi-model category with a fixed set of cofibrant generators, making it a cellularized category. Let* $\mathbf{R}$ *be a Reedy category that admits positive sections. Let* $\mathfrak{F} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$ *be a cellularized functor. Then the commutative diagram of functors of* ∞*-categories*

$$\begin{tikzcd} \mathbf{R} \arrow[r] \arrow[d, hook] \arrow[drr, Rightarrow, "1"'] & \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \arrow[r, "F"] & \mathbf{C} \arrow[r] & \mathcal{C} \\ & & & \mathbf{C} \arrow[u] \\ \mathbf{C\bar{e}ll}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}) \arrow[r] & \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \arrow[urr, "F"] & & \end{tikzcd} \tag{11.40}$$

is a left Kan extension.

Remark 11.2.3.5. Observe that, in the case where we are given 1-categorical transformation of functors $F \Rightarrow G$, the proof of Theorem 11.2.3.3 significantly simplifies, and we can drop the assumption that $\mathbf{C}$ is locally presentable and simplicial.

Having this remark in mind, we make the following two definitions.

Definition 11.2.3.6. Let $\mathbf{C}$ be a properly generated simple homotopy theory. Let $\mathfrak{F}, \mathfrak{G} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$ be two finite cellularized functors. A natural transformation of functors of 1-categories $\eta: F \Rightarrow G$ is a *simple equivalence of cellularized functors*, if it fulfills one of the two equivalent properties stated in Theorem 11.2.3.3.

Let us take the time, to draw the connection between Definition 11.2.3.6 and Theorem 11.2.3.3 more closely.

Remark 11.2.3.7. Suppose we are in the setting of Theorem 11.2.3.3. Restriction along the Yoneda embedding $\mathbf{R} \hookrightarrow \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ induces a map

$$\mathrm{ho}(\mathcal{C}^{\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}})(F, G) \to \mathrm{ho}\mathcal{C}^{\mathbf{R}}(F|_{\mathbf{R}}, G|_{\mathbf{R}}).$$

We may furthermore compose this map with the series of bijections

$$\mathrm{ho}\mathcal{C}^{\mathbf{R}}(F|_{\mathbf{R}}, G|_{\mathbf{R}}) \cong \mathrm{ho}\mathbf{C}^{\mathbf{R}}(F|_{\mathbf{R}}, G|_{\mathbf{R}}) = \mathfrak{ho}_c\mathbf{C}^{\mathbf{R}}(\mathfrak{F}|_{\mathbf{R}}, \mathfrak{G}|_{\mathbf{R}}).$$

In this fashion, we obtain a map

$$R: \mathrm{ho}(\mathcal{C}^{\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}})(F, G) \to \mathfrak{ho}_c\mathbf{C}^{\mathbf{R}}(\mathfrak{F}|_{\mathbf{R}}, \mathfrak{G}|_{\mathbf{R}})$$

associating to any natural transformation of the associated functors of ∞-categories $\eta: F \to G$ a morphism in $\mathfrak{ho}_c\mathbf{C}^{\mathbf{R}}(\mathfrak{F}|_{\mathbf{R}}, \mathfrak{G}|_{\mathbf{R}})$ between cellularized diagrams. Theorem 11.2.3.3 together with Corollary 11.1.2.8 state that $\eta_{\mathfrak{X}}: \mathfrak{F}(\mathfrak{X}) \to \mathfrak{G}(\mathfrak{X})$ is a simple equivalence, for every finite cell complex $\mathfrak{X} \in \mathbf{C\bar{e}ll}_c(\mathbf{C})$, if and only if $R(\eta): \mathfrak{F}|_{\mathbf{R}} \to \mathfrak{G}|_{\mathbf{R}}$ is a simple equivalence. If $\mathbf{R}$ is an elegant Reedy category, and thus $\mathbf{C\bar{e}ll}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}) = \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, then it follows from Lemma 11.2.3.4 that R is even a bijection. Hence, we may rephrase Theorem 11.2.3.3 as stating that R restricts to a bijection

$$\begin{tikzcd} \{\eta: F \Rightarrow G \in \mathrm{ho}(\mathcal{C}^{\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}}) \mid \eta_X: \mathfrak{F}(X) \to \mathfrak{G}(X) \text{ is a simple equivalence, for } X \text{ finite}\} \arrow[d, "\cong"] \\ \{\alpha: \mathfrak{F}|_{\mathbf{R}} \to \mathfrak{G}|_{\mathbf{R}} \in \mathfrak{ho}_c\mathbf{C}^{\mathbf{R}} \mid \alpha^r \text{ is a simple equivalence, for } r \in \mathbf{R}\} \end{tikzcd} \tag{11.41}$$

Now, suppose that $\mathbf{R}$ is finite, and hence that $\mathfrak{F}|_{\mathbf{R}}$ and $\mathfrak{G}|_{\mathbf{R}}$ are finite cell complexes. Then it follows by Corollary 11.1.2.8, that a morphism $\alpha: \mathfrak{F}|_{\mathbf{R}} \to \mathfrak{G}|_{\mathbf{R}}$ as described in the source of R is a simple equivalence of cell complexes of diagrams. As every simple equivalence in the homotopy category associated to a Whitehead model category lifts to a zig-zag of simple equivalence on the level of 1-categories, any element of the target of this bijection can be expressed in the form

$$\mathfrak{F}|_{\mathbf{R}} \xrightarrow{s_0} \tilde{\mathfrak{E}} \xleftarrow{s_1} \mathfrak{G}|_{\mathbf{R}}$$

with s_0 and s_1 simple equivalences. Under Corollary 8.3.6.3 and Theorem 11.2.3.3, this zigzag is lifted to a zigzag of simple equivalences of finite cellularized functors

$$\mathfrak{F} \xrightarrow{\overline{s_0}} \mathfrak{E} \xleftarrow{\overline{s_1}} \mathfrak{G}\,.$$

In particular, it follows that under these assumptions, any natural transformation

$$\eta\colon F \Rightarrow G \in (\mathcal{C}^{\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}})$$

as in the source of R, can be expressed in terms of a zig-zag of simple equivalences of finite cellularized functors

$$\mathfrak{F} \xrightarrow{\simeq} \mathfrak{E} \xleftarrow{\simeq} \mathfrak{G}\,.$$

If $\mathbf{R}$ is elegant, but not finite, then one can at least expose such a zig-zag of weak equivalences with $\mathfrak{E}$ not necessarily finite, which evaluates to η_X, for $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, after passing to homotopy categories. If elegance is also dropped, this still holds for cell complexes $X \in \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$.

Thus, the following nomenclature seems justified.

Definition 11.2.3.8. Let $\mathbf{R}$ be an elegant Reedy category that locally has finitely many degeneracies and faces. Let $\mathbf{C}$ be a properly generated simplicial Whitehead model category, the underlying category of which is locally presentable. Let $\mathfrak{F}, \mathfrak{G} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$ be two finite cellularized functors. A natural transformation of the associated functors of ∞-categories $\eta\colon F \Rightarrow G$ is called an *∞-categorical simple equivalence* of cellularized functors if it fulfills one of the two equivalent properties stated in Theorem 11.2.3.3.

Remark 11.2.3.9. It follows by Lemma 11.2.3.4, that whenever $\mathbf{R}$ is elegant, then any ∞-categorical equivalence of cellularized functors is an isomorphism of functors, in the ∞-categorical sense. We will also speak of an ∞-categorical equivalence between cellularized functors when $\mathbf{R}$ is not necessarily elegant, but just admits positive sections. In this case, one should be careful to note that Lemma 11.2.3.4 does not guarantee that such a transformation η is an isomorphism of functors ∞-categories on all of $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$. This is true only after restricting to absolute cell complexes. In any case, it follows that this usage of nomenclature is compatible with the one used in Definition 10.3.2.7.

Let us apply this result to derive a well-known classical example.

Example 11.2.3.10. Let $\mathbf{R} = \Delta$ and $\mathbf{C}$ be the (Whitehead model category of) simplicial sets. Consider the identity functor

$$1\colon \mathbf{sSet} \to \mathbf{sSet}$$

as well as the subdivision functor

$$\mathrm{sd}\colon \mathbf{sSet} \to \mathbf{sSet},$$

given by left Kan extension of the $\Delta \to \mathbf{sSet}$ mapping $[n]$ to the nerve of the poset of non-empty subsets of $[n]$, ordered by inclusion. Both functors preserve colimits and monomorphisms and are thus (uniquely) cellularized. Now, let $\eta\colon \mathrm{sd} \to 1$ be given by the last vertex map. For every simplex $\Delta^n \in \mathbf{sSet}$, the last vertex map $\mathrm{sd}\Delta^n \to \Delta^n$ is a simple equivalence. This is either immediate if one already assumes to know that the Whitehead model category $\mathbf{sSet}$ produces classical simple homotopy theory, or it can be derived by an easy elementary combinatorial argument (see [Waa21, Prop. 2.3.19], for a proof). Hence, one immediately obtains from Theorem 11.2.3.3 that

$$\mathrm{sd}X \to X$$

is a simple equivalence in **sSet**, for every finite simplicial set X. If one already supposes to know that the simple homotopy theory of **sSet** is just classical simple homotopy theory, this is of course a well-known result. We will, however, use it in Section 12.2 to derive precisely this claim.

It turns out that for most of the examples we are interested in, one does not need to construct a natural transformation between the cellularized functors by hand. Instead, one can use the following lemma.

Lemma 11.2.3.11. *Let $\mathbf{R}$ be an elegant Reedy category. Let $\mathbf{C}$ be a simplicial combinatorial semi-model category. Let $F, G\colon \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \to \mathbf{C}$ be two left adjoint functors. Now, suppose that the following conditions hold.*

1. *For each pair $r, r' \in \mathbf{R}$ and $X \in \mathbf{C}$, the (derived) mapping spaces $\mathcal{C}(F(\mathbf{R}^r), F(\mathbf{R}^{r'}))$, $\mathcal{C}(G(\mathbf{R}^r), G(\mathbf{R}^{r'}))$, $\mathcal{C}(F(\mathbf{R}^r), G(\mathbf{R}^{r'}))$ and $\mathcal{C}(G(\mathbf{R}^r), F(\mathbf{R}^{r'}))$ are empty, or contractible.*
2. *For each $r \in \mathbf{R}$, the mapping spaces $\mathcal{C}(F(\mathbf{R}^r), G(\mathbf{R}^r))$ and $\mathcal{C}(G(\mathbf{R}^r), F(\mathbf{R}^r))$ are non-empty.*

Then there exists a natural transformation (unique up to homotopy)

$$\begin{tikzcd} \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\simeq\Downarrow"] & \mathcal{C} \end{tikzcd} \tag{11.42}$$

of the associated functor of ∞-categories. This natural transformation is an isomorphism of functors of ∞-categories.

Proof. By Lemma 11.2.3.4 the functors of ∞-categories

$$\mathbf{C\overline{el}l}_c(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}) \hookrightarrow \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \xrightarrow{F,G} \mathbf{C} \to \mathcal{C}$$

are both given by left Kan extension along $\mathbf{R} \hookrightarrow \mathbf{C\overline{el}l}_c(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}})$. As $\mathbf{R}$ is elegant, the equality $\mathbf{C\overline{el}l}_c(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}) = \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$ holds. Hence, it follows that homotopy classes of natural transformation F and G (thought of as functors of ∞-categories with target $\mathcal{C}$), are in one-to-one correspondence with homotopy classes of transformations of the functors of ∞-categories $F|_{\mathbf{R}}$ and $G|_{\mathbf{R}}$ (see, for example, [Cis19, Proposition 6.4.9.], using that $\mathcal{C}$ is cocomplete, by the existence of homotopy colimits given by the injective model structure). Now, observe that by the first assumption, $F|_{\mathbf{R}}$ and $G|_{\mathbf{R}}$ factor through a full subcategory $\mathcal{P}$ of $\mathcal{C}$, such that all mapping spaces are either empty or contractible. Such a category is equivalent to a poset (given by choosing one representative of each isomorphism class, and passing to the homotopy category). Two functors F', G' into an ∞-category equivalent to a poset admit at most one natural transformation $F' \Rightarrow G'$, up to homotopy, and such a natural transformation exists under the second condition in the statement of the lemma. Finally, the claim on isomorphisms is immediate from the uniqueness statement we have just made, inverting the roles of F and G. □

Remark 11.2.3.12. In most examples which we have encountered so far, the conditions of Lemma 11.2.3.11 are fulfilled because the objects $F(\mathbf{R}^r), G(\mathbf{R}^r)$ are equivalent, and −1-*truncated (subterminal).* Recall that this means that the mapping spaces $\mathcal{C}(X, F(\mathbf{R}^r))$ are contractible or empty, for each $X \in \mathbf{C}$. For example, if $\mathbf{C}$ is **sSet** or **Top**, Y being subterminal means that Y is equivalent either to the point or the empty space.

We can now combine Theorem 11.2.3.3 with Lemma 11.2.3.11 to obtain a general identification principle for cellularized functors, up to simple equivalence.

Theorem 11.2.3.13 (Simple acyclic models)**.** *Let* $\mathbf{R}$ *be an elegant Reedy category that has locally finitely many degeneracies and faces. Let* $\mathbf{C}$ *be a properly generated Whitehead model category, such that the underlying model category is simplicial and locally presentable. Suppose we are given two finite cellularized functors* $\mathfrak{F}, \mathfrak{G} \in \mathbf{CellCat}(\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}, \mathbf{C})$*. Suppose furthermore that the following conditions hold:*

- *For each pair* $r, r' \in \mathbf{R}$ *and* $I, J \in \{F, G\}$*, the derived mapping spaces* $\mathcal{C}(I(\mathbf{R}^r), J(\mathbf{R}^{r'}))$ *are empty or contractible and the mapping spaces* $\mathcal{C}(F(\mathbf{R}^r), G(\mathbf{R}^r))$ *and* $\mathcal{C}(G(\mathbf{R}^r), F(\mathbf{R}^r))$ *are non-empty.*
- *For each* $r \in \mathbf{R}$*, the unique morphism in* $\mathfrak{ho}_c\mathbf{C}$*,* $\mathfrak{F}(\mathbf{R}^r) \to \mathfrak{G}(\mathbf{R}^r)$*, is a simple equivalence (for example, because* $\mathrm{Wh}_{\mathbf{C}}(\mathfrak{F}(\mathbf{R}^r)) = 0 = \mathrm{Wh}_{\mathbf{C}}(\mathfrak{G}(\mathbf{R}^r))$ *holds).*

Then there exists an essentially unique natural transformation of functors of ∞*-categories* $F \Rightarrow G$*. This natural transformation is an* ∞*-categorical simple equivalence of cellularized functors.*

Proof. By Lemma 11.2.3.11 a natural transformation of the associated functors of ∞-categories exists. Furthermore, by Lemma 8.3.5.8, the requirements of Theorem 11.2.3.3 are met. Hence, the assertion follows. □

Chapter 12

Relating combinatorial and topological generalized simple homotopy theories

A typical situation in simple homotopy theory, already encountered in Whitehead's original articles [Whi39; Whi50], is that one wants to pass from one model for a specific simple homotopy theory to another. For example, one could define classical simple homotopy theory in terms of simplicial sets, or in terms of CW-complexes. It is a well-known fact (and for example, a consequence of the Kan-Quillen equivalence between topological spaces and simplicial sets) that these different models lead to the same homotopy theory. The question is, however, whether they also lead to the same *simple homotopy theory*. For example, one may ask: "Is the Whitehead group defined in terms of inclusions of subsimplicial sets and horn inclusions the same as the one defined in the language of CW-complexes?". In this chapter, we are going to prove the following two main results. Firstly, we will prove a general theorem that gives conditions under which one may transfer the structure of a Whitehead model category to another Whitehead model category along a left adjoint, such that the induced cellularized functor becomes a weak equivalence of Whitehead model categories (see Theorem 12.1.0.4). We will refer to this statement as the *change of models* theorem, from here on out. In particular, it follows from this that the realization functor $\mathbf{sSet} \to \mathbf{Top}$ induces a weak equivalence of Whitehead model categories. Proving the change of models theorem will take up the largest part of this chapter. Then, we will use this result to show that the Whitehead model categories $\mathbf{sSet}$ and $\mathbf{Top}$ give rise to classical simple homotopy theory, as introduced by Whitehead, and discussed in great detail in [Coh73] (Theorem 12.2.0.4).

12.1 A theorem on transferred Whitehead model structures

Our proof of the change of models theorem will make use of additional simplicial structure on Whitehead model categories. Simplicial Whitehead model categories are defined as follows.

Definition 12.1.0.1. A simplicial Whitehead model category consists of the following data:

- A Whitehead model category $\mathbf{C}$;
- The structure of a simplicial semi-model category on $\mathbf{C}$ (in the sense of Definition 7.4.1.2);
- The structure of a cellularized bifunctor on the simplicial action

$$- \otimes - \colon \mathbf{sSet} \times \mathbf{C} \to \mathbf{C};$$

Such that the following holds:

1. For every boundary inclusion $\partial\Delta^n \to \Delta^n$, the induced cellularized relative functor
$$\partial\Delta^n \otimes - \hookrightarrow \Delta^n \otimes -$$
is a W-functor.

2. For every horn inclusion $\Lambda^n_k \to \Delta^n$, the induced cellularized relative functor
$$\Lambda^n_k \otimes - \hookrightarrow \Delta^n \otimes -$$
is a simple cellularized relative functor.

Notation 12.1.0.2. Following the notation conventions of Notation 7.2.1.4, we will denote simplicial Whitehead model categories in the form $\underline{\mathbf{C}}$, and the associated underlying semi-model category by $\mathbf{C}$.

Observe that in a simplicial Whitehead model category, one may use the cellularization of $\partial\Delta^1 \otimes - \hookrightarrow \Delta^1 \otimes -$ as a simple cylinder.

Example 12.1.0.3. The simplicial category of simplicial sets equipped with the Kan-Quillen model structure, and horn inclusions as expansions defines a simplicial combinatorial homotopy theory. That $\Delta^n \times -$ is a W-functor and $\Lambda^n_k \times -$ a simple cellularized functor is verified, for example, in great detail in [Mos19].

Theorem 12.1.0.4. *Let $\underline{\mathbf{C}}$ be a simplicial Whitehead model category. Suppose that every generating boundary inclusion $b \in \mathbb{B}_{\mathbf{C}}$ has cofibrant source. Furthermore, let $\underline{\mathbf{D}}$ be a simplicial semi-model category, and*
$$L\colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}\,.$$
be a simplicial left Quillen functor. Now, suppose that the following holds:

- *$L(\mathbb{B}_{\mathbf{C}})$ defines the structure of a cellularized category on $\mathbf{D}$.*
- *Equipping this cellularized category with the class of expansions $L(\mathbb{E}_{\mathbf{C}})$ (with the cell structures induced by L) defines the structure of a Whitehead model category on $\mathbf{D}$ (that is compatible with the semi-model structure).*
- *The functor of ∞-categories $\mathcal{C} \to \mathcal{D}$ induced by L is fully faithful.*

Then, L canonically inherits the structure of a W-functor $\mathfrak{L}\colon \mathbf{C} \to \mathbf{D}$ that is a weak equivalence of Whitehead model categories, with respect to the induced structure on $\mathbf{D}$.

Example 12.1.0.5. We may take $\underline{\mathbf{C}} = \underline{\mathbf{sSet}}$, $\underline{\mathbf{D}} = \underline{\mathbf{Top}}$ and $L = |-|$. The resulting structure of a cellularized category with expansions on $\mathbf{Top}$ is precisely the one described in Example 10.1.2.3. Hence, we obtain a weak equivalence of Whitehead model categories
$$|-|\colon \mathbf{sSet} \xrightarrow{\simeq} \mathbf{Top}.$$
In particular, we obtain an isomorphism of Whitehead groups
$$\mathrm{Wh}_{\mathbf{sSet}}(X) \cong \mathrm{Wh}_{\mathbf{Top}}(|X|)$$
for every finite simplicial set $X \in \mathbf{sSet}$.

The proof of this theorem is the content of this section.

Remark 12.1.0.6. The new part of Theorem 12.1.0.4 is not that L induces an equivalence of the associated homotopy theories (or equivalently a Quillen equivalence). By assumption, it is fully faithful. This shows that we may think of the associated ∞-category $\mathcal{C}$ as a full subcategory of $\mathcal{D}$. Furthermore, as L defines a left Quillen functor, this inclusion of ∞-categories preserves colimits. Now, by the assumption that the semi-model category $\mathbf{D}$ is cofibrantly generated by arrows in the image of L, it follows that every object in $\mathcal{D}$ can be written in terms of pushouts and transfinite compositions of objects in $\mathcal{C}$. As $\mathcal{C} \to \mathcal{D}$ preserves colimits and is faithful, it thus follows that $\mathcal{C} \to \mathcal{D}$ is also essentially surjective, making it an equivalence of categories. The new part of Theorem 12.1.0.4 is that one of the two equivalent conditions in Lemma 10.3.2.12 for a W-functor that is a Quillen equivalence, to define a weak equivalence of Whitehead model categories holds. Namely, we show that $\mathfrak{L}$ induces a τ-equivalence of Whitehead frameworks, or in other words, induces isomorphisms on Whitehead monoids (see Proposition 9.2.0.7).

Remark 12.1.0.7. In fact, it follows from the proof below that it suffices that we are given the cellularized simplicial structure up to simplices of dimension $n \leq 2$, and that L induces a fully faithful functor of $(2,1)$-categories, obtained by truncating homotopically. Much of the rather technical proof we provide below points towards there being an easier and more conceptual proof if one generally takes an ∞-categorical as opposed to a model categorical approach to generalized simple homotopy theories. While we will not pursue this interesting avenue of investigation, incorporating higher simple homotopy theory as pursued by Hatcher and Waldhausen ([Hat75; WJR13]) here, this is certainly a promising direction of future research.

Proving that $\mathfrak{L}$ defines a τ-equivalence (i.e., induces isomorphisms on Whitehead groups) will take up the remainder of this section. Let us begin by fixing up some useful notational conventions.

Notation 12.1.0.8. In the following, we will often encounter the situation where we glue a structured cell complex $\mathfrak{a}\colon A \to B$ to an absolute cell complex $\mathfrak{X}\colon \varnothing \to X$, along a morphism $f\colon A \to X$. That is, in the language of cobase changes and vertical compositions, we consider the structured cell complex

$$f_{\mathrm{i}}\mathfrak{a} \circ \mathfrak{X}.$$

We found that this notation, while formally correct, can be somewhat distracting. We will, instead, denote the resulting structured cell complex by

$$\mathfrak{X} \cup_f B.$$

From a purely formal point of view, this notation is clearly lacking, as it does not make any mention of the data of $\mathfrak{a}$, which is necessary for this construction. This is, however, not so different from the standard notation for pushouts $X \cup_A B$, which also omits the crucial data of the morphism from the notation. In our context, the structured relative cell complex $\mathfrak{a}$ will always be uniquely identifiable through B, so there will not be a risk of confusion. We will, however, encounter the situation of having several different choices of morphism $A \to X$, which is why we keep track of this morphism with the notation $\cup_f$. Whether we are referring to the absolute structured cell complex or its underlying object in $\mathbf{C}$ (given by the pushout) will be indicated by writing $\mathfrak{X} \cup_f B$ or $X \cup_f B$, respectively.

Notation 12.1.0.9. We will also not add to notational overload by choosing several different tensor notations for the simplicial actions on different categories, and their cellularized versions. What tensor is meant will always be clear from the arguments.

Next, let us make some immediate observations about the setup in Theorem 12.1.0.4.

Observation 12.1.0.10. Under the assumptions of Theorem 12.1.0.4, the left Quillen functor L canonically inherits the structure of a cellularized functor. Namely, for $b \in \mathbb{B}_{\mathbf{C}}$, $L(b)$ is, by definition, an element of $\mathbb{B}_{\mathbf{D}} = L(\mathbb{B}_{\mathbf{C}})$, and hence admits a canonical one-cell structure. Furthermore, by the definition of the generating expansions in $\mathbf{D}$ as $L(\mathbb{E}_{\mathbf{C}})$, it follows that the thus obtained cellularized functor

$$\mathfrak{L}\colon \mathbf{C} \to \mathbf{D}$$

is a W-functor. In particular, it descends to a functor

$$\mathfrak{L}\colon \mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}_c\mathbf{D}$$

which preserves simple equivalences. The bifunctor

$$- \otimes -\colon \mathbf{sSet} \times \mathbf{D} \to \mathbf{D}$$

inherits a unique cellularization from the cellularization of the simplicial action on $\underline{\mathbf{C}}$, which makes the natural isomorphism $L(- \otimes -) \cong - \otimes L(-)$ an isomorphism of cellularized bifunctors. Explicitly, given a boundary inclusion $i\colon\partial\Delta^n \to \Delta^n$ in $\mathbf{sSet}$ and $(L(b)\colon L(\partial D) \to L(D)) \in \mathbb{B}_{\mathbf{D}}$, simply equip $i\hat{\otimes}L(b)$ with the cell structure of $L(i\hat{\otimes}\mathfrak{b})$, transported under the canonical isomorphism

$$\begin{aligned}
&L(\Delta^n \otimes \partial D \cup_{\partial\Delta^n \otimes \partial D} \partial\Delta^n \otimes D \to \Delta^n \otimes D)\\
&\cong \big(L(\Delta^n \otimes \partial D) \cup_{L(\partial\Delta^n \otimes \partial D)} L(\partial\Delta^n \otimes D) \to \Delta^n \otimes L(D)\big)\\
&\cong \big(\Delta^n \otimes L(\partial D) \cup_{\partial\Delta^n \otimes L(\partial D)} (\partial\Delta^n \otimes L(D)) \to \Delta^n \otimes L(D)\big).
\end{aligned}$$

It then follows from Lemma 10.2.1.6 and Remark 10.2.1.2 that this cellularization of the simplicial action on $\mathbf{D}$ equips $\mathbf{D}$ with the structure of a simplicial Whitehead model category.

The core idea of the proof is simple. Using Proposition 9.2.0.7 and Lemma 10.3.2.12, it suffices to construct an inverse to the map of Whitehead monoids

$$\widetilde{\mathrm{Wh}}_{\mathfrak{L}}\colon \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X}) \to \widetilde{\mathrm{Wh}}_{\mathbf{D}}(\mathfrak{L}(\mathfrak{X}))\,.$$

Given a structured cell complex $\mathfrak{X} \in \mathbf{Cell}(\mathbf{C})$, and the inclusion of a finite subcomplex $a\colon \mathfrak{L}(\mathfrak{X}) \hookrightarrow \mathfrak{Y}$, we need to find an inclusion of a finite structured cell complex $\tilde{a}\colon \mathfrak{X} \hookrightarrow \tilde{\mathfrak{Y}}$, and a simple equivalence $\mathfrak{L}(\tilde{\mathfrak{Y}}) \xrightarrow{\gamma} \mathfrak{Y}$, which identifies $\langle \mathfrak{F}(\tilde{a})\rangle$ with $\langle a \rangle$. Suppose, for a second, that we are in the special scenario where $\mathfrak{X}$ and $\mathfrak{L}(\mathfrak{X})$ are fibrant, and that a is given by gluing in a single cell $L(b)\colon L(\partial D \to D)$, along a morphism $f\colon L(\partial D) \to L(X)$, i.e., that a is $\mathfrak{L}(\mathfrak{X}) \hookrightarrow \mathfrak{L}(\mathfrak{X}) \cup_f L(D)$. Using the fact that L defines a fully faithful functor of ∞-categories, as well as $\partial D, L(\partial D)$ being cofibrant and $\mathfrak{X}, L(\mathfrak{X})$ bifibrant, it follows that f may be approximated, up to a simplicial homotopy $H\colon \Delta^1 \otimes \partial D \to L(X)$, by a morphism $L(\tilde{f})$, with $\tilde{f}\colon \partial D \to X$. Let $\tilde{\mathfrak{Y}} = \mathfrak{X} \cup_{\tilde{f}} D$, and consider the zig-zag under $\mathfrak{X}$

$$\begin{tikzcd}
 & & \mathfrak{L}(\mathfrak{X}) \arrow[dll, hook, "a"'] \arrow[d, hook] \arrow[drr, "L(\tilde{a})"] & & \\
\mathfrak{Y} & \mathfrak{L}(\mathfrak{X}) \cup_f \{0\} \otimes L(D) \arrow[l, "\cong"'] \arrow[r, hook] & \mathfrak{L}(\mathfrak{X}) \cup_H \Delta^1 \otimes L(D) & \mathfrak{L}(\mathfrak{X}) \cup_{L(\tilde{f})} \{1\} \otimes L(D) \arrow[l, hook'] \arrow[r, "\cong"] & \mathfrak{L}(\mathfrak{X} \cup_{\tilde{f}} D)
\end{tikzcd} \tag{12.1}$$

Both horizontals are inclusions of subcomplexes, with relative cell structure obtained via the cobase change diagram.

$$\begin{tikzcd}
\{0\} \otimes L(D) \cup_{\{0\} \otimes L(\partial D)} \Delta^1 \otimes L(D) \arrow[r] \arrow[d, hook, "i_0\hat{\otimes}\mathfrak{L}(\mathfrak{b})"'] & \mathfrak{L}(\mathfrak{X}) \cup_f \{0\} \otimes L(D) \arrow[d, hook] \\
\Delta^1 \otimes L(D) \arrow[r] & \mathfrak{L}(\mathfrak{X}) \cup_H \Delta^1 \otimes L(D)
\end{tikzcd} \tag{12.2}$$

and the analogous diagram for i_1 and $\tilde{f}$. In particular, by the assumption that $(\{i\} \hookrightarrow \Delta^1) \otimes -$ is a simple cellularized relative functor, for $i = 0, 1$, it follows that both inclusions are simple equivalences. Hence, the morphism $\mathfrak{Y} \to \mathfrak{L}(\tilde{\mathfrak{Y}})$ induced by the zig-zag is a simple equivalence, which identifies $\langle a \rangle$ and $\langle L(\tilde{a})\rangle$. The difficulty of the actual proof is to show that this construction can be performed inductively, without fibrancy assumptions, and in a well-defined manner.

Construction 12.1.0.11. The way we extend the argument which we have just made is as follows. Let $\mathbf{I}$ be the walking isomorphism, i.e., the category with two objects 0 and 1 and a unique isomorphism between the two. Consider the path fibration $\mathbf{P}_{\mathfrak{L}} \to \mathfrak{ho}_c\mathbf{D}$ given by the pullback

$$\begin{tikzcd} \mathbf{P}_{\mathfrak{L}} \arrow[r] \arrow[d] & (\mathfrak{ho}_c\mathbf{D})^{\mathbf{I}} \arrow[d, "\mathrm{ev}_0"] \\ \mathfrak{ho}_c\mathbf{C} \arrow[r, "\mathfrak{L}"'] & \mathfrak{ho}_c\mathbf{D}\,. \end{tikzcd} \tag{12.3}$$

Explicitly, this is the category with objects given by triples $(\mathfrak{X},\mathfrak{Y},\omega\colon\mathfrak{L}(\mathfrak{X}) \xrightarrow{\cong} \mathfrak{Y})$ and morphisms $(\mathfrak{X}_0,\mathfrak{Y}_0,\omega_0) \to (\mathfrak{X}_1,\mathfrak{Y}_1,\omega_1)$ given by pairs $(\alpha\colon\mathfrak{X}_0 \to \mathfrak{X}_1, \beta\colon\mathfrak{Y}_0 \to \mathfrak{Y}_1)$, such that the diagram

$$\begin{tikzcd} \mathfrak{L}(\mathfrak{X}_0) \arrow[r, "\mathfrak{L}(\alpha)"] \arrow[d, "\omega_0"'] & \mathfrak{L}(\mathfrak{X}_1) \arrow[d, "\omega_1"] \\ \mathfrak{Y}_0 \arrow[r, "\beta"'] & \mathfrak{Y}_1 \end{tikzcd} \tag{12.4}$$

commutes. Observe that, as we assumed that L is fully faithful as a functor of ∞-categories, by Theorem 10.2.2.1, the functor $\mathfrak{L}\colon\mathfrak{ho}_c\mathbf{C} \to \mathfrak{ho}_c\mathbf{C}$ is also fully faithful. In particular, it follows that β and α as above uniquely determine each other. Hence, we will often just write α, when we are referring to a morphism. Now, consider the full subcategory $\mathbf{P}_{\mathfrak{L},s} \subset \mathbf{P}_{\mathfrak{L}}$ given only by such objects $(\mathfrak{X},\mathfrak{Y},\omega)$, where $\omega\colon\mathfrak{L}(\mathfrak{X}) \to \mathfrak{Y}$ is a simple equivalence. We will call the restriction $\mathbf{P}_{\mathfrak{L},s} \to \mathfrak{ho}_c\mathbf{C}$ the *simple path fibration*. The simple path fibration provides a canonical factorization of $\mathfrak{L}$

$$\begin{tikzcd} & \mathbf{P}_{\mathfrak{L},s} \arrow[dr, "F"] & \\ \mathfrak{ho}_c\mathbf{C} \arrow[ur, hook, "\rho"] \arrow[rr, "\mathfrak{L}"'] & & \mathfrak{ho}_c\mathbf{D} \end{tikzcd} \tag{12.5}$$

where ρ is given by mapping $\mathfrak{X}$ to $(\mathfrak{X},\mathfrak{L}(\mathfrak{X}),1_{\mathfrak{L}(\mathfrak{X})})$, and F is the obvious forgetful functor. The evaluation at 0-map $r\colon\mathbf{P}_{\mathfrak{L}} \to \mathfrak{ho}_c\mathbf{C}$ defines a retraction of ρ, which comes equipped with a canonical natural isomorphism

$$\rho \circ r \cong 1_{\mathbf{P}_{\mathfrak{L}}}$$

given by

$$(1_{\mathfrak{X}},\omega)\colon(\mathfrak{X},\mathfrak{L}(\mathfrak{X}),1_{\mathfrak{L}(\mathfrak{X})}) \to (\mathfrak{X},\mathfrak{Y},\omega)\,.$$

Define a morphism in $\mathbf{P}_{\mathfrak{L},s}$ to be a simple equivalence, if and only if it descends to simple equivalences in $\mathfrak{ho}_c\mathbf{C}$ and $\mathfrak{ho}_c\mathbf{D}$. Observe that, as $\mathfrak{L}$ preserve simple equivalence, it follows by the two-out-of-three property for simple equivalence that this is the case if and only if the morphism descends to a simple equivalence in $\mathfrak{ho}_c\mathbf{C}$.

