RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 29.11.2016

Exercise sheet 6 - looong

Proper actions and complex projective spaces
or: How I learned to stop worrying and love the Levi-Civita connection

To be handed in by Monday, December 12th, 1pm, will be discussed on December 14th

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Exercise 1. Consider a smooth action of a Lie group G on a manifold M. Prove that

(a) If G is compact, then the action is proper. If M is compact and G is not compact, then the action is not proper.

(b) The action is proper if and only if for all sequences $p_n \in M$ and $g_n \in G$, if p_n and $g_n p_n$ are convergent, then a subsequence of g_n converges.

(c) Let Γ be a (countable) group of diffeomorphisms of a manifold M. Equip Γ with the discrete topology, turning it into a 0–dimensional Lie group. Show that Γ is properly discontinuous (and acts freely) iff $\Gamma \times M \to M$ is a free and proper group action.

Exercise 2. Consider S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$ and define an action of S^1 ond S^{2n+1} by

$$z \cdot (z_1, \ldots, z_{n+1}) = (zz_1, \ldots, zz_{n+1})$$

for $z \in S^1 \subset \mathbb{C}$.

(a) Show that this action is smooth and its orbits are disjoint unit circles in $\mathbb{C}^{n+1}$.

(b) Show that this action is free and proper and that the orbit space S^{2n+1}/S^1 is diffeomorphic to $\mathbb{C}P^n$.
Hint: Consider the composition $\mathbb{C}^{n+1} \setminus \{0\} \to S^{2n+1} \to \mathbb{C}P^n$, where the first map is given by identifying real multiples.

Exercise 3. In this exercise, we derive a formula relating the curvature of image and preimage of a Riemannian submersion that we will apply in the next exercise to calculate the sectional curvatures of $\mathbb{C}P^n$. First, the setup:
A *submersion* is a differentiable map $f : M \to N$ between manifolds of dimensions m and $n \leq m$ such that the differential $\mathrm{d}f$ has rank n at every point. The preimages $f^{-1}(p) = F_p$ are then submanifolds of M of dimension $m - n$, and vectors in $\mathrm{T}M$ tangent to such submanifolds are called *vertical*.
When both manifolds are Riemannian, we obtain a splitting

$$\mathrm{T}_x M = (\mathrm{T}_x M)^v \oplus (\mathrm{T}_x M)^h$$

at every point $x \in M$, where $(\mathrm{T}_x M)^v$ denotes the subspace of vertical directions and $(\mathrm{T}_x M)^h = ((\mathrm{T}_x M)^v)^\perp$ gives us a preferred choice of horizontal directions. A *Riemannian submersion* is a submersion between Riemannian manifolds such that

$$\mathrm{d}f|_{(\mathrm{T}_x M)^h} : (\mathrm{T}_x M)^h \to \mathrm{T}_{f(x)} N$$

is an isometry at every point.
This allows us to define horizontal lifts of vector fields: If $X \in \mathcal{V}(N)$ is a vector field, we denote by $\overline{X} \in \mathcal{V}(M)$ the vector field on M such that $\overline{X}_x \in (\mathrm{T}_x M)^h$ and $\mathrm{d}f_x(\overline{X}) = X_{f(x)}$ (if you feel motivated, you can prove that it is differentiable/smooth if X has this property).

(a) Let $\overline{\nabla}$ and ∇ be the Levi-Civita-connections of M and N, and $X \in \mathcal{V}(N)$. Show that

$$\overline{\nabla}_{\overline{X}} \overline{Y} = \overline{\nabla_X Y} + \frac{1}{2} \left[\overline{X}, \overline{Y}\right]^v,$$

where Z^v denotes the vertical component of the vector Z.
Hint: Use the Koszul formula, orthogonality of the horizontal and vertical components and the equality $[X, Y] = \left[\mathrm{d}f(\overline{X}), \mathrm{d}f(\overline{Y})\right] = \mathrm{d}f\left[\overline{X}, \overline{Y}\right]$ to obtain

$$\left\langle \overline{\nabla}_{\overline{X}} \overline{Y}, \overline{Z} \right\rangle = \langle \nabla_X Y, Z \rangle, \quad 2\left\langle \overline{\nabla}_{\overline{X}} \overline{Y}, V \right\rangle = \left\langle \left[\overline{X}, \overline{Y}\right], V \right\rangle,$$

where V denotes any vertical vector (field).

(b) Let $\overline{K}$ and K denote the sectional curvatures of M and N, and let $X, Y \in \mathcal{V}(N)$ be orthonormal. Prove that

$$K(X, Y) = \overline{K}\left(\overline{X}, \overline{Y}\right) + \frac{3}{4} \left\| \left[\overline{X}, \overline{Y}\right]^v \right\|^2.$$

Hint: Remember that the connection is metric and torsion-free.

Exercise 4. In this exercise, we continue our description of complex projective spaces. Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ denote the projection from before.

(a) Show that the action of S^1 on S^{2n+1} from exercise 2 is by isometries. Conclude that this allows us to define a metric on $\mathbb{C}P^n$ by requiring that the restriction $\mathrm{d}\pi_p : (\ker(\mathrm{d}\pi_p))^\perp \to \mathrm{T}_{\pi(p)}\mathbb{C}P^n$ is an isometry for every $p \in S^{2n+1}$. This metric on $\mathbb{C}P^n$ is called the *Fubini-Study-metric*; it is the unique metric on $\mathbb{C}P^n$ that makes the projection from S^{2n+1} to $\mathbb{C}P^n$ a Riemannian submersion.

(b) For orthonormal vectors $X, Y \in \mathrm{T}_{\pi(p)}\mathbb{C}P^n$, we consider horizontal lifts $\overline{X}, \overline{Y} \in \mathrm{T}_p S^{2n+1}$ and the angle ϕ between $\overline{X}$ and $i\overline{Y}$, where complex multiplication is induced by the embedding $S^{2n+1} \subset \mathbb{C}^{n+1}$:

$$\cos(\phi) = \left\langle \overline{X}, i\overline{Y} \right\rangle$$

Show that we have

$$K(X, Y) = 1 + 3\cos^2\phi.$$

In particular, K varies between 1 and 4, the maximum is attained when X, Y form a real basis of a complex line, and the minimum is attained when Y is orthogonal to the complex line spanned by X.
Hint: To compute $\left\| \left[\overline{X}, \overline{Y}\right]^v \right\|$, you can use once more that the connection is metric and torsion-free.