RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 15.11.2016

Exercise sheet 4 - short

Properly discontinuous groups
or: The one after the long one

Not to be handed in, will be discussed on November 23rd

Exercise 1. This exercise is about the two defining properties (PD 1) and (PD 2) of a properly discontinuous group. Show that:

(a) If Γ is a group of isometries of a Riemannian manifold, then (PD 1) implies (PD 2).

(b) for $M = \mathbb{R}^2 \setminus \{0\}$ the group Γ generated by the diffeomorphism

$$\phi(x, y) = \left(\frac{x}{2}, 2y\right)$$

satisfies (PD 1) but not (PD 2).

Exercise 2. Let $S^3 \subset \mathbb{R}^4$ be the unit sphere. We identify $\mathbb{R}^4$ with $\mathbb{C}^2$ via the mapping

$$(x_1, y_1, x_2, y_2) \mapsto (x_1 + iy_1, x_2 + iy_2)$$

so that $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Let $\phi : S^3 \to S^3$ be given by

$$\phi(z_1, z_2) = \left(\mathrm{e}^{2\pi i/q} z_1, \mathrm{e}^{2\pi ir/q} z_2\right),$$

where q and r are relatively prime integers and $q \geq 2$.

(a) Show that ϕ generates a properly discontinuous group Γ of isometries of S^3 that acts freely.

(b) Show that all geodesics of S^3/Γ are closed, but can have different length.

Remark: The manifold S^3/Γ is called a *lens space*.

Exercise 3. Let (M, g) and (N, h) be Riemannian manifolds, M complete, and let $f \colon M \to N$ be a surjective local diffeomorphism with the additional property that

$$h(\mathrm{d}f_p(v), \mathrm{d}f_p(v)) \geq g(v, v) \qquad \forall p \in M, v \in T_pM.$$

Prove that M has the *path lifting property*, that is, for a smooth curve $\alpha \colon [0, 1] \to N$ and a point $q \in M$ with $f(q) = \alpha(0)$, there exists a curve $\widetilde{\alpha} \colon [0, 1] \to M$ with $\widetilde{\alpha}(0) = q$ and $f \circ \widetilde{\alpha} = \alpha$.