As we have illustrated in the introduction of this proof, showing the claim ultimately comes down to solving certain lifting problems along $\mathbf{P}_{\mathfrak{L},s} \to \mathfrak{ho}_c\mathbf{D}$. In order to deal with some annoyances when it comes to the non-uniqueness of lifts, let us introduce an intermediary category.

Construction 12.1.0.12. Denote by $\mathbf{S}$ the wide subcategory $\mathbf{P}_{\mathfrak{L},s}$ only given by such morphisms (α,β), where $\beta = 1_{\mathfrak{Y}}$, for some $\mathfrak{Y} \in \mathfrak{ho}_c\mathbf{C}$, and $\alpha \in \mathfrak{X}$ is a simple equivalence. Observe that this is an essentially discrete category, i.e., that it is equivalent to a category in which the only morphisms are the identities. Indeed, we may write $\mathbf{S}$ as the disjoint union of the subcategories $\mathbf{S}_{\mathfrak{Y}} \subset F_{\mathfrak{Y}}$, given only by such morphisms α that are simple equivalences. Here $F_{\mathfrak{Y}}$ denotes the fiber of F at $\mathfrak{Y}$. It can be equivalently described as the category whose objects are triples $(\mathfrak{X},\omega\colon\mathfrak{L}(\mathfrak{X}) \xrightarrow{\cong} \mathfrak{Y})$ and whose morphisms are given by arrows $\alpha\colon\mathfrak{X}_0 \to \mathfrak{X}_1$ in $\mathfrak{ho}_c\mathbf{C}$, such that $\mathfrak{L}(\alpha) = \omega_1^{-1}\omega_0$. As $\mathfrak{L}$ is fully faithful, these equalities uniquely determine α and

show that α defines a morphism if and only if α^{-1} defines a morphism. Hence $\mathbf{S}$ is a category with at most one morphism between two objects, and every morphism an isomorphism, i.e., a category that is equivalent to a discrete category.
Given an object $x \in \mathbf{P}_{\mathfrak{L},s}$, we will denote by $[x]$ its isomorphism class in $\mathbf{S}$. Given a morphism $\alpha: x \to y$ in $\mathbf{P}_{\mathfrak{L},s}$, we will denote by $[\alpha]$ the equivalence class of α, under pre- and post-composition with morphisms in $\mathbf{S}$, and call this the simple fiber homotopy class of $[\alpha]$.
One elementary consequence of the (essential) discreteness of $\mathbf{S}$ is that we may collapse its path components in $\mathbf{P}_{\mathfrak{L},s}$, and still obtain a well-defined, equivalent category. That is, we define $\mathbf{E}$ to be the category whose objects are path components of $\mathbf{S}$ and whose morphisms are simple fiber homotopy classes. Composition is defined by composing appropriate choices of representatives, and it is not hard to verify from the essential discreteness of $\mathbf{S}$ that this still defines a well-defined category. A particular consequence of the collapsing functor $\mathbf{P}_{\mathfrak{L},s} \to \mathbf{E}$ being an equivalence of categories is that, given a morphism $\alpha: [x_0] \to [x_1]$, there exists a unique morphism $x_0 \to x_1$ representing α. By abuse of notation, we will denote this morphism by the same notation. Let us call a morphism $[\alpha] \in \mathbf{E}$ a simple equivalence if any (and hence all) of its representatives $\alpha \in \mathfrak{ho}_c\mathbf{C}$ is a simple equivalence.

Notation 12.1.0.13. To clean up notation a little bit, we will denote objects of $\mathbf{E}$ by $[x] \in \mathbf{E}$, where x always refers to a triple denoted $(\mathfrak{X}, \mathfrak{Y}, \omega) \in \mathbf{P}_{\mathfrak{L},s}$. If x is equipped with some subscript (writing x_0), then so will the notation of the associated triple (writing $(\mathfrak{X}_0, \mathfrak{Y}_0, \omega_0)$). We proceed analogously with ornaments such as $\tilde{x}$ and x'.

Observation 12.1.0.14. Using this definition, we obtain an induced diagram of functors

$$\begin{tikzcd} \mathbf{P}_{\mathfrak{L},s} \arrow[d, "\pi"] \arrow[dd, bend left] \\ \mathbf{E} \arrow[d, dashed, "p"] \\ \mathfrak{ho}_c\mathbf{D} \end{tikzcd} \tag{12.6}$$

with the first vertical an equivalence of categories.

Observation 12.1.0.15. Observe that, since $\mathbf{P}_{\mathfrak{L},s} \xrightarrow{F} \mathfrak{ho}_c\mathbf{D}$ is faithful, and $\mathbf{P}_{\mathfrak{L},s} \to \mathbf{E}$ is an equivalence of categories, $p: \mathbf{E} \to \mathfrak{ho}_c\mathbf{D}$ is again a faithful functor. In other words, if we are given a morphism $\beta \in \mathfrak{ho}_c\mathbf{D}$, then a lift of β under p is uniquely determined by its source and target.

Notation 12.1.0.16. In the following, given $\mathfrak{Y} \in \mathfrak{ho}_c\mathbf{D}$, we denote by $\mathbf{E}_{\mathfrak{Y}}$ the fiber category of $p: \mathbf{E} \to \mathfrak{ho}_c\mathbf{D}$ at $\mathfrak{Y}$. Explicitly, this is the category whose objects are given by equivalence classes $[x]$ with x of the form $(\mathfrak{X}, \mathfrak{Y}, \omega)$ and with morphisms given by equivalence classes of tuples of the form $(\alpha, 1_{\mathfrak{Y}})$.

Observation 12.1.0.17. Given $\mathfrak{Y} \in \mathfrak{ho}_c\mathbf{D}$ and $[\tilde{x}] \in \mathbf{E}_{\mathfrak{Y}}$, the class of objects $\mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}})$ is in bijection with $\mathrm{Wh}_{\mathbf{C}}(\tilde{\mathfrak{X}})$. A bijection is constructed as follows: Given another element $[x] \in \mathbf{E}_{\mathfrak{Y}}$, denote by $\alpha(x): \tilde{\mathfrak{X}} \to \mathfrak{X}$ the unique morphism in $\mathfrak{ho}_c\mathbf{C}$ mapping to $\omega \circ \tilde{\omega}^{-1}$ under $\mathfrak{L}$. Now consider the map

$$\begin{aligned} \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}}) &\to \mathrm{Wh}_{\mathbf{C}}(\tilde{\mathfrak{X}}) \\ [x] &\mapsto \langle \alpha(x) \rangle. \end{aligned}$$

It follows by Theorem 10.2.3.9 that this assignment defines a well-defined bijection. In particular, the class of objects of $\mathbf{E}_{\mathfrak{Y}}$ forms a set.

Definition 12.1.0.18. Consider the map of sets

$$\begin{aligned} \mathrm{Ob}(\mathbf{E}_{-}): \mathrm{Ob}(\mathfrak{ho}_c\mathbf{D}) &\to \mathbf{Set} \\ \mathfrak{Y} &\mapsto \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}}) \end{aligned}$$

mapping an object $\mathfrak{Y}$ to the fiber $\mathrm{Ob}(p)^{-1}(\mathfrak{Y})$. By *a coherent family of lifts along* p, we mean an extension of this map to a functor

$$l: \mathfrak{ho}_c\mathbf{D} \to \mathbf{Set}$$

such that

1. For every morphism $\beta: \mathfrak{Y}_0 \to \mathfrak{Y}_1 \in \mathfrak{ho}_c\mathbf{D}$ and $[x_0] \in \mathbf{E}_{\mathfrak{Y}_0}$, there exists a (necessarily unique) morphism $\tilde{\beta}_{[x_0]}: [x_0] \to l(\beta)([x_0])$ in $\mathbf{E}$, such that $p(\tilde{\beta}_{[x_0]}) = \beta$.
2. Whenever $\gamma: \mathfrak{Y}_0 \to \mathfrak{Y}_1$ is a simple equivalence, then, for every $[x_0] \in \mathbf{E}_{\mathfrak{Y}_0}$, the unique lift $\tilde{\gamma}: [x_0] \to l([x_0])$ is a simple equivalence.

Construction 12.1.0.19. Given such a coherent family of lifts l, a morphism $\beta: \mathfrak{Y}_0 \to \mathfrak{Y}_1 \in \mathfrak{ho}_c\mathbf{D}$ and $[x_0] \in \mathbf{E}_{\mathfrak{Y}_0}$, we will denote by $\tilde{\beta}_{[x_0]}: [x_0] \to l(\beta)[x_0]$ the unique lift of β. If we fix $x_0 \in [x_0]$, then any two representatives of the underlying morphism of $\tilde{\beta}_{[x_0]}$ of the form

$$\tilde{\alpha}_{[x_0]}: x_0 \to x_1$$

only differ by a post-composition with a simple equivalence. We denote by $\alpha_{x_0}: \mathfrak{X}_0 \to \mathfrak{X}_1$ the underlying morphism in $\mathfrak{ho}_c\mathbf{C}$ (defined up to composition with a simple equivalence).

Lemma 12.1.0.20. *Suppose that we are given a coherent family of lifts l and suppose that l has the additional property that for every morphism $a: \mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1$ in $\mathbf{C\vec{e}ll}_c(\mathbf{C})$, it holds that*

$$l(\mathfrak{L}(a))([\rho(\mathfrak{X}_0)]) = [\rho(\mathfrak{X}_1)]\,.$$

Then $\widetilde{\mathrm{Wh}}_{\mathfrak{L}}$ is a natural isomorphism.

Proof. Recall that, given $\beta: \mathfrak{Y}_0 \to \mathfrak{Y}_1 \in \mathfrak{ho}_c\mathbf{D}$ and x_0 with $[x_0] \in \mathbf{E}_{\mathfrak{Y}_0}$, the associated morphism α_{x_0} as constructed in Construction 12.1.0.19 is well defined up to composition with a simple equivalence. Consequently, for any $\mathfrak{X} \in \mathfrak{ho}_c\mathbf{C}$, we obtain a well defined map

$$\phi: \widetilde{\mathrm{Wh}}_{\mathbf{D}}(\mathfrak{L}(\mathfrak{X})) \to \widetilde{\mathrm{Wh}}_{\mathbf{C}}(\mathfrak{X}) \tag{12.7}$$

$$\langle\beta\rangle \mapsto \langle\alpha_{[\rho(\mathfrak{X})]}\rangle\,. \tag{12.8}$$

By definition, the morphism $\alpha_{\rho(\mathfrak{X}_0)}$ fits into a commutative diagram

$$\begin{tikzcd} \mathfrak{L}(\mathfrak{X}_0) \arrow[r, "\mathfrak{L}(\alpha_{\rho(\mathfrak{X}_0)})"] \arrow[d, "1"'] & \mathfrak{L}(\mathfrak{X}_1) \arrow[d, "\omega_1"] \\ \mathfrak{L}(\mathfrak{X}_0) \arrow[r, "\beta"'] & \mathfrak{Y}_1 \end{tikzcd} \tag{12.9}$$

with the right vertical given by a simple equivalence. In particular, it holds that

$$\widetilde{\mathrm{Wh}}_{\mathfrak{L}}\langle\alpha_{\rho(\mathfrak{X}_0)}\rangle = \langle\mathfrak{L}(\alpha_{\rho(\mathfrak{X}_0)})\rangle = \langle\omega_1^{-1} \circ \beta\rangle = \langle\beta\rangle\,.$$

Consequently, ψ defines a section to $\widetilde{\mathrm{Wh}}_{\mathfrak{L}}$. Now, recall that we additionally assumed that for any morphism $a: \mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1$ in $\mathbf{C\vec{e}ll}_c(\mathbf{C})$, it holds that

$$l(\mathfrak{L}(a))([\rho(\mathfrak{X}_0)]) = [\rho(\mathfrak{X}_1)]\,.$$

Then, by the faithfulness of p, it follows that $\widetilde{\mathfrak{L}(a)}_{[\rho(\mathfrak{X}_0)]} = [\rho(a)]$. Hence, in this case we may choose $\tilde{\alpha}_{[x_0]}$ (using notation as in Construction 12.1.0.19) as a. Consequently, it holds that

$$\psi(\widetilde{\mathrm{Wh}}_{\mathfrak{L}}\langle a\rangle) = \psi\langle\mathfrak{L}(a)\rangle = \langle a\rangle\,.$$

In particular, ψ defines a two sided inverse to $\widetilde{\mathrm{Wh}}_{\mathfrak{L}}$, showing that the latter is an isomorphism. ∎

So far, we have reduced the proof of Theorem 12.1.0.4 to proving the existence of a coherent family of lifts as in Definition 12.1.0.18. Next, let us reduce to proving the existence of such a family for a very specific class of morphisms in $\mathfrak{ho}_c\mathbf{C}$.

Lemma 12.1.0.21. *Suppose, we are given an extension of the map*

$$\begin{aligned}\ddot{\mathbf{Cell}}_c(\mathbf{D}) &\to \mathbf{Set} \\ \mathfrak{Y} &\mapsto \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}})\end{aligned}$$

to a functor

$$l: \ddot{\mathbf{Cell}}_c(\mathbf{D}) \to \mathbf{Set}$$

such that, for every morphism $a: \mathfrak{Y} \hookrightarrow \mathfrak{Y}'$ in $\ddot{\mathbf{Cell}}_c(\mathbf{C})$, and $[x_0] \in \mathbf{E}_{\mathfrak{Y}}$, there exists a (necessarily unique) morphism $\beta: [x_0] \to l(a)[x_1]$, such that $p(\beta) = a$ in $\mathfrak{ho}_c\mathbf{D}$. Suppose, furthermore, that whenever a is an expansion, then the associated lift is a simple equivalence. Then l descends to a functor on $\mathfrak{ho}_c\mathbf{D}$ that defines coherent family of lifts.

Proof. Recall that $\mathfrak{ho}_c\mathbf{D}$ is given by the localization of $\ddot{\mathbf{Cell}}_c(\mathbf{D})$ at finite expansions. As any such finite expansion can be decomposed into a composition of elementary expansions, one may equivalently obtain $\mathfrak{ho}_c\mathbf{D}$ by localizing $\ddot{\mathbf{Cell}}_c(\mathbf{D})$ at elementary expansions. Let $a: \mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}_1$ in $\ddot{\mathbf{Cell}}_c(\mathbf{C})$ be an elementary expansion. Consider the induced map of fibers of $p: \mathbf{E} \to \mathfrak{ho}_c\mathbf{D}$

$$l_a: \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_0}) \to \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_1}).$$

If $a = e$ is an elementary expansion, this map is a bijection. To see this, fix some $\tilde{x}_0 \in \mathbf{P}_{\mathfrak{L},s}$ representing an element on the left hand side and $\tilde{x}_1$ representing $l(e)[\tilde{x}_0]$. Let $\eta: \tilde{\mathfrak{X}}_0 \to \tilde{\mathfrak{X}}_1$ be the unique lift of the morphism

$$\mathfrak{L}(\tilde{\mathfrak{X}}_0) \xrightarrow{\tilde{\omega}_0} \mathfrak{Y}_0 \xrightarrow{e} \mathfrak{Y}_1 \xrightarrow{\tilde{\omega}_1^{-1}} \mathfrak{L}(\tilde{\mathfrak{X}}_1)$$

to $\mathfrak{ho}_c\mathbf{C}$. It presents $\tilde{e}_{[\tilde{x}_0]}$. In particular η is a simple equivalence. Now, consider the diagram

$$\begin{tikzcd} \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_0}) \arrow[r] \arrow[d] & \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_1}) \arrow[d] \\ \mathrm{Wh}_{\mathbf{C}}(\tilde{\mathfrak{X}}_0) \arrow[r, "\eta_*"'] & \mathrm{Wh}_{\mathbf{C}}(\tilde{\mathfrak{X}}_1) \end{tikzcd} \tag{12.10}$$

with verticals given by

$$[x_i] \mapsto \langle \alpha(x_i) \rangle$$

where $\alpha(x_i): \tilde{\mathfrak{X}}_i \to \mathfrak{X}_i$ is the unique morphism mapping to $\omega_i^{-1} \circ \tilde{\omega}_i$. Observe that by definition of $\mathbf{E}$, $\alpha(x_i)$ only depends on the choice of representative of $[x_i]$ up to composition with a simple equivalence. In particular, the verticals are independent of this choice. By Observation 12.1.0.17, the verticals of this square are bijections. To see that $l(e)$ is a bijection, it thus suffices to show, that Diagram (12.10) commutes. Indeed, as η was assumed to be simple, it follows by Lemma 9.1.3.11 that the inverse of η_* is given by η^*. Hence, it suffices to see that

$$\begin{tikzcd} \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_0}) \arrow[r] \arrow[d] & \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_1}) \arrow[d] \\ \mathrm{Wh}_{\mathbf{C}}(\tilde{\mathfrak{X}}_0) & \mathrm{Wh}_{\mathbf{C}}(\tilde{\mathfrak{X}}_1) \arrow[l, "\eta^*"'] \end{tikzcd} \tag{12.11}$$

commutes. For $[x_0]$ in $\mathbf{E}_{\mathfrak{Y}_0}$ and $[x_1]$ the target of $\tilde{e}_{[x_0]}$, η is constructed precisely so that, by the fully faithfulness of $\mathfrak{L}$, the associated diagram

$$\begin{tikzcd} \mathfrak{X}_0 \arrow[r, "\tilde{e}_{[x_0]}"] & \mathfrak{X}_1 \\ \tilde{\mathfrak{X}}_0 \arrow[r, "\eta"'] \arrow[u, "\alpha(x_0)"] & \tilde{\mathfrak{X}}_1 \arrow[u, "\alpha(x_1)"'] \end{tikzcd} \tag{12.12}$$

commutes. As $\tilde{e}_{[x_0]}$ was assumed to be a simple equivalence, this ensures the identity

$$\langle \alpha(x_0) \rangle = \langle \alpha(x_1) \circ \eta \rangle = \eta^* \langle [x_0] \rangle \,.$$

We have now shown that any expansion e is mapped by l into a bijection. Consequently, it follows that l descends to a functor on $\mathfrak{ho}_c\mathbf{D}$

$$\begin{tikzcd} \mathbf{C\ddot{e}ll}_c(D) \arrow[r, "l"] \arrow[d] & \mathbf{Set} \\ \mathfrak{ho}_c\mathbf{D}, \arrow[ur, dashed] & \end{tikzcd} \tag{12.13}$$

which we denote the same by abuse of notation. Now, recall that by Lemma 9.1.3.2, every morphism of $\mathfrak{ho}_c\mathbf{D}$ is of the form $e^{-1}a$, for $a \in \mathbf{C\ddot{e}ll}_c(D)$ and $e \in \mathbf{C\ddot{e}ll}_c(D)$ an expansion. Observe that then for any $[y_0]$ in the fiber of the source of e^{-1}, it follows that $l(e^{-1})[y_0] = l(e)^{-1}[y_0]$. Let $\tilde{\eta}_{l(e^{-1})[y_0]} \colon l(e)^{-1}[y_0] \to [y_0]$ be the unique lift of e. Then $\tilde{\eta}_{l(e^{-1})[y_0]}$ specifies the unique lift of e^{-1} from $[y_0] \to l(e)^{-1}[y_0]$. In particular, this lift is again a simple equivalence. Denote by $\tilde{\alpha}_{[x_0]} \colon [x_0] \to l(a)[x_0]$ the lift of a which exists by assumption. Then

$$\tilde{\eta}^{-1}_{l(e^{-1})l(a)[x_0]} \circ \tilde{\alpha}_{[x_0]}$$

specifies a morphism $[x_0] \to l(e^{-1})l(a)[x_0] = l(e^{-1}a)[x_0]$ which lifts $e^{-1} \circ a$. Finally, observe that if $\gamma \colon \mathfrak{Y}_0 \to \mathfrak{Y}_1 \in \mathfrak{ho}_c\mathbf{D}$ is a simple equivalence, then by Definition 9.1.3.8, we may write it in the form

$$e^{-1} \circ a$$

where a is also an expansion. Consequently, for any $[x_0] \in \mathbf{E}_{\mathfrak{Y}_0}$ the associated lift

$$\tilde{\eta}^{-1}_{l(e^{-1})l(a)[x_0]} \circ \tilde{\alpha}_{[x_0]}$$

is a composition of two simple equivalences. This finishes the proof that $l \colon \mathfrak{ho}_c\mathbf{D} \to \mathbf{Set}$ defines a coherent family of lifts. □

Let us now construct a functor as in Lemma 12.1.0.21. For notational reasons, it will be preferable to also use the notation $\overline{x} \in \mathbf{E}$, to refer to elements of $\mathbf{E}$, instead of only using equivalence class notation with brackets.

Construction 12.1.0.22. Suppose we are given the following data

1. A relative structured cell complex $\mathfrak{a} \colon A \hookrightarrow B$ in $\mathbf{RCell(C)}$, where A is cofibrant and filtration compact.
2. A morphism $f \colon L(A) \to Y$ in $\mathbf{D}$, where $\mathfrak{Y} \in \mathbf{C\ddot{e}ll}_c(D)$ is a finite structured cell complex.
3. An element $\overline{x} \in \mathbf{E}_{\mathfrak{Y}}$.

We will now construct a lift of the morphism

$$\mathfrak{Y} \hookrightarrow \mathfrak{Y} \cup_f L(B)$$

in $\mathfrak{ho}_c\mathbf{C}$ at $\overline{x}$ along p, which we denote $l(\mathfrak{a}, f, \overline{x})$. Observe that, as such a lift is uniquely determined by its target, we may as well construct an element $\overline{x}_1$ of $\mathbf{E}_{\mathfrak{Y} \cup_f L(B)}$, together with a morphism $\overline{x} \to \overline{x}_1$. We will be slightly sloppy in language insofar as we sometimes do not distinguish between the lift and its target. To construct such a lift, we make the following choices:

1. A fibrant replacement $e \colon \mathfrak{Y} \hookrightarrow \hat{\mathfrak{Y}}$ given by an expansion to an (infinite) structured cell complex $\hat{\mathfrak{Y}}$ (which exists by Lemma 10.2.2.2).

2. A choice of representative $x = (\mathfrak{X}, \omega\colon \mathfrak{L}(\mathfrak{X}) \to \mathfrak{Y})$, of $\overline{x}$.
3. A morphism $\tilde{f}\colon A \to X$ in $\mathbf{C}$, such that $\omega \circ L(\tilde{f}) = f$.
4. A representation of $e \circ \omega$ in terms of a morphism $w\colon L(X) \to \hat{Y}$.
5. A choice of homotopy $H\colon \Delta^1 \otimes L(A) \to \hat{Y}$ from $w \circ L(\tilde{f})$ to $e \circ f$.

Observe that such choices can indeed be made. Fixing any pair $(\mathfrak{X}', \omega') \in \overline{x}$, it follows from the assumption that L induces a fully faithful functor on homotopy categories that there exists a morphism $\alpha\colon A \to X$ such that $\omega \circ L(\alpha) = f$. Now, we may choose a fibrant replacement of $\mathfrak{X}'$, given by an expansion $\iota\colon \mathfrak{X}' \hookrightarrow \hat{\mathfrak{X}}$. Then, since A was assumed to be cofibrant, $\iota \circ \alpha$ can be represented by a morphism $\hat{f}\colon A \to \hat{X}$ in $\mathbf{C}$. As A was assumed to be filtration compact, it follows by Lemma 10.2.2.2 that $\hat{f}$ factors through a finite expansion $e\colon \mathfrak{X}' \hookrightarrow \mathfrak{X}$. Denote the resulting morphism $A \to X$ by $\tilde{f}$. Then, setting $\omega = \omega' \circ e^{-1}$, it follows that $x = (\mathfrak{X}, \omega)$ is as described above. Finally, since $\mathfrak{L}(\mathfrak{X})$ is an absolute cell complex and thus cofibrant, we may represent $e \circ \omega$ by w as above. Similarly, as by construction $w \circ L(\tilde{f}) = e \circ f$ in $\mathrm{ho}\mathbf{D}$, it follows again from the cofibrancy of $L(A)$ and the fibrancy of $\hat{Y}$ that a homotopy H as specified above exists. We now proceed analogously to how we did in the introduction of this proof, and define $\mathfrak{X}_1 = \mathfrak{X} \cup_{\tilde{f}} B$ and ω_1 as defined via the zig-zag of weak equivalences

$$L(\mathfrak{X} \cup_{\tilde{f}} B) \to \hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(B) \hookrightarrow \hat{\mathfrak{Y}} \cup_{e \circ L(\tilde{f})} \Delta^1 \otimes L(B) \hookleftarrow \hat{\mathfrak{Y}} \cup_{w \circ f} \{0\} \otimes L(B) \hookleftarrow \mathfrak{Y} \cup_f B \tag{12.14}$$

A priori, this defines a morphism in $\mathfrak{ho}D$. However, by Theorem 10.2.2.1, it can also be presented by a morphism in $\mathfrak{ho}_c\mathbf{D}$. To see that ω_1 is a simple equivalence, observe first that since A is filtration compact and cofibrant, $L(A)$ is also filtration compact. Indeed, under this assumption A can be exposed as a retract of a finite cell complex. Hence, $L(A)$ is also a retract of a finite cell complex and compactness of $L(A)$ follows by Lemmas 8.1.6.7 and 8.1.6.15. Consequently, the homotopy H factors through a finite subexpansion $\mathfrak{Y} \hookrightarrow \mathfrak{Y}' \hookrightarrow \hat{\mathfrak{Y}}$ of e. By abuse of notation, we will denote all resulting factorizations through Y' by the same name. We obtain a new diagram

$$\begin{tikzcd}
L(\mathfrak{X} \cup_{\tilde{f}} B) \arrow[r] \arrow[dr, hook] & \hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(B) \arrow[r, hook] & \hat{\mathfrak{Y}} \cup_H \Delta^1 \otimes L(B) & \hat{\mathfrak{Y}} \cup_{e \circ f} \{1\} \otimes L(B) \arrow[l, hook'] & \mathfrak{Y} \cup_f B \arrow[l, hook'] \arrow[dl, hook'] \\
& \mathfrak{Y}' \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(B) \arrow[r, hook] \arrow[u, hook, "\simeq"] & \mathfrak{Y}' \cup_H \Delta^1 \otimes L(B) \arrow[u, hook', "\simeq"] & \mathfrak{Y}' \cup_{e \circ f} \{1\} \otimes L(B) \arrow[l, hook'] \arrow[u, hook, "\simeq"] &
\end{tikzcd} \tag{12.15}$$

with verticals given by acyclic cofibrations. In particular, the upper horizontal composition (taking inverse wherever necessary) defines the same morphism in $\mathfrak{ho}_c\mathbf{D}$ as the lower composition. Hence it suffices to show that the bottom row is given by simple equivalences. The first morphism is a simple equivalence by Corollary 10.2.3.13. Similary, the remaining two morphisms are given by cobase changes of inclusions of the form $\hat{\mathfrak{i}}_i(\mathfrak{a})\colon \Delta^1 \otimes A \cup_{\{i\} \otimes A} \{i\} \otimes B$, and hence are simple equivalences by the simplicity of the cylinder $\Delta^1 \otimes -$. To summarize, we have shown that $x_1 = (\mathfrak{X} \cup_{\tilde{f}} B, \omega_1)$ defines an element of $\mathbf{E}_{\mathfrak{Y} \cup_f L(B)}$. Finally, a quick diagram chase shows that the diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathfrak{X}) \arrow[r, hook] \arrow[d, "\omega"'] & \mathfrak{L}(\mathfrak{X} \cup_{\tilde{f}} B) \arrow[d, "\omega_1"] \\
\mathfrak{Y} \arrow[r, hook] & \mathfrak{Y} \cup_f B
\end{tikzcd} \tag{12.16}$$

commutes in $\mathfrak{ho}_c\mathbf{D}$, which shows that there does indeed exists a lift of $\mathfrak{Y} \hookrightarrow \mathfrak{Y} \cup_f B$ from $\overline{x}$ to $[x_1]$. We thus define

$$l(\mathfrak{a}, f, \overline{x}, e, x, w, \tilde{f}, H) := [x_1]\,.$$

We will now need to see that this definition is independent from the choices $(e, x, w, \tilde{f}, H)$.

Now, let us move on with the proof of Theorem 12.1.0.4, and show independence of the construction in Construction 12.1.0.22 from the list of choices we made.

Lemma 12.1.0.23. *Suppose that we are given choices $(e, x, w, \tilde{f}, H)$ as in Construction 12.1.0.22. Suppose, furthermore, we consider a finite expansion $\hat{e}: \mathfrak{X} \hookrightarrow \hat{\mathfrak{X}}$, and denote by $\hat{\omega}$ a weak equivalence making the diagram*

$$\begin{tikzcd} L(X) \arrow[r, hook, "L(\hat{e})"] \arrow[d, "w"'] & L(\hat{X}) \arrow[dl, dashed, "\hat{w}"] \\ \hat{Y} & \end{tikzcd} \tag{12.17}$$

commute. (This morphism exists because the upper horizontal is an acyclic cofibration between cofibrant objects and $\hat{Y}$ is fibrant.) Denote $\hat{x} = (\hat{\mathfrak{X}}, \omega \circ L(\hat{e})^{-1})$. Then

$$l(\mathfrak{a}, f, \overline{x}, e, x, w, \tilde{f}, H) = l(\mathfrak{a}, f, \overline{x}, e, \hat{x}, \hat{w}, \hat{e} \circ \tilde{f}, H)\,.$$

Proof. Observe that the diagram

$$\begin{tikzcd} L(\mathfrak{X} \cup_{\tilde{f}} B) \arrow[r] \arrow[d] & \hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(B) \arrow[d, equal] \\ L(\hat{\mathfrak{X}} \cup_{\hat{e} \circ \tilde{f}} B) \arrow[r] & \hat{\mathfrak{Y}} \cup_{\hat{w} \circ L(\hat{e} \circ \tilde{f})} \{0\} \otimes L(B) \end{tikzcd} \tag{12.18}$$

commutes. It follows that if we denote by $\hat{\omega}_1$ the isomorphism $\mathfrak{L}(\hat{\mathfrak{X}}) \to \mathfrak{Y} \cup_f B$ associated to $((\mathfrak{a}, f, \overline{x}, e, \hat{x}, \hat{w}, \hat{e} \circ \tilde{f}, H))$, then the diagram

$$\begin{tikzcd} \mathfrak{L}(\mathfrak{X} \cup_{\tilde{f}} B) \arrow[rr] \arrow[dr, "\omega_1"'] & & \mathfrak{L}(\hat{\mathfrak{X}} \cup_{\hat{e} \circ \tilde{f}} B) \arrow[dl, "\hat{\omega}_1"] \\ & \mathfrak{Y} \cup_f L(B) & \end{tikzcd} \tag{12.19}$$

in $\mathfrak{hoD}$ commutes. The morphism $\mathfrak{X} \cup_{\tilde{f}} B \to \hat{\mathfrak{X}} \cup_{\hat{e} \circ \tilde{f}} B$ is the cobase change of an expansion, and thus a simple equivalence. Consequently, $(\mathfrak{X} \cup_{\tilde{f}} B, \omega_1)$ and $(\hat{\mathfrak{X}} \cup_{\hat{e} \circ \tilde{f}} B, \hat{\omega}_1)$ define the same element in $\mathbf{E}$. □

This allows us to always increase $\mathfrak{X}$ by finite expansions, without changing the associated class in $\mathbf{E}$.

