RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 17.1.2017

Exercise sheet 9 (Bonus)

The End

To be handed in by Friday, January 27th, 1pm, will be discussed on Monday, January 30th (probably)
Note: Points from this exercise sheet will count as bonus points

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Exercise 1. Recall that $\mathrm{PSL}(2,\mathbb{R})$ can be identified with the group $\mathrm{Isom}_+(\mathbb{H}^2)$ of orientation-preserving isometries of $\mathbb{H}^2$. Show that the classification of isometries as hyperbolic, parabolic and elliptic in terms of the displacement function can be formulated in terms of the trace: Let $I \neq A \in \mathrm{PSL}(2,\mathbb{R})$. Show that:

1. A is hyperbolic iff $|\operatorname{Tr}(A)| > 2$ or equivalently iff A is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \lambda \neq 1$.
2. A is parabolic iff $|\operatorname{Tr}(A)| = 2$ or equivalently iff A is conjugate to $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$.
3. A is elliptic iff $|\operatorname{Tr}(A)| < 2$ or equivalently iff A is conjugate to $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ for some $\theta \neq k \cdot \pi,\ k \in \mathbb{Z}$.

Exercise 2. Let M be a Hadamard manifold. For any subset $A \subset M$, let $\operatorname{conv}(A)$ be the convex hull of A,

$$\operatorname{conv}(A) := \bigcap_{\substack{A \subset B \\ B \text{ convex}}} B.$$

(a) Let $c_1(A)$ be the set of all points that lie on a geodesic segment with endpoints in A, and define $c_{i+1}(A) := c_1(c_i(A))$. Show that

$$\operatorname{conv}(A) = \bigcup_{i \in \mathbb{N}} c_i(A)$$

(b) Let $K \subset M$ be a compact subset. For any $p \in M$, denote by $r(p)$ the minimal radius such that $\overline{B_{r(p)}(p)}$ contains K.
Show that there is a unique point $q \in M$ such that $r(q)$ is minimal. Furthermore, show that we have $q \in \overline{\operatorname{conv}(K)}$.

(c) Let G be a compact subgroup of the isometry group $\mathrm{Isom}(M)$. Show that there exists a fixed point $p \in M$ of G.

Exercise 3. This exercise is about asymptotic rays in the hyperbolic space $\mathbb{H}^n$.

(a) Consider the Poincaré ball model of $\mathbb{H}^n$. Show that two rays are asymptotic if and only if their closures in $\overline{\mathbb{B}^n}$ intersect the boundary of $\mathbb{B}^n$ in the same point.

(b) Consider the hyperboloid model of the hyperbolic space. For a ray γ denote by $H_\gamma \subset \mathbb{R}^{1,n}$ the corresponding 2-dimensional subspace. Show that if two rays γ, γ' are asymptotic, then $H_\gamma \cap H_{\gamma'} \subset \mathbb{R}^{1,n}$ is a line in the light cone.

Exercise 4. Recall the non-inverse Toponogov Theorem:

Theorem. Let M be a complete manifold with curvature bound $\sec_M \geq \kappa$. Let $\triangle(a, b, c) \subset M$ be a geodesic triangle where a and b are minimal. If $\kappa > 0$ assume also that $\ell(c) \leq \frac{\pi}{\sqrt{\kappa}}$. Then there exists a geodesic triangle $\triangle(\overline{a}, \overline{b}, \overline{c})$ in the connected 2-dimensional space M^2_κ of constant curvature κ such that

- $\ell(a) = \ell(\overline{a})$, $\ell(b) = \ell(\overline{b})$, $\ell(c) = \ell(\overline{c})$.
- $\overline{\alpha} \leq \alpha, \overline{\beta} \leq \beta$.

Unless $\kappa > 0$ and some side of the triangle has length $\pi/\sqrt{\kappa}$, the triangle in M^2_κ is uniquely determined.

Let M be a complete manifold with nonnegative sectional curvature $K \geq 0$. Let $\gamma, \sigma : [0, \infty) \longrightarrow M$ be two geodesics with $\gamma(0) = \sigma(0)$. Let γ be a geodesic ray, that is, $d(\gamma(0), \gamma(t)) = t$ for all $t \in [0, \infty)$.

(a) Show that if $\sphericalangle(\gamma'(0), \sigma'(0)) < \frac{1}{2}\pi$, then σ goes to ∞: $\lim_{t\to\infty} d(\sigma(0), \sigma(t)) = \infty$.

(b) Find a counterexample to show that the statement is not true if $\sphericalangle(\gamma'(0), \sigma'(0)) = \frac{1}{2}\pi$.

(c) Find a counterexample to show that the statement is not true if we allow $K < 0$.