RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 13.12.2016

Exercise sheet 7

The Rauch comparison theorem - all smoke and mirrors?

Not to be handed in, will be discussed on December 21st

Exercise 1. Let M be a complete Riemannian manifold with non-positive sectional curvature. Let $p \in M$, $X \in \mathrm{T}_pM$, $Y \in \mathrm{T}_X(\mathrm{T}_pM) \cong \mathrm{T}_pM$. Show that we have

$$|(\mathrm{d}\exp_p)_X(Y)| \geq |Y|.$$

Conclude from this that for any curve γ in T_pM, the inequality

$$L(\gamma) \leq L(\exp_p \circ \gamma)$$

holds.

Exercise 2. Let M be a complete simply connected Riemannian manifold with sectional curvature $K \leq 0$ (but not necessarily constant). Let $A, B, C \in M$ be three arbitrary points and Δ the unique geodesic triangle in M with vertices A, B, C. Let α, β, γ be the angles at the vertices A, B, C and a, b, c be the lengths of the sides opposite the vertices A, B, C. Show that

(a) $a^2 + b^2 - 2ab\cos\gamma \leq c^2$ with strict inequality if $K < 0$.

(b) $\alpha + \beta + \gamma \leq \pi$ with strict inequality if $K < 0$.

Hint: Pick a vertex (e.g. A) and map the triangle into T_AM using $\exp_A^{-1}$. Compare the side lengths with those of the Euclidean triangle in T_AM with the same endpoints. Then use (a) to prove (b).

Exercise 3. Let M be a complete Riemannian manifold with sectional curvature $K \leq k$, where k is a positive constant. Let $p, q \in M$ and let γ_0, γ_1 be two distinct geodesics joining p to q with $l(\gamma_0) \leq l(\gamma_1)$. Assume that γ_0 is homotopic to γ_1 with homotopy H_t, $t \in [0,1]$. Prove that there exists $t_0 \in [0,1]$ such that

$$l(\gamma_0) + l(H_{t_0}) \geq \frac{2\pi}{\sqrt{k}}.$$

Hint: Use Rauch's comparison theorem to show that $\exp_p$ has no critical points in the $\frac{\pi}{\sqrt{k}}$-ball around 0. Next, show that there is a $t_0 \in [0,1]$ such that H_{t_0} cannot be lifted to T_pM. Combine these two facts to show that H_{t_0} leaves the $\frac{\pi}{\sqrt{k}}$-ball around p.