Lemma 12.1.0.24. *Provided one has made a choice of x such that a $\tilde{f}$ as described exists, then the construction in Construction 12.1.0.22 is independent of the choice of $\tilde{f}$ and H.*

Proof. So suppose we are given $(\tilde{f}, H)$ and $(\tilde{f}', H')$ as above and denote the associated elements of $\mathbf{P}_{\mathfrak{L},s}$ by $x_1 = (\mathfrak{X}_1, \omega_1)$ and $x_1' = (\mathfrak{X}_1', \omega_1')$. We may glue the homotopies H and H' at $e \circ f$ to obtain a morphism $\Lambda^2_2 \otimes L(A) \to \hat{Y}$. Now, using fibrancy of $\hat{Y}$ and cofibrancy of $L(A)$, let $R: \Delta^2 \otimes L(A) \to Y$ be a filler to this morphism, and consider its restriction to the face opposite to 2, $H'': \Delta^1 \otimes L(A) \to \hat{Y}$. H'' provides a homotopy between $w \circ L(\tilde{f})$ and $w \circ L(\tilde{f}')$. Denote by $\hat{e}: \mathfrak{X} \hookrightarrow \hat{\mathfrak{X}}$ a fibrant replacement of X through a transfinite composition of elementary expansions. Denote by $\hat{w}$ an expansion of w to $L(\hat{X})$. Now, consider the following associated (solid) commutative diagram.

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "(\hat{e}\tilde{f}, \hat{e}\tilde{f}')"] \arrow[dd] & \underline{\mathbf{C}}(A, \hat{X}) \arrow[d, "L"] \\ & \underline{\mathbf{D}}(L(A), L(\hat{X})) \arrow[d, "\hat{w} \circ -"] \\ \Delta^1 \arrow[r, "H''"] \arrow[uur, dashed, "\tilde{H}"] & \underline{\mathbf{D}}(L(A), \hat{Y}) \end{tikzcd} \tag{12.20}$$

The composition

$$\underline{\mathbf{C}}(A,\hat{X}) \xrightarrow{L} \underline{\mathbf{D}}(L(A),L(\hat{X})) \xrightarrow{\hat{w}\circ -} \underline{\mathbf{D}}(L(A),\hat{Y})$$

is a weak homotopy equivalence. To see this, observe that $\hat{\omega}\colon L(\hat{X}) \to \hat{Y}$ defines a fibrant replacement of $L(\hat{X})$ in $\mathbf{D}$. In particular, as A is cofibrant and $\hat{X}$ is fibrant, it follows that the composition in question computes the map induced by L on derived mapping spaces, i.e., on the mapping spaces in the ∞-categorical sense (see, for example, [GH05] for the semi-model category case). By the assumption that L induces a fully faithful functor of ∞-categories, this map is a weak homotopy equivalence.
It is a classical fact that the right vertical composition in Diagram (12.20) being a homotopy equivalence of Kan-complexes implies that the diagram admits a diagonal, making the upper left triangle commute on the nose, and the lower right triangle commute up to homotopy relative to $\partial\Delta^1$ (see, for example, [Vog10]). In other words, we obtain a homotopy $\tilde{H}\colon \Delta^1 \otimes A \to \hat{X}$ between $\tilde{f}$ and $\tilde{f}'$ as well as a homotopy $R'\colon \Delta^1 \otimes (\Delta^1 \otimes L(A)) \to \hat{Y}$ between $\hat{w}\circ L(\tilde{H})$ and H'', relative to $\partial\Delta^1 \otimes L(A)$. Using a compactness argument, as we have now done several times in this section, we may again factor $\tilde{H}$ through a finite subexpansion of $\mathfrak{X} \hookrightarrow \hat{\mathfrak{X}}$ and hence assume that $\hat{\mathfrak{X}}$ is finite. Furthermore, using Lemma 12.1.0.23, we may then assume without loss of generality that $\hat{X} = X$. Now, consider the morphism specified by

$$\mathfrak{X} \cup_{\tilde{f}} B \hookrightarrow \mathfrak{X} \cup_{\tilde{H}} (\Delta^1 \otimes B) \hookleftarrow \mathfrak{X} \cup_{\tilde{f}'} B$$

in $\mathfrak{ho}_c\mathbf{C}$. As a cobase change of a simple equivalence, both morphisms are given by simple equivalences. We claim that it maps to $(\omega_1')^{-1} \circ \omega_1$, under L, which shows that $[x_1] = [x_1']$ in $\mathbf{E}$. Indeed, this follows from the commutative diagram in $\mathbf{C\overline{e}ll}(D)$ of weak equivalences in $\mathbf{D}$. We will use notation such as $\{1\}$ for Δ^0 and $\Delta^{\{1,2\}}$ for Δ^1 here, to indicate how we consider the latter as embedded in potential larger simplices such as Δ^2.

$$\begin{tikzcd}
\mathfrak{L}(\mathfrak{X} \cup_{\tilde{f}} \{0\} \otimes \{0\} \otimes B) \arrow[r, hook] \arrow[d] & \mathfrak{L}(\mathfrak{X} \cup_{\tilde{H}} \{0\} \otimes \Delta^1 \otimes B) \arrow[d] & L(\mathfrak{X} \cup_{\tilde{f}'} \{0\} \otimes \{1\} \otimes B) \arrow[l, hook'] \arrow[d] \\
\hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f})} \{0\} \otimes \{0\} \otimes L(B) \arrow[r, hook] \arrow[d, hook] & \hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{H})} \{0\} \otimes \Delta^1 \otimes L(B) \arrow[d, hook] & \hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f}')} \{0\} \otimes \{1\} \otimes L(B) \arrow[l, hook'] \arrow[d, hook] \\
\hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f})} \Delta^1 \otimes \{0\} \otimes L(B) \arrow[r, hook] & \hat{\mathfrak{Y}} \cup_{R'} \Delta^1 \otimes \Delta^1 \otimes L(B) & \hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f}')} \Delta^1 \otimes \{1\} \otimes L(B) \arrow[l, hook'] \\
\hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f})} \{1\} \otimes \{0\} \otimes L(B) \arrow[r, hook] \arrow[u, hook] \arrow[d, equal] & \hat{\mathfrak{Y}} \cup_{H''} \{1\} \otimes \Delta^1 \otimes L(B) \arrow[u, hook] \arrow[d, equal] & \hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f}')} \{1\} \otimes \{1\} \otimes L(B) \arrow[l, hook'] \arrow[u, hook] \arrow[d, equal] \\
\hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f})} \{0\} \otimes L(B) \arrow[r, hook] \arrow[d, hook] & \hat{\mathfrak{Y}} \cup_{H''} \Delta^{\{0,1\}} \otimes L(B) \arrow[d, hook] & \hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f}')} \{1\} \otimes L(B) \arrow[l, hook'] \arrow[d, hook] \\
\hat{\mathfrak{Y}} \cup_{H} \Delta^{\{0,1\}} \otimes L(B) \arrow[r, hook] & \hat{\mathfrak{Y}} \cup_{R} \Delta^{2} \otimes L(B) & \hat{\mathfrak{Y}} \cup_{H'} \Delta^{\{1,2\}} \otimes L(B) \arrow[l, hook'] \\
& \hat{\mathfrak{Y}} \cup_f \{2\} \otimes L(B) \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] & \\
& \mathfrak{Y} \cup_f L(B) \arrow[u, hook, "\simeq"] &
\end{tikzcd} \tag{12.21}$$

Passing to $\mathfrak{ho}D$, note first that the composition in the left vertical (inverting arrows appropriately)

$$\hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f})} \{0\} \otimes \{0\} \otimes L(B) \hookrightarrow \hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f})} \Delta^1 \otimes \{0\} \otimes L(B) \hookleftarrow \hat{\mathfrak{Y}} \cup_{w\circ L(\tilde{f})} \{1\} \otimes \{0\} \otimes L(B)$$

is the identity , and the analogous result at the right hand side holds. It follows from this that the composition following the left vertical, and finally going diagonally to the right and then down is ω_1, and one similarly obtains ω_1' on the right hand side. Hence, it follows from commutativity of the diagram that the top horizontal composition specifies $(\omega_1')^{-1} \circ \omega_1$. $\square$

Given this new information, we will just write

$$l(\mathfrak{a}, f, \overline{x}, e, x, w)$$

from here on out.

Lemma 12.1.0.25. *The construction in Construction 12.1.0.22 is independent of w.*

Proof. Let G be a homotopy from w to w'. Glue H and $G \circ (\Delta^1 \otimes \tilde{f})$ along $e \circ f$ to obtain a horn $\Lambda^2_1 \otimes L(A) \to \hat{Y}$, and denote by $R\colon \Delta^2 \otimes L(A) \to \hat{Y}$ a filler to this horn, and by H' its face opposing the 1-vertex. Next, consider the commutative diagram of weak equivalences

$$\begin{tikzcd}
\mathfrak{L}(\mathfrak{X} \cup_{\tilde{f}} \{1\} \otimes B) \arrow[r, hook] \arrow[d] & \mathfrak{L}(\Delta^1 \otimes (\mathfrak{X} \cup_f B)) \arrow[d, "{(G, G \circ (\tilde{f} \otimes \Delta^1))}"] & L(\mathfrak{X} \cup_{\tilde{f}} \{2\} \otimes B) \arrow[l, hook'] \arrow[d] \\
\hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f})} L(\{1\} \otimes B) \arrow[r, hook] \arrow[d, hook] & \hat{\mathfrak{Y}} \cup_{G \circ \tilde{f} \otimes \Delta^1} (\Delta^{\{1,2\}} \otimes L(B)) \arrow[d, hook] & \hat{\mathfrak{Y}} \cup_{w' \circ L(\tilde{f})} \{2\} \otimes L(B) \arrow[l, hook'] \arrow[d, hook] \\
\hat{\mathfrak{Y}} \cup_H (\Delta^{\{0,1\}} \otimes L(B)) \arrow[r, hook] & \hat{\mathfrak{Y}} \cup_R (\Delta^2 \otimes L(B)) & \hat{\mathfrak{Y}} \cup_{H'} (\Delta^{\{0,2\}} \otimes L(B)) \arrow[l, hook'] \\
 & \hat{\mathfrak{Y}} \cup_f \{0\} \otimes L(B) \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] & \\
 & \mathfrak{Y} \cup_f B \arrow[u, hook] &
\end{tikzcd} \tag{12.22}$$

in $\mathbf{C\overline{e}ll(D)}$. Observe that the path from the top, going down all the way on the left defines ω_1. The upper horizontal defines a simple equivalence (the identity even). The path from the top right, following the right all the way down is ω_1' (constructed as in Construction 12.1.0.22), for the choices $(e, x, w', H', \tilde{f})$. Hence, it follows by the invariance of choice of homotopy in Lemma 12.1.0.24 that

$$l(\mathfrak{a}, f, \overline{x}, e, x, w) = l(\mathfrak{a}, f, \overline{x}, e, x, w')\,.$$

$\square$

Using this insight, we will just write

$$l(\mathfrak{a}, f, \overline{x}, e, x)$$

from here on out. Next, let us prove independence of the choice of x.

Lemma 12.1.0.26. *Construction 12.1.0.22, is independent of the choice of x.*

Proof. Suppose we are given two representatives $x, x' \in \overline{x}$. In particular, this means that the unique lift of ${\omega_1'}^{-1} \circ \omega_1$ to a morphism $\mathfrak{X} \to \mathfrak{X}'$ in $\mathfrak{ho}_c\mathbf{C}$ is a simple equivalence. Increasing $\mathfrak{X}$ by an expansion, if necessary, which by Lemma 12.1.0.23 does not change $l(f, \mathfrak{a}, \overline{x}, e, x)$ it follows from Definition 9.1.3.8, that we may assume that $\mathfrak{X} \to \mathfrak{X}'$ is presented by an expansion $\mathfrak{X} \hookrightarrow \mathfrak{X}'$. Now, the claim follows from Lemmas 12.1.0.23 to 12.1.0.25. $\square$

Finally, let us show independence of the choice of e.

Lemma 12.1.0.27. *Construction 12.1.0.22 is independent of the choice of e.*

Proof. Observe that any two such fibrant replacements $e_i \colon \mathfrak{X} \hookrightarrow \hat{\mathfrak{Y}}_i$, $i = 1, 2$, fit into a commutative diagram

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, hook] \arrow[d, hook] & \hat{\mathfrak{Y}}_1 \arrow[d, hook] \\ \hat{\mathfrak{Y}}_2 \arrow[r, hook] & \hat{\mathfrak{Y}} \end{tikzcd} \tag{12.23}$$

in $\mathbf{C\bar{\bar{e}}ll}(\mathbf{C})$ with all morphisms given by (transfinite expansions). To see this, simply take the pushout of e_1 and e_2 (which exists by Corollary 8.1.4.8) and replace the latter fibrantly. It thus follows that we only need to consider the case where e_2 is obtained from e_1 by composing with an expansion e'. Then, we obtain a commutative diagram

$$\begin{tikzcd}[column sep=small] L(\mathfrak{X} \cup_{\tilde{f}} B) \arrow[r] \arrow[dr, hook] & \hat{\mathfrak{Y}}_2 \cup_{e' \circ w \circ L(\tilde{f})} \{0\} \otimes L(B) \arrow[r, hook] & \hat{\mathfrak{Y}}_2 \cup_{e' \circ H} \Delta^1 \otimes L(B) & \hat{\mathfrak{Y}}_2 \cup_{e' \circ e \circ f} \{1\} \otimes L(B) \arrow[l, hook'] & \mathfrak{Y} \cup_f B \arrow[l, hook'] \arrow[dl, hook] \\ & \hat{\mathfrak{Y}}_1 \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(B) \arrow[r, hook] \arrow[u, hook, "\simeq"] & \hat{\mathfrak{Y}}_1 \cup_H \Delta^1 \otimes L(B) \arrow[u, hook, "\simeq"] & \hat{\mathfrak{Y}}_1 \cup_{e \circ f} \{1\} \otimes L(B) \arrow[l, hook'] \arrow[u, hook, "\simeq"] & \end{tikzcd} \tag{12.24}$$

with verticals given by acyclic cofibrations. The upper horizontal defines $l(\mathfrak{a}, f, \overline{x}, e_1)$ and the path along the lower horizontal defines $l(\mathfrak{a}, f, \overline{x}, e_2)$. In particular, using the fact that all verticals define isomorphisms in $\mathfrak{ho}\mathbf{D}$, it follows by commutativity of the diagram that $l(\mathfrak{a}, f, \overline{x}, e_1) = l(\mathfrak{a}, f, \overline{x}, e_2)$. ☐

This finishes our proof of the independence of the construction of l from choices. We are furthermore going to need the following lemmata to obtain functoriality.

Lemma 12.1.0.28. *Suppose we are given a structured relative cell complex $\mathfrak{a} \colon A \hookrightarrow B$ in $\mathbf{RCell}(\mathbf{C})$ and a morphism $g \colon A \to A'$, such that A and A' both fulfill the requirements of Construction 12.1.0.22. Suppose, furthermore, we are given a morphism $f \colon A' \to Y$, with $\mathfrak{Y} \in \mathbf{C\bar{\bar{e}}ll}_c(D)$. Consider the diagram of pushout squares*

$$\begin{tikzcd} L(A) \arrow[r, hook, "L(a)"] \arrow[d, "L(g)"'] & L(B) \arrow[d] \\ L(A') \arrow[r, hook, "L(b)"'] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & L(B') \arrow[d] \\ Y \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & Y \cup_{f \circ L(g)} L(B) \end{tikzcd} \tag{12.25}$$

which identifies $\mathfrak{Y} \cup_{f \circ L(g)} L(B)$ with $\mathfrak{Y} \cup_f L(B')$. Then, for any $\overline{x} \in \mathbf{E}_{\mathfrak{Y}}$ the equality

$$l(\mathfrak{a}, f \circ L(g), \overline{x}) = l(g_{\text{!`}}\mathfrak{a}, f, \overline{x})$$

holds.

Proof. Choose $e, w, \tilde{f}, H$ associated to $\mathfrak{a}$, f, $\overline{x}$ as in Construction 12.1.0.22. We then obtain a commutative diagram in $\mathbf{C\bar{\bar{e}}ll}(\mathbf{D})$

$$\begin{tikzcd}[column sep=small] L(\mathfrak{X} \cup_{\tilde{f} \circ g} B) \arrow[r] \arrow[d, "\cong"] & \hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f} \circ g)} \{0\} \otimes L(B) \arrow[r] \arrow[d, "\cong"'] & \hat{\mathfrak{Y}} \cup_{H \circ (\Delta^1 \otimes g)} \Delta^1 \otimes L(B) \arrow[d, "\cong"'] & & \\ L(\mathfrak{X} \cup_{\tilde{f}} B') \arrow[r] & \hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(B') \arrow[r, hook] & \hat{\mathfrak{Y}} \cup_H \Delta^1 \otimes L(B') & \hat{\mathfrak{Y}} \cup_{e \circ f} \{1\} \otimes L(B') \arrow[l, hook'] & \mathfrak{Y} \cup_f B' \arrow[l, hook'] \arrow[ull, hook'] \end{tikzcd} \tag{12.26}$$

with all verticals isomorphisms. The left vertical is induced by the canonical isomorphism $\mathfrak{X} \cup_{\tilde{f} \circ g} B \cong \mathfrak{X} \cup_{\tilde{f}} B'$ in $\mathbf{C\bar{\bar{e}}ll}_c(\mathbf{C})$ (stemming from functoriality of cobase change). In particular, it defines a simple equivalence. The upper composition defines a representative of $l(\mathfrak{a}, f \circ L(g), \overline{x})$. The lower composition defines a representative of $l(g_{\text{!`}}\mathfrak{a}, f, \overline{x})$. As the two representatives only differ by a simple equivalence in $(\mathbf{P}_{\mathfrak{L},s})_{\mathfrak{Y}}$, the two classes agree. ☐

Lemma 12.1.0.29. *Suppose we are given two finite structured cell complexes $\mathfrak{a}\colon A \to B \in$ $\mathbf{RCell}(C)$ into two relative cell complexes $\mathfrak{a}_1, \mathfrak{a}_2$, $\mathfrak{a} = \mathfrak{a}_2 \circ \mathfrak{a}_1$, where A is as in Construction 12.1.0.22. Suppose furthermore, we are given a morphism $f\colon A \to Y$, for a finite structured cell complex $\mathfrak{Y}$. Consider the diagram of pushout squares*

$$\begin{tikzcd}
L(A) \arrow[r, hook, "L(a_1)"] \arrow[d, "f"] & L(A') \arrow[r, hook, "L(a_2)"] \arrow[d, "f'"'] & L(B) \arrow[d] & \\
Y \arrow[r, hook] & Y \cup_f L(A') \arrow[r, hook] & (Y \cup_f L(A')) \cup_{f'} L(B) \arrow[r, equal] & Y \cup_f L(B)
\end{tikzcd} \tag{12.27}$$

in $\mathbf{D}$, which equips the inclusion of subcomplexes $\mathfrak{Y} \hookrightarrow \mathfrak{Y} \cup_f L(B)$ with a factorization

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r, hook] \arrow[rr, hook, bend right] & \mathfrak{Y} \cup_f L(A') \arrow[r, hook] & \mathfrak{Y} \cup_f L(B)
\end{tikzcd} \tag{12.28}$$

in $\ddot{\mathbf{Cell}}_c(D)$. Then, for any $\overline{x} \in \mathbf{E}_{\mathfrak{Y}}$, the equality

$$l(\mathfrak{a}, f, \overline{x}) = l(\mathfrak{a}_2, f', l(\mathfrak{a}_1, f, \overline{x}))$$

holds.

Proof. Choose $e, x, w, \tilde{f}, H$ for $\mathfrak{a}, f, \overline{x}$ as in Construction 12.1.0.22. Observe that this is also a compatible family of choices for $\mathfrak{a}_1, f$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathfrak{X} \cup_{\tilde{f}} A') \arrow[r, hook] \arrow[d] & \mathfrak{L}(\mathfrak{X} \cup_{\tilde{f}} B) \arrow[d] \\
\hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(A') \arrow[r, hook] \arrow[d, hook] & \hat{\mathfrak{Y}} \cup_{w \circ L(\tilde{f})} \{0\} \otimes L(B) \arrow[d, hook] \\
\hat{\mathfrak{Y}} \cup_H \Delta^1 \otimes L(A') \arrow[r] & \hat{\mathfrak{Y}} \cup_H \Delta^1 \otimes L(A') \\
\hat{\mathfrak{Y}} \cup_{e \circ f} \{1\} \otimes L(A') \arrow[r] \arrow[u, hook] & \hat{\mathfrak{Y}} \cup_{e \circ f} \{1\} \otimes L(B) \arrow[u, hook] \\
\mathfrak{Y} \cup_f L(A') \arrow[r] \arrow[u, hook] & \mathfrak{Y} \cup_f L(B) \arrow[u, hook]
\end{tikzcd} \tag{12.29}$$

with the verticals specifying simple equivalences in $\mathfrak{ho}_c\mathbf{D}$. We need to show that if we apply $l(\mathfrak{a}_2, f', -)$ to the element of $\mathbf{E}$ specified by the left vertical that this produces the morphism specified by the right vertical. We will denote all associated choices as in Construction 12.1.0.22 in the form $-'$. Choose a fibrant replacement by an expansion $e'\colon \mathfrak{Y} \cup_f L(A') \hookrightarrow \hat{\mathfrak{Y}}'$. Now, consider the solid diagram

$$
\begin{tikzcd}
\mathfrak{L}(\mathfrak{X}\cup_{\tilde f} A') \arrow[d, "g"'] & \\
\hat{\mathfrak{Y}} \cup_{w\circ L(\tilde f)} \{0\}\otimes L(A') \arrow[d, hook, "j"'] & \\
\hat{\mathfrak{Y}} \cup_H \Delta^1 \otimes L(A') \arrow[ddr, dashed] & \\
\hat{\mathfrak{Y}} \cup_{e\circ f} \{1\}\otimes L(A') \arrow[u, hook] & \\
\mathfrak{Y}\cup_f L(A') \arrow[u, hook] \arrow[r, hook] & \hat{\mathfrak{Y}}'
\end{tikzcd}
\tag{12.30}
$$

As all upward pointing vertical morphisms are acyclic cofibrations between cofibrant objects, it follows that a dashed arrow as indicated exists, which makes the diagram commute. We may hence choose w' (representing the left vertical) as the composition all the way from the top to $\hat{\mathfrak{Y}}'$. Notably, this leaves us with a canonical choice for $\tilde{f}'$ and H'. Namely, take $\tilde{f}'$ as

$$A' \to X \cup_{\tilde f} A'$$

and the homotopy H' from $e'\circ f'$ to $w'\circ L(\tilde{f}')$ as the composition

$$\Delta^1 \otimes L(A') \to \hat{\mathfrak{Y}} \cup_H \Delta^1 \otimes L(A') \to \hat{\mathfrak{Y}}'\,.$$

With these choices, we obtain a commutative diagram in $\mathbf{C}$:

$$
\begin{tikzcd}
\mathfrak{L}((\mathfrak{X}\cup_{\tilde f} A')\cup_{\tilde f'} B) \arrow[r, equal] \arrow[d] & \mathfrak{L}((\mathfrak{X}\cup_{\tilde f} A')\cup_{\tilde f'} B) \arrow[r, equal] \arrow[d] & \mathfrak{L}(\mathfrak{X}\cup_{\tilde f} B) \arrow[d] \\
\hat{\mathfrak{Y}}'\cup_{w'\circ L(\tilde f')}\{0\}\otimes L(B) \arrow[d, hook] & (\hat{\mathfrak{Y}}\cup_{w\circ L(\tilde f)}\{0\}\otimes L(A'))\cup_{\{0\}\otimes L(A')}\{0\}\otimes L(B) \arrow[l] \arrow[r, equal] \arrow[d, hook] & \hat{\mathfrak{Y}}\cup_{w\circ L(\tilde f)}\{0\}\otimes L(B) \arrow[d, hook] \\
\hat{\mathfrak{Y}}'\cup_{H'}\Delta^1\otimes L(B) & (\hat{\mathfrak{Y}}\cup_H\Delta^1\otimes L(A'))\cup_{\Delta^1\otimes L(A')}\Delta^1\otimes L(B) \arrow[l] \arrow[r, equal] & \hat{\mathfrak{Y}}\cup_H\Delta^1\otimes L(B) \\
\hat{\mathfrak{Y}}'\cup_{e'\circ f'}\{1\}\otimes L(B) \arrow[u, hook] & (\hat{\mathfrak{Y}}\cup_f\{1\}\otimes L(A))\cup_{f'}\{1\}\otimes L(B) \arrow[l] \arrow[r, equal] \arrow[u, hook] & \hat{\mathfrak{Y}}\cup_{e\circ f} L(B) \arrow[u, hook] \\
(\mathfrak{Y}\cup_f L(A))\cup_{f'} L(B) \arrow[r, equal] \arrow[u, hook] & (\mathfrak{Y}\cup_f L(A))\cup_{f'} L(B) \arrow[r, equal] \arrow[u, hook] & \mathfrak{Y}\cup_f L(B) \arrow[u, hook]
\end{tikzcd}
\tag{12.31}
$$

A diagram chase using the stability of acyclic cofibrations with cofibrant source under cobase change shows that all horizontals in this diagram are acyclic cofibrations. Passing to $\mathfrak{ho}\mathbf{D}$, the left vertical defines $l(\mathfrak{a}_2, f', l(\mathfrak{a}_1, f, \overline{x}))$, and the right vertical defines $l(\mathfrak{a}, f, \overline{x})$. By commutativity of this diagram, these two arrows agree (up to canonical isomorphism of cobase changes in $\mathbf{C\breve{e}ll}_c(\mathbf{C})$), which finishes the proof. ☐

We may now finish the proof of the theorem.

Proof of Theorem 12.1.0.4. By Proposition 9.2.0.7, it suffices to show that $\widetilde{\mathrm{Wh}}_{\mathfrak{F}}$ is an isomorphism. By Lemma 12.1.0.20, we may furthermore reduce to constructing a coherent family of lifts, such that the additional requirement of Lemma 12.1.0.20 holds. By Lemma 12.1.0.21, it suffices to construct a functor

$$l\colon \mathbf{C\breve{e}ll}_c(\mathbf{D}) \to \mathbf{Set}$$

as in Lemma 12.1.0.21, which has the property that, for every morphism $a\colon \mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1$, the unique morphism in $\mathbf{E}$ given by lifting $\mathfrak{L}(a)$ to a morphism $[\rho(\mathfrak{X}_0)] \to l(a)[\rho(\mathfrak{X}_0)]$ is given by $\rho(a)$. Let us now define such a functor. On objects, l is already uniquely specified by $l(\mathfrak{Y}) = \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}})$.
Now, on morphisms we use Lemma 10.1.1.5. Recall that Lemma 10.1.1.5 states that the category $\mathbf{C\ddot{e}ll}_c(\mathbf{D})$ is the free category on the elementary inclusions (inclusions of subcomplexes with one or zero cells) subject to certain relations arising from changing the order in which cells are glued in. Hence, it suffices to define l on elementary inclusions, and then verify compatibility with these exchange relations.
Let $c\colon \mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}_1$ be an elementary inclusion. By definition, the associated relative cell complex $\mathfrak{c}$ is given by a cobase change of a relative cell complex $L(\mathfrak{a})\colon L(A) \to L(B)$, where either $\mathfrak{a} \in \mathbb{B}_{\mathbf{C}}$, or $A = B = \varnothing$. In particular, there is a canonical morphism $\sigma\colon L(B) \to Y_1$ (given either by the initial morphism) or by the unique cell of $\mathfrak{c}$, such that $L(A) \to L(B) \xrightarrow{\sigma} Y_1$ factors through Y_0, and such that the diagram

$$\begin{tikzcd} L(A) \arrow[r, dashed, "f"] \arrow[d, hook, "\mathfrak{a}"] & Y_0 \arrow[d, hook, "\mathfrak{c}"] \\ L(B) \arrow[r, "\sigma"'] & Y_1 \end{tikzcd} \tag{12.32}$$

is a cobase change. That is, we may write c as $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}_0 \cup_f L(B)$. Now set

$$\begin{aligned} l(c)\colon \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_0}) &\to \mathrm{Ob}(\mathbf{E}_{\mathfrak{Y}_1}) \\ \overline{x} &\mapsto l(\mathfrak{a}, f, \overline{x})\,. \end{aligned}$$

Supposing we have shown that this assignment descends to a well-defined functor on $\mathbf{C\ddot{e}ll}_c(\mathbf{D})$ – i.e., that we have compatibility with the relations in Lemma 10.1.1.5 – it follows from Construction 12.1.0.22 that the existence of lift condition of Lemma 12.1.0.21 holds. Let us now verify compatibility with the relations of Lemma 10.1.1.5. Clearly, in case c is the identity, then $l(\mathfrak{a}, f, \overline{x}) = \overline{x}$ (by making appropriate trivial choices in the construction of $l(\mathfrak{a}, f, \overline{x})$). Now, to see compatibility with the exchange relations, we are going to use the following stronger statement, which is an immediate consequence of an inductive application of Lemma 12.1.0.29. Suppose that $A \overset{\mathfrak{a}}{\hookrightarrow} B \in \mathbf{RCell}(\mathbf{C})$ is as in Construction 12.1.0.22. Choosing an appropriate filtration-presentation of $\mathfrak{a}$, we may write $\mathfrak{a}$ as a vertical composition

$$\mathfrak{a}_n \circ \cdots \circ \mathfrak{a}_o$$

of cobase changes of boundary inclusions b_i, $i > 0$ and possibly an isomorphism $\mathfrak{a}_o$. Given a morphism $f\colon L(A) \to \mathfrak{Y}$ in $\mathbf{D}$, we obtain an induced diagram of cobase changes

$$
\begin{tikzcd}
\varnothing \arrow[d, "\mathfrak{L}(\mathfrak{b}_0)"'] \arrow[dr] & & \\
\varnothing \arrow[dr] & L(A) \arrow[r, "f"] \arrow[d, "\mathfrak{L}(\mathfrak{a}_0)"', "\cong"] & Y \arrow[d, hook, "\cong"] \\
L(\partial D_1) \arrow[r, "L(g_1)"'] \arrow[d, hook, "\mathfrak{L}(\mathfrak{b}_1)"'] & L(B^0) \arrow[r, "f_1"'] \arrow[d, hook, "\mathfrak{L}(\mathfrak{a}_1)"'] & Y' \arrow[d, hook] \\
D_1 \arrow[r] & L(B^1) \arrow[r, "f_2"'] \arrow[d, hook] & Y \cup_{f_1 \circ L(g_1)} L(D^0) \arrow[d, hook] \\
\dots \arrow[r] & \dots \arrow[r] \arrow[d, hook] & \dots \arrow[d, hook] \\
\partial D_n \arrow[r] \arrow[d, "\mathfrak{L}(\mathfrak{b}_n)"'] & L(B^n) \arrow[r, "f_n"] \arrow[d, "\mathfrak{L}(\mathfrak{a}_n)"'] & Y \cup_{f_n \circ L(g_n)} B \arrow[d, hook] \\
D_n \arrow[r] & L(B^{n+1}) \arrow[r] & Y \cup_f B
\end{tikzcd}
\tag{12.33}
$$

Under these conditions, it follows from Lemma 12.1.0.28 and an inductive application of Lemma 12.1.0.29 that

$$l(\mathfrak{a}, f, \overline{x}) = \big(l(\mathfrak{b}_n, f_n \circ L(g_n), -) \circ \dots \circ l(\mathfrak{b}_1, f_1 \circ L(g_1), -) \circ l(\mathfrak{b}_0, f, -)\big)(\overline{x})\,. \tag{12.34}$$

Observe that it follows from this that the definition of $l\colon \mathbf{C\vec{e}ll}_c(\mathbf{C}) \to \mathbf{Set}$ as above does compatibly extend from elementary inclusions to any morphism $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}_1$ given by a cobase change as in Construction 12.1.0.22. Now, suppose we are given a pushout diagram

$$
\begin{tikzcd}
\mathfrak{Y}_0 \arrow[r, hook, "c_1"] \arrow[d, hook, "c_2"'] & \mathfrak{Y}_2 \arrow[d, hook, "c_2'"] \\
\mathfrak{Y}_1 \arrow[r, hook, "c_1'"'] \arrow[ur, phantom, "\ulcorner"] & \mathfrak{Y}
\end{tikzcd}
\tag{12.35}
$$

in $\mathbf{Cell(D)}$ with c_1 and c_2 given by elementary inclusions. Suppose that c_1 is given by a cobasechange of $L(\mathfrak{b}_1)$ along $f_1\colon L(A_1) \to Y_0$, for a structured relative cell complex $\mathfrak{b}_1\colon A_1 \hookrightarrow B_1$, and analogously c_2 is given a cobasechange of $L(\mathfrak{b}_2)$ along $f_1\colon L(A_2) \to Y_0$, for a structured relative cell complex $\mathfrak{b}_2\colon A_2 \hookrightarrow B_2$. Then it follows from Eq. (12.34) that

$$l(c_2') \circ l(c_1) = l(\mathfrak{b}_1 \sqcup \mathfrak{b}_2, (f_1, f_2), -) = l(c_2) \circ l(c_1')$$

which shows compatibility with the relations of Lemma 10.1.1.5. To summarize, we have constructed a well defined functor l on $\mathbf{C\vec{e}ll}_c(\mathbf{D})$, uniquely determined by

$$l(c)(\overline{x}) = l(\mathfrak{a}, f, \overline{x})$$

for $\mathfrak{Y} \in \mathfrak{ho}_c\mathbf{D}$, $\overline{x} \in \mathbf{E}_{\mathfrak{Y}}$, $\mathfrak{a}\colon A \hookrightarrow B$, $f\colon L(A) \to Y$ and $c\colon \mathfrak{Y} \to \mathfrak{Y} \cup_f L(B)$ as in Construction 12.1.0.22. In particular, we obtain that for every expansion e the associated lift $[x_0] \to l(e)[x_0]$ is given by a simple equivalence (represented by an expansion even). Finally, by making appropriate choices in Construction 12.1.0.22), one also obtains that for any inclusion of subcomplexes $a\colon \mathfrak{X} \hookrightarrow \mathfrak{X}'$ the value of $l(L(a))$ at $[\rho(\mathfrak{X})]$ is given by $[\rho(\mathfrak{X}')]$. This finishes the proof. □

12.2 Connection to the classical framework

From what we have explained so far, there are two obvious candidates to model the simple homotopy theory of spaces, namely the structures of Whitehead model categories on simplicial sets and on topological spaces, defined in Example 10.1.2.3. For the remainder of this subsection, when we refer to **sSet** or **Top** as Whitehead model categories, it will always be with respect to these structures. There is a third, more classical theory available. Namely, one can consider the Whitehead framework that is defined as follows. In the following, we use the notation $\mathring{D}^n$ to refer to the interior of an n-dimensional disk. We will also generally identify disks D^n with the realization of the n-simplex $|\Delta^n|$, using some fixed homeomorphism throughout this section.

Construction 12.2.0.1. See Example 8.1.1.13 for some of the notation used below. Denote by $\widetilde{\mathbf{CW}}$ the following pre-Whitehead framework.

- Objects are filtered spaces X, together with a finite set of pairwise disjoint subspaces Z of X, such that
  1. $X = \bigcup_{e \in Z} e$;
  2. There exists the structure of a CW-complex $\mathfrak{X}$ on X (as defined in Example 8.1.1.13), such that
  $$Z = \{\sigma(\mathring{D}^n) \mid \sigma\colon D^n \to X \in \mathfrak{C}_{\mathfrak{X}}\}.$$
- Morphisms between two such tuples (X_1, Z_1) and (X_2, Z_2) are given by maps $f\colon X_1 \to X_2$, such that there exist compatible (in the sense we just described) choices of cell structures $\mathfrak{X}_1$ and $\mathfrak{X}_2$, such that $f\colon X_1 \to X_2$ defines an inclusion of structured cell complexes $f\colon \mathfrak{X}_1 \hookrightarrow \mathfrak{X}_2$.
- Cobase change squares are given by such commutative squares in $\widetilde{\mathbf{CW}}$, for which the underlying diagram of filtered spaces is a pushout.
- Expansions are given by finite compositions of morphisms of the following type:
  1. Isomorphisms;
  2. Morphisms $e\colon (X_0, Z_0) \hookrightarrow (X_1, Z_1)$, such that there exist compatible cell structures $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1$, making e an inclusion of subcomplexes, such that the associated structured relative cell complex of filtered spaces $\mathfrak{e}\colon X_0 \hookrightarrow X_1$ fits into a cobase change square
  $$\begin{tikzcd} H^n \arrow[r] \arrow[d, hook, "\mathfrak{e}^n"'] & X_0 \arrow[d, hook, "\mathfrak{e}"] \\ \tilde{E}^{n+1} \arrow[r] & X_1 \end{tikzcd} \tag{12.36}$$
  for some $n \in \mathbb{N}$, where $\mathfrak{e}^n$ is the following structured relative cell complex:
  (a) $\tilde{E}^{n+1}$ is the filtered space
  $$\varnothing = \dots = \varnothing \subset S^{n-1} \subset S^n_- \subset S^n \subset D^{n+1} = \dots$$
  with D^{n+1} in degree $n+1$, S^n_- the lower hemisphere of S^n, given by all vectors with the final entry non-positive, and S^{n-1} embedded into D^{n+1} by expanding a vector in $\mathbb{R}^n$ by 0 in the final coordinate. H^n is given by the filtered subspace
  $$\varnothing = \dots = \varnothing \subset S^{n-1} \subset S^n_- = S^n_- = \dots$$
  given by the lower hemisphere S^n_- and its boundary.
  (b) $\mathfrak{e}^n\colon H^n \hookrightarrow \tilde{E}^{n+1}$, is the obvious inclusion, equipped with two cells, one given by including E_n as the upper hemisphere, and one given by $E^{n+1} \to \tilde{E}^{n+1}$, with underlying map the identity on D^{n+1}.

It is not hard to verify that the assumptions of a pre-Whitehead framework are fulfilled. In fact, this is the original example of a Whitehead framework, as was explained in [Eck06; Sie70]. The resulting Whitehead group $\mathrm{Wh}_{\widetilde{\mathbf{CW}}}(X,Z)$ is precisely the geometric construction of the classical Whitehead group as it was discussed in great detail in [Coh73]. (Cohen uses cubes instead of disks, but these two approaches clearly translate into each other in a straightforward manner).

The really important point to note here is that $\widetilde{\mathbf{CW}}$ is different from the Whitehead frameworks arising from cellularized categories in the sense that cell structures are only part of the data insofar as they specify a decomposition into open cells. A priori, this category allows for significantly more morphisms. For example, any self-homeomorphism of a disk that fixes the boundary will give rise to all sorts of automorphisms in $\widetilde{\mathbf{CW}}$. The advantage of this additional degree of flexibility is demonstrated in [Coh73], where it is used all over the place. The disadvantage is, of course, that at least a priori, it makes being a simple equivalence a significantly weaker condition than the ones we have considered so far. It turns out that, nevertheless, this Whitehead framework is τ-equivalent to the ones arising from **sSet** and **Top**. To prove this, we are going to make use of another intermediary framework.

Construction 12.2.0.2. One may proceed entirely analogously with structured cell complexes in **Top**, to obtain a pre-Whitehead framework $\widetilde{\mathbf{Top}}$, given as follows:

1. Objects are spaces X, together with a finite set of pairwise disjoint subspaces Z of X,
   (a) $X = \bigcup_{e \in Z} e$;
   (b) There exists the structure of a structured cell complex in **Top** $\mathfrak{X}$ on X (as defined in Example 8.1.1.13), such that
   $$Z = \{\sigma(\mathring{D}^n) \mid \sigma\colon D^n \to X \in \mathfrak{C}_{\mathfrak{X}}\}.$$
2. Morphisms between two such tuples (X_0, Z_0) and (X_1, Z_1) are given by maps $f\colon X_0 \to X_1$, such that there exist compatible (in the sense we just described) choices of cell structures $\mathfrak{X}_0$ and $\mathfrak{X}_1$, such that $f\colon \mathfrak{X}_0 \to \mathfrak{X}_1$ defines the inclusion of a subcomplex in $\mathbf{Cell}(\mathbf{Top})$.
3. Cobase change squares are given by such squares in $\widetilde{\mathbf{Top}}$, for which the underlying square in **Top** is a pushout.
4. Expansions are given by such morphisms $e\colon (X_0, Z_0) \to (X_0, Z_0)$, such that there exist compatible choices of cell structures $\mathfrak{X}_0$ and $\mathfrak{X}_1$ making $e\colon \mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1$ an expansion in the Whitehead framework $\mathbf{W}(\mathbf{Top})$.

Construction 12.2.0.3. There is an obvious forgetful functor $\widetilde{\mathbf{CW}} \to \widetilde{\mathbf{Top}}$ that is easily seen to define a functor of pre-Whitehead frameworks. Furthermore, consider the functor

$$|-|_f\colon \breve{\mathbf{Cell}}_c(\mathbf{sSet}) \to \widetilde{\mathbf{CW}},$$

associating to a finite simplicial set the filtered space given by its filtration by skeletons, together with the decomposition of its realization into the interiors of non-degenerate simplices. It induces a functor of Whitehead frameworks $\mathbf{W}(\mathbf{sSet}) \to \widetilde{\mathbf{CW}}$ (clearly, cobase change squares are preserved and it is a classical fact that horn inclusions define simple homotopy equivalences). Together with the obvious forgetful functors, and the cellularized realization functor $|-|\colon \mathbf{sSet} \to \mathbf{Top}$, we obtain a commutative diagram of (pre)-Whitehead frameworks

$$\begin{tikzcd} \mathbf{W}(\mathbf{sSet}) \arrow[r] \arrow[d] & \mathbf{W}(\mathbf{Top}) \arrow[d] \\ \widetilde{\mathbf{CW}} \arrow[r] & \widetilde{\mathbf{Top}}. \end{tikzcd} \tag{12.37}$$

Let us now prove the following results, which show that we may equivalently use the Whitehead model categories **sSet** or **Top** to describe classical simple homotopy theory:

Theorem 12.2.0.4. *All functors of (pre-)Whitehead frameworks in the commutative diagram*

$$\begin{tikzcd} \mathbf{W}(\mathbf{sSet}) \arrow[r] \arrow[d] & \mathbf{W}(\mathbf{Top}) \arrow[d] \\ \widetilde{\mathbf{CW}} \arrow[r] & \widetilde{\mathbf{Top}}\,. \end{tikzcd} \tag{12.38}$$

are τ-equivalences.

Before we provide a proof, let us perform a sanity check, in the form of the following lemmata.

Lemma 12.2.0.5. *Let $n \geq 0$. Any weak equivalence*

$$f\colon |\partial\Delta^n| \to |\partial\Delta^n|$$

in **Top** *is a simple equivalence (in the sense of a simple equivalence in the Whitehead model category* **Top***).*

Proof. Observe that as the underlying space of $|\partial\Delta^n|$ is homeomorphic to S^{n-1}, there are exactly two such auto-equivalences, up to homotopy. The first is given by the identity; the second (the orientation-inverting one, which we denote by -1 here) can be defined by affinely extending the map

$$\begin{aligned} f\colon (\partial\Delta^n)_0 = [n] &\to [n] = (\partial\Delta^n)_0 \\ 0 &\mapsto 1 \\ 1 &\mapsto 0 \\ k &\mapsto k, \text{for } k > 1\,. \end{aligned}$$

Geometrically speaking, thinking of $|\partial\Delta^n|$ as embedded in $\mathbb{R}^{[n]}$, this is the restriction of the map that permutes the first two components. Now, consider the following commutative diagram, with verticals given by the barycentric subdivision homeomorphisms, and the upper horizontal uniquely defined by commutativity.

$$\begin{tikzcd} |\mathrm{sd}\partial\Delta^n| \arrow[r, dotted, "\rho"] \arrow[d, "\cong"] & |\mathrm{sd}\partial\Delta^n| \arrow[d, "\cong"'] \\ |\partial\Delta^n| \arrow[r, "-1"] & |\partial\Delta^n|\,. \end{tikzcd} \tag{12.39}$$

The barycentric subdivision homeomorphisms are homotopic to the realizations of the last vertex map. As these are realizations of simple equivalences in **sSet** (see Example 11.2.3.10) and the cellularization of $|-|$ is a W-functor, it follows that the horizontals are simple equivalences in **Top**. Consequently, it suffices to see that the upper horizontal ρ is a simple equivalence. Finally, observe that the upper horizontal is the realization of a simplicial map. Namely, the map

$$\mathrm{sd}\partial\Delta^n \to \mathrm{sd}\partial\Delta^n$$

defined on the vertices of $\mathrm{sd}\partial\Delta^n$, given by flags $\mathcal{I} \subset [n]$ as

$$\mathcal{I} \mapsto f(\mathcal{I})\,.$$

This map is bijective on non-degenerate simplices, and hence an isomorphism of simplicial sets. In particular, it is a simple equivalence in **sSet**, showing that ρ is a simple equivalence in **Top**. $\square$

Next, let us prove that the cell structure on a structured cell complex in **Top** is only relevant insofar as it determines a set of open cells.

Lemma 12.2.0.6. *Let $\mathfrak{X}$ be a finite cell complex in* **Top**. *Suppose we are given another cell structure on X, which has the same open cells as $\mathfrak{X}$, denoted by $\mathfrak{X}'$. Then the identity on X induces a simple equivalence*

$$\mathfrak{X} \to \mathfrak{X}'$$

in **Top**.

Proof. We proceed via induction over the number of cells of $\mathfrak{X}$. In the case of 0 cells, there is nothing to be shown. Now, for the inductive step, let $\mathring{e}_m \subset X$ be an open cell of maximal dimension in $\mathfrak{X}$ and $\mathfrak{X}'$. Then, it follows by inductive assumption that the identity on $A = X \setminus \mathring{e}$ induces a simple equivalence between the associated restricted structured cell complexes $\mathfrak{A}$ and $\mathfrak{A}'$. Now, consider the cell structure $\mathfrak{X}''$ on X, given by using the characteristic maps of $\mathfrak{X}$, for all open cells in A, and the characteristic map of $\mathfrak{X}'$ for $\mathring{e}_m$. Then we obtain a commutative diagram (pushout in **Top**, trivially so, as all horizontals are identities),

$$\begin{tikzcd}
\mathfrak{A} \arrow[r] \arrow[d, hook] & \mathfrak{A}' \arrow[d, hook] \\
\mathfrak{X}'' \arrow[r] & \mathfrak{X}'
\end{tikzcd} \tag{12.40}$$

where both verticals define inclusions of subcomplexes and the relative cell structure on the right is given by the cobase change of the relative cell structure on the left. Furthermore, the upper horizontal is a simple equivalence. Hence, we may use Corollary 10.2.3.13, from which it follows that the lower horizontal defines a simple equivalence. The diagram

$$\begin{tikzcd}
& \mathfrak{X}'' \arrow[dr] & \\
\mathfrak{X} \arrow[ur] \arrow[rr] & & \mathfrak{X}'
\end{tikzcd} \tag{12.41}$$

in $\mathfrak{ho}_c$**Top** is trivially commutative, as all underlying morphisms are given by the identity. Hence, by the two-out-of-three property for simple equivalences, it suffices to see that $\mathfrak{X} \to \mathfrak{X}''$ is a simple equivalence. In this fashion, we have reduced to the case of changing a single characteristic map of a cell which can be glued in the final step of a filtration-presentation. In other words, we have two pushout diagrams

$$\begin{tikzcd}
|\partial\Delta^n| \arrow[r, "\partial\sigma_i"] \arrow[d, hook] & A \arrow[d, hook] \\
|\Delta^n| \arrow[r, "\sigma_i"'] & X
\end{tikzcd} \tag{12.42}$$

for $i = 1, 2$, with the same verticals, and want to show that the identity on X induces a simple equivalence

$$\mathfrak{A} \cup_{\partial\sigma_1} |\Delta^n| \to \mathfrak{A} \cup_{\partial\sigma_2} |\Delta^n| .$$

Denote by $b \in |\Delta^n|$ the barycenter. Now, observe first that, without loss of generality, we can always change σ_i by a homeomorphism on $|\Delta^n|$ which fixes the boundary and is homotopic to the identity relative to the boundary, to assume that $\sigma_1(b) = \sigma_2(b)$. Denote by $\mathring{e} \subset X$ the open cells given by σ_1, σ_2, and by abuse of notation, by $b \in \mathring{e}$ the point corresponding to the barycenter of $|\Delta^n|$ (under both σ_1 and σ_2). Next, consider the following diagrams of pushout squares

$$\begin{tikzcd}
\mathring{e} \setminus \{b\} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "Q"] & \mathring{e} \arrow[d, hook] \arrow[dr, phantom, "\Longleftarrow"] & |\partial\Delta^n| \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "Q'_i \ulcorner"] & |\Delta^n| \arrow[d, hook] \arrow[dr, phantom, "\Longrightarrow"] & |\partial\Delta^n| \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "Q_i \ulcorner"] & |\Delta^n| \arrow[d] \\
X \setminus \{b\} \arrow[r, hook] & X & M_{\partial\sigma_i} \arrow[r, hook] & M_{\partial\sigma_i} \cup_{|\partial\Delta^n|} |\Delta^n| & A \arrow[r, hook] & X .
\end{tikzcd} \tag{12.43}$$

□

Let us describe the morphisms of the square indicated by arrows of the form $\Rightarrow$. As all squares are pushout, it suffices to specify the morphisms between the objects in the spans. The arrows pointing to the right are given by the identity at $|\partial\Delta^n|, |\Delta^n|$, and by the collapsing the cylinder map at $M_{\partial\sigma_i}$. If we consider $|\partial\Delta^n|$ as equipped with its canonical cell structure, then it follows by Example 10.2.3.6 that $\mathfrak{M}_{\partial\sigma_i} \cup_{|\partial\Delta^n|} |\Delta^n| \to \mathfrak{X}_i$ is a simple equivalence. The arrows pointing to the left are given as follows: For $i = 1, 2$, let

$$f_i\colon |\Delta^n| \to \mathring{e}$$

be the map given by

$$x \mapsto \sigma_i(\frac{1}{2}x + \frac{1}{2}b)\,.$$

Clearly, this map maps $|\partial\Delta^n|$ to $\mathring{e} \setminus \{b\}$. f_i, and its restriction to $|\partial\Delta^n|$, specify the maps at $|\Delta^n|$ and $|\partial\Delta^n|$. Finally, the map $M_{\partial\sigma_i} \to X \setminus b$ is given by

$$[a] \mapsto [a]$$

for $a \in A$, and by

$$[(x,t)] \mapsto \sigma_i(t(\frac{1}{2}x + \frac{1}{2}b) + (1-t)x)$$

for $(x,t) \in [0,1] \times |\partial\Delta^n|$. Continuity and well-definedness of this map are easily verified through the universal property of the pushout. Now, it is not hard to see, using appropriate retractions, that all arrows associated to objects in the spans which we have specified are weak equivalences. The left pointing morphism with target X is even a homeomorphism, and the right pointing arrow a simple equivalence in **Top**. Hence, we may agglomerate all of these squares into a diagram of weak equivalences of functors $\Delta^1 \times \Delta^1 \to \mathbf{Top}$.

$$\begin{tikzcd} Q_1' \arrow[r, Rightarrow] \arrow[d, Rightarrow] & Q & Q_2' \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\ Q_1 & & Q_2\,. \end{tikzcd} \tag{12.44}$$

As all arrows between these squares are weak equivalences, we can complete this diagram to a commutative diagram of isomorphisms in $\mathrm{ho}(\mathbf{Top}^{\Delta^1 \times \Delta^1})$

$$\begin{tikzcd} Q_1' \arrow[r, Rightarrow] \arrow[d, Rightarrow] & Q & Q_2' \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\ Q_1 \arrow[ur, Rightarrow, dashed] \arrow[rr, Rightarrow, dashed] & & Q_2\,. \arrow[ul, Rightarrow, dashed, "\eta"] \end{tikzcd} \tag{12.45}$$

If we evaluate this diagram at $\{(1,1)\}$, we obtain a commutative diagram

$$\begin{tikzcd} M_{\partial\sigma_1} \cup_{|\partial\Delta^n|} |\Delta^n| \arrow[r, "\cong"] \arrow[d, "\simeq"'] & X & M_{\partial\sigma_2} \cup_{|\partial\Delta^n|} |\Delta^n| \arrow[l, "\cong"'] \arrow[d, "\simeq"] \\ X \arrow[ur, dashed] \arrow[rr, dashed] & & X \arrow[ul, dashed] \end{tikzcd} \tag{12.46}$$

in ho**Top**, with the upper horizontals presented by homeomorphisms. Now, observe that by the construction of the horizontal homeomorphisms and the vertical collapse maps, the compositions $X \cong M_{\partial\sigma_i} \cup_{|\partial\Delta^n|} |\Delta^n| \to X$ are homotopic to the identity on X. It follows that all dotted arrows in Diagram (12.46) are given by the identity on X in ho**Top**. Now, denote by $\mathbf{Q}$ the Reedy category, obtained by equipping $\Delta^1 \times \Delta^1$ with the Reedy model structure indicated below.

$$\begin{tikzcd} (0,0) \arrow[r, "-"] \arrow[d, "+"'] \arrow[dr] & (1,0) \arrow[d, "+"] \\ (0,1) \arrow[r, "-"'] & (1,1)\,. \end{tikzcd} \tag{12.47}$$

Next, consider $|\partial\Delta^n|$ as equipped with its canonical cell structure. Then equipping Q_1 and Q_2 with the associated cell structures

$$\begin{tikzcd} {|\partial\Delta^n|} \arrow[r] \arrow[d, hook] & \mathfrak{A} \arrow[d, hook] \\ {|\Delta^n|} \arrow[r] & \mathfrak{X}_i \end{tikzcd} \tag{12.48}$$

defines cell structures on Q_1 and Q_2 in $\mathbf{Top}^{\mathbf{Q}}$, denoted $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Then there are natural isomorphisms of cell complexes

$$(\mathfrak{Q}_i)^{(1,1)} \cong \mathbf{Q}^{(1,1)} \circledast \mathfrak{Q}_i\,,$$

using notation as in Section 8.3. By Corollary 11.2.1.8, it suffices to see that $\eta^{(0,0)}$, $\eta^{(0,1)}$ and $\eta^{(1,0)}$ are simple equivalences. At $(1,0)$, η is given by a map $|\Delta^n| \to |\Delta^n|$. Any such map is homotopic to the identity, and hence a simple equivalence. At $(0,1)$, η is given by the identity on $\mathfrak{A}$, and thus a simple equivalence. At $(0,0)$, η is a simple equivalence by Lemma 12.2.0.5. This finishes the proof.

proof of Theorem 12.2.0.4. We already know that the upper horizontal is a τ-equivalence, by Example 12.1.0.5. Let us first show that the remaining associated functors on localizations are all fully faithful. We will denote the associated localizations in the form $\mathfrak{ho}_c\mathbf{C}$. Using the forgetful functors into topological spaces, we obtain a commutative diagram

$$\begin{tikzcd} \mathfrak{ho}_c\widetilde{\mathbf{CW}} \arrow[r] \arrow[dr] & \mathfrak{ho}_c\widetilde{\mathbf{Top}} \arrow[d] & \mathfrak{ho}_c\mathbf{Top} \arrow[l] \arrow[dl] \\ & \mathrm{ho}\mathbf{Top}\,. & \end{tikzcd} \tag{12.49}$$

It suffices to see that all downwards pointing arrows are fully faithful. For the right hand diagonal arrow, this follows by Theorem 10.2.2.1. For the left hand diagonal, this is classical, and was explained (for example) in [Eck06; Sie70]. The argument is essentially a more elementary version of the proof of Theorem 10.2.2.1, and can easily be derived from the latter. The same argument applies to the middle vertical, replacing CW-complexes by structured cell complexes of spaces. This actually simplifies the argument, as there is no more need for cellular approximation. Now, to see essential surjectivity in the diagram

$$\begin{tikzcd} \mathfrak{ho}_c\mathbf{sSet} \arrow[r] \arrow[d] & \mathfrak{ho}_c\mathbf{Top} \arrow[d] \\ \mathfrak{ho}_c\widetilde{\mathbf{CW}} \arrow[r] & \mathfrak{ho}_c\widetilde{\mathbf{Top}} \end{tikzcd} \tag{12.50}$$

observe that the right hand vertical is essentially surjective by definition of $\widetilde{\mathbf{Top}}$. The horizontal is already known to be essentially surjective. Consequently, the diagonal composition is also essentially surjective, which implies that the lower horizontal is essentially surjective. As the lower vertical is fully faithful, it follows from essential surjectivity of the diagonal that the left vertical is also essentially surjective. It remains to show that all functors induce isomorphisms on Whitehead monoids. So, given a simplicial set X, consider the associated commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{Wh}}_{\mathbf{sSet}}(X) \arrow[r] \arrow[d] & \widetilde{\mathrm{Wh}}_{\mathbf{Top}}(|X|) \arrow[d] \\ \widetilde{\mathrm{Wh}}_{\widetilde{\mathbf{CW}}}(|X|_f) \arrow[r] & \widetilde{\mathrm{Wh}}_{\widetilde{\mathbf{Top}}}(|X|, Z) \end{tikzcd} \tag{12.51}$$

where Z is the decomposition of $|X|$ into its open, non-degenerate simplices. As $\widetilde{\mathbf{CW}}$ is a Whitehead framework, the elements of $\widetilde{\mathrm{Wh}}_{\widetilde{\mathbf{CW}}}(|X|_f)$ can be identified with homotopy classes of maps of CW-complexes $|X|_f \to \mathfrak{Y}$, modulo post-composition with simple equivalences (see Theorem 9.1.3.4). It is a classical fact that every CW-complex has the simple homotopy type of a simplicial complex, and hence of a simplicial set. Consequently, using Theorem 9.1.3.4, every element of $\widetilde{\mathrm{Wh}}_{\widetilde{\mathbf{CW}}}(|X|_f)$ is of the form $\langle |X|_f \to |Y|_f \rangle$, for some finite simplicial set Y. As we have already seen that $\mathfrak{ho}_c\mathbf{sSet} \to \mathfrak{ho}_c\widetilde{\mathbf{CW}}$ is full, this shows that the left hand vertical in Diagram (12.51) is a surjection (again using Theorem 9.1.3.4). We already know the top horizontal to be an isomorphism. Consequently, to see that all maps are isomorphisms, it suffices to see that the right vertical is an isomorphism. Then it follows that the left vertical is injective, and hence also an isomorphism, which also implies that the lower horizontal is an isomorphism. Using Construction 10.1.2.8, which shows that $\widetilde{\mathrm{Wh}}_{\mathbf{Top}}(|X|)$ does not depend on the cell structure of $|X|$, it immediately follows that the right hand vertical is surjective. Hence, it remains to be shown that the right hand vertical is injective. This is a straightforward consequence of Lemma 12.2.0.6 and Theorem 10.2.3.9. $\square$

Chapter 13

Decomposing the diagrammatic stratified Whitehead group

Note to the reader In this chapter, we finally apply the new tools in generalized simple homotopy theory developed in the previous chapters to stratified homotopy theory. We thus recommend that the reader be familiar with the notation of stratified homotopy theory introduced in Part II. All notation that we use on the stratified side can either be found in Chapter 7 or already in Chapter 1.

13.1 Simple diagrammatic stratified homotopy theory

In [Waa21], we developed a simple homotopy theory for the category of stratified simplicial sets $s\mathbf{Strat}_P$ over a finite poset P, equipped with the diagrammatic model structure (also called the Douteau model structure, see Definition 1.2.3.9 and Theorem 1.2.3.14, for example). The stratified Whitehead group $\mathrm{Wh}_P(\mathcal{X})$ associated to a finite stratified simplicial set $\mathcal{X}$ over P, defined there, was given as follows. Recall first that a stratified horn inclusion $\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}} \in s\mathbf{Strat}_P$, with $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$ is called an *admissible horn inclusion*, if p_k is repeated in $\Delta^{\mathcal{J}}$. Equivalently, this is the case if and only if the stratified realization $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$ is a stratified homotopy equivalence (see Definition 3.2.9.1).

1. As a set $\mathrm{Wh}_P(\mathcal{X})$ is a quotient set of acyclic cofibrations of finite stratified simplicial sets $a\colon \mathcal{X} \hookrightarrow \mathcal{Y}$ in the Douteau model structure on $s\mathbf{Strat}_P$. The equivalence relation is generated by composition with pushouts of admissible horn inclusions.

2. The group structure on $\mathrm{Wh}_P(\mathcal{X})$ is induced by pushouts (see [Waa21]) for details.

In [Waa21], we developed the resulting simple homotopy theory by elementary means. Let us now interpret this theory in terms of the general framework of Whitehead model categories, which we developed in Chapter 10. As a consequence, we will obtain a topological version of the purely combinatorially defined Whitehead group $\mathrm{Wh}_P(\mathcal{X})$, and compute the latter in terms of classical Whitehead groups of generalized links and strata. For the remainder of this chapter, let P be a fixed poset.

Let us investigate the simple homotopy theory developed in [Waa21] through the tools provided by our language of Whitehead model categories. We will end up using essentially all of the core results of Chapters 10 to 12 in the process.

13.1.1 The diagrammatic Whitehead model structure on stratified simplicial sets

We begin by investigating the purely combinatorial setting of stratified simplicial sets. Let us begin with a few elementary observations.

Observation 13.1.1.1. As we have frequently done in Chapters 3 and 7, we can equivalently think of $s\mathbf{Strat}_P$ as the category of presheaves $\mathbf{Set}^{\Delta_P^{op}}$ on the category of flags in P, $\Delta_P = \Delta_{/\mathrm{N}(P)}$. We consider Δ as a Reedy category equipped with the classical Reedy structure. Δ forms an elegant Reedy category (see Section 8.3.5). It follows that Δ_P inherits the structure of an elegant Reedy category (see Example 8.3.5.4) from Δ. Given a flag $\mathcal{J} \in \Delta_P$, the associated boundary inclusion

$$\iota^{\mathcal{J}} : \partial\Delta_P^{\mathcal{J}} \to \Delta_P^{\mathcal{J}}$$

(see Recollection 8.3.2.4) is simply the boundary inclusion of the stratified simplex associated to $\mathcal{J}$, which we have usually just denoted by

$$\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}} .$$

It follows by Corollary 8.3.5.5 that every stratified simplicial $\mathcal{X}$ set over P admits a unique cell structure with respect to these inclusions. The cells of this cell structure are precisely the non-degenerate stratified simplices of $\mathcal{X}$ (see Corollary 8.3.4.11). Furthermore, the inclusions of subcomplexes between such objects are precisely monomorphisms. Consequently, as long as we are working with stratified simplicial sets, cell structures are entirely intrinsic, and there is no need to make explicit mention of them.

Recall the definition of a simplicial Whitehead model category (Definitions 10.2.2.3 and 12.1.0.1) as well as proper generation (Definition 11.1.1.1). We then obtain the following result.

Theorem 13.1.1.2. *Equipping $\underline{s\mathbf{Strat}}_P$ with the following two sets of maps defines a properly generated simplicial Whitehead model category:*

1. *The set of generating boundary inclusions is given by*

$$\mathbb{B}_P := \{\partial\Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}} \mid \mathcal{J} \in \Delta_P\} .$$

2. *The set of generating expansions is given by*

$$\mathbb{E}_P := \left\{\Lambda_k^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}} \mid \mathcal{J} = [p_0 \leq \cdots \leq p_n] \in \Delta_P, 0 \leq k \leq n \text{ fulfills } p_k \in \{p_{k-1}, p_{k+1}\}\right\} .$$

Proof. That equipping $s\mathbf{Strat}_P$ with the class of boundary inclusions $\mathbb{B}_P$ defines a cellularized category follows by Construction 8.3.3.3 together with Observation 13.1.1.1. That the two classes above define cofibrant generators for a combinatorial model category was recalled, for example, in Recollection 5.2.1.1, and first proven in [Dou21a]. In particular, the requirements for proper generation are fulfilled. Any compactness assumptions are immediate from the finiteness of all simplicial sets involved. That the simplicial structure on $s\mathbf{Strat}_P$ fulfills the requirements of a simplicial Whitehead model category was shown in [Waa21, Prop. 2.2.33.]. In particular, the simplicial cylinder $(\partial\Delta^1 \otimes - \to \Delta^1 \otimes -, \Delta^1 \otimes - \to \Delta^0 \otimes -)$ defines a simple cylinder. □

We now have all of the abstract tools developed in Chapter 9 available to study the stratified simple homotopy theory which we originally defined in [Waa21].

Notation 13.1.1.3. In this section, we will denote by $\underline{s\mathbf{Strat}}_P^{\mathfrak{d}}$ not only the simplicial model category defining the diagrammatic homotopy theory of stratified simplicial sets over P, but also the simplicial Whitehead model category specified in Theorem 13.1.1.2. To keep notation concise, we will denote the associated Whitehead monoids and Whitehead groups by, respectively, $\widetilde{\mathrm{Wh}}_P(\mathcal{X})$ and $\mathrm{Wh}_P(\mathcal{X})$, for a finite P-stratified simplicial set $\mathcal{X}$. We will call this Whitehead group (monoid) the *diagrammatic stratified Whitehead group (monoid)*. At times, we will omit the *diagrammatic*, as this is the only stratified theory which we investigate in this chapter. We will generally refer to simple equivalences in the Whitehead model category $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ as *simple diagrammatic equivalences*. When the context is clear, we will also just speak of simple equivalences.

Let us quickly recall what this explicitly means for simple diagrammatic equivalences.

Remark 13.1.1.4. It is immediate from the definition and Observation 13.1.1.1 that the Whitehead model category $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ agrees with the one defined in [Waa21]. In particular, a diagrammatic equivalence of finite stratified simplicial sets in $\omega \colon \mathcal{X} \to \mathcal{Y} \in \mathrm{hos}\mathbf{Strat}^{\mathfrak{d}}_P$ is a simple diagrammatic equivalence if and only if it can be written as a zig-zag of finitely many cobase changes of admissible horn inclusions. In this sense, the Whitehead torsion $\langle \omega \rangle \in \mathrm{Wh}_P(\mathcal{X})$ is a complete obstruction to expressing ω in terms of a sequence of such elementary combinatorial moves.

13.1.2 A topological stratified diagrammatic Whitehead group

For many intents and purposes, it can be useful to have the tools of simple homotopy theory available not only in the purely combinatorial scenario of stratified simplicial sets, but also in the more flexible scenario of structured stratified cell complexes. Let us now develop the topological analogue to the Whitehead model structure on stratified simplicial sets which we developed in the previous section.

Lemma 13.1.2.1. *The set of stratified boundary inclusions*

$$\{ |\partial \Delta^{\mathcal{J}} \hookrightarrow \Delta^{\mathcal{J}}|_s \mid \mathcal{J} \in \Delta_P \}$$

defines the structure of a cellularized category on $\mathbf{Strat}_P$.

Proof. That $\mathbf{Strat}_P$ is a cocomplete category was already used all over Chapters 3 and 7, and recalled there separately. The only non-obvious part to verify for the definition of a cellularized category is that any pushout diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{A}' \arrow[r, hook] & \mathcal{B}' \end{tikzcd} \tag{13.1}$$

of stratified spaces, with upper horizontal a relative stratified cell complex, is also pullback. As $\mathbf{Strat}_P$ is simply the overcategory $\mathbf{Top}_{/P}$, where P is equipped with the Alexandrov topology, it follows from the general fact that the forgetful functor, $\mathbf{C}_{/X} \to \mathbf{C}$, from an overcategory reflects pullbacks that it suffices to verify that the underlying diagram of topological spaces

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] \\ A' \arrow[r, hook] & B' \end{tikzcd} \tag{13.2}$$

is pullback. As the forgetful functor $\mathbf{Top}_{/P} \to \mathbf{Top}$ is left adjoint, it follows that this is a pushout diagram with upper horizontal given by a topological relative cell complex. Such a diagram is pullback by Example 8.1.1.12. $\square$

Notation 13.1.2.2. When we refer to $\mathbf{Strat}_P$ as a cellularized category, it will always be with respect to the class of generating boundary inclusions specified in Lemma 13.1.2.1.

Construction 13.1.2.3. By construction of the cellularization of $\mathbf{Strat}_P$, the stratified realization functor

$$|-|_s\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Strat}_P$$

obtains a tautological cellularization, given by equipping

$$|\partial\Delta^{\mathcal{J}} \to \Delta^{\mathcal{J}}|,$$

for $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, with the one cell structure defined by $1_{|\Delta^{\mathcal{J}}|_s}$. When we refer to $|-|_s$ as a cellularized functor, or to a cell structure on the realization of a stratified simplicial set, it will generally be with respect to this cellularization of $|-|_s$. In particular, we obtain a canonical relative cell structure on the realization of the stratified horn inclusion $|\Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}}|_s$, with two cells; one cell given by the total simplex $|\Delta^n|$, and one by the face opposite to the k-th vertex.

Notation 13.1.2.4. We denote by $\mathbf{Strat}^{\mathfrak{d}}_P$ the cellularized category with expansions, obtained by equipping the cellularized category $\mathbf{Strat}_P$ with the set of generating elementary expansions given by the structured relative cell complexes

$$\{|\Lambda^{\mathcal{J}}_k \to \Delta^{\mathcal{J}}|_s \mid \mathcal{J} = [p_0 \leq \cdots \leq p_n] \in \Delta_P, 0 \leq k \leq n, \Lambda^{\mathcal{J}}_k \hookrightarrow \Delta^{\mathcal{J}} \text{ is admissible}\}.$$

Observe that this notation is consistent with the notation for the semi-model category $\mathbf{Strat}^{\mathfrak{d}}_P$ insofar as the generating boundary inclusions and elementary expansions are precisely the cofibrant generators for the semi-model category $\mathbf{Strat}^{\mathfrak{d}}_P$ which we specified in Theorem 7.4.2.6.

Theorem 13.1.2.5. *$\mathbf{Strat}^{\mathfrak{d}}_P$ is a properly generated Whitehead model category. It extends to a simplicial Whitehead model category on $\underline{\mathbf{Strat}}^{\mathfrak{d}}_P$ (under the construction detailed in Observation 12.1.0.10).*

Proof. Let us first verify that the set of boundary inclusions defines a cellularized category. Among the axioms in Definition 10.2.2.3, only the compactness axioms and the existence of a simple cylinder are not an immediate consequence of Theorem 7.4.2.6. To see that the source and target of every generating boundary inclusion and expansion are filtration compact, observe that by Corollary 8.1.6.10 and Proposition 8.1.6.9, this follows if we can show that every map from a finite stratified cell complex into a relative structured stratified cell complex factors through a finite relative subcomplex. The existence of such a factorization can be verified entirely on the level of the underlying topological spaces. That the analogous assertion holds for classical topological spaces is classical, and shown, for example, in [Hir03, p. 10.7.4.]. It remains to show the existence of a simple cylinder. In fact, by Observation 12.1.0.10, we even obtain that the simplicial structure on $\mathbf{Strat}^{\mathfrak{d}}_P$ with its cellularization as in Observation 12.1.0.10 defines the structure of a simplicial Whitehead model category, and we may simply use the simplicial cylinder as a simple cylinder. ☐

Notation 13.1.2.6. We will generally refer to simple equivalences in the Whitehead model category $\underline{\mathbf{Strat}}^{\mathfrak{d}}_P$ as *simple diagrammatic equivalences*. When the context is clear, we will also just speak of simple equivalences.

We may now apply Theorem 12.1.0.4 together with Theorem 7.4.2.11 to derive the following theorem.

Theorem 13.1.2.7. *The cellularized functor*

$$|-|_s\colon \mathrm{s}\mathbf{Strat}^{\mathfrak{d}}_P \to \mathbf{Strat}^{\mathfrak{d}}_P$$

defines a weak equivalence of Whitehead model categories. In particular, it follows that for every finite stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, the induced morphism of Whitehead groups

$$\mathrm{Wh}_{|-|_s}\colon \mathrm{Wh}_P(\mathcal{X}) \to \mathrm{Wh}_{\mathbf{Strat}^{\mathfrak{d}}_P}(|\mathcal{X}|_s)$$

is an isomorphism.

Theorem 13.1.2.7 justifies the following notation.

Notation 13.1.2.8. Given a structured stratified cell complex $\mathfrak{X}$ in $\mathbf{Strat}^{\mathfrak{d}}_P$, we also denote the associated Whitehead group $\mathrm{Wh}_{\mathbf{Strat}^{\mathfrak{d}}_P}(\mathfrak{X})$ by $\mathrm{Wh}_P(\mathfrak{X})$.

13.2 Computation of the diagrammatic stratified Whitehead group

In [Waa21], we asked the question of whether the stratified Whitehead groups $\mathrm{Wh}_P(\mathcal{X})$ can be computed entirely in terms of classical Whitehead groups associated to strata and homotopy links. Having all of the new technology developed in Chapters 10 and 11 available, obtaining a positive answer to this question turns out to be a fairly easy task. The strategy of computation will be to entirely transform the question into a question of the associated diagrams of (homotopy) links. Then we can use the methods for the computation of simple homotopy theories of diagrams which we developed in Chapter 11.

Remark 13.2.0.1. For the remainder of this section, we will only be considering cellularized categories of the form $\mathbf{Set}^{\mathbf{R}^{\mathrm{op}}}$, where $\mathbf{R}$ is an elegant Reedy category (see Section 8.3.5). In particular, as we already discussed in Observation 13.1.1.1, in this kind of framework (relative) cell structures are always intrinsic, every object naturally carries a unique cell structure, and a morphism is a relative cell complex if and only if it is a monomorphism. This also implies that being a cellularized functor between such presheaf categories does not involve any choice of additional structure. A functor

$$L\colon \mathbf{Set}^{\mathbf{R}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{S}^{\mathrm{op}}}$$

with $\mathbf{R}$ and $\mathbf{S}$ elegant Reedy categories, admits a (necessarily unique) cellularization, if and only if it preserves colimits and monomorphisms. Observe that any such functor is necessarily left adjoint, with right adjoint given by

$$X \mapsto \{r \mapsto \mathbf{Set}^{\mathbf{S}^{\mathrm{op}}}(L(\mathbf{R}^r), X)\},$$

extending to morphisms in the obvious way.

Construction 13.2.0.2. Recall that we denote by $\mathrm{sd}(P)$, the subdivision of a poset P, given by the set of regular flags in P ordered by inclusion. Considered as a category, $\mathrm{sd}(P)$ admits the structure of a Reedy category where we set $\mathrm{sd}(P)^+ = \mathrm{sd}(P)$. In particular, $\mathrm{sd}(P)$ has no (non-trivial) degeneracy maps, and is thus an elegant Reedy category.

13.2.1 Cellular link functors

The crucial idea to compute the stratified Whitehead groups is to work with versions of the simplicial homotopy link functor that are W-functors. In this section, we give an axiomatic approach to what properties such a functor needs to fulfill, and show that these characterize the functor up to essentially unique simple equivalence.

Construction 13.2.1.1. Recall that we denote by $\mathbf{Diag}_P$ the simplicial category of simplicial presheaves $\mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$, equipped with the injective model structure, inherited from the Kan-Quillen model structure on $\mathbf{sSet}$. By [BR12, Proposition 3.15] this is equivalently the Reedy model structure on $\mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$. We may thus use Proposition 11.1.1.8 to lift the Whitehead model category structure on $\mathbf{sSet}$ (given by boundary inclusions of simplices and horn inclusions) to $\mathbf{Diag}_P$. When we refer to $\mathbf{Diag}_P$ as a Whitehead model category, it will always be with respect to this structure.

Observation 13.2.1.2. It follows by Corollary 8.3.5.7 that Remark 13.2.0.1 also applies to the Whitehead model categories $\mathbf{Diag}_P$, for P a poset. Hence, there is no need to specify cell structures in this context. Finite cell complexes in $\mathbf{Diag}_P$ are precisely such diagrams $D \in \mathbf{Diag}_P$, for which $D_{\mathcal{I}}$ is finite, for each $\mathcal{I} \in \mathrm{sd}(P)$ and empty, for all but finitely many $\mathcal{I} \in \mathrm{sd}(P)$. It follows by Corollary 11.1.2.8 that a morphism $\omega\colon D \to D' \in \mathrm{ho}\mathbf{Diag}_P$ between two such finite diagrams is a simple equivalence, if and only if the induced morphisms in the homotopy category of finite simplicial sets $\omega_{\mathcal{I}}\colon D_{\mathcal{I}} \to D'_{\mathcal{I}}$ are simple equivalences, for each $\mathcal{I} \in \mathrm{sd}(P)$.

The core idea to computing the stratified Whitehead groups $\mathrm{Wh}_P(\mathcal{X})$ is to use the diagrammatic homotopy link functor

$$\mathrm{HoLink}_{\mathcal{I}}\colon \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \to \mathbf{Diag}_P$$

(see, for example, Recollection 5.2.2.1) to translate the computation into a purely diagrammatic one. $\mathrm{HoLink}_{\mathcal{I}}$ does generally not preserve colimits (unless we are working over a discrete poset) and thus is not a cellularized functor. We have already seen in Chapter 5 that there are left adjoint models for the homotopy link functor available. It will be useful to freely be able to switch between such models. Let us therefore give a general definition of the types of functors that we will consider.

Definition 13.2.1.3. A functor

$$L\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$$

is called a *cellular link functor* if it fulfills the following properties:

1. L is a finite cellularized functor.
2. For each flag $\mathcal{J} \in \Delta_P$, with underlying regular flag $\mathcal{I}$, it holds that $L(\Delta^{\mathcal{J}})_{\mathcal{I}'}$ is contractible, whenever $\mathcal{I}' \subset \mathcal{I}$, and is empty, whenever $\mathcal{I}' \in \mathrm{sd}(P)$ is not a subset of $\mathcal{I}$.

Notation 13.2.1.4. Given a cellular link functor $L\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$ and a flag $\mathcal{I} \in \mathrm{sd}(P)$, we denote by

$$L_{\mathcal{I}}\colon \mathrm{s}\mathbf{Strat}_P \xrightarrow{L} \mathbf{Diag}_P \xrightarrow{\mathrm{ev}_{\mathcal{I}}} \mathbf{sSet}$$

the composition of L with evaluation at $\mathcal{I}$.

Remark 13.2.1.5. As we have explained in Remark 13.2.0.1, L as in Definition 13.2.1.3 being a cellularized functor is equivalent to it preserving monomorphisms and colimits. It follows by Lemma 11.2.3.2 that L being finite is equivalent to $L(\Delta^{\mathcal{J}})$ being a finite diagram, for each $\mathcal{J} \in \Delta_P$.

It will also be useful to observe the following equivalent characterization of the second defining condition of a cellular link functor.

Recollection 13.2.1.6. Given a simplicial set $X \in \mathbf{sSet}$ and a presheaf $F \in \mathbf{Set}^{\mathrm{sd}(P)^{\mathrm{op}}}$, recall that the notation $F \circledcirc X$ (see Section 8.3) refers to the presheaf on $\mathrm{sd}(P) \times \Delta$, given by

$$(\mathcal{I}, n) \mapsto \bigsqcup_{i \in F_{\mathcal{I}}} X_n$$

with the obvious functoriality induced by the one of F and X. Under the exponential law for functor categories

$$\mathbf{Set}^{\mathrm{sd}(P)^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \cong \mathbf{sSet}^{\mathrm{sd}(P)^{\mathrm{op}}}$$

we can identify this presheaf with an element of $\mathbf{Diag}_P$. Here, we will mainly refer to the special case where $F = \mathrm{sd}(P)^{\mathcal{I}}$, for some flag $\mathcal{I} \in \mathrm{sd}(P)$, and $X = \star = \Delta^0$. In this case, $\mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star$ is the unique diagram that fulfills

$$\mathcal{I}' \mapsto \begin{cases} \varnothing & \text{, if } \mathcal{I}' \not\subset \mathcal{I} \\ \star & \text{, if } \mathcal{I}' \subset \mathcal{I} \end{cases}.$$

Observation 13.2.1.7. Given $\mathcal{I} \in \mathrm{sd}(P)$, the diagram $\mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star$ is fibrant. It follows that for any $D \in \mathbf{Diag}_P$, there is a canonical equivalence

$$\underline{\mathbf{Diag}}_P(D, \mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star) \simeq \mathcal{D}\mathbf{iag}_P(D, \mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star)$$

between the simplicial and the derived mapping spaces, computing the mapping spaces in the ∞-category $\mathcal{D}\mathbf{iag}_P$. Observe that the simplicial mapping spaces

$$\underline{\mathbf{Diag}}_P(D, \mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star)$$

are empty, or a point, and are non-empty if and only if $D_{\mathcal{I}'}$ is empty, whenever $\mathcal{I}' \notin \mathcal{I}$. It follows that $\mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star$ defines the (essentially) unique subterminal object in $\mathcal{D}\mathbf{iag}_P$ that has the following characterizing property.

- For any $D \in \mathbf{Diag}_P$, $\mathcal{D}\mathbf{iag}_P(D, \mathrm{sd}^{\mathcal{I}} \circledcirc \star)$ is non-empty empty, if and only if $D_{\mathcal{I}'}$ is empty, for each $\mathcal{I}' \notin \mathcal{I}$.

Lemma 13.2.1.8. *Let $\mathcal{I} \in \mathrm{sd}(P)$, and let $D \in \mathbf{Diag}_P$ be a finite diagram. Then the following are equivalent:*

1. *$D_{\mathcal{I}'}$ is contractible, whenever $\mathcal{I}' \subset \mathcal{I}$, and empty otherwise.*
2. *There is a (necessarily unique) simple equivalence*

$$D \xrightarrow{\simeq} \mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star\,.$$

Proof. That the latter condition implies the former is immediate by Observation 13.2.1.7. For the converse, it follows by the description of simplicial mapping spaces in Observation 13.2.1.7 that such a unique morphism exists. By assumption on D, at each $\mathcal{I} \in \mathrm{sd}(P)$, this morphism has both source and target empty, or both source and target contractible. In particular, it is given by a simple equivalence, for each $\mathcal{I} \in P$. Hence, it follows by Corollary 11.1.2.8 that the unique morphism defines a simple equivalence of diagrams. ☐

Let us now give three examples of cellular link functors. All of these are derived from different combinatorial models for the regular neighborhood of strata in a stratified simplicial complex.

Example 13.2.1.9. In Construction 5.2.2.4, we constructed a cellular link functor

$$\mathrm{Link}^m \colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P,$$

via left Kan-extension, by mapping the stratified simplex $\Delta^{\mathcal{J}}$ to the diagram

$$\mathcal{I} \mapsto \prod_{p \in \mathcal{I}} \Delta^{\mathcal{J}_p}$$

with structure morphisms given by the obvious projections between products. Here, $\Delta^{\mathcal{J}_p}$ denotes the trivially stratified simplex given by the subflag $\mathcal{J}_p \subset \mathcal{J}$, of all entries that correspond to p. It follows by Propositions 5.2.2.9 and 5.2.2.11 and Recollection 5.2.2.1 that this construction defines a left Quillen equivalence between the diagrammatic model structure on stratified simplicial sets and the injective model structure on diagrams. In particular, it defines a cellularized functor, which is evidently finite. Clearly,

$$\prod_{p \in \mathcal{I}} \Delta^{\mathcal{J}_p}$$

is empty, if and only if $\mathcal{I}$ is not a subset of the regular flag which $\mathcal{J}$ degenerates from, and is contractible otherwise. The basic idea behind this link functor is best illustrated in the scenario of two strata, $P = \{p < q\}$, and a stratified simplicial complex $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$. Then the simplicial neighborhood of $X_p \subset X$, $\mathrm{N}_{X_p}(X) \subset X$, is given by a subcomplex of the simplicial join

$$\mathrm{N}_{X_p}(X) \subset X_p \star X_q\,.$$

Passing to realizations, one obtains an inclusion

$$\mathrm{N}_{X_p}(X) \subset |X_p \star X_q| \cong |X_p| \star |X_q|$$

into the topological join, which is given by the pushout

$$\begin{tikzcd} {|X_p| \times |X_q| \times \{0\} \sqcup |X_p| \times |X_q| \times \{1\}} \arrow[r] \arrow[d] & {|X_p| \times |X_q| \times [0,1]} \arrow[d] \\ {|X_p| \times \{0\} \sqcup |X_q| \times \{1\}} \arrow[r] & {|X_p| \star |X_q|\,.} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{13.3}$$

One then obtains a boundary B for a regular neighborhood of $|X_p| \subset |X|$, by intersecting $|\mathrm{N}_{X_p}(X)| \subset |X_p \star X_q|$ with the inverse image of $\frac{1}{2}$ under the join coordinate map $|X_p| \star |X_q| \to [0,1]$. Observe that this inverse image is naturally homeomorphic to $|X_p| \times |X_q| \cong |X_p \times X_q|$. $\mathrm{Link}^m_{\{p<q\}}(\mathcal{X})$ then provides an explicit triangulation for B in terms of a subcomplex of $X_p \times X_q$.

Example 13.2.1.10. In Definition 3.2.5.4, we provided another cellular link functor (we only gave a pointwise definition there). At a flag $\mathcal{I}$, it is given by associating a stratified simplicial set $\mathcal{X}$ to P, the fiber over the barycenter of the first barycentric subdivision of the nerve of P at $\mathcal{I}$. In other words, it is given by the pullback.

$$\begin{tikzcd} (\mathrm{sd}X)_{\mathcal{I}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{sd}X \arrow[d, "\mathrm{sd}s_{\mathcal{X}}"] \\ \star \arrow[r, hook, "\mathcal{I}"'] & \mathrm{N}(\mathrm{sd}(P)) \cong \mathrm{sdN}(P)\,. \end{tikzcd} \tag{13.4}$$

We will denote this simplicial set by $(\mathrm{sd}\mathcal{X})_{\mathcal{I}}$, in order to keep track of the stratification. Combinatorially speaking, $(\mathrm{sd}\mathcal{X})_{\mathcal{I}}$ is the subcomplex of $\mathrm{sd}X$, spanned by the vertices coming from such stratified simplices which degenerate from $\mathcal{I}$. In the case of two strata, $P = \{p < q\}$, $(\mathrm{sd}\mathcal{X})_{\{p<q\}}$ is simply the classical construction for the boundary of a regular neighborhood of $\mathcal{X}$, and $(\mathrm{sd}\mathcal{X})_{\{p\}}$ and $(\mathrm{sd}\mathcal{X})_{\{q\}}$ are given by the barycentric subdivision of the simplicial strata.

From this perspective, it is not surprising that this construction admits a functoriality in $\mathcal{I}$ (in fact, it was already described in [Hat75]). For the sake of completeness, let us repeat the construction here: As $(\mathrm{sd}-)_{\mathcal{I}}$ is a colimit and monomorphism preserving functor, it is uniquely determined by its values on stratified simplices and via left Kan extension, we only need to expose functoriality in $\mathcal{I}$ after restriction to Δ_P. For a stratified simplex $\Delta^{\mathcal{J}}$, with $\mathcal{J} = [p_0 \leq \cdots \leq p_n]$, it follows from the fact that the nerve preserves limits, that we can identify the defining pullback diagram of $(\mathrm{sd}\Delta^{\mathcal{J}})_{\mathcal{I}}$ with the nerve of the pullback diagram of posets

$$\begin{tikzcd} (\mathrm{sd}\mathcal{J})^{-1}(\mathcal{I}) \arrow[r, hook] \arrow[d] & \mathrm{sd}[n] \arrow[d, "\mathrm{sd}\mathcal{J}"] \\ \{\mathcal{I}\} \arrow[r, hook] & \mathrm{sd}(P)\,. \end{tikzcd} \tag{13.5}$$

In other words, $(\mathrm{sd}\Delta^{\mathcal{J}})_{\mathcal{I}}$ is given by the nerve of the subposet of $\mathrm{sd}[n]$ given by such subsets $S \subset [n]$, for which $\mathcal{I} = \{p_i \mid i \in S\}$. If $\mathcal{I}' \subset \mathcal{I}$, we obtain a map of posets via

$$\begin{aligned} (\mathrm{sd}\mathcal{J})^{-1}(\mathcal{I}) &\to (\mathrm{sd}\mathcal{J})^{-1}(\mathcal{I}') \\ S &\mapsto (\mathrm{sd}\mathcal{J})^{-1}(\mathcal{I}') \cap S\,. \end{aligned}$$

This map can easily be seen to be natural in Δ_P, hence, passing to nerves and left Kan-extending, we obtain a natural transformation

$$\rho_{\mathcal{I}' \subset \mathcal{I}}(\mathrm{sd}\mathcal{X})_{\mathcal{I}} \to (\mathrm{sd}\mathcal{X})_{\mathcal{I}'}\,.$$

These natural transformations are also clearly contravariantly functorial in $\mathcal{I}$, and hence agglomerate into a functor

$$\begin{aligned} \mathrm{Link}^{\mathrm{sd}} \colon \mathrm{s}\mathbf{Strat}_P &\to \mathbf{Diag}_P \\ \mathcal{X} &\mapsto \begin{cases} \mathcal{I} & \mapsto (\mathrm{sd}\mathcal{X})_{\mathcal{I}} \\ (\mathcal{I}' \subset \mathcal{I}) & \mapsto \rho_{\mathcal{I}' \subset \mathcal{I}}\,. \end{cases} \end{aligned}$$

To see that $\mathrm{Link}^{\mathrm{sd}}$ preserves colimits and monomorphisms, we only need to see that this is the fact flagwise. For any such regular flag $\mathcal{I} \in \mathrm{sd}(P)$, $\mathrm{Link}^{\mathrm{sd}}_{\mathcal{I}}$ is a composition of subdivision and base change, both of which preserve colimits and monomorphisms, and hence has the required property. Finiteness of $\mathrm{Link}^{\mathrm{sd}}$ is immediate by construction. Finally, to see that the homotopy theoretic conditions on $\mathrm{Link}^{\mathrm{sd}}$ hold, observe that the poset $(\mathrm{sd}\mathcal{J})^{-1}(\mathcal{I})$ is empty, if and only if $\mathcal{I}$ is contained in the non-degenerate flag which $\mathcal{J}$ degenerates from, and that whenever it is non-empty, it contains a maximal element, given by the flag $\mathcal{J}^{-1}(\mathcal{I})$. In particular, this implies that its nerve $\mathrm{Link}^{\mathrm{sd}}_{\mathcal{I}}(\Delta^{\mathcal{J}}) \cong \mathrm{N}((\mathrm{sd}\mathcal{J})^{-1}(\mathcal{I}))$ is contractible.

Example 13.2.1.11. A third example of a cellular link functor was provided in [Dou21b]. It was denoted Sd_P there. We will use the notation $\mathrm{Link}^N_{\mathcal{I}}$ here. Given a regular flag $\mathcal{I}$, denote by $\mathrm{Link}^N_{\mathcal{I}}(\Delta^{\mathcal{I}})$ the subcomplex of $\mathrm{sdN}(P) \cong \mathrm{N}(\mathrm{sd}(P))$, given by the nerve of the poset given by such flags $\mathcal{I}' \in \mathrm{sd}(P)$ that contain $\mathcal{I}$. Then, for a general stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, $\mathrm{Link}^N_{\mathcal{I}}(\mathcal{X})$ is defined as the subsimplicial set of $\mathrm{sd}X$, defined via the pullback square

$$\begin{tikzcd}
\mathrm{Link}^N_{\mathcal{I}}(\mathcal{X}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{sd}X \arrow[d, "\mathrm{sd}s_{\mathcal{X}}"] \\
\mathrm{Link}^N_{\mathcal{I}}(\Delta^{\mathcal{I}}) \arrow[r, hook, "\mathcal{I}"'] & \mathrm{N}(\mathrm{sd}(P))\,.
\end{tikzcd} \tag{13.6}$$

Whenever $\mathcal{I}' \subset \mathcal{I}$, then there is a containment $\mathrm{Link}^N_{\mathcal{I}}(\mathcal{X}) \subset \mathrm{Link}^N_{\mathcal{I}'}(\mathcal{X})$. Hence, the construction agglomerates into a functor

$$\mathrm{Link}^N \colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$$
$$\mathcal{X} \mapsto \begin{cases} \mathcal{I} & \mapsto \mathrm{Link}^N_{\mathcal{I}}(\mathcal{X}) \\ (\mathcal{I}' \subset \mathcal{I}) & \mapsto (\mathrm{Link}^N_{\mathcal{I}}(\mathcal{X}) \subset \mathrm{Link}^N_{\mathcal{I}'}(\mathcal{X}))\,. \end{cases}$$

with the obvious functoriality on morphisms. If $\mathcal{I} = [p]$ is singleton, then $\mathrm{Link}^N_p(\mathcal{X})$ is the standard model for the regular neighborhood $N(X_p) \subset \mathrm{sd}X$ of the simplicial stratum $X_p \subset X$. For a more general flag $\mathcal{I}$, $\mathrm{Link}^N_{\mathcal{I}}(\mathcal{X})$ can be computed as the intersection of such neighborhoods

$$\mathrm{Link}^N_{\mathcal{I}}(\mathcal{X}) = \bigcap_{p \in \mathcal{I}} N(X_p)\,.$$

Applying this observation to simplices, it is not hard to show that this construction defines a cellular link functor.

It turns out that for the purpose of simple homotopy theory, the choice of concrete combinatorial model is essentially irrelevant. Following the notational convention of Notation 7.2.1.4, we denote the ∞-category obtained by localizing $\mathbf{Diag}_P$ at pointwise weak equivalences by $\mathcal{D}\mathbf{iag}_P$.

Proposition 13.2.1.12. *Let $L, L' \colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$ be two cellular link functors. Then there exists a unique (up to homotopy) natural transformation of functors of ∞-categories*

$$\begin{tikzcd}
 & \mathbf{Diag}_P \arrow[dr] \arrow[dd, Rightarrow, "\eta"] & \\
\mathrm{s}\mathbf{Strat}_P \arrow[ur, "L"] \arrow[dr, "L'"'] & & \mathcal{D}\mathbf{iag}_P\,. \\
 & \mathbf{Diag}_P \arrow[ur] &
\end{tikzcd} \tag{13.7}$$

η is an ∞-categorical simple equivalence of cellularized functors.

Proof. We apply Theorem 11.2.3.13, where we treat $\mathrm{s}\mathbf{Strat}_P$ as a category of presheaves on Δ_P. Since **sSet** is a properly generated Whitehead model category, so is $\mathbf{Diag}_P$ (Proposition 11.1.1.8). $\mathbf{Diag}_P$ is locally presentable and admits the structure of a simplicial model category. Hence, we only need to verify the requirements on mapping spaces and Whitehead groups. It follows by Corollary 11.1.2.8 that simple equivalences in $\mathbf{Diag}_P$ are precisely the flag-wise simple equivalences. Using this, the requirements on mapping spaces and Whitehead groups follow from Observation 13.2.1.7 and Lemma 13.2.1.8. □

Proposition 13.2.1.12 is extremely useful in so far, as it often allows us to verify a property for some choice of cellular link functor, and it immediately follows for all link functors. First off, let us observe that any link functor is a W-functor.

Proposition 13.2.1.13. *Every cellular link functor* $L\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$ *defines a W-functor of Whitehead model categories*

$$\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \to \mathbf{Diag}_P.$$

In particular, it follows by Proposition 10.3.1.4 that L *also defines a left Quillen functor between the associated model categories.*

Proof. Suppose that we expose a single cellular link functor which preserves weak equivalences. Then, it follows by Proposition 13.2.1.12 and the fact that a morphism in a model category is a weak equivalence, if and only if the associated morphisms in its localization at weak equivalences is an isomorphism, that all link functors preserve weak equivalences. As both model categories involved are cofibrant, a left adjoint functor between them is left Quillen, if and only if it preserves cofibrations and weak equivalences. The monomorphisms in both structures are precisely the relative cell complexes. Consequently, any cellular functor between the two categories preserves cofibrations, which shows that under the assumption at the beginning of this proof every cellular link functor is left Quillen. By Proposition 10.3.1.4, it now suffices to show that the induced functor of homotopy categories of finite cell complexes

$$L\colon \mathfrak{ho}_c\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \to \mathfrak{ho}_c\mathbf{Diag}_P$$

preserves simple equivalences. By the two-out-of-three property for simple equivalences, this property is invariant under ∞-categorical simple equivalence of cellularized functors. Using Proposition 13.2.1.12, we have thus reduced the proof to showing the existence of a single cellular link functor which is a W-functor. Let us show that $\mathrm{Link}^{\mathrm{sd}}$ which we recalled in Example 13.2.1.10 is such a cellularized functor. By Corollary 11.1.2.8 and Proposition 10.3.1.4, it suffices to see that $\mathrm{Link}^{\mathrm{sd}}_{\mathcal{I}}$ is a W-functor, for each $\mathcal{I} \in \Delta_P$. By Lemma 10.1.3.7, it suffices to show that $\mathrm{Link}^{\mathrm{sd}}_{\mathcal{I}}$ maps admissible horn inclusions into simple equivalences. In fact, in the proof of Lemma 3.4.3.2, we have even shown that the functor $\mathrm{Link}^{\mathrm{sd}}$ maps admissible horn inclusions into expansions. □

From Propositions 5.2.2.11 and 13.2.1.12 we can obtain another characterization of cellular link functors.

Corollary 13.2.1.14. *A finite cellularized functor* $L\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$ *is a cellular link functor if and only if the induced functor of* ∞*-categories obtained by composing with* $\mathbf{Diag}_P \to \mathcal{D}\mathbf{iag}_P$ *is isomorphic to* $\mathrm{HoLink}\colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P \to \mathcal{D}\mathbf{iag}_P$.

Next, we will show that cellular link functors define an (∞-categorical) homotopy equivalence of Whitehead model categories in the sense of Definition 10.3.2.9.

Theorem 13.2.1.15. *Let* $L\colon \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \to \mathbf{Diag}_P$ *be a cellular link functor. Denote by* $I\colon \mathbf{Diag}_P \to \mathrm{s}\mathbf{Strat}_P$ *the left adjoint to the simplicial homotopy link functor, given by*

$$D \mapsto \int^{\mathcal{I}} D_{\mathcal{I}} \otimes \Delta^{\mathcal{I}}.$$

I defines a W-functor. There are essentially unique natural transformations of the associated functors of ∞-categories

$$I \circ L \Rightarrow 1_{s\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P} \text{ and } L \circ I \Rightarrow 1_{\mathcal{D}\mathbf{iag}_P}.$$

These natural transformations are ∞-categorical simple equivalences. In particular, it holds that L and I are ∞-categorical homotopy equivalences of Whitehead model categories.

Proof. I is a left Quillen functor, and hence preserves cofibrations and colimits. Consequently, it follows by Remark 13.2.0.1 that I is a cellularized functor. I is clearly finite. To show that I is a W-functor, it remains to show that I preserves simple equivalences, or by Lemma 10.1.3.7 that I maps generating elementary expansions into simple equivalences. It follows by Corollary 11.1.2.8 that the canonical simplicial structure on $\mathbf{Diag}_P$ equips the latter with the structure of a simplicial Whitehead model category. The generating elementary expansions of $\mathbf{Diag}_P$ are given by the Leibniz tensors

$$(\Lambda^n_k \to \Delta^n)\hat{\otimes}(\partial \mathrm{sd}(P)^{\mathcal{I}} \hookrightarrow \mathrm{sd}(P)^{\mathcal{I}}).$$

Observe that I defines a simplicial functor. Hence, up to isomorphisms of relative cell complexes, the generating acyclic fibrations above are mapped to

$$(\Lambda^n_k \to \Delta^n)\hat{\otimes} I(\partial \mathrm{sd}(P)^{\mathcal{I}} \hookrightarrow \mathrm{sd}(P)^{\mathcal{I}}) = (\Lambda^n_k \to \Delta^n)\hat{\otimes}(\partial \Delta^{\mathcal{I}} \to \Delta^{\mathcal{I}}).$$

As $\underline{\mathrm{s}\mathbf{Strat}}^{\mathfrak{d}}_P$ is a simplicial Whitehead model category, it follows that this map is a simple equivalence. To prove the uniqueness and existence part on the claimed natural transformations, we use Lemma 11.2.3.11. We first expose a unique natural transformation $I \circ L \Rightarrow 1_{s\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P}$. Let $\mathcal{J} \in \Delta_P$. By Lemma 13.2.1.8 it follows that $L(\Delta^{\mathcal{J}})$ is weakly equivalent to $\mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star$, where $\mathcal{I}$ is the underlying regular flag of $\mathcal{I}$. Consequently, $I \circ L(\Delta^{\mathcal{J}})$ is weakly equivalent to $\Delta^{\mathcal{I}} \simeq \Delta^{\mathcal{J}} = 1(\Delta^{\mathcal{J}})$. Observe that $\Delta^{\mathcal{I}}$ is subterminal in the simplicial category $\underline{\mathrm{s}\mathbf{Strat}}_P$, and fibrant, and hence a subterminal object in $s\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P$. Hence, the requirements of Lemma 11.2.3.11 are fulfilled and we obtain the existence of an essentially unique natural transformation

$$\begin{tikzcd} & \mathrm{s}\mathbf{Strat}_P \arrow[dr] \arrow[dd, phantom, "\Downarrow"] & \\ \mathrm{s}\mathbf{Strat}_P \arrow[ur, "L\circ I"] \arrow[dr, "1"'] & & s\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \\ & \mathrm{s}\mathbf{Strat}_P \arrow[ur] & \end{tikzcd} \quad . \tag{13.8}$$

By the universal property of the localization, the latter descends to an essentially unique natural transformation of functors of ∞-categories

$$\begin{tikzcd} s\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \arrow[r, bend left, "I\circ L"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & s\mathcal{S}\mathbf{trat}^{\mathfrak{d}}_P \end{tikzcd} . \tag{13.9}$$

An essentially entirely analogous argument produces the converse natural transformation of functors of ∞-categories $L \circ I \Rightarrow 1$. Under the isomorphism of categories $\mathbf{Diag}_P \cong \mathbf{Set}^{\Delta^{\mathrm{op}} \times \mathrm{sd}(P)^{\mathrm{op}}}$, the representable presheaves are precisely given by the diagrams of the form

$$\mathrm{sd}(P)^{\mathcal{I}} \circledcirc \Delta^n \cong \Delta^n \otimes (\mathrm{sd}(P)^{\mathcal{I}} \circledcirc \star)\,.$$

It follows by Theorem 11.1.2.6 that the Whitehead groups of these diagrams are trivial. Hence, it follows Theorem 11.2.3.13 that the transformation $L \circ I \Rightarrow 1$ is a simple equivalence. It remains to be shown that the transformation $I \circ L \Rightarrow 1$ is a simple equivalence. By uniqueness of this natural transformation, together with Proposition 13.2.1.12 it suffices to prove this claim for a fixed cellular link functor L. We use the cellular link functor Link^N of Example 13.2.1.11. In [Dou21b], it was shown that the composition of functors of 1-categories $I \circ \mathrm{Link}^N$ is naturally isomorphic to the stratified subdivision functor $\mathrm{sd}_P \colon \mathrm{s}\mathbf{Strat}_P \to \mathrm{s}\mathbf{Strat}_P$ of [Dou21b]. In [Waa21, Prop. 2.3.19], we showed that the last vertex map transformation $\mathrm{sd}_P \to 1_{\mathrm{s}\mathbf{Strat}_P}$ is a simple equivalence of cellularized functors. Consequently, it also descends to an ∞-categorical simple equivalence $I \circ \mathrm{Link}^N \cong \mathrm{sd}_P \Rightarrow 1_{\mathrm{s}\mathbf{Strat}_P}$. By the essential uniqueness, it thus follows that any ∞-categorical natural transformation $I \circ \mathrm{Link}^N \Rightarrow 1_{\mathrm{s}\mathbf{Strat}_P}$ is a simple equivalence. □

13.2.2 The decomposition theorem

In Theorem 11.1.2.6, we computed the Whitehead group of a diagram indexed over a finite Reedy category in terms of the product of the Whitehead groups of its pointwise values. We may now combine this result with Theorem 13.2.1.15, to compute the stratified Whitehead groups $\mathrm{Wh}_P(\mathcal{X})$.

Corollary 13.2.2.1. *Let* $\mathrm{Link} \colon \mathrm{s}\mathbf{Strat}_P \to \mathbf{Diag}_P$ *be a cellular link functor. Then, the associated W-functors* $\mathrm{Link}_{\mathcal{I}} \colon \mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \to \mathbf{sSet}$ *, for* $\mathcal{I} \in \mathrm{sd}(P)$*, induce an isomorphism*

$$\begin{aligned} \mathrm{Wh}_P(\mathcal{X}) &\xrightarrow{\cong} \bigoplus_{\mathcal{I} \in \mathrm{sd}(P)} \mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big) \\ \langle \omega \rangle &\mapsto \sum_{\mathcal{I} \in \mathrm{sd}(P)} \langle \mathrm{Link}_{\mathcal{I}}(\omega) \rangle \,. \end{aligned}$$

Proof. In the case where P is a finite poset the claimed isomorphism is simply the composition of the isomorphism on Whitehead groups

$$\mathrm{Wh}_{\mathrm{Link}} \colon \mathrm{Wh}_P(\mathcal{X}) \to \mathrm{Wh}_{\mathbf{Diag}_P}\big(\mathrm{Link}(\mathcal{X})\big)$$

with the isomorphism

$$\mathrm{Wh}_{\mathbf{Diag}_P}\big(\mathrm{Link}(\mathcal{X})\big) \cong \prod_{\mathcal{I} \in \mathrm{sd}(P)} \mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big) = \bigoplus_{\mathcal{I} \in \mathrm{sd}(P)} \mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big)$$

of Theorem 11.1.2.6. For the infinite case, observe the following. Given a subposet $Q \subset P$, denote by

$$i_! \colon \mathrm{s}\mathbf{Strat}_Q \hookrightarrow \mathrm{s}\mathbf{Strat}_P$$

the left adjoint to the restriction functor $\mathcal{X} \mapsto \mathcal{X}|_Q$, given by postcomposing the stratification map with $Q \to P$. Observe that this functor defines a fully faithful inclusion of categories, and that it preserves cofibrations and admissible boundary inclusions. In particular, it defines a W-functor. More than that, $i_!$ has the property that its image is closed under weak equivalences in $\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$. It follows from this, that for any finite $\mathcal{Y} \in \mathrm{s}\mathbf{Strat}_Q$ the induced map of Whitehead groups

$$\mathrm{Wh}_Q(\mathcal{Y}) \to \mathrm{Wh}_P(i_! \mathcal{Y})$$

is an isomorphism. Next, observe that for any finite stratified simplicial set $\mathcal{X}$, there is a finite poset $Q \overset{i}{\hookrightarrow} P$, such that the counit of adjunction $i_!(\mathcal{X}|_Q) \to \mathcal{X}$ is an isomorphism (the identity even). Hence, one obtains a canonical isomorphism

$$\mathrm{Wh}_{i_!} \colon \mathrm{Wh}_Q(\mathcal{X}|_Q) \to \mathrm{Wh}_P(i_!(\mathcal{X}|_Q)) = \mathrm{Wh}_P(\mathcal{X}).$$

Furthermore, for any flag $\mathcal{I} \in \mathrm{sd}(P)$ that is not contained in Q, it holds that $\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) = \varnothing$ and hence that $\mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big) = 0$. Consequently, the canonical morphism

$$\bigoplus_{\mathcal{I} \in \mathrm{sd}(Q)} \mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big) \to \bigoplus_{\mathcal{I} \in \mathrm{sd}(P)} \mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big)$$

is an isomorphism. Now, consider the commutative diagram

$$\begin{tikzcd} \mathrm{Wh}_Q(\mathcal{X}|_Q) \arrow[r, dashed] \arrow[d, "\cong"'] & \bigoplus_{\mathcal{I}\in \mathrm{sd}(Q)} \mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big) \arrow[d, "\cong"] \\ \mathrm{Wh}_P(\mathcal{X}) \arrow[r] & \bigoplus_{\mathcal{I}\in \mathrm{sd}(P)} \mathrm{Wh}\big(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})\big)\,, \end{tikzcd} \tag{13.10}$$

with lower horizontal as in the statement of the theorem, and the dashed map uniquely determined by commutativity. Next, observe that the composition

$$\mathrm{s}\mathbf{Strat}_Q \xhookrightarrow{i_!} \mathrm{s}\mathbf{Strat}_P \xrightarrow{\mathrm{Link}} \mathbf{Diag}_P \xrightarrow{(-)|_{\mathrm{sd}(Q)}} \mathbf{Diag}_Q$$

fulfills the defining properties of a cellular link functor. The dashed horizontal in the last commutative diagram is precisely the map in the statement of the theorem associated to this link functor. Hence, it follows from the finite case that it is an isomorphism. Consequently, so is the lower horizontal. ☐

Notation 13.2.2.2. Given the isomorphism in Corollary 13.2.2.1, we will always denote the projection to the $\mathcal{I}$-th component of $\langle \omega \rangle \in \mathrm{Wh}_P(\mathcal{X})$ by $\langle \omega \rangle_{\mathcal{I}}$. Observe that, up to the unique identification of cellular homotopy links guaranteed by Proposition 13.2.1.12, this expression is well-defined.

13.3 Some final observations

We can summarize the key insights of what we have determined about diagrammatic stratified simple homotopy theory so far in terms of the following two statements.

1. Stratified realization induces an equivalence of Whitehead model categories

   $$\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}} \simeq \mathbf{Strat}_P^{\mathfrak{d}},$$

   between the simplicial and the topological setting.

2. Any choice of cellular link functor induces a natural isomorphism

   $$\mathrm{Wh}_P(\mathcal{X}) \cong \bigoplus_{\mathcal{I}\in \mathrm{sd}(P)} \mathrm{Wh}(\mathrm{Link}_{\mathcal{I}}(\mathcal{X})),$$

   where $\mathcal{X}$ ranges over finite P-stratified simplicial sets.

We can now use these results to derive a few immediate consequences about the diagrammatic topological stratified simple homotopy theory.

Observation 13.3.0.1. Let $\mathfrak{Y}$ in $\mathbf{Strat}_P^{\mathfrak{d}}$ be a finite structured stratified cell complex with underlying stratified space $\mathcal{Y}$. Then it follows by Theorem 13.1.2.7 that there exists a finite stratified simplicial set $\mathcal{X} \in \mathrm{s}\mathbf{Strat}_P$, together with a simple equivalence

$$|\mathcal{X}|_s \simeq \mathfrak{Y}.$$

If we now fix some cellular link functor Link, then it follows from Corollary 13.2.1.14 that we obtain canonical equivalences

$$\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \simeq \mathrm{HoLink}_{\mathcal{I}}(\mathcal{X}) \simeq \mathrm{HoLink}_{\mathcal{I}}(\mathrm{Sing}_s(\mathcal{Y})) \cong \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y})\,,$$

for $\mathcal{I} \in \mathrm{sd}(P)$. Here, we consider $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y})$ as a simplicial set, by passing to singular simplices as in Recollection 7.3.1.1. Observe that it also follows by Theorem 13.1.2.7 that for any other choice of stratified simplicial set and simple equivalence

$$|\mathcal{X}'|_s \simeq \mathcal{Y}$$

the induced composition

$$\mathrm{Link}_{\mathcal{I}}(\mathcal{X}) \simeq \mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{Y}) \simeq \mathrm{Link}_{\mathcal{I}}(\mathcal{X}')$$

is a simple equivalence. In this sense, any choice of finite cell structure on a stratified topological space $\mathcal{Y}$ (or more generally a choice of presentation of the diagrammatic homotopy type up to simple equivalence) determines a choice of simple homotopy type for all generalized homotopy links. Combining Theorem 13.1.2.7 with Corollary 13.2.2.1 produces isomorphisms

$$\mathrm{Wh}_P(\mathcal{Y}) \cong \mathrm{Wh}_P(|\mathcal{X}|_s) \cong \mathrm{Wh}_P(\mathcal{X}) \cong \bigoplus_{\mathcal{I}\in \mathrm{sd}(P)} \mathrm{Wh}(\mathrm{Link}_{\mathcal{I}}(\mathcal{X}))$$

Under Theorem 10.2.3.9, we may interpret this result as stating that a choice of presentation of the diagrammatic stratified homotopy type of $\mathcal{Y}$ in terms of a finite stratified cell complex is, modulo simple equivalences, the same as a choice of presentation of each homotopy link in terms of a finite cell complex (CW-complex, simplicial set, simplicial complex).

Observation 13.3.0.2. Taking the perspective that finite cell structures on a stratified space $\mathcal{X} \in \mathbf{Strat}_P^{\mathfrak{d}}$ correspond to fixing a presentation in terms of a finite cell complex for all homotopy links of $\mathcal{X}$, we may characterize simple homotopy equivalences between finite structured stratified cell complexes as such maps that induce simple homotopy equivalences on all generalized homotopy links (with respect to the induced presentations).

Remark 13.3.0.3. One can also deduce through an inductive argument involving Theorems 13.1.2.7 and 13.2.1.15 that a stratified space $\mathcal{X}$ over a finite poset P has the diagrammatic homotopy type of a finite stratified cell complex, if and only if for each $\mathcal{I} \in \mathrm{sd}(P)$, the homotopy link $\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$ has the weak homotopy type of a finite cell complex.

As a first corollary, we obtain the Whitehead groups of a stratified simplex.

Corollary 13.3.0.4. *Let $\mathcal{J} \in \Delta_P$. Then*

$$\mathrm{Wh}_P(|\Delta^{\mathcal{J}}|_s) = 0.$$

Proof. As Whitehead groups are invariant under weak equivalences and every stratified simplex is weakly equivalent to a stratified simplex $|\Delta^{\mathcal{I}}|_s$, where $\mathcal{I} \in \Delta_P$ is regular, we only need to cover the case of a regular flag. We may then use the link functor of Example 13.2.1.10, which produces

$$\mathrm{Link}_{\mathcal{I}'}(\Delta^{\mathcal{I}}) = \begin{cases} \star & \text{, if } \mathcal{I}' \subset \mathcal{I} \\ \varnothing & \text{, if } \mathcal{I}' \not\subset \mathcal{I}. \end{cases}$$

Consequently, it follows that

$$\mathrm{Wh}_P(|\Delta^{\mathcal{I}}|_s) = \mathrm{Wh}_P(\Delta^{\mathcal{I}}) = \bigoplus_{\mathcal{I}'\in \mathrm{sd}(P)} \mathrm{Wh}(\mathrm{Link}_{\mathcal{I}'}\Delta^{\mathcal{I}}) = 0\,.$$

□

As a particular corollary of Corollary 13.3.0.4, we obtain the invariance of simple stratified homotopy types under subdivisions of cell structures:

Recollection 13.3.0.5. Recall that a subdivision of a stratified cell complex $\mathfrak{X}$ in $\mathbf{Strat}_P$ is a cell structure $\mathfrak{X}'$ on $\mathcal{X}$ of the form

$$\mathfrak{C}_{\mathfrak{X}'} = \{\sigma \circ \tau \mid \sigma\colon |\Delta^{\mathcal{I}}|_s \to \mathcal{X} \in \mathfrak{C}_{\mathfrak{X}}, \tau \in \mathfrak{C}_\sigma\}$$

where $\mathfrak{C}_\sigma$, for each $\sigma \in \mathfrak{C}_{\mathfrak{X}}$, is a choice of relative cell structure on $|\partial\Delta^{\mathcal{I}} \to \Delta^{\mathcal{J}}|_s$ (see, Definition 11.2.2.3). We will slightly extend this definition in the sense that we also call a stratum-preserving homeomorphism between stratified cell complexes $\phi\colon \mathfrak{X} \to \mathfrak{Y}$ a subdivision, if the cell structure $\mathfrak{Y}' = \phi\mathfrak{C}_{\mathfrak{X}}$ on $\mathcal{Y}$ obtained by transporting the cell structure on $\mathfrak{X}$ to $\mathcal{Y}$ along ϕ is a subdivision of $\mathfrak{Y}$.

As an immediate corollary of Corollary 13.3.0.4 and Proposition 11.2.2.4, we obtain:

Corollary 13.3.0.6. *Let $\mathfrak{X}$ be a finite structured stratified cell complex and let $\phi\colon \mathfrak{X}' \xrightarrow{\cong} \mathfrak{X}$ be a subdivision of $\mathfrak{X}$. Then ϕ is a simple diagrammatic equivalence.*

Remark 13.3.0.7. One should be careful to note that the definition of subdivision on Recollection 13.3.0.5 is significantly more general than just being a subdivision in the classical, piece-wise linear sense. In particular, one is generally allowed to subdivide in ways which are highly non-linear and furthermore can subdivide the interior of a cell, without subdividing any of the cells intersecting its boundary.

13.3.1 Connection with the piecewise linear setting

Much of the investigations of stratified spaces which admit some type of triangulation were performed not in the framework of simplicial sets, but instead in the piecewise linear world, using the language of polyhedra (see [Sto72] and [Wei94] for an overview). Let us quickly make the connection to this framework. The transition is fairly elementary and we will not give too many details here.

Definition 13.3.1.1. We call a simplicial set $X \in \mathbf{sSet}$ an *(ordered) simplicial complex*, if each non-degenerate simplex of X is uniquely determined by its set of vertices, i.e., if for each $n \in \mathbb{N}$, the map

$$\begin{aligned} f\colon \bigsqcup_{n\in\mathbb{N}} X_{n,\text{n.d.}} &\to X_0 \\ \sigma &\mapsto \{\sigma_i \in X_0 \mid i \in [n]\} \end{aligned}$$

– assigning to a non-degenerate simplex its set of vertices – is injective.

Remark 13.3.1.2. Note that if X is a simplicial complex, then it follows in particular that every face τ of a non-degenerate simplex $\sigma\colon \Delta^n \to X$ is non-degenerate. Indeed, if this were not the case, then σ would have a set of vertices that is of cardinality smaller than $n+1$, and one can show that there exists a non-degenerate face σ' of σ such that σ' and σ have the same vertices. It is not hard to see that the full subcategory of such X is equivalent to the category of what is classically referred to as ordered abstract simplicial complexes and order-preserving simplicial maps.

Definition 13.3.1.3. Given simplicial complex $X \in \mathbf{sSet}$, a *simplicial subdivision of* X consists of a simplicial complex X', together with a homeomorphism

$$\phi\colon |X'| \xrightarrow{\cong} |X|$$

such that, for every non-degenerate simplex σ' of X', the following holds. There exists a non-degenerate simplex σ of X such that the following factorization exists

$$\begin{tikzcd} {|\Delta^n|} \arrow[r, hook, dashed] \arrow[d, hook, "{|\sigma'|}"'] & {|\Delta^m|} \arrow[d, hook, "{|\sigma|}"] \\ {|X|} \arrow[r, "\cong", "\phi"'] & {|X'|} \end{tikzcd} \tag{13.11}$$

and such that the dashed map is an affine map.

Remark 13.3.1.4. If X is finite, then we can embed $|X|$ into $\mathbb{R}^{X_0}$ by affinely embedding the (non-degenerate) simplices of $|X|$ via

$$\begin{aligned} |\Delta^n| &\to \mathbb{R}^{X_0} \\ e_i &\mapsto e_{\sigma_i} \end{aligned}$$

where we consider $|\Delta^n|$ as embedded in $\mathbb{R}^{[n]}$, and denote by $e_s \in \mathbb{R}^S$, for a set S and $s \in S$, the standard basis vector given by 1 at s and 0 everywhere else. Then the definition of subdivision in Definition 13.3.1.3 is equivalent to the classical definition of subdivision of simplicial complexes in the sense of [RS12] in the sense that the images of the non-degenerate simplices of $|X'|$ under ϕ in $|X| \subset \mathbb{R}^{X_0}$ define a subdivision of the (classical) simplicial structure on $|X|$ (given by the non-degenerate simplices of X).

Definition 13.3.1.5. Given simplicial complexes $X, Y \in \mathbf{sSet}$, we say that a continuous map

$$f \colon |X| \to |Y|$$

is *piecewise linear* if there exists a simplicial subdivision $\phi \colon |X'| \xrightarrow{\cong} |X|$ such that the induced map $\phi \circ f \colon |X'| \to |Y|$ has the following property: For every non-degenerate simplex σ' of X', there exists a non-degenerate simplex τ of Y such that the following factorization exists

$$\begin{tikzcd} |\Delta^n| \arrow[r, dashed] \arrow[d, hook, "|\sigma'|"'] & |\Delta^m| \arrow[d, hook, "|\tau|"] \\ |X| \arrow[r, "\cong", "\phi\circ f"'] & |X'| \end{tikzcd} \tag{13.12}$$

and such that the dashed map is affine.

Remark 13.3.1.6. In the case of finite simplicial complexes X and Y it is not hard to see that this definition of piecewise linearity agrees with the classical one (see for example [RS12]) modulo appropriate embeddings into Euclidean space.

Remark 13.3.1.7. Using stellar subdivisions, one can see that every subdivision of a subcomplex $K \subset X$, $|K'|_s \cong |K|$ can be extended to a subdivision of X. It follows that being piecewise linear is a property that can be verified after restriction to simplices. In particular, using the compactness of simplices, checking piecewise linearity can be reduced to a verification on finite simplicial sets. It follows from this that most classical statements on piecewise linearity, such as being stable under composition and inversion of homeomorphisms, translate to our slightly more general setting.

Remark 13.3.1.8. The thus-defined category of simplicial complexes and piecewise linear maps embeds fully faithfully into what Zeeman defined to be the category of polyspaces in [ZI63].

Notation 13.3.1.9. We will generalize most of the language introduced for the non-stratified scenario here to the stratified scenario, by referring to the underlying non-stratified objects. In this sense, by a stratified simplicial complex, we mean a stratified simplicial set, whose underlying simplicial set is a simplicial complex. We proceed analogously for piecewise linear maps etc.

Any piecewise linear stratum-preserving homeomorphism defines a simple equivalence in the Whitehead model category $\mathbf{Strat}_P^{\mathfrak{d}}$:

Corollary 13.3.1.10. *Let $\mathcal{X}, \mathcal{Y} \in \mathrm{s}\mathbf{Strat}_P$ be finite stratified simplicial complexes. Let*

$$\phi \colon |\mathcal{X}|_s \to |\mathcal{Y}|_s$$

be a stratum-preserving homeomorphism that is piecewise linear with respect to the induced polyhedral structure on $|X|$ and $|Y|$. Then ϕ is a simple diagrammatic equivalence, with respect to the induced cell structures on source and target.

Proof. It is a classical fact that any piecewise linear map between realizations of finite simplicial complexes can be made simplicial, up to a subdivision of source and target ([RS12, Theo. 2.14]). Subdividing barycentrically once more, we may ensure that the resulting subdivisions come from ordered simplicial complexes, which define simplicial sets. Hence, we obtain a commutative diagram of stratified cell complexes

$$\begin{tikzcd} |\mathcal{X}|_s \arrow[r, "\phi"] \arrow[d, "\cong"'] & |\mathcal{Y}|_s \arrow[d, "\cong"] \\ |\mathcal{X}'|_s \arrow[r, "|f'|"'] & |\mathcal{Y}'|_s \end{tikzcd} \tag{13.13}$$

where the two verticals are given by subdivisions in the classical simplicial complex sense, and $f' : \mathcal{X}' \to \mathcal{Y}'$ is a map of stratified simplicial sets. By commutativity of the diagram, it follows that $|f'|_s$ is an isomorphism. Consequently, the underlying topological map is a homeomorphism, and it follows from the fact that $|-|$ is a conservative functor that $f' : \mathcal{X} \to \mathcal{Y}$ is an isomorphism of stratified simplicial sets. In particular, it is an isomorphism of cell complexes (with respect to the canonical unique cell structures) and thus a simple equivalence. Hence, it follows from Corollary 13.3.0.6 and the two-out-of-three property for simple equivalences that ϕ is a simple equivalence. □

There is another useful consequence to Corollary 13.3.0.6. Namely, it allows us to define a Whitehead group for stratified polyhedra, where the cell structure is only well defined up to PL homeomorphism. It will be useful to also have a notion of polyhedron available, which allows for flexible triangulations. We will emulate the definition used in [Sto72; ZI63].

Definition 13.3.1.11. By a *triangulation of a stratified space* $\mathcal{X} \in \mathbf{Strat}_P^{\mathfrak{d}}$, we mean a pair consisting of a stratified simplicial complex $\mathcal{K} \in \mathrm{s}\mathbf{Strat}_P$ together with a stratum-preserving homeomorphism

$$\phi : |\mathcal{K}|_s \xrightarrow{\cong} \mathcal{X}.$$

We say that two triangulations $(\mathcal{K}, \phi)$ and $(\mathcal{K}', \phi')$ are *related*, if the induced stratum-preserving homeomorphism

$$|\mathcal{K}|_s \xrightarrow{\phi} \mathcal{X} \xrightarrow{\phi'^{-1}} |\mathcal{K}'|_s$$

(and hence also its inverse) is piecewise linear.

Observation 13.3.1.12. Observe that given a triangulation of a stratified space $|\mathcal{K}|_s \cong \mathcal{X}$, the space is compact if and only if $\mathcal{X}$ is finite.

Definition 13.3.1.13. By a *stratified finite polyhedron*, we mean a compact stratified topological space $\mathcal{X} \in \mathbf{Strat}_P$ together with a maximal set of related triangulations.

Remark 13.3.1.14. There is a slightly set-theoretical issue here, in the sense that the class of all triangulations $|\mathcal{K}'|_s \cong \mathcal{X}$ related to a fixed triangulation is not a set. Hence, from this set-theoretic perspective, such a set cannot exist. This is of course easily circumvented, by fixing some set-sized category of finite sets beforehand (for example, all subsets of $\mathbb{N}$) and then defining a finite simplicial set to be a presheaf valued in this category. Equivalently, we can take the quotient set where we mod out by the relation of isomorphism of finite simplicial sets. The resulting quotient set has only countably many elements. We will freely ignore these set-theoretic issues here, noting that the ∞-categorical methods which we have made excessive use of often assume the existence of larger cardinals anyway.

We will generally drop the *finite* from the name for polyhedra, and assume that for our purposes all polyhedra are compact.

Notation 13.3.1.15. At this point, we have used the calligraphic font for stratified things, fraktur font for cell complexes and bold font for categories. Having run out of fonts at this point, we are also going to use fraktur font for polyhedra. This is at least not entirely wrong, in-so-far, as wherever there is a polyhedron a family of cell structures is included.

Remark 13.3.1.16. In the case where $P = [n]$, a stratified polyhedron is essentially the same thing as what Stone calls a filtered (compact) polyhedron in [Sto72]. The subtle differences are the following:
We use simplicial sets as a model for simplicial complexes, while Stone uses geometric simplicial complexes embedded in some Euclidean space. As we elaborated in Remark 13.3.1.4 and Remark 13.3.1.6, in the finite scenario these two theories are equivalent (as categories) after passing to the piecewise linear (PL for short) world. The advantage for our language is that all simplicial complexes are intrinsically ordered and that the subcomplexes $\mathcal{X}_{\leq p} \subset \mathcal{X}$, for $p \in P$ are full subcomplexes. Stone usually ensures these conditions by subdividing triangulations once barycentrically, before stating any kind of result. Clearly, up to PL isomorphisms, these conditions can always be achieved.

Construction 13.3.1.17. We can now associate to a stratified polyhedron $\mathfrak{X}$ a stratified Whitehead group as follows: The obvious thing to do is to fix some triangulation $\phi\colon |\mathcal{K}|_s \xrightarrow{\cong} \mathcal{X}$ and set

$$\mathrm{Wh}_P(\mathfrak{X}) \coloneqq \mathrm{Wh}_P(|\mathcal{K}|_s).$$

While there is clearly a choice being made here, for any other choice of triangulation $\phi'\colon |\mathcal{K}|_s \xrightarrow{\cong} \mathcal{X}$, the induced stratum-preserving homeomorphism $\phi'^{-1} \circ \phi$ defines a canonical isomorphism

$$\mathrm{Wh}_P(|\mathcal{K}|_s) \xrightarrow{(\phi'^{-1}\circ\phi)_*} \mathrm{Wh}_P(|\mathcal{K}'|_s).$$

For the sake of cleanliness, it can be nice to not have to constantly refer to a specific choice of triangulation. We can circumvent this, by considering the indiscrete (small) category $\mathbf{T}(\mathfrak{X})$ whose objects are triangulations of $\mathfrak{X}$, and which has exactly one morphism between any two objects. We then consider the diagram of abelian groups defined by

$$\begin{gathered} (\mathcal{K}, \phi) \mapsto \mathrm{Wh}_P(|\mathcal{K}|_s) \\ \big((\mathcal{K}, \phi) \to (\mathcal{K}', \phi')\big) \mapsto (\phi'^{-1} \circ \phi)_* \, . \end{gathered}$$

and set $\mathrm{Wh}_P(\mathfrak{X})$ to the colimit of this diagram. As this is a colimit over a diagram over an indiscrete category with all morphisms given by isomorphisms, the canonical associated morphisms

$$\mathrm{Wh}_P(|\mathcal{K}|_s) \to \mathrm{Wh}_P(\mathfrak{X})$$

are all isomorphisms, and we can safely identify $\mathrm{Wh}_P(\mathfrak{X})$ with the Whitehead group of any of its triangulations. In this fashion, this construction also becomes covariantly functorial in morphisms between polyhedra in $\mathrm{ho}\mathbf{Strat}^{\mathfrak{c}}_P$, by pre- and postcomposing the latter appropriately and using the functoriality of the Whitehead group on the associated morphisms of triangulations.

The crucial point to be made here is that not only the choice of Whitehead group functor for polyhedra is independent of the choice of triangulation, but also the choice of Whitehead torsion:

Construction 13.3.1.18. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stratified polyhedra, and let $\alpha\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of the underlying stratified spaces in $\mathrm{ho}\mathbf{Strat}^{\mathfrak{c}}_P$. The Whitehead torsion of α, denoted

$$\langle \alpha \rangle \in \mathrm{Wh}_P(\mathfrak{X})$$

is defined as follows. Fix two triangulations $\phi\colon |\mathcal{K}|_s \cong \mathcal{X}$ and $\psi\colon |\mathcal{L}|_s \cong \mathcal{Y}$ of $\mathfrak{X}$ and $\mathfrak{Y}$. Then

$$\psi^{-1} \circ \alpha \circ \phi\colon |\mathcal{K}|_s \xrightarrow{\phi} \mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\psi^{-1}} |\mathcal{L}|_s$$

is a morphism between structured stratified cell complexes, and hence has a well defined Whitehead torsion in $\langle \psi^{-1} \circ \alpha \circ \phi \rangle \in \mathrm{Wh}_P(\mathcal{K})$. We define $\langle \alpha \rangle$ as the image of $\langle \psi^{-1} \circ \alpha \circ \phi \rangle$ under the canonical isomorphism

$$\mathrm{Wh}_P(|\mathcal{K}|_s) \cong \mathrm{Wh}_P(\mathfrak{X}) \, .$$

Let us verify the invariance of this construction of Whitehead torsion of the choices of triangulations made.

Proof. Observe first that this class is invariant of the choice of ψ. For any other choice ψ', the two morphisms $\psi^{-1} \circ \alpha \circ \phi$ and $\psi'^{-1} \circ \alpha \circ \phi$ differ by the postcomposition with the piecewise linear stratified homeomorphism $\psi'^{-1} \circ \psi$. By Corollary 13.3.1.10, the latter is a simple equivalence. Hence, it follows that

$$\langle \psi^{-1} \circ \alpha \circ \phi \rangle = \langle \psi'^{-1} \circ \alpha \circ \phi \rangle \, .$$

To see invariance of the choice of ϕ, observe that for any other choice $(\mathcal{K}', \phi')$, the identifications of Whitehead groups of triangulations with the Whitehead group of $\mathfrak{X}$ fit into a commutative diagram

$$\begin{tikzcd} \mathrm{Wh}_P(|\mathcal{K}|_s) \arrow[rr, "(\phi'^{-1} \circ \phi)_*"] \arrow[dr, "\cong"'] & & \mathrm{Wh}_P(|\mathcal{K}'|_s). \arrow[dl, "\cong"] \\ & \mathrm{Wh}_P(\mathfrak{X}) & \end{tikzcd} \tag{13.14}$$

Hence, we only need to verify that

$$(\phi'^{-1} \circ \phi)_* \langle \psi^{-1} \circ \alpha \circ \phi \rangle = \langle \psi^{-1} \circ \alpha \circ \phi' \rangle.$$

Indeed, as $(\phi'^{-1} \circ \phi)$ is a simple diagrammatic equivalence, it follows by Lemma 9.1.3.11 that

$$(\phi'^{-1} \circ \phi)_* \langle \psi^{-1} \circ \alpha \circ \phi \rangle = \langle \psi^{-1} \circ \alpha \circ \phi \circ (\phi'^{-1} \circ \phi)^{-1} \rangle = \langle \psi^{-1} \circ \alpha \circ \phi' \rangle$$

as was to be shown. $\square$

13.3.2 Stratum-preserving homeomorphisms of stratified cell complexes are generally not simple diagrammatic equivalences.

In the previous section, we have seen that every stratum-preserving PL homeomorphism is a simple diagrammatic equivalence. One should be careful, however, not to draw the following conclusion in analogy to the classical scenario of CW-complexes: It is a classical result due to Chapman that any homeomorphism of finite CW-complexes is a simple homotopy equivalence (see [Cha74]). Hence, in the classical topological scenario, one can even associate a well-defined Whitehead torsion to any map of spaces which have the homeomorphism type of a CW-complex, without any need for fixing cell structures on target and source. In the stratified world, this is no longer the case. In fact, without a choice of cell structure, one can generally not even assign a well-defined simple homotopy type to strata.

This is certainly a fact well known to the expert in slightly different language, which is handled with techniques involving cosheaves of Whitehead spectra in Quinn's study of cobordism theorems for homotopically stratified spaces in [Qui88]. We will spell out the argument in some detail in the remainder of this section, as it illustrates well the possible pitfalls when working with simple stratified homotopy types. It may also be seen as a first step from the homotopy-theoretic approach to simple stratified homotopy theory to more geometric approaches. For a detailed outlook into how these scenarios connect, see Section 2.5.2.

Recollection 13.3.2.1. In the following, we will think of cobordisms as arrows in the piecewise linear cobordism category (of some fixed dimension n), where objects are given by closed piecewise linear manifolds of dimension n and morphisms from M to M' are given by piecewise linear homeomorphism classes of arrows

$$i \colon M \sqcup M' \to W,$$

where W is a compact PL $n+1$-manifold with boundary, such that i defines a PL homeomorphism onto the boundary of W. The identity on M is given by the inclusion of the boundary of the cylinder $M \sqcup M \hookrightarrow M \times [0,1]$. Composition is given by gluing of cobordisms as indicated in the diagram below.

$$\begin{tikzcd}
 & & & M' \arrow[dl] \arrow[dr] & & \\
 & & M \sqcup M' \arrow[dl] \arrow[dr] & M \sqcup M'' \arrow[d, hook] & M' \sqcup M'' \arrow[dl] \arrow[dr] & \\
 & W_1 \arrow[drr] & & M \sqcup M' \sqcup M'' \arrow[d, dashed] & & W_2 \arrow[dll] \\
 & & & W_1 \cup_{M'} W_2 & &
\end{tikzcd} \tag{13.15}$$

We will use sloppy notation insofar, as we will just refer to a cobordism $M \xrightarrow{W} M'$, when referring to a morphism, and keep the fact that we are actually dealing with an equivalence class of boundary inclusions implicit. We will also often omit the inclusions into the boundary, and just think of M and M' as submanifolds of W.

Recollection 13.3.2.2. It is a consequence of the PL version of the s-cobordism theorem of Mazur, Stallings and Barden (see, for example, [Coh73]), that in dimension $n \geq 5$ every h-cobordism $M \xrightarrow{W} M'$ defines an invertible morphism in the PL cobordism category. To see this, denote by $i_M : M \hookrightarrow W$ and $i_{M'} : M' \hookrightarrow W$ the two associated boundary inclusions. As these inclusions are assumed to be homotopy equivalences, the two maps (the first being a map of sets) of Whitehead groups

$$i_M^* : \mathrm{Wh}(W) \to \mathrm{Wh}(M)$$
$$(i_{M'})_* : \mathrm{Wh}(M') \to \mathrm{Wh}(W)$$

define bijections (we use the fact that PL isomorphisms are simple equivalences to suppress any mention of triangulations here). It follows that there is an element of $\alpha \in \mathrm{Wh}(M')$ such that $i_M^*(i_{M'})_*\alpha = 0$. By the realization part of the s-cobordism theorem, we can represent α in terms of the inclusion $j_{M'} : M' \hookrightarrow V$ into a piecewise linear h-cobordism $M' \xrightarrow{V} M''$. By definition of $(i_{M'})_*$, it follows that

$$(i_{M'})_* \langle j_{M'} \rangle$$

is given by the Whitehead torsion of the inclusion $W \hookrightarrow W \cup_{M'} V$. Observe that the composition of inclusions

$$M \hookrightarrow W \hookrightarrow W \cup_{M'} V$$

is the (left) boundary inclusion associated to the composition of cobordisms $M \xrightarrow{W} M' \xrightarrow{V} M''$. By construction, we have

$$0 = i_M^*(i_{M'})_* \langle j_{M'} \rangle = \langle M \hookrightarrow W \hookrightarrow W \cup_{M'} V \rangle = \langle M \hookrightarrow (V \circ W) \rangle.$$

Hence, it follows by the s-cobordism theorem that $V \circ W$ is PL homeomorphic to a cylinder $M \times [0,1]$ (relative to M). In particular, we have a PL homeomorphism $M'' \cong M$, and may assume without loss of generality (by replacing $M'' \hookrightarrow V$ by $M \cong M'' \hookrightarrow V$) that $M'' = M$. Then the PL homeomorphism $W \circ V \cong M \times [0,1]$ is relative to $M \sqcup M$, and hence it follows that $V \circ W$ defines the identity in the cobordism category. Repeating the argument with the h-cobordism V, it follows that V also has a right inverse, which is necessarily given by W.

Recollection 13.3.2.3. Given a PL h-cobordism of 5-manifolds

$$M \xrightarrow{W} M'$$

the space $W \setminus M$ is homeomorphic to $M \times [0,1)$ relative to M. This is a consequence of the following Eilenberg swindle: It is a classical fact in PL topology (and a consequence of the existence of collar neighborhoods) that the gluing $W \cup_{M'} M' \times [0,\infty)$ is homeomorphic to $W \smallsetminus M'$ (relative to M). We may rewrite $M' \times [0,\infty)$ as the colimit

$$\varinjlim_n M' \times [0,n]$$

along the obvious diagram of inclusions. Then, using the inverse cobordism V to W, we may rewrite this colimit as

$$\varinjlim_n \big((W \circ V) \circ \cdots \circ (W \circ V)\big),$$

where the composition in the n-th step is $2n$-fold. Putting all of this information together, we obtain a PL homeomorphism

$$\begin{aligned} W \smallsetminus M' &\cong W \cup_{M'} M' \times [0,\infty) \\ &\cong W \cup_{M'} \varinjlim_n \big((W \circ V \circ \cdots \circ W \circ V\big) \\ &\cong \varinjlim_n \big(W \circ V \circ \cdots \circ W \circ V \circ W\big). \\ &\cong \varinjlim_n \big((W \circ V \circ \cdots \circ W\big) \circ V \circ W) \\ &\cong \varinjlim_n \big(W \circ V \circ \cdots \circ W\big) \circ M \times [0,1] \\ &\cong \varinjlim_n \big((W \circ V \circ \cdots \circ W\big) \end{aligned}$$

relative to M. Performing an Eilenberg swindle on the colimit in the last step, we obtain

$$\begin{aligned} \varinjlim_n \big(W \circ V \circ \cdots \circ W\big) &\cong \varinjlim_n \big(V \circ \cdots \circ W\big) \\ &\cong \varinjlim_n \big((V \circ W) \circ \dots (V \circ W)\big) \\ &\cong \varinjlim_n M \times [0,n] \\ &\cong M \times [0,\infty) \cong M \times [0,1)\,. \end{aligned}$$

We can now give a counterexample that shows that not every stratified homeomorphism defines a simple diagrammatic equivalence.

Example 13.3.2.4. Let M be a closed PL manifold of dimension greater or equal to 5 and let $M \xrightarrow{W} M'$ be a PL h-cobordism with non-trivial Whitehead torsion. Fix any triangulation of W in terms of a simplicial complex $K \in \mathbf{sSet}$, such that there is a subcomplex L of K triangulating M and L' triangulating M'. Subdividing barycentrically once, we may assume that K is such that there exists a map $s\colon K \to \mathrm{N}(\{p < q\})$ with $s^{-1}(p) = L'$.
We denote by $\mathcal{K}$ the thus obtained stratified simplicial set with lower stratum L'. Denote by

$$\mathcal{C}M := M \times |\Delta^{\{p<q\}}|_s / M \times |\Delta^{\{p\}}|_s\,,$$

the stratified cone one M over a poset with two elements $\{p < q\}$. The triangulation $|L|_s \cong M$, equips the stratified cone with a stratified cell structure arising from the stratum-preserving homeomorphism

$$|L \times \Delta^{\{p<q\}} / L \times \Delta^{\{p\}}|_s \cong M \times |\Delta^{\{p<q\}}|_s / M \times |\Delta^{\{p\}}|_s \cong \mathcal{C}M.$$

There is, however, another cell structure available, arising from $\mathcal{K}$. To see this, observe that we may treat W as a stratified space over $\{p < q\}$, taking M' to be the p-stratum. Then there is a stratum-preserving homeomorphism

$$W/M' \cong \mathcal{C}M,$$

obtained by fixing a homeomorphism $\phi: W \setminus M' \cong M \times [0,1)$ and then passing to one-point-compactifications. A priori, this produces cones with the teardrop topology (see [Qui88]). However, as M is compact and Hausdorff, teardrop topology and quotient topologies agree. Using this stratum-preserving homeomorphism, we obtain another cell structure on $\mathcal{C}M$ via the stratum-preserving homeomorphisms

$$|\mathcal{K}/\mathcal{K}_p|_s \cong W/M' \cong \mathcal{C}M.$$

We claim that the identity map between these two cell structures is not a simple diagrammatic equivalence, or in other words, that the induced stratum-preserving homeomorphism

$$\phi: |L \times \Delta^{\{p<q\}}/L \times \Delta^{\{p\}}|_s \cong |\mathcal{K}/\mathcal{K}_p|_s$$

is not a simple diagrammatic equivalence. Under Theorem 13.1.2.7, we may equivalently show that the associated isomorphism in $\mathrm{hos}\mathbf{Strat}_P$,

$$\omega: L \times \Delta^{\{p<q\}}/L \times \Delta^{\{p\}} \simeq \mathcal{K}/\mathcal{K}_p.$$

is not a simple diagrammatic equivalence. By Theorem 11.1.2.6, it suffices to show that ω does not descend to a simple equivalence on some cellular link. In fact, ω does not even induce simple equivalences on q-strata. We use the cellular link functor of Example 13.2.1.9. Then Link_q is simply the functor sending a stratified simplicial set over $\{p < q\}$ to its q-stratum. Applying Link_q, we obtain a weak equivalence of simplicial sets

$$\omega_q: L = \mathrm{Link}_q(L \times \Delta^{\{p<q\}}) \simeq \mathcal{K}_q = \mathrm{Link}_q(\mathcal{K}).$$

To see that ω is not a simple diagrammatic equivalence, it suffices to show that ω_q is not a simple equivalence. Using the natural weak equivalence $(|-|_s)_p \simeq |-_p|$ (which follows, for example, by Theorem 3.1.0.4), it follows that the realization of this weak equivalence fits into a homotopy commutative diagram

$$\begin{tikzcd} {|L|} \arrow[rr, "\simeq"] \arrow[dr, hook, "\simeq"'] & & {|\mathcal{K}_q|} \arrow[dl, hook', "\simeq"] \\ & (\mathcal{C}M)_q . & \end{tikzcd} \tag{13.16}$$

This uniquely determines the homotopy class of $|L| \simeq |K|$ as the homotopy class of the realization of the inclusion $L \hookrightarrow \mathcal{K}_q$. Now, observe that by construction of K as a barycentric subdivision of a simplicial complex, we are in the following situation (this follows, for example, from the techniques discussed in [Sto72, Ch. 1, Sec. 3]):

1. The subcomplex N_p given by all simplices contained in a simplex intersecting $\mathcal{K}_p$ non-trivially defines a collar neighborhood of M' in W.
2. The subcomplex K_q given by all simplices not intersecting $\mathcal{K}_p$ realizes to a manifold with boundary $M \sqcup M'$.

We may thus think of $|\mathcal{K}_q|$ as a cobordism from M to M'. In fact, this cobordism is PL homeomorphic to W, relative to $M \sqcup M'$, by the chain of PL isomorphisms

$$|\mathcal{K}_q| \cong M' \times [0,1] \circ |\mathcal{K}_q| \cong |\mathcal{K}_q| \cup_{\mathcal{K}_q \cap N_p} |N_p| \cong |K| \cong W.$$

It follows that up to PL homeomorphisms in source and target, we may identify $|\omega_q|: |L| \simeq |\mathcal{K}_q|$ with the inclusion $M \hookrightarrow W$. By the assumption that $\langle M \hookrightarrow W \rangle \neq 0$ and the fact that every PL homeomorphism is a simple equivalence, it follows that $|\omega_q|$ is not a simple equivalence, and hence that ω_q is not a simple equivalence.

List of notation of Part III

Important general categories

$\star$	Terminal category	P. 397
$[1]$	Category $\{0 \to 1\}$	P. 397
$\mathbf{Set}$	Category of sets	P. 397
Δ	Category of finite linear posets of the form $[n] := \{0, \cdots, n\}$, for $n \in \mathbb{N}$	P. 397
$\mathbf{sSet}$	Category of simplicial sets	P. 397
$\underline{\mathbf{sSet}}$	Simplicial category of simplicial sets	P. 397, 398
$\mathbf{Top}$	Category of (compactly generated or Δ-generated) spaces	P. 397
$\underline{\mathbf{Top}}$	Simplicial category of (compactly generated or Δ-generated) spaces	P. 397, 398
$\mathbf{Ab}$	Category of abelian groups	P. 397
$\mathbf{AbMon}$	Category of unital abelian monoids	P. 397
$\mathbf{Ch}_{\geq 0}(R)$	Category of non-negatively graded chain-complexes of R-modules	P. 397
$\mathbf{Filt}$	Category of filtered topological spaces	P. 405
$\mathbf{Cat}$	2-Category of (sufficiently small) categories	P. 398
$\mathbf{SymMonCat}$	2-Category of (sufficiently small) symmetric monoidal categories	P. 484
$\mathbf{Leib}$	A certain bicategory using Leibniz composition of functors	P. 432
$\mathbf{CellCat}$	2-category of cellularized categories and absolute cellularized functors	P. 438
$\mathbf{CellCat}^{\to}$	Bi-category of cellularized categories and relative cellularized functors	P. 437
$\mathbf{WES}$	Category of pre-Whitehead frameworks	P. 493

Constructions on categories

$\mathrm{Ob}(\mathbf{C})$	Class or set of objects of $\mathbf{C}$	P. 398
$\mathbf{C}^{\mathrm{op}}$	Opposite category of $\mathbf{C}$	P. 398
$\mathbf{C}(X, Y)$	Object of morphisms from X to Y in $\mathbf{C}$	P. 398
$\mathbf{Fun}(\mathbf{C}, \mathbf{D})$	Category of functors from $\mathbf{C}$ to $\mathbf{D}$	P. 398
$\mathbf{D}^{\mathbf{C}}$	Alternative notation for category of functors from $\mathbf{C}$ to $\mathbf{D}$	P. 398
$\mathbf{C}^{[1]}$	Category of arrows in $\mathbf{C}$	P. 398
$F_{/X}$, $F_{X/}$	Comma-categories associated to a functor F and object X	P. 399
$\mathbf{C}_{/X}$, $\mathbf{C}_{X/}$	Over- and undercategory of $X \in \mathbf{C}$	P. 399

$\mathbf{C}^{\simeq}$	Groupoid core of $\mathbf{C}$, given by the wide subcategory of isomorphisms	P. 399
F^*	Precomposition functor associated to a functor F	P. 399
$F_!$	Left adjoint to pre-composition functor F^*	P. 399
$\mathbf{C}[W^{-1}]$	∞- or 1-categorical localization of $\mathbf{C}$ at W (depending on context)	P. 399

Notation associated to categories with cobase changes and pre-Whitehead frameworks $\mathbf{W}$

$Y_1 \cup_X^Q Y_2$	Notation for a gluing operation induced by cobasechange squares	P. 482
$a_1 \oplus_Q a_2$	Notation for a certain monoidal structure induced on undercategories	P. 482
$f_¡$	The cobase change functor associated to f	P. 478
$\mathbf{Wh}_{\mathbf{W}}(X)$	A certain category whose path-components give rise to the Whitehead monoid of X	P. 485
$\widetilde{\mathrm{Wh}}_{\mathbf{W}}(X)$	Whitehead monoid of X	P. 485
$\mathrm{Wh}_{\mathbf{W}}(X)$	Whitehead group of X	P. 491
f_*	Notation for covariant functoriality on Whitehead groups and monoids associated to f	P. 487, 488, 504, 523
f^*	Notation for contravariant functoriality on the underlying sets of Whitehead groups and monoids associated to f	P. 492
$\langle \alpha \rangle$	Whitehead torsion of α	P. 491, 504, 523

Notation associated to structured cell complexes

$\mathfrak{C}_{\mathfrak{c}}$	Set of characteristic maps of $\mathfrak{c}$	P. 404
$\mathfrak{C}(f)$	Map of sets of characteristic maps of $\mathfrak{c}$ induced by structure preserving morphisms f	P. 408
$f\mathfrak{C}_{\mathfrak{c}}$	Set of postcompositions of characteristic maps of $\mathfrak{c}$ with a map f	P. 408, 409
$\mathfrak{d} \circ \mathfrak{c}$	Structured relative cell complex obtained through vertical composition of relative structured cell complex $\mathfrak{c}$ and $\mathfrak{d}$	P. 409
$\mathfrak{c} \circ \mathfrak{X}$	Structured absolute cell complex obtained through extension of a structured absolute cell $\mathfrak{X}$ by a structured relative cell complex $\mathfrak{c}$	P. 409
$\mathfrak{X} \cup_{\mathfrak{A}} \mathfrak{B}, \mathfrak{X} \cup_A B$	Notations for certain gleanings of structured cell complexes	P. 512, 561
$\mathfrak{M}_f$	Cellularized version of mapping cylinder	P. 511
$\overline{\mathfrak{M}}_H$	Modified cellularized mapping cylinder of homotopy H	P. 512
$\mathrm{Wh}_{\mathbf{C}}(\mathfrak{X})$	Whitehead group of a finite structured cell complex $\mathfrak{X}$ in a Whitehead model category $\mathbf{C}$	P. 505

Constructions associated to a cellularized category, a category with expansions or a Whitehead model category $\mathbf{C}$

$\mathbb{B}_{\mathbf{C}}$	Set of generating boundary inclusions	P. 406
$\mathbb{E}_{\mathbf{C}}$	Set of generating elementary expansions	P. 502
$\hat{-}$	Notation used for all sorts of Leibniz operations	P. 431, 432

$\mathbf{Cell}(\mathbf{C})$	Category of absolute structured cell complexes	P. 407
$\mathbf{RCell}(\mathbf{C})$	Category of relative structured cell complexes	P. 407
$\mathbf{RCell}(\mathbf{C})_A$	Category of relative structured cell complexes with source A	P. 410
$\mathbf{C\overset{\rightleftarrows}{e}ll}_c(\mathbf{C})$	Wide subcategory of $\mathbf{Cell}(\mathbf{C})$ given inclusions of finite structured cell complexes	P. 499
$\mathbf{C\overset{\rightleftarrows}{e}ll}(\mathbf{C})$	Wide subcategory of $\mathbf{Cell}(\mathbf{C})$ given inclusions of not-necessarily finite structured cell complexes	P. 499
$\mathbf{C\overline{e}ll}(\mathbf{C})$	Category with objects in $\mathbf{Cell}(\mathbf{C})$ but morphisms in $\mathbf{C}$	P. 511
$\mathbf{C\overline{e}ll}_c(\mathbf{C})$	Category with objects finite complexes in $\mathbf{Cell}(\mathbf{C})$ but morphisms in $\mathbf{C}$	P. 511
$\mathbf{CellCat}(\mathbf{C},\mathbf{D})$	Category of absolute cellularized functors from $\mathbf{C}$ to $\mathbf{D}$	P. 437
$\mathbf{CellCat}^{\rightarrow}(\mathbf{C},\mathbf{D})$	Category of relative cellularized functors from $\mathbf{C}$ to $\mathbf{D}$	P. 437
$\mathbf{CellBiFun}(\mathbf{C}\times\mathbf{D},\mathbf{E})$	Category of cellular bifunctors on $\mathbf{C}\times\mathbf{D}$ with target $\mathbf{E}$	P. 445
$\mathbf{W}(\mathbf{C})$	(Pre)-Whitehead framework associated to $\mathbf{C}$	P. 503
$\mathrm{Wh}_{\mathfrak{F}}$	Natural transformation of Whitehead groups associated to a cellularized functor $\mathfrak{F}$	P. 508
$\mathrm{Pres}_{\mathbf{C}}(X)$	Set of presentations of the homotopy type of X in terms of finite structured cell complexes, modulo simple equivalences	P. 524
$\widetilde{\mathrm{Pres}}_{\mathbf{C}}(X)$	An extension of $\mathrm{Pres}_{\mathbf{C}}(X)$ that incorporates non-isomorphisms	P. 524
$\mathfrak{ho}_c\mathbf{C}$	Homotopy category of finite structured cell complexes	P. 522
$\mathfrak{ho}\mathbf{C}$	Homotopy category of not-necessarily finite structured cell complexes	P. 522
$\mathcal{C}\mathbf{ell}(\mathbf{C})$	∞-category of structured cell complexes	P. 529
$\mathcal{S}\mathbf{im}(\mathbf{C})$	∞-category of finite structured cell complexes and simple equivalences	P. 530

Notation associated to diagram categories, especially over a Reedy category R

$\mathbf{el}(X)$	Category of elements of X	P. 447, 462
$\mathbf{el}_{n.d.}(X)$	Category of non-degenerate elements of X and their faces	P. 472
$\mathbf{R}^+,\mathbf{R}^-$	Wide subcategories of face and degeneracies	P. 449
f^+,f^-	Unique face f^+ and degeneracy f^- such that $f=f^+\circ f^-$	P. 450
$\mathbf{R}^r,\mathbf{R}_r$	Yoneda embeddings of $r\in\mathbf{R}$	P. 447
$\partial\mathbf{R}^r,\partial\mathbf{R}_r$	Certain subdiagrams of $\mathbf{R}^r$ and $\mathbf{R}_r$	P. 451
ι^r,ι_r	Inclusions of, respectively, $\partial\mathbf{R}^r,\partial\mathbf{R}_r$ into $\mathbf{R}^r,\mathbf{R}_r$	P. 454
$\iota_{r,\bullet}^{\bullet,s}$	Inclusion $\partial(\mathbf{R}_r\times\mathbf{S}^s)\to\mathbf{R}_r\times\mathbf{S}^s$	P. 452
L^r,L_r	Notation for the latching functors at $r\in\mathbf{R}$	P. 451, 465
$s\hat{L}^r(c),s\hat{L}_r(c)$	Notation for source of relative latching maps	P. 459
sk_n	n-skeleton functor	P. 451, 465
$S\star X$	Notation for the S-fold coproduct of an object X	P. 443, 456
$W\circledast X$	Notation for the weighted colimit of X with weights W	P. 447, 456
$X\circledcirc Y$	Notation for the composition tensor of X and Y	P. 448, 456
$\mathfrak{C}_{\mathfrak{c},r}$	Cells of a relative cell complex $\mathfrak{c}$ of diagrams on $\mathbf{R}$ of type $r\in\mathbf{R}$	P. 458

Notation related to the stratified setting

Δ_P	Category of flags of P	P. 338
$\mathrm{sd}(P)$	Category of regular flags of P	P. 339
$\mathbf{Strat}_P$	Category of P-stratified spaces	P. 340
$\underline{\mathbf{Strat}}_P$	Simplicial category of P-stratified spaces	P. 340
$\mathbf{Strat}_P^{\mathfrak{d}}$	Category of P-stratified spaces equipped with diagrammatic semi-model structure	P. 357, 588
$\underline{\mathbf{Strat}}_P^{\mathfrak{d}}$	Simplicial category of P-stratified spaces equipped with diagrammatic semi-model structure	P. 357
$\mathrm{s}\mathbf{Strat}_P$	Category of P-stratified simplicial sets	P. 341
$\underline{\mathrm{s}\mathbf{Strat}}_P$	Simplicial category of P-stratified simplicial sets	P. 341
$\mathrm{s}\mathbf{Strat}_P^{\mathfrak{d}}$	Category of P-stratified simplicial sets equipped with Douteau-Henriques model structure	P. 347, 586
$\underline{\mathrm{s}\mathbf{Strat}}_P^{\mathfrak{d}}$	Simplicial category of P-stratified simplicial sets equipped with Douteau-Henriques model structure	P. 347
$\mathcal{H}o\mathrm{Link}_{\mathcal{I}}(\mathcal{X})$	$\mathcal{I}$-th generalized homotopy link of $\mathcal{X}$	P. 343
$\mathcal{H}o\mathrm{Link}$	Homotopy link diagram functor	P. 343
$\mathrm{HoLink}_{\mathcal{I}}(\mathcal{X})$	$\mathcal{I}$-th generalized simplicial homotopy link of $\mathcal{X}$	P. 347
HoLink	Simplicial homotopy link diagram functor	P. 347
$\mathrm{Link}^{\mathrm{sd}}, \mathrm{Link}^m, \mathrm{Link}^N$	Several cellular link functors	P. 591–593
$\mathrm{Wh}_P(\mathfrak{X})$	Diagrammatic stratified Whitehead group of $\mathfrak{X}$	P. 586, 588, 602

Bibliography

[And74] Douglas R. Anderson. "The Whitehead torsion of a fiber-homotopy equivalence". *Michigan Math. J.* 21 (1974), pp. 171–180. URL: http://projecteuclid.org/euclid.mmj/1029001264.

[Are46] Richard Arens. "Topologies for Homeomorphism Groups". *American Journal of Mathematics* 68.4 (1946), pp. 593–610. URL: http://www.jstor.org/stable/2371787 (visited on 07/08/2022).

[AFR19] David Ayala, John Francis, and Nick Rozenblyum. "A stratified homotopy hypothesis". *J. Eur. Math. Soc. (JEMS)* 21.4 (2019), pp. 1071–1178. DOI: 10.4171/JEMS/856. URL: https://doi-org.ezp.sub.su.se/10.4171/JEMS/856.

[AFT17] David Ayala, John Francis, and Hiro Lee Tanaka. "Local structures on stratified spaces". *Advances in Mathematics* 307 (2017), pp. 903–1028. DOI: https://doi.org/10.1016/j.aim.2016.11.032. URL: https://www.sciencedirect.com/science/article/pii/S0001870816316097.

[Ban07] Markus Banagl. *Topological invariants of stratified spaces*. Berlin, Germany: Springer, 2007.

[BMS24] Markus Banagl, Tim Mäder, and Filip Sadlo. "Stratified simple homotopy type: Theory and computation". *Advances in Applied Mathematics* 160 (2024), p. 102753. DOI: https://doi.org/10.1016/j.aam.2024.102753. URL: https://www.sciencedirect.com/science/article/pii/S019688582400085X.

[BK12a] Clarck Barwick and Daniel M. Kan. "A characterization of simplicial localization functors and a discussion of DK equivalences". *Indag. Math. (N.S.)* 23.1-2 (2012), pp. 69–79. DOI: 10.1016/j.indag.2011.10.001. URL: https://doi.org/10.1016/j.indag.2011.10.001.

[BK12b] Clarck Barwick and Daniel M. Kan. "Relative categories: another model for the homotopy theory of homotopy theories". *Indag. Math. (N.S.)* 23.1-2 (2012), pp. 42–68. DOI: 10.1016/j.indag.2011.10.002. URL: https://doi.org/10.1016/j.indag.2011.10.002.

[Bar07] Clark Barwick. "On Reedy Model Categories". *arXiv e-prints* (2007). arXiv: 0708.2832 [math.AT].

[Bar10] Clark Barwick. "On left and right model categories and left and right Bousfield localizations". *Homology Homotopy Appl.* 12.2 (2010), pp. 245–320. URL: http://projecteuclid.org/euclid.hha/1296223884.

[BGH18] Clark Barwick, Saul Glasman, and Peter Haine. "Exodromy". *arXiv e-prints* (July 2018). arXiv: 1807.03281 [math.AT].

[BHS64] H. Bass, A. Heller, and R. G. Swan. "The Whitehead group of a polynomial extension". *Inst. Hautes Études Sci. Publ. Math.* 22 (1964), pp. 61–79. URL: http://www.numdam.org/item?id=PMIHES_1964__22__61_0.

[BDW23] Michael Batanin, Florian De Leger, and David White. "Model structures on operads and algebras from a global perspective". *arXiv preprint arXiv:2311.07320* (2023).

[BW24] Michael Batanin and David White. "Left Bousfield localization without left properness". *Journal of Pure and Applied Algebra* 228.6 (2024), p. 107570.

[Bau21] Ulrich Bauer. "Ripser: efficient computation of Vietoris–Rips persistence barcodes". *Journal of Applied and Computational Topology* 5.3 (2021), pp. 391–423.

[BL15] Ulrich Bauer and Michael Lesnick. "Induced matchings and the algebraic stability of persistence barcodes". *J. Comput. Geom.* 6.2 (2015), pp. 162–191. DOI: `10.20382/jocg.v6i2a9`.

[BH11] Paul Bendich and John Harer. "Persistent intersection homology". *Found. Comput. Math.* 11.3 (2011), pp. 305–336. DOI: `10.1007/s10208-010-9081-1`.

[BWM12] Paul Bendich, Bei Wang, and Sayan Mukherjee. "Local homology transfer and stratification learning". *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*. New York, NY, USA: Association for Computing Machinery, 2012, pp. 1355–1370. DOI: `10.1137/1.9781611973099.107`.

[Ber07a] Julia E. Bergner. "A model category structure on the category of simplicial categories". *Trans. Amer. Math. Soc.* 359.5 (2007), pp. 2043–2058. DOI: `10.1090/S0002-9947-06-03987-0`. URL: `https://doi.org/10.1090/S0002-9947-06-03987-0`.

[Ber07b] Julia E. Bergner. "Three models for the homotopy theory of homotopy theories". *Topology* 46.4 (2007), pp. 397–436. DOI: `10.1016/j.top.2007.03.002`. URL: `https://doi.org/10.1016/j.top.2007.03.002`.

[BR12] Julia E. Bergner and Charles Rezk. *Reedy categories and the* Θ*-construction*. 2012. arXiv: `1110.1066 [math.AT]`. URL: `https://arxiv.org/abs/1110.1066`.

[BL07] Andreas Bernig and Alexander Lytchak. "Tangent spaces and Gromov-Hausdorff limits of subanalytic spaces". *J. Reine Angew. Math.* 608 (2007), pp. 1–15. DOI: `10.1515/CRELLE.2007.050`.

[BL21] Håvard Bakke Bjerkevik and Michael Lesnick. l^p*-distances on multiparameter persistence modules*. arXiv preprint available at arXiv:2106.13589. 2021.

[BP19] Jean-Daniel Boissonnat and Siddharth Pritam. "Computing Persistent Homology of Flag Complexes via Strong Collapses". *Proceedings of the 35th International Symposium on Computational Geometry (SoCG 2019)*. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019, 55:1–55:15. DOI: `10.4230/LIPIcs.SoCG.2019.55`. URL: `https://drops.dagstuhl.de/opus/volltexte/2019/10497/`.

[Bor+08] A. Borel et al. *Intersection cohomology*. Modern Birkhäuser Classics. Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. x+234.

[Bor48] Karol Borsuk. "On the imbedding of systems of compacta in simplicial complexes". *Fund. Math.* 35 (1948), pp. 217–234. DOI: `10.4064/fm-35-1-217-234`.

[BW20] David Bramer and Guo-Wei Wei. "Atom-specific persistent homology and its application to protein flexibility analysis". *Comput. Math. Biophys.* 8 (2020), pp. 1–35. DOI: `10.1515/cmb-2020-0001`.

[BQ79] William Browder and Frank Quinn. "A surgery theory for G-manifolds and stratified sets". *Manuscripta Math.* 28.2 (1979), pp. 231–246. DOI: `10.1007/BF01647924`. URL: `https://doi.org/10.1007/BF01647924`.

[BSS20] Peter Bubenik, John A. Scott, and Don Stanley. *An algebraic wasserstein distance for generalized persistence modules*. arXiv preprint available at arXiv:1809.09654v2. 2020.

[CM20] Pierre Cagne and Paul-André Melliès. "On bifibrations of model categories". *Adv. Math.* 370 (2020), pp. 107205, 43. DOI: `10.1016/j.aim.2020.107205`. URL: `https://doi-org.ezp.sub.su.se/10.1016/j.aim.2020.107205`.

[Car09] Gunnar Carlsson. "Topology and data". *Bull. Amer. Math. Soc. (N.S.)* 46.2 (2009), pp. 255–308. DOI: 10.1090/S0273-0979-09-01249-X.

[Cha74] T. A. Chapman. "Topological invariance of Whitehead torsion". *Amer. J. Math.* 96 (1974), pp. 488–497. DOI: 10.2307/2373556. URL: https://doi.org/10.2307/2373556.

[CST24] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré. "Homotopy truncations of homotopically stratified spaces". *Proc. Amer. Math. Soc.* 152.3 (2024), pp. 1319–1332. DOI: 10.1090/proc/16612. URL: https://doi.org/10.1090/proc/16612.

[CT20] David Chataur and Daniel Tanré. "Natural operations in Intersection Cohomology". *arXiv e-prints* (Mar. 2020). arXiv: 2005.04960 [math.AT].

[CL05] Frédéric Chazal and André Lieutier. "Weak feature size and persistant homology: computing homology of solids in $\mathbb{R}^n$ from noisy data samples". SCG '05. New York, NY, USA: Association for Computing Machinery, 2005, pp. 255–262. DOI: 10.1145/1064092.1064132.

[Cha+09] Frédéric Chazal et al. "Proximity of Persistence Modules and Their Diagrams". *Proceedings of the Twenty-Fifth Annual Symposium on Computational Geometry.* SCG '09. Aarhus, Denmark: Association for Computing Machinery, 2009, pp. 237–246. DOI: 10.1145/1542362.1542407.

[Cis06] Denis-Charles Cisinski. "Les préfaisceaux comme modèles des types d'homotopie". *Astérisque* 308 (2006), pp. xxiv+390.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra.* Vol. 180. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2019, pp. xviii+430. DOI: 10.1017/9781108588737. URL: https://doi.org/10.1017/9781108588737.

[Coh73] M.M. Cohen. *A Course in Simple-homotopy Theory.* Graduate texts in mathematics. Springer-Verlag, 1973. URL: https://books.google.de/books?id=MFrvAAAAMAAJ.

[CEH07] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. "Stability of persistence diagrams". *Discrete Comput. Geom.* 37.1 (2007), pp. 103–120. DOI: 10.1007/s00454-006-1276-5.

[Cos00] Michel Coste. *An introduction to o-minimal geometry.* Pisa, Italy: Istituti editoriali e poligrafici internazionali, 2000.

[CGN16] Justin Curry, Robert Ghrist, and Vidit Nanda. "Discrete Morse theory for computing cellular sheaf cohomology". *Foundations of Computational Mathematics* 16 (2016), pp. 875–897.

[Cza12] Małgorzata Czapla. "Definable triangulations with regularity conditions". *Geom. Topol.* 16.4 (2012), pp. 2067–2095. DOI: 10.2140/gt.2012.16.2067.

[DK70] B. J. Day and G. M. Kelly. "On topological quotient maps preserved by pullbacks or products". *Proc. Cambridge Philos. Soc.* 67 (1970), pp. 553–558. DOI: 10.1017/s0305004100045850. URL: https://doi.org/10.1017/s0305004100045850.

[DT58] Albrecht Dold and René Thom. "Quasifaserungen und unendliche symmetrische Produkte". *Annals of Mathematics* 67.2 (1958), pp. 239–281.

[Dou19a] Sylvain Douteau. "Étude homotopique des espaces stratifiés". PhD thesis. Université de Picardie Jules Verne, July 2019. URL: https://arxiv.org/pdf/1908.01366.pdf.

[Dou19b] Sylvain Douteau. *Stratified Homotopy Theory.* arXiv preprint available at arXiv:1908.01366. 2019.

[Dou21a] Sylvain Douteau. “A simplicial approach to stratified homotopy theory”. *Trans. Amer. Math. Soc.* 374.2 (2021), pp. 955–1006. DOI: 10.1090/tran/8264. URL: https://doi.org/10.1090/tran/8264.

[Dou21b] Sylvain Douteau. “A stratified Kan-Quillen equivalence”. *arXiv e-prints* (Feb. 2021). arXiv: 2102.04876 [math.AT].

[Dou21c] Sylvain Douteau. “Homotopy theory of stratified spaces”. *Algebr. Geom. Topol.* 21.1 (2021), pp. 507–541. DOI: 10.2140/agt.2021.21.507. URL: https://doi.org/10.2140/agt.2021.21.507.

[DW22] Sylvain Douteau and Lukas Waas. “From homotopy links to stratified homotopy theories”. *arXiv e-prints* (2022). arXiv: 2112.02394 [math.AT].

[Dri86] Lou van den Dries. “A generalization of the Tarski-Seidenberg theorem, and some nondefinability results”. *Bull. Amer. Math. Soc. (N.S.)* 15.2 (1986), pp. 189–193. DOI: 10.1090/S0273-0979-1986-15468-6.

[Dri98] Lou van den Dries. *Tame topology and o-minimal structures*. Vol. 248. London Mathematical Society Lecture Note Series. Cambridge, UK: Cambridge University Press, 1998. DOI: 10.1017/CBO9780511525919.

[Dug01] Daniel Dugger. “Universal homotopy theories”. *Adv. Math.* 164.1 (2001), pp. 144–176. DOI: 10.1006/aima.2001.2014. URL: https://doi-org.ezp.sub.su.se/10.1006/aima.2001.2014.

[Dug03] Daniel Dugger. *Notes on Delta-generated spaces*. https://pages.uoregon.edu/ddugger/delta.html. Accessed 2022-09-27. 2003.

[Dug08] Daniel Dugger. “A primer on homotopy colimits”. 2008. URL: https://pages.uoregon.edu/ddugger/hocolim.pdf.

[DK80a] William G. Dwyer and Daniel M. Kan. “Calculating simplicial localizations”. *J. Pure Appl. Algebra* 18.1 (1980), pp. 17–35. DOI: 10.1016/0022-4049(80)90113-9. URL: https://doi.org/10.1016/0022-4049(80)90113-9.

[DK80b] William G. Dwyer and Daniel M. Kan. “Function complexes in homotopical algebra”. *Topology* 19.4 (1980), pp. 427–440. DOI: 10.1016/0040-9383(80)90025-7. URL: https://doi.org/10.1016/0040-9383(80)90025-7.

[DK80c] William G. Dwyer and Daniel M. Kan. “Simplicial localizations of categories”. *J. Pure Appl. Algebra* 17.3 (1980), pp. 267–284. DOI: 10.1016/0022-4049(80)90049-3. URL: https://doi.org/10.1016/0022-4049(80)90049-3.

[DK87] William G. Dwyer and Daniel M. Kan. “Equivalences between homotopy theories of diagrams”. *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*. Vol. 113. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1987, pp. 180–205.

[DK84] William. G. Dwyer and Daniel M. Kan. “Realizing diagrams in the homotopy category by means of diagrams of simplicial sets”. *Proc. Amer. Math. Soc.* 91.3 (1984), pp. 456–460. DOI: 10.2307/2045321.

[Eck06] Benno Eckmann. “Simple homotopy type and categories of fractions”. *Mathematical Survey Lectures* (2006), pp. 77–91. DOI: 10.1007/978-3-540-33791-1_7. URL: https://doi.org/10.1007/978-3-540-33791-1_7.

[EK00] Katsuya Eda and Kazuhiro Kawamura. “The singular homology of the Hawaiian earring”. *J. London Math. Soc. (2)* 62.1 (2000), pp. 305–310. DOI: 10.1112/S0024610700001071. URL: https://doi.org/10.1112/S0024610700001071.

[ELZ00] Herbert Edelsbrunner, David Letscher, and Afra Zomorodian. “Topological persistence and simplification”. *41st Annual Symposium on Foundations of Computer Science (Redondo Beach, CA, 2000)*. Los Alamitos, CA, USA: IEEE Comput. Soc. Press, 2000, pp. 454–463. DOI: 10.1109/SFCS.2000.892133.

[Ehr51] Charles Ehresmann. “Les connexions infinitésimales dans un espace fibré différentiable”. *Colloque de topologie (espaces fibrés), Bruxelles, 1950*. Liège and Paris: Georges Thone and Masson & Cie, 1951, pp. 29–55.

[FR08] L. Fajstrup and J. Rosický. “A convenient category for directed homotopy”. *Theory Appl. Categ.* 21 (2008), No. 1, 7–20.

[FW16] Brittany Terese Fasy and Bei Wang. “Exploring persistent local homology in topological data analysis”. *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. New York, NY, USA: Institute of Electrical and Electronics Engineers (IEEE), 2016, pp. 6430–6434. DOI: 10.1109/ICASSP.2016.7472915.

[FR01] Steve Ferry and Andrew Ranicki. “A survey of Wall’s finiteness obstruction”. *Surveys on surgery theory, Vol. 2*. Vol. 149. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 2001, pp. 63–79.

[For98] Robin Forman. “Morse Theory for Cell Complexes”. *Advances in Mathematics* 134.1 (1998), pp. 90–145. DOI: 10.1006/aima.1997.1650. URL: https://www.sciencedirect.com/science/article/pii/S0001870897916500.

[Fre09] Benoit Fresse. *Modules over operads and functors*. Vol. 1967. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2009, pp. x+308. DOI: 10.1007/978-3-540-89056-0. URL: https://doi.org/10.1007/978-3-540-89056-0.

[Fre10] Benoit Fresse. “Props in model categories and homotopy invariance of structures” (2010).

[Fri03] Greg Friedman. “Stratified fibrations and the intersection homology of the regular neighborhoods of bottom strata”. *Topology Appl.* 134.2 (2003), pp. 69–109. DOI: 10.1016/S0166-8641(03)00088-9.

[Fri20] Greg Friedman. *Singular Intersection Homology*. New Mathematical Monographs. Cambridge University Press, 2020.

[Fu85] Joseph Howland Guthrie Fu. “Tubular neighborhoods in Euclidean spaces”. *Duke Math. J.* 52.4 (1985), pp. 1025–1046. DOI: 10.1215/S0012-7094-85-05254-8.

[Gau21] Philippe Gaucher. “Left properness of flows”. *Theory Appl. Categ.* 37 (2021), Paper No. 19, 562–612. DOI: 10.1002/num.22542. URL: https://doi.org/10.1002/num.22542.

[Ghr08] Robert Ghrist. “Barcodes: the persistent topology of data”. *Bull. Amer. Math. Soc. (N.S.)* 45.1 (2008), pp. 61–75. DOI: 10.1090/S0273-0979-07-01191-3.

[GH05] Paul G. Goerss and Michael J. Hopkins. “Moduli problems for structured ring spectra”. *preprint* 1086 (2005).

[GJ12] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*. Progress in Mathematics. Birkhäuser Basel, 2012. URL: https://books.google.de/books?id=iFv2BwAAQBAJ.

[Gor78] Mark Goresky. “Triangulation of stratified objects”. *Proc. Amer. Math. Soc.* 72.1 (1978), pp. 193–200. DOI: 10.2307/2042563. URL: https://doi.org/10.2307/2042563.

[GM80] Mark Goresky and Robert MacPherson. “Intersection homology theory”. *Topology* 19.2 (1980), pp. 135–162. DOI: 10.1016/0040-9383(80)90003-8. URL: https://doi.org/10.1016/0040-9383(80)90003-8.

[GM83] Mark Goresky and Robert MacPherson. “Intersection homology. II”. *Invent. Math.* 72.1 (1983), pp. 77–129. DOI: 10.1007/BF01389130.

[Gro21] Alexander Grothendieck. “Pursuing stacks”. *arXiv preprint arXiv:2111.01000* (2021).

[GR04] Alexander Grothendieck and Michele Raynaud. *Revêtements étales et groupe fondamental (SGA 1)*. 2004. arXiv: math/0206203 [math.AG]. URL: https://arxiv.org/abs/math/0206203.

[Hai23] Peter J. Haine. "On the homotopy theory of stratified spaces". *arXiv e-prints* (2023). arXiv: 1811.01119 [math.AT].

[HPT24] Peter J. Haine, Mauro Porta, and Jean-Baptiste Teyssier. "Exodromy beyond conicality". *arXiv preprint arXiv:2401.12825* (2024). URL: https://arxiv.org/abs/2401.12825.

[Han51] Olof Hanner. "Some theorems on absolute neighborhood retracts". *Ark. Mat.* 1 (1951), pp. 389–408. DOI: 10.1007/BF02591376. URL: https://doi.org/10.1007/BF02591376.

[Hat75] Allen Hatcher. "Higher simple homotopy theory". *Ann. of Math. (2)* 102.1 (1975), pp. 101–137. DOI: 10.2307/1970977. URL: https://doi.org/10.2307/1970977.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge, UK: Cambridge University Press, 2002.

[HN22] Martin Helmer and Vidit Nanda. "Conormal Spaces and Whitney Stratifications". *Found. Comput. Math.* (2022), pp. 1–36. DOI: 10.1007/s10208-022-09574-8.

[Hen] Andre Henriques. "A model category for stratified spaces". Note available at http://andreghenriques.com/PDF/Model_Cat_Stratified_spaces.pdf.

[Hes+17] Kathryn Hess et al. "A necessary and sufficient condition for induced model structures". *J. Topol.* 10.2 (2017), pp. 324–369. DOI: 10.1112/topo.12011. URL: https://doi-org.ezp.sub.su.se/10.1112/topo.12011.

[Hin16] Vladimir Hinich. "Dwyer-Kan localization revisited". *Homology Homotopy Appl.* 18.1 (2016), pp. 27–48. DOI: 10.4310/HHA.2016.v18.n1.a3. URL: https://doi.org/10.4310/HHA.2016.v18.n1.a3.

[Hir69] Heisuke Hironaka. "Normal cones in analytic Whitney stratifications". *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), pp. 127–138. URL: http://www.numdam.org/item?id=PMIHES_1969__36__127_0.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. Providence, RI, USA: American Mathematical Society, 2003. DOI: 10.1090/surv/099.

[Hof+17] Christoph Hofer et al. "Deep Learning with Topological Signatures". *Advances in Neural Information Processing Systems*. Ed. by I. Guyon et al. Vol. 30. Red Hook, NY, USA: Curran Associates, Inc., 2017, pp. 1633–1643. URL: https://proceedings.neurips.cc/paper/2017/file/883e881bb4d22a7add958f2d6b052c9f-Paper.pdf.

[Hov07] M. Hovey. *Model Categories*. Mathematical surveys and monographs. American Mathematical Society, 2007. URL: https://books.google.de/books?id=Kfs4uuiTXN0C.

[Hug99a] Bruce Hughes. "Stratifications of mapping cylinders". *Topology and its Applications* 94.1-3 (1999), pp. 127–145. DOI: 10.1016/S0166-8641(98)00028-5.

[Hug99b] Bruce Hughes. "Stratified path spaces and fibrations". *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 129 (Jan. 1999). DOI: 10.1017/S0308210500021405.

[Jan24] Mikala Ørsnes Jansen. "Stratified homotopy theory of topological infinity-stacks: A toolbox". *Journal of Pure and Applied Algebra* 228.11 (2024), p. 107710. DOI: https://doi.org/10.1016/j.jpaa.2024.107710. URL: https://www.sciencedirect.com/science/article/pii/S0022404924001075.

[JY20] Niles Johnson and Donald Yau. "2-Dimensional Categories" (2020). arXiv: 2002.06055 [math.CT]. URL: https://arxiv.org/abs/2002.06055.

[Joy] André Joyal. "Quasi-categories vs Simplicial categories". *in preparation* (). URL: http://www.math.uchicago.edu/~may/IMA/Incoming/Joyal/QvsDJan9(2007).pdf.

[JT07] André Joyal and Myles Tierney. "Quasi-categories vs Segal spaces". *Categories in algebra, geometry and mathematical physics*. Vol. 431. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 277–326. DOI: 10.1090/conm/431/08278. URL: https://doi.org/10.1090/conm/431/08278.

[JT08] André Joyal and Myles Tierney. "Notes on simplicial homotopy theory". *Preprint* (2008).

[KP86] Klaus Heiner Kamps and Timothy Porter. *Abstract homotopy and simple homotopy theory*. University of Wales, 1986.

[Kan57] Daniel M. Kan. "On c. s. s. complexes". *Amer. J. Math.* 79 (1957), pp. 449–476. DOI: 10.2307/2372558. URL: https://doi-org.ezp.sub.su.se/10.2307/2372558.

[KS69] R. C. Kirby and L. C. Siebenmann. "On the triangulation of manifolds and the Hauptvermutung". *Bull. Amer. Math. Soc.* 75 (1969), pp. 742–749. DOI: 10.1090/S0002-9904-1969-12271-8. URL: https://doi.org/10.1090/S0002-9904-1969-12271-8.

[KY21] Inbar Klang and Sarah Yeakel. "Isovariant homotopy theory and fixed point invariants". *arXiv e-prints* (Oct. 2021). arXiv: 2110.07853 [math.AT].

[Lan21] Markus Land. *Introduction to infinity-categories*. Compact Textbooks in Mathematics. Birkhäuser/Springer, Cham, 2021, pp. ix+296. DOI: 10.1007/978-3-030-61524-6. URL: https://doi.org/10.1007/978-3-030-61524-6.

[Les15] Michael Lesnick. "The theory of the interleaving distance on multidimensional persistence modules". *Found. Comput. Math.* 15.3 (2015), pp. 613–650. DOI: 10.1007/s10208-015-9255-y.

[Loi98] Ta Lê Loi. "Verdier and strict Thom stratifications in o-minimal structures". *Illinois J. Math.* 42.2 (1998), pp. 347–356. URL: http://projecteuclid.org/euclid.ijm/1256045049.

[Loi16] Ta Lê Loi. "Łojasiewicz inequalities in o-minimal structures". *Manuscripta Math.* 150.1-2 (2016), pp. 59–72. DOI: 10.1007/s00229-015-0806-y.

[Łoj65] Stanislaw Łojasiewicz. *Ensembles semi-analytiques*. IHES notes, available at http://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf. 1965.

[Lüc87] Wolfgang Lück. "The transfer maps induced in the algebraic K0-and K1-groups by a fibration II". *Journal of Pure and Applied Algebra* 45.2 (1987), pp. 143–169.

[LM24] Wolfgang Lück and Tibor Macko. *Surgery theory—foundations*. Vol. 362. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With contributions by Diarmuid Crowley. Springer, Cham, [2024] ©2024, pp. xv+956. DOI: 10.1007/978-3-031-56334-8. URL: https://doi.org/10.1007/978-3-031-56334-8.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[Lur17] Jacob Lurie. *Higher Algebra*. http://www.math.harvard.edu/~lurie/papers/HA.pdf. 2017.

[Lur23] Jacob Lurie. *Kerodon*. https://kerodon.net. 2023.

[Lyt04] Alexander Lytchak. "Differentiation in metric spaces". *Algebra i Analiz* 16.6 (2004), pp. 128–161. DOI: 10.1090/S1061-0022-05-00888-5.

[Mac13] Saunders Mac Lane. *Categories for the working mathematician.* Vol. 5. Springer Science & Business Media, 2013.

[MW24] Tim Mäder and Lukas Waas. "From samples to persistent stratified homotopy types*". *J. Appl. Comput. Topol.* 8.3 (2024), pp. 761–838. URL: `https://doi.org/10.1007/s41468-024-00170-z`.

[MKS04] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory: Presentations of groups in terms of generators and relations.* Courier Corporation, 2004.

[Man16] Ciprian Manolescu. "Lectures on the triangulation conjecture". *arXiv preprint arXiv:1607.08163* (2016). Version 3, revised February 20, 2024. URL: `https://arxiv.org/abs/1607.08163`.

[Mat73] John Mather. "Stratifications and mappings". *Dynamical Systems (Proc. Sympos. Univ. Bahia, Salvador, 1971).* Ed. by M.M. PEIXOTO. New York, NY, USA: Academic Press, 1973, pp. 195–232. DOI: `https://doi.org/10.1016/B978-0-12-550350-1.50021-7`.

[Mat12] John Mather. "Notes on topological stability". *Bull. Amer. Math. Soc. (N.S.)* 49.4 (2012), pp. 475–506. DOI: `10.1090/S0273-0979-2012-01383-6`.

[May99] J. Peter May. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999, pp. x+243.

[May16] J. Peter May. "Finite spaces and larger contexts". *Unpublished book* (2016).

[Mil21] Yuriy Mileyko. "Another look at recovering local homology from samples of stratified sets". *J. Appl. Comput. Topol.* 5.1 (2021), pp. 55–97. DOI: `10.1007/s41468-020-00062-y`.

[Mil94] Chris Miller. "Expansions of the real field with power functions". *Annals of Pure and Applied Logic* 68.1 (1994), pp. 79–94. DOI: `10.1016/0168-0072(94)90048-5`.

[Mil09] David A. Miller. "Popaths and Holinks". *arXiv: Algebraic Topology* (2009). URL: `https://api.semanticscholar.org/CorpusID:2948510`.

[Mil13] David A. Miller. "Strongly stratified homotopy theory". *Trans. Amer. Math. Soc.* 365.9 (2013), pp. 4933–4962. DOI: `10.1090/S0002-9947-2013-05795-9`. URL: `https://doi.org/10.1090/S0002-9947-2013-05795-9`.

[Mil59] John Milnor. "On spaces having the homotopy type of a CW-complex". *Transactions of the American Mathematical Society* 90 (1959), pp. 272–280. URL: `https://api.semanticscholar.org/CorpusID:123048606`.

[Mil68] John Milnor. *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968, pp. iii+122.

[Mos19] Sean Moss. "Another approach to the Kan–Quillen model structure". *Journal of Homotopy and Related Structures* 15.1 (Sept. 2019), pp. 143–165. DOI: `10.1007/s40062-019-00247-y`. URL: `http://dx.doi.org/10.1007/s40062-019-00247-y`.

[Nan19] Stephen J. Nand-Lal. "A Simplicial Approach to Stratified Homotopy Theory". available at https://livrepository.liverpool.ac.uk/3036209/. PhD thesis. University of Liverpool, 2019.

[Nan20] Vidit Nanda. "Local cohomology and stratification". *Found. Comput. Math.* 20.2 (2020), pp. 195–222. DOI: `10.1007/s10208-019-09424-0`.

[NSW08] Partha Niyogi, Stephen Smale, and Shmuel Weinberger. "Finding the homology of submanifolds with high confidence from random samples". *Discrete Comput. Geom.* 39.1-3 (2008), pp. 419–441. DOI: `10.1007/s00454-008-9053-2`.

[nLa23] nLab authors. *transferred model structure*. https://ncatlab.org/nlab/show/transferred+model+structure. Revision 48. June 2023.

[nLa24a] nLab authors. *adjoint functor theorem*. https://ncatlab.org/nlab/show/adjoint+functor+theorem. Revision 78. Dec. 2024.

[nLa24b] nLab authors. *category with weak equivalences*. https://ncatlab.org/nlab/show/category+with+weak+equivalences. Revision 39. Dec. 2024.

[nLa24c] nLab authors. *Eilenberg-Zilber category*. https://ncatlab.org/nlab/show/Eilenberg-Zilber+category. Revision 8. Dec. 2024.

[nLa24d] nLab authors. *elegant Reedy category*. https://ncatlab.org/nlab/show/elegant+Reedy+category. Revision 16. Sept. 2024.

[nLa24e] nLab authors. *end*. https://ncatlab.org/nlab/show/end. Revision 65. Sept. 2024.

[nLa24f] nLab authors. *final functor*. https://ncatlab.org/nlab/show/final+functor. Revision 46. Aug. 2024.

[nLa24g] nLab authors. *Grothendieck fibration*. https://ncatlab.org/nlab/show/Grothendieck+fibration. Revision 117. Aug. 2024.

[nLa24h] nLab authors. *homotopy hypothesis*. https://ncatlab.org/nlab/show/homotopy+hypothesis. Revision 82. Dec. 2024.

[nLa24i] nLab authors. *left Bousfield localization of model categories*. https://ncatlab.org/nlab/show/left+Bousfield+localization+of+model+categories. Revision 75. Feb. 2024.

[nLa24j] nLab authors. *localization*. https://ncatlab.org/nlab/show/localization. Revision 2. Dec. 2024.

[nLa24k] nLab authors. *simplicial model category*. https://ncatlab.org/nlab/show/simplicial+model+category. Revision 49. Dec. 2024.

[nLa25a] nLab authors. *bicategory*. https://ncatlab.org/nlab/show/bicategory. Revision 66. Jan. 2025.

[nLa25b] nLab authors. *factorization lemma*. https://ncatlab.org/nlab/show/factorization+lemma. Revision 30. Jan. 2025.

[nLa25c] nLab authors. *Grothendieck universe*. https://ncatlab.org/nlab/show/Grothendieck+universe. Revision 58. Jan. 2025.

[nLa25d] nLab authors. *locally presentable category*. https://ncatlab.org/nlab/show/locally+presentable+category. Revision 102. Jan. 2025.

[nLa25e] nLab authors. *Quillen adjunction*. https://ncatlab.org/nlab/show/Quillen+adjunction. Revision 39. Jan. 2025.

[nLa25f] nLab authors. *Quillen equivalence*. https://ncatlab.org/nlab/show/Quillen+equivalence. Revision 39. Jan. 2025.

[nLa25g] nLab authors. *symmetric monoidal category*. https://ncatlab.org/nlab/show/symmetric+monoidal+category. Revision 54. Jan. 2025.

[NV23] Guglielmo Nocera and Marco Volpe. "Whitney stratifications are conically smooth". *Selecta Mathematica* 29.5 (Sept. 15, 2023), p. 68. DOI: 10.1007/s00029-023-00877-4. URL: https://doi.org/10.1007/s00029-023-00877-4.

[Oli88] Robert Oliver. *Whitehead groups of finite groups*. Vol. 132. Cambridge University Press, 1988.

[Oud15] Steve Y. Oudot. *Persistence theory: from quiver representations to data analysis*. Vol. 209. Providence, RI, USA: American Mathematical Society, 2015. DOI: 10.1090/surv/209.

[Pfl01] Markus J. Pflaum. *Analytic and geometric study of stratified spaces*. Vol. 1768. Lecture Notes in Mathematics. Berlin, Germany: Springer, 2001.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967, iv+156 pp. (not consecutively paged).

[Qui88] Frank Quinn. "Homotopically stratified sets". *J. Amer. Math. Soc.* 1.2 (1988), pp. 441–499. DOI: 10.2307/1990924. URL: https://doi.org/10.2307/1990924.

[Rab22] Martin Rabel. "Homotopically Stratified Cobordism Theories". PhD thesis. 2022. DOI: 10.11588/heidok.00031297.

[Rez01] Charles Rezk. "A model for the homotopy theory of homotopy theory". *Trans. Amer. Math. Soc.* 353.3 (2001), pp. 973–1007. DOI: 10.1090/S0002-9947-00-02653-2. URL: https://doi.org/10.1090/S0002-9947-00-02653-2.

[Rez10] Charles Rezk. "A cartesian presentation of weak n–categories". *Geometry & Topology* 14.1 (2010), pp. 521–571. DOI: 10.2140/gt.2010.14.521. URL: https://doi.org/10.2140/gt.2010.14.521.

[Rez17] Charles Rezk. *Compactly generated spaces*. 2017. URL: https://ncatlab.org/nlab/files/Rezk_CompactlyGeneratedSpaces.pdf.

[Rie17] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017. URL: https://books.google.de/books?id=6B9MDgAAQBAJ.

[Rie14] Emily Riehl. *Categorical homotopy theory*. Vol. 24. New Mathematical Monographs. Cambridge University Press, Cambridge, 2014, pp. xviii+352. DOI: 10.1017/CBO9781107261457. URL: https://doi.org/10.1017/CBO9781107261457.

[RV13] Emily Riehl and Dominic Verity. "The theory and practice of Reedy categories". *Theory and Applications of Categories* 29 (Apr. 2013).

[RS12] Colin P. Rourke and Brian Joseph Sanderson. *Introduction to piecewise-linear topology*. Springer Science & Business Media, 2012.

[Sch71] C.H. Schwartz. "Theoreme d'approximation simpliciale lie a une stratification". *Mathematica (Cluj)* 13 (1971), pp. 301–334.

[Shi05] Masahiro Shiota. "Whitney triangulations of semialgebraic sets". *Ann. Polon. Math.* 87 (2005), pp. 237–246. DOI: 10.4064/ap87-0-20.

[Sie72] Laurence Carl Siebenmann. "Deformation of homeomorphisms on stratified sets. I, II". *Comment. Math. Helv.* 47 (1972), 123–136, ibid. 47 (1972), 137–163. DOI: 10.1007/BF02566793.

[Sie70] Laurent C. Siebenmann. "Infinite simple homotopy types". *Indagationes Mathematicae (Proceedings)*. Vol. 73. Elsevier. 1970, pp. 479–495.

[SMS18] Vin de Silva, Elizabeth Munch, and Anastasios Stefanou. "Theory of interleavings on categories with a flow". *Theory Appl. Categ.* 33 (2018), Paper No. 21, 583–607.

[SW14] Primoz Skraba and Bei Wang. "Approximating local homology from samples". *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*. New York, NY, USA: Association for Computing Machinery, 2014, pp. 174–192. DOI: 10.1137/1.9781611973402.13.

[Spa89] Edwin H. Spanier. *Algebraic topology*. Springer Science & Business Media, 1989.

[Spi04] Markus Spitzweck. "Operads, algebras and modules in model categories and motives". PhD thesis. Universitäts-und Landesbibliothek Bonn, 2004.

[Ste13] Marc Stephan. "On equivariant homotopy theory for model categories". *arXiv preprint arXiv:1308.0856* (2013).

[Sto+20] Bernadette J. Stolz et al. "Geometric anomaly detection in data". *Proc. Natl. Acad. Sci. USA* 117.33 (2020), pp. 19664–19669. DOI: 10.1073/pnas.2001741117.

[Sto72] David A. Stone. Berlin, Heidelberg: Springer Berlin Heidelberg, 1972. DOI: 10.1007/BFb0058585. URL: https://doi.org/10.1007/BFb0058585.

[Str72] Arne StrØm. “The homotopy category is a homotopy category”. *Archiv der Mathematik* 23.1 (Dec. 1, 1972), pp. 435–441. DOI: 10.1007/BF01304912. URL: https://doi.org/10.1007/BF01304912.

[Tho69] René Thom. “Ensembles et morphismes stratifiés”. *Bull. Amer. Math. Soc.* 75 (1969), pp. 240–284. DOI: 10.1090/S0002-9904-1969-12138-5.

[Tre09] David Treumann. “Exit paths and constructible stacks”. *Compositio Mathematica* 145.6 (2009), pp. 1504–1532. DOI: 10.1112/S0010437X09004229.

[Vog10] R. M. Vogt. “The HELP-Lemma and its converse in Quillen model categories” (2010). arXiv: 1004.5249 [math.AT]. URL: https://arxiv.org/abs/1004.5249.

[Vol22] Marco Volpe. “Verdier duality on conically smooth stratified spaces”. *arXiv preprint arXiv:2206.02728* (2022).

[Waa21] Lukas Waas. “Stratified Homotopy Theory and a Whitehead Group for Stratified Spaces”. *Master Thesis arXiv:2102.06971* (2021).

[Waa24a] Lukas Waas. “Combinatorial models for stratified homotopy theory” (2024). arXiv: 2403.06280 [math.AT].

[Waa24b] Lukas Waas. “On the homotopy links of stratified cell complexes” (2024). arXiv: 2403.06272 [math.AT].

[Waa24c] Lukas Waas. “Presenting the topological stratified homotopy hypothesis” (2024). arXiv: 2403.07686 [math.AT].

[WWY24] Lukas Waas, Jon Woolf, and Shoji Yokura. “On stratifications and poset-stratified spaces”. *arXiv preprint arXiv:2407.17690* (2024).

[WJR13] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes. *Spaces of PL manifolds and categories of simple maps*. Vol. 186. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2013, pp. vi+184. DOI: 10.1515/9781400846528. URL: https://doi.org/10.1515/9781400846528.

[Wei94] Shmuel Weinberger. *The topological classification of stratified spaces*. Chicago, 1994.

[WY18] David White and Donald Yau. “Bousfield localization and algebras over colored operads”. *Applied Categorical Structures* 26.1 (2018), pp. 153–203.

[Whi39] J. H. C. Whitehead. “Simplicial spaces, nuclei and m-groups”. *Proceedings of the London mathematical society* 2.1 (1939), pp. 243–327.

[Whi49a] J. H. C. Whitehead. “Combinatorial homotopy. I”. *Bull. Amer. Math. Soc.* 55 (1949), pp. 213–245. DOI: 10.1090/S0002-9904-1949-09175-9.

[Whi49b] J. H. C. Whitehead. “Combinatorial homotopy. II”. *Bull. Amer. Math. Soc.* 55 (1949), pp. 453–496. DOI: 10.1090/S0002-9904-1949-09213-3.

[Whi50] J. H. C. Whitehead. “Simple homotopy types”. *American Journal of Mathematics* 72.1 (1950), pp. 1–57.

[Whi65a] Hassler Whitney. “Local properties of analytic varieties”. *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*. Princeton, NJ, USA: Princeton Univ. Press, 1965, pp. 205–244. DOI: 10.1515/9781400874842-014.

[Whi65b] Hassler Whitney. “Tangents to an analytic variety”. *Ann. of Math. (2)* 81 (1965), pp. 496–549. DOI: 10.2307/1970400. URL: https://doi.org/10.2307/1970400.

[Woo09] Jon Woolf. “The fundamental category of a stratified space”. *J. Homotopy Relat. Struct.* 4.1 (2009), pp. 359–387.

[ZI63] E.C. Zeeman and IHES. *Seminar on Combinatorial Topology*. Seminar on Combinatorial Topology Teile 1-4. Institut des hautes études scientifiques, 1963. URL: https://books.google.de/books?id=2FvvAAAAMAAJ.

[ZC05] Afra Zomorodian and Gunnar Carlsson. "Computing persistent homology". *Discrete Comput. Geom.* 33.2 (2005), pp. 249–274. DOI: 10.1007/s00454-004-1146-